2019

H3 Math

1.	ASRJC	
2.	EJC	
3.	RI	
4.	TJC	

ANDERSON SERAGOON JUNIOR COLLEGE
JC2 Preliminary Examinations 2019
Higher 3

H3 MATHEMATICS 9820/01

Paper 1 20 September 2019

3 hours

Additional Materials: Answer Booklet

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, class, syllabus and paper number on the answer booklet you hand in.
Write in dark blue or black pen on both sides of the answer booklet.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, ensure that you have submitted all your work.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **1** blank page.

Anderson Serangoon Junior College

1 **(a)** Show that $\gcd\left(a+b, a^2-ab+b^2\right)=\gcd\left(a+b, 3ab\right)$ for non-zero integers a and b.
If $\gcd\left(a,b\right)=1,$ show that $\gcd\left(a+b, a^2-ab+b^2\right)$ is either 1 or 3. [7]

(b) Show that $\gcd\left(n!+1, \left(n+1\right)!+1\right)=1$, where n is a positive integer. [4]

2 **(i)** It is given that all the terms in the two sequences $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ are non-zero, where $n \geq 1$. By considering the function $\mathrm{f}(x)=\sum_{r=1}^{n}(p_r x-q_r)^2$, prove that

$$\left(\sum_{r=1}^{n} {p_r}^2\right)\left(\sum_{r=1}^{n} {q_r}^2\right) \geq \left(\sum_{r=1}^{n} p_r \cdot q_r\right)^2,$$

and that equality holds if and only if $\dfrac{q_1}{p_1}=\dfrac{q_2}{p_2}=\cdots=\dfrac{q_n}{p_n}$. [4]

(ii) Use the result in **(i)** to show that if a, b, c and d are positive real numbers, then

$$\left(ad^3+ba^3+cb^3+dc^3\right)\left(ab+bc+cd+da\right) \geq abcd(a+b+c+d)^2.$$ [3]

(iii) Hence show that $\left(ad^3+ba^3+cb^3+dc^3\right)\left(ab+bc+cd+da\right) \geq 16\left(abcd\right)^{\frac{3}{2}}$.
Find a condition relating a, b, c and d for equality to hold, explaining your answer. [4]

3 **(a)** Find $\displaystyle\int \frac{\cos x}{\sin x-\cos^2 x}\,\mathrm{d}x.$ [4]

(b) **(i)** For a twice differentiable function f, if $\mathrm{f}(x)$ satisfies the differential equation $\mathrm{f}''(x)=\lambda \mathrm{f}'(x)\mathrm{f}(x)$ for some real constant λ, express $\displaystyle\int [\mathrm{f}'(x)]^2 \cdot [\mathrm{f}(x)]^N \,\mathrm{d}x$ in terms of $\mathrm{f}(x)$, $\mathrm{f}'(x)$, λ and N, where N is a positive integer. [4]

(ii) Verify that the above result can be applied when $\mathrm{f}(x)=\tan 4x$.

Hence find the exact value of $\displaystyle\int_0^{\frac{\pi}{16}} \frac{\sin^{246} 4x}{\cos^{250} 4x}\,\mathrm{d}x.$ [4]

4 For two integers x and y, let their highest common factor be denoted by $\text{hcf}(x, y)$ and their lowest common multiple by $\text{lcm}(x, y)$.

Given that a and b are positive integers,

(i) show that there exists integers x and y such that $\text{hcf}(x, y) = a$ and $\text{lcm}(x, y) = b$ if and only if a is a divisor of b. [4]

(ii) if a is a divisor of b and r is the number of distinct primes in the prime factorisation of $\frac{b}{a}$, show that the number of distinct ordered pairs of positive integers, (x, y), such that $\text{hcf}(x, y) = a$ and $\text{lcm}(x, y) = b$, is 2^r. [5]

5 Happie Newz Café sells 3 types of muffins, namely chocolate, strawberry and vanilla. The café owner bakes a large number of muffins of each type. All muffins of the same flavour are taken to be identical to one another.

(i) The café owner wants to put 20 muffins in a row on a glass panel so as to entice customers to buy muffins. Find the number of ways she can do so if

(a) there are no restrictions, [1]

(b) no 2 muffins of the same type are adjacent. [2]

(ii) Mr Tay wants some muffins for his 6 students. Find the number of ways that he can do so if

(a) every student gets at least 1 muffin and no student gets 2 or more muffins of the same type, [2]

(b) every student gets exactly 4 muffins and no 2 students get the same combination of muffins. [3]

(iii) At the end of the day, the café owner is left with 6 chocolate, 7 strawberry and 8 vanilla muffins. She wants to throw away these muffins into the 4 different rubbish bins in the vicinity of the café. She wants to ensure that at least 1 muffin is thrown into each bin so as to avoid detection by the members of public that she is wasting food. Find the number of ways she can do so. [5]

6 **(i)** Let $L_{n,k}$ denote the number of ways to distribute n distinct objects into k identical boxes, such that no box is empty and the ordering of objects within each box matters.

(a) Find $L_{n,n}$ and $L_{n,1}$, giving your answer in terms of n whenever possible. [2]

(b) Explain why $L_{n,n-1} = n^2 - n$ for $n \geq 2$. [2]

(c) Show that $L_{n,k} = L_{n-1,k-1} + (n+k-1)L_{n-1,k}$ for $n \geq k \geq 2$. [3]

(ii) Let $l_{n,k}$ denote the number of ways to distribute n distinct objects into k distinct boxes, such that no box is empty and the ordering of objects within each box matters.

(a) Use part **(i)(c)** to show that $l_{n,k} = k\,l_{n-1,k-1} + (n+k-1)l_{n-1,k}$ for $n \geq k \geq 2$. [2]

(b) Explain why $\dfrac{l_{n,k}}{n!} = \dbinom{n-1}{k-1}$. [2]

(c) Hence, by using bijective principle, or otherwise, show that for $n \geq k$,

$$\sum_{j=1}^{k} \binom{k}{j}\binom{n-1}{j-1} = \frac{(n+k-1)!}{n!(k-1)!}$$ [5]

7 Two sequences $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are given by

$$u_1 = 1, \quad v_1 = 1, \quad \text{and}$$
$$u_{n+1} = u_n + 3v_n, \quad v_{n+1} = 2u_n + 7v_n,$$

for positive integers n. The sequence $r_1, r_2, r_3, \ldots$ is such that $r_n = \dfrac{u_n}{v_n}$ for positive integers n.

(i) Use induction to prove that $2{u_n}^2 - 3{v_n}^2 + 6u_nv_n = 5$ for all positive integers n. [4]

It is given that as $n \to \infty$, $v_n \to \infty$ and $r_n \to L$ for some real constant L.

(ii) Using the results in **(i)** or otherwise, show that $L = \dfrac{1}{2}\left(-3+\sqrt{15}\right)$. [3]

(iii) Describe the behaviour of the sequence $r_1, r_2, r_3, \ldots$, justifying your answer by considering $r_{n+1} - r_n$. Hence, show that $\sqrt{15}$ lies within the interval $\left(\dfrac{15}{2r_n+3}, 2r_n+3\right)$ for all positive integers n. [4]

(iv) Deduce that $\dfrac{213}{55} < \sqrt{15} < \dfrac{275}{71}$. [2]

8 For any positive integer x, $\phi(x)$ is defined to be the number of positive integers not exceeding x which are coprime to x. For example, $\phi(4)=2$ since positive integers not exceeding 4 and coprime to 4 are 1 and 3.

(i) Find $\phi(p)$ when p is prime. [1]

(ii) Show that, if p is prime and r is a positive integer, then $\phi(p^r)=p^{r-1}(p-1)$. [2]

It is given that $\phi(mn)=\phi(m)\phi(n)$ for positive integers m and n such that $\gcd(m,n)=1$.

(iii) Show that $\phi(x)$ is even for all $x \geq 3$. [5]

(iv) Find all positive integers x such that $\phi(3x)=\phi(2x)$. [7]

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1

(a) Let $\gcd\left(a+b, a^2-ab+b^2\right)=k$.

Then $k\,|\,(a+b)$ and $k\,|\left(a^2-ab+b^2\right)$

ie $k\,|\,(a+b)^2 \Rightarrow k\,|\left(a^2+2ab+b^2\right)$

$$\Rightarrow k\,|\left(\left(a^2+2ab+b^2\right)-\left(a^2-ab+b^2\right)\right)$$

$$\Rightarrow k\,|\,3ab$$

$$\Rightarrow \gcd\left(a+b, a^2-ab+b^2\right) \le \gcd(a+b, 3ab)$$

Let $\gcd(a+b, 3ab)=h$

Then $h\,|\,(a+b)$ and $h\,|\,3ab$

ie $h\,|\,(a+b)^2 \Rightarrow h\,|\left(a^2+2ab+b^2\right)$

$$\Rightarrow h\,|\left(\left(a^2+2ab+b^2\right)-3ab\right)$$

$$\Rightarrow h\,|\left(a^2-ab+b^2\right)$$

$$\Rightarrow \gcd(a+b, 3ab) \le \gcd\left(a+b, a^2-ab+b^2\right)$$

Hence $\gcd\left(a+b, a^2-ab+b^2\right)=\gcd(a+b, 3ab)$.

Let $s=\gcd\left(a+b, a^2-ab+b^2\right)=\gcd(a+b, 3ab)$

Then $s\,|\,(a+b)$ and $s\,|\,3ab$

If $s\,|\,a$, then $s\,|\left((a+b)-a\right) \Rightarrow s\,|\,b \Rightarrow s=1$ since $\gcd(a,b)=1$.

Similarly, if $s\,|\,b$, then $s\,|\left((a+b)-b\right) \Rightarrow s\,|\,a \Rightarrow s=1$ since $\gcd(a,b)=1$.

If $s \nmid a$ and $s \nmid b$, then from $s\,|\,3ab$ and $\gcd(a,b)=1$, we have $s\,|\,3$.

Therefore s = 1 or 3 ie $\gcd\left(a+b, a^2-ab+b^2\right)=1$ or 3.

(b) Let $\gcd\left(n!+1, (n+1)!+1\right)=k$

$k\,|\,(n!+1) \Rightarrow n!+1=sk$ for some $s \in Z$

$k\,|\left((n+1)!+1\right) \Rightarrow (n+1)!+1=tk$ for some $t \in Z$

$$\Rightarrow (n!)(n+1)+1=tk$$

$$\Rightarrow (sk-1)(n+1)+1=tk$$

$$\Rightarrow skn+sk-n-1+1=tk$$

$$\Rightarrow n=k(sn+s-t)$$

$$\Rightarrow k\,|\,n$$

$$\Rightarrow k\,|\,n!$$

$$\Rightarrow k\,|\left((n!+1)-n!\right)$$

$$\Rightarrow k\,|\,1$$

Therefore k = 1 ie $\gcd\left(n!+1, (n+1)!+1\right)=1$.

2 **(i)**

$$\mathrm{f}(x)=\sum_{r=1}^{n}(p_r x-q_r)^2$$
$$=\sum_{r=1}^{n}(p_r{}^2x^2-2p_rq_rx+q_r{}^2)$$
$$=\left(\sum_{r=1}^{n}p_r{}^2\right)x^2-\left(2\sum_{r=1}^{n}p_rq_r\right)x+\left(\sum_{r=1}^{n}q_r{}^2\right)$$

Since $(p_rx-q_r)^2\geq 0,\ \mathrm{f}(x)\geq 0$ for all *x*. Therefore, discriminant ≤ 0.

$$\left(2\sum_{r=1}^{n}p_rq_r\right)^2-4\left(\sum_{r=1}^{n}p_r{}^2\right)\left(\sum_{r=1}^{n}q_r{}^2\right)\leq 0$$
$$\left(\sum_{r=1}^{n}p_r{}^2\right)\left(\sum_{r=1}^{n}q_r{}^2\right)\geq\left(\sum_{r=1}^{n}p_rq_r\right)^2$$

Equality holds if and only if the graph of $y=\mathrm{f}(x)$ touches the *x*-axis, i.e. $p_rx-q_r=0$ for $r=1,2,\ldots,n \quad\Leftrightarrow x=\frac{q_1}{p_1}=\frac{q_2}{p_2}=\cdots=\frac{q_n}{p_n}$.

(ii) Using part **(i)** result with *n* = 4,

$$\left[\left(\frac{a}{\sqrt{cd}}\right)^2+\left(\frac{b}{\sqrt{ad}}\right)^2+\left(\frac{c}{\sqrt{ab}}\right)^2+\left(\frac{d}{\sqrt{bc}}\right)^2\right]\left[\left(\sqrt{cd}\right)^2+\left(\sqrt{ad}\right)^2+\left(\sqrt{ab}\right)^2+\left(\sqrt{bc}\right)^2\right]\geq(a+b+c+d)^2$$
$$\Leftrightarrow\quad\left(\frac{d^2}{bc}+\frac{a^2}{cd}+\frac{b^2}{ad}+\frac{c^2}{ab}\right)\left(ab+bc+cd+da\right)\geq(a+b+c+d)^2$$
$$\Leftrightarrow\quad\left(ad^3+ba^3+cb^3+dc^3\right)\left(ab+bc+cd+da\right)\geq abcd(a+b+c+d)^2$$

(iii) By AM-GM inequality, $a+b+c+d\geq 4\cdot\left(abcd\right)^{\frac{1}{4}}$.

Therefore, $abcd(a+b+c+d)^2\geq abcd\cdot 16(abcd)^{1/2}=16\left(abcd\right)^{3/2}$, and the result follows.

For equality to hold for the AM-GM inequality, $a=b=c=d$.

For equality to hold for the Cauchy-Schwarz inequality, $\frac{cd}{a}=\frac{ad}{b}=\frac{ab}{c}=\frac{bc}{d}$, which will be satisfied when $a=b=c=d$.

3 **(a)** $\int\frac{\cos x}{\sin x-\cos^2 x}\,\mathrm{d}x=\int\frac{\cos x}{\sin^2 x+\sin x-1}\,\mathrm{d}x$

$$=\int\frac{\cos x}{\left(\sin x+\frac{1}{2}\right)^2-\frac{5}{4}}\,\mathrm{d}x$$
$$=\frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\ln\left|\frac{\sin x+\frac{1}{2}-\frac{\sqrt{5}}{2}}{\sin x+\frac{1}{2}+\frac{\sqrt{5}}{2}}\right|+c=\frac{1}{\sqrt{5}}\ln\frac{\left|2\sin x+1-\sqrt{5}\right|}{2\sin x+1+\sqrt{5}}+c$$

(b)

$$\int [f'(x)]^2[f(x)]^N \,dx = \int f'(x)\cdot f'(x)[f(x)]^N \,dx$$

$$= \frac{1}{N+1}[f(x)]^{N+1}f'(x) - \frac{1}{N+1}\int\left([f(x)]^{N+1}f''(x)\right)dx \quad \text{[integrate by parts]}$$

$$= \frac{1}{N+1}[f(x)]^{N+1}f'(x) - \frac{1}{N+1}\int\left(\lambda f'(x)[f(x)]^{N+2}\right)dx$$

$$= \frac{1}{N+1}[f(x)]^{N+1}f'(x) - \frac{\lambda}{(N+1)(N+3)}[f(x)]^{N+3} + c$$

$f(x) = \tan 4x$, $f'(x) = 4\sec^2 4x$, $f''(x) = 32\sec^2 4x\tan 4x = 8f'(x)f(x)$. So $\lambda = 8$.

$$\int_0^{\frac{\pi}{16}} \frac{\sin^{246} 4x}{\cos^{250} 4x}\,dx = \frac{1}{16}\int_0^{\frac{\pi}{16}} [4\sec^2 4x]^2(\tan 4x)^{246}\,dx$$

$$= \frac{1}{16}\left[\frac{1}{247}\tan^{247} 4x\cdot 4\sec^2 4x - \frac{8\tan^{249} 4x}{247\times 249}\right]_0^{\frac{\pi}{16}}$$

$$= \frac{1}{16}\left[\frac{1}{247}(1)(4)(2) - \frac{8}{247\times 249}\right]$$

$$= \frac{1}{2}\left(\frac{249-1}{247\times 249}\right) = \frac{124}{61503}$$

4 $(\Leftarrow)$ If $a \mid b \Rightarrow b = ak$ for some $k \in Z$.

Let $x = a$ and $y = b = ak$

Then $hcf(x,y) = (a, ak) = a$ and $lcm(x,y) = lcm(a, ak) = ak = b$.

$(\Rightarrow)$ There exists integers x, y such that $hcf(x,y) = a$ and $lcm(x,y) = b$.

Then $x = as$ and $y = at$ where $\gcd(s,t) = 1$

$lcm(x,y) = lcm(as, at) = b \Rightarrow b = ast$

Therefore a is a divisor of b.

Let $\frac{b}{a} = p_1^{k_1}p_2^{k_2}.....p_r^{k_r}$ where $p_1, p_2, ...p_r$ are the r distinct prime factors of $\frac{b}{a}$ and $k_1, k_2, ..., k_r$ are non-negative integers. Then $b = ap_1^{k_1}p_2^{k_2}.....p_r^{k_r}$

$hcf(x,y) = a$ and $lcm(x,y) = b \Rightarrow x = as,\quad y = at,\quad b = ast$ where $\gcd(s,t) = 1$.

$ast = ap_1^{k_1}p_2^{k_2}.....p_r^{k_r}$

$st = p_1^{k_1}p_2^{k_2}.....p_r^{k_r}$

Since $\gcd(s,t) = 1$, then s is the product of some prime factors each of the form $p_i^{k_i}$ where t is the product of the remaining prime factors of the form $p_j^{k_j}$

Each $p_i^{k_i}$ may or may not be a factor of s which means that there are two choices for each $p_i^{k_i}$ with regard to s. Thus the total number of possible integers s is 2^r with t taking the leftovers.

Hence the number of distinct ordered pairs of positive integers (x, y) such that $hcf(x,y) = a$ and $lcm(x,y) = b$ is 2^r.

5

(i)(a) $3^{20} = 3,486,784,401$

(i)(b) $3 \times 2^{19} = 1,572,864$

(ii)(a) No. of ways to distribute the muffins to a student

$$= \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 7 \text{ (OR } 2^3 - 1\text{)}$$

Answer $= 7^6 = 117,649$

(ii)(b) No. of ways to distribute 4 identical objects (muffins) into 3 distinct boxes (flavours)

$$= \binom{4+2}{2} = 15 \text{ (OR, } 3 + 2 \times \binom{3}{2} + \binom{3}{2} + 3 = 15\text{)}$$

Answer $= 15 \times 14 \times \cdots \times 10 = 3,603,600$ (OR, ${}^{15}P_6$)

(iii) Total number of ways to throw away the muffins $= \binom{6+3}{3}\binom{7+3}{3}\binom{8+3}{3} = 1,663,200$

Let A_i be the set comprising of all possible combinations the muffins can be disposed into the bins, such that auntie did not throw any muffin into the i^{th} bin.

$$|A_i| = \binom{6+2}{2}\binom{7+2}{2}\binom{8+2}{2} = 46,110$$

$$|A_i \cup A_j| = \binom{6+1}{1}\binom{7+1}{1}\binom{8+1}{1} = 504 \text{ (OR } 7 \times 8 \times 9\text{)}$$

$$|A_i \cup A_j \cup A_k| = \binom{6+0}{0}\binom{7+0}{0}\binom{8+0}{0} = 1$$

$$\text{Answer } = 1,663,200 - \binom{4}{1}|A_i| + \binom{4}{2}|A_i \cup A_j| - \binom{4}{3}|A_i \cup A_j \cup A_k|$$

$$= 1,484,780$$

6

(i)(a) $L_{n,n} = 1$

$L_{n,1} = n!$

(i)(b) First, choose 2 objects from the *n* distinct objects to be “together” in a box. This gives us $\binom{n}{2}$ ways of doing do.

Next, we can give ordering to the 2 chosen objects within that particular box. By multiplicative principle, we have $\binom{n}{2} \times 2! = n^2 - n$ ways

(ii)(c)

Case 1: Object A is alone in a box

We can then distribute the remaining $n-1$ objects into the remaining $k-1$ boxes. This gives us $L_{n-1,k-1}$ ways of doing so.

Case 2: Object A is not alone

First, we set aside this object A and arrange the remaining $n-1$ books into the *k* boxes. We have $L_{n-1,k}$ ways of doing so.

Next, we want to slot this object A amongst the $n-1$ books in the *k* boxes. Note that we can put this object A on top (to the left) of any of the $n-1$ objects or we can also put object A right at the bottom (or rightmost) of any of the *k* objects. So, total of $(n+k-1)$ ways of slotting in this object A.

By multiplicative principle, we have $(n+k-1)L_{n-1,k}$ for case 2.

Note that case 1 and case 2 are mutually exclusive. By addition principle, we conclude that

$L_{n,k} = L_{n-1,k-1} + (n+k-1)L_{n-1,k}$

(ii)(a)

Where ordering within each box matters, we have $l_{n,k} = (k!)L_{n,k}$

$$\text{ie. } L_{n,k} = \frac{l_{n,k}}{k!}$$

From (iii),

$L_{n,k} = L_{n-1,k-1} + (n+k-1)L_{n-1,k}$

$L_{n,k} = L_{n-1,k-1} + (n+k-1)L_{n-1,k}$

$\frac{l_{n,k}}{k!} = \frac{l_{n-1,k-1}}{(k-1)!} + (n+k-1)\frac{I_{n-1,k}}{k!}$

Therefore, $l_{n,k} = k\,l_{n-1,k-1} + (n+k-1)l_{n-1,k}$, $n \geq k \geq 2$

(ii)(b) Dividing by $n!$, we remove the permutations of distinction of objects as well as the ordering of objects within each box.

Hence $\frac{l_{n,k}}{n!}$ gives the number of ways to distribute *n* identical objects into *k* distinct boxes, such that no box is empty.

To ensure no boxes are empty, we first place one object into each of the *k* boxes. We then distribute the remaining $(n-k)$ objects, this is akin to arrangement $(n-k)$ zeros and $(k-1)$ ones, thus giving $\binom{(n-k)+(k-1)}{k-1}$ ways.

Therefore, $\frac{l_{n,k}}{n!} = \binom{n-1}{k-1}$.

(ii)(c) Let *A* be the set of all possible permutations of *n* distinct elements into *k* ordered partitions, where partitions may be empty.

For the case where $n=5$ and $k=3$, $\left((2,1),(3),(5,4)\right)$ means box 1, 2 and 3 contain objects 1 and 2, object 3, objects 4 and 5 respectively. In terms of arrangements within each boxes, in box 1, object 2 comes first before object 1; whereas in box 3, object 5 comes first before object 4.

We can see that $\left|A\right| = \sum_{j=1}^{k}\binom{k}{j} l_{n,j} = n!\sum_{j=1}^{k}\binom{k}{j}\binom{n-1}{j-1}$ which counts the number of ways to distribute n distinct objects into k distinct boxes such that the ordering of objects within each box is important and that boxes may be empty.

Let B be all possible permutations of the $(n+k-1)$ numbers involving n distinct integers from $\{1,2,\cdots,n\}$, as well as $(k-1)$ zeros. We can see that there exists a bijection f between A and B. In the case where $n=5$ and $k=3$,

$$\mathrm{f}\left((2,1),(3),(5,4)\right) = 2103054$$
$$\mathrm{f}\left((1,3,5,4),(),(2)\right) = 1354002$$

Now, arranging $\{1,2,\cdots,n;\underbrace{0,0,\cdots,0}_{k\ \text{zeros}}\}$ gives us $\left|B\right| = \frac{(n+k-1)!}{(k-1)!}$.

Since $\left|A\right| = \left|B\right|$, we have $n!\sum_{j=1}^{k}\binom{k}{j}\binom{n-1}{j-1} = \frac{(n+k-1)!}{(k-1)!}$.

Therefore, $\sum_{j=1}^{k}\binom{k}{j}\binom{n-1}{j-1} = \frac{(n+k-1)!}{n!(k-1)!}$.

7

(i) Let P_n be the statement $2u_n{}^2 - 3v_n{}^2 + 6u_nv_n = 5$ for $n \in \mathbb{Z}^+$.

For n = 1, $\text{LHS} = 2u_1{}^2 - 3u_1{}^2 + 6u_1v_1 = 2(1) - 3(1) + 6 = 5 = \text{RHS}$. So P_1 is true.

Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e. $2u_k{}^2 - 3v_k{}^2 + 6u_kv_k = 5$.

$$\begin{aligned}
&\text{LHS of } P_{k+1} = 2u_{k+1}{}^2 - 3v_{k+1}{}^2 + 6u_{k+1}v_{k+1} \\
&= 2\left(u_k{}^2 + 6u_kv_k + 9v_k{}^2\right) - 3\left(4u_k{}^2 + 28u_kv_k + 49v_k{}^2\right) + 6\left(2u_k{}^2 + 13u_kv_k + 21v_k{}^2\right) \\
&= 2u_k{}^2 - 3v_k{}^2 + 6u_kv_k \\
&= 5. \quad \text{So } P_k \text{ is true } \Rightarrow P_{k+1} \text{ is true.}
\end{aligned}$$

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by the Principle of Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

[**(ii)** For $2u_n{}^2 - 3v_n{}^2 + 6u_nv_n = 5$, dividing throughout by $v_n{}^2$ yields $2r_n{}^2 - 3 + 6r_n = \frac{5}{v_n{}^2}$.

Since $\lim_{n\to\infty}\left(v_n\right) = \infty$ and $\lim_{n\to\infty}\left(r_n\right) = L$,

$$2L^2 - 3 + 6L = 0 \quad \Rightarrow \quad L = \frac{1}{2}\left(-3+\sqrt{15}\right) \text{ or } \frac{1}{2}\left(-3-\sqrt{15}\right).$$

However, as $r_n > 0$, its limit L must be non-negative. Therefore, $L = \frac{1}{2}\left(-3+\sqrt{15}\right)$.

(iii) $$r_{n+1}-r_n=\frac{u_n+3v_n}{2u_n+7v_n}-\frac{u_n}{v_n}$$
$$=\frac{u_nv_n+3v_n{}^2-2u_n{}^2-7u_nv_n}{v_n(2u_n+7v_n)}$$
$$=-\frac{2u_n+6u_nv_n-3v_n{}^2}{v_n(2u_n+7v_n)}$$
$$=-\frac{5}{v_n(2u_n+7v_n)}<0 \quad \text{(since } u_n,\ v_n>0\text{)}.$$

So the sequence $\{r_n\}$ is strictly decreasing and tends to the limit $\frac{1}{2}\left(-3+\sqrt{15}\right)$.

$\therefore\ r_n>\frac{1}{2}\left(-3+\sqrt{15}\right)$

$\Rightarrow 2r_n+3>\sqrt{15}$ ---- (2)

$\Rightarrow \frac{1}{2r_n+3}<\frac{1}{\sqrt{15}}$

$\Rightarrow \frac{15}{2r_n+3}<\sqrt{15}$ ---- (3)

By taking intersection of (2) and (3), we have the result.

(iv) $r_3=\frac{u_3}{v_3}=\frac{31}{71}$. So $\frac{15}{2r_3+3}<\sqrt{15}<2r_3+3 \quad \Rightarrow \frac{213}{55}<\sqrt{15}<\frac{275}{71}$.

8

(i) $\phi(p)=p-1$ since all the numbers 1, 2, 3, …, $p-1$ are all relatively prime to the prime number *p*.

(ii) The only positive integers not exceeding p^r which are not coprime to p^r are $1p,\ 2p,\ 3p,\ ...,\ p^{r-1}, p^r$. As there are p^{r-1} of such integers, $\phi\left(p^r\right)=p^r-p^{r-1}=p^{r-1}\left(p-1\right)$.

(iii) Case (1): x is a power of 2 only ie $x=2^r, r\geq 2$.

Then $\phi\left(x\right)=\phi\left(2^r\right)=2^{r-1}\left(2-1\right)=2^{r-1}$ which is even.

Case (2) *x* is not a power of 2 only, then it has an odd prime *p* as a factor. Thus $x=p^rq$ where $r\geq 1$ and $\gcd\left(p^r,q\right)=1$.

By the above result,

$\phi\left(x\right)=\phi\left(p^rq\right)=\phi\left(p^r\right)\phi\left(q\right)=p^{r-1}\left(p-1\right)\phi\left(q\right)$

Since p is an odd prime, then $2\,|\left(p-1\right)$ ie $\phi\left(x\right)$ is even

Hence $\phi\left(x\right)$ is even for all $x\geq 3$.

(iv) Consider $n=2^t3^s m$ where t and s are non-negative integers and m is a positive integer which does not have 2 or 3 as factors, ie $\gcd(6,m)=1$

$\phi(3x)=\phi(2x)$

$\phi\left(3.2^t3^s m\right)=\phi\left(2.2^t3^s m\right)$

$\phi\left(2^t3^{s+1}m\right)=\phi\left(2^{t+1}3^s m\right)$

$\phi\left(2^t3^{s+1}\right)\phi(m)=\phi\left(2^{t+1}3^s\right)\phi(m)$ since m is coprime to both 2^t3^{s+1} and $2^{t+1}3^s$

$\phi\left(2^t3^{s+1}\right)=\phi\left(2^{t+1}3^s\right)$

$\phi\left(2^t\right)\phi\left(3^{s+1}\right)=\phi\left(2^{t+1}\right)\phi\left(3^s\right)$ since powers of 2's and powers of 3's are coprime

<u>Case 1: For $s\geq 1, t\geq 1,$</u>

$\phi\left(2^t\right)\phi\left(3^{s+1}\right)=\phi\left(2^{t+1}\right)\phi\left(3^s\right)$

$2^{t-1}(2-1)3^s(3-1)=2^t(2-1)3^{s-1}(3-1)$

$2^{t-1}3^s=2^t3^{s-1}$

$3=2 \quad (\textit{no solution})$

<u>Case 2: For $s\geq 1, t=0,$</u>

$\phi\left(2^t\right)\phi\left(3^{s+1}\right)=\phi\left(2^{t+1}\right)\phi\left(3^s\right)$

$\phi(1)\phi\left(3^{s+1}\right)=\phi(2)\phi\left(3^s\right)$

$3^s(3-1)=3^{s-1}(3-1)$

$3=1 \quad (\textit{no solution})$

<u>Case 3: For $s=0, t\geq 1,$</u>

$\phi\left(2^t\right)\phi\left(3^{s+1}\right)=\phi\left(2^{t+1}\right)\phi\left(3^s\right)$

$\phi\left(2^t\right)\phi(3)=\phi\left(2^{t+1}\right)\phi(1)$

$2^{t-1}(2-1)(2)=2^t(2-1)(1)$

$2^t=2^t$

ie true for every $t\geq 1$ and $s=0$ and positive integers m where $\gcd(6,m)=1$.

<u>Case 4: For $s=0, t=0,$</u>

$\phi\left(2^t\right)\phi\left(3^{s+1}\right)=\phi\left(2^{t+1}\right)\phi\left(3^s\right)$

$\phi(1)\phi(3)=\phi(2)\phi(1)$

$(1)(2)=(1)(1)$

$2=1 \quad (\textit{no solution})$

Therefore the positive values of x are all even numbers which do not have 3 as a factor.

EUNOIA JUNIOR COLLEGE
JC2 Preliminary Examination 2019
General Certificate of Education Advanced Level
Higher 3

MATHEMATICS 9820/01

24 September 2019

Paper 1 [100 marks]

3 hours

Additional Materials: Answer Booklet
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

1 **(i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{(c+t)x}{1-t^2}, \qquad \textbf{(A)}$$

where c is a constant. [5]

(ii) Make the substitution $y=tx$ in the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{ax+by}{bx+ay}, \qquad \textbf{(B)}$$

where a and b are constants and $a\neq 0$, and show that the resulting equation is equivalent to equation **(A)**. [3]

(iii) Hence show that equation **(B)** has solutions of the form

$$|x-y|^{b+a}=D|x+y|^{b-a},$$

where D is an arbitrary constant. [3]

2 **(i)** Show that $\int_a^b \mathrm{f}(x)\,\mathrm{d}x=\int_a^b \mathrm{f}(a+b-x)\,\mathrm{d}x$ for any function f which is continuous on the domain $[a,b]$. [1]

(ii) Using the result in **(i)**, show that $\int_0^{\frac{\pi}{4}}\ln(1+\tan\theta)\,\mathrm{d}\theta=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2}{1+\tan\theta}\right)\mathrm{d}\theta.$ [2]

(iii) Hence, by using an appropriate substitution, find the exact value of $\int_0^1\frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x$. [3]

(iv) Use the result in **(i)** to also find the exact value of $\int_0^{\frac{\pi}{4}}\frac{\cos x}{\cos(x-\frac{\pi}{4})}\,\mathrm{d}x$. [3]

3 **(a)** Let x, y, z be real numbers such that $0 < x < y < z$. Show that x^2, y^2, z^2 are three consecutive terms of an arithmetic sequence if and only if $\frac{1}{y+z}$, $\frac{1}{x+z}$, $\frac{1}{x+y}$ are three consecutive terms of another arithmetic sequence. [3]

(b) An infinite sequence of integers, a_n, $n \in \mathbb{Z}^+$, is defined by $a_1 = a_2 = 1$ and

$$a_{n+1} = \frac{a_n^2 + 2}{a_{n-1}},\ n \geq 2.$$

(i) Show that $\frac{a_{n+2} + a_n}{a_{n+1}} = \frac{a_{n+1} + a_{n-1}}{a_n}$ for [2]

(ii) Hence or otherwise, show that $a_{n+1} = 4a_n - a_{n-1}$ and explain why a_n is an odd integer for all $n \in \mathbb{Z}^+$. [2]

(c) Prove that there is no arithmetic progression which has $1, \sqrt{2}$, $\sqrt{3}$ among its terms. [5]

4 Let n be a positive integer greater than 2 and x_0, x_1, x_2, ..., x_n be real numbers with $x_0 = 1$.

(i) By using the Cauchy-Schwarz Inequality or otherwise, show that

$$\sum_{i=1}^{n} (x_{i-1} - x_i)^2 + x_n^2 \geq \frac{1}{n+1}.$$ [3]

(ii) Given that $\sum_{i=1}^{n} (x_{i-1} - x_i)^2 + x_n^2 = \frac{1}{n+1}$, find x_k for $0 \leq k \leq n$, $k \in \mathbb{Z}$, leaving your answer in terms of n and k. [3]

5 A factory supervisor intends to distribute work to his three subordinates. He has 24 indistinguishable cards each representing one unit of work, and he intends to distribute them into three different colored boxes, red, green and blue, representing each of his subordinates.

(i) Find the number of ways he can distribute the cards into the boxes without restriction. [1]

The supervisor wishes to split the work relatively evenly amongst his three subordinates so that each subordinate has at most 10 units of work.

(ii) Find the number of ways the supervisor can distribute the workload. [3]

The red box represents the workload of Sue, who is pregnant, and the supervisor will like to reduce her workload such that she will only have a workload at most half of any of her other two colleagues. For example, if Sue is given 3 units of work, then her colleagues must have at least 6 units of work each.

(iii) Find the number of ways the supervisor can distribute the workload. [3]

Peter is given r units of work. He places the r indistinguishable cards into n indistinguishable boxes for him to randomly choose one box for each of the subsequent days.

Let the number of ways for him to do so be denoted by $P(r,\ n)$.

(iv) State the recurrence relation between $P(r,\ n)$, $P(r-1,\ n-1)$ and $P(r-n,\ n)$. [1]

(v) Using the result in **(iv)** or otherwise, show that $P(r,\ n)=\sum_{k=1}^{n}P(r-n,\ k)$ for $r>n\geq 1$. [2]

(vi) Using **(iv)** or **(v)**, find the value of $P(10,\ 3)$. [2]

6 You have an unlimited supply of 1×1, 1×2 and 2×2 tiles. Tiles of the same size are indistinguishable.

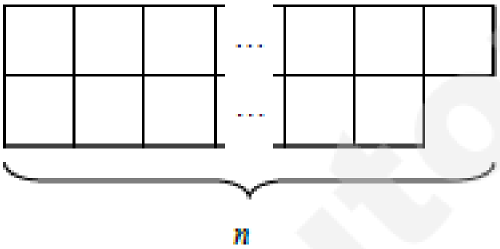
(i) Let T_n is the number of ways of tiling a $1\times n$ path.

(a) State the value of T_1 and T_2. [1]

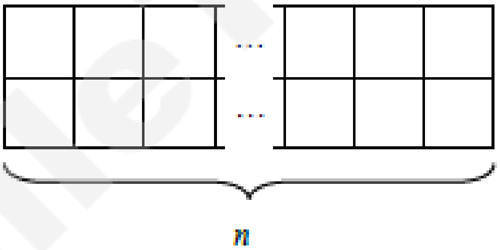
(b) Write down an appropriate recurrence relation between T_{n+2}, T_{n+1} and T_n. [1]

Consider the tilings of a $2\times n$ path. (The 1×2 tiles can be rotated in the tilings.)

Let P_n be the number of tilings of



Let Q_n be the number of tilings of



(ii) Show that $P_{n+1} = P_n + Q_n$ for $n \geq 1$. Explain your reasoning clearly. [2]

(iii) Show that $Q_{n+1} = 2P_{n+1} + 2Q_{n-1}$ for $n \geq 2$. Explain your reasoning clearly. [4]

(iv) Use (ii) and (iii) to show that $P_{n+2} + 2P_{n-1} = 3P_{n+1} + 2P_n$ for $n \geq 2$. [2]

It is given that the solution to the above recurrence relation is

$$P_n = -\frac{(-1)^n}{7} + \frac{1+2\sqrt{2}}{14}(2+\sqrt{2})^n + \frac{1-2\sqrt{2}}{14}(2-\sqrt{2})^n.$$

(v) Find the number of distinct ways of tiling a $2\times n$ path. [2]

7 **(a)** Show that if $7\left|(a^2+b^2)\right.$ then $7|a$ and $7|b$ where *a* and *b* are positive integers. [3]

(b) Given that *n* is an integer greater than 1, by considering the cases when *n* is even and *n* is odd, show that n^4+4^n cannot be a prime number. [4]

(c) Show that for every positive integer *n*, $16^n+10n-1$ is divisible by 25. [5]

8 Let $L_1(x)=\dfrac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$.

(i) State the values of $L_1(x_1)$, $L_1(x_2)$ and $L_1(x_3)$. [2]

(ii) Hence or otherwise, obtain expressions of quadratic polynomials L_2 and L_3 such that $L_i(x_i)=1$, and $L_i(x_j)=0$, $i\neq j$, $i=2, 3$, and $j=1, 2, 3$. [2]

It is known that if *p* is a polynomial of degree *m*, and $p(x_i)=0$, for $i=1, 2, \ldots, m+1$, where $x_1<x_2<x_3<\ldots<x_n<x_{m+1}$, then $p(x)=0$, for all $x\in\mathbb{R}$.

(iii) Suppose there are 3 distinct values x_1, x_2, x_3 and any real values y_1, y_2, y_3. Using the above result, explain why there is at most one quadratic polynomial P such that $P(x_i)=y_i$ for $i=1, 2, 3$. [3]

(iv) Obtain the quadratic polynomial P as a combination of y_i and L_i such that $P(x_i)=y_i$ for $i=1, 2, 3$, and justify your answer. [2]

(v) Let $x_i=i$ for $i=1, 2, 3$, and $\{y_1, y_2, y_3\}=\{1,2,3\}$. Find all possible quadratic polynomials Q satisfying $Q(x_i)=y_i$, $y_i\neq i$, for $i=1,2,3$, with appropriate justification to be provided. [You need not simplify your answer] [4]

9 Let $S(n)$ denote the sum of all divisors of the number *n*. For example,

$$S(6)=1+2+3+6=12\,.$$

(i) Find

(a) $S(p)$ in terms of *p*, [1]

(b) $S(pq)$ in terms of *p* and *q*, [1]

(c) $S(p^m q^n)$ in terms of *p*, *q*, *m* and *n*, [2]

where *p*, *q* are distinct prime numbers and *m*, *n* are positive integers.

Aliquot divisors of a number are divisors of the number excluding the number itself (i.e. the sum of all proper divisors of the number). For example, the aliquot divisors of 6 are 1, 2 and 3 respectively.

Two numbers are called amicable if each equals the sum of the aliquot divisors of the other, and the two number form an amicable pair.

(ii) Verify that 220 and 284 is an amicable pair. [2]

Assume that two amicable numbers, *M* and *N*, has the form $M=apq$ and $N=ar$, where *p*, *q*, *r* are distinct prime numbers, *a* (not necessarily prime) is the common divisor of *M* and *N*, and *a* is co-prime to *p*, *q* and *r*.

(iii) Show that $a(pq+r)=(p+1)(q+1)S(a)$. [2]

(iv) Express $r+1$ in terms of *p* and *q* only, and use **(iii)** to show that

$$(p+1)(q+1)\left[2a-S(a)\right]=a(p+q+2).$$ [3]

The equation in **(iv)** can be further manipulated to the following equality:

$$\left(p+1-\frac{a}{2a-S(a)}\right)\left(q+1-\frac{a}{2a-S(a)}\right)=\left(\frac{a}{2a-S(a)}\right)^2$$

(v) Find the only possible solution for *p* and *q* for $a=4$, and hence find *M* and *N*. [2]

End of Paper

2019 Year 6 H3 Math Prelim Exam Solutions

Qn	Solution	Remarks
1(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{(c+t)x}{1-t^2}$ $\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{(c+t)}{1-t^2}$ $\int\frac{1}{x}\ \mathrm{d}x=\int\frac{c}{1-t^2}-\frac{1}{2}\frac{-2t}{1-t^2}\ \mathrm{d}t$ $\ln\|x\|-d=\frac{c}{2}\ln\left\|\frac{1+t}{1-t}\right\|-\frac{1}{2}\ln\left\|1-t^2\right\|$ $\ln\left\|\frac{x}{k}\right\|=\ln\left\|\frac{1+t}{1-t}\right\|^{\frac{c}{2}}-\ln\left\|(1+t)(1-t)\right\|^{\frac{1}{2}}$ $\left\|\frac{x}{k}\right\|=\frac{\|1+t\|^{\frac{c-1}{2}}}{\|1-t\|^{\frac{c+1}{2}}}$ $x=\frac{p\|1+t\|^{\frac{c-1}{2}}}{\|1+t\|^{\frac{c+1}{2}}}$, where p is an arbitrary constant.	
(ii)	$y=tx$ $\frac{\mathrm{d}y}{\mathrm{d}x}=t+x\frac{\mathrm{d}t}{\mathrm{d}x}$ $t+x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{ax+b(tx)}{bx+a(tx)}$ $t+x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{a+b(t)}{b+at}$ $x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{a+b(t)}{b+at}-t$ $x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{a-at^2}{b+at}$ $\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{b+at}{a(1-t^2)}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\left(\frac{b}{a}+t\right)x}{1-t^2}$ and the resulting equation is equivalent to equation (A) with $c=\frac{b}{a}$.	

1(iii)	$\left\lvert\frac{x}{k}\right\rvert=\frac{\left\lvert1+\frac{y}{x}\right\rvert^{\frac{\frac{b}{a}-1}{2}}}{\left\lvert1-\frac{y}{x}\right\rvert^{\frac{\frac{b}{a}+1}{2}}}$ $\lvert x\rvert\lvert x-y\rvert^{\frac{\frac{b}{a}+1}{2}}\lvert x\rvert^{-\frac{\frac{b}{a}+1}{2}}=\lvert k\rvert\lvert x+y\rvert^{\frac{\frac{b}{a}-1}{2}}\lvert x\rvert^{-\frac{\frac{b}{a}-1}{2}}$ $\lvert x\rvert\lvert x-y\rvert^{\frac{\frac{b}{a}+1}{2}}=\lvert k\rvert\lvert x+y\rvert^{\frac{\frac{b}{a}-1}{2}}\lvert x\rvert^{\frac{\frac{b}{a}+1}{2}-\frac{\frac{b}{a}-1}{2}}$ $\lvert x\rvert\lvert x-y\rvert^{\frac{b+a}{2a}}=\lvert k\rvert\lvert x+y\rvert^{\frac{b-a}{2a}}\lvert x\rvert$ $\lvert x\rvert\lvert x-y\rvert^{b+a}=\lvert k\rvert^{2a}\lvert x+y\rvert^{b-a}\lvert x\rvert$ $\lvert x-y\rvert^{b+a}=D\lvert x+y\rvert^{b-a}$	
		Total Marks: 11

Qn	Solution	Remarks
2(i)	Using the substitution $y=a+b-x,$ we have $\int_a^b \mathrm{f}(x)\ \mathrm{d}x=\int_b^a -\mathrm{f}(a+b-y)\ \mathrm{d}y$ $=\int_a^b \mathrm{f}(a+b-y)\ \mathrm{d}y$ $=\int_a^b \mathrm{f}(a+b-x)\ \mathrm{d}x$ (shown)	
(ii)	$\int_0^{\frac{\pi}{4}} \ln\left(1+\tan\theta\right)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\tan\left(\frac{\pi}{4}\right)-\tan\theta}{1+\tan\left(\frac{\pi}{4}\right)\tan\theta}\right)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right)\ \mathrm{d}\theta$ (shown)	
(iii)	$\int_0^1 \frac{\ln\left(1+x\right)}{1+x^2}\ \mathrm{d}x=\int_0^{\frac{\pi}{4}} \ln\left(1+\tan\theta\right)\ \mathrm{d}\theta$ by using the substitution $x=\tan\theta.$ Also, we have $\int_0^1 \frac{\ln\left(1+x\right)}{1+x^2}\ \mathrm{d}x=\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right)\ \mathrm{d}\theta.$ Summing up both equalities, we have $2\int_0^1 \frac{\ln\left(1+x\right)}{1+x^2}\ \mathrm{d}x=\int_0^{\frac{\pi}{4}} \ln\left(1+\tan\theta\right)\ \mathrm{d}\theta+\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} \ln\left(1+\tan\theta\right)+\ln 2-\ln\left(1+\tan\theta\right)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} \ln 2\ \mathrm{d}\theta=\frac{\pi}{4}\ln 2$ $\therefore \int_0^1 \frac{\ln\left(1+x\right)}{1+x^2}\ \mathrm{d}x=\frac{\pi}{8}\ln 2$	

2 **(iv)**	$\int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos\left(x-\frac{\pi}{4}\right)}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{\cos\left(\frac{\pi}{4}-x\right)}{\cos\left(\left(\frac{\pi}{4}-x\right)-\frac{\pi}{4}\right)}\,\mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \frac{\cos\left(\frac{\pi}{4}\right)\cos x+\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}{\cos(-x)}\,\mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \frac{\cos\left(\frac{\pi}{4}\right)\cos x+\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}{\cos(-x)}\,\mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x}{\cos x}\,\mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\tan x\,\mathrm{d}x$ $= \left[\frac{x}{\sqrt{2}}+\frac{1}{\sqrt{2}}\ln\lvert\sec x\rvert\right]_0^{\frac{\pi}{4}}$ $= \left(\frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}\ln\left\lvert\sec\left(\frac{\pi}{4}\right)\right\rvert\right)-0$ $= \frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}\ln\left\lvert\sqrt{2}\right\rvert$ $= \frac{\pi}{4\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln 2$	
		Total Marks: 9

Qn	Solution	Remarks
3(a)	$\dfrac{1}{y+z},\ \dfrac{1}{x+z},\ \dfrac{1}{x+y}$ are three consecutive numbers of an AP $\Leftrightarrow \dfrac{1}{x+z}-\dfrac{1}{y+z}=\dfrac{1}{x+y}-\dfrac{1}{x+z}$ $\Leftrightarrow \dfrac{(y+z)-(x+z)}{(x+z)(y+z)}=\dfrac{(x+z)-(x+y)}{(x+y)(x+z)}$ $\Leftrightarrow \dfrac{y-x}{(x+z)(y+z)}=\dfrac{z-y}{(x+y)(x+z)}$ $\Leftrightarrow \dfrac{(y-x)(x+y)}{(x+y)(x+z)(y+z)}=\dfrac{(z-y)(y+z)}{(x+y)(x+z)(y+z)}$ $\Leftrightarrow \dfrac{y^2-x^2}{(x+y)(x+z)(y+z)}=\dfrac{z^2-y^2}{(x+y)(x+z)(y+z)}$ $\Leftrightarrow y^2-x^2=z^2-y^2$ $\Leftrightarrow x^2,\ y^2,\ z^2$ are three consecutive numbers of an AP	
(b) **(i)**	$\dfrac{a_{n+2}+a_n}{a_{n+1}}=\dfrac{\dfrac{a_{n+1}^2+2}{a_n}+a_n}{a_{n+1}}$ $=\dfrac{a_{n+1}^2+2+a_n^2}{a_{n+1}a_n}$ $=\dfrac{a_{n+1}+\dfrac{a_n^2+2}{a_{n+1}}}{a_n}$ $=\dfrac{a_{n+1}+a_{n-1}}{a_n},\ n\ge 2$	
(b) **(ii)**	$a_3=\dfrac{a_2^2+2}{a_1}=\dfrac{1^2+2}{1}=3$ $\dfrac{a_{n+1}+a_{n-1}}{a_n}=\ldots=\dfrac{a_3+a_1}{a_2}=\dfrac{3+1}{1}=4$ $a_{n+1}=4a_n-a_{n-1},\ n\ge 2.$ Therefore a_{n+1} has the same parity as a_{n-1} since $4a_n$ is even. Since both a_1 and a_2 are both odd, by induction, is an odd integer for all $n\in\mathbb{Z}^+$. Alternatively, let P(n) be the statement that is odd for $n\in\mathbb{Z}^+$. P(1) and P(2) are both true trivially by definition.	

	Suppose a_k and a_{k+1} are both odd for some positive integer k. Then $a_{k+2}=\frac{a_{k+1}^2+2}{a_k}=\frac{(\text{odd})^2+2}{\text{odd}}=\frac{\text{odd}}{\text{odd}}\neq \text{even}$ Hence $\mathrm{P}(k),\mathrm{P}(k+1)$ both true $\Rightarrow \mathrm{P}(k+2)$ is true. Therefore, since P(1) and P(2) are both true, and $\mathrm{P}(k),\mathrm{P}(k+1)$ both true $\Rightarrow \mathrm{P}(k+2)$ is true, by the Principle of Mathematical Induction, a_n is an odd integer for all $n\in\mathbb{Z}^+$.	
(c)	Suppose there is an arithmetic progression a_1, a_2, ... with common difference d that has 1, $\sqrt{2}$, $\sqrt{3}$ among its terms. Then there exist distinct positive integers m, n and p such that $a_m=1$, $a_n=\sqrt{2}$, $a_p=\sqrt{3}$. Thus we have $\sqrt{2}-1=a_n-a_m=(n-m)d$ and $\sqrt{3}-\sqrt{2}=a_p-a_n=(p-n)d$, so $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}=\frac{n-m}{p-n}$ is rational. Since $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}=(\sqrt{3}-\sqrt{2})(\sqrt{2}+1)=\sqrt{6}-2+\sqrt{3}-\sqrt{2}$ thus $a=\sqrt{6}+\sqrt{3}-\sqrt{2}$ is a rational number. Then, $a+\sqrt{2}=\sqrt{6}+\sqrt{3}$ $(a+\sqrt{2})^2=(\sqrt{6}+\sqrt{3})^2$ $a^2+2a\sqrt{2}+2=6+2\sqrt{18}+3$ $2a\sqrt{2}-6\sqrt{2}=7-a^2$ $(2a-6)\sqrt{2}=7-a^2$ Since $a=3$ does not satisfy the equality, we can divide throughout by $(2a-6)$ so $\sqrt{2}=\frac{7-a^2}{2a-6}\in\mathbb{Q}$ which is a contradiction. Hence the supposition does not hold and the result required is shown.	
		Total Marks: 12

Qn	**Solution**	**Remarks**
4(i)	Applying Cauchy-Schwarz Inequality $\left(\sum_{i=1}^{n+1}a_i^2\right)\left(\sum_{i=1}^{n+1}b_i^2\right)\geq\left(\sum_{i=1}^{n+1}a_ib_i\right)^2$ with $a_i=x_{i-1}-x_i$ for $i=1,\ 2,\ ...,\ n,\ \ a_{n+1}=x_n$ and $b_i=1$ for $i=1,\ 2,\ ...,\ n+1,$ we have $\left(\left(\sum_{i=1}^{n}\left(x_{i-1}-x_i\right)^2\right)+x_n^2\right)\left(\sum_{i=1}^{n+1}1^2\right)\geq\left(\left(\sum_{i=1}^{n}\left(x_{i-1}-x_i\right)\right)+x_n\right)^2=1$ $\left(\left(\sum_{i=1}^{n}\left(x_{i-1}-x_i\right)^2\right)+x_n^2\right)\geq\frac{1}{n+1}$	
(ii)	Since equality holds, we have $a_i=mb_i=m$ (constant). Thus, $a_1=a_2=...=a_{n+1}.$ Since $\sum_{i=1}^{n+1}a_i=1,$ therefore we have $a_i=\frac{1}{n+1}$ for $i=1,\ 2,\ ...,\ n+1,$ Thus, $x_n=\frac{1}{n+1},$ $x_{n-1}=a_n+x_n=\frac{2}{n+1},$ $x_{n-2}=a_{n-1}+x_{n-1}=\frac{3}{n+1},$ and so on so forth. Generalizing, we have $x_k=\frac{n+1-k}{n+1},\ \ k=0,\ 1,\ 2,\ ...,\ n.$	
		Total Marks: 6

<table>
<tr><th>Qn</th><th>Solution</th><th>Mark Scheme</th></tr>
<tr><td>5(i)</td><td>Number of ways $=\binom{24+3-1}{3-1}=325$</td><td>B1</td></tr>
<tr><td>(ii)</td><td>
The problem is
$x_1+x_2+x_3=24$
$0\le x_i\le 10,\ i=1,2,3$
with $x_1,\ x_2,\ x_3$ being the number of cards in the red, green and blue boxes respectively.

<u>Method 1</u>
Let A_i denote the event where $0\le x_i\le 10$.
Number of ways
$|A_1\cap A_2\cap A_3|$
$=|S|-|A_1'\cup A_2'\cup A_3'|$
$=|S|-\left(\sum_{i=1}^{3}|A_i'|-\sum_{\substack{i\neq j\\ i,j\in\{1,2,3\}}}|A_i'\cap A_j'|+|A_1'\cap A_2'\cap A_3'|\right)$
$=325-\left(3\binom{13+3-1}{3-1}-3\binom{2+3-1}{3-1}+0\right)$
$=325-\left(3(105)-3(6)+0\right)=28$

<u>Method 2</u>
$x_i\ge 4$ since $x_i\le 3$ for any i will result in
$x_1+x_2+x_3<24$ for $0\le x_i\le 10,\ i=1,2,3.$
<table>
<tr><th>x_1</th><th>x_2+x_3</th><th>$(x_2,\ x_3)$ or corresponding cases for x_2</th><th>Number of Ways</th></tr>
<tr><td>4</td><td>20</td><td>$(10,\ 10)$</td><td>1</td></tr>
<tr><td>5</td><td>19</td><td>$(9,\ 10),\ (10,\ 9)$</td><td>2</td></tr>
<tr><td>6</td><td>18</td><td>$8\le x_2\le 10$</td><td>3</td></tr>
<tr><td>7</td><td>17</td><td>$7\le x_2\le 10$</td><td>4</td></tr>
<tr><td>8</td><td>16</td><td>$6\le x_2\le 10$</td><td>5</td></tr>
<tr><td>9</td><td>15</td><td>$5\le x_2\le 10$</td><td>6</td></tr>
<tr><td>10</td><td>14</td><td>$4\le x_2\le 10$</td><td>7</td></tr>
</table>
Therefore number of ways = 1 + 2 + … + 7 = 28
</td><td></td></tr>
</table>

(ii)	**Method 3** Number of ways $=$ (none with 10 units) + (one with 10 units) + (two with 10 units) $=\left(\underset{(8,8,8)}{1}+\underset{(6,9,9)^*}{3}+\underset{(7,8,9)^*}{6}\right)+\underbrace{5}_{(5,9)\ \ldots\ (9,5)}\times{}^3C_1+\underset{(4,10,10)^*}{{}^3C_2}$ $=10+15+3=28$	
(iii)	Sue can take either no work, or up to 4 units of work, since it is not possible to have Sue take 5 or more units of work as the total amount of work is 24 units (5 + 10 + 10 =25 > 24). If Sue takes *k* units of work where *k* = 0 to 4, then the other 2 must take at least 2*k* units of work from the remaining $(24-k)$ units of work, no. of ways $=\binom{24-k-4k+2-1}{2-1}=25-5k$ since *k* units of work is assigned to Sue, and we place 2*k* units of work each into the other two boxes. Number of ways supervisor can distribute the workload $=\sum_{k=0}^{4}\left(25-5k\right)=75$	
(iv)	$P(r,\ n)=P(r-1,\ n-1)+P(r-n,\ n)$	
(v)	$P(r,\ n)$ $=P(r-1,\ n-1)+P(r-n,\ n)$ (applying **(iv)** on $P(r,\ n)$) $=\left[P(r-2,\ n-2)+P\left((r-1)-(n-1),\ (n-1)-1\right)\right]$ $+P(r-n,\ n)$ (applying **(iv)** on $P(r-1,\ n-1)$) $=P(r-2,\ n-2)+\left[P(r-n,\ n-1)+P(r-n,\ n)\right]$ (rearrangement) $=P\left(r-(n-1),\ 1\right)$ $+\left[P\left(r-n,\ n-(n-2)\right)+\ldots+P(r-n,\ n)\right]$ (applying **(iv)** on $P(r-2,\ n-2),\ldots$, until $P(r-n+1,\ 1)$) $=P\left(r-n+1,\ 1\right)+\left[P\left(r-n,\ 2\right)+\ldots+P(r-n,\ n)\right]$ $=P\left(r-n,\ 1\right)+\left[P\left(r-n,\ 2\right)+\ldots+P(r-n,\ n)\right]$ $(\because\ P\left(r-n+1,\ 1\right)=P\left(r-n,\ 1\right)=1)$ $=\sum_{k=1}^{n}P(r-n,\ k)$ (shown)	

(vi)	Method 1 (Using **(v)**) $P(10,\ 3)=\sum_{k=1}^{3}P(7,\ k)$ $=P(7,\ 1)+P(7,\ 2)+P(7,\ 3)$ $=1+\sum_{k=1}^{2}P(5,\ k)+\sum_{k=1}^{3}P(4,\ k)$ $=1+P(5,\ 1)+P(5,\ 2)+P(4,\ 1)+P(4,\ 2)+P(4,\ 3)$ $=1+1+P(5,\ 2)+1+P(4,\ 2)+1$ $=4+\sum_{k=1}^{2}P(3,\ k)+\sum_{k=1}^{2}P(2,\ k)$ $=4+P(3,\ 1)+P(3,\ 2)+P(2,\ 1)+P(2,\ 2)$ $=4+1+1+1+1=8$ Method 2 (Using **(iv)**) $P(10,\ 3)$ $=P(9,\ 2)+P(7,\ 3)$ $=\left(P(8,\ 1)+P(7,\ 2)\right)+\left(P(6,\ 2)+P(4,\ 3)\right)$ $=1+\left(P(6,\ 1)+P(5,\ 2)\right)+\left(P(5,\ 1)+P(4,\ 2)\right)+1$ $=1+1+\left(P(4,\ 1)+P(3,\ 2)\right)+1+\left(P(3,\ 1)+P(2,\ 2)\right)+1$ $=8$	
		Total Marks: 12

Qn	Solution	Remarks
6(i) **(a)**	$T_1 = 1,\ T_2 = 2.$	
(i) **(b)**	$T_{n+2} = T_{n+1} + T_n$ for $n \geq 1$.	
(ii)	Consider the "odd" tile / last tile in a tiling of P_{n+1}. It can only be covered by a 1×1 or a 1×2 tile. If it is covered by a 1×1 tile, the rest form a tiling of Q_n. If it is covered by a 1×2 tile, the rest form a tiling of P_n. Thus $P_{n+1} = P_n + Q_n$.	
(iii)	Consider the last column of 2 tiles in a tiling of Q_{n+1}. The following cases are possible: • 2×2 tile: The rest form a tiling of Q_{n-1}. • 1×2 tile (vertical): The rest form a tiling of Q_n • Two 1×1 tiles: The rest form a tiling of Q_n. • Two 1×2 tiles (horizontal): The rest form a tiling of Q_{n-1}. • One 1×1 tile and one 1×2 tile (horizontal): The rest form a tiling of P_n. Note that this case counts twice (depending on which tile covers the top line and which tile covers the bottom line). Thus $Q_{n+1} = 2Q_n + 2Q_{n-1} + 2P_n$ $= 2P_{n+1} + 2Q_{n-1}$ using the result from (**ii**). **Alternative** Tiling of Q_{n+1} can be obtained from • Q_{n-1} using a 2×2 tile or two horizontal 1×2 tiles. • Q_n using one vertical 1×2 tile • P_{n+1} using one 1×1 tile The first two cases involve non-1×1 tiles on the last column, and the last case involves 1×1 tiles on the last column. However, the last case we must consider the double counting of the subcase where tiles from the last column are both 1×1 tiles, and the number of ways to do so is Q_n. Thus $Q_{n+1} = 2Q_{n-1} + Q_n + \left(2P_{n+1} - Q_n\right) = 2P_{n+1} + 2Q_{n-1}$	

(iv)	Add $P_{n+1}+2P_{n-1}$ to both sides of (**iii**): $P_{n+1}+2P_{n-1}+Q_{n+1}=P_{n+1}+2P_{n-1}+2P_{n+1}+2Q_{n-1}$ $P_{n+2}+2P_{n-1}=3P_{n+1}+2P_{n}$ (using result from (**ii**))	
(v)	Number of tilings of $2\times n$ path is Q_n So $Q_n=P_{n+1}-P_n$ $=\frac{2}{7}\left(-1\right)^n+\frac{1+2\sqrt{2}}{14}\left(2+\sqrt{2}\right)^n\left(2+\sqrt{2}-1\right)$ $+\frac{1-2\sqrt{2}}{14}\left(2-\sqrt{2}\right)^n\left(2-\sqrt{2}-1\right)$ $=\frac{2}{7}\left(-1\right)^n+\frac{5+3\sqrt{2}}{14}\left(2+\sqrt{2}\right)^n+\frac{5-3\sqrt{2}}{14}\left(2-\sqrt{2}\right)^n$	
		Total Marks: 12

<table>
<tr><th>Qn</th><th>Solution</th><th>Remarks</th></tr>
<tr><td>7(a)</td><td>
We show by proving the contrapositive statement:
if $7 \nmid a$ or $7 \nmid b$ then $7 \nmid (a^2+b^2)$.

For positive integers n not divisible by 7,

<table>
<tr><th>n (mod 7)</th><th>n^2 (mod 7)</th></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>4</td></tr>
<tr><td>3</td><td>2</td></tr>
<tr><td>4</td><td>2</td></tr>
<tr><td>5</td><td>4</td></tr>
<tr><td>6</td><td>1</td></tr>
</table>

Therefore $(a^2+b^2)(\bmod 7)$ can only take values
1 (from 4 + 4), 2 (from 1 + 1), 3 (from 1 + 2),
4 (from 2 + 2), 5 (from 1 + 4) 6 (from 2 + 4) but never 0.

Hence the contrapositive statement is shown.
</td><td></td></tr>
<tr><td>(b)</td><td>
When n is even, n^4+4^n is even and hence not a prime.

When n is odd, i.e. $n=2k+1$ with integer $k \geq 1$, (since $n>1$)
$$\begin{aligned} n^4+4^n &= \left(n^2+2^n\right)^2-2\left(n^2\right)\left(2^n\right) \\ &= \left(n^2+2^n\right)^2-\left(n^2\right)\left(2^{2k+2}\right) \\ &= \left(n^2+2^n\right)^2-\left(2^{k+1}n\right)^2 \\ &= \left(n^2+2^n-2^{k+1}n\right)\left(n^2+2^n+2^{k+1}n\right) \end{aligned}$$
The smaller factor is $\left(n^2+2^n-2^{k+1}n\right)$, and
$$\begin{aligned} n^2+2^n-2^{k+1}n &= n^2-2^{k+1}n \qquad\qquad +2^n \\ &= n^2-2\left(2^k\right)n+2^{2k}-2^{2k}+2^{2k+1} \\ &= \left(n-2^k\right)^2+2^{2k}>1 \end{aligned}$$
so n^4+4^n cannot be prime.
</td><td></td></tr>
</table>

7(c)

<u>Method 1: Consider the values of</u>
16^n, $10n$ and -1 (mod 25)

n	16^n (mod 25)	$10n$ (mod 25)	-1	Sum (mod 25)
1	16	10	–1	0
2	6	20	–1	0
3	21	5	–1	0
4	11	15	–1	0
5	1	0	–1	0
6	16	10	–1	0
7	6	20	–1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Since the table repeats cyclically for every 5 values of *n*, $16^n+10n-1$ is divisible by 25, for $n\in\mathbb{Z}^+$.

<u>Method 2: Mathematical Induction</u>
Let P(*n*) be the statement that $16^n+10n-1$ is divisible by 25, for $n\in\mathbb{Z}^+$.

P(1) is true since $16^1+10(1)-1=25$ which is divisible by 25.

Suppose P(*k*) is true for some positive integer *k*.
Then for $\text{P}(k+1)$,

$$16^{k+1}+10(k+1)-1$$
$$=16\left(\left(16^k\right)+10k-1\right)-160k+16+10(k+1)-1$$
$$=16\left(\left(16^k\right)+10k-1\right)-150k+25$$

which is divisible by 25.

Hence $\text{P}(k)$ is true $\Rightarrow$ $\text{P}(k+1)$ is true.

Therefore, since P(1) is true, and $\text{P}(k)$ is true $\Rightarrow$ $\text{P}(k+1)$ is true, by the Principle of Mathematical Induction, $16^n+10n-1$ is divisible by 25, for $n\in\mathbb{Z}^+$.

Total Marks: 12

Qn	Solution	Remarks
8(i)	$L_1(x_1)=1,$ $L_1(x_2)=0,\ L_1(x_3)=0.$	
(ii)	$L_2(x)=\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$ $L_3(x)=\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$	
(iii)	Suppose there exist two quadratic polynomials p_1, p_2 (of degree less or equal to $n-1=3-1=2)$, with $p_1(x_i)=p_2(x_i)=y_i$, for $i=1,\ 2,\ 3.$ Then the difference polynomial $q=p_1-p_2$ is a polynomial of degree less or equal to $n-1=3-1=2$ that satisfy $h(x_i)=0,$ for $i=1,\ 2,\ 3.$ Since the number of zeroes of a nonzero polynomial is equal to its degree, it follows that $q(x)=p_1(x)-p_2(x)=0,\ \ x\in\mathbb{R},$ i.e. $p_1(x)=p_2(x),$ for all $x\in\mathbb{R}.$	
(iv)	$P(x)=y_1L_1(x)+y_2L_2(x)+y_3L_3(x),\ \ x\in\mathbb{R}.$ This is because since $L_i(x_i)=1,\ \ L_i(x_j)=0$ for $i\neq j,$ $i,j=1,\ 2,\ 3$ from the earlier parts, we have $P(x_i)=y_i$, for $i=1,\ 2,\ 3.$	
(v)	Let $X=Y=\{1,\ 2,\ 3\}$ and a function $\mathrm{f}:X\to X$, satisfying $\mathrm{f}(x_i)=y_i,\ \ y_i\neq i$ for $i=1,\ 2,\ 3,$ where $x_i=i,$ and $y_i\in Y=X$, for $i=1,\ 2,\ 3.$ Note that the above translates to a derangement problem, where the number of such possible mappings is given by $D_3=3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right)=2.$ In fact, the 2 derangements are given by Derangement 1: $\mathrm{f}(1)=2,\ \ \mathrm{f}(2)=3,\ \ \mathrm{f}(3)=1,$ Derangement 2: $\mathrm{f}(1)=3,\ \ \mathrm{f}(2)=1,\ \ \mathrm{f}(3)=2.$ We now show that we can extend this derangement property of function f defined on $\{1,\ 2,\ 3\}$ to a quadratic polynomial Q, which is defined on the real line by finding explicitly 2 quadratic polynomials Q_1, Q_2 such that they satisfy $Q_1(1)=2,\ \ Q_1(2)=3,\ \ Q_1(3)=1,$ $Q_2(1)=3,\ \ Q_2(2)=1,\ \ Q_2(3)=2.$	

To do so, we can use either of the following two methods:

Method 1

We can find two quadratic polynomials $Q_1,\ Q_2$ such that they satisfy

$$Q_1(1)=2,\ Q_1(2)=3,\ Q_1(3)=1,$$
$$Q_2(1)=3,\ Q_2(2)=1,\ Q_2(3)=2.$$

from parts **(iii)** and the interpolatory property **(iv)**, we get

$$Q_1(x)=2L_1(x)+3L_2(x)+1L_3(x),$$
$$Q_2(x)=3L_1(x)+1L_2(x)+2L_3(x),$$

where

$$L_1(x)=\frac{(x-2)(x-3)}{(1-2)(1-3)}=\frac{1}{2}(x-2)(x-3),$$
$$L_2(x)=\frac{(x-1)(x-3)}{(2-1)(2-3)}=-(x-1)(x-3),$$
$$L_3(x)=\frac{(x-1)(x-2)}{(3-1)(3-2)}=\frac{1}{2}(x-1)(x-2).$$

Method 2

Let $Q_1(x)=a_2x^2+a_1x+a_0$, and

$$Q_2(x)=b_2x^2+b_1x+b_0.$$

We solve for coefficients a_2,a_1,a_0 and b_2,b_1,b_0 such that

$$Q_1(1)=2,\ Q_1(2)=3,\ Q_1(3)=1,$$

or $Q_2(1)=3,\ Q_2(2)=1,\ Q_2(3)=2.$

We obtain the resulting system of linear equations

$$a_2(1)^2+a_1(1)+a_0=2,$$
$$a_2(2)^2+a_1(2)+a_0=3,$$
$$a_2(3)^2+a_1(3)+a_0=1.$$

$$b_2(1)^2+b_1(1)+b_0=3,$$
$$b_2(2)^2+b_1(2)+b_0=1,$$
$$b_2(3)^2+b_1(3)+b_0=2.$$

$$a_2=-\frac{3}{2},\ a_1=\frac{11}{2},\ a_0=-2,$$
$$b_2=\frac{3}{2},\ b_1=-\frac{13}{2},\ b_0=8,$$

Therefore, $Q_1(x)=-\frac{3}{2}x^2+\frac{11}{2}x-2,$

$$Q_2(x)=\frac{3}{2}x^2-\frac{13}{2}x+8.$$

Total Marks: 13

Qn	Solution	Remarks
9(i) **(a)**	$S(p)=p+1$	
(i) **(b)**	$S(pq)=S(p)S(q)=(p+1)(q+1)$	
(i) **(c)**	$S(p^m q^n)=S(p^m)S(q^n)$ $=(1+p+\ldots+p^m)(1+q+\ldots+q^n)$ $=\dfrac{(p^{m+1}-1)(q^{n+1}-1)}{(p-1)(q-1)}$	
(ii)	$220=2^2\times5\times11$ The proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110, and Sum of all proper divisors of 220 $=1+2+4+5+10+11+20+22+44+55+110$ $=284$ $284=2^2\times71$ The proper divisors of 2924 are 1, 2, 4, 71 and 142, and Sum of all proper divisors of 284 $=1+2+4+71+142=220$	
(iv)	$a(pq+r)=M+N$ $=S(M)$ $=S(apq)$ $=S(a)S(p)S(q)$ $=(S(a))(p+1)(q+1)$	
(v)	Since M , N form a pair of amicable numbers, $S(M)-M=N$, and $S(N)-N=M$. Simplifying, we get $S(M)=S(N)$, and thus $S(apq)=S(ar)$ $(p+1)(q+1)S(a)=(r+1)S(a)$ $r+1=(p+1)(q+1)$ $(2a-S(a))(p+1)(q+1)$ $=2a(p+1)(q+1)-S(a)(p+1)(q+1)$ $=2a(p+1)(q+1)-a(pq+r)$ $=2a(p+1)(q+1)-a(pq+(p+1)(q+1)-1)$ $=2a(pq+p+q+1)-a(2pq+p+q)$ $=(2apq+2ap+2aq+2a)-(2apq+ap+aq)$ $=(ap+aq+2a)=a(p+q+2)$ (shown)	

(vi) $a=4,\ S(4)=1+2+4=7.$

$$\left(p+1-\frac{a}{2a-S(a)}\right)\left(q+1-\frac{a}{2a-S(a)}\right)=\left(\frac{a}{2a-S(a)}\right)^2$$

becomes

$$\left(p+1-\frac{4}{2(4)-7}\right)\left(q+1-\frac{4}{2(4)-7}\right)=\left(\frac{4}{2(4)-7}\right)^2$$

$$(p-3)(q-3)=16$$

WLOG let $p \le q$,

$p-3$	$q-3$	p	q	r
1	16	4	19	99
2	8	5	11	71
4	4	7	7	63

Since p, q, r are distinct primes,
so the only solution is $p=5,\ q=11,\ r=71,$ and
$M=4\times5\times11=220,\ N=4\times71=284.$

Total Marks: 13

H

TEMASEK JUNIOR COLLEGE,
SINGAPORE

JC 2
Preliminary Examination 2019

CANDIDATE NAME		CG	

MATHEMATICS

9820/01

Higher 3

20 September 2019

Paper 1

3 hours

Additional Materials:
Answer Booklet
Graph Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.
Write in dark blue or black pen on both sides of the answer booklet.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all you work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **3** blank pages.

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PASSION PURPOSE DRIVE

1 **(i)** Euclid's Lemma states that if $a \mid bc$ and $\gcd(a,b)=1$, then $a \mid c$. Let b and c be integers and p be a prime. Prove that if $p \mid bc$, then either $p \mid b$ or $p \mid c$. [3]

1(i)	If $p \mid b$, then the result is proved. Suppose then that $p \nmid b$. Since the only positive integer divisors of p are 1 and p, it follows that $\gcd\left(p,b\right)=1$. Thus, by Euclid's Lemma, $p \mid c$ and the proof is complete.

(ii) Let $n \in \mathbb{Z}^+$. Prove that $\sqrt{n}$ is a rational number if and only if $\sqrt{n}$ is an integer. [6]

1(ii)	If $\sqrt{n}$ is an integer, then it is obvious that $\sqrt{n}$ is rational. We need to prove the converse, i.e. $\sqrt{n}$ is rational $\Rightarrow$ $\sqrt{n}$ is an integer. Assume by contradiction that there exists some positive integer n such that $\sqrt{n}$ is rational but not an integer. Then, $\sqrt{n}=\frac{a}{b}$ in the simplest form, where $a,\ b>0$, and a is not a multiple of b. Hence, $\gcd\left(a,b\right)=1$. Since $\frac{a}{b}$ is not an integer, $b \geq 2$. Therefore, $n=\frac{a^2}{b^2} \Rightarrow a^2=nb^2$. Clearly, since $b \geq 2$, b will have a prime factor, say p. Thus, $p \mid nb^2$ and so $p \mid a^2$. By the result in **(i)**, $p \mid a$. This implies that a and b have a common factor p, which is a contradiction that $\gcd\left(a,b\right)=1$.

2 For any non-negative integer n, let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \,\mathrm{d}x$.

(i) By considering $x \le \tan x \le \frac{4}{\pi} x$ for $x \in \left[0, \frac{1}{4}\pi\right]$, show that

$$\frac{1}{n+1}\left(\frac{\pi}{4}\right)^{n+1} \le I_n \le \frac{1}{n+1}\left(\frac{\pi}{4}\right).$$ [3]

2(i)	$x \le \tan x \le \frac{4x}{\pi} \Rightarrow x^n \le \tan^n x \le \left(\frac{4}{\pi}\right)^n x^n$ for $x \in \left[0, \frac{\pi}{4}\right]$ $\int_0^{\frac{\pi}{4}} x^n \,\mathrm{d}x \le \int_0^{\frac{\pi}{4}} \tan^n x \,\mathrm{d}x \le \left(\frac{4}{\pi}\right)^n \int_0^{\frac{\pi}{4}} x^n \,\mathrm{d}x$ i.e. $\left[\frac{x^{n+1}}{n+1}\right]_0^{\frac{\pi}{4}} \le I_n \le \left(\frac{4}{\pi}\right)^n \left[\frac{x^{n+1}}{n+1}\right]_0^{\frac{\pi}{4}} \Rightarrow \frac{1}{n+1}\left(\frac{\pi}{4}\right)^{n+1} \le I_n \le \frac{1}{n+1}\left(\frac{4}{\pi}\right)^n \left(\frac{\pi}{4}\right)^{n+1}$ $\therefore \frac{1}{n+1}\left(\frac{\pi}{4}\right)^{n+1} \le I_n \le \frac{1}{n+1}\left(\frac{\pi}{4}\right)$ (shown)

(ii) Hence, or otherwise, find $\lim_{n \to \infty} I_n$. [1]

(ii)	Since $I_n \ge 0$ and $\lim_{n \to \infty} \frac{1}{n+1}\left(\frac{\pi}{4}\right) = 0 \therefore \lim_{n \to \infty} I_n = 0$

(iii) Show that $I_n + I_{n-2} = \frac{1}{n-1}$ for $n = 2, 3, 4, \ldots$. [2]

(iii)	$I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^n x \,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \tan^{n-2} x \,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \tan^{n-2} x\left(\tan^2 x + 1\right)\mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\left(\sec^2 x\right)\mathrm{d}x$ $= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{n-1}$

For $n = 1, 2, 3, \cdots$, let $a_n = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{2r-1}$.

(iv) Using **(iii)**, or otherwise, express a_n in terms of I_{2n} . [3]

(iv)	$a_n = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{2r-1} = \sum_{r=1}^{n} (-1)^{r+1}\left(I_{2r} + I_{2r-2}\right)$ *Using the result in **(iii)**

	$$\begin{aligned} &I_2 + I_0 \\ = \quad &-I_4 - I_2 \\ &+I_6 + I_4 \\ &+\cdots \\ &(-1)^n (I_{2n-2} + I_{2n-4}) \\ &(-1)^{n+1} (I_{2n} + I_{2n-2}) \\ = &(-1)^{n+1} I_{2n} + I_0 = (-1)^{n+1} I_{2n} + \frac{\pi}{4} \end{aligned}$$

(v) Evaluate $\lim_{n\to\infty} a_n$. [2]

(v)	$\lim_{n\to\infty} a_n = 0 + \frac{\pi}{4} = \frac{\pi}{4}$ since $\lim_{n\to\infty} I_n = 0$

3 **(a)** A geometric progression $\{u_n\}$ has first term a and common ratio r. The sequence $\{v_n\}$ comprises the terms of the geometric progression modulo N, where a, r and N are coprime positive integers.

Prove that there are only non-zero terms in $\{v_n\}$. [3]

3(a)	Let $N = p_1^{m_1} p_2^{m_2} p_3^{m_3} ... p_q^{m_q}$ for some $q \in \mathbb{Z}^+$. Since a, r and N are coprime, for each $i \in \mathbb{Z}^+$, p_i is not a factor of a and p_i is not a factor of r. Hence p_i is not a factor of ar^{n-1} for all $i \in \mathbb{Z}^+, n \in \mathbb{Z}^+$. Therefore N is not a factor of ar^{n-1} for all $n \in \mathbb{Z}^+$ and $\{v_n\}$ does not contain zeroes.

(b) An arithmetic progression has first term b and common difference d. The sequence $\{w_n\}$ comprises the terms of the arithmetic progression modulo M, where b, d and M are coprime positive integers.

(i) Prove that $\{w_n\}$ is periodic with period M. [5]

(ii) Deduce the period of $\{w_n\}$ if d is a factor of M where $d \neq 1$ and $d \neq M$. [3]

(b)(i)	Consider $\{w_t, w_{t+1}, \ldots, w_{t+M}\}$ for $t \in \mathbb{Z}^+$. Since there are only M possible values in $\{w_n\}$, by the Pigeonhole Principle, there must be at least 2 distinct terms, say w_j and w_k, where $t \le j < k \le t + M$, such that $w_j = w_k$ and therefore $w_k - w_j \equiv 0$. Hence $(b + (k - 1)d) - (b + (j - 1)d) = (k - j)d \equiv 0 \pmod{M}$. Since d and M are coprime, $M \mid (k - j)$. However, since $t \le j < k \le t + M$, $(k - j) = M \Rightarrow j = t$, $k = t + M$ and $w_t = w_{t+M}$. Since this result is true for all $t \in \mathbb{Z}^+$, $\{w_n\}$ is periodic with period M (shown).
(b)(ii)	From **(i)**, $(k - j)d \equiv 0 \pmod{M}$. Let $M = d \times f$: then $(k - j) \equiv 0 \pmod{f}$, where $f = \frac{M}{d}$ Hence $f \mid (k - j)$, and $k - j = f, 2f, \ldots, df\ (= M)$. Therefore $\{w_n\}$ is periodic with period $f = \frac{M}{d}$.

4 The power series method is used to seek a power series solution to certain differential equations. In general, such a solution assumes a power series with unknown coefficients, then substitutes that solution into the differential equation to find a recurrence relation for the coefficients.

Duffing's equation is a type of non-linear oscillator equation as follows.

$$y'' + y + \varepsilon y^3 = 0 \ , \ y(0) = 1 \ , \ y'(0) = 0 \ .$$

where $y'(t) = \frac{\mathrm{d}y}{\mathrm{d}t}$ and $y''(t) = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$.

A solution $y(t)$ of this equation can be found by expanding $y(t)$ as a power series in ε , i.e.

$$y(t) = \sum_{n=0}^{\infty} \varepsilon^n y_n(t)$$

where $y_0(0) = 1$, $y_0'(0) = 0$ and $y_n(0) = y_n'(0) = 0$ for $n \geq 1$.

(i) By substituting the power series of $y(t)$ into Duffing's equation, show that $y_0'' + y_0 = 0$ and $y_1'' + y_1 = -(y_0)^3$. [3]

4(i)	$y(t) = \sum_{n=0}^{\infty} \varepsilon^n y_n(t)$ $y'(t) = \sum_{n=0}^{\infty} \varepsilon^n y_n'(t)$ $y''(t) = \sum_{n=0}^{\infty} \varepsilon^n y_n''(t)$ Hence, $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y + \varepsilon y^3 = 0$ $\sum_{n=0}^{\infty} \varepsilon^n y_n''(t) + \sum_{n=0}^{\infty} \varepsilon^n y_n(t) + \varepsilon \left(\sum_{n=0}^{\infty} \varepsilon^n y_n(t) \right)^3 = 0$ $y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon \left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \right)^3 + \ldots = 0$ $y_0'' + y_0 + \varepsilon \left[(y_0)^3 + y_1'' + y_0 \right] + \ldots = 0$ By comparing the coefficients, $y_0'' + y_0 = 0$ and $y_1'' + y_1 = -(y_0)^3$ (shown)

(ii) Given that $y = \mathrm{e}^{\alpha x}(A\cos\beta x + B\sin\beta x)$ is the general solution of $ay'' + by' + cy = 0$ whose auxiliary equation $am^2 + bm + c = 0$ has complex roots $m = \alpha \pm \beta\mathrm{i}$, find the particular equation of the differential equation $y_0'' + y_0 = 0$. [3]

(ii)	$y_0'' + y_0 = 0$ has the auxiliary equation $m^2 + 1 = 0$. $m = \pm\mathrm{i}$ Hence, the general solution is $y_0 = \mathrm{e}^{0x}(A\cos x + B\sin x) = A\cos x + B\sin x$. Given that $y_0(0) = 1$, $A = 1$. Given that $y_0{}'(0) = 0$, $B = 0$ $\therefore \ y_0 = \cos(t)$ is the particular solution.

(iii) By considering the trigonometric identity $\cos^3(t)=\frac{1}{4}\cos(3t)+\frac{3}{4}\cos(t)$, verify that

$$y_1(t)=A\cos(t)+B\sin(t)+C\cos(3t)+Dt\sin(t)$$

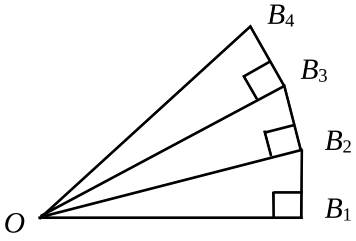
is a particular solution to $y_1''+y_1=-y^3{}_0$, where *A*, *B*, *C* and *D* are constants to be determined. [4]

(iii)	$y_1''+y_1=-\left(y_0\right)^3$ $\Rightarrow y_1''+y_1=-\left(\cos\left(t\right)\right)^3$ $\Rightarrow y_1''+y_1=-\frac{1}{4}\cos\left(3t\right)-\frac{3}{4}\cos\left(t\right)$ ---(*) Since $y_1\left(t\right)=A\cos\left(t\right)+B\sin\left(t\right)+C\cos\left(3t\right)+Dt\sin\left(t\right)$, $y_1'\left(t\right)=-A\sin\left(t\right)+B\cos\left(t\right)-3C\sin\left(3t\right)+Dt\cos\left(t\right)+D\sin\left(t\right)$ $y_1''\left(t\right)=-A\cos\left(t\right)-B\sin\left(t\right)-9C\cos\left(3t\right)-Dt\sin\left(t\right)+D\cos\left(t\right)+D\cos\left(t\right)$ $y_1''\left(t\right)=-A\cos\left(t\right)-B\sin\left(t\right)-9C\cos\left(3t\right)-Dt\sin\left(t\right)+2D\cos\left(t\right)$ Substituting into the differential equation (*), $-A\cos\left(t\right)-B\sin\left(t\right)-9C\cos\left(3t\right)-Dt\sin\left(t\right)+2D\cos\left(t\right)=-\frac{1}{4}\cos\left(3t\right)-\frac{3}{4}\cos\left(t\right)$ $+A\cos\left(t\right)+B\sin\left(t\right)+C\cos\left(3t\right)+Dt\sin\left(t\right)$ $-8C\cos\left(3t\right)+2D\cos\left(t\right)=-\frac{1}{4}\cos\left(3t\right)-\frac{3}{4}\cos\left(t\right)$ So $C=\frac{1}{32}$ and $D=-\frac{3}{8}$. Substituting the initial conditions $y_1\left(0\right)=y_1'\left(0\right)=0$, we have $A=\frac{1}{32}$ and $B=0$. Hence, the particular solution is $y_1\left(t\right)=\frac{1}{32}\cos\left(3t\right)-\frac{1}{32}\cos\left(t\right)-\frac{3}{8}\sin\left(t\right)$.

(iv) Hence, find the first order solution to Duffing's equation in increasing powers of ε. [1]

(iv)	$y\left(t\right)=y_0+\varepsilon y_1+\dots$ $y\left(t\right)=\cos\left(t\right)+\varepsilon\left(\frac{1}{32}\cos\left(3t\right)-\frac{1}{32}\cos\left(t\right)-\frac{3}{8}\sin\left(t\right)\right)+\dots$

5 A series of right-angled triangles is constructed such that for all positive integers *i*, $\angle B_iOB_{i+1} = \alpha$, $\angle OB_iB_{i+1} = \frac{1}{2}\pi$ and all triangles $\triangle OB_iB_{i+1}$ are similar. The first 3 triangles are shown below.



(i) Given that $OB_1 = d_1$, show that $OB_n = \sec^{n-1}\alpha\ d_1$ for all $n \geq 2$. [2]

5(i)	$\cos\alpha = \dfrac{OB_i}{OB_{i+1}}$ for all $i \in \mathbb{Z}^+$ $OB_n = OB_{n-1}(\sec\alpha) = OB_{n-2}(\sec^2\alpha) = ... = OB_1(\sec^{n-1}\alpha) = \sec^{n-1}\alpha\ d_1$

(ii) Denoting the area of $\triangle OB_iB_{i+1}$ by A_i, prove by mathematical induction that

$$\sum_{r=1}^{n} A_r = \tfrac{1}{2}d_1^2\cot\alpha(\sec^{2n}\alpha - 1),\ n \in \mathbb{Z}^+. \quad [6]$$

(ii)	Let P_n be the statement $\displaystyle\sum_{r=1}^{n} A_r = \frac{1}{2}d_1^2\cot\alpha\left(\sec^{2n}\alpha - 1\right), n \in \mathbb{Z}^+$ Let *n* = 1: $A_1 = \dfrac{1}{2}(OB_1)(OB_2)\sin\alpha$ $= \dfrac{1}{2}d_1\left(d_1\sec\alpha\right)\sin\alpha$ $= \dfrac{1}{2}d_1^2\cot\alpha\tan^2\alpha$ $= \dfrac{1}{2}d_1^2\cot\alpha\left(\sec^2\alpha - 1\right) = \text{RHS}$ Hence P_1 is true. Let P_n be true for some *n* = *k*, $k \in \mathbb{Z}^+$ i.e. $\displaystyle\sum_{r=1}^{k} A_r = \frac{1}{2}d_1^2\cot\alpha\left(\sec^{2k}\alpha - 1\right)$ Then $\displaystyle\sum_{r=1}^{k+1} A_r = \sum_{r=1}^{k} A_r + A_{k+1}$ $= \dfrac{1}{2}d_1^2\cot\alpha\left(\sec^{2k}\alpha - 1\right) + \dfrac{1}{2}OB_{k+1}OB_{k+2}\sin\alpha$ $= \dfrac{1}{2}d_1^2\cot\alpha\left(\sec^{2k}\alpha - 1\right) + \dfrac{1}{2}\left(\sec^k\alpha\ d_1\right)\left(\sec^{k+1}\alpha\ d_1\right)\sin\alpha$ from **(i)** $= \dfrac{1}{2}d_1^2\cot\alpha\left(\sec^{2k}\alpha - 1\right) + \dfrac{1}{2}d_1^2\sec^{2k+1}\alpha\sin\alpha$ $= \dfrac{1}{2}d_1^2\cot\alpha\left(\sec^{2k}\alpha - 1\right) + \dfrac{1}{2}d_1^2\cot\alpha\sec^{2k}\alpha\tan^2\alpha$ $= \dfrac{1}{2}d_1^2\cot\alpha\left(\sec^{2k}\alpha - 1\right) + \dfrac{1}{2}d_1^2\cot\alpha\sec^{2k}\alpha\left(\sec^2\alpha - 1\right)$

	$=\frac{1}{2}d_1^2\cot\alpha\left(\sec^{2k+2}\alpha-\sec^{2k}\alpha+\sec^{2k}\alpha-1\right)$ $=\frac{1}{2}d_1^2\cot\alpha\left(\sec^{2k+2}\alpha-1\right)$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true. Hence by the principle of mathematical induction, P_n is true for all $n\in\mathbb{Z}^+$

(iii) Prove the result in **(ii)** using another method. [4]

(iii)	$A_i=\frac{1}{2}\left(OB_i\right)\left(OB_{i+1}\right)\sin\alpha=\frac{1}{2}\left(\sec^{i-1}\alpha\ d_1\right)\left(\sec^{i}\alpha\ d_1\right)\sin\alpha=\frac{1}{2}d_1^2\left(\sec^{2i-1}\alpha\right)\sin\alpha$ Hence $\sum_{r=1}^{n}A_r=\frac{1}{2}d_1^2\sin\alpha\sum_{r=1}^{n}\sec^{2r-1}\alpha$ which is a G.P. $=\frac{1}{2}d_1^2\sin\alpha\frac{\sec\alpha\left(\sec^{2n}\alpha-1\right)}{\sec^2\alpha-1}$ $=\frac{1}{2}d_1^2\tan\alpha\frac{\left(\sec^{2n}\alpha-1\right)}{\tan^2\alpha}$ $=\frac{1}{2}d_1^2\cot\alpha\left(\sec^{2n}\alpha-1\right)$ (shown)

6 **(a)** Find the number of paths in the array shown below that spell out the word TEMASEK. [3]

```
                        T
                        |
                    T - E - T
                    |   |   |
                T - E - M - E - T
                |   |   |   |   |
            T - E - M - A - M - E - T
            |   |   |   |   |   |   |
        T - E - M - A - S - A - M - E - T
        |   |   |   |   |   |   |   |   |
    T - E - M - A - S - E - S - A - M - E - T
    |   |   |   |   |   |   |   |   |   |   |
T - E - M - A - S - E - K - E - S - A - M - E - T
    |   |   |   |   |   |   |   |   |   |   |
    T - E - M - A - S - E - S - A - M - E - T
        |   |   |   |   |   |   |   |   |
        T - E - M - A - S - A - M - E - T
            |   |   |   |   |   |   |
            T - E - M - A - M - E - T
                |   |   |   |   |
                T - E - M - E - T
                    |   |   |
                    T - E - T
                        |
                        T
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6(a)	$4\times 2^6-4=252$

(b) A teacher has 9 identical pens to distribute among a H3 Mathematics class of 6 students. Find the number of ways this can be done if each student gets at least one pen. [2]

(b)	$\binom{8}{3}=56$

The teacher wishes to assign the 6 students to 3 different consultation slots. Find the number of ways this can be done if no consultation slot is empty. [4]

(b)	$3^6-3\times 2^6+3-0=540$

The 6 students, assumed to be of different heights, then queue up in line at the Temasek Café to purchase snacks. Use the principle of inclusion and exclusion to find the number of ways that they can arrange themselves in line such that no three consecutive students are in ascending order of height, from front to back. [6]

(b)	Let A_1 be the event that the first, second, and third people are in ordered height, A_2 be the event that the second, third, and fourth people are in ordered height, A_3 be the event that the third, fourth, and fifth people are in ordered height, and A_4 be the event that the fourth, fifth and sixth people are in ordered height. By a combination of complementary counting and PIE, we have that our answer will be

$$\left|A_1' \cap A_2' \cap A_3' \cap A_4'\right| = |S| - \left|A_1 \cup A_2 \cup A_3 \cup A_n\right|$$

$$= |S| - \sum_{i=1}^{4}|A_i| + \sum_{i<j}\left|A_i \cap A_j\right| - \sum_{i<j<k}\left|A_i \cap A_j \cap A_k\right| + \left|A_1 \cap A_2 \cap A_3 \cap A_4\right|$$

$$|S| = 6! = 720$$

$$\sum_{i=1}^{4}|A_i| = \binom{6}{3} \times 3! \times 4 = 480$$

$$\sum_{i<j}\left|A_i \cap A_j\right| = \left|A_1 \cap A_2\right| + \left|A_1 \cap A_3\right| + \left|A_1 \cap A_4\right| + \left|A_2 \cap A_3\right| + \left|A_2 \cap A_4\right| + \left|A_3 \cap A_4\right|$$

$$= \binom{6}{4} \times 2! + \binom{6}{5} + \binom{6}{3} + \binom{6}{4} \times 2! + \binom{6}{5} + \binom{6}{4} \times 2!$$

$$= 122$$

$$\sum_{i<j<k}\left|A_i \cap A_j \cap A_k\right| = \left|A_1 \cap A_2 \cap A_3\right| + \left|A_1 \cap A_2 \cap A_4\right| + \left|A_1 \cap A_3 \cap A_4\right| + \left|A_2 \cap A_3 \cap A_4\right|$$

$$= \binom{6}{5} + 1 + 1 + \binom{6}{5}$$

$$= 14$$

$$\left|A_1 \cap A_2 \cap A_3 \cap A_4\right| = 1$$

$$\left|A_1' \cap A_2' \cap A_3' \cap A_4'\right| = 720 - 480 + 122 - 14 + 1 = 349$$

7 **(a)** Let S be a set of n integers. By using the pigeonhole principle, or otherwise, prove that S contains a subset such that the sum of its elements is divisible by n. [4]

7(a)	Let $S=\{a_1,a_2,...,a_n\}$. Consider the following sums $s_1=a_1,$ $s_2=a_1+a_2,$ $\vdots$ $s_n=a_1+a_2+...+a_n$ There are in total n numbers. If any of the s_k is divisible by n, then our subset is $\{a_1,a_2,...,a_k\}$ and we are done. If none of the s_k, $1\le k\le n$, is divisible by n, then the residues of s_k modulo n are $1,2,...,n-1$. We have n sums and $n-1$ residues. By (PP), there exists two sums s_i and s_j, $1\le i<j\le n$, which have the same residue. Then s_j-s_i is divisible by n. Take the subset $\{a_{i+1},a_{i+2},...,a_j\}$ and we are done.

(b) A *linear Diophantine equation* takes the form $ax+by=c$, where a, b and c are given integers. Solving a linear Diophantine equation means that only integer solutions for x and y are sought. For example, consider the equation $3x+4y=1$. If the values of x and y are integers, i.e. $(-1,1)$, $(3,-2)$, $(7,-5)$ and so on, a family of solutions may be described.

(i) Prove that the linear Diophantine equation $ax+by=c$ has a solution if and only if $d\,|\,c$, where $d=\gcd(a,b)$. [4]

(b)(i)	$(\Rightarrow)$ Assume (x_0,y_0) is a solution, i.e. $ax_0+by_0=c$. Since $d\,|\,a$ and $d\,|\,b$, $d\,|\,(ax_0+by_0)$ and so $d\,|\,c$. $(\Leftarrow)$ Assume $d\,|\,c$. Then $c=dk$ for some integer k. Also, $d=au+bv$ for some integers u and v. Thus we have $c=dk=a(uk)+b(vk)$ which means that the equation $ax+by=c$ has a solution (namely uk and vk).

(ii) Hence prove that if (x_0,y_0) is found to be a solution to the linear Diophantine equation $ax+by=c$, then the general solution is given by

$$x=x_0+\frac{b}{d}t,\quad y=y_0-\frac{a}{d}t,\ \text{ where } t\in\mathbb{Z}. \qquad [3]$$

(ii)	Let x and y be integer solutions to the equation. Since $ax+by=c=ax_0+by_0$, we have $a(x-x_0)+b(y-y_0)=0$ and so dividing by d we get $\frac{a}{d}(x-x_0)=-\frac{b}{d}(y-y_0)$. Since $\frac{a}{d}$ and $\frac{b}{d}$ are relatively prime, $(x-x_0)$ must be divisible by $\frac{b}{d}$ and so $x=x_0+\frac{b}{d}t$ for some integer t. Similarly, we have $y=y_0-\frac{a}{d}t$ for some integer t.

(iii) Two linear Diophantine equations are given below. For the one that can be solved, find the solutions (x, y) such that $x > 0$, $y > 0$. Explain why the other cannot be solved. [4]

(A) $12x + 21y = 80$

(B) $4x + 7y = 97$

(iii)	(A) $12x + 21y = 80 \quad \Rightarrow \quad 3(4x + 7y) = 80$ However 3 does not divide 80 and so equation has no solution. (B) We have $\gcd(4,7) = 1$ and notice that $4(2) + 7(-1) = 1$. Then a solution is $(2 \times 97, -1 \times 97)$ and the general solution can be written as $x = 2 \times 97 + 7t$ and $y = -97 - 4t$. We want $2 \times 97 + 7t > 0$ and $-97 - 4t > 0$. This gives us $-\frac{2 \times 97}{7} < t < -\frac{97}{4}$ which means $t = -25, -26, -27$. The solutions are (5, 11), (12, 7) and (19, 3).

8 **(a)** Let a_i, b_i be non-zero real numbers, $i = 1, 2, \cdots, n$.

(i) Prove the Cauchy Schwarz's inequality

$$\left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i^{\,2}\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$$ [4]

8(a)(i)	We can consider $\displaystyle\sum_{i=1}^{n}\left(a_i t - b_i\right)^2 > 0$ for all $t \in \mathbb{R}$, $\displaystyle\therefore \left(\sum_{i=1}^{n} a_i^{\,2}\right)t^2 - 2\left(\sum_{i=1}^{n} a_i b_i\right)t + \sum_{i=1}^{n} b_i^{\,2} > 0$ for all $t \in \mathbb{R}$. When $\displaystyle\sum_{i=1}^{n} a_i^{\,2} = 0$, the inequality is trivial (become equal). So we need only to consider $\displaystyle\sum_{i=1}^{n} a_i^{\,2} > 0$. For the quadratic function to be non-negative for all values of *t*, we have discriminant < 0. $\displaystyle 4\left(\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i^{\,2}\right) < 0$ $\displaystyle \Rightarrow \left(\sum_{i=1}^{n} a_i b_i\right)^2 < \left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i^{\,2}\right)$ Equality holds if and only if $a_i = kb_i$ for all real constant *k*. **Alternatively,** Let $a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \\ a_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \\ b_n \end{pmatrix}$ Since $a \bullet b = \lvert a\rvert\lvert b\rvert\cos\theta \Rightarrow \lvert a \bullet b\rvert = \lvert a\rvert\lvert b\rvert\lvert\cos\theta\rvert$ and $\lvert\cos\theta\rvert \leq 1$, we have $\lvert a \bullet b\rvert \leq \lvert a\rvert\lvert b\rvert$ i.e. $\lvert a_1b_1 + a_2b_2 + \cdots a_nb_n\rvert \leq \sqrt{a_1^{\,2} + a_2^{\,2} + \cdots a_n^{\,2}}\sqrt{b_1^{\,2} + b_2^{\,2} + \cdots b_n^{\,2}}$ $\Rightarrow \left(a_1b_1 + a_2b_2 + \cdots a_nb_n\right)^2 \leq \left(a_1^{\,2} + a_2^{\,2} + \cdots a_n^{\,2}\right)\left(b_1^{\,2} + b_2^{\,2} + \cdots b_n^{\,2}\right)$ $\displaystyle \therefore \left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i^{\,2}\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$

(ii) If $p \leq \dfrac{b_i}{a_i} \leq q$, show that $pqa_i^{\,2} - (p+q)a_ib_i + b_i^{\,2} \leq 0$ and deduce that

$$(p+q)\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} b_i^{\,2} + pq\sum_{i=1}^{n} a_i^{\,2} \quad .$$ [3]

(a)(ii)	Given $p \le \frac{b_i}{a_i} \le q \Rightarrow \left(p-\frac{b_i}{a_i}\right) \le 0$ i.e. $\frac{1}{a_i}\left(pa_i - b_i\right) \le 0$ $\left(q-\frac{b_i}{a_i}\right) \ge 0$ i.e. $\frac{1}{a_i}\left(qa_i - b_i\right) \ge 0$ We have $\frac{1}{a_i}\left(pa_i - b_i\right)\ \frac{1}{a_i}\left(qa_i - b_i\right) \le 0$ $\Rightarrow \frac{1}{a_i^{2}}\left(pqa_i^{2} - \left(p+q\right)a_ib_i + b_i^{2}\right) \le 0$ $\Rightarrow \left(pqa_i^{2} - \left(p+q\right)a_ib_i + b_i^{2}\right) \le 0$ $\Rightarrow \sum_{i=1}^{n}\left(pqa_i^{2} - \left(p+q\right)a_ib_i + b_i^{2}\right) \le 0$ $\Rightarrow pq\sum_{i=1}^{n}\left(a_i^{2}\right) - \left(p+q\right)\sum_{i=1}^{n}\left(a_ib_i\right) + \sum_{i=1}^{n}\left(b_i^{2}\right) \le 0$ $\therefore \left(p+q\right)\sum_{i=1}^{n}a_ib_i \ge \sum_{i=1}^{n}b_i^{2} + pq\sum_{i=1}^{n}a_i^{2}$

(iii) If $0 < m \le a_i \le M$ and $0 < m \le b_i \le M$, by using **(ii)** or otherwise, show that

$$\left(\sum_{i=1}^{n}a_i^{2}\right)\left(\sum_{i=1}^{n}b_i^{2}\right) \le \frac{1}{4}\left(\frac{M}{m}+\frac{m}{M}\right)^{2}\left(\sum_{i=1}^{n}a_ib_i\right)^{2}.$$

[3]

(a)(iii)	From **(ii)** $\therefore \left(\frac{m}{M}+\frac{M}{m}\right)\sum_{i=1}^{n}a_ib_i \ge \sum_{i=1}^{n}b_i^{2} + \left(\frac{m}{M}\right)\left(\frac{M}{m}\right)\sum_{i=1}^{n}a_i^{2}$ Since $\sum_{i=1}^{n}b_i^{2} + \sum_{i=1}^{n}a_i^{2} \ge 2\sqrt{\sum_{i=1}^{n}a_i^{2}\sum_{i=1}^{n}b_i^{2}}$ ($\because AM \ge GM$) We have $\Rightarrow \left(\frac{m}{M}+\frac{M}{m}\right)\sum_{i=1}^{n}a_ib_i \ge \sum_{i=1}^{n}b_i^{2} + \sum_{i=1}^{n}a_i^{2} \ge 2\sqrt{\sum_{i=1}^{n}a_i^{2}\sum_{i=1}^{n}b_i^{2}}$ Squaring both sides: $\frac{1}{4}\left(\frac{m}{M}+\frac{M}{m}\right)^{2}\left\{\sum_{i=1}^{n}a_ib_i\right\}^{2} \ge \sum_{i=1}^{n}a_i^{2}\sum_{i=1}^{n}b_i^{2}$ (shown)

(b) Using **(a)** or otherwise, show that

$$\left(n+\frac{1}{9}\right)^{2} < \left\{\sum_{i=1}^{n}\left(1+\frac{1}{3^{i}}\right)^{2}\right\}\left\{\sum_{i=1}^{n}\left(1-\frac{1}{3^{i+1}}\right)^{2}\right\} < \frac{169}{144}\left(n+\frac{1}{3}\right)^{2}.$$

[6]

(b)	Let $a_i = 1+\frac{1}{3^{i}}, b_i = 1-\frac{1}{3^{i+1}}$ $\Rightarrow 1-\frac{1}{3^{2}} \le a_i, b_i \le 1+\frac{1}{3^{1}}$ $\Rightarrow \frac{8}{9} \le a_i, b_i \le \frac{4}{3}$

Take $m=\frac{8}{9}, M=\frac{4}{3}$

By **(a)(iii)**,
$$\begin{aligned}\left(\sum_{i=1}^{n}\left(1+\frac{1}{3^i}\right)^2\right)\left(\sum_{i=1}^{n}\left(1-\frac{1}{3^{i+1}}\right)^2\right) &\le \frac{1}{4}\left(\frac{\frac{4}{3}}{\frac{8}{9}}+\frac{\frac{8}{9}}{\frac{4}{3}}\right)^2\left(\sum_{i=1}^{n}\left(1+\frac{1}{3^i}\right)\left(1-\frac{1}{3^{i+1}}\right)\right)^2\\ &=\frac{169}{144}\left(n+\frac{2}{3}\sum_{i=1}^{n}\left(\frac{1}{3^i}\right)-\frac{1}{3}\sum_{i=1}^{n}\left(\frac{1}{9^i}\right)\right)^2\\ &<\frac{169}{144}\left(n+\frac{2}{3}\sum_{i=1}^{n}\left(\frac{1}{3^i}\right)\right)^2\\ &<\frac{169}{144}\left(n+\frac{2}{3}\frac{\frac{1}{3}}{1-\frac{1}{3}}\right)^2=\frac{169}{144}\left(n+\frac{1}{3}\right)^2\end{aligned}$$

By **(a)(i)**,
$$\begin{aligned}\left(\sum_{i=1}^{n}\left(1+\frac{1}{3^i}\right)^2\right)\left(\sum_{i=1}^{n}\left(1-\frac{1}{3^{i+1}}\right)^2\right) &\ge \left(\sum_{i=1}^{n}\left(1+\frac{1}{3^i}\right)\left(1-\frac{1}{3^{i+1}}\right)\right)^2\\ &=\left\{n+\frac{2}{3}\sum_{i=1}^{n}\frac{1}{3^i}-\frac{1}{3}\sum_{i=1}^{n}\frac{1}{9^i}\right\}^2\\ &>\left\{n+\frac{2}{3}\sum_{i=1}^{n}\frac{1}{3^i}-\frac{1}{3}\sum_{i=1}^{n}\frac{1}{3^i}\right\}^2\\ &=\left\{n+\frac{1}{3}\sum_{i=1}^{n}\frac{1}{3^i}\right\}^2\\ &\ge\left(n+\frac{1}{3}\cdot\frac{1}{3}\right)^2=\left(n+\frac{1}{9}\right)^2\end{aligned}$$