2019

JC1 H2 Math

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ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

4 Oct 2019

H2 Mathematics Paper (100 marks)

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
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12	
Total	

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1 The graph of $y = f(x)$ undergoes transformations in the following order:

I. Reflection in the x-axis

II. Translation in the positive x-direction by 4 units

III. Scaling parallel to the x-axis by a scale factor of $\frac{1}{3}$

The equation of the resulting graph is $e^{2y} = 7 - 3x$.

Find the equation of the original graph in the form $y = f(x)$. [3]

2 The curve of a cubic polynomial $y = f(x)$ has y-intercept 5 and a turning point at $(1,1)$. Given that it passes through $(2,7)$, find the equation of the curve. [4]

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3 A curve C has equation $(\ln y)^2 - 3xy + x^2 - 4 = 0$.

(i) Show that $\frac{dy}{dx} = \frac{(3y - 2x)y}{2\ln y - 3xy}$. [3]

(ii) It is given that C contains the point A with coordinates $(\alpha, 1)$ where $\alpha < 0$. Find the value of α and hence the equation of the normal to C at the point A. [3]

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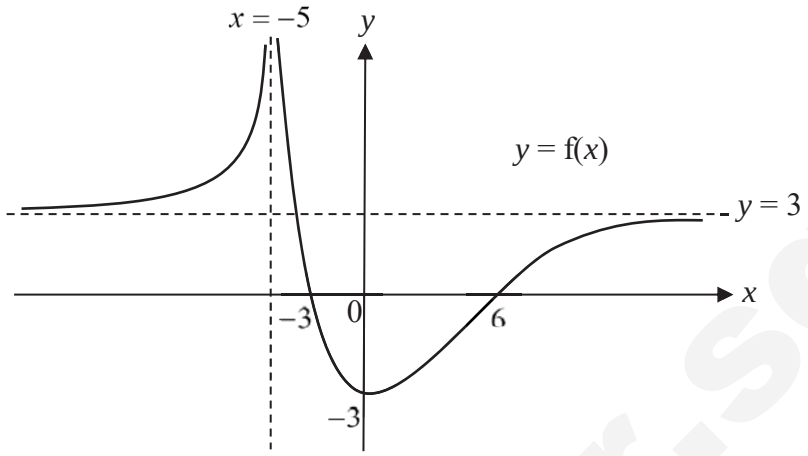
4 The diagram shows a sketch of the curve $y = \mathrm{f}(x)$. The curve cuts the x-axis at $(-3,\ 0)$ and $(6,\ 0)$. It has a stationary point at $(0, -3)$ and asymptotes $x = -5$ and $y = 3$.



On two separate diagrams, sketch the graphs of the following equations. Show clearly the equations of asymptotes and the coordinates of the points of intersection with the axes, if any.

(a) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

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(b) $y = \left|\mathrm{f}\left(-x\right)\right|$. [3]

5 **(a)** Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin 2x - 2x\cos 2x\right)$. Hence, or otherwise, find $\int x^2 \cos 2x \, \mathrm{d}x$. [4]

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(b) If $0 < a < 1$, find $\int_0^1 (a+x)|a-x|\,\mathrm{d}x$, leaving your answer in terms of a. [3]

6 **(a)** It is given that

$$\mathrm{f}(x) = \begin{cases} 2+\sqrt{9-(x-3)^2} & \text{for } 0 < x < 6, \\ 2 & \text{for } 6 \le x \le 12. \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+12)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-11 \le x \le 12$. [3]

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(ii) Find the exact value of $\int_{-9}^{10} f(x)\,dx$. [2]

(b) The curve C has equation $y = \dfrac{2x^2+3}{x^2-6}$.

Using an algebraic method, find the set of values of y that C can take. [3]

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7 Referred to the origin O, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero and non-parallel vectors denoting the position vectors of the points A, B and C respectively.

(i) Given that $\mathbf{a}\times\mathbf{b}=3\mathbf{a}\times\mathbf{c}$, show that $\mathbf{b}-3\mathbf{c}=\lambda\mathbf{a}$ where λ is a scalar. [2]

The point M is the mid-point of OC and the point N lies on OB produced such that $3ON = 5OB$. The point P lies on MN such that MP: $MN = 2 : 3$.

(ii) Show that the position vector of P is $\frac{10}{9}\mathbf{b}+\frac{1}{6}\mathbf{c}$. [1]

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It is given that **b** is a unit vector, $|\mathbf{c}| = \sqrt{2}$, $|\mathbf{b} - 3\mathbf{c}| = 5$ and the angle between **b** and **c** is $45°$.

(iii) Find the exact length of projection of $\overrightarrow{OP}$ on $\overrightarrow{OA}$. [4]

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8 **(a)** The curve C is defined by the equations

$$x = 2\operatorname{cosec} t, \quad y = 5\cos t \sin^3 t, \quad \text{for } 0 < t \leq \frac{\pi}{2}.$$

The region R is bounded by C, the x-axis and $x = 4$ in the first quadrant. Find the exact area of the region R. [6]

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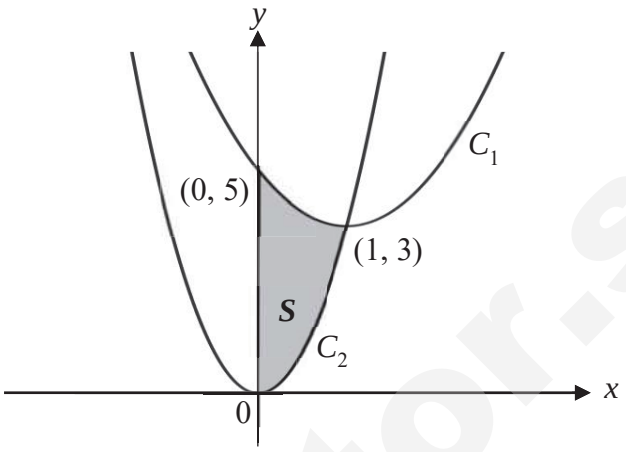
(b) The diagram below shows the curves C_1 and C_2 with equations $y = 2(x-1)^2 + 3$ and $y = 3x^2$ respectively. The region in the first quadrant enclosed by the curves and the y-axis is denoted by S. Find the volume of the solid generated when the region S is rotated through 2π radians about the y-axis, giving your answer correct to 4 decimal places.



[3]

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9 **(a)** The function g is defined by

$$g : x \mapsto ax + b, \quad x \in \mathbb{R}, x > 0,$$

where a and b are positive real numbers.

Show that g^2 exists and hence determine the range of g^2, leaving your answer in terms of a and b. [3]

(b) Function h is defined by

$$h : x \mapsto \frac{x+7}{x-1}, \quad x \in \mathbb{R}, x \neq 1.$$

(i) Find $h^{-1}(x)$ and state the domain of h^{-1}. [3]

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(ii) Find the exact values of c such that $\mathrm{h}^{2018}(c) = \mathrm{h}^{-1}(c)$. Explain your answers clearly. [3]

(c) Function f is defined by

$$\mathrm{f} : x \mapsto 2x^2 - \lambda x + 5, \quad x \in \mathbb{R},$$

where λ is a non-zero constant.

(i) Give a reason why f^{-1} does not exists. [1]

(ii) For the function f defined above, the range of f is $[-3, \infty)$. If the domain of f is restricted to the set of all positive real numbers, f^{-1} exists. Find the value of λ. [2]

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10 The point P has position vector $2\mathbf{i}+5\mathbf{j}+\mathbf{k}$. The plane Π_1 has equation $\mathbf{r}\cdot(-\mathbf{i}+2\mathbf{k})=10$.

(i) Find the coordinates of N, the foot of perpendicular from P to Π_1. [3]

The point A with position vector $-4\mathbf{i}+\alpha\mathbf{j}+3\mathbf{k}$, where $\alpha\in\mathbb{R}$, lies on plane Π_1.
The plane Π_2 has equation $\mathbf{r}\cdot(\mathbf{i}+3\mathbf{k})=5$.

(ii) Show that point A lies on the plane Π_2. Hence, using an algebraic method, find a vector equation of l, the line of intersection of Π_1 and Π_2. Leave your answer in terms of α. [3]

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The plane Π_3 is the image of the plane Π_2 about the plane Π_1. Given that point P lies on plane Π_2,

(iii) find an equation of plane Π_3 in scalar product form. [3]

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The line l_1 has equation $x = -4, y = z + 2$.

(iv) Find the position vectors of the points on the line l_1 such that the shortest distance from those points to the plane Π_1 is $\frac{4\sqrt{5}}{5}$ units. [4]

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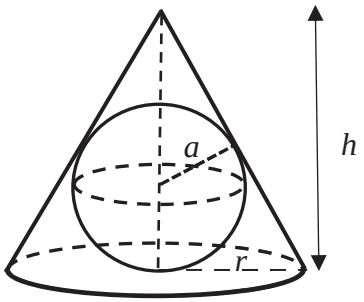
11 (a)



A decorative artifact, as shown in the diagram above, is in the shape of a cone with radius r and height h with a sphere of fixed radius a inscribed in it.

(i) Show that the volume of the cone, V, is given by $V = \dfrac{\pi a^2 h^2}{3(h-2a)}$.

[Volume of cone, $V = \dfrac{1}{3}\pi r^2 h$] [3]

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(ii) Use differentiation to find, in terms of a, the minimum value of V. Leave your answer in exact form. [5]

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(b) In a triangle PQR, $PQ = 3$ cm and $PR = 2$ cm. If angle QPR is increasing at a constant rate of 0.1 radians per second, find the rate of increase of the length QR at the instant when angle QPR is $\frac{\pi}{3}$ radians. [4]

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12 Ben and his wife, Jerrie, intend to save for their child's education in the future.

(a) Jerrie decides to save \$200 on 1st January 2020. On the first day of each subsequent month she will save \$50 more than in the previous month, so that she saves \$250 on 1st February 2020, \$300 on 1st March 2020, and so on. Find the earliest date for Jerrie to save at least \$22000 in total. [4]

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(b) Ben decides to put \$400 into a bank account on 1st January 2020. On the first day of each subsequent month from February 2020, he puts another \$150 into the account. The interest rate is 1% per month, such that on the last day of each month, the amount in the account on that day is increased by 1%.

(i) Show that the value of Ben's account on the last day of the nth month (where January 2020 is the 1st month, February 2020 is the 2nd month, and so on) is $15400(1.01)^n - 15150$. [3]

(ii) Hence, calculate the earliest date for Ben to save at least \$59900 in his bank account. [3]

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(iii) Ben wants to save \$88000 in his bank account by 2nd January 2038 for his child's university education. Given that the initial amount deposited on 1st January 2020 remains at \$400 and the interest rate is 1% per month, how much does Ben need to put into his savings account every month, from February 2020, instead, in order to have at least \$88000 in his savings account by 2nd January 2038? [3]

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[Turn Over

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1 The graph of $y = \mathrm{f}(x)$ undergoes transformations in the following order:

I. Reflection in the x-axis
II. Translation in the positive x-direction by 4 units
III. Scaling parallel to the x-axis by a scale factor of $\frac{1}{3}$

The equation of the resulting graph is $\mathrm{e}^{2y} = 7 - 3x$.
Find the equation of the original graph in the form $y = \mathrm{f}(x)$. [3]

Solution

Method 1

Reverse III: Scaling parallel to the x-axis by a scale factor of 3. (Replace x by $\frac{x}{3}$)

Equation becomes: $\mathrm{e}^{2y} = 7 - x$

Reverse II: Translation in the negative x-direction by 4 units. (Replace x by $x+4$)

Equation becomes: $\mathrm{e}^{2y} = 7 - (x+4)$

$\mathrm{e}^{2y} = 3 - x$

Reverse I: Reflection in the x-axis (replace y by $-y$)

Equation becomes $\mathrm{e}^{-2y} = 3 - x$

$\Rightarrow y = \mathrm{f}(x) = -\frac{1}{2}\ln(3-x)$

Method 2

After the 3 transformations, $y = \mathrm{f}(x)$ becomes $y = -\mathrm{f}(3x-4)$

As the final curve is $y = -\frac{1}{2}\ln(7-3x)$

So $\mathrm{f}(3x-4) = \frac{1}{2}\ln(7-3x)$

Let $z = 3x-4 \Rightarrow x = \frac{z+4}{3}$

$\mathrm{f}(z) = \frac{1}{2}\ln\left[7 - 3\left(\frac{z+4}{3}\right)\right] = \frac{1}{2}\ln(3-z)$

$y = \mathrm{f}(x) = \frac{1}{2}\ln(3-x)$

Method 3

After the 3 transformations, $y = \mathrm{f}(x)$ becomes $y = -\mathrm{f}(3x-4)$

As the final curve is $y = -\frac{1}{2}\ln(7-3x)$

So $\mathrm{f}(3x-4) = \frac{1}{2}\ln(7-3x)$

$\mathrm{f}(3x-4) = \frac{1}{2}\ln[-(3x-4)+3]$

$\mathrm{f}(x) = \frac{1}{2}\ln(3-x)$

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2	The curve of a cubic polynomial $y = \mathrm{f}(x)$ has y-intercept 5 and a turning point at $(1,1)$. Given that it passes through $(2,7)$, find the equation of the curve.	[4]
	Solution	
	Let the polynomial be $\mathrm{f}(x) = ax^3 + bx^2 + cx + 5$. At $(1,1)$, $a + b + c = -4 \cdots\cdots (1)$ At $(2,7)$, $8a + 4b + 2c = 2 \Rightarrow 4a + 2b + c = 1 \cdots\cdots (2)$ $\mathrm{f}'(x) = 3ax^2 + 2bx + c$ When $x = 1$, $3a + 2b + c = 0 \cdots\cdots (3)$ Using G.C, $a = 1, b = 2, c = -7$ Equation of curve is $\mathrm{f}(x) = x^3 + 2x^2 - 7x + 5$	
3	A curve C has equation $(\ln y)^2 - 3xy + x^2 - 4 = 0$.	
	(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(3y - 2x)y}{2\ln y - 3xy}$.	[3]
	(ii) It is given that C contains point A with coordinates $(\alpha, 1)$ where $\alpha < 0$. Find the value of α and hence the equation of the normal to C at the point A.	[3]
	Solution	

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(i) $(\ln y)^2 - 3xy + x^2 - 4 = 0.$

Differentiate w.r.t. x:

$$2\ln y\left(\frac{1}{y}\right)\frac{dy}{dx} - \left(3x\frac{dy}{dx} + 3y\right) + 2x = 0$$

$$\left(\frac{2\ln y}{y} - 3x\right)\frac{dy}{dx} = 3y - 2x$$

$$\frac{dy}{dx} = \frac{(3y-2x)y}{2\ln y - 3xy}$$

(ii) At A, $y = 1$. Substitute into equation of C:

$$-3x + x^2 - 4 = 0$$

$$(x-4)(x+1) = 0$$

$x = 4$ or -1

Since $x < 0$, $x = -1 \Rightarrow \alpha = -1$

$$\frac{dy}{dx} = \frac{3-2(-1)}{-3(-1)} = \frac{5}{3}$$

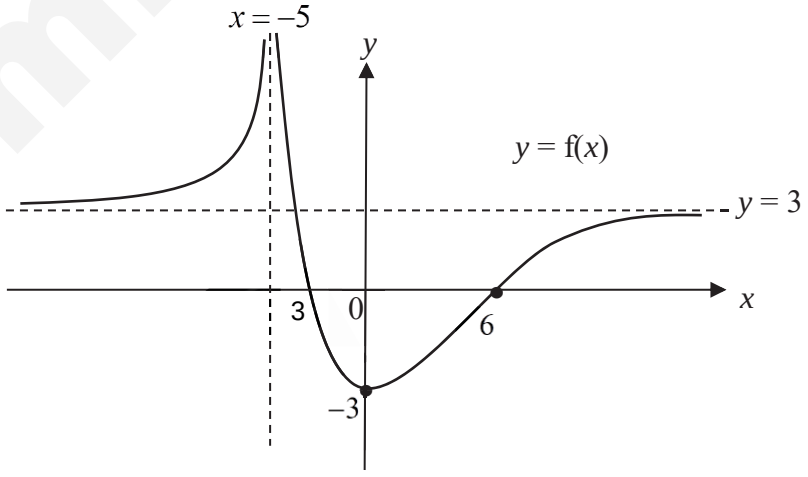
Gradient of normal $= -\frac{3}{5}$

Equation of normal to C at A is

$$y - 1 = -\frac{3}{5}(x+1)$$

$$y = -\frac{3}{5}x + \frac{2}{5}$$

4 The diagram shows a sketch of the curve $y = \mathrm{f}(x)$. The curve cuts the x-axis at $(-3,\ 0)$ and $(6,\ 0)$. It has a stationary point at $(0, -3)$ and asymptotes $x = -5$ and $y = 3$.



On separate diagrams, sketch the graphs of

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \left|\mathrm{f}(-x)\right|$, [3]

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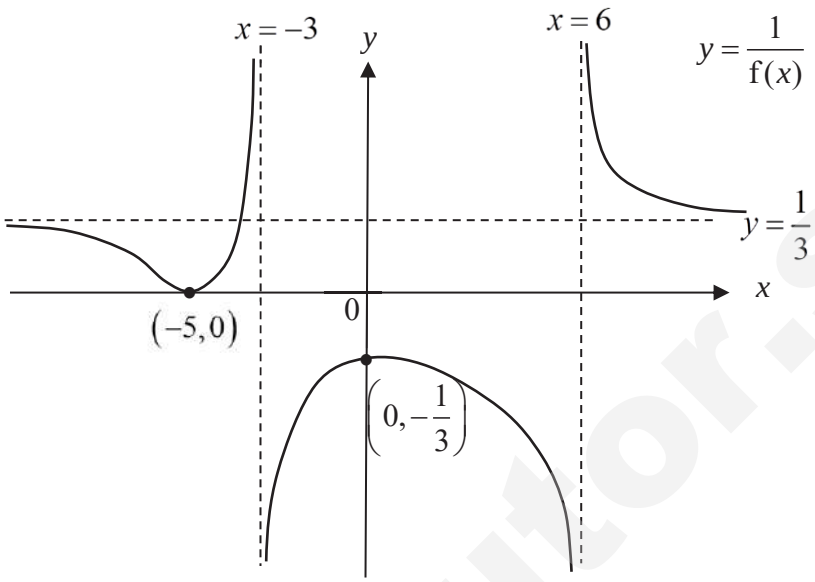
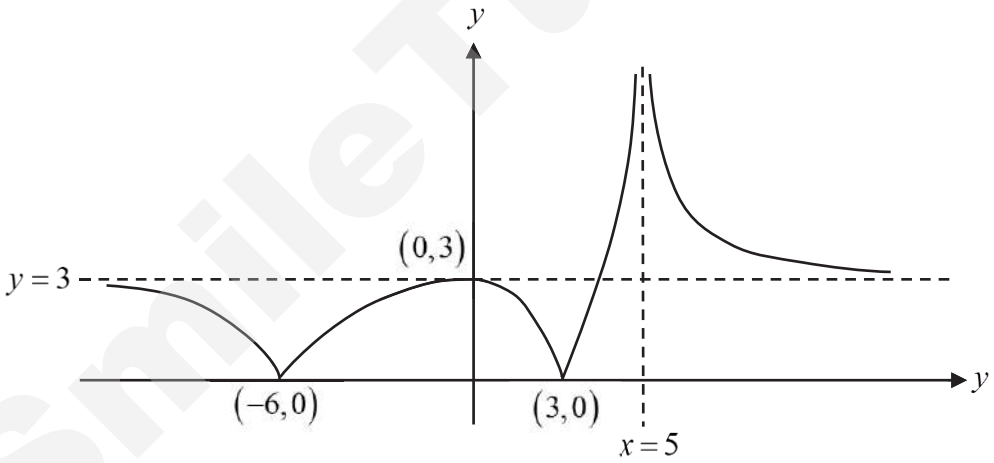
	indicating clearly, in each case, the equations of asymptotes and the coordinates of the points of intersection with the axes, if any.	
	Solution	
	(i)  	
	(ii)  	
5	**(a)** Find $\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x - 2x\cos 2x)$. Hence, or otherwise, find $\int x^2 \cos 2x \, \mathrm{d}x$.	[4]
	(b) If $0 < a < 1$, find $\int_0^1 (a+x)\lvert a-x\rvert \, \mathrm{d}x$, leaving your answers in terms of a.	[3]
	Solution	

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(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sin 2x - 2x\cos 2x\right)$

$= 2\cos 2x - [2x(-2\sin 2x) + 2\cos 2x]$

$= 2\cos 2x + 4x\sin 2x - 2\cos 2$

$= \dfrac{x^2 \sin 2x}{2} - \dfrac{1}{4}\sin 2x + \dfrac{1}{2}x\cos 2x + C$

(b) $\int_0^1 (a+x)|a-x|\,\mathrm{d}x,\ 0<a<1$

$= \int_0^a (a+x)(a-x)\,\mathrm{d}x + \int_a^1 (a+x)[-(a-x)]\,\mathrm{d}x$

$= \int_0^a a^2 - x^2\,\mathrm{d}x - \int_a^1 a^2 - x^2\,\mathrm{d}x$

$= \left[a^2x - \dfrac{x^3}{3}\right]_0^a - \left[a^2x - \dfrac{x^3}{3}\right]_a^1$

$= \left[\left(a^3 - \dfrac{a^3}{3}\right) - 0\right] - \left[\left(a^2 - \dfrac{1}{3}\right) - \left(a^3 - \dfrac{a^3}{3}\right)\right]$

$= \dfrac{4a^3}{3} - a^2 + \dfrac{1}{3}$

6 **(a)** It is given that

$$\mathrm{f}(x) = \begin{cases} 2 + \sqrt{9-(x-3)^2} & \text{for } 0 < x < 6, \\ 2 & \text{for } 6 \le x \le 12. \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+12)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-11 \le x \le 12$. [3]

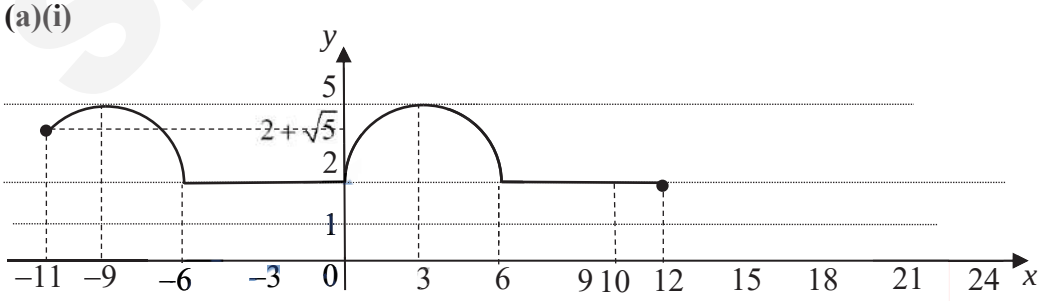
(ii) Find the exact value of $\int_{-9}^{10} \mathrm{f}(x)\,\mathrm{d}x$. [2]

(b) The curve C

Using an algebraic method, find the set of values of y that C can take. [3]

Solution

(a)(i)



(ii) $\int_{-9}^{10} \mathrm{f}(x)\,\mathrm{d}x$ = Area under curve bounded by x-axis from $x = -9$ to $x = 10$

$= 3\left(\dfrac{1}{4}\pi(3)^2\right) + 19(2)$

$= \dfrac{27}{4}\pi + 38$

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(b) $y = \dfrac{2x^2+3}{x^2-6}$

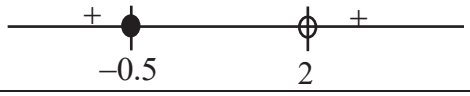
$x^2y - 6y = 2x^2 + 3$

$(y-2)x^2 - 6y - 3 = 0$

For the equation to have real solutions, discriminant ≥ 0 , $y \neq 2$

$0 - 4(y-2)(-6y-3) \geq 0$

$(y-2)(2y+1) \geq 0, y \neq 2$



hence, $\left\{ y \in \mathbb{R} : y < -\dfrac{1}{2} \text{ or } y > 2 \right\}$.

7 Referred to the origin *O*, **a**, **b** and **c** are non-zero and non-parallel vectors denoting the position vectors of the points *A*, *B* and *C* respectively.

(i) Given that $\mathbf{a} \times \mathbf{b} = 3\mathbf{a} \times \mathbf{c}$, show that $\mathbf{b} - 3\mathbf{c} = \lambda \mathbf{a}$ where λ is a scalar. [2]

The point *M* is the mid-point of *OC* and the point *N* lies on *OB* produced such that $3ON = 5OB$. The point *P* lies on *MN* such that $MP : MN = 2 : 3$.

(ii) Show that the position vector of *P* is $\dfrac{10}{9}\mathbf{b} + \dfrac{1}{6}\mathbf{c}$. [1]

It is given that **b** is a unit vector, $|\mathbf{c}| = \sqrt{2}$, $|\mathbf{b} - 3\mathbf{c}| = 5$ and the angle between **b** and **c** is $45°$.

(iii) Find the exact length of projection of $\overrightarrow{OP}$ on $\overrightarrow{OA}$. [4]

Solution

(i)

$\mathbf{a} \times \mathbf{b} = 3\mathbf{a} \times \mathbf{c}$

$(\mathbf{a} \times \mathbf{b}) - (3\mathbf{a} \times \mathbf{c}) = \mathbf{0}$

$\mathbf{a} \times (\mathbf{b} - 3\mathbf{c}) = \mathbf{0}$

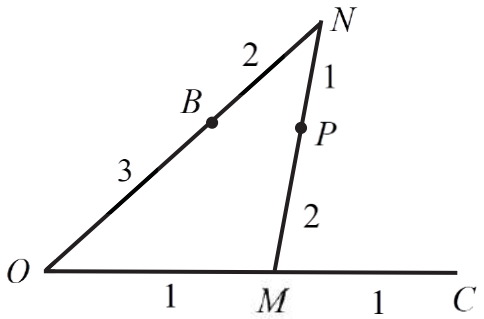
a is parallel to **b** – 3**c**, hence $\mathbf{b} - 3\mathbf{c} = \lambda \mathbf{a}$.

(ii) By ratio theorem,

$\overrightarrow{OP} = \dfrac{2\overrightarrow{ON} + \overrightarrow{OM}}{3}$

$= \dfrac{1}{3}\left(2 \times \dfrac{5}{3}\mathbf{b} + \dfrac{1}{2}\mathbf{c}\right)$

$= \dfrac{10}{9}\mathbf{b} + \dfrac{1}{6}\mathbf{c}$ or $\dfrac{20\mathbf{b} + 3\mathbf{c}}{18}$



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(iii) length of projection $= \left|\overrightarrow{OP}\cdot\frac{\overrightarrow{OA}}{\left|\overrightarrow{OA}\right|}\right| = \frac{1}{18}\left|\frac{(20\mathbf{b}+3\mathbf{c})\cdot\mathbf{a}}{|\mathbf{a}|}\right|$

$= \frac{1}{18}\left|\frac{(20\mathbf{b}+3\mathbf{c})\cdot\lambda(\mathbf{b}-3\mathbf{c})}{|\lambda||\mathbf{b}-3\mathbf{c}|}\right|$

$= \frac{1}{18}\left|\frac{20\mathbf{b}\cdot\mathbf{b}-57\mathbf{c}\cdot\mathbf{b}-9\mathbf{c}\cdot\mathbf{c}}{|\mathbf{b}-3\mathbf{c}|}\right|$

$= \frac{1}{18}\left|\frac{20|\mathbf{b}|^2-57\mathbf{c}\cdot\mathbf{b}-9|\mathbf{c}|^2}{|\mathbf{b}-3\mathbf{c}|}\right|$

$= \frac{1}{18}\left|\frac{20-18-57|\mathbf{c}||\mathbf{b}|\cos 45^\circ}{5}\right|$

$= \frac{1}{18}\left|\frac{-55}{5}\right|$

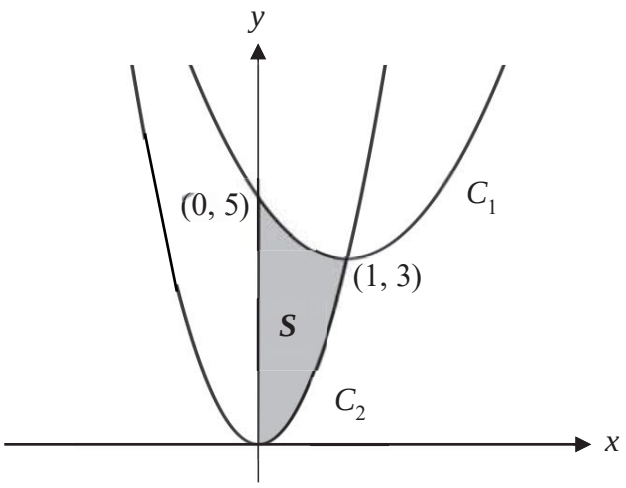
$= \frac{11}{18}$

8 **(a)** The curve C is defined by the equations

$$x = 2\operatorname{cosec} t, \quad y = 5\cos t \sin^3 t, \quad \text{for } 0 < t \le \frac{\pi}{2}.$$

The region R is bounded by C, the x – axis and $x = 4$ in the first quadrant. Find the exact area of the region R. [6]

(b) The diagram below shows the curves C_1 and C_2 with equations $y = 2(x-1)^2 + 3$ and $y = 3x^2$ respectively. The region in the first quadrant enclosed by the curves and the y-axis is denoted by S. Find the volume of the solid generated when the region S is rotated through 2π radians about the y-axis, giving your answer correct to 4 decimal places.



[3]

Solution

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(a) $x = 2\operatorname{cosec} t, \quad y = 5\cos t \sin^3 t, \quad \text{for } 0 < t \leq \frac{\pi}{2}.$

When $x = 2$, $2 = 2\operatorname{cosec} t$.

$\operatorname{cosec} t = 1$

$\sin t = 1$

$t = \frac{\pi}{2}$

When $x = 2$, $4 = 2\operatorname{cosec} t$.

$\operatorname{cosec} t = 2$

$\sin t = \frac{1}{2}$

$t = \frac{\pi}{6}$

$\frac{dx}{dt} = -2\cot t \operatorname{cosec} t$

$\text{Area} = \int_2^4 y \, dx$

$= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \left(5\cos t \sin^3 t\right) \left(-2\cot t \operatorname{cosec} t\right) dt$

$= -10\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \left(5\cos t \sin^3 t\right)\left(\frac{\cos t}{\sin t}\right)\left(\frac{1}{\sin t}\right) dt$

$= 10\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 t \sin t \, dt$

$= -10\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 t(-\sin t) \, dt$

$= -10\left[\frac{\cos^3 t}{3}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= -\frac{10}{3}\left[0 - \left(\frac{\sqrt{3}}{2}\right)^3\right]$

$= \frac{10}{3}\left(\frac{3\sqrt{3}}{8}\right)$

$= \frac{5\sqrt{3}}{4}$

(b)

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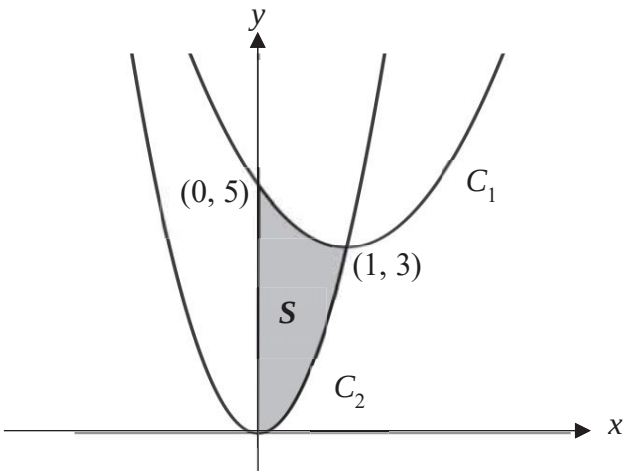

Required Volume

$$= \pi \int_0^3 \frac{y}{3}\,dy + \pi \int_3^5 \left(1 - \sqrt{\frac{y-3}{2}}\right)^2 dy$$

$$= 5.759586 \text{ units}^3$$

$$= 5.7596 \text{ units}^3 \text{ (to 4 dp)}$$

9 **(a)** The function g is defined by

$$g : x \mapsto ax + b, \quad x \in \mathbb{R}, x > 0,$$

where a and b are positive real numbers.

Show that g^2 exists and hence determine the range of g^2, leaving your answer in terms of a and b. [3]

(b) Function h is defined by

$$h : x \mapsto \frac{x+7}{x-1}, \quad x \in \mathbb{R}, x \neq 1.$$

(i) Find $h^{-1}(x)$ and state the domain of h^{-1}. [3]

(ii) Find the exact values of c such that $h^{2018}(c) = h^{-1}(c)$. Explain your answers clearly. [3]

(c) Function f is defined by

$$f : x \mapsto 2x^2 - \lambda x + 5, \quad x \in \mathbb{R},$$

where λ is a non-zero constant.

(i) Give a reason why f^{-1} does not exists. [1]

(ii) For the function f defined above, the range of f is $[-3, \infty)$. If the domain of f is restricted to the set of all positive real numbers, f^{-1} exists. Find the value of λ. [2]

Solution

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(a) For g^2 to exist, $R_g \subseteq D_g$.

From the graph of g, $R_g = (b, \infty)$

$$D_g = (0, \infty)$$

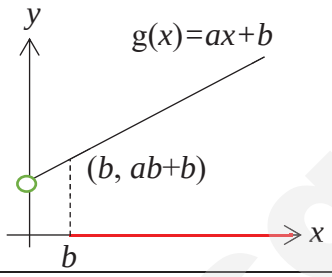
Since $b > 0$, $R_g \subseteq D_g$. Hence g^2 exists.

Consider the graph of g.

By restricting the domain of g to the range of g,

$R_g = (b, \infty)$

$R_{g^2} = (ab + b, \infty)$



(bi) $y = \dfrac{x+7}{x-1}$

$xy - y = x + 7$

$x(y-1) = 7 + y$

$x = \dfrac{7+y}{y-1}$

$h^{-1}(x) = \dfrac{x+7}{x-1}$

$R_h = D_{h^{-1}} = \mathbb{R} \setminus \{-1\}$

(ii)

$h^{2018}(c) = h^{-1}(c)$

$hh^{2018}(c) = hh^{-1}(c)$

$\underbrace{hh^{-1}hh^{-1}...hh^{-1}}h(c) = c \quad (\because h = h^{-1})$

2019 times

$h(c) = c$

$\dfrac{c+7}{c-1} = c$

$c + 7 = c^2 - c$

$c^2 - 2c - 7 = 0$

$c = \dfrac{2 \pm \sqrt{4 - 4(-7)}}{2}$

$c = 1 \pm 2\sqrt{2}$

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(c)(i) $f(x) = 2x^2 - \lambda x + 5$

$$= 2\left(x^2 - \frac{\lambda}{2}x\right) + 5$$

$$= 2\left(x - \frac{\lambda}{4}\right)^2 + 5 - \frac{\lambda^2}{8}$$

Method 1 (Counter-example)

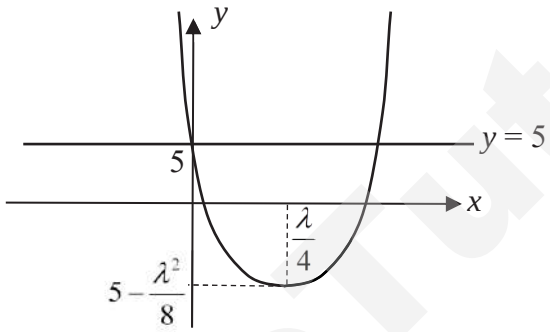
$f(0) = 5, \quad f\left(\frac{\lambda}{2}\right) = 2\left(\frac{\lambda}{2} - \frac{\lambda}{4}\right)^2 + 5 - \frac{\lambda^2}{8} = 5$

Since $f(0) = f\left(\frac{\lambda}{2}\right)$ but $0 \neq \frac{\lambda}{2}$, $0, \frac{\lambda}{2} \in \mathbb{R}$,

$\Rightarrow$ f is not 1-1

$\Rightarrow$ inverse function of f does not exists.

Method 2 (Graphical)



Since the horizontal line $y = 5$ cuts the graph of f at 2 points, f is not 1-1.
Hence f^{-1} does not exists.

(ii) Range of $f = \left[5 - \frac{\lambda^2}{8}, \infty\right) = [-3, \infty)$

$$5 - \frac{\lambda^2}{8} = -3$$

$$\lambda^2 = 8^2$$

$$\lambda = \pm 8$$

Since f^{-1} exists when domain of f is restricted to the set of positive real numbers,

$\therefore \lambda = -8$

10	The point P has position vector $2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$. The plane Π_1 has equation $\mathbf{r} \cdot (-\mathbf{i} + 2\mathbf{k}) = 10$.	
	(i) Find the coordinates of N, the foot of perpendicular from P to Π_1.	[3]
	The point A with position vector $-4\mathbf{i} + \alpha\mathbf{j} + 3\mathbf{k}$, where $\alpha \in \mathbb{R}$, lies on plane Π_1. The plane Π_2 has equation $\mathbf{r} \cdot (\mathbf{i} + 3\mathbf{k}) = 5$.	
	(ii) Show that point A lies on the plane Π_2. Hence, using an algebraic method, find a vector equation of l, the line of intersection of Π_1 and Π_2. Leave your answer in terms of α.	[3]
	The plane Π_3 is the image of the plane Π_2 about the plane Π_1. Given that point P lies on plane Π_2,	

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	(iii) find an equation of plane Π_3 in scalar product form.	[3]
	The line l_1 has equation $x = -4, y = z + 2$.	
	(iv) Find the position vectors of the points on the line l_1 such that the shortest distance from those points to the plane Π_1 is $\frac{4\sqrt{5}}{5}$ units.	[4]
	Solution	
	(i) $l_{PN} : \underset{\sim}{r} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R}$ Since N lies on both l_{PN} and Π_1, then we have $\begin{pmatrix} 2-\lambda \\ 5 \\ 1+2\lambda \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 10$ for some $\lambda \in \mathbb{R}$ $\Rightarrow -2 + \lambda + 2 + 4\lambda = 10$ $\lambda = 2$ Hence $\overrightarrow{ON} = \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$ So the coordinates of N are $(0,5,5)$	
	(ii) Since $\begin{pmatrix} -4 \\ \alpha \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = -4(1) + \alpha(0) + 3(3) = 5$ Hence point A lies on Π_2 $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $l : \underset{\sim}{r} = \begin{pmatrix} -4 \\ \alpha \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \beta \in \mathbb{R}$	

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(iii) Let P' be the image of P about the plane Π_1

$$\overrightarrow{ON} = \frac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$$

$$\overrightarrow{OP'} = 2\begin{pmatrix}0\\5\\5\end{pmatrix} - \begin{pmatrix}2\\5\\1\end{pmatrix} = \begin{pmatrix}-2\\5\\9\end{pmatrix}$$

A vector parallel to $\Pi_3 = \begin{pmatrix}-2\\5\\9\end{pmatrix} - \begin{pmatrix}-4\\\alpha\\3\end{pmatrix} = \begin{pmatrix}2\\5-\alpha\\6\end{pmatrix}$

$$\begin{pmatrix}2\\5-\alpha\\6\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-6\\0\\2\end{pmatrix}$$

Equation of Π_3: $\mathbf{r} \cdot \begin{pmatrix}-3\\0\\1\end{pmatrix} = \begin{pmatrix}-4\\\alpha\\3\end{pmatrix} \cdot \begin{pmatrix}-3\\0\\1\end{pmatrix} = 15$

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(iv) $l_1 : \underset{\sim}{r} = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \ \gamma \in \mathbb{R}$

Let the point B on the plane Π_1 be $(-10, 0, 0)$.

$$\Rightarrow \frac{\left| \left[\begin{pmatrix} -4 \\ \gamma \\ \gamma - 2 \end{pmatrix} - \begin{pmatrix} -10 \\ 0 \\ 0 \end{pmatrix} \right] \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right|}{\sqrt{5}} = \frac{4\sqrt{5}}{5} \text{ for some } \gamma \in \mathbb{R}$$

$$\Rightarrow \frac{\left| \begin{pmatrix} 6 \\ \gamma \\ \gamma - 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right|}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$$

$$\Rightarrow |2\gamma - 10| = 4$$

$$\Rightarrow 2\gamma - 10 = \pm 4$$

$$\Rightarrow \gamma = 7 \text{ or } 3$$

Method 2

$$\Rightarrow \frac{\left| \begin{pmatrix} -4 \\ \gamma \\ \gamma - 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - 10 \right|}{\sqrt{5}} = \frac{4\sqrt{5}}{5} \text{ for some } \gamma \in \mathbb{R}$$

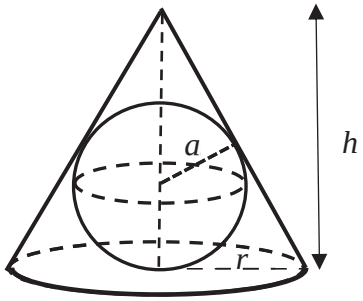
$$\Rightarrow |2\gamma - 10| = 4$$

$$\Rightarrow 2\gamma - 10 = \pm 4$$

$$\Rightarrow \gamma = 7 \text{ or } 3$$

So the position vectors are $\begin{pmatrix} -4 \\ 7 \\ 5 \end{pmatrix}$ or $\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$

11 **(a)**



A decorative artifact, as shown in the diagram above, is in the shape of a cone with radius r and height h with a sphere of fixed radius a inscribed in it.

(i) Show that the volume of the cone, V, is given by $V = \dfrac{\pi a^2 h^2}{3(h - 2a)}$. [3]

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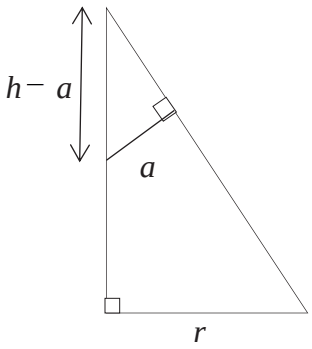
[Volume of cone, $V = \frac{1}{3}\pi r^2 h$]

(ii) Use differentiation to find, in terms of a, the minimum value of V. Leave your answer in exact form. [5]

(b) In a triangle PQR, $PQ = 3$ cm and $PR = 2$ cm. If angle QPR is increasing at a constant rate of 0.1 radians per second, find the rate of increase of the length QR at the instant when angle QPR is $\frac{\pi}{3}$ radians. [4]

Solution

(a)



(i) $\frac{h-a}{\sqrt{r^2+h^2}} = \frac{a}{r}$ or $\frac{h}{\sqrt{(h-a)^2-a^2}} = \frac{r}{a}$

$(h-a)(r) = a\sqrt{r^2+h^2}$

$\Rightarrow (h-a)^2(r)^2 = a^2(r^2+h^2)$

$r^2(h^2-2ah+a^2-a^2) = a^2h^2$

$r^2 = \frac{a^2h^2}{h^2-2ah} = \frac{a^2h}{h-2a}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{a^2h}{h-2a}\right)h$

$V = \frac{\pi a^2 h^2}{3(h-2a)}$

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(ii) $\dfrac{\mathrm{d}V}{\mathrm{d}h} = \dfrac{\pi a^2}{3}\left[\dfrac{(h-2a)(2h)-h^2}{(h-2a)^2}\right]$

$= \dfrac{\pi a^2}{3}\left[\dfrac{h^2-4ah}{(h-2a)^2}\right] = \dfrac{\pi a^2 h}{3}\left[\dfrac{h-4a}{(h-2a)^2}\right]$

When V is minimum, $\dfrac{\mathrm{d}V}{\mathrm{d}h} = 0$

$h = 0$ (rej since h>0), $\quad h = 4a$

$\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2} = \dfrac{\pi a^2}{3}\left[\dfrac{(h-2a)^2(2h-4a)-2\left(h^2-4ah\right)(h-2a)}{(h-2a)^4}\right]$

$= \dfrac{\pi a^2}{3}\left[\dfrac{2(h-2a)\left[(h-2a)^2-\left(h^2-4ah\right)\right]}{(h-2a)^4}\right]$

$= \dfrac{8\pi a^4}{3(h-2a)^3}$

Since $h = 4a$, $(h-2a)^3 > 0$, $\therefore \dfrac{\mathrm{d}^2V}{\mathrm{d}h^2} = \dfrac{8\pi a^4}{3(h-2a)^3} = \dfrac{\pi a}{3} > 0$

$\therefore$ V is minimum at $h = 4a$

$V = \dfrac{\pi a^2 h^2}{3(h-2a)} = \dfrac{16\pi a^4}{3(2a)} = \dfrac{8\pi a^3}{3}$

(b) Let $QR = x$ cm and $\angle QPR = \theta$ radians.

Using cosine rule,

$x^2 = 3^2 + 2^2 - 2(3)(2)\cos\theta$

$x^2 = 13 - 12\cos\theta$

$2x\dfrac{\mathrm{d}x}{\mathrm{d}t} = 12\sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ or $2x\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 12\sin\theta$

When $\theta = \dfrac{\pi}{3}$, $\quad x = \sqrt{7}$

$2\sqrt{7}\dfrac{\mathrm{d}x}{\mathrm{d}t} = 12\left(\dfrac{\sqrt{3}}{2}\right)(0.1)$ or $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\mathrm{d}x}{\mathrm{d}\theta}\times\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{0.3\sqrt{3}}{\sqrt{7}} \approx 0.196$ cm/s

12 Ben and his wife, Jerrie, intend to save for their child's education in the future.

(a) Jerrie decides to save \$200 on 1st January 2020. On the first day of each subsequent month she will save \$50 more than in the previous month, so that she saves \$250 on 1st February 2020, \$300 on 1st March 2020, and so on. Find the earliest date for Jerrie to save at least \$22000 in total. [4]

(b) Ben decides to put \$400 into a bank account on 1st January 2020. On the first day of each subsequent month from February 2020, he puts another \$150 into the account. The interest rate is 1% per month, such that on the last day of each month, the amount in the account on that day is increased by 1%.

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(i) Show that the value of Ben's account on the last day of the nth month (where January 2020 is the 1st month, February 2020 is the 2nd month, and so on) is $15400(1.01)^n - 15150$. [3]

(ii) Hence, calculate the earliest date for Ben to save at least \$59900 in his bank account. [3]

(iii) Ben wants to save \$88000 in his bank account by 2nd January 2038 for his child's university education. Given that the initial amount deposited on 1st January 2020 remains at \$400 and the interest rate is 1% per month, how much does Ben need to put into his savings account every month, from February 2020, instead, in order to have at least \$88000 in his savings account by 2nd January 2038? [3]

Suggested Solutions

(a)

Month	Amount invested in that month (\$)
1	200
2	250
3	300
…	…

Total amount at the end of the nth month

$= \frac{n}{2}[2(200)+(n-1)(50)]$

$= 200n+25n(n-1)$

$= 25n^2+175n$

$25n^2+175n \geq 22000$

$n^2+7n-880 \geq 0$

Consider $n^2+7n-880=0$

Using GC, $n=-33.37$ or 26.37

$n \leq -33.37$ or $n \geq 26.37$

(rejected since $n \geq 1$)

Least $n = 27$

Hence, 1 March 2022 is the **earliest date** for Jerrie to save at least \$22000.

(b)(i)

Month	**Amount, in dollars (Beginning of Month)**	Amount, in dollars (End of Month)
1	400	$400(1.01)$
2	$400(1.01)+150$	$400(1.01)^2+150(1.01)$
3	$400(1.01)^2+150(1.01)+150$	$400(1.01)^3+150(1.01)^2+150(1.01)$
…	…	…

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	n	$400(1.01)^{n-1}$ $+150(1.01)^{n-2}+...+150$	$400(1.01)^{n}$ $+150(1.01)^{n-1}+...+150(1.01)$		

Total amount at the end of nth month

$= 400(1.01)^{n}+150(1.01)^{n-1}+...+150(1.01)$

$= 400(1.01)^{n}+150\left[1.01+1.01^{2}+...+(1.01)^{n-1}\right]$

$= 400(1.01)^{n}+150\left[\dfrac{1.01\left(1.01^{n-1}-1\right)}{1.01-1}\right]$

$= 400(1.01)^{n}+15000\left(1.01^{n}-1.01\right)$

$= 15400(1.01)^{n}-15150$

(b)(ii)

$15400(1.01)^{n}-15150 \geq 59900$

$(1.01)^{n} \geq \dfrac{1501}{308}$

$n \geq \dfrac{\ln\left(\dfrac{1501}{308}\right)}{\ln 1.01}$

$n \geq 159.17$

When $n = 159$,

Amount at the end of 159th month = $15400(1.01)^{159}-15150 \approx \$\ 59773.69$

So, by the end of March 2033, Ben would have $\$59773.69$. Hence, on 1 April 2033, Ben would have $\$59773.69+\$150=\$59923.69\ (>\$59900)$

(first exceed \$59900)

(b)(iii)

Let x be the monthly amount that Ben needs to deposit from February 2020 onwards.

From (bi),

Total amount at the start of the 217th month (by 2 January 2038)

$= 400(1.01)^{216}+x\left[\dfrac{1\left(1.01^{216}-1\right)}{1.01-1}\right]$

$=400(1.01)^{216}+100x\left(1.01^{216}-1\right)$

$400(1.01)^{216}+100x\left(1.01^{216}-1\right) \geq 88000$

$x \geq 111.59$

Hence, Ben needs to put in at least \$111.59 per month, from February 2020 onwards.

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INDEX NUMBER

MATHEMATICS 9758/01

Paper 1 **7 October 2019**

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on all the work you hand in.
Write in dark blue or black pen.

You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Question	Marks
1	/3
2	/5
3	/5
4	/5
5	/6
6	/10
7	/12
8	/9
9	/12
10	/9
11	/12
12	/12

This document consists of **28** printed pages.

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1 A banker is helping a client to invest \$100,000 in stocks. He shortlisted three stocks; high risk stock *H*, moderate risk stock *M* and low risk stock *L* with a projected rate of return of 12%, 8% and 4% per annum respectively.

He aimed to help his client achieve an overall projected rate of return of 10% per annum from the \$100,000 investment with a portfolio such that the amount invested in stock *L* is equal to the total sum of 20% of the amount invested in stock *H* and 10% of the amount invested in stock *M*. Assuming that all the money were invested, determine how much did the banker invest in each of the three stocks. [3]

2 Without using a calculator, solve

(i) $1+\frac{6}{x} \geq x$, [3]

(ii) $1+\frac{6}{|x|} \geq |x|$. [2]

3 Find the general solution of the differential equation $\frac{\mathrm{d}m}{\mathrm{d}v} = \frac{m^2+m+1}{m+1}$. [5]

4 The sum, S_n, of the first *n* terms of a sequence $u_1, u_2, u_3, \ldots$ is given by

$$S_n = kn^2 + (5-k)n,$$

where *k* is an unknown constant.

(i) Find u_n in terms of *n* and *k*. [2]

(ii) Hence show that the sequence is an arithmetic progression. [1]

The *r*th term, v_r, of another sequence $v_1, v_2, v_3, \ldots$, is given by $v_r = \mathrm{e}^{u_r}$.

(iii) Show that this sequence is geometric and determine the values of *k* for which the sum to infinity exists. [2]

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5 **(i)** State a sequence of transformations that will transform the curve with equation $y = |x|^3$ onto the curve with equation $y = 2|x-5|^3 + m$, where m is a positive constant. [3]

Functions f and h are defined by

$$\mathrm{f} : x \mapsto 2|x-5|^3 + m, \ x \in \mathbb{R},$$

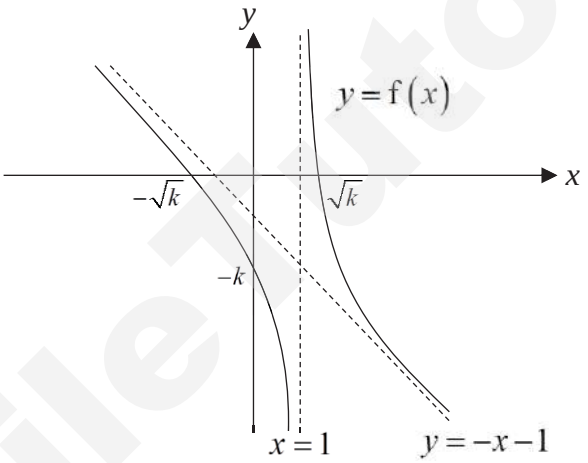
$$\mathrm{h} : x \mapsto \ln(x+1), \ x \in \mathbb{R}, \ x > -1,$$

where m is a positive constant.

(ii) Explain why the composite function hf exists. [1]

(iii) Find the domain and range of hf. [2]

6 **(a)**



The diagram shows the graph of $y = \mathrm{f}(x)$ with asymptotes $y = -x-1$ and $x = 1$. The coordinates of the axial intercepts are $\left(-\sqrt{k}, 0\right)$, $\left(\sqrt{k}, 0\right)$ and $(0, -k)$, where k is a positive constant.

On separate diagrams, sketch the graph of

(i) $y = \mathrm{f}'(x)$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

(b) Curve C has equation $(x+y)^2 + 2(x-y)^2 = 24$. Find the equation of a tangent to C such that this tangent crosses the positive x- and positive y-axes and makes an angle of 45° to the horizontal. [4]

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7 (a) Given that θ is sufficiently small, show that $\tan\left(\frac{\pi}{3}+\theta\right) \approx \sqrt{3}+4\theta$. [3]

(b) It is given that $y = \sec^2 x$.

(i) Show that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6y^2 - 4y$. [3]

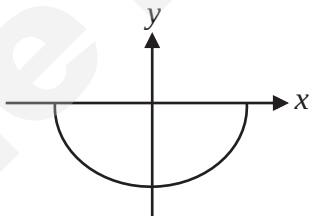
(ii) By further differentiation of this result, find the first three non-zero terms in the Maclaurin series for $\sec^2 x$. [4]

(iii) Hence find the first two non-zero terms in the Maclaurin series for $\tan^2 2x$. [2]

8 (i) Show that $\int_0^{\frac{\pi}{2}} \mathrm{e}^{-2x} \cos 2x \, \mathrm{d}x = \frac{1}{4}\left[1+\mathrm{e}^{-\pi}\right]$. [4]

(ii) The region bounded by the curve $y = \mathrm{e}^{-x}\cos x$, $y = 1$ and $x = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis. Using the result in **(i)**, find the exact volume of the solid formed. [5]

9



The diagram shows a curve C with parametric equations

$$x = h\cos\theta, \quad y = k\sin\theta,$$

where h and k are positive constants and $-\pi \le \theta \le 0$.

(i) Using the parametric equations of curve C, find the coordinates of the points where C crosses the axes. [2]

(ii) The region enclosed by C and the x-axis is denoted by A. Find the area of A in terms of h, k and π. [6]

(iii) Find the Cartesian equation of C, by expressing y in terms of x. [2]

(iv) A curve D has equation $y = -\sqrt{m-x^2}$. State the conditions relating m, h and k such that curve D is enclosed within region A. [2]

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10 **(i)** It is given that $\mathrm{f}(r) = r\left(\frac{1}{2}\right)^r$. Show that $\mathrm{f}(r) - \mathrm{f}(r-1) = \left(\frac{1}{2}\right)^{r-1} - r\left(\frac{1}{2}\right)^r$. [1]

(ii) Hence use the method of differences to show

$$\sum_{r=1}^{n} r\left(\frac{1}{2}\right)^r = 2 - \left(\frac{1}{2}\right)^{n-1} - n\left(\frac{1}{2}\right)^n. \quad [4]$$

The function g, with domain the set of positive integers, is given by $\mathrm{g}(n) = \sum_{r=1}^{n} r\left(\frac{1}{2}\right)^r$.

(iii) If n_1 and n_2 are two positive integers such that $n_2 > n_1$, by considering $\mathrm{g}(n_2) - \mathrm{g}(n_1)$, explain clearly why g has an inverse and find $\mathrm{g}^{-1}\left(\frac{251}{128}\right)$. [4]

11 A research team would like to examine the growth of a certain bacteria in a controlled environment. Beginning with a sample amounting to A_0 of this bacteria, the researchers noted down the amount of bacteria found in this environment at the end of each day, such that A_1 represents the amount present at the end of the first day and A_2 for the second day.

(i) It was found that the amount of bacteria present at the end of kth day, denoted by A_k, relies on the amount present at the end of the previous day, with the relationship $A_k = A_{k-1} + 360k^2$.

Use this relation to find expressions for A_1, A_2 and A_3, leaving each in terms of A_0. Hence, or otherwise, show that the amount of bacteria present at the end of nth day, A_n is given by $A_n = A_0 + an^3 + bn^2 + cn$, where a, b and c are constants to be determined. [You may use the result $\sum_{r=1}^{N} r^2 = \frac{1}{6}N(N+1)(2N+1)$.] [5]

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The research team later discovered an antibody to curb the growth of the bacteria. With the introduction of the antibody, the amount of bacteria present (measured in cells per ml) in the new controlled environment, $P(n)$ is given by $P(n)=\sum_{r=1}^{2n}(450-nr)$, where n is a positive integer that denotes the number of days from which the antibody is administered.

(ii) **(a)** Find the number of days after the antibody is administered before it effectively reduces the amount of bacteria present. [3]

(b) Calculate the total number of days required for the bacteria to be completely wiped out in the controlled environment. [2]

Based on the chemical composition of the antibody, the team formulated a synthetic medication. During the clinical trial where the synthetic medication is used, the amount of bacteria present (measured in cells per ml) in the experimental controlled environment, $Q(n)$ can be modelled by $Q(n)=1617-20(n-7)^2$, where n is a positive integer that denotes the number of days from which the medication is administered.

(iii) Assuming the initial amount of bacteria present in both controlled environments are the same, comment with justification whether the antibody or the synthetic medication is more effective in reducing the amount of bacteria present. [2]

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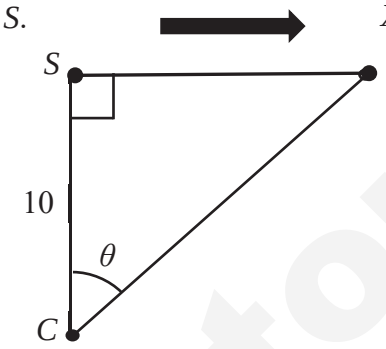
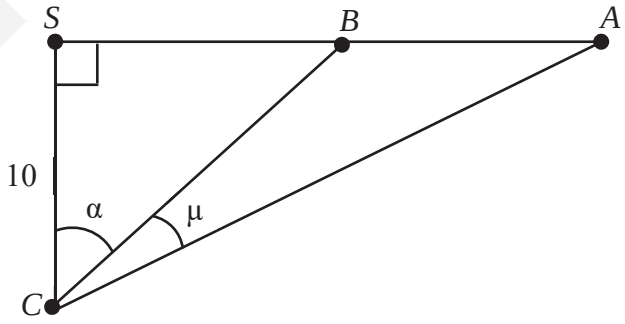
12 **(i)** A cameraman is filming a race at a family carnival. A runner runs along a straight track at a constant speed of p m/s. X represents the location of the runner who starts running from the starting point S.

The cameraman positions himself at a point C which is 10 m from the starting point S. CS is perpendicular to the track as shown in the diagram below. θ is the angle through which the cameraman has to rotate his camera to film the runner who starts running from S. Find the rate of change of θ, in terms of p, at the instant when the runner is 10 m from the S. [4]



(ii) Another cameraman who is also positioned at C, wishes to film two other runners, A and B. Both runners start from S at the same time and run along the straight track. The first runner A runs at a constant speed of $2q$ m/s and the second runner B runs at a constant speed of q m/s. The angle through which the cameraman has to rotate his camera to film the runners are α and μ where $\angle SCB = \alpha$ and $\angle BCA = \mu$ as shown in the diagram below.



By finding an expression for $\tan(\alpha + \mu)$, t seconds after the race has started, show that

$$\mu = \tan^{-1}\left(\frac{qt}{5}\right) - \tan^{-1}\left(\frac{qt}{10}\right). \quad [2]$$

Hence, use differentiation, to find the maximum value of μ as the two runners run and show that the value obtained is a maximum. [6]

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2019 ACJC H2 Math Promo P1 Marking Scheme

Qn	Solutions
1	Let the amount invested in high risk stock, moderate risk stock and low risk stock be H, L and M respectively. $H+M+L=100\,000$ ----(1) $0.2H+0.1M-L=0$ ----(2) $0.12H+0.08M+0.04L=10\,000$ ----(3) Using G.C, $H=\$65\,000, M=\$20\,000, L=\$15\,000.$
2	**(i)** $1+\frac{6}{x}\geq x, \quad x\neq 0$ $\frac{x+6}{x}\geq x$ $\frac{x+6}{x}-x\geq 0$ $\frac{-x^2+x+6}{x}\geq 0$ $\frac{(-x+3)(x+2)}{x}\geq 0$ $(-x+3)(x+2)x\geq 0$ $x\leq -2 \quad \text{or} \quad 0<x\leq 3$ **(ii)** Replace x with $\lvert x\rvert$, $\lvert x\rvert\leq -2$ or $0<\lvert x\rvert\leq 3$ (Reject as $\lvert x\rvert>0$) $\lvert x\rvert\leq 3$ $-3\leq x\leq 3, \ x\neq 0.$

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3

$$\frac{\mathrm{d}m}{\mathrm{d}v} = \frac{m^2+m+1}{m+1}$$

$$\int \frac{m+1}{m^2+m+1}\,\mathrm{d}x = \int 1\,\mathrm{d}m$$

$$\int \frac{\frac{1}{2}(2m+1)+\frac{1}{2}}{m^2+m+1}\,\mathrm{d}x = \int 1\,\mathrm{d}m$$

$$\frac{1}{2}\int \frac{(2m+1)}{m^2+m+1}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{m^2+m+1} = \int 1\,\mathrm{d}m$$

$$\frac{1}{2}\int \frac{(2m+1)}{m^2+m+1}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{\left(\sqrt{\frac{3}{4}}\right)^2+\left(m+\frac{1}{2}\right)^2} = \int 1\,\mathrm{d}m$$

$$\frac{1}{2}\ln\left(m^2+m+1\right)+\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{\frac{3}{4}}}\right)\tan^{-1}\left(\frac{m+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right) = m+C$$

$$\frac{1}{2}\ln\left(m^2+m+1\right)+\left(\frac{1}{\sqrt{3}}\right)\tan^{-1}\left(\frac{2m+1}{\sqrt{3}}\right) = m+C.$$

4

(i)

$S_n = kn^2+(5-k)n$

$S_{n-1} = k(n-1)^2+(5-k)(n-1)$

$= kn^2-2kn+k+5n-5-kn+k$

$= kn^2-3kn+2k+5n-5$

$u_n = S_n - S_{n-1}$

$= kn^2+5n-kn-\left(kn^2-3kn+5n-5+2k\right)$

$= 2kn-2k+5$

(ii)

$u_{n-1} = 2k(n-1)-2k+5$

$u_n - u_{n-1} = 2kn-2k+5-\left(2kn-2k-2k+5\right) = 2k$

Since $u_n - u_{n-1} = 2k$ which is a constant, hence the sequence is an arithmetic progression.

(iii)

Given $v_r = \mathrm{e}^{u_r}$,

$v_{r-1} = \mathrm{e}^{u_{r-1}}$

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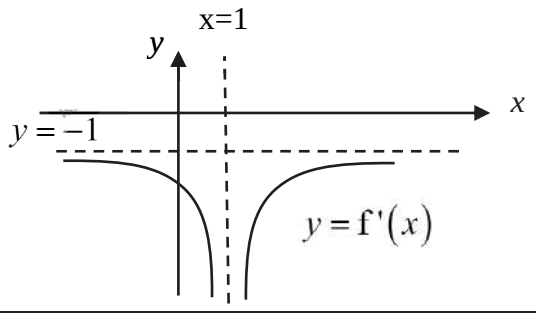
	$\frac{v_r}{v_{r-1}} = \frac{e^{u_r}}{e^{u_{r-1}}}$ $= e^{u_r - u_{r-1}}$ $= e^{2k}$ Since $\frac{v_r}{v_{r-1}} = e^{2k}$ which is a constant, this sequence is geometric. For sum to infinity to exist, $\left\|e^{2k}\right\| < 1$ $e^{2k} < 1$ $2k \ln e < \ln 1$ $k < 0.$
5	**(i)** $y = \|x\|^3$ ↓ Translation of 5 units in the positive x-direction. $y = \|x-5\|^3$ ↓ Scaling by a factor of 2 parallel to the y-axis. $y = 2\|x-5\|^3$ ↓ Translation of m units in the positive y-direction. $y = 2\|x-5\|^3 + m$ **(ii)** $R_f = [m, \infty),\ D_h = (-1, \infty)$ Since $m > 0$ hence $R_f \subseteq D_h$. $\therefore$ hf exists. **(iii)** $D_{hf} = D_f = \mathbb{R}$ To find the range of hf, we use the R_f as the restricted domain of h and read off the corresponding range from the graph of $y = h(x)$. $R_{hf} = [\ln\ m+1\ , \infty)$.
6	**(a)** **(i)** 

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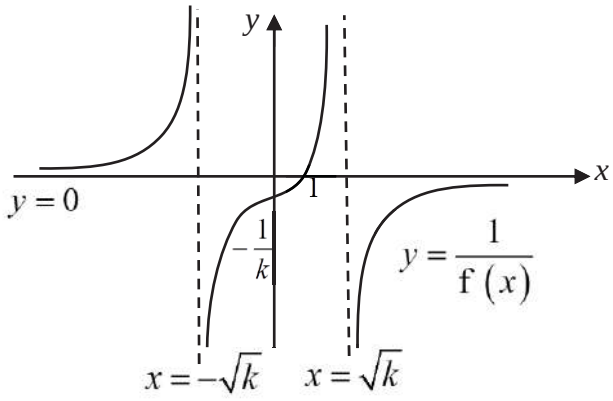
(ii)



(b)

$$(x+y)^2+2(x-y)^2=24---(1)$$

$$2(x+y)\left(1+\frac{dy}{dx}\right)+4(x-y)\left(1-\frac{dy}{dx}\right)=0$$

$$(x+y)\left(1+\frac{dy}{dx}\right)+2(x-y)\left(1-\frac{dy}{dx}\right)=0---(2)$$

Since the tangent crosses the positive x- and positive y-axes and makes an angle of 45° to the horizontal, $\frac{dy}{dx}=-\tan 45^\circ=-1$.

Sub. $\frac{dy}{dx}=-1$ into eqn. (2),

$$(x+y)(1-1)+2(x-y)(1+1)=0$$

$$x=y---(3)$$

Sub. (3) into (1),

$$(x+x)^2+2(x-x)^2=24$$

$$(2x)^2=24$$

$$x=\pm\sqrt{6}$$

Since $x>0$, $\therefore x=\sqrt{6}$ and $y=\sqrt{6}$.

Hence equation of tangent:

$$y-\sqrt{6}=(-1)\left(x-\sqrt{6}\right)$$

$$y=-x+2\sqrt{6}.$$

7 **(a)**

$$\tan\left(\frac{\pi}{3}+\theta\right)=\frac{\tan\frac{\pi}{3}+\tan\theta}{1-\tan\frac{\pi}{3}\tan\theta}$$

$$=\frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}$$

$$\approx\left(\sqrt{3}+\theta\right)\left(1-\sqrt{3}\theta\right)^{-1}$$

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$$\approx \left(\sqrt{3}+\theta\right)\left(1+(-1)\left(-\sqrt{3}\theta\right)+\ldots\right)$$
$$\approx \sqrt{3}+3\theta+\theta=\sqrt{3}+4\theta\,.$$

(b)(i)

$y=\sec^2 x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2(\sec x)(\sec x\tan x)$$
$$=2\sec^2 x\tan x$$
$$=2y\tan x$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left(y\cdot\sec^2 x+2(\tan x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right)$$
$$=2\left(y^2+2(\tan x)\,(2y\tan x)\right)$$
$$=2y^2+4y\tan^2 x$$
$$=2y^2+4y\left(\sec^2 x-1\right)$$
$$=2y^2+4y(y-1)$$
$$=6y^2-4y\,.\text{(Shown)}$$

(ii)

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=6\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)-4\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$=12y\frac{\mathrm{d}y}{\mathrm{d}x}-4\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=12\left(y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right)-4\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$
$$=12y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+12\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-4\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

When $x=0,\ y=1,\ \frac{\mathrm{d}y}{\mathrm{d}x}=0,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2,\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0,\ \frac{\mathrm{d}^4y}{\mathrm{d}x^4}=16$

$$\sec^2 x=1+\frac{2}{2!}x^2+\frac{16}{4!}x^4+\ldots$$
$$\approx 1+x^2+\frac{2}{3}x^4$$

(iii)

$$\sec^2 2x\approx 1+(2x)^2+\frac{2}{3}(2x)^4$$
$$=1+4x^2+\frac{32}{3}x^4$$

Since $\tan^2 2x=\sec^2 2x-1$

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$\tan^2 2x \approx 1 + 4x^2 + \frac{32}{3}x^4 - 1$

$= 4x^2 + \frac{32}{3}x^4$

8 **(i) Method 1:**

$\int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x \, dx$

$= \left[(\cos 2x)\left(-\frac{1}{2}e^{-2x} \right) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2}e^{-2x} \right)(-2\sin 2x) \, dx$

$= \left[(\cos \pi)\left(-\frac{1}{2}e^{-\pi} \right) - (\cos 0)\left(-\frac{1}{2}e^{0} \right) \right] - \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2}e^{-2x} \right)(-2\sin 2x) \, dx$

$= \left[\frac{1}{2}e^{-\pi} + \frac{1}{2} \right] - \int_0^{\frac{\pi}{2}} e^{-2x} \sin 2x \, dx ---(1)$

Consider $\int_0^{\frac{\pi}{2}} e^{-2x} \sin 2x \, dx$

$= \left[(\sin 2x)\left(-\frac{1}{2}e^{-2x} \right) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2}e^{-2x} \right)(2\cos 2x) \, dx$

$= \left[(\sin \pi)\left(-\frac{1}{2}e^{-\pi} \right) - (\sin 0)\left(-\frac{1}{2}e^{0} \right) \right] + \int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x \, dx$

$= 0 + \int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x \, dx ----(2)$

Sub. (2) into (1).

$\int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x \, dx = \left[\frac{1}{2}e^{-\pi} + \frac{1}{2} \right] - \int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x \, dx$

$2\int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x \, dx = \frac{1}{2}e^{-\pi} + \frac{1}{2}$

$\int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x \, dx = \frac{1}{4}\left[e^{-\pi} + 1 \right]$ (Shown)

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8

(i) Method 2:

$$\int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,dx$$

$$=\left[\left(e^{-2x}\right)\left(\frac{1}{2}\sin 2x\right)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\sin 2x\right)\left(-2e^{-2x}\right)dx$$

$$=\left[\left(e^{-\pi}\right)\left(\frac{1}{2}\sin \pi\right)-\left(e^{0}\right)\left(\frac{1}{2}\sin 0\right)\right]-\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\sin 2x\right)\left(-2e^{-2x}\right)dx$$

$$=0+\int_0^{\frac{\pi}{2}} e^{-2x}\sin 2x\,dx---(1)$$

Consider $\int_0^{\frac{\pi}{2}} e^{-2x}\sin 2x\,dx$

$$=\left[\left(e^{-2x}\right)\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\left(-\frac{1}{2}\cos 2x\right)\left(-2e^{-2x}\right)dx$$

$$=\left[\left(e^{-\pi}\right)\left(-\frac{1}{2}\cos \pi\right)-\left(e^{0}\right)\left(-\frac{1}{2}\cos 0\right)\right]-\int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,dx$$

$$=\left[\frac{1}{2}e^{-\pi}+1\right]-\int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,dx----(2)$$

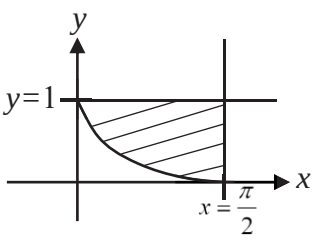
Sub. (2) into (1).

$$\int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,dx=0+\left[\left[\frac{1}{2}e^{-\pi}+1\right]-\int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,dx\right]$$

$$2\int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,dx=\frac{1}{2}e^{-\pi}+\frac{1}{2}$$

$$\int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,dx=\frac{1}{4}\left[e^{-\pi}+1\right]\ \text{(Shown)}$$

(ii)



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$$\text{Required Volume} = \pi\left(1^2\right)\left(\frac{\pi}{2}\right) - \pi\int_0^{\frac{\pi}{2}} y^2 \, \mathrm{d}x$$

$$= \frac{\pi^2}{2} - \pi\int_0^{\frac{\pi}{2}} \left(\mathrm{e}^{-x}\cos x\right)^2 \, \mathrm{d}x$$

$$= \frac{\pi^2}{2} - \pi\int_0^{\frac{\pi}{2}} \mathrm{e}^{-2x}\cos^2 x \, \mathrm{d}x$$

$$= \frac{\pi^2}{2} - \pi\int_0^{\frac{\pi}{2}} \mathrm{e}^{-2x}\left(\frac{1+\cos 2x}{2}\right) \mathrm{d}x$$

$$= \frac{\pi^2}{2} - \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \mathrm{e}^{-2x} + \left(\mathrm{e}^{-2x}\cos 2x\right) \mathrm{d}x$$

$$= \frac{\pi^2}{2} - \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \mathrm{e}^{-2x} \, \mathrm{d}x - \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \mathrm{e}^{-2x}\cos 2x \, \mathrm{d}x$$

$$= \frac{\pi^2}{2} - \frac{\pi}{2}\left[\frac{\mathrm{e}^{-2x}}{-2}\right]_0^{\frac{\pi}{2}} - \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \mathrm{e}^{-2x}\cos 2x \, \mathrm{d}x$$

$$= \frac{\pi^2}{2} + \frac{\pi}{4}\left[\mathrm{e}^{-2x}\right]_0^{\frac{\pi}{2}} - \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \mathrm{e}^{-2x}\cos 2x \, \mathrm{d}x$$

$$= \frac{\pi^2}{2} + \frac{\pi}{4}\left[\mathrm{e}^{-\pi} - 1\right] - \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \mathrm{e}^{-2x}\cos 2x \, \mathrm{d}x$$

$$= \frac{\pi^2}{2} + \frac{\pi}{4}\left[\mathrm{e}^{-\pi} - 1\right] - \frac{\pi}{2}\left(\frac{1}{4}\left[1+\mathrm{e}^{-\pi}\right]\right)$$

$$= \frac{\pi^2}{2} + \frac{\pi}{4}\mathrm{e}^{-\pi} - \frac{\pi}{4} - \frac{\pi}{8} - \frac{\pi}{8}\mathrm{e}^{-\pi}$$

$$= \frac{\pi^2}{2} + \frac{\pi}{8}\mathrm{e}^{-\pi} - \frac{3\pi}{8}$$

$$= \frac{\pi}{8}\left[4\pi + \mathrm{e}^{-\pi} - 3\right] \text{ units}^3.$$

9

(i)

Finding x-intercepts:

Using $y = k\sin\theta$, when $y = 0$,

$k\sin\theta = 0$

$\theta = 0, -\pi \quad (\text{since } -\pi \le \theta \le 0)$

When $\theta = 0$, $x = h\cos 0 = h$

When $\theta = -\pi$, $x = h\cos(-\pi) = -h$

Hence x-intercepts are $(-h, 0)$ and $(h, 0)$.

Finding y-intercepts:

Using $x = h\cos\theta$, when $x = 0$,

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$h\cos\theta = 0$

$\theta = -\dfrac{\pi}{2}$ (since $-\pi \le \theta \le 0$)

When $\theta = -\dfrac{\pi}{2}$, $y = k\sin\left(-\dfrac{\pi}{2}\right) = -k$

Hence y-intercept is $(0, -k)$.

(ii)

$x = h\cos\theta$

$\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -h\sin\theta$

When $x = 0$, $\cos\theta = 0 \Rightarrow \theta = \dfrac{-\pi}{2}$ $\left(-\pi \le \theta \le 0\right)$

When $x = h$, $\cos\theta = 1 \Rightarrow \theta = 0$ $\left(-\pi \le \theta \le 0\right)$

$A = -2\int_0^h y\,\mathrm{d}x$

$= -2\int_{\frac{-\pi}{2}}^{0} (k\sin\theta)(-h\sin\theta)\,\mathrm{d}\theta$

$= 2kh\int_{\frac{-\pi}{2}}^{0} (\sin\theta)(\sin\theta)\,\mathrm{d}\theta$

$= 2kh\int_{\frac{-\pi}{2}}^{0} \left(\sin^2\theta\right)\mathrm{d}\theta$

$= 2kh\int_{\frac{-\pi}{2}}^{0} \left(\dfrac{1-\cos 2\theta}{2}\right)\mathrm{d}\theta$

$= kh\int_{\frac{-\pi}{2}}^{0} (1-\cos 2\theta)\,\mathrm{d}\theta$

$= kh\left[\theta - \dfrac{\sin 2\theta}{2}\right]_{\frac{-\pi}{2}}^{0}$

$= kh\left[\left(0 - \dfrac{\sin 0}{2}\right) - \left(\dfrac{-\pi}{2} - \dfrac{\sin(-\pi)}{2}\right)\right] = \dfrac{kh\pi}{2}\text{ units}^2.$

(iii)

$x = h\cos\theta$ $\qquad$ $y = k\sin\theta$

$\cos\theta = \dfrac{x}{h} - - - (1)$ $\qquad$ $\sin\theta = \dfrac{y}{k} - - - (2)$

$(1)^2 + (2)^2$:

$\cos^2\theta + \sin^2\theta = \dfrac{x^2}{h^2} + \dfrac{y^2}{k^2}$

$1 = \dfrac{x^2}{h^2} + \dfrac{y^2}{k^2}$

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	$\frac{x^2}{h^2}+\frac{y^2}{k^2}=1$ $y^2=k^2\left(1-\frac{x^2}{h^2}\right)$ $y=\pm\sqrt{k^2\left(1-\frac{x^2}{h^2}\right)}$ $y=-k\sqrt{1-\frac{x^2}{h^2}}$ (since $k>0$ and from diagram $y\le 0$) **(iv)** $y=-\sqrt{m-x^2}$ $y^2+x^2=\left(\sqrt{m}\right)^2$ Since $y<0$, curve D is a semicircle with centre (0,0) and radius $\sqrt{m}$, lying below the x-axis. If $h>k$, $\sqrt{m}\le k$. If $h<k$, $\sqrt{m}\le h$.
10	**(i)** $\mathrm{f}(r)-\mathrm{f}(r-1)=r\left(\frac{1}{2}\right)^r-(r-1)\left(\frac{1}{2}\right)^{r-1}$ $=r\left(\frac{1}{2}\right)^r-r\left(\frac{1}{2}\right)^{r-1}+\left(\frac{1}{2}\right)^{r-1}$ $=\left(\frac{1}{2}\right)^r[r-2r]+\left(\frac{1}{2}\right)^{r-1}$ $=\left(\frac{1}{2}\right)^{r-1}-r\left(\frac{1}{2}\right)^r$ (Shown) **(ii)** $\mathrm{f}(r)-\mathrm{f}(r-1)=\left(\frac{1}{2}\right)^{r-1}-r\left(\frac{1}{2}\right)^r$ $\sum_{r=1}^{n}[\mathrm{f}(r)-\mathrm{f}(r-1)]=\sum_{r=1}^{n}\left[\left(\frac{1}{2}\right)^{r-1}-r\left(\frac{1}{2}\right)^r\right]$

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Consider $\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]=\mathrm{f}(1)-\mathrm{f}(0)+$

$\mathrm{f}(2)-\mathrm{f}(1)+$

$\mathrm{f}(3)-\mathrm{f}(2)+\ldots$

$\mathrm{f}(2)-\mathrm{f}(1)+$

$\mathrm{f}(n-1)-\mathrm{f}(n\text{-}2)+$

$\mathrm{f}(n)-\mathrm{f}(n\text{-}1)$

$=\mathrm{f}(n)-\mathrm{f}(0)$

$=n\left(\frac{1}{2}\right)^{n}-0$

$$\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]=\sum_{r=1}^{n}\left[\left(\frac{1}{2}\right)^{r-1}-r\left(\frac{1}{2}\right)^{r}\right]$$

$$n\left(\frac{1}{2}\right)^{n}-0=\sum_{r=1}^{n}\left[\left(\frac{1}{2}\right)^{r-1}\right]-\sum_{r=1}^{n}\left[r\left(\frac{1}{2}\right)^{r}\right]$$

$$\sum_{r=1}^{n}r\left(\frac{1}{2}\right)^{r}=\sum_{r=1}^{n}\left[\left(\frac{1}{2}\right)^{r-1}\right]-n\left(\frac{1}{2}\right)^{n}$$

$$=\left[\frac{(1)\left(1-0.5^{n}\right)}{1-0.5}\right]-n\left(\frac{1}{2}\right)^{n}$$

$$=2\left(1-0.5^{n}\right)-n\left(\frac{1}{2}\right)^{n}$$

$$=2-\left(\frac{1}{2}\right)^{n-1}-n\left(\frac{1}{2}\right)^{n}\ \text{(Shown)}$$

(iii)

$$\mathrm{g}\left(n_2\right)-\mathrm{g}\left(n_1\right)=\sum_{r=1}^{n_2}r\left(\frac{1}{2}\right)^{r}-\sum_{r=1}^{n_1}r\left(\frac{1}{2}\right)^{r}$$

Since $n_2 > n_1$,

$$\mathrm{g}\left(n_2\right)-\mathrm{g}\left(n_1\right)=\sum_{r=1}^{n_2}r\left(\frac{1}{2}\right)^{r}-\sum_{r=1}^{n_1}r\left(\frac{1}{2}\right)^{r}=\sum_{r=n_1+1}^{n_2}r\left(\frac{1}{2}\right)^{r}>0$$

$\therefore \mathrm{g}\left(n_2\right)>\mathrm{g}\left(n_1\right)$.

Hence g is a one to one function as $\mathrm{g}\left(n_2\right)>\mathrm{g}\left(n_1\right)$.

$\therefore$ g^{-1} exists.

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$$\mathrm{g}(n)=\sum_{r=1}^{n} r\left(\frac{1}{2}\right)^{r}$$

$$\mathrm{g}(n)=2-\left(\frac{1}{2}\right)^{n-1}-n\left(\frac{1}{2}\right)^{n}$$

$$n=\mathrm{g}^{-1}\left(2-\left(\frac{1}{2}\right)^{n-1}-n\left(\frac{1}{2}\right)^{n}\right)$$

To find $\mathrm{g}^{-1}\left(\frac{251}{128}\right)$,

$$2-\left(\frac{1}{2}\right)^{n-1}-n\left(\frac{1}{2}\right)^{n}=\frac{251}{128}$$

Using GC,

$n=8.$

11 **(i)**

Using $A_k = A_{k-1} + 360k^2$,

$$A_1 = A_0 + 360(1)^2 = A_0 + 360$$

$$A_2 = A_1 + 360(2)^2 = A_0 + 360(1)^2 + 360(2)^2 = A_0 + 1800$$

$$A_3 = A_2 + 360(3)^2 = A_0 + 360(1)^2 + 360(2)^2 + 360(3)^2 = A_0 + 5040$$

$$\begin{aligned} A_n &= A_0 + 360(1)^2 + 360(2)^2 + 360(3)^2 + \ldots + 360(n)^2 \\ &= A_0 + 360\left[(1)^2 + (2)^2 + (3)^2 + \ldots + (n)^2\right] \\ &= A_0 + 360\left[\sum_{r=1}^{n} r^2\right] \\ &= A_0 + 360\left[\frac{n}{6}(n+1)(2n+1)\right] \\ &= A_0 + 60n\left[2n^2 + 3n + 1\right] \\ &= A_0 + 120n^3 + 180n^2 + 60n. \end{aligned}$$

$a = 120, b = 180, c = 60.$

11 **(ii)(a)**

$$\begin{aligned} P(n) &= \sum_{r=1}^{2n} (450 - nr) \\ &= \sum_{r=1}^{2n} (450) - n\sum_{r=1}^{2n} (r) \\ &= (450)(2n) - n\left[\frac{2n}{2}(1+2n)\right] \\ &= 900n - n^2 - 2n^3. \end{aligned}$$

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$P(n) = 900n - n^2 - 2n^3$

Using G.C to find maximum point of graph $y = P(n)$,

$n = 12.08$ (since $n>0$)

Number of days the amount of bacteria present reduces is 13.

Alternative Method (Using G.C):

$n = 11,\ P(11) = 7117$

$n = 12,\ P(12) = 7200$

$n = 12.5, P(12) = 7187.5$

$n = 13,\ P(13) = 7137$

Key into G.C,

$Y_1 = \sum_{A=1}^{2X}(450\text{-}XA)$ *and press* [2nd][graph] *to look at table generated.*

From G.C, it can be observed that the bacteria start to reduce between the 12th and 13th day.

Since n is a positive integer, the number of days the amount of bacteria present reduces is 13.

(ii)(b)

$P(n) = 0$

$900n - n^2 - 2n^3 = 0$

Using G.C to solve $900n - n^2 - 2n^3 = 0$,

$n = 20.96$ (since $n>0$)

Number of days the amount of bacteria to be wiped out is 21.

(iii)

$Q(n) = 1617 - 20(n-7)^2$

The maximum point of graph $y = Q(n)$ occurs when $n = 7$.

Since it only takes 7 days for the amount of bacteria present to be reduced when using the synthetic medication as compared to the 13 days required when using the antibody, hence the synthetic medication is more effective in reducing the amount of bacteria present.

12	**(i)**

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Let x be the distance run in t seconds

$$\frac{dx}{dt} = p$$

$$\tan\theta = \frac{x}{10}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$$

$$when\ x = 10, \tan\theta = \frac{10}{10} = 1$$

$$\theta = \frac{\pi}{4}$$

$$\frac{d\theta}{dt} = \frac{1}{10}\frac{p}{\sec^2\left(\frac{\pi}{4}\right)} = \frac{p}{10}\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{p}{20}\ radians/s$$

(ii)

$$\tan(\alpha+\mu) = \frac{2qt}{10} = \frac{qt}{5}$$

$$\tan\alpha = \frac{qt}{10}$$

$$\alpha + \mu = \tan^{-1}\left(\frac{qt}{5}\right)$$

$$\mu = \tan^{-1}\left(\frac{qt}{5}\right) - \tan^{-1}\left(\frac{qt}{10}\right)$$

(iii)

$$Let\ \frac{d\mu}{dt} = \frac{\frac{q}{5}}{1+\left(\frac{qt}{5}\right)^2} - \frac{\frac{q}{10}}{1+\left(\frac{qt}{10}\right)^2} = 0$$

$$\frac{q}{5}\left(1+\left(\frac{qt}{10}\right)^2\right) - \frac{q}{10}\left(1+\left(\frac{qt}{5}\right)^2\right) = 0$$

$$\frac{q}{5} + \frac{q^3}{500}t^2 - \frac{q}{10} - \frac{q^3}{250}t^2 = 0$$

$$\frac{q}{10} = \frac{q^3}{500}t^2$$

$$t = \sqrt{\frac{50}{q^2}} = \frac{5\sqrt{2}}{q}$$

$$when\ t = \frac{\sqrt{50}}{q},\ \mu = \tan^{-1}\left(\frac{q}{5}.\frac{\sqrt{50}}{q}\right) - \tan^{-1}\left(\frac{q}{10}.\frac{\sqrt{50}}{q}\right)$$

$$= \tan^{-1}\sqrt{2} - \tan^{-1}\frac{\sqrt{2}}{2} = 0.340 radians$$

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$$\frac{d^2\mu}{dt^2} = \frac{-\frac{2q^3t}{125}}{\left(1+\left(\frac{qt}{5}\right)^2\right)^2} - \frac{-\frac{2q^3t}{1000}}{\left(1+\left(\frac{qt}{10}\right)^2\right)^2}$$

$when\ t^2 = \frac{50}{q^2}$

$$\frac{d^2\mu}{dt^2} = \frac{-\frac{2q^3}{125}\left(\frac{\sqrt{50}}{q}\right)}{(1+2)^2} - \frac{-\frac{2q^3}{1000}\left(\frac{\sqrt{50}}{q}\right)}{\left(1+\frac{1}{2}\right)^2}$$

$$= 2q^2\sqrt{50}\left(\frac{-1}{125(9)} + \frac{4}{1000(9)}\right) = -\frac{\sqrt{50}}{1125}q^2 < 0$$

Hence value obtained is a maximum.

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CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC1 Promotional Examination

CANDIDATE NAME			
CLASS		INDEX NUMBER	

MATHEMATICS 9758/01

Paper 1 07 October 2019

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Question	1	2	3	4	5	6	7	8	9	10	11	12	Total
Marks													
Total	3	6	7	7	7	7	8	10	9	10	12	14	100

This document consists of **28** printed pages and **2** blank pages.

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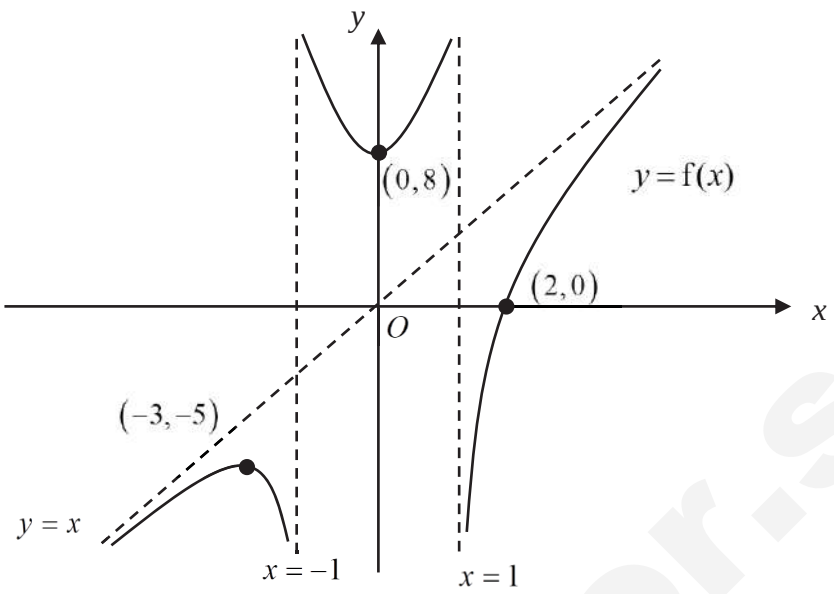
1



The diagram shows the graph of $y = \mathrm{f}(x)$. The graph crosses the x-axis at $x = 2$, and the graph has a minimum point $(0,8)$ and a maximum point $(-3,-5)$. The lines $x = -1$, $x = 1$ and $y = x$ are the asymptotes of the graph.

Sketch the graph of $y = \mathrm{f}'(x)$, indicating clearly the equations of any asymptotes and the coordinates of any points of intersection with the x- and y-axes. [3]

2 The curve C has an equation $y = \dfrac{ax^2 + bx + a}{x^2 - b}$ where a and b are constants.

(i) Given that two of the asymptotes of C are $x = 2$ and $y = 3$, determine the values of a and b. [2]

(ii) Sketch the curve C, giving the coordinates of the point(s) where the curve crosses the axes, stationary points and the equations of all asymptotes. [4]

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3 The function f is defined by

$$\mathrm{f}: x \mapsto x^2 - 4x - 5, x \in \mathbb{R}, x \leq 2.$$

(i) Show that f has an inverse. [2]

The function g is defined by

$$\mathrm{g}: x \mapsto \mathrm{e}^x + 1, x \in \mathbb{R}, x \leq 0.$$

(ii) Determine whether the composite function gf exists. [2]

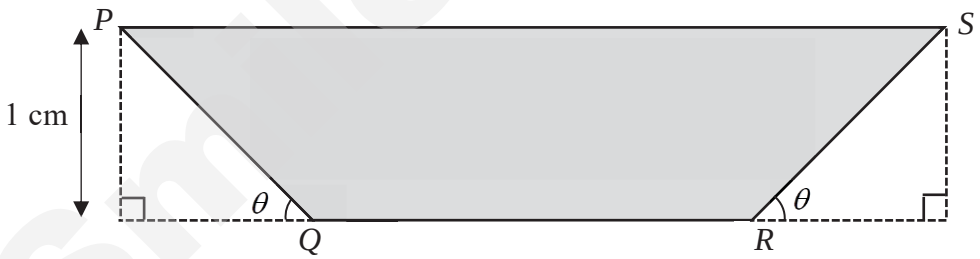
(iii) It is known that fg exists. Find an expression for fg(x). Hence or otherwise, find the range of fg. [3]

4 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following, simplifying your answers.

(a) $y = \cos^{-1}\sqrt{1-x^2}, \ 0 < x < 1$ [3]

(b) $x^2 + xy + y^2 = \sin x$ [4]

5



A regular trapezium $PQRS$ with height 1 cm is such that $PQ + QR + RS = 4$ cm. PQ and RS are each inclined to the line QR at an angle θ, where $\frac{\pi}{6} < \theta < \frac{\pi}{2}$, as shown in the diagram.

(i) By considering the length of QR, show that the area, A cm^2, of the trapezium $PQRS$ is $4 - 2\,\mathrm{cosec}\,\theta + \cot\theta$. [3]

(ii) Hence, find the maximum area of the trapezium $PQRS$, giving your answer in the form $M + N\sqrt{3}$, where M and N are integers. [4]

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6 It is given that $\mathrm{f}(x) = \dfrac{1}{\sqrt[3]{8-3x}}$.

(i) Find the binomial expansion for $\mathrm{f}(x)$, up to and including the term in x^2. Give the coefficients as exact fractions in their simplest form.
State the range of values of x for which the expansion is valid. [4]

(ii) By putting $x = \dfrac{1}{16}$ into the expansion found in part **(i)**, find an approximate value of $\sqrt[3]{16}$. Leave your answer in the form of $\dfrac{a}{b}$ in its lowest term, where a and b are positive integers to be determined. [3]

7 **(a)** Without using a calculator, solve the inequality

$$\frac{x^2+3x-5}{x-2} \leq 1.$$

[4]

(b) Sketch the graphs of $y = x-3$ and $y = \ln(x+2)$ on the same diagram, indicating clearly the coordinates of any points of intersection with the axes, the coordinates of the points of intersection of the graphs and equations of any asymptotes.

Hence solve

$$\ln(x+2) \leq x-3.$$

[4]

8 **(i)** Find $\sum_{r=1}^{n} 4^{-2r}$ in terms of n. [2]

(ii) Express $\dfrac{2}{(2r+1)(2r+3)}$ in partial fractions. Hence find $\sum_{r=1}^{n} \dfrac{1}{(2r+1)(2r+3)}$ in terms of n. [5]

(iii) Deduce that $\sum_{r=1}^{n} \left[4^{-2(r+1)} + \dfrac{1}{(2r+1)(2r+3)} \right] < \dfrac{41}{240}$. [3]

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9 Planes p_1 and p_2 are respectively represented by cartesian equations

$$x+2y+2z=3,$$
$$-2x-y+2z=-6,$$

and they meet in a line l.

(i) Find a vector equation for line l. [2]

Plane p_3 is represented by cartesian equation

$$M(x+2y+2z-3)+(-2x-y+2z+6)=0$$, where M is a constant.

(ii) Write down a normal vector to plane p_3, in terms of M. [1]

(iii) Explain why plane p_3 contains line l, for any value of M. [2]

(iv) Show that the cosine of the acute angle between planes p_1 and p_3 is

$$\frac{M}{\sqrt{M^2+1}}, \text{ if } M>0.$$ [4]

10 Relative to the origin O, the position vectors of the points A and B are $\begin{pmatrix}0\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}\alpha\\1\\-1\end{pmatrix}$, $\alpha<0$ respectively.

(i) Show that area of triangle $OAB=\frac{1}{2}\sqrt{1+5\alpha^2}$. [2]

(ii) The point C lies on AB produced such that $3AB=BC$, and that $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{OC}$. Find the value of α. [3]

(iii) Using your answers in parts (i) and (ii), deduce the exact perpendicular distance from B to OA. Hence find $\overrightarrow{ON}$, where N is the foot of perpendicular from B to OA. [5]

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11 A curve C has parametric equations

$$x = \frac{1}{\sqrt{t+1}}, \quad y = \mathrm{e}^{t^2}, \text{ where } t > -1.$$

(i) Sketch C, stating the equations of asymptotes and coordinates of any turning points. [3]

(ii) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -4t\mathrm{e}^{t^2}\sqrt{(t+1)^3}$. [2]

(iii) The normal at $t = \frac{5}{4}$ meets the y-axis at the point P. Given that the point Q on the curve C has a parameter q, find the value of q such that the length PQ is a minimum. Find the minimum length PQ. [7]

12 **(a)** To purchase a grand piano, Ken took a loan of \$12,000 from a finance company on 1st October 2019. The company charges a compound interest at a rate of 0.5% on the outstanding amount owed on the last day of every month, while Ken decides to repay the loan by paying back a fixed instalment of \$$P$ on the first day of each month starting from 1st November 2019.

(i) How much would Ken owe after paying the second instalment on 1st December 2019? [2]

(ii) Show that the amount Ken owes after paying the n th instalment is

$$\$\left[12000(1.005^n) - 200P(1.005^n - 1)\right].$$ [3]

(iii) Ken intends to repay the loan completely by 1st December 2021. Find the minimum monthly instalment that Ken needs to pay, giving your answer to the nearest cent. [2]

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(b) The fundamental frequencies of sound tones produced by successive keys on a piano follow a geometric progression. On a modern 88-key piano, the 49th key produces a standardised frequency of 440 hertz (cycles per second). The 61st key produces a frequency of 880 hertz.

(i) Determine the frequency produced by the first key on the piano. [2]

(ii) Hence, show that the frequency produced by the nth key on the piano is given by $u_n = k(2^{\frac{n-1}{m}})$ hertz, where k and m are constants to be found.

Explain why the logarithm of the frequencies produced by successive piano keys follow an arithmetic progression. [3]

In music, a chord is a harmonic set of tones produced by three or more keys.
The major triad chord is produced by the nth, $(n+4)$th and $(n+7)$th keys sounded together.

(iii) Show that the ratio of ascending frequencies produced in a major triad chord, is independent of n. [2]

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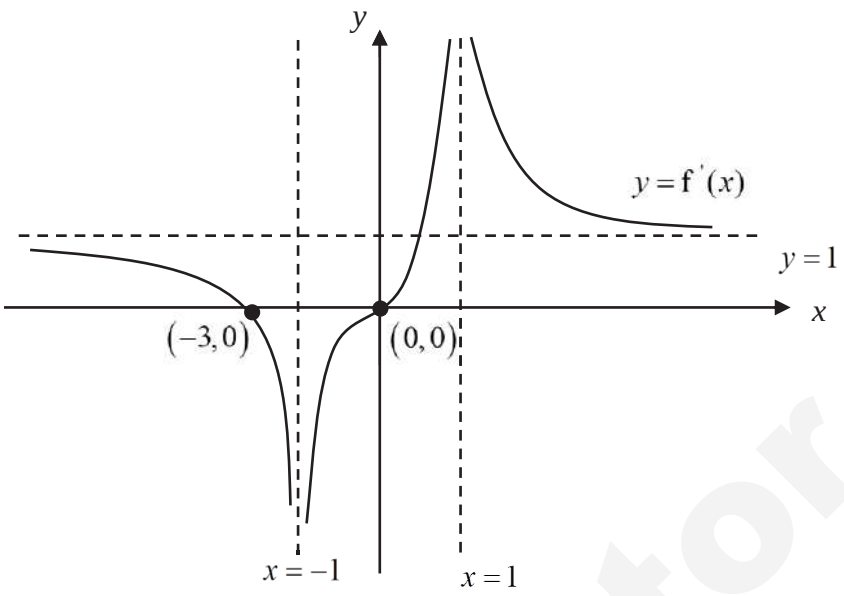
2019 JC1 H2 Promotional Examination Suggested Mark Scheme

1



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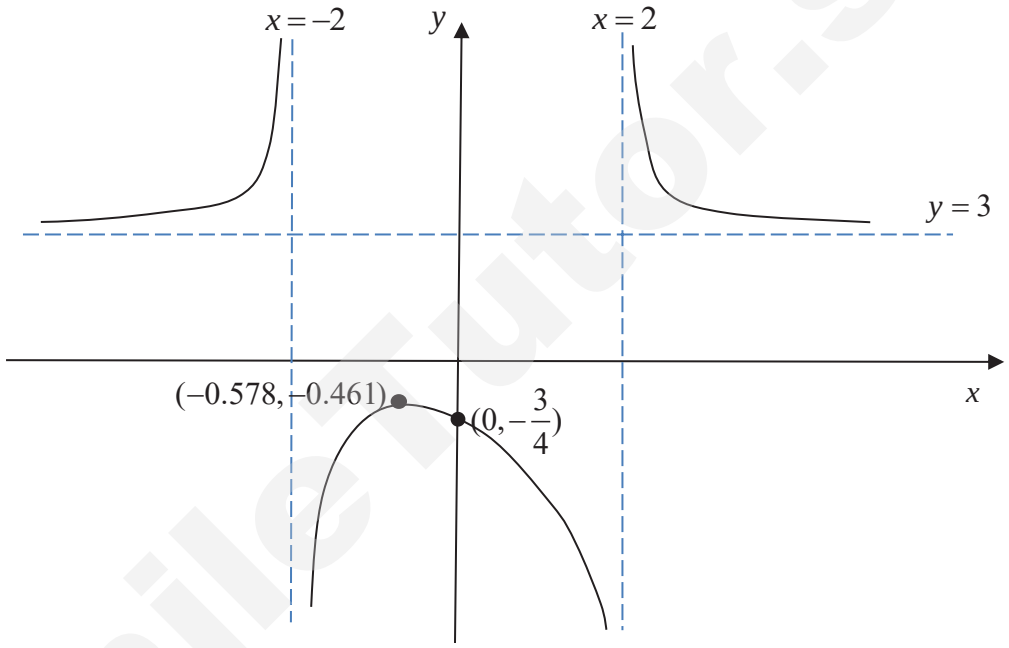
2	**a(i)**	$x^2-b=\left(x+\sqrt{b}\right)\left(x-\sqrt{b}\right)$ So, $x=\sqrt{b}$ and $x=-\sqrt{b}$ are asymptotes. Hence, $\sqrt{b}=2\Rightarrow b=4$. $y=\dfrac{ax^2+bx+a}{x^2-b}=a+\dfrac{bx+ab+a}{x^2-b}$ So, $y=a$ is an asymptote. Hence, $a=3$.
6m	**(ii)**	

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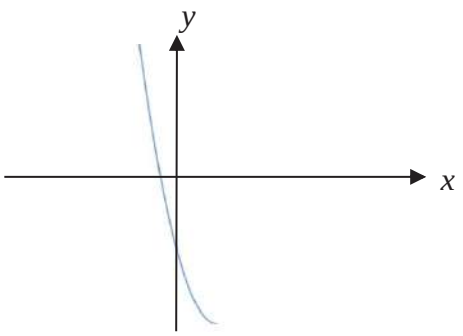
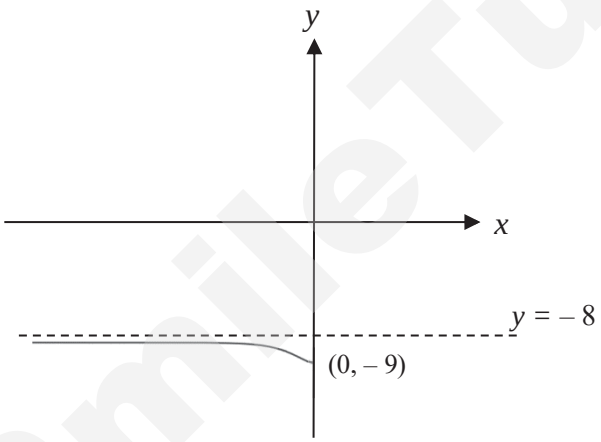
3 **7m**	**(i)**	 Since any horizontal line intersects the graph of f at most once, f is one-one and it has an inverse. (shown)
	(ii)	$R_f = [-9, \infty)$ Since $R_f = [-9, \infty) \not\subseteq D_g$, therefore gf does not exists.
	(iii)	$fg : x \mapsto (e^x + 1)^2 - 4(e^x + 1) - 5$ $= e^{2x} - 2e^x - 8, x \in \mathbb{R}, x \leq 0$  $R_{fg} = [-9, -8)$

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4	**(a)**	$\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \dfrac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \cdot (-2x)$ $= -\dfrac{1}{\sqrt{x^2}} \cdot \dfrac{1}{2\sqrt{1-x^2}} \cdot (-2x)$ $= \dfrac{1}{x} \cdot \dfrac{1}{\sqrt{1-x^2}} \cdot x \qquad (\because x > 0)$ $= \dfrac{1}{\sqrt{1-x^2}}$
	(b)	5 $\dfrac{d}{dx}\left(x^2+xy+y^2\right) = \dfrac{d}{dx}\sin x$ $2x + x \cdot \dfrac{dy}{dx} + y(1) + 2y\dfrac{dy}{dx} = \cos x$ $\dfrac{dy}{dx}(x+2y) = \cos x - 2x - y$ $\dfrac{dy}{dx} = \dfrac{\cos x - 2x - y}{x+2y}$

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5 **(i)**

$$\sin\theta = \frac{1}{PQ} \quad \Rightarrow \quad PQ = \operatorname{cosec}\theta$$

$$QR = 4 - PQ - RS = 4 - 2\operatorname{cosec}\theta$$

$$PS = QR + 2\left(\cot\theta\right) = 4 - 2\ \operatorname{cosec}\theta + 2\cot\theta$$

$$A = \frac{1}{2}\left(PS + QR\right)(1)$$

$$= \frac{1}{2}\left(4 - 2\ \operatorname{cosec}\theta + 2\cot\theta + 4 - 2\ \operatorname{cosec}\theta\ \right)$$

$$= 4 - 2\ \operatorname{cosec}\theta + \cot\theta \quad (\text{shown})$$

7m **(ii)**

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = 2\ \operatorname{cosec}\theta\cot\theta - \operatorname{cosec}^2\theta$$

For maximum area, $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 0$.

$$2\ \operatorname{cosec}\theta\cot\theta - \operatorname{cosec}^2\theta = 0$$

$$2\ \operatorname{cosec}\theta\left(\operatorname{cosec}\theta\cos\theta\right) - \operatorname{cosec}^2\theta = 0$$

$$\operatorname{cosec}^2\theta\left(2\cos\theta - 1\right) = 0$$

$$2\cos\theta - 1 = 0 \ \text{ or } \ \operatorname{cosec}\theta = 0 \quad (\text{No solution})$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

θ	$\left(\frac{\pi}{3}\right)^-$	$\frac{\pi}{3}$	$\left(\frac{\pi}{3}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}\theta}$	>0	0	<0
	/	—	\

Hence, A is a maximum.

$$\text{Maximum Area} = 4 - 2\ \operatorname{cosec}\frac{\pi}{3} + \cot\frac{\pi}{3}$$

$$= 4 - \frac{4}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= 4 - \frac{3}{\sqrt{3}}$$

$$= 4 - \sqrt{3}$$

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6	(i)	$(8-3x)^{-\frac{1}{3}}$ $=8^{-\frac{1}{3}}\left(1-\frac{3}{8}x\right)^{-\frac{1}{3}}$ $=\frac{1}{2}\left(1-\frac{3}{8}x\right)^{-\frac{1}{3}}$ $=\frac{1}{2}\left(1+\left(-\frac{1}{3}\right)\left(-\frac{3}{8}x\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2}\left(-\frac{3}{8}x\right)^2+\ldots\right)$ $=\frac{1}{2}\left(1+\frac{1}{8}x+\frac{2}{64}x^2+\ldots\right)$ $=\frac{1}{2}+\frac{1}{16}x+\frac{1}{64}x^2+\ldots$ $\left\|-\frac{3}{8}x\right\|<1$ $\Leftrightarrow -\frac{8}{3}<x<\frac{8}{3}$
		Method 2: $f(x)=(8-3x)^{-\frac{1}{3}}$ $f'(x)=(8-3x)^{-\frac{4}{3}}$ $f''(x)=4(8-3x)^{-\frac{7}{3}}$ $f(0)=\frac{1}{2}$ $f'(0)=\frac{1}{16}$ $f''(0)=\frac{1}{32}$ $(8-3x)^{-\frac{1}{3}}=\frac{1}{2}+\frac{1}{16}x+\frac{1}{64}x^2+\ldots$ $\left\|-\frac{3}{8}x\right\|<1\Leftrightarrow -\frac{8}{3}<x<\frac{8}{3}$

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7m	**(ii)**	$\dfrac{1}{\sqrt[3]{8-3\left(\frac{1}{16}\right)}} = \dfrac{1}{2}+\dfrac{1}{16}\left(\dfrac{1}{16}\right)+\dfrac{1}{64}\left(\dfrac{1}{16}\right)^2+\ldots$ $\dfrac{1}{\sqrt[3]{\frac{125}{16}}} = \dfrac{1}{2}+\dfrac{1}{256}+\dfrac{1}{16384}+\ldots$ $\dfrac{\sqrt[3]{16}}{5} \approx \dfrac{8192+64+1}{16384}$ $\sqrt[3]{16} \approx \dfrac{41285}{16384}$

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7 **8m**

(a)

$$\frac{x^2+3x-5}{x-2} \le 1$$

$$\frac{x^2+3x-5}{x-2} - 1 \le 0$$

$$\frac{x^2+3x-5-(x-2)}{x-2} \le 0$$

$$\frac{x^2+2x-3}{x-2} \le 0$$

$$\frac{(x+3)(x-1)}{x-2} \le 0$$

Hence, $x \le -3$ or $1 \le x < 2$.

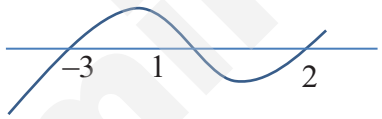
Method 2:

$$\frac{x^2+3x-5}{x-2} \le 1$$

$$(x^2+3x-5)(x-2) \le (x-2)^2$$

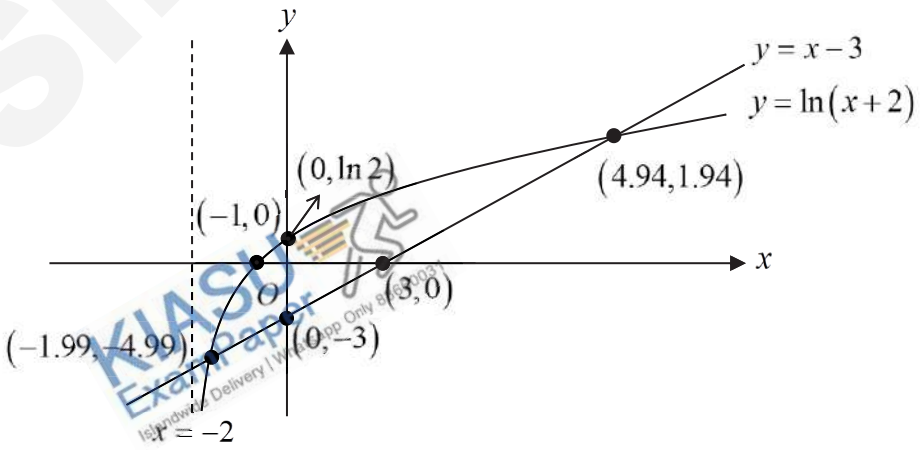
$$(x^2+3x-5)(x-2) - (x-2)^2 \le 0$$

$$(x-2)(x-1)(x+3) \le 0$$



$x \le -3$ or $1 \le x < 2$.

(b)



Hence, $-2 < x \le -1.99$ or $x \ge 4.94$.

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8	**(i)**	$\sum_{r=1}^{n} 4^{-2r} = \dfrac{\frac{1}{4^2}\left(1-\left(\frac{1}{4^2}\right)^n\right)}{1-\frac{1}{4^2}}$ $= \frac{1}{15}\left(1-\frac{1}{2^{4n}}\right)$
10m	**(ii)**	$\frac{2}{(2r+1)(2r+3)} = \frac{A}{2r+1} + \frac{B}{2r+3}$ when $r=-\frac{1}{2}$, $A = \frac{2}{2\left(-\frac{1}{2}\right)+3} = 1$ when $r=-\frac{3}{2}$, $B = \frac{2}{2\left(-\frac{3}{2}\right)+1} = -1$ $\frac{2}{(2r+1)(2r+3)} = \frac{1}{2r+1} - \frac{1}{2r+3}$ $\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)} = \frac{1}{2}\sum_{r=1}^{n} \frac{2}{(2r+1)(2r+3)}$ $= \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r+1} - \frac{1}{2r+3}\right)$ $= \frac{1}{2}\left[\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+ \ldots \left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)+\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)\right]$ $= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2n+3}\right]$ $= \frac{1}{6} - \frac{1}{2(2n+3)}$

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(iii)

$$\sum_{r=1}^{n}\left[4^{-2(r+1)}+\frac{1}{(2r+1)(2r+3)}\right]$$

$$=\sum_{r=1}^{n}\left[4^{-2(r+1)}\right]+\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)}$$

$$=\sum_{r=1}^{n}\left[4^{-2}4^{-2r}\right]+\frac{1}{6}-\frac{1}{2(2n+3)}$$

$$=\frac{1}{16}\sum_{r=1}^{n}4^{-2r}+\frac{1}{6}-\frac{1}{2(2n+3)}$$

$$=\frac{1}{16}\frac{1}{15}\left(1-\frac{1}{2^{4n}}\right)+\frac{1}{6}-\frac{1}{2(2n+3)}$$

$$=\frac{1}{16(15)}+\frac{1}{6}-\frac{1}{2^{4n}}-\frac{1}{2(2n+3)}$$

$$<\frac{41}{240}\ \text{(deduced)}\qquad \text{since } \frac{1}{2^{4n}}>0 \text{ and } \frac{1}{2(2n+3)}>0$$

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9 **(i)**

Solving the cartesian equations representing planes π_1 and π_2 (on the GC) produces

$$\begin{cases} x = 3 + 2\lambda \\ y = -2\lambda \\ z = \lambda \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad l : \underset{\sim}{r} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

(ii)

$p_3 \ : \ M(x+2y+2z-3)+(-2x-y+2z+6)=0$

$(M-2)x+(2M-1)y+(2M+2)z=3M-6$

$\underset{\sim}{r} \cdot \begin{pmatrix} M-2 \\ 2M-1 \\ 2M+2 \end{pmatrix} = 3M-6$, A normal vector to plane p_3 is $\begin{pmatrix} M-2 \\ 2M-1 \\ 2M+2 \end{pmatrix}$.

9m **(iii)**

<u>(Elegant) method</u> :

Let (x, y, z) be a point on line l.

$\because$ Line l is the common line of intersection between planes p_1 and p_2, (x, y, z) is on plane p_1 as well as plane p_2.

$\therefore \quad x+2y+2z=3,$

$\quad -2x-y+2z=-6.$

The expression

$M(\underline{x+2y+2z}-3)+(\underline{\underline{-2x-y+2z}}+6)$

$= M(\underline{3}-3)+(\underline{\underline{-6}}+6)$

$= \quad M(0) \quad + \quad (0)$

$= \quad 0$

$\therefore$ (x, y, z) lies on plane p_3.

$\because$ Any point on line l lies on plane p_3, $\therefore$ line l lies on plane p_3.

<u>Alternative standard method</u> :

Direction vector for line l, $\underset{\sim}{m} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Normal vector for plane p_3, $\underset{\sim}{n_3} = \begin{pmatrix} M-2 \\ 2M-1 \\ 2M+2 \end{pmatrix}$.

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$\because \ \underset{\sim}{m} \cdot \underset{\sim}{n}_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} M-2 \\ 2M-1 \\ 2M+2 \end{pmatrix}$

$= 2(M-2) - 2(2M-1) + (2M+2) \ = 0$,

line l is parallel to plane p_3.

Point (3, 0, 0), a point on line l.

$\because$ The expression

$M\left(3 + 2(0) + 2(0) - 3\right) + \left(-2(3) - 0 + 2(0) + 6\right)$

$= M(3-3) + (-6+6)$

$= M(0) + 0 \qquad = 0$,

this point is also on plane p_3.

$\because$ Line l // plane p_3, and

l contains a point (3, 0, 0) that's also on p_3,

$\therefore$ line l lies on plane p_3.

(iv)

Normal vector for plane π_1, $\underset{\sim}{n}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Normal vector for plane π_3, $\underset{\sim}{n}_3 = \begin{pmatrix} M-2 \\ 2M-1 \\ 2M+2 \end{pmatrix}$

Let θ be the angle formed between normal vectors $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_3$.

$\therefore \ \cos\theta = \dfrac{\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_3}{|\underset{\sim}{n}_1||\underset{\sim}{n}_3|}$

$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} M-2 \\ 2M-1 \\ 2M+2 \end{pmatrix}$

$= (M-2) + 2(2M-1) + 2(2M+2)$

$= 9M$

$|\underset{\sim}{n}_1| = \sqrt{1^2 + 2^2 + 2^2} = 3$

$|\underset{\sim}{n}_3| = \sqrt{(M-2)^2 + (2M-1)^2 + (2M+2)^2}$

$= \sqrt{(M^2 - 4M + 4) + (4M^2 - 4M + 1) + (4M^2 + 8M + 4)}$

$= \sqrt{9M^2 + 9}$

$= 3\sqrt{M^2 + 1}$

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		$\cos\theta = \dfrac{\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_3}{\left\|\underset{\sim}{n}_1\right\|\left\|\underset{\sim}{n}_3\right\|}$ $= \dfrac{9M}{3\left(3\sqrt{M^2+1}\right)} = \dfrac{M}{\sqrt{M^2+1}} > 0 \quad \text{if } M > 0$ $\therefore$ The cosine of the acute angle between planes p_1 and p_3 is $\dfrac{M}{\sqrt{M^2+1}}$.

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10 **(i)**

Area of triangle $OAB = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$

$= \frac{1}{2}\left|\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} \alpha \\ 1 \\ -1 \end{pmatrix}\right|$

$= \frac{1}{2}\left|\begin{pmatrix} -1-(-2) \\ -(0+2\alpha) \\ 0-\alpha \end{pmatrix}\right|$

$= \frac{1}{2}\left|\begin{pmatrix} 1 \\ -2\alpha \\ -\alpha \end{pmatrix}\right|$

$= \frac{1}{2}\sqrt{1+(-2\alpha)^2+(-\alpha)^2}$

$= \frac{1}{2}\sqrt{1+5\alpha^2}$

10m **(ii)**

By the Ratio Theorem, $\overrightarrow{OB} = \frac{3\overrightarrow{OA}+\overrightarrow{OC}}{4}$

$\begin{pmatrix} \alpha \\ 1 \\ -1 \end{pmatrix} = \frac{1}{4}\left[3\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + \overrightarrow{OC}\right]$

$\overrightarrow{OC} = \begin{pmatrix} 4\alpha \\ 4-3 \\ -4+6 \end{pmatrix} = \begin{pmatrix} 4\alpha \\ 1 \\ 2 \end{pmatrix}$

$\overrightarrow{OB}$ is perpendicular to $\overrightarrow{OC}$, $\begin{pmatrix} \alpha \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 4\alpha \\ 1 \\ 2 \end{pmatrix} = 4\alpha^2 - 1 = 0$

$\alpha = -\frac{1}{2}\left(\because \alpha < 0\right)$

(iii)

Exact perpendicular distance from B to $OA = \left|\overrightarrow{BN}\right|$

From (ii), $\alpha = -\frac{1}{2} \Rightarrow$ Area of triangle $OAB = \frac{1}{2}\sqrt{1+5\alpha^2} = \frac{1}{2}\sqrt{1+5\left(\frac{1}{4}\right)}$

$\frac{1}{2}\sqrt{\frac{9}{4}} = \frac{1}{2}\left|\overrightarrow{BN}\right|\left|\overrightarrow{OA}\right|$

$\frac{3}{2} = \left|\overrightarrow{BN}\right|\sqrt{1+4}$

$\left|\overrightarrow{BN}\right| = \frac{3}{2\sqrt{5}}$

$\left|\overrightarrow{OB}\right|^2 = \left|\overrightarrow{ON}\right|^2 + \left|\overrightarrow{BN}\right|^2$

$\left(\frac{1}{4}+1+1\right) = \left|\overrightarrow{ON}\right|^2 + \frac{9}{4(5)}$

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		$\left\|\overrightarrow{ON}\right\|^2 = \frac{9}{5}$ $\overrightarrow{ON} = \frac{3}{\sqrt{5}} \frac{\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{1+4}} = \begin{pmatrix} 0 \\ \frac{3}{5} \\ -\frac{6}{5} \end{pmatrix}$

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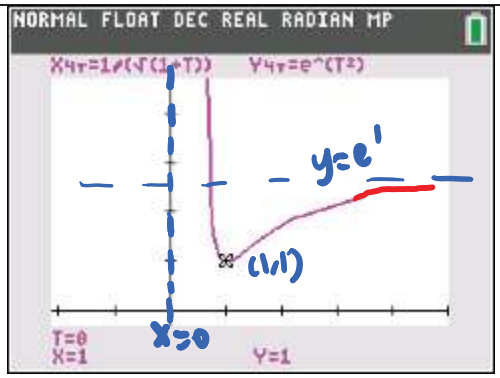
11 **(i)**



As $x \to \infty, t \to -1 \Rightarrow y \to e^{(-1)^2} = e$

12m **(ii)**

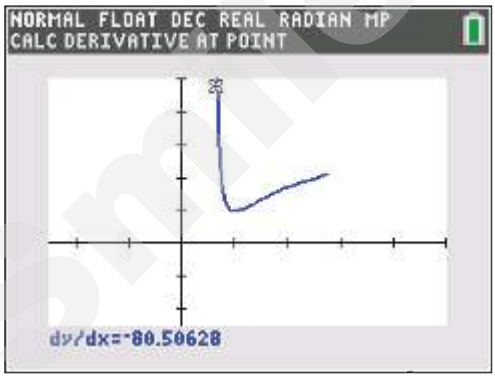
$$x = \frac{1}{\sqrt{t+1}} \Rightarrow \frac{dx}{dt} = -\frac{1}{2\sqrt{(t+1)^3}}$$

$$y = e^{t^2} \Rightarrow \frac{dy}{dt} = 2te^{t^2}$$

$$\frac{dy}{dx} = -\frac{2te^{t^2}}{\frac{1}{2\sqrt{(t+1)^3}}} = -4te^{t^2}\sqrt{(t+1)^3} \text{ (shown)}$$

(iii)

Gradient of normal at $t = \frac{5}{4}$ is $-\frac{1}{\left.\frac{dy}{dx}\right|_{t=\frac{5}{4}}} = \frac{1}{80.50628} = 0.012421$



When $t = \frac{5}{4}$, the point on C is $\left(\frac{2}{3}, e^{\frac{25}{16}}\right)$ or $\left(\frac{2}{3}, 4.7707\right)$

The normal at $t = \frac{5}{4}$ meets the y-axis at the point P:

At y-axis:

$$y - e^{\frac{25}{16}} = 0.012421\left(0 - \frac{2}{3}\right)$$

$$y = 4.76245$$

P: $(0, 4.76245)$

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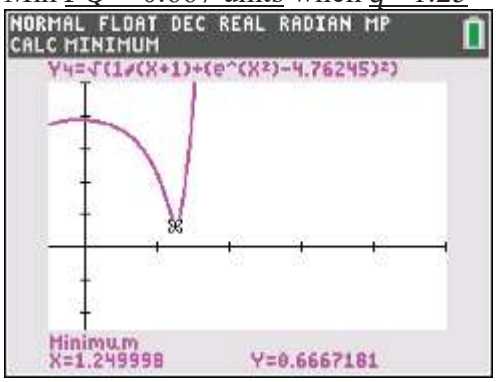
Given Q: $\left(\frac{1}{\sqrt{q+1}}, e^{q^2}\right)$

$$PQ = \sqrt{\left(\frac{1}{\sqrt{q+1}}\right)^2 + \left(e^{q^2} - 4.76245\right)^2}$$

Min PQ = 0.667 units when q= 1.25



OR

For PQ to be a minimum, PQ must be perpendicular to the tangent of Q [B1],

thus Q must be $q = \frac{5}{4}$ [B1]

Thus Min PQ = $\sqrt{\left(\frac{1}{\sqrt{\frac{5}{4}+1}}\right)^2 + \left(e^{\frac{25}{16}} - 4.76245\right)^2}$ =0.667 [B1]

OR

For PQ to be a minimum, PQ must be perpendicular to the tangent of Q [B1]

Thus

$$\frac{1}{-4\left(\frac{5}{4}\right)e^{\left(\frac{5}{4}\right)}\sqrt{\left(\frac{5}{4}+1\right)^3}}\left(-4qe^{q^2}\sqrt{(q+1)^3}\right) = -1$$

Solving gives q = 1.25 [B1]

Thus Min PQ = $\sqrt{\left(\frac{1}{\sqrt{\frac{5}{4}+1}}\right)^2 + \left(e^{\frac{25}{16}} - 4.76245\right)^2}$ =0.667 [B1]

OR

For PQ to be a minimum, gradient of the line PQ must 0

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		P: $(0, 4.76245)$, Q: $\left(\frac{1}{\sqrt{q+1}}, e^{q^2}\right)$, thus $\frac{dy}{dx} = \frac{e^{q^2} - 4.7625}{\frac{1}{\sqrt{q+1}}}$ Solving for $\frac{dy}{dx} = \frac{e^{q^2} - 4.7625}{\frac{1}{\sqrt{q+1}}} = 0$ $\Rightarrow q = 1.25$ (to 3 sf) [B1] Thus Min PQ = $\sqrt{\left(\frac{1}{\sqrt{1.2493+1}}\right)^2 + \left(e^{\frac{25}{16}} - 4.76245\right)^2}$ =0.667

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12 **(a) (i)**

Month	Outstanding amt. owed	
	1st day of month aft. instalment	Last day of month aft. interest
Oct-2019	12000	$12000(1.005)$
Nov-2019	$12000(1.005) - P$	$12000(1.005^2) - P(1.005)$
Dec-2019	$12000(1.005^2) - P(1.005) - P$	

Amount owed upon paying the 2nd instalment (in \$)
$= 12000(1.005^2) - P(1.005) - P$

14m **(a) (ii)**

Instal-ment	Outstanding amt. owed	
	1st day of month aft. instalment	Last day of month aft. interest
0	12000	$12000(1.005)$
1	$12000(1.005) - P$	$12000(1.005^2) - P(1.005)$
2	$12000(1.005^2) - P(1.005) - P$	$12000(1.005^3) - P(1.005^2) - P(1.005)$
3	$12000(1.005^3) - P(1.005^2) - P(1.005) - P$	…
…	…	…
n	$12000(1.005^n) - P(1.005^{n-1}) \ldots - P(1.005) - P$	

Amount owed upon paying the nth instalment (in \$)

$$= 12000(1.005^n) \underbrace{- P(1.005^{n-1}) \ldots \; P(1.005) - P}$$

Geometric series, n terms
Starting from right : first term $= -P$, common ratio $= 1.005$

$$= 12000(1.005^n) - \frac{P(1.005^n - 1)}{1.005 - 1}$$

$$= 12000(1.005^n) - 200P(1.005^n - 1) \quad \text{(shown)}$$

(a) (iii)

Intended last instalment on 1st December 2021.
- 2019 : 2 instalments
- 2020 : 12 instalments
- 2021 : 12 instalments

Total no. of instalments = 26

Amount owed upon paying the 26th instalment (in \$)

$= 12000(1.005^{26}) - 200P(1.005^{26} - 1) \leq 0$ if loan is repaid completely

formulate eqn.

$$\therefore P \geq \frac{12000(1.005^{26})}{200(1.005^{26} - 1)} = 60\left(\frac{1.005^{26}}{1.005^{26} - 1}\right)$$

$$= 493.3395\ldots$$

Minimum monthly instalment needed = \$493.34
(nearest cent)

(b) (i)

Let a hertz be the frequency of the tone produced by the 1st key on the piano, and r be the common ratio of the frequency between successsive keys.

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		$\because$ the 49th key produces 440 hertz, $ar^{48} = 440$ (1) $\because$ the 61st key produces 440 hertz, $ar^{60} = 880$ (2) Eliminating a, $\frac{(2)}{(1)}$: $r^{12} = 2$, $r = 2^{\frac{1}{12}}$ From (1): $a = \frac{440}{r^{48}} = \frac{440}{\left(2^{\frac{1}{12}}\right)^{48}}$ $= \frac{440}{2^4} = \frac{440}{16} = 27.5$ (frequency of 1st key in hertz)	
	(b) (ii)	Frequency produced by the nth key on the piano (in hertz), $u_n = ar^{n-1} = 27.5\ (2^{\frac{1}{12}})^{n-1}$ $= 27.5\ (2^{\frac{n-1}{12}})$ Sequence of logarithm of the frequencies produced by successive piano keys : $\ln(u_1),\ \ln(u_2),\ \ln(u_3),\ \ldots,\ \ln(u_n),\ \ldots$ The difference between consecutive terms of the sequence $= \ln(u_n) - \ln(u_{n-1})$ $= \ln\left(\frac{u_n}{u_{n-1}}\right)$, using properties of the ln function $= \ln(r)$, $\because\ u_1,\ u_2,\ u_3,\ \ldots,\ u_n,\ \ldots$ is in a G.P., with $r = 2^{\frac{1}{12}}$ $= \ln(2^{\frac{1}{12}})$, a constant $\therefore\ \ln(u_1),\ \ln(u_2),\ \ln(u_3),\ \ldots,\ \ln(u_n),\ \ldots$ is in an A.P.	
	(b) (iii)	Ratio of ascending frequencies produced in a major triad chord, sounded by the nth, $(n+4)^{\text{th}}$ and $(n+7)^{\text{th}}$ keys is $= ar^{n-1} : ar^{(n+4)-1} : ar^{(n+7)-1}$ $= ar^{n-1}(1) : ar^{n-1}(r^4) : ar^{n-1}(r^7)$ $= 1 : r^4 : r^7$ $= 1 : (2^{\frac{1}{12}})^4 : (2^{\frac{1}{12}})^7$ $= 1 : 2^{\frac{1}{3}} : 2^{\frac{7}{12}}$, which is independent of n. (shown)	

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Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Promotional Examination
Year 5

MATHEMATICS (Higher 2) 9758/01

Paper 1 3 October 2019

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Name, Index Number and Class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer all the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Total
Score													
Max Score	6	4	6	8	7	7	8	9	10	11	12	12	100

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1 **(i)** Obtain the expansion of $(1+x)^{\frac{1}{2}}$ up to and including the term in x^2. [1]

(ii) In the triangle ABC, $AC = 1$, $BC = \sqrt{3}$ and angle $ACB = \theta + \frac{\pi}{6}$ radians. Given that θ is a sufficiently small angle, show that

$$AB \approx 1 + p\theta + q\theta^2,$$

where p and q are constants to be determined in exact form. [5]

2 Using the result $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$, show that $\sum_{r=1}^{n} (r-n)(2^{-r}+1)$ can be expressed in the form $C\left(1 - \frac{1}{2^n}\right) + Dn(n+1)$, where C and D are constants to be determined. [4]

3 The equations of the lines l_1 and l_2 are as follows:

$$l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ \alpha \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}, \qquad l_2 : \frac{x+2}{-4} = y - 4, \ z = 4.$$

where α is a constant.

(i) Write down a vector equation of line l_2. [1]

The two lines l_1 and l_2 intersect at point P.

(ii) Find the value of α and the position vector of P. [3]

(iii) Find the acute angle between the lines l_1 and l_2. [2]

4 A curve C is defined by the parametric equations

$$x = \frac{t}{1+t^2}, \ y = \frac{t}{1-t^2}, \text{ where } 0 \le t < 1.$$

(i) Find the equation of the tangent to C at the point where $x = \frac{2}{5}$. [4]

(ii) Find the exact area of the finite region bounded by C, the tangent found in part **(i)** and the x-axis. [4]

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5 Express

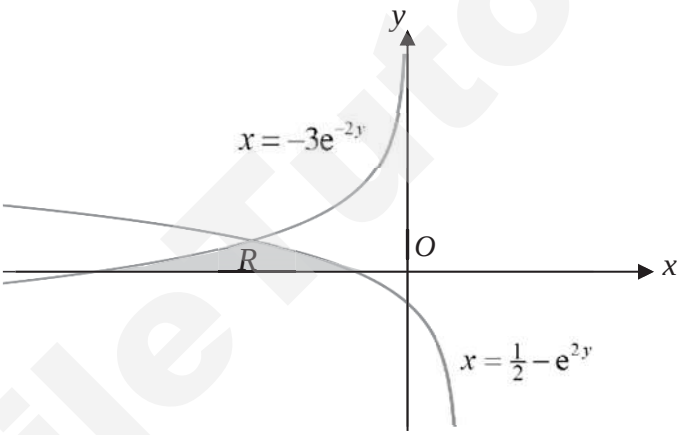
$$\frac{2r+1}{r(r+1)(r+2)}$$

in partial fractions and hence use the method of differences to find the sum of the series

$$\frac{5}{24}+\frac{7}{60}+\frac{9}{120}+...+\frac{2n+1}{n(n+1)(n+2)}.$$ [5]

Give a reason why the series $\frac{5}{12}+\frac{7}{30}+\frac{9}{60}+...$ converges, and write down its value. [2]

6 The shaded region R is bounded by the curves $x=-3e^{-2y}$, $x=\frac{1}{2}-e^{2y}$ and the x-axis as shown in the diagram below.



(i) Find the exact area of the region R. [5]

(ii) Find the volume of the solid of revolution formed when R is rotated through 4 right angles about the y-axis, giving your answer correct to 2 decimal places. [2]

7 With respect to the origin O, the position vectors of two points A and B are given by $\mathbf{a}$ and $\mathbf{b}$ respectively where $|\mathbf{a}|=1$, $|\mathbf{b}|=2$ and $\mathbf{a}\cdot\mathbf{b}=1$. Point C lies on AB such that $AC=2CB$. Point N is the foot of perpendicular from C to the line OB.

(i) Find the position vector of the point C in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(ii) Show that the length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OB}$ is $\frac{3}{2}$ and deduce the position vector of N. [4]

(iii) Find the value of λ such that C, D and N are collinear where $\overrightarrow{OD}=\mathbf{a}+\lambda\mathbf{b}$. [3]

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8 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{2x}{x-2}, \text{ for } x \in \mathbb{R},\ x \neq 2.$$

(i) Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. [2]

(ii) Solve the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [1]

The function g is defined as follows.

$$\mathrm{g}: x \mapsto \frac{2}{x-3}, \text{ for } x \in \mathbb{R},\ x \neq 3,\ x \neq 4.$$

(iii) Find $\mathrm{fg}(x)$. [1]

(iv) Solve the inequality $\mathrm{fg}(x) < x$ for all x in the domain of fg. [3]

(v) Find the range of fg. [2]

9 **(a)** **(i)** Find the values of a, b and c such that $x = (ax+b)(x+1) + c(x^2+2x+2)$ for all real values of x. [2]

(ii) Hence find $\displaystyle\int \frac{x}{(x+1)(x^2+2x+2)}\,\mathrm{d}x$. [4]

(b) **(i)** By using a graphical approach, solve $2x + \sqrt{x+5} > 0$. [1]

(ii) Given that $\displaystyle\int_{-2}^{2} \left|2x + \sqrt{x+5}\right| \mathrm{d}x = p\sqrt{3} + \tfrac{1}{3}q\left(\sqrt{7}-1\right)$ for some real values of p and q, find the values of p and q. [3]

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10 The curve C_1 is defined by $y = 2x + 1 + \dfrac{b}{x+a}, x \neq -a$, where a and b are constants. Given that C_1 has two stationary points, what can be said about the values of a and b? [3]

It is given that $a = b = 1$.

Sketch C_1, indicating in your graph any points where C_1 crosses the axes and the equations of any asymptotes. [2]

(a) The curve C_2 is defined by $(x+p)^2 + y^2 = R^2$, where p is a real constant and $R > 0$. Sketch the graph of C_2 on the same diagram as C_1. [1]

It is given that C_1 and C_2 intersect. By considering the stationary points of C_1, find the minimum value of R as p varies and state the corresponding value of p. Express both your answers in exact form. [3]

(b) Sketch the graph of $y = \mathrm{f}'(x)$, where $\mathrm{f}(x) = \left|2x + 1 + \dfrac{1}{x+1}\right|$. [2]

11 An interior design company can only furnish one house at a time. For a residential project A, it takes 160 hours to furnish the first house. Each subsequent house takes k hours shorter to furnish than the previous one due to the similar design.

(i) If the company takes a total of 3800 hours to furnish the first 25 houses, find the exact time it would need to furnish the next 25 houses. [4]

(ii) The company would like to achieve a target completion time of less than 140 hours for the nth house. Find the value of n when the target is first met. [2]

To reduce the overall furnishing time, the company decides to double its manpower. For project B, a similarly designed residential project like A, it still takes 160 hours to furnish the first house. However, the time taken to furnish each subsequent house is 4% shorter than the previous one.

(iii) Find the number of houses that can be fully furnished in 3800 hours. If productivity is defined as the number of houses fully furnished per man-hour, calculate the percentage improvement in productivity for project B compared to A based on the houses fully furnished in 3800 hours.

[Note : Man-hour for a task = number of people assigned × time taken to complete the task] [4]

(iv) For project B, show that the total time taken to furnish will not exceed 4000 hours, no matter how many houses there are. Briefly comment on your answer in the context of this question. [2]

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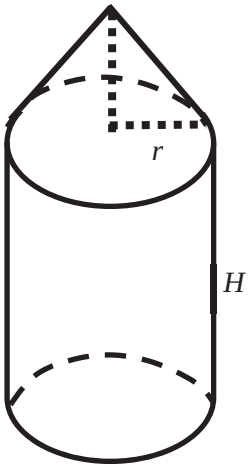
12 [A circular cone with base radius r, vertical height h and slant height l, has curved surface area πrl and volume $\frac{1}{3}\pi r^2 h$.]



A capsule made of metal sheet of fixed volume p cm^3 is made up of three parts.

- The top is modelled by the curved surface of a circular cone of radius r cm. The ratio of its height to its base radius is $4:3$.
- The body is modelled by the curved surface of a cylinder of radius r cm and height H cm.
- The base is modelled by a circular disc of radius r cm.

The cost of making the body of the capsule is \$$k$ per cm^2, while that of the top and the base of the capsule is \$$2k$ per cm^2, where k is a constant. The total cost of making the capsule is \$$C$. Assume the metal sheet is made of negligible thickness.

(i) Show that $H = \frac{p}{\pi r^2} - \frac{4r}{9}$. [2]

(ii) Express C in the form $\frac{A}{r} + Br^2$, where A and B are expressions in terms of k and p. Use differentiation to show that C has a minimum as r varies. [8]

(iii) Hence determine the ratio of H to r when C is a minimum. [2]

End of Paper

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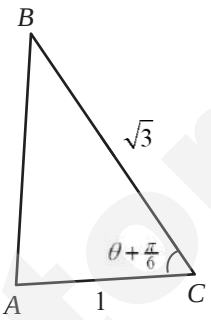
2019 Year 5 H2 Math Promotional Examination solution

Qn	Suggested Solution
1(i)	$(1+x)^{\frac{1}{2}}$ $=1+\frac{1}{2}x-\frac{1}{8}x^2+...$
ii)	Using cosine rule, $AB^2=1^2+\left(\sqrt{3}\right)^2-2(1)(\sqrt{3})\cos\left(\theta+\frac{\pi}{6}\right)$ $=4-2\sqrt{3}\left(\cos\theta\cos\frac{\pi}{6}-\sin\theta\sin\frac{\pi}{6}\right)$ $=4-2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta\right)$ $=4-3\cos\theta+\sqrt{3}\sin\theta$ $\approx 4-3\left(1-\frac{1}{2}\theta^2\right)+\sqrt{3}\theta$ $=1+\sqrt{3}\theta+\frac{3}{2}\theta^2$ $AB=\left(1+\sqrt{3}\theta+\frac{3}{2}\theta^2\right)^{\frac{1}{2}}$ $=1+\frac{1}{2}\left(\sqrt{3}\theta+\frac{3}{2}\theta^2\right)-\frac{1}{8}\left(\sqrt{3}\theta+\frac{3}{2}\theta^2\right)^2+...$ $=1+\frac{\sqrt{3}}{2}\theta+\frac{3}{4}\theta^2-\frac{3}{8}\theta^2+...$ $\approx 1+\frac{\sqrt{3}}{2}\theta+\frac{3}{8}\theta^2$ where $p=\frac{\sqrt{3}}{2}$, $q=\frac{3}{8}$



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Qn	Suggested Solution
2	$\sum_{r=1}^{n}(r-n)(2^{-r}+1)=\sum_{r=1}^{n}\left(\frac{r}{2^r}+r-\frac{n}{2^r}-n\right)$ $=\left(2-\frac{n+2}{2^n}\right)+\left(\frac{1}{2}n(n+1)\right)-n\left(\frac{\frac{1}{2}\left(1-(\frac{1}{2})^n\right)}{1-\frac{1}{2}}\right)-n^2$ $=2-\frac{n}{2^n}-\frac{2}{2^n}+\frac{1}{2}n(n+1)-n\left(1-\frac{1}{2^n}\right)-n^2$ $=2\left(1-\frac{1}{2^n}\right)-\frac{n}{2^n}+\frac{1}{2}n(n+1)-n(n+1)+\frac{n}{2^n}$ $=2\left(1-\frac{1}{2^n}\right)-\frac{1}{2}n(n+1)$ where $C=2,\ D=-\frac{1}{2}$

Qn	Suggested Solution
3(i)	$l_2:\mathbf{r}=\begin{pmatrix}-2\\4\\4\end{pmatrix}+\mu\begin{pmatrix}-4\\1\\0\end{pmatrix},\ \mu\in\mathbb{R}$
(ii)	Consider $\begin{pmatrix}3\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\\alpha\\1\end{pmatrix}=\begin{pmatrix}-2\\4\\4\end{pmatrix}+\mu\begin{pmatrix}-4\\1\\0\end{pmatrix},$ $\lambda\begin{pmatrix}3\\\alpha\\1\end{pmatrix}+\mu\begin{pmatrix}4\\-1\\0\end{pmatrix}=\begin{pmatrix}-5\\3\\1\end{pmatrix}$ Solving: $3+4\mu=-5\Rightarrow\mu=-2,\lambda=1,\alpha=1,$ $\therefore\overrightarrow{OP}=\begin{pmatrix}6\\2\\4\end{pmatrix}.$
(iii)	Let θ be the angle between the two lines. $\cos\theta=\frac{\left\lvert\begin{pmatrix}3\\1\\1\end{pmatrix}.\begin{pmatrix}-4\\1\\0\end{pmatrix}\right\rvert}{\sqrt{11}\sqrt{17}}=\frac{11}{\sqrt{11}\sqrt{17}}$ $\theta=\cos^{-1}\left(\frac{\sqrt{11}}{\sqrt{17}}\right)=36.448^o=36.4^o\ (1\text{ d.p.})$

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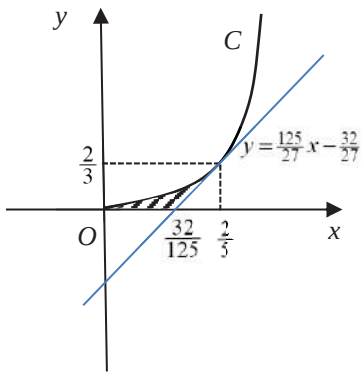
Qn	Suggested Solutions
4(i)	$x=\dfrac{t}{1+t^2},\ y=\dfrac{t}{1-t^2}$ $\dfrac{dx}{dt}=\dfrac{1-t^2}{\left(1+t^2\right)^2}\quad \dfrac{dy}{dt}=\dfrac{1+t^2}{\left(1-t^2\right)^2}$ $\dfrac{dy}{dx}=\dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)}=\dfrac{1+t^2}{\left(1-t^2\right)^2}\div\dfrac{1-t^2}{\left(1+t^2\right)^2}=\left(\dfrac{1+t^2}{1-t^2}\right)^3$ When $x=\frac{2}{5}$, $\frac{2}{5}=\dfrac{t}{1+t^2}$ $2t^2-5t+2=0$ $(2t-1)(t-2)=0$ $t=\frac{1}{2}$ or $t=2$ (reject $\because 0\le t<1$) When $t=\frac{1}{2}, \dfrac{dy}{dx}=\left(\dfrac{1+\frac{1}{2}^2}{1-\frac{1}{2}^2}\right)^3=\dfrac{125}{27}, y=\dfrac{\frac{1}{2}}{1-\frac{1}{2}^2}=\frac{2}{3}$ The equation of the tangent at $x=\frac{2}{5}$ is : $y-\frac{2}{3}=\frac{125}{27}\left(x-\frac{2}{5}\right)$ $y=\frac{125}{27}x-\frac{32}{27}$
(ii)	When $x=0,\ t=0$ When $x=\frac{2}{5},\ t=\frac{1}{2}$ x-intercept of tangent : $y=0$ $\frac{125}{27}x-\frac{32}{27}=0$ $\therefore x=\frac{32}{125}$ 

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	Required area = Area under C – Area of triangle $= \int_0^{\frac{2}{5}} y \, \mathrm{d}x - \frac{1}{2}\left(\frac{2}{5} - \frac{32}{125}\right)\left(\frac{2}{3}\right)$ $= \int_0^{\frac{1}{2}} \left(\frac{t}{1-t^2}\right)\left(\frac{1-t^2}{\left(1+t^2\right)^2}\right) \mathrm{d}t - \frac{6}{125}$ $= \int_0^{\frac{1}{2}} \left(\frac{t}{\left(1+t^2\right)^2}\right) \mathrm{d}t - \frac{6}{125}$ $= \frac{1}{2}\int_0^{\frac{1}{2}} 2t\left(1+t^2\right)^{-2} \mathrm{d}t - \frac{6}{125}$ $= -\frac{1}{2}\left[\frac{1}{1+t^2}\right]_0^{\frac{1}{2}} - \frac{6}{125}$ $= \frac{1}{10} - \frac{6}{125}$ $= \frac{13}{250}$ unit^2

	Suggested Solution
5	$\frac{2r+1}{r(r+1)(r+2)} = \frac{1}{2r} + \frac{1}{r+1} - \frac{3}{2(r+2)}$ (by cover-up rule)

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	Suggested Solution
	$\frac{5}{24}+\frac{7}{60}+\frac{9}{120}+\ldots+\frac{2n+1}{n(n+1)(n+2)}$ $=\sum_{r=2}^{n}\frac{2r+1}{r(r+1)(r+2)}$ $=\sum_{r=2}^{n}\left(\frac{1}{2r}+\frac{1}{r+1}-\frac{3}{2(r+2)}\right)$ $=\frac{1}{2}\sum_{r=2}^{n}\left(\frac{1}{r}+\frac{2}{r+1}-\frac{3}{r+2}\right)$ $=\frac{1}{2}\left[\frac{1}{2}+\frac{2}{3}-\frac{3}{4}\right.$ $+\frac{1}{3}+\frac{2}{4}-\frac{3}{5}$ $+\frac{1}{4}+\frac{2}{5}-\frac{3}{6}$ $+\quad\vdots$ $+\frac{1}{n-2}+\frac{2}{n-1}-\frac{3}{n}$ $+\frac{1}{n-1}+\frac{2}{n}-\frac{3}{n+1}$ $\left.+\frac{1}{n}+\frac{2}{n+1}-\frac{3}{n+2}\right]$ $=\frac{1}{2}\left[\frac{1}{2}+\frac{2}{3}+\frac{1}{3}-\frac{3}{n+1}+\frac{2}{n+1}-\frac{3}{n+2}\right]$ $=\frac{1}{2}\left[\frac{3}{2}-\frac{1}{n+1}-\frac{3}{n+2}\right]$
	As $n\to\infty$, $\frac{1}{2}\left[\frac{3}{2}-\frac{1}{n+1}-\frac{3}{n+2}\right]\to\frac{1}{2}\left[\frac{3}{2}-0-0\right]=\frac{3}{4}$ $\frac{5}{12}+\frac{7}{30}+\frac{9}{60}+\ldots=2\left(\frac{5}{24}+\frac{7}{60}+\frac{9}{120}+\ldots\right)$ $=2\sum_{r=2}^{\infty}\frac{2r+1}{r(r+1)(r+2)}=2\times\frac{3}{4}=\frac{3}{2}$ (finite value) $\therefore$ the series converges and the sum to infinity is $\frac{3}{2}$.

Qn	Suggested Solution

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6(i)	$\frac{1}{2}-e^{2y}=-3e^{-2y}$ $e^{2y}-\frac{1}{2}-3e^{-2y}=0$ $2e^{4y}-e^{2y}-6=0$ $\left(e^{2y}-2\right)\left(2e^{2y}+3\right)=0$ $e^{2y}=2$ or $e^{2y}=-\frac{3}{2}$(rej. since $e^{2y}>0$) $2y=\ln 2$ $y=\frac{\ln 2}{2}$ Exact area of region $R=-\int_0^{\frac{\ln 2}{2}}-3e^{-2y}-\left(\frac{1}{2}-e^{2y}\right)dy$ $=-\left[-3\left(\frac{e^{-2y}}{-2}\right)-\frac{1}{2}y+\frac{e^{2y}}{2}\right]_0^{\frac{\ln 2}{2}}$ $=-\left[\left(\frac{3}{2}\left(\frac{1}{2}\right)-\frac{\ln 2}{4}+1\right)-\left(\frac{3}{2}+\frac{1}{2}\right)\right]$ $=\frac{\ln 2+1}{4}$ unit^2
(ii)	Volume of solid of revolution $=\pi\int_0^{\frac{\ln 2}{2}}\left(-3e^{-2y}\right)^2-\left(\frac{1}{2}-e^{2y}\right)^2\ dy$ $=4.24$ unit^3 (2 d.p.)

Qn	Suggested Solution

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7(i)

$\overrightarrow{OC} = \frac{1}{3}(\mathbf{a}+2\mathbf{b})$

(ii)

Length of projection of OC onto OB

$$= \frac{|\overrightarrow{OC}\bullet\mathbf{b}|}{|\mathbf{b}|} = \frac{\left|\frac{1}{3}(\mathbf{a}+2\mathbf{b})\bullet\mathbf{b}\right|}{2}$$

$$= \frac{\frac{1}{3}|\mathbf{a}\bullet\mathbf{b}+2\mathbf{b}\bullet\mathbf{b}|}{2} = \frac{\frac{1}{3}|\mathbf{a}\bullet\mathbf{b}+2|\mathbf{b}|^2|}{2}$$

$$= \frac{1}{6}|1+2(2)^2|$$

$$= \frac{3}{2}$$

$\overrightarrow{ON} = \frac{3}{2}\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{3}{4}\mathbf{b}$

(iii)

$\overrightarrow{CN} = \frac{3}{4}\mathbf{b} - \frac{1}{3}(\mathbf{a}+2\mathbf{b}) = -\frac{1}{3}\mathbf{a} + \frac{1}{12}\mathbf{b}$

$\overrightarrow{ND} = (\mathbf{a}+\lambda\mathbf{b}) - \frac{3}{4}\mathbf{b} = \mathbf{a} + (\lambda - \frac{3}{4})\mathbf{b}$

Since the points C, N and D are collinear,

$\overrightarrow{ND} = \alpha\overrightarrow{CN}$ for some $\alpha \in \mathbb{R}$

i.e. $\mathbf{a} + \left(\lambda - \frac{3}{4}\right)\mathbf{b} = \alpha\left(-\frac{1}{3}\mathbf{a} + \frac{1}{12}\mathbf{b}\right)$

Since $\mathbf{a}$ is not parallel to $\mathbf{b}$ and $\mathbf{a}$ and $\mathbf{b}$ are non-zero,

$1 = -\frac{\alpha}{3}$ and $\lambda - \frac{3}{4} = \frac{\alpha}{12}$

$\therefore \alpha = -3$ and $\lambda = \frac{1}{2}$

Alternative

$\overrightarrow{ND}\bullet\mathbf{b} = 0$

$\overrightarrow{ND}\bullet\mathbf{b} = [(\mathbf{a}+\lambda\mathbf{b}) - \frac{3}{4}\mathbf{b}]\bullet\mathbf{b} = [\mathbf{a} + \left(\lambda - \frac{3}{4}\right)\mathbf{b}]\bullet\mathbf{b} = 0$

$\mathbf{a}\bullet\mathbf{b} + \left(\lambda - \frac{3}{4}\right)\mathbf{b}\bullet\mathbf{b} = 0$

$\therefore 1 + 4(\lambda - \frac{3}{4}) = 0$

Thus $\lambda = \frac{1}{2}$.

Qn	Suggested Solution
8(i)	Let $y = f(x) = \frac{2x}{x-2}$

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Qn	Suggested Solution
	$xy-2y=2x$ $x(y-2)=2y$ $x=\dfrac{2y}{y-2}$ $\mathrm{f}^{-1}(y)=\dfrac{2y}{y-2}$ $\therefore \mathrm{f}^{-1}(x)=\dfrac{2x}{x-2}$ Domain of $\mathrm{f}^{-1} = \mathbb{R}\setminus\{2\}$.
(ii)	$\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ Since both graphs intersects at all points in the domain of f $x\in\mathbb{R}\setminus\{2\}$
(iii)	$\mathrm{fg}(x) = \mathrm{f}\left(\dfrac{2}{x-3}\right)$ $=\dfrac{2\left(\dfrac{2}{x-3}\right)}{\dfrac{2}{x-3}-2}$ $=\dfrac{\dfrac{4}{x-3}}{\dfrac{2-2(x-3)}{x-3}}$ $=\dfrac{4}{8-2x}$ $=\dfrac{2}{4-x}$
(iv)	$\mathrm{fg}(x)<x$

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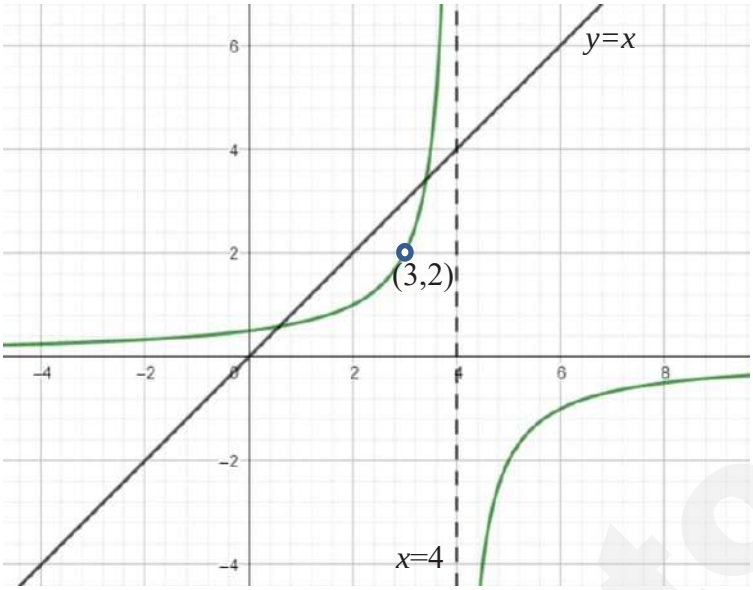
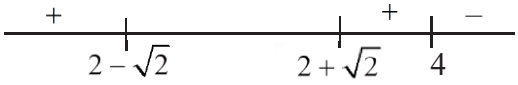
Qn	Suggested Solution
	$\frac{2}{4-x} < x$  Intersection points between the 2 graphs. ie $(0.586, 0.586)$ and $(3.41, 3.41)$ For the inequality to be satisfied, $x > 4$ or $0.586 < x < 3$ or $3 < x < 3.41$ $x > 4$ or $x \in (0.586, 3.41) \setminus \{3\}$ **Alternative** $\frac{2}{4-x} < x$ $\frac{2}{4-x} - x < 0$ $\frac{2-4x+x^2}{4-x} < 0$ $\frac{(x-2)^2-2}{4-x} < 0$ $\frac{(x-2-\sqrt{2})(x-2+\sqrt{2})}{4-x} < 0$  $x > 4$ or $2-\sqrt{2} < x < 3$ or $3 < x < 2+\sqrt{2}$ $x > 4$ or $x \in (2-\sqrt{2}, 2+\sqrt{2}) \setminus \{3\}$
(v)	For the range of fg

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Qn	Suggested Solution
	$D_g \xrightarrow{g} R_g = \mathbb{R}\setminus\{0,2\} \xrightarrow{f} R_{fg} = \mathbb{R}\setminus\{0,2\}$

Qn	Suggested Solution
9(a)(i)	$x = (ax+b)(x+1)+c(x^2+2x+2)$ $= (a+c)x^2+(a+b+2c)x+(b+2c)$ Comparing coefficients, $a + c = 0$ $a + b + 2c = 1$ $b + 2c = 0$ Solving, we have $a=1, b=2, c=-1$. **Alternative** Sub $x=-1, -1 = 0+c(1-2+2) \Rightarrow c=-1$ Compare coeff x^2: $a+c=0 \Rightarrow a-1=0 \Rightarrow a=1$ Compare coeff x: $a+b+2c=1 \Rightarrow 1+b-2=1 \Rightarrow b=2$ $\therefore a=1, b=2, c=-1$
(ii)	$\int \frac{x}{(x+1)(x^2+2x+2)}\,dx$ $= \int \frac{(x+2)(x+1)-(x^2+2x+2)}{(x+1)(x^2+2x+2)}\,dx$ $= \int \frac{x+2}{x^2+2x+2} - \frac{1}{x+1}\,dx$ $= \int \frac{x+1}{x^2+2x+2} + \frac{1}{(x+1)^2+1} - \frac{1}{x+1}\,dx$ $= \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+2}\,dx + \int \frac{1}{(x+1)^2+1} - \frac{1}{x+1}\,dx$ $= \frac{1}{2}\ln\left((x+1)^2+1\right) + \tan^{-1}(x+1) - \ln\lvert x+1\rvert + c$
(b)(i) (ii)	

Qn	Suggested Solution
10	$y = 2x+1+\frac{b}{x+a}, x \neq -a$

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Qn	Suggested Solution
	$\frac{dy}{dx} = 2 - \frac{b}{(x+a)^2}, x \neq -a.$ Given that C_1 has two stationary points, $\frac{dy}{dx} = 2 - \frac{b}{(x+a)^2} = 0$ has a solution. $(x+a)^2 = \frac{b}{2}$ has a solution ---- (*) Thus we need $b > 0$ (Note $b = 0$ gives C_1: $y = 2x + 1$ which has no stationary points) $a \in \mathbb{R}$ **Alternative 1** $\frac{dy}{dx} = 2 - \frac{b}{(x+a)^2} = 0$ has a solution. $2(x+a)^2 - b = 0$ has a solution $2x^2 + 4ax + 2a^2 - b = 0$ has a solution Discriminant = $16a^2 - 4(2)(2a^2 - b) = 8b \geq 0$ Thus $b > 0$ (Note $b \neq 0$) and $a \in \mathbb{R}$. **Alternative 2** For a rational function curve $y = 2x + 1 + \frac{b}{x+a}, x \neq -a$ to have stationary points, need $y \to (2x+1)^+$ as $x \to \infty$ Thus $b > 0$ and $a \in \mathbb{R}$.

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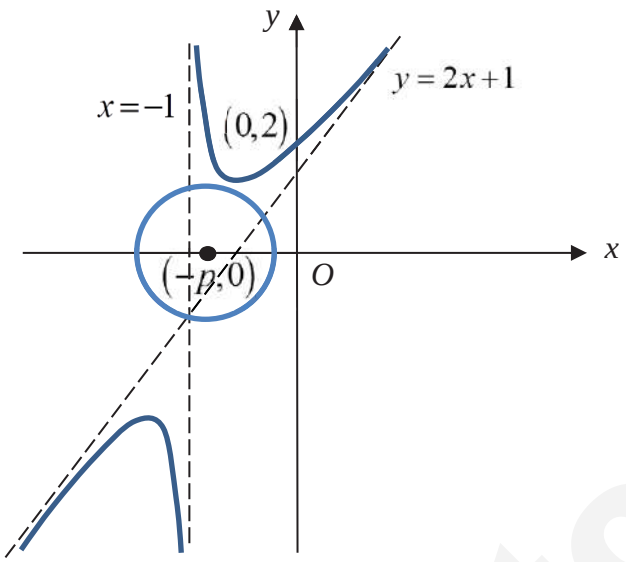
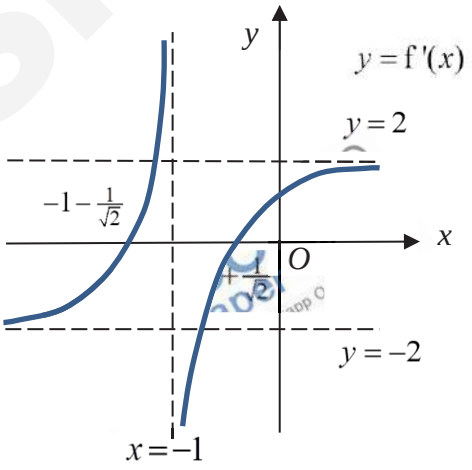
Qn	Suggested Solution
	
(i)	$(x+p)^2+y^2=R$ is a circle centre $(-p,0)$ and radius R. From (*), for stationary points of C_1, $(x+1)^2=\frac{1}{2}\Rightarrow x=-1\pm\frac{1}{\sqrt{2}}$ $x=-1+\frac{1}{\sqrt{2}}, y=2\left(-1+\frac{1}{\sqrt{2}}\right)+1+\frac{1}{\frac{1}{\sqrt{2}}}=-1+2\sqrt{2}$ Min R for C_1 and C_2 to intersect occurs when the centre of the circle is right beneath the point $\left(-1+\frac{1}{\sqrt{2}},-1+2\sqrt{2}\right)$. Thus min $R=2\sqrt{2}-1$ Corresponding value of $p=-\left(-1+\frac{1}{\sqrt{2}}\right)=1-\frac{1}{\sqrt{2}}$
(ii)	

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Qn	Suggested Solution
11(i)	Project A: Total time to furnish first 25 houses, $S_{25} = 3800 = \frac{25}{2}[2(160) - k(25-1)]$ $320 - 24k = 304$ $k = \frac{2}{3}$ Time to furnish next 25 houses $= S_{50} - S_{25}$ $= \frac{50}{2}\left[2(160) - \frac{2}{3}(50-1)\right] - 3800$ $= 3383\frac{1}{3}$ h or 3383h 20min **Alternative** Time needed to furnish the 26th house, $u_{26} = 160 - \frac{2}{3}(26-1) = 143\frac{1}{3}$h Total time needed to furnish the next 25 houses $= \frac{25}{2}\left[2(143\frac{1}{3}) - \frac{2}{3}(25-1)\right]$ $= 3383\frac{1}{3}$ h or 3383h 20min
(ii)	$160 - \frac{2}{3}(n-1) < 140$ $n - 1 > 30$ $n > 31$ $\therefore n \geq 32$ For target time of < 140h to be first met, $n = 32$
(iii)	$S_n = \frac{160(1-0.96^n)}{1-0.96} = 3800$ $1 - 0.96^n = 0.95$ $0.96^n = 0.05$ $n = \frac{\ln 0.05}{\ln 0.96} = 73.4$ Number of houses that can be fully furnished = 73 Let x be the number of man used for project A Productivity for $A = = \frac{\text{houses fully furnished}}{\text{man-hours}} = \frac{25}{3800x}$ Productivity for $B = = \frac{\text{houses fully furnished}}{\text{man-hours}} = \frac{73}{3800 \times 2x}$

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	Improvement in productivity $= \dfrac{\dfrac{73}{7600x} - \dfrac{25}{3800x}}{\dfrac{25}{3800x}} = 46\%$
(iv)	$S_n = \dfrac{160(1-0.96^n)}{1-0.96} < \dfrac{160}{1-0.96}$ since $0.96^n < 1,\ \forall n \in \mathbb{Z}^+$ $S_\infty = \dfrac{160}{1-0.96} = 4000$ $\therefore$ the total time taken will not exceed 4000h. As the geometric series is convergent, the **time taken** for furnishing a house will eventually be **negligible in the long run**. But this is **not possible in reality** since a **substantial amount of time will be required to furnish a house.**

Qn	**Suggested Solutions**
12(i)	Height of cone, $h = \dfrac{4r}{3}$ Volume $p = \pi r^2 H + \dfrac{1}{3}\pi r^2 h = \pi r^2 H + \dfrac{4}{9}\pi r^3$ $H = \dfrac{1}{\pi r^2}\left(p - \dfrac{4}{9}\pi r^3\right) = \dfrac{p}{\pi r^2} - \dfrac{4r}{9}$ (shown)
(ii)	Slant height of the cone $l = \sqrt{r^2 + \left(\dfrac{4r}{3}\right)^2} = \sqrt{\dfrac{25r^2}{9}} = \dfrac{5r}{3}$ $C = k\left(2\pi rH\right) + 2k\left(\pi rl\right) + 2k\left(\pi r^2\right)$ $= 2k\pi r\left(\dfrac{p}{\pi r^2} - \dfrac{4r}{9}\right) + 2k\pi r\left(\dfrac{5r}{3}\right) + 2k\pi r^2$ $= \dfrac{2kp}{r} - \dfrac{8k\pi r^2}{9} + \dfrac{10k\pi r^2}{3} + 2k\pi r^2$ $= \dfrac{2kp}{r} + \dfrac{40k\pi r^2}{9}$ $A = 2kp,\ B = \dfrac{40k\pi}{9}$ $\dfrac{\mathrm{d}C}{\mathrm{d}r} = -\dfrac{2kp}{r^2} + \dfrac{80k\pi r}{9}$ For minimum C, $\dfrac{\mathrm{d}C}{\mathrm{d}r} = -\dfrac{2kp}{r^2} + \dfrac{80k\pi r}{9} = 0$ $\dfrac{2kp}{r^2} = \dfrac{80k\pi r}{9}$ $r^3 = \dfrac{18kp}{80k\pi} = \dfrac{9p}{40\pi}$

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Method 1 [2nd derivative]

$\frac{\mathrm{d}^2C}{\mathrm{d}r^2} = \frac{4kp}{r^3} + \frac{80k\pi}{9}. \because\ k, p > 0, \frac{\mathrm{d}^2C}{\mathrm{d}r^2} > 0.$

When $r^3 = \frac{9p}{40\pi}$, C is minimum.

Method 2 [1st derivative]

$\frac{\mathrm{d}C}{\mathrm{d}r} = -\frac{2kp}{r^2} + \frac{80k\pi r}{9} = -kr\left(\frac{2p}{r^3} + \frac{80\pi}{9}\right)$**

r	$\sqrt[3]{\frac{9p}{40\pi}}^{-}$	$\sqrt[3]{\frac{9p}{40\pi}}$	$\sqrt[3]{\frac{9p}{40\pi}}^{+}$
$\frac{\mathrm{d}C}{\mathrm{d}r}$	–ve	0	+ve
Tangent	\	–	/

When $r = \sqrt[3]{\frac{9p}{40\pi}}$, C is minimum.

(iii)

$$\frac{H}{r} = \left(\frac{p}{\pi r^2} - \frac{4r}{9}\right)\left(\frac{1}{r}\right)$$
$$= \frac{p}{\pi r^3} - \frac{4}{9}$$
$$= \frac{p}{\pi\left(\frac{9p}{40\pi}\right)} - \frac{4}{9}$$
$$= \frac{40}{9} - \frac{4}{9}$$
$$= 4$$

$\therefore H : r = 4 : 1$

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JC1 Promotional Examination 2019

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME			
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MATHEMATICS 9758/01

02 October 2019

Paper 1 [100 marks] **3 hours**

Candidates answer on the Question Paper

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **28** printed pages (including this cover page) and **2** blank page.

For markers' use:												
Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**	**Q11**	**Q12**	**Total**

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1 At the start of the year, Mr Toh invested in three types of savings bonds, namely "Ucare", "Ushare" and "Ugain". The amount invested in "Ugain" is equal to the sum of the amounts invested in the other 2 bonds. In addition, the sum of the amount invested in "Ushare" and twice the amount invested in "Ugain" is 8 times the amount invested in "Ucare".

At the end of the year, "Ucare", "Ushare" and "Ugain" paid out interest at a rate of 2.5%, 1.75% and 3% respectively. Mr Toh received a total of $657.60 in interest.

Express this information as 3 linear equations and hence find the amount invested in savings bond "Ushare". [4]

2 Sketch the curve $y = 3 + \left|\frac{2x+1}{x-1}\right|$, stating the equations of the asymptotes.

Hence, solve the inequality $\left|\frac{2x+1}{x-1}\right| > x+5$. [5]

3 The curve C has equation $a^2x^2 - y^2 - 2(ax - 2y + 2) = 0$ where a is a constant such $a \in \mathbb{R} \setminus \{0\}$.

Show that $\frac{dy}{dx} = \frac{a(ax-1)}{y-2}$.

Hence, find the coordinates of the points on C at which the tangent is parallel to y-axis. [6]

4 Referred to the origin O, the points A, B, and C have position vectors **a**, **b**, and **c** respectively.

(i) Given that non-zero numbers λ and μ are such that $\lambda\mathbf{a} + \mu\mathbf{b} + \mathbf{c} = \mathbf{0}$ and $\lambda + \mu + 1 = 0$ with $\mu > 0$. Show that A, B, and C are collinear and find the ratio $CA : CB$ in terms of μ. [4]

(ii) F is another point such that the line passing through A, B, and C does not contain it. Find $\frac{\left|\overrightarrow{BF} \times \overrightarrow{BC}\right|}{\left|\overrightarrow{AF} \times \overrightarrow{AC}\right|}$ in terms of μ. [2]

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5 **(a)** Find $\sum_{r=n}^{2n}\left(3^r+n\right)$. [3]

(b) Express $\ln\left(\frac{r^2}{r^2-1}\right)$ as $A\ln(r-1)+B\ln r+C\ln(r+1)$, where A, B and C are integers to be determined.

Hence, find $\sum_{r=2}^{n}\ln\left(\frac{r^2}{r^2-1}\right)$, leaving your answer as a single logarithmic function in terms of n. [4]

6 The sum, S_n, of the first n terms of a sequence $\{u_n\}$ is given by $S_n=n^2-(k-2)n$, where k is a non-zero real constant.

(i) Prove that the sequence $\{u_n\}$ is an arithmetic sequence. [3]

(ii) Given that u_8, u_4 and u_2 are the first 3 terms in a geometric sequence, find the value of k. [2]

(iii) Give a reason why the geometric series converges and find the value of the sum to infinity. [2]

7 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{bx}{ax-b}, \quad \text{for } x \in \mathbb{R}, x \neq \frac{b}{a},$$

where a and b are non-zero constants.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Hence show that $\mathrm{f}^2(x)=x$, and write down $\mathrm{f}^n(x)$ where n is an odd number. [2]

The function g is defined by $\mathrm{g}: x \mapsto 2+\mathrm{e}^{-x}$, for $x \in \mathbb{R}, x \geq 0$. If $a=2$ and $b=1$ for function f,

(iii) explain why the composite function gf does not exist. [1]

(iv) find an expression for $\mathrm{fg}(x)$ and state the domain and exact range of fg. [3]

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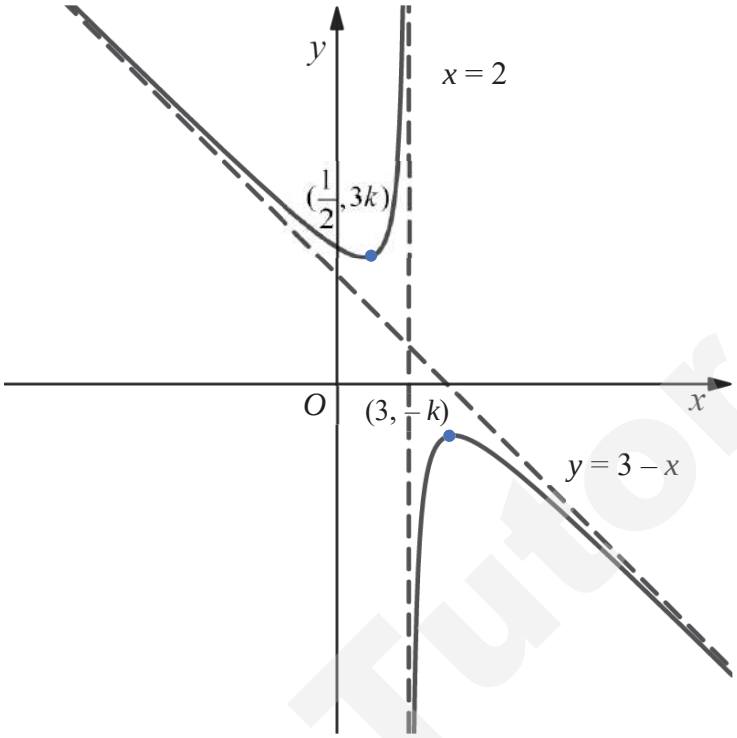
8 The diagram shows the curve $y = \mathrm{f}(x)$. The curve has a minimum point $(\frac{1}{2}, 3k)$ and a maximum point $(3, -k)$ where $k > 0$. It has asymptotes $x = 2$ and $y = 3 - x$.



Sketch, on separate diagrams, the graph of

(i) $y = \mathrm{f}(x+2) + k$, [3]

showing clearly the stationary points, axial intercepts and equations of all asymptotes.

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

showing clearly the stationary points, axial intercepts and equations of all asymptotes.

(iii) $y = \mathrm{f}'(x)$, [3]

showing clearly the stationary points, axial intercepts and equations of all asymptotes.

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9 **(a)** Find $\int \frac{1-4x}{4x^2+1}\,\mathrm{d}x$. [4]

(b) Use the substitution $u = \sqrt{x}$ to show that $\int_0^{\frac{\pi^2}{4}} \sqrt{x}\sin\sqrt{x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} u^2 \sin u\,\mathrm{d}u$. [2]

Hence, evaluate the integral $\int_0^{\frac{\pi^2}{4}} \sqrt{x}\sin\sqrt{x}\,\mathrm{d}x$ exactly. [4]

10 The curve C is defined parametrically by the equations $x = 2\mathrm{cosec}\,\theta$, $y = 2\cot\theta$, where $0 < \theta < \frac{\pi}{2}$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec\theta$. [2]

(ii) Find the equation of the tangent to the curve at the point $P(2\mathrm{cosec}\,p, 2\cot p)$ and show that the equation of the normal to the curve at P is $y = 4\cot p - x\cos p$. [3]

(iii) The tangent and normal at P cut the x-axis at T and N respectively.

Show that T has coordinates $(2\sin p, 0)$.

By finding the x-coordinates of N in the simplest form, evaluate $|ON||OT|$. [4]

(iv) Find the rate of change of y at $(2\sqrt{2}, 2)$, given that x is increasing at a constant rate of 1 unit per second. [3]

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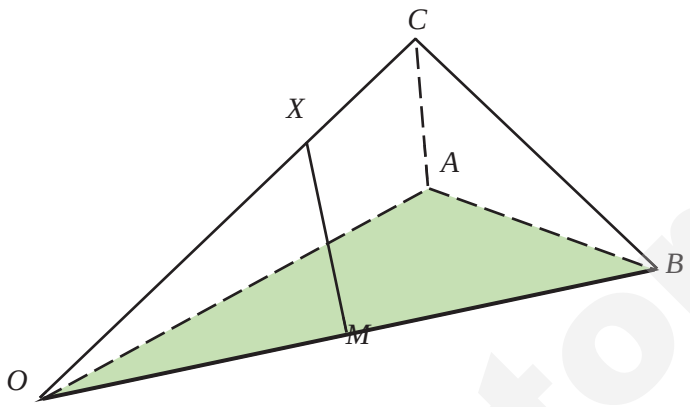
11 Engineers are building a greenhouse on an inclined grass slope. Points (x, y, z) are defined relative to a base point $O(0,0,0)$, where units are in metres.

The greenhouse is in the shape of a tetrahedron. The corners O, A and B lie on the grass slope, while the point C is the highest point of the tetrahedron (also called apex), as shown in the diagram below.



The coordinates of A, B and C are $(100,100,10)$, $(160,20,8)$ and $(80,59,20)$ respectively.

(i) Show that the Cartesian equation of the grass slope is $3x+4y-70z=0$. [2]

(ii) Find the angle of inclination of the grass slope. [2]

(iii) To secure the greenhouse, the architect plans to build a central pillar from the apex directly to the floor of the greenhouse such that the central pillar is perpendicular to the horizontal plane. Find the point where the central pillar meets the grass slope. [2]

Unfortunately, the workers misunderstand the instructions and build the central pillar from the apex to the floor of the greenhouse such that the pillar is perpendicular to the grass slope.

(iv) Find the length of this central pillar and use this length to find the volume enclosed by the greenhouse $OABC$. [4]

[The volume of a tetrahedron is given by $\frac{1}{3}\times(\text{base area})\times(\text{height})$.]

Another straight supporting beam is to be constructed from point M, the midpoint of OB, to a point X on OC such that XM is perpendicular to OB.

(v) Find the coordinates of X, correct to 1 decimal place. [3]

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12 In a model making competition, each contestant is given a square cardboard of side n cm to make a model. John takes part in the Pyramid Category and he is given a $n \times n$ cardboard to make a pyramid with the maximum volume. John recalled his calculus during his schooling days and came out with a design as shown in the diagrams below.

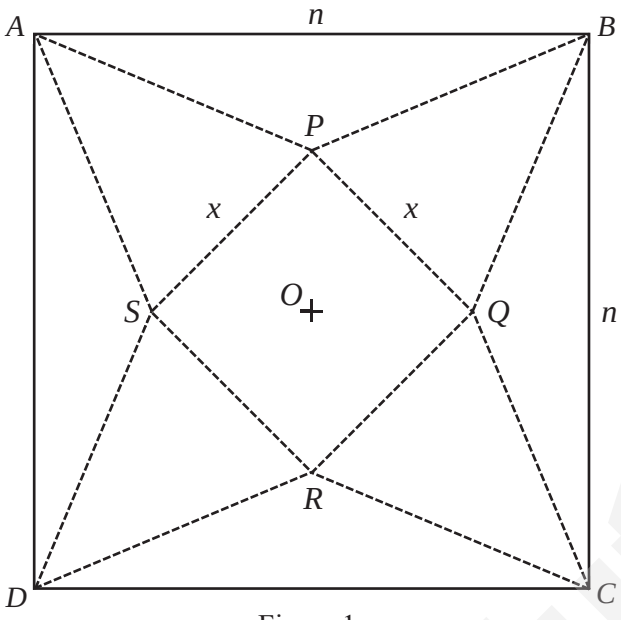

Figure 1

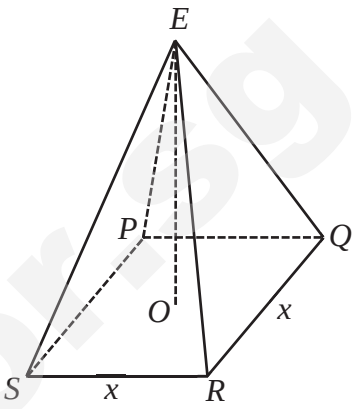

Figure 2

Figure 1 shows a square piece of cardboard $ABCD$ of side n cm, where n is a positive constant. Four triangles (ABP, BCQ, CDR, DAS) are removed from each side of $ABCD$. The remaining shape is now folded along PQ, QR, RS and SP to form a pyramid (as shown in Figure 2) with a square base $PQRS$ of side x cm. The point O is the centre of both $ABCD$ and $PQRS$. The vertex E is the point where A, B, C and D meet. Let V be the volume of the pyramid.

(i) Show that $OE^2 = \frac{1}{2}\left(n^2 - \sqrt{2}nx\right)$. [3]

(ii) Show that $V^2 = \frac{x^4}{18}\left(n^2 - \sqrt{2}nx\right)$. Hence, by using differentiation, find in terms of n, the exact value of x which John uses to obtain a maximum value of V. [7]

(iii) John's pyramid has a volume that is greater than 45 cm^3. Find the smallest $n \times n$ cardboard given, where $n \in \mathbb{Z}^+$. [2]

[The volume of a pyramid is given by $\frac{1}{3} \times (\text{base area}) \times (\text{height})$.]

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JC1 Promotional Examination 2019

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME			
CLASS		INDEX NO.	

MATHEMATICS 9758/01

02 October 2019

Paper 1 [100 marks] **3 hours**

Candidates answer on the Question Paper

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

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Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
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The number of marks is given in brackets [ ] at the end of each question or part question.

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For markers' use:												
Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**	**Q11**	**Q12**	**Total**

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1 At the start of the year, Mr Toh invested in three types of savings bonds, namely "Ucare", "Ushare" and "Ugain". The amount invested in "Ugain" is equal to the sum of the amounts invested in the other 2 bonds. In addition, the sum of the amount invested in "Ushare" and twice the amount invested in "Ugain" is 8 times the amount invested in "Ucare".

At the end of the year, "Ucare", "Ushare" and "Ugain" paid out interest at a rate of 2.5%, 1.75% and 3% respectively. Mr Toh received a total of $657.60 in interest.

Express this information as 3 linear equations and hence find the amount invested in savings bond "Ushare". [4]

Suggested solution

Let c, s and g be the amount of investments in saving bonds 'Ucare', 'Ushare' and 'Ugain' respectively.

Amount invested in bond 'Ugain' equals the sum of the amounts invested in the other 2 bonds:

$c+s=g$

$c+s-g=0$ **----- (1)**

The sum of the amount invested in "Ushare" and twice the amount invested in "Ugain" is 8 times the amount invested in "Ucare".

$s+2g=8c$

$8c-s-2g=0$ **-----(2)**

Total interest of $657.60:

$0.025c+0.0175s+0.03g=657.60$ **----(3)**

Solving simultaneously,

$c=\$4384,\quad s=\$8768 \quad \text{and} \quad g=\$13152$

Total amount of investments for Ushare is $8768

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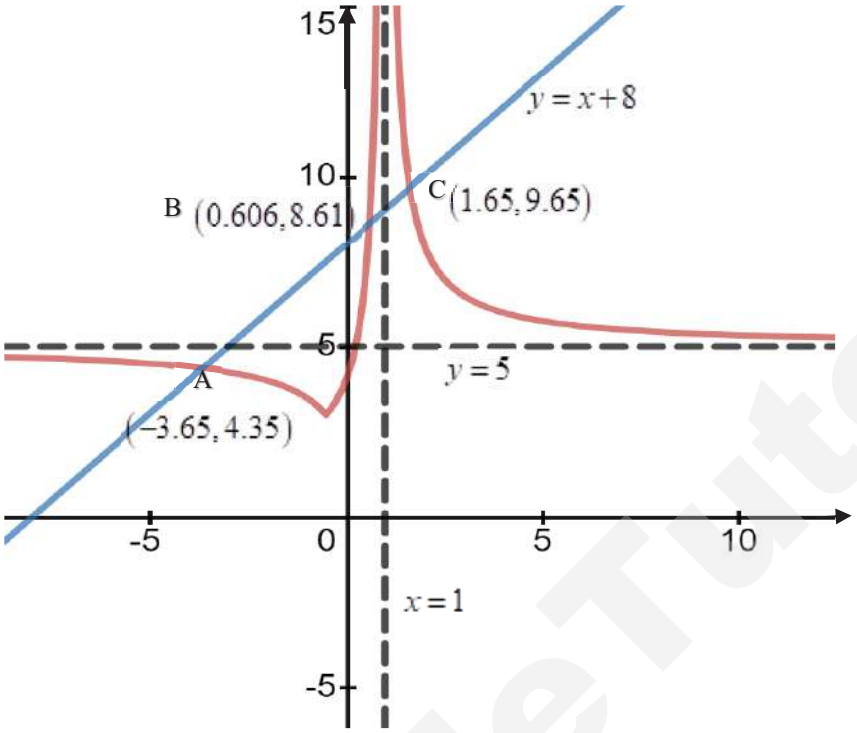
2 Sketch the curve $y = 3 + \left|\frac{2x+1}{x-1}\right|$, stating the equations of the asymptotes.

Hence, solve the inequality $\left|\frac{2x+1}{x-1}\right| > x + 5$. [5]

Suggested solution



Note that since $y = \frac{2x+1}{x-1} = 2 + \frac{3}{x-1}$, the horizontal asymptote is given by $y = 5$.

$\left|\frac{2x+1}{x-1}\right| > x + 5 \Rightarrow 3 + \left|\frac{2x+1}{x-1}\right| > x + 8$

Sketch the graph of $y = x + 8$ on the same axes.

By GC, the intersection points are given by:

$A\ (-3.65, 4.35)$

$B\ (0.606, 8.61)$

$C\ (1.65, 9.65)$

Therefore, the solution is:

$x < -3.65$ or $0.606 < x < 1$ or $1 < x < 1.65$

(Note that $x = 1$ is not part of the solution)

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3 The curve C has equation $a^2x^2 - y^2 - 2(ax - 2y + 2) = 0$ where a is a constant such $a \in \mathbb{R} \setminus \{0\}$.

Show that $\frac{dy}{dx} = \frac{a(ax-1)}{y-2}$.

Hence, find the coordinates of the points on C at which the tangent is parallel to y-axis. [6]

Suggested solution

$a^2x^2 - y^2 - 2(ax - 2y + 2) = 0$

$2a^2x - 2y\frac{dy}{dx} - 2(a - 2\frac{dy}{dx}) = 0$

$2a^2x - 2y\frac{dy}{dx} - 2a + 4\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2 - y) = a - a^2x$

$\frac{dy}{dx} = \frac{a - a^2x}{2 - y} = \frac{a(ax-1)}{y-2}$

The point on C at which the tangent is parallel to y-axis

$\Rightarrow \frac{dx}{dy} = 0$

$\Rightarrow 2 - y = 0$

$\Rightarrow y = 2$

$a^2x^2 - y^2 - 2(ax - 2y + 2) = 0$

$a^2x^2 - 2^2 - 2(ax - 2(2) + 2) = 0$

$a^2x^2 - 2ax = 0$

$ax[ax - 2] = 0$

$x = \frac{2}{a}$ or $x = 0$

$\therefore (0, 2)$ and $\left(\frac{2}{a}, 2\right)$ are the **two points which its** tangent is parallel to to the y-axis.

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4 Referred to the origin O, the points A, B, and C have position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively.

(i) Given that non-zero numbers λ and μ are such that $\lambda\mathbf{a} + \mu\mathbf{b} + \mathbf{c} = \mathbf{0}$ and $\lambda + \mu + 1 = 0$ with $\mu > 0$. Show that A, B, and C are collinear and find the ratio $CA : CB$ in terms of μ. [4]

(ii) F is another point such that the line passing through A, B, and C does not contain it. Find $\dfrac{\left|\overrightarrow{BF} \times \overrightarrow{BC}\right|}{\left|\overrightarrow{AF} \times \overrightarrow{AC}\right|}$ in terms of μ. [2]

Suggested solution

(i)

Method 1

We first show A, B, and C are collinear by $\overrightarrow{CA} = k\,\overrightarrow{CB}$, where k is a constant to be found in terms of μ, and note that we can then also use the value of constant k to determine the ratio CA: CB later.

Using $\lambda\mathbf{a} + \mu\mathbf{b} + \mathbf{c} = \mathbf{0}$, **and** $\lambda + \mu + 1 = 0$

$$\begin{aligned}\overrightarrow{CA} &= \mathbf{a} - \mathbf{c} \\ &= \mathbf{a} + \lambda\mathbf{a} + \mu\mathbf{b} \\ &= \mathbf{a} + (-\mu - 1)\mathbf{a} + \mu\mathbf{b} \\ &= \mu(\mathbf{b} - \mathbf{a})\end{aligned} \qquad \begin{aligned}\overrightarrow{CB} &= \mathbf{b} - \mathbf{c} \\ &= \mathbf{b} + \lambda\mathbf{a} + \mu\mathbf{b} \\ &= \mathbf{b} + (-\mu - 1)\mathbf{a} + \mu\mathbf{b} \\ &= (\mu + 1)(\mathbf{b} - \mathbf{a})\end{aligned}$$

Thus, $\overrightarrow{CA} = \dfrac{\mu}{\mu + 1}\overrightarrow{CB}$,

since $\overrightarrow{CA}$ is a scalar multiple of $\overrightarrow{CB}$, and C is a common point, so A, B, and C are collinear.

$$\left|\overrightarrow{CA}\right| = \frac{\mu}{\mu + 1}\left|\overrightarrow{CB}\right|$$

$$\frac{CA}{CB} = \frac{\mu}{\mu + 1}$$

Thus $CA : CB = \mu : 1 + \mu$

Method 2

$$(-\mu - 1)\,\mathbf{a} + \mu\mathbf{b} + \mathbf{c} = 0 \qquad \lambda = -\mu - 1$$

$$\mu\,(\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{a}) = \mathbf{0}$$

$$\mu\overrightarrow{AB} + \overrightarrow{AC} = 0$$

$$\overrightarrow{AB} = -\frac{1}{\mu}\overrightarrow{AC}$$

$\Rightarrow$ Since $\overrightarrow{AB}$ is a scalar multiple of $\overrightarrow{AC}$, and A is a common point, A, B, and C are collinear.

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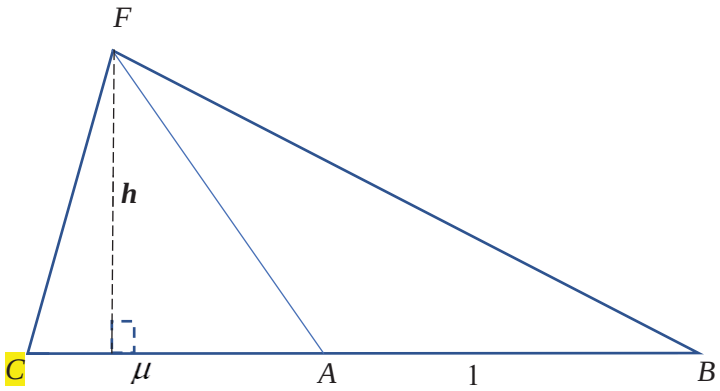
$$(-\mu - 1)\mathbf{a} + \mu\mathbf{b} + \mathbf{c} = 0$$

$$\mathbf{a} = \frac{-\mu\mathbf{b} - \mathbf{c}}{-\mu - 1}$$

$$\mathbf{a} = \frac{\mu\mathbf{b} + \mathbf{c}}{\mu + 1}$$

Thus, A divides BC in the ratio $1:\mu \Rightarrow CA : CB = \mu : 1 + \mu$

(ii)



$$\frac{\left|\overrightarrow{BF} \times \overrightarrow{BC}\right|}{\left|\overrightarrow{AF} \times \overrightarrow{AC}\right|} = \frac{\frac{1}{2} \times \left|\overrightarrow{BF} \times \overrightarrow{BC}\right|}{\frac{1}{2} \times \left|\overrightarrow{AF} \times \overrightarrow{AC}\right|}$$

$$= \frac{\text{area } \Delta BFC}{\text{area } \Delta AFC}$$

$$= \frac{1+\mu}{\mu} \ (\because \text{ the two triangles have the same height.})$$

Method II:

Consider the fixed height $\boldsymbol{h}$:

$$\boldsymbol{h} = \frac{\left|\overrightarrow{BF} \times \overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|} = \frac{\left|\overrightarrow{AF} \times \overrightarrow{AC}\right|}{\left|\overrightarrow{AC}\right|}$$

$$\Rightarrow \frac{\left|\overrightarrow{BF} \times \overrightarrow{BC}\right|}{\left|\overrightarrow{AF} \times \overrightarrow{AC}\right|} = \frac{\left|\overrightarrow{BC}\right|}{\left|\overrightarrow{AC}\right|} = \frac{1+\mu}{\mu} \text{ (from (i))}$$

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5 **(a)** Find $\sum_{r=n}^{2n}\left(3^r+n\right)$. [3]

(b) Express $\ln\left(\frac{r^2}{r^2-1}\right)$ as $A\ln(r-1)+B\ln r+C\ln(r+1)$, where A, B and C are integers to be determined.

Hence, find $\sum_{r=2}^{n}\ln\left(\frac{r^2}{r^2-1}\right)$, leaving your answer as a single logarithmic function in terms of n. [4]

Suggested solution

$$\sum_{r=n}^{2n}\left(3^r+n\right)=\sum_{r=n}^{2n}3^r+\sum_{r=n}^{2n}n$$
$$=\frac{3^n\left(3^{n+1}-1\right)}{2}+n(n+1)$$
$$=\frac{3^n\left(3^{n+1}-1\right)+2n(n+1)}{2}$$

$$\ln\left(\frac{r^2}{r^2-1}\right)=\ln\left(\frac{r^2}{(r-1)(r+1)}\right)$$
$$=\ln r^2-\ln(r-1)-\ln(r+1)$$
$$=-\ln(r-1)+2\ln r-\ln(r+1)$$

$$\sum_{r=2}^{n}\ln\left(\frac{r^2}{r^2-1}\right)=\sum_{r=2}^{n}\left[-\ln(r-1)+2\ln r-\ln(r+1)\right]$$
$$=[-\ln 1+2\ln 2-\ln 3$$
$$-\ln 2+2\ln 3-\ln 4$$
$$-\ln 3+2\ln 4-\ln 5$$
$$-\ln 4+2\ln 5-\ln 6$$
$$+\ldots$$
$$-\ln(n-3)+2\ln(n-2)-\ln(n-1)$$
$$-\ln(n-2)+2\ln(n-1)-\ln(n)$$
$$-\ln(n-1)+2\ln(n)-\ln(n+1)]$$
$$=-\ln 1+\ln 2+\ln(n)-\ln(n+1)$$
$$=\ln\left(\frac{2n}{n+1}\right)$$

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6 The sum, S_n, of the first n terms of a sequence $\{u_n\}$ is given by $S_n = n^2 - (k-2)n$, where k is a non-zero real constant.

(i) Prove that the sequence $\{u_n\}$ is an arithmetic sequence. [3]

(ii) Given that u_8, u_4 and u_2 are the first 3 terms in a geometric sequence, find the value of k. [2]

(iii) Give a reason why the geometric series converges and find the value of the sum to infinity. [2]

Suggested solution

(i) $S_n = n^2 - (k-2)n$

$$\begin{aligned} u_n &= S_n - S_{n-1} \\ &= n^2 - (k-2)n - \left[(n-1)^2 - (k-2)(n-1)\right] \\ &= n^2 - kn + 2n - \left[n^2 - 2n + 1 - kn + 2n + k - 2\right] \\ &= 2n + 1 - k \end{aligned}$$

$$\begin{aligned} & u_n - u_{n-1} \\ &= 2n + 1 - k - \left[2(n-1) + 1 - k\right] \\ &= 2 \quad \text{is a constant independent of } n \end{aligned}$$

$\therefore$ the sequence $\{u_n\}$ is an arithmetic sequence

(ii) $u_8 = 2(8) + 1 - k = 17 - k$

$u_4 = 2(4) + 1 - k = 9 - k$

$u_2 = 2(2) + 1 - k = 5 - k$

$$\frac{9-k}{17-k} = \frac{5-k}{9-k}$$

$$(9-k)^2 = (17-k)(5-k)$$

$$81 - 18k + k^2 = 85 - 22k + k^2$$

$$4k - 4 = 0$$

$$k = 1$$

Alternative Solution:

Given that u_n is a AP, so $u_n = a + 2(n-1)$, where a is the first term in the AP.

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$u_8 = a + 2(7) = a + 14$

$u_4 = a + 2(3) = a + 6$

$u_2 = a + 2(1) = a + 2$

Since $u_8,\ u_4,\ u_2$ are first three terms in a GP,

$$\frac{a+6}{a+14} = \frac{a+2}{a+6}$$

$$(a+6)^2 = (a+2)(a+14)$$

$$a^2 + 12a + 36 = a^2 + 16a + 28$$

$$4a = 8$$

$$a = 2$$

$S_1 = a$

$1^2 - (k-2)(1) = 2$

$k = 1$

(iii) $r = \dfrac{9-k}{17-k} = \dfrac{8}{16} = \dfrac{1}{2}$

Since $|r| = \dfrac{1}{2} < 1$, the geometric series is convergent and

$$\therefore S_\infty = \frac{u_8}{1-r} = \frac{16}{1-\frac{1}{2}} = 32$$

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7 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{bx}{ax-b}, \quad \text{for } x \in \mathbb{R},\ x \neq \frac{b}{a},$$

where a and b are non-zero constants.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Hence show that $\mathrm{f}^2(x) = x$, and write down $\mathrm{f}^n(x)$ where n is an odd number. [2]

The function g is defined by $\mathrm{g}: x \mapsto 2 + \mathrm{e}^{-x}, \quad \text{for } x \in \mathbb{R},\ x \geq 0$. If $a = 2$ and $b = 1$ for function f,

(iii) explain why the composite function gf does not exist. [1]

(iv) find an expression for $\mathrm{fg}(x)$ and state the domain and exact range of fg. [3]

Suggested solution

(i) $\mathrm{f}: x \mapsto \frac{bx}{ax-b}, \quad \text{for } x \in \mathbb{R},\ x \neq \frac{b}{a},$

Let $y = \frac{bx}{ax-b} \Rightarrow y(ax-b) = bx$

$\Rightarrow axy - bx = by$

$\Rightarrow x = \frac{by}{ay-b}$

$\mathrm{f}^{-1}: x \mapsto \frac{bx}{ax-b}, \quad \text{for } x \in \mathbb{R},\ x \neq \frac{b}{a},$

(ii) Since $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, $\mathrm{f}^2(x) = \mathrm{f}\left[\mathrm{f}^{-1}(x)\right] = x$

$\mathrm{f}^3(x) = \mathrm{f}\left[\mathrm{f}^2(x)\right] = \mathrm{f}(x)$, therefore for $\mathrm{f}^n(x) = \mathrm{f}(x) = \frac{bx}{ax-b}$ for n odd,

(iii) If $a = 2$ and $b = 1$ for function f,

$\mathrm{f}: x \mapsto \frac{x}{2x-1}, \quad \text{for } x \in \mathbb{R},\ x \neq \frac{1}{2},$

$\mathrm{R_f} = \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$

$\mathrm{D}_g = [0, \infty)$

As $\mathrm{R_f} \not\subset \mathrm{D}_g$, gf does not exist.

(iv) $\mathrm{fg}(x) = \mathrm{f}\left(2 + \mathrm{e}^{-x}\right)$

$= \frac{2 + \mathrm{e}^{-x}}{2\left(2 + \mathrm{e}^{-x}\right) - 1}$

$= \frac{2 + \mathrm{e}^{-x}}{3 + 2\mathrm{e}^{-x}}, \quad x \geq 0$

$\mathrm{D_{fg}} = \mathrm{D_g} = [0, \infty)$

Range of $\mathrm{R_{fg}} = \left[\frac{3}{5}, \frac{2}{3}\right)$

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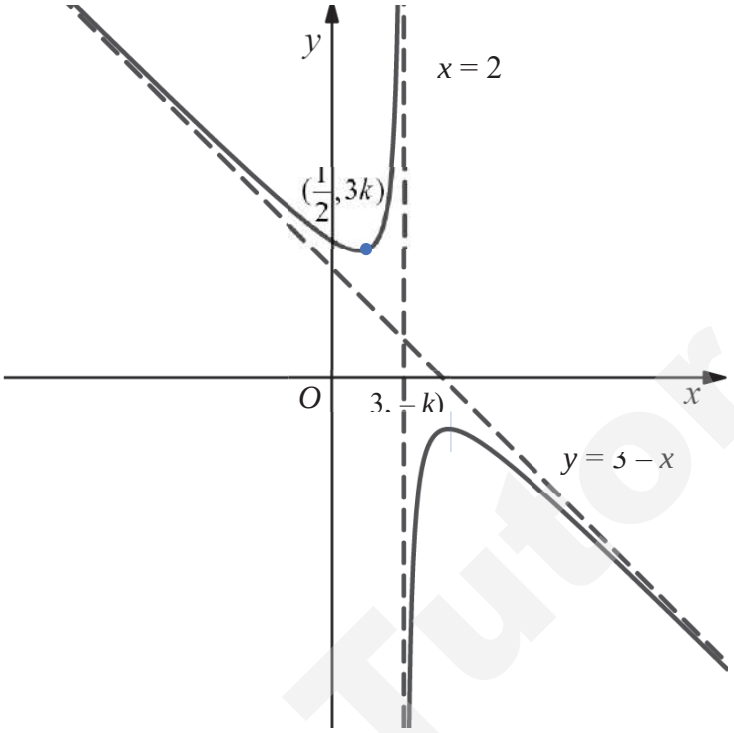
8 The diagram shows the curve $y = f(x)$. The curve has a minimum point $(\frac{1}{2}, 3k)$ and a maximum point $(3, -k)$ where $k > 0$. It has asymptotes $x = 2$ and $y = 3 - x$.



Sketch, on separate diagrams, the graph of

(i) $y = f(x+2) + k$, [3]

showing clearly the stationary points, axial intercepts and equations of all asymptotes.

(ii) $y = \dfrac{1}{f(x)}$, [3]

showing clearly the stationary points, axial intercepts and equations of all asymptotes.

(iii) $y = f'(x)$, [3]

showing clearly the stationary points, axial intercepts and equations of all asymptotes.

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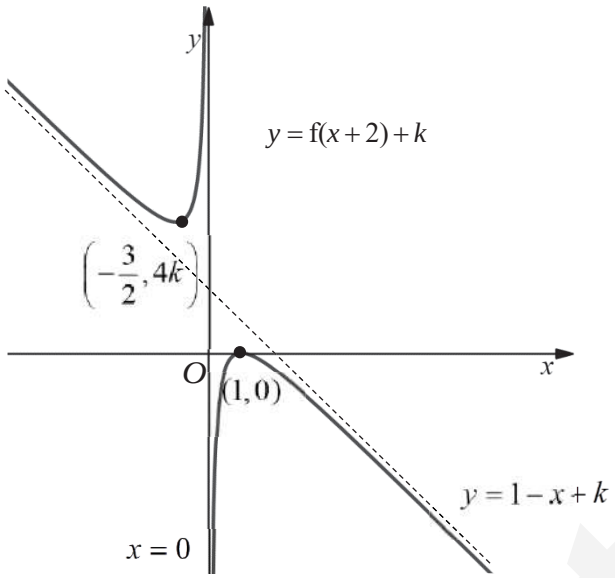
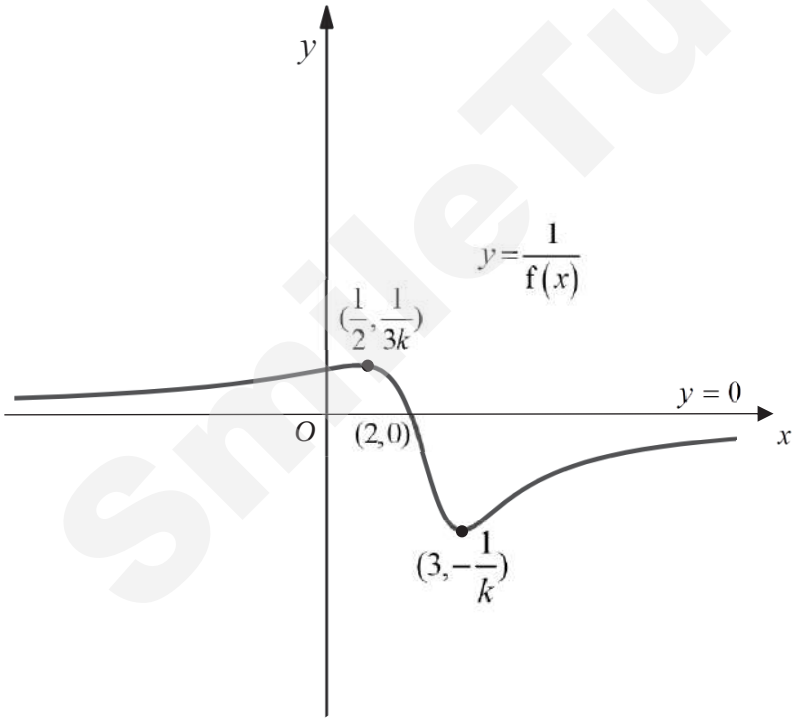
Suggested solution

(i)



(ii)



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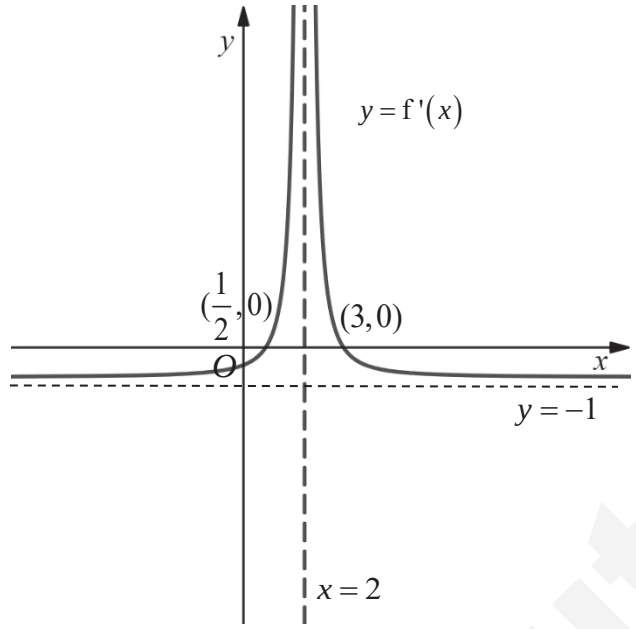
(iii)



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9 **(a)** Find $\int \frac{1-4x}{4x^2+1}\,\mathrm{d}x$. [4]

(b) Use the substitution $u = \sqrt{x}$ to show that $\int_0^{\frac{\pi^2}{4}} \sqrt{x}\sin\sqrt{x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} u^2 \sin u\,\mathrm{d}u$. [2]

Hence, evaluate the integral $\int_0^{\frac{\pi^2}{4}} \sqrt{x}\sin\sqrt{x}\,\mathrm{d}x$ exactly. [4]

Suggested solution

(a) $$\int \frac{1-4x}{4x^2+1}\,\mathrm{d}x = \int \left(\frac{1}{4x^2+1} - \frac{4x}{4x^2+1}\right)\mathrm{d}x$$
$$= \frac{1}{2}\int \frac{2}{(2x)^2+1}\,\mathrm{d}x - \frac{1}{2}\int \frac{8x}{4x^2+1}\,\mathrm{d}x$$
$$= \frac{1}{2}\tan^{-1}2x - \frac{1}{2}\ln\left(4x^2+1\right) + C,$$
where C is an arbitrary constant.

Alternative:

$$\int \frac{1}{4x^2+1}\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{x^2+\left(\frac{1}{2}\right)^2}\,\mathrm{d}x = \frac{1}{4}\left(\frac{1}{\frac{1}{2}}\right)\tan^{-1}\frac{x}{\frac{1}{2}}$$
$$= \frac{1}{2}\tan^{-1}2x + C$$

(b) Given substitution: Let $u = \sqrt{x} \Rightarrow u^2 = x$

When $x = 0, u = 0$ $x = \frac{\pi^2}{4}, u = \frac{\pi}{2}$

Differentiating with respect to x, we have $2u\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\therefore \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2u} \quad \left(\text{or } \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}\right)$$

Hence, $\int_0^{\frac{\pi^2}{4}} \sqrt{x}\sin\sqrt{x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} u^2\sin u\,\mathrm{d}u$

$2\int_0^{\frac{\pi}{2}} u^2\sin u\,\mathrm{d}u = 2\left[u^2(-\cos u)\right]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} 2u(-\cos u)\,\mathrm{d}u$ (By parts) =

$2\left[-u^2\cos u\right]_0^{\frac{\pi}{2}} + 4\int_0^{\frac{\pi}{2}} u\cos u\,\mathrm{d}u$ **(simplify)**

$= 2\left[-u^2\cos u\right]_0^{\frac{\pi}{2}} + 4\left[\left[u\sin u\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\sin u\,\mathrm{d}u\right]$ =

$2\left[-u^2\cos u + 2u\sin u + 2\cos u\right]_0^{\frac{\pi}{2}}$ =

$2\left(\left(0+2\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2}+0\right)-(0+0+2)\right) = 2(\pi-2)$

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10 The curve C is defined parametrically by the equations $x = 2\text{cosec}\theta$, $y = 2\cot\theta$, where $0 < \theta < \frac{\pi}{2}$.

(i) Show that $\frac{dy}{dx} = \sec\theta$. [2]

(ii) Find the equation of the tangent to the curve at the point $P(2\text{cosec } p, 2\cot p)$ and show that the equation of the normal to the curve at P is $y = 4\cot p - x\cos p$. [3]

(iii) The tangent and normal at P cut the x-axis at T and N respectively.

Show that T has coordinates $(2\sin p, 0)$.

By finding the x-coordinates of N in the simplest form, evaluate $|ON||OT|$. [4]

(iv) Find the rate of change of y at $(2\sqrt{2}, 2)$, given that x is increasing at a constant rate of 1 unit per second. [3]

Suggested solution

(i) $x = 2\ \text{cosec}\theta$ $\quad y = 2\cot\theta$

$\frac{dx}{d\theta} = -2\ \text{cosec}\theta\cot\theta$ $\quad \frac{dy}{d\theta} = -2\ \text{cosec}^2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}$$
$$= \frac{-2\ \text{cosec}^2\theta}{-2\ \text{cosec}\theta\cot\theta}$$
$$= \text{cosec}\theta\tan\theta$$
$$= \frac{1}{\sin\theta}\frac{\sin\theta}{\cos\theta}$$
$$= \sec\theta$$

(ii)Equation of tangent at point $P\ (2\cos\text{ec } p, 2\cot p)$ with $\theta = p$:

$y - 2\cot p = \sec p(x - 2\text{cosec } p)$

$y = 2(\cot p - \text{cosec } p\sec p) + x\sec p$

Equation of normal at point $P\ (2\cos\text{ec } p, 2\cot p)$ with $\theta = p$:

$y - 2\cot p = -\frac{1}{\sec p}(x - 2\text{cosec } p)$

$y = 2\cot p - (\cos p)\left(x - \frac{2}{\sin p}\right)$

$y = 4\cot p - x\cos p$

(iii)At point $T, y = 0$

$0 = 2(\cot p - \text{cosec } p\sec p) + x\sec p$

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$$x = \frac{2(\cot p - \operatorname{cosec} p \sec p)}{-\sec p}$$
$$= -2\,(\cos^2 p \operatorname{cosec} p - \operatorname{cosec} p)$$
$$= 2\operatorname{cosec} p(1 - \cos^2 p)$$
$$= 2\operatorname{cosec} p(\sin^2 p)$$
$$= 2\sin p$$

At point N, $y = 0$

$$0 = 4\cot p - x\cos p$$

$$x = \frac{4\cot p}{\cos p}$$
$$= 4\operatorname{cosec} p$$

$$|ON||OT| = |2\sin t||4\ \operatorname{cosec} t|$$
$$= 8$$

(iv) $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$
$$= (\sec\theta)(1)$$

At $(x, y) = (2\sqrt{2}, 2)$, $\theta = \frac{\pi}{4}$,

$$\frac{dy}{dt} = \sec\frac{\pi}{4} = \sqrt{2}.$$

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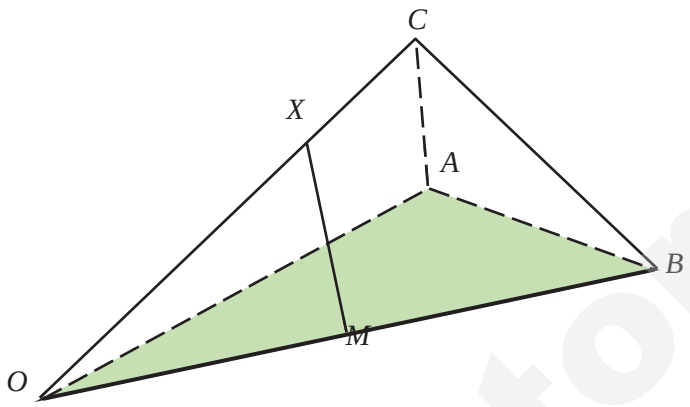
11 Engineers are building a greenhouse on an inclined grass slope. Points (x, y, z) are defined relative to a base point $O(0,0,0)$, where units are in metres.

The greenhouse is in the shape of a tetrahedron. The corners O, A and B lie on the grass slope, while the point C is the highest point of the tetrahedron (also called apex), as shown in the diagram below.



The coordinates of A, B and C are $(100,100,10)$, $(160,20,8)$ and $(80,59,20)$ respectively.

(i) Show that the Cartesian equation of the grass slope is $3x+4y-70z=0$. [2]

(ii) Find the angle of inclination of the grass slope. [2]

(iii) To secure the greenhouse, the architect plans to build a central pillar from the apex directly to the floor of the greenhouse such that the central pillar is perpendicular to the horizontal plane. Find the point where the central pillar meets the grass slope. [2]

Unfortunately, the workers misunderstand the instructions and build the central pillar from the apex to the floor of the greenhouse such that the pillar is perpendicular to the grass slope.

(iv) Find the length of this central pillar and use this length to find the volume enclosed by the greenhouse $OABC$. [4]

[The volume of a tetrahedron is given by $\frac{1}{3}\times(\text{base area})\times(\text{height})$.]

Another straight supporting beam is to be constructed from point M, the midpoint of OB, to a point X on OC such that XM is perpendicular to OB.

(v) Find the coordinates of X, correct to 1 decimal place. [3]

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Suggested solution

(i) $\overrightarrow{OA}\times\overrightarrow{OB}=\begin{pmatrix}100\\100\\10\end{pmatrix}\times\begin{pmatrix}160\\20\\8\end{pmatrix}=\begin{pmatrix}600\\800\\-14000\end{pmatrix}=200\begin{pmatrix}3\\4\\-70\end{pmatrix}$

Since the origin lies on the plane *OAB*.

Plane *OAB*: $\mathbf{r}\cdot\begin{pmatrix}3\\4\\-70\end{pmatrix}=0$

$3x+4y-70z=0$ (shown)

(ii) Angle of inclination

$$=\cos^{-1}\frac{\left|\begin{pmatrix}3\\4\\-70\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}3\\4\\-70\end{pmatrix}\right|\left|\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}=\cos^{-1}\frac{70}{\sqrt{4925}}=4.1°\text{ (to 1 d.p.)}$$

(iii) **Method I**: Let F be a point directly below C. Hence coordinate of F is $(80,59,f)$.

Since *F* is on plane *OAB*,

$$\begin{pmatrix}80\\59\\f\end{pmatrix}\cdot\begin{pmatrix}3\\4\\-70\end{pmatrix}=0\Rightarrow 476-70f=0\Rightarrow f=\frac{34}{5}=6.8$$

Hence the coordinates of the point *F* is $(80,59,\frac{34}{5})$

Method II: Let the line from apex parallel to **k** be L_c.

$$L_c:\ \mathbf{r}=\begin{pmatrix}80\\59\\20\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}\quad\lambda\in\mathbb{R}$$

At point of intersection with plane *OAB*,

$$\begin{pmatrix}80\\59\\20+\lambda\end{pmatrix}\cdot\begin{pmatrix}3\\4\\-70\end{pmatrix}=0\Rightarrow -924-70\lambda=0\Rightarrow\lambda=-\frac{66}{5}$$

Hence the coordinates of the point *F* is $(80,59,\frac{34}{5})$

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(iv) Perpendicular distance from C to the plane OAB

$$= \frac{\left|\overrightarrow{OC}\cdot\begin{pmatrix}3\\4\\-70\end{pmatrix}\right|}{\left|\begin{pmatrix}3\\4\\-70\end{pmatrix}\right|} = \frac{\left|\begin{pmatrix}80\\59\\20\end{pmatrix}\cdot\begin{pmatrix}3\\4\\-70\end{pmatrix}\right|}{\left|\begin{pmatrix}3\\4\\-70\end{pmatrix}\right|}$$

$$= \frac{924}{\sqrt{4925}} = \frac{924}{5\sqrt{197}} = 13.16645 \approx 13.2 \text{ m (to 3.s.f.)}$$

Volume enclosed by the greenhouse

$$= \frac{1}{3}\left(\text{Area of } \Delta OAB\right)\left(\text{Perpendicular distance from } C \text{ to plane } OAB\right)$$

$$= \frac{1}{3}\left(\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|\right)\left(\frac{924}{\sqrt{4925}}\right)$$

$$= \frac{1}{6}\left|200\begin{pmatrix}3\\4\\-70\end{pmatrix}\right|\left(\frac{924}{\sqrt{4925}}\right)$$

$$= 30800 \text{ m}^3$$

(v) Since X lies on OC, $\overrightarrow{OX} = t\begin{pmatrix}80\\59\\20\end{pmatrix}$, for some $t \in \mathbb{R}$.

Since XM is perpendicular to OB, $\overrightarrow{XM}\cdot\overrightarrow{OB} = 0$

$$\left(\overrightarrow{OM} - \overrightarrow{OX}\right)\cdot\overrightarrow{OB} = 0$$

$$\left(\begin{pmatrix}80\\10\\4\end{pmatrix} - t\begin{pmatrix}80\\59\\20\end{pmatrix}\right)\cdot\begin{pmatrix}160\\20\\8\end{pmatrix} = 0$$

$$13032 - 14140t = 0$$

$$t = \frac{3258}{3535}$$

Therefore, $\overrightarrow{OX} = \frac{3258}{3535}\begin{pmatrix}80\\59\\20\end{pmatrix}$.

Coordinates of X are $(73.7, 54.4, 18.4)$.

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12 In a model making competition, each contestant is given a square cardboard of side n cm to make a model. John takes part in the Pyramid Category and he is given a $n \times n$ cardboard to make a pyramid with the maximum volume. John recalled his calculus during his schooling days and came out with a design as shown in the diagrams below.

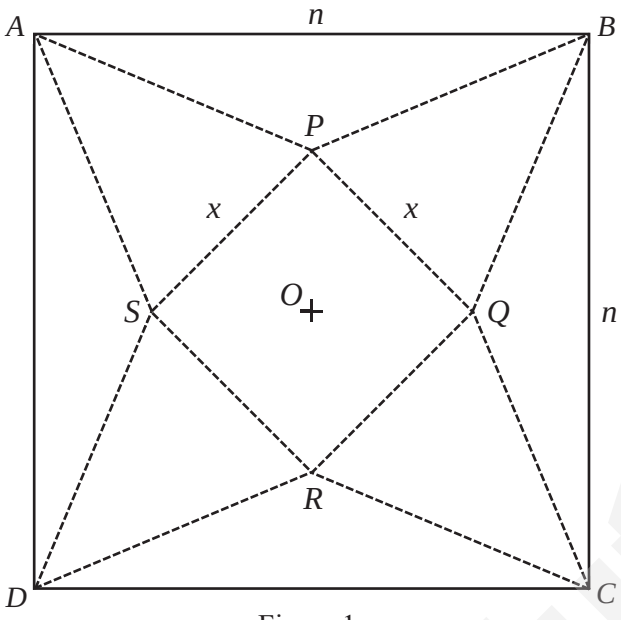

Figure 1

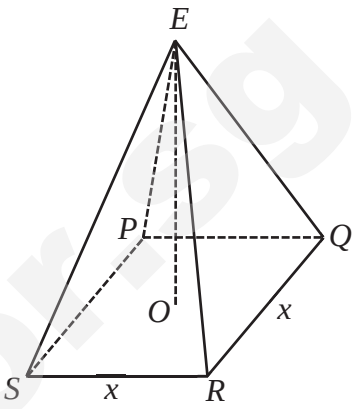

Figure 2

Figure 1 shows a square piece of cardboard $ABCD$ of side n cm, where n is a positive constant. Four triangles (ABP, BCQ, CDR, DAS) are removed from each side of $ABCD$. The remaining shape is now folded along PQ, QR, RS and SP to form a pyramid (as shown in Figure 2) with a square base $PQRS$ of side x cm. The point O is the centre of both $ABCD$ and $PQRS$. The vertex E is the point where A, B, C and D meet. Let V be the volume of the pyramid.

(i) Show that $OE^2 = \frac{1}{2}\left(n^2 - \sqrt{2}nx\right)$. [3]

(ii) Show that $V^2 = \frac{x^4}{18}\left(n^2 - \sqrt{2}nx\right)$. Hence, by using differentiation, find in terms of n, the exact value of x which John uses to obtain a maximum value of V. [7]

(iii) John's pyramid has a volume that is greater than 45 cm^3. Find the smallest $n \times n$ cardboard given, where $n \in \mathbb{Z}^+$. [2]

[The volume of a pyramid is given by $\frac{1}{3} \times (\text{base area}) \times (\text{height})$.]

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Suggested solution

(i) Method I

$BD = \sqrt{n^2 + n^2} = \sqrt{2}\,n$

Let M be the midpoint of SR, $\therefore DM = EM = \dfrac{\sqrt{2}n - x}{2}$

$EM^2 = OE^2 + OM^2$

$OE^2 = EM^2 - OM^2$

$$= \left(\frac{\sqrt{2}n - x}{2}\right)^2 - \left(\frac{x}{2}\right)^2$$

$$= \frac{1}{4}\left(2n^2 - 2\sqrt{2}nx + x^2 - x^2\right)$$

$$= \frac{1}{4}\left(2n^2 - 2\sqrt{2}nx\right)$$

$$= \frac{1}{2}\left(n^2 - \sqrt{2}nx\right) \quad \text{(shown)}$$

Method II (Not recommended)

$PR = \sqrt{x^2 + x^2} = \sqrt{2}\,x$

Let M be the midpoint of DC, $\therefore RM = \dfrac{n - \sqrt{2}\,x}{2}$

$$DR^2 = DM^2 + RM^2 = \left(\frac{n}{2}\right)^2 + \left(\frac{n - \sqrt{2}\,x}{2}\right)^2 = ER^2$$

$ER^2 = OE^2 + OR^2$

$OE^2 = ER^2 - OR^2$

$$= \left[\left(\frac{n}{2}\right)^2 + \left(\frac{n - \sqrt{2}\,x}{2}\right)^2\right] - \left(\frac{\sqrt{2}\,x}{2}\right)^2$$

$$= \frac{n^2}{4} + \frac{n^2 - 2\sqrt{2}\,nx + 2x^2}{4} - \frac{2x^2}{4}$$

$$= \frac{1}{4}\left(2n^2 - 2\sqrt{2}\,nx\right)$$

$$= \frac{1}{2}\left(n^2 - \sqrt{2}\,nx\right) \quad \text{(shown)}$$

(ii)

$V = \dfrac{1}{3}(\text{base area})(\text{height, } OE)$

$$= \frac{1}{3}x^2\sqrt{\frac{1}{2}\left(n^2 - \sqrt{2}nx\right)}$$

$$\Rightarrow V^2 = \frac{x^4}{18}\left(n^2 - \sqrt{2}nx\right) \quad \text{(shown)}$$

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Method I (Implicit differentiation and 2nd derivative test)

$$V^2 = \frac{x^4}{18}\left(n^2 - \sqrt{2}nx\right) = \frac{n^2}{18}x^4 - \frac{\sqrt{2}n}{18}x^5$$

Diff wrt x,

$$2V\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{n^2}{18}\left(4x^3\right) - \frac{\sqrt{2}n}{18}\left(5x^4\right)$$

$$= \frac{2n^2}{9}x^3 - \frac{5\sqrt{2}n}{18}x^4$$

When $\frac{\mathrm{d}V}{\mathrm{d}x} = 0 \Rightarrow \frac{2n^2}{9}x^3 - \frac{5\sqrt{2}n}{18}x^4 = 0$

$$\Rightarrow \frac{nx^3}{18}\left(4n - 5\sqrt{2}x\right) = 0$$

$$\Rightarrow x = \frac{4n}{5\sqrt{2}} \quad \text{or} \quad x = 0 \quad \left(\text{rej. } \because x > 0\right)$$

$$= \frac{2\sqrt{2}}{5}n$$

Consider: $2V\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{2n^2}{9}x^3 - \frac{5\sqrt{2}n}{18}x^4$

Diff wrt x,

$$2\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right) + 2V\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2n^2}{9}\left(3x^2\right) - \frac{5\sqrt{2}n}{18}\left(4x^3\right)$$

$$2\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 + 2V\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2n^2}{3}x^2 - \frac{10\sqrt{2}n}{9}x^3$$

Substituting $x = \frac{2\sqrt{2}n}{5}, \frac{\mathrm{d}V}{\mathrm{d}x} = 0$,

$$2(0)^2 + 2V\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2n^2}{3}\left(\frac{2\sqrt{2}n}{5}\right)^2 - \frac{10\sqrt{2}n}{9}\left(\frac{2\sqrt{2}n}{5}\right)^3$$

$$2V\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2n^2}{3}\left(\frac{8n^2}{25}\right) - \frac{10\sqrt{2}n}{9}\left(\frac{16\sqrt{2}n^3}{125}\right)$$

$$= \frac{16n^4}{75} - \frac{64n^4}{225}$$

$$= -\frac{16n^4}{225}$$

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -\frac{8n^4}{225V} < 0 \quad \left(\text{since } V, n > 0\right)$$

$\therefore V$ is maximum when $x = \frac{2\sqrt{2}}{5}n$.

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Method II (Making V the subject, differentiating and using 1st derivative test) – Not recommended

$$V = \frac{1}{3}x^2\sqrt{\frac{1}{2}\left(n^2 - \sqrt{2}nx\right)}$$

$$= \frac{1}{3\sqrt{2}}x^2\left(n^2 - \sqrt{2}nx\right)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{1}{3\sqrt{2}}\left\{2x\left(n^2 - \sqrt{2}nx\right)^{\frac{1}{2}} + x^2\left[\frac{1}{2}\left(n^2 - \sqrt{2}nx\right)^{-\frac{1}{2}}\left(-\sqrt{2}n\right)\right]\right\}$$

$$= \frac{1}{3\sqrt{2}}\left\{2x\left(n^2 - \sqrt{2}nx\right)^{\frac{1}{2}} + x^2\left[\frac{-\sqrt{2}n}{2\left(n^2 - \sqrt{2}nx\right)^{\frac{1}{2}}}\right]\right\}$$

$$= \frac{x}{6\sqrt{2}}\left\{\frac{4\left(n^2 - \sqrt{2}nx\right)}{\left(n^2 - \sqrt{2}nx\right)^{\frac{1}{2}}} + x\left[\frac{-\sqrt{2}n}{\left(n^2 - \sqrt{2}nx\right)^{\frac{1}{2}}}\right]\right\}$$

$$= \frac{x}{6\sqrt{2}}\left[\frac{4n^2 - 4\sqrt{2}nx - \sqrt{2}nx}{\left(n^2 - \sqrt{2}nx\right)^{\frac{1}{2}}}\right]$$

$$= \frac{nx}{6\sqrt{2}}\left[\frac{4n - 5\sqrt{2}x}{\left(n^2 - \sqrt{2}nx\right)^{\frac{1}{2}}}\right]$$

When $\frac{\mathrm{d}V}{\mathrm{d}x} = 0 \Rightarrow x\left(4n - 5\sqrt{2}x\right) = 0$

$$\Rightarrow x = \frac{4n}{5\sqrt{2}} \quad \text{or} \quad x = 0 \quad \left(\text{rej. } \because x > 0\right)$$

$$= \frac{2\sqrt{2}}{5}n$$

x	$\left(\frac{2\sqrt{2}}{5}n\right)^-$	$\frac{2\sqrt{2}}{5}n$	$\left(\frac{2\sqrt{2}}{5}n\right)^+$
Explanation	$4n - 5\sqrt{2}x > 0$ $x > 0$	$4n - 5\sqrt{2}x = 0$	$4n - 5\sqrt{2}x < 0$ $x > 0$
$\frac{\mathrm{d}V}{\mathrm{d}x}$	+ve	0	–ve
Slope	/	–	\

$\therefore V$ is maximum when $x = \frac{2\sqrt{2}}{5}n$.

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(iii) Method I (Inequalities without GC table method)

V is maximum when $x = \frac{2\sqrt{2}}{5}n, V > 45 \Rightarrow V^2 > 45^2$

From part **(ii)**, $V^2 = \frac{x^4}{18}\left(n^2 - \sqrt{2}nx\right) = \frac{n^2}{18}x^4 - \frac{\sqrt{2}n}{18}x^5$

$$\frac{n^2}{18}\left(\frac{2\sqrt{2}}{5}n\right)^4 - \frac{\sqrt{2}n}{18}\left(\frac{2\sqrt{2}}{5}n\right)^5 > 45^2$$

$$\frac{32}{5625}n^6 - \frac{128}{28125}n^6 > 2025$$

$$\frac{32}{28125}n^6 > 2025$$

$$n > 11.008 \quad (5\text{s.f.})$$

Least integer n is 12. Smallest cardboard given is $12\text{cm} \times 12\text{cm}$.

Method II (Inequalities using GC table method) - Recommended

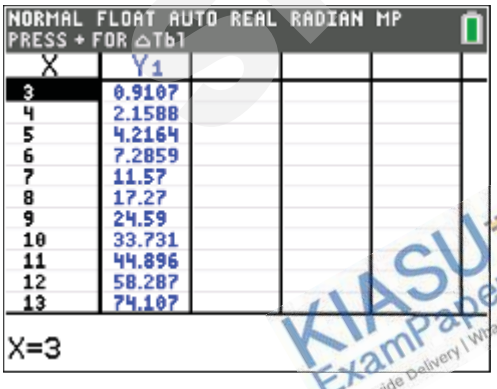
V is maximum when $x = \frac{2\sqrt{2}}{5}n, V > 45$

From part **(ii)**, $V^2 = \frac{x^4}{18}\left(n^2 - \sqrt{2}nx\right) \Rightarrow V = \sqrt{\frac{x^4}{18}\left(n^2 - \sqrt{2}nx\right)}$

$$\sqrt{\frac{n^2}{18}\left(\frac{2\sqrt{2}}{5}n\right)^4 - \frac{\sqrt{2}n}{18}\left(\frac{2\sqrt{2}}{5}n\right)^5} > 45$$

Using GC,
When $n = 11, V = 44.9 < 45$
When $n = 12, V = 58.3 > 45$

Smallest cardboard given is $12\text{cm} \times 12\text{cm}$.



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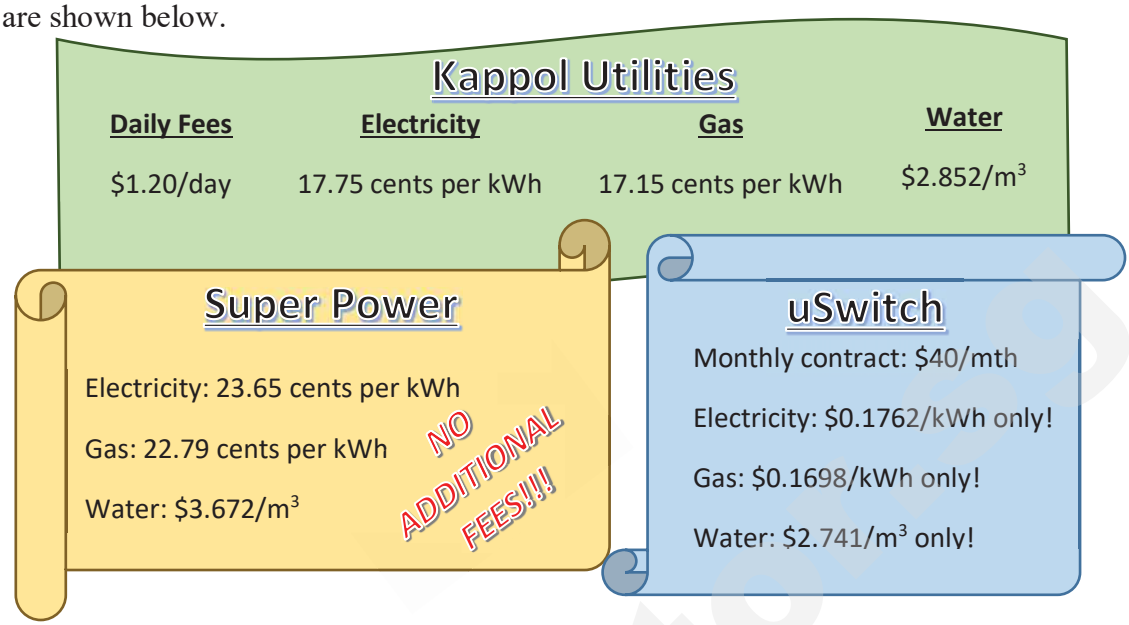
1 The rates from three different utilities retailers Kappol Utilities, Super Power and uSwitch are shown below.



Based on Mr Lim's utilities consumption for the 30 days in the month of June, Mr Lim found that if he subscribed all his electricity, gas and water services from a single utilities retailer, he would have to pay Kappol Utilities, Super Power and uSwitch a bill of \$143.06, \$140.78 and \$144.96 respectively.

Find the amount of electricity and gas in kWh, and water in m^3 that Mr Lim used in the month of June. [3]

2 **(i)** Differentiate $\tan^{-1}\left(x^k\right)$, where k is a positive integer, leaving your answer in terms of k. [1]

(ii) Hence find $\int x^{k-1}\tan^{-1}\left(x^k\right)\mathrm{d}x$ in terms of k. [3]

3 **(i)** Show that the x-coordinate of the point(s) of intersection between a horizontal line $y=k$ where $k\in\mathbb{R}$ and the curve $y=\dfrac{ax+b}{x^2+1}$ where $a,b\in\mathbb{R}$ would satisfy the equation

$$kx^2-ax+(k-b)=0\ .$$ [1]

(ii) Given that $-1\le k\le 4$, find the values of a and b. [4]

4 An arithmetic sequence $a_1, a_2, a_3, \ldots$ has common difference d, where $d\neq 0$ and $a_1\neq 0$.

Another geometric sequence $b_1, b_2, b_3, \ldots$ has common ratio r, where $r>0,\ r\neq 1$. It is given that $a_1=b_1$, $a_3=b_3$ and $a_7=b_5$.

If $a_n=b_m$ where $n,m\in\mathbb{Z}^+$, find n in terms of m. [6]

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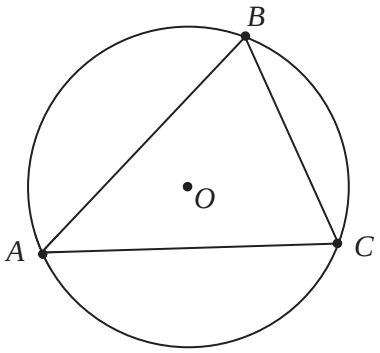

The diagram above shows a circle centred at O with radius 1 unit.

It is given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = -\frac{2}{3}\mathbf{a} - \frac{1}{3}\mathbf{b}$.

(a) Show that the area of triangle ABC is given by $|\mathbf{a} \times \mathbf{b}|$ square units. Given that $\angle ACB = \frac{\pi}{3}$, find the exact area of triangle ABC. [5]

(b) The point M lies on CB produced such that the area of triangle AMC is three times that of the area of triangle ABC. Find $\overrightarrow{OM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. [2]

6 **(i)** Find $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sec^n \theta\right)$. [1]

(ii) Using the substitution $x = 2\tan^2 \theta$, show that $\int_0^2 x\sqrt{1+\frac{x}{2}}\,\mathrm{d}x$ can be expressed as

$$k\int_a^b \left(\sec^5 \theta \tan\theta - \sec^3 \theta \tan\theta\right)\mathrm{d}\theta,$$

where a, b and k are constants to be determined. [4]

(iii) Hence find the exact value of $\int_0^2 x\sqrt{1+\frac{x}{2}}\,\mathrm{d}x$, expressing your answer in the form $A\left(\sqrt{2}+B\right)$, where A and B are constants to be determined. [2]

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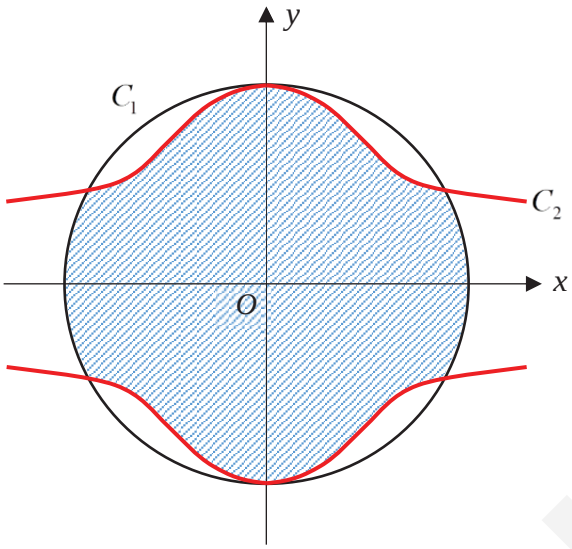
7



The curves C_1 and C_2 have equations $x^2 + y^2 = 4$ and $y^2 = \frac{4}{1+x^2}$ respectively.

(i) Find the exact coordinates of the points of intersection of C_1 and C_2. [2]

(ii) Find the area of the shaded region, giving your answer correct to 3 decimal places. [2]

(iii) Find the exact volume of the solid obtained when the shaded region is rotated through π radians about the x-axis. [3]

8 **(i)** Show that $\frac{1}{(2r+3)\sqrt{2r+1} + (2r+1)\sqrt{2r+3}} = \frac{1}{2}\left(\frac{1}{\sqrt{2r+1}} - \frac{1}{\sqrt{2r+3}}\right)$. [2]

(ii) Given that $S_n = \sum_{r=1}^{n} \frac{1}{(2r+3)\sqrt{2r+1} + (2r+1)\sqrt{2r+3}}$, find S_n in terms of n. [3]

(iii) Find the smallest value of n for which the difference between S_n and S_∞ is less than 0.05. [3]

9 The parametric equations of a curve are given by

$$x = \sin t \cos t, \quad y = \cos\left(t + \frac{\pi}{4}\right), \text{ where } 0 \le t \le \frac{\pi}{2}.$$

(i) Show that $\frac{dy}{dx} = \frac{1}{\sqrt{2}(\sin t - \cos t)}$. [2]

(ii) Hence find the equation of the tangent parallel to the y-axis. [2]

The curve cuts the y-axis at the points P and Q.

(iii) Find the exact coordinates of P and Q. [2]

(iv) The point R on the curve has coordinates $\left(\sin\theta\cos\theta, \cos\left(\theta + \frac{\pi}{4}\right)\right)$.

Show that the area of triangle PQR is given by $\frac{\sqrt{2}}{4}\sin 2\theta$. Hence find the value of θ for which the area of triangle PQR is a maximum. (You need not prove that it is a maximum.) [2]

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10 **(i)** Describe a sequence of transformations which transforms the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{3x+5}{x+2}$. [3]

(ii) Sketch the graphs of $y = \frac{3x+5}{x+2}$ and $16(x+3)^2 + 9(y-4)^2 = 144$ on a single diagram, indicating clearly any axial intercepts, points of intersection of the two graphs and the equations of asymptotes. [5]

(iii) Hence find the set of values of x that satisfies the inequality

$$\frac{3x+5}{x+2} > 4 - \sqrt{\frac{144-16(x+3)^2}{9}}.$$ [2]

11 The function f is defined by

$$\mathrm{f} : x \mapsto 2\left|4-(x-\mathrm{e})^2\right|, \text{ where } x \in \mathbb{R}.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$, showing clearly the exact coordinates of any axial intercepts and turning points. [2]

(ii) Show that f^{-1} does not exist. [1]

(iii) The domain of f is restricted to $a \le x < \mathrm{e}$, where $a \in \mathbb{R}$.

State the smallest exact value of a for which f^{-1} exists. Hence find f^{-1} in similar form. [5]

The function g is defined by

$$\mathrm{g} : x \mapsto \frac{x}{x+1}, \text{ where } x \in \mathbb{R}, x \neq -1.$$

(iv) Using the domain in part **(iii)**, show that gf^{-1} exists and find the exact range of gf^{-1}. [3]

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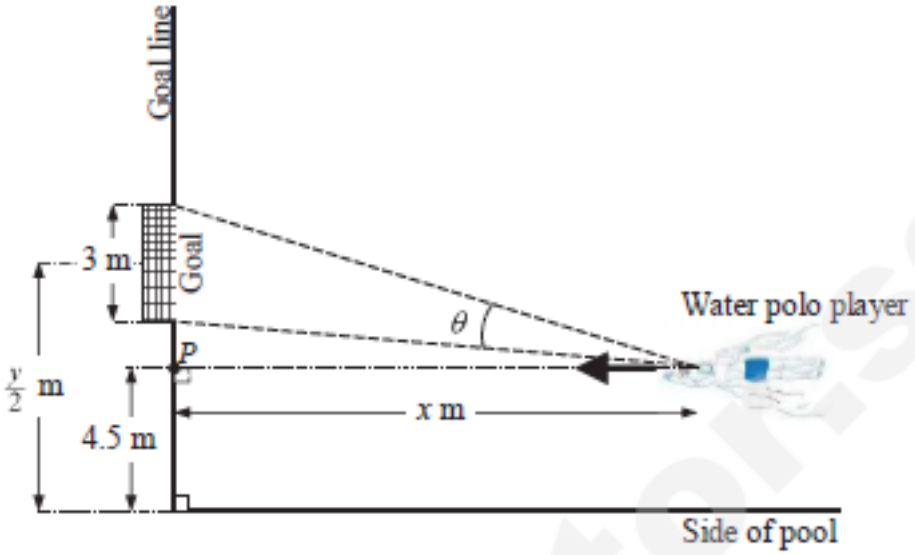
12



The diagram (not drawn to scale) shows part of a rectangular water polo pool with a fixed width of y m. A goal 3 m wide is placed on the goal line at one end of the pool with the centre of the goal $\frac{y}{2}$ m from the side of the pool. A water polo player at a distance of x m perpendicular to the goal line and a distance of 4.5 m away from the side of the pool swims directly towards a point P on the goal line. A visual angle θ of the goal is the angle subtended at the eye of the water polo player by the goal.

(i) Show that $\tan\theta = \dfrac{12x}{4x^2 + A}$ where $A = (y-6)(y-12)$. [3]

(ii) The desired visual angle of the goal is obtained when θ is a maximum. Find by differentiation, the value of x such that the desired visual angle of the goal is obtained. Leave your answer in exact form in terms of A. [4]

For the rest of the question, let $y = 20$.

(iii) Show that the water polo player needs to be $2\sqrt{7}$ m away from the goal line in order to obtain the desired visual angle. [2]

(iv) The water polo player swims at a constant speed of 50 m per minute. Find the rate of change of θ at the instant when the water polo player is 15 m away from the goal line. [3]

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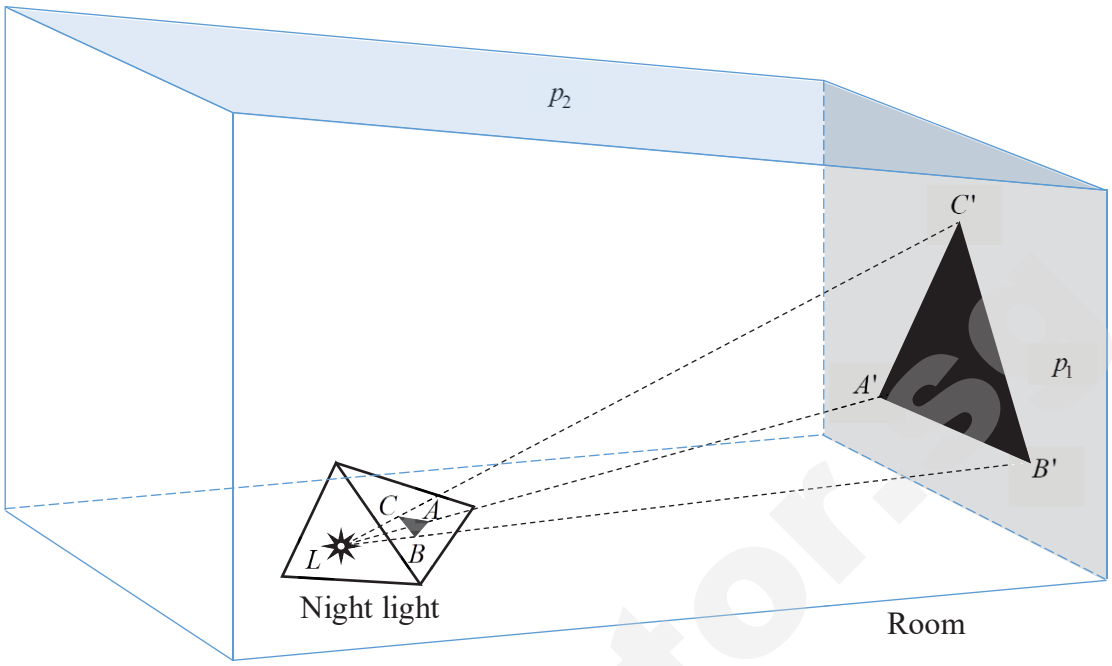

In the diagram (not drawn to scale), a night light with a light emitting diode L housed in a transparent acrylic case is placed in a room. A triangular sticker is stuck onto one face of the acrylic case. Referred to the origin O, the three vertices of the triangular sticker have coordinates $A(2, 3, 3)$, $B(4, 3, 4)$ and $C(2, 2, 5)$, where the units of measurement are in centimetres. When the night light is switched on, a triangular shadow $A'B'C'$ of the triangular sticker is cast on a vertical plane wall p_1 on one side of the room.

It is given that $\overrightarrow{OA'} = \begin{pmatrix} -3 \\ 13 \\ 8 \end{pmatrix}$ and $\overrightarrow{OB'} = \begin{pmatrix} 9 \\ 13 \\ 14 \end{pmatrix}$.

(i) Show that $\overrightarrow{OL} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. [4]

(ii) Given that p_1 is parallel to the x-z plane, find $\overrightarrow{OC'}$. [4]

The slanted ceiling p_2 is a plane with equation $y + 4z = 1053$.

(iii) Find the shortest distance between L and p_2. [2]

(iv) Find the line l when p_2 meets p_1. [2]

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Qn	Solution
1	Let e, g, w be amount of electricity and gas in kWh and water in m^3 used by Mr Lim in June respectively. $0.1775e + 0.1715g + 2.852w = 143.06 - 36$ $0.1775e + 0.1715g + 2.852w = 107.06 \quad ----(1)$ $0.2365e + 0.2279g + 3.672w = 140.78 \quad ----(2)$ $0.1762e + 0.1698g + 2.741w = 144.96 - 40$ $0.1762e + 0.1698g + 2.741w = 104.96 \quad ----(3)$ By G.C., $e = 266.4558551 = 266$ $g = 115.0629925 = 115$ $w = 14.03603875 = 14.0$ Mr Lim used 266kWh of electricity, 115kWh of gas and 14.0 m^3 of water in June.
2(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}\left(x^k\right) = \frac{kx^{k-1}}{1+x^{2k}}$
2(ii)	$u = \tan^{-1}\left(x^k\right) \qquad v' = x^{k-1}$ $u' = \frac{kx^{k-1}}{1+x^{2k}} \qquad v = \frac{x^k}{k}$ $\int x^{k-1}\tan^{-1}\left(x^k\right)\mathrm{d}x$ $= \frac{x^k}{k}\tan^{-1}\left(x^k\right) - \int \frac{x^{2k-1}}{1+x^{2k}}\mathrm{d}x$ $= \frac{x^k}{k}\tan^{-1}\left(x^k\right) - \frac{1}{2k}\int \frac{2kx^{2k-1}}{1+x^{2k}}\mathrm{d}x$ $= \frac{x^k}{k}\tan^{-1}\left(x^k\right) - \frac{1}{2k}\ln\left(1+x^{2k}\right) + c$ since $1 + x^{2k} > 0$
3(i)	$k = \frac{ax+b}{x^2+1}$ $kx^2 + k = ax + b$ $kx^2 - ax + (k-b) = 0$ ----------(1) (shown)
3(ii)	For (1) to have real roots, $(-a)^2 - 4k(k-b) \geq 0$ $a^2 - 4k^2 + 4kb \geq 0$ $4k^2 - 4bk - a^2 \leq 0$ Find critical values

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Qn	Solution
	$k = \dfrac{4b \pm \sqrt{16b^2 - 4(4)(-a^2)}}{2(4)}$ $k = \dfrac{b \pm \sqrt{b^2 + a^2}}{2}$ Therefore $4k^2 - 4bk - a^2 \le 0$ $\dfrac{b - \sqrt{b^2 + a^2}}{2} \le k \le \dfrac{b + \sqrt{b^2 + a^2}}{2}$ Since $-1 \le k \le 4$, $\dfrac{b}{2} = \dfrac{-1+4}{2} \Rightarrow b = 3$ $\dfrac{b + \sqrt{b^2 + a^2}}{2} = 4 \Rightarrow \sqrt{9 + a^2} = 5 \Rightarrow a = \pm 4$
4	Let $a_1 = b_1 = a$ $a_3 = b_3 \quad \Rightarrow \quad a + 2d = a r^2$ ----------(1) $a_7 = b_5 \quad \Rightarrow \quad a + 6d = a r^4$ ----------(2) Eliminate d, $(1) \times 3 - (2)$: $2a = 3ar^2 - ar^4$ $a(r^4 - 3r^2 + 2) = 0$ Since $a \neq 0$, $r^4 - 3r^2 + 2 = 0$ ----------(3) $\Rightarrow r^2 = 1, \quad \text{or} \quad r^2 = 2$ Since $r > 0$, $r \neq 1$, therefore, $r = \sqrt{2}$. Sub $r = \sqrt{2}$ into (1): $a + 2d = 2a \quad \Rightarrow \quad d = \dfrac{a}{2}$ If $a_n = b_m$:

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Qn	Solution
	$a+(n-1)d = ar^{m-1}$ $a+(n-1)\frac{a}{2} = ar^{m-1}$ $1+\frac{(n-1)}{2} = r^{m-1}$ $\frac{(n+1)}{2} = 2^{\frac{1}{2}(m-1)}$ $n = 2^{\frac{1}{2}(m+1)} - 1$
5(a)	Area of $ABC = \frac{1}{2}\left\lvert\overrightarrow{AC}\times\overrightarrow{BC}\right\rvert$ $= \frac{1}{2}\left\lvert\left(-\frac{2}{3}\mathbf{a}-\frac{1}{3}\mathbf{b}-\mathbf{a}\right)\times\left(-\frac{2}{3}\mathbf{a}-\frac{1}{3}\mathbf{b}-\mathbf{b}\right)\right\rvert$ $= \frac{1}{2}\left\lvert\left(-\frac{5}{3}\mathbf{a}-\frac{1}{3}\mathbf{b}\right)\times\left(-\frac{2}{3}\mathbf{a}-\frac{4}{3}\mathbf{b}\right)\right\rvert$ $= \frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left\lvert(5\,\mathbf{a}+\mathbf{b})\times(\mathbf{a}+2\,\mathbf{b})\right\rvert$ $= \frac{1}{9}\left\lvert5\,\mathbf{a}\times\mathbf{a}+\mathbf{b}\times\mathbf{a}+10\,\mathbf{a}\times\mathbf{b}+2\,\mathbf{b}\times\mathbf{b}\right\rvert$ $= \frac{1}{9}\left\lvert9\mathbf{a}\times\mathbf{b}\right\rvert$ $= \left\lvert\mathbf{a}\times\mathbf{b}\right\rvert$ (shown) Since $\angle AOB = 2\times\angle ACB$ since $\angle$ at centre $= 2\times\angle$ at circumference. Area of $ABC = \left\lvert\boldsymbol{a}\times\boldsymbol{b}\right\rvert$ $= \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\left\lvert\sin\frac{2\pi}{3}\right\rvert\lvert\hat{\boldsymbol{n}}\rvert$ $= (1)(1)\left(\frac{\sqrt{3}}{2}\right)(1)$ $= \frac{\sqrt{3}}{2}$ units^2

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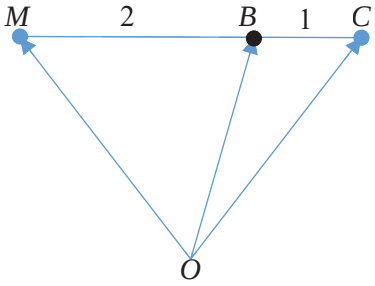
Qn	Solution
5(b)	Since $\overrightarrow{MC}-3\overrightarrow{BC}$, $\overrightarrow{OB}=\dfrac{\overrightarrow{OM}+2\overrightarrow{OC}}{3}$ $\overrightarrow{OM}=3\overrightarrow{OB}-2\overrightarrow{OC}$ $=3\boldsymbol{b}-2\left(-\dfrac{2}{3}\boldsymbol{a}-\dfrac{1}{3}\boldsymbol{b}\right)$ $=\dfrac{4}{3}\boldsymbol{a}+\dfrac{11}{3}\boldsymbol{b}$ Alternatively, $\overrightarrow{OM}=\overrightarrow{OC}+\overrightarrow{CM}$ $=\overrightarrow{OC}+3\overrightarrow{CB}$ $=3\overrightarrow{OB}-3\overrightarrow{OC}$ 
6(i)	$\dfrac{\mathrm{d}}{\mathrm{d}\theta}\sec^n\theta=n\sec^{n-1}\theta\left(\sec\theta\tan\theta\right)$ $=n\sec^n\theta\tan\theta$
6(ii)	$x=2\tan^2\theta\ \Rightarrow\ \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=4\tan\theta\sec^2\theta$ When $x=0,\ \tan\theta=0\ \Rightarrow\ \theta=0$ When $x=2,\ \tan\theta=1\ \Rightarrow\ \theta=\dfrac{\pi}{4}$ $\displaystyle\int_0^2 x\sqrt{1+\frac{x}{2}}\,\mathrm{d}x$ $=\displaystyle\int_0^{\frac{\pi}{4}}2\tan^2\theta\sqrt{1+\tan^2\theta}\left(4\tan\theta\sec^2\theta\right)\mathrm{d}\theta$ $=\displaystyle\int_0^{\frac{\pi}{4}}8\tan^3\theta\left\lvert\sec\theta\right\rvert\sec^2\theta\,\mathrm{d}\theta$ $=\displaystyle\int_0^{\frac{\pi}{4}}8\tan^3\theta\sec^3\theta\,\mathrm{d}\theta\ \left(\because\sec\theta>0\text{ for }0<\theta<\dfrac{\pi}{4}\right)$ $=8\displaystyle\int_0^{\frac{\pi}{4}}\tan\theta\left(\sec^2\theta-1\right)\sec^3\theta\,\mathrm{d}\theta$ $=8\displaystyle\int_0^{\frac{\pi}{4}}\sec^5\theta\tan\theta-\sec^3\theta\tan\theta\,\mathrm{d}\theta$ where $k=8,\ a=0\ ,\ b=\dfrac{\pi}{4}$

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Qn	Solution
6(iii)	$\int_0^2 x\sqrt{1+\frac{x}{2}}\ \mathrm{d}x$ $=8\int_0^{\frac{\pi}{4}} \sec^5\theta\tan\theta-\sec^3\theta\tan\theta\ \mathrm{d}\theta$ $=8\left[\frac{\sec^5\theta}{5}-\frac{\sec^3\theta}{3}\right]_0^{\frac{\pi}{4}}$ $=8\left(\frac{\left(\sqrt{2}\right)^5}{5}-\frac{\left(\sqrt{2}\right)^3}{3}\right)-8\left(\frac{1}{5}-\frac{1}{3}\right)$ $=\frac{16}{15}\left(\sqrt{2}+1\right)$ $A=\frac{16}{15}, B=1$
7(i)	To find exact points of intersection, sub C_2 into C_1 : $x^2+\frac{4}{1+x^2}=4$ $x^2+x^4+4=4+4x^2$ $x^4-3x^2=0$ $x^2\left(x^2-3\right)=0$ $x=0$ or $x=\pm\sqrt{3}$ (or using G.C,) When $x=0, y^2=4\Rightarrow y=\pm2$ When $x=\pm\sqrt{3}, y^2=1\Rightarrow y=\pm1$ $\therefore (0,2),(0,-2),\left(\sqrt{3},1\right),\left(-\sqrt{3},1\right),\left(\sqrt{3},-1\right),\left(-\sqrt{3},-1\right)$
7(ii)	**Method 1**: Using $\int x\ \mathrm{d}y$ Area of shaded region $=4\left[\int_0^1\sqrt{4-y^2}\ \mathrm{d}y+\int_1^2\sqrt{\frac{4}{y^2}-1}\ \mathrm{d}y\right]$ $=11.260\ \text{units}^2\ (\text{to 3d.p})$ OR $=4\pi-4\int_1^2\sqrt{4-y^2}-\sqrt{\frac{4}{y^2}-1}\ \mathrm{d}y$ **Method 2**: Using $\int y\ \mathrm{d}x$

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Qn	Solution
	Area of shaded region $= 4\left[\int_0^{\sqrt{3}} \frac{2}{\sqrt{1+x^2}}\,\mathrm{d}x + \int_{\sqrt{3}}^{2} \sqrt{4-x^2}\,\mathrm{d}x\right]$ $= 11.260 \text{ units}^2 \text{ (to 3d.p)}$ OR $= 4\pi - 2\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2} - \sqrt{\frac{4}{1+x^2}}\,\mathrm{d}x$
7(iii)	$V_x = 2\pi\left[\int_0^{\sqrt{3}} \frac{4}{1+x^2}\,\mathrm{d}x + \int_{\sqrt{3}}^{2} 4-x^2\,\mathrm{d}x\right]$ $= 2\pi\left[\left[4\tan^{-1}x\right]_0^{\sqrt{3}} + \left[4x - \frac{x^2}{3}\right]_{\sqrt{3}}^{2}\right]$ $= 2\pi\left(\frac{4\pi}{3} + 8 - \frac{8}{3} - 4\sqrt{3} + \sqrt{3}\right)$ $= \frac{2\pi}{3}\left(4\pi + 16 - 9\sqrt{3}\right) \text{ units}^3$ OR $V_x = \frac{4}{3}\pi(2)^3 - \pi\left[\int_{-\sqrt{3}}^{\sqrt{3}} 4-x^2 - \frac{4}{1+x^2}\,\mathrm{d}x\right]$
8(i)	From RHS: $\frac{1}{2}\left(\frac{1}{\sqrt{2r+1}} - \frac{1}{\sqrt{2r+3}}\right) = \frac{1}{2}\frac{\sqrt{2r+3}-\sqrt{2r+1}}{\sqrt{2r+1}\sqrt{2r+3}}$ $= \frac{1}{2}\frac{\left(\sqrt{2r+3}-\sqrt{2r+1}\right)\left(\sqrt{2r+3}+\sqrt{2r+1}\right)}{\sqrt{2r+1}\sqrt{2r+3}\left(\sqrt{2r+3}+\sqrt{2r+1}\right)}$ $= \frac{1}{2}\frac{(2r+3)-(2r+1)}{(2r+3)\sqrt{(2r+1)} + (2r+1)\sqrt{(2r+3)}}$ $= \frac{1}{(2r+3)\sqrt{(2r+1)} + (2r+1)\sqrt{(2r+3)}}$ =LHS OR From LHS: $\frac{1}{(2r+3)\sqrt{(2r+1)} + (2r+1)\sqrt{(2r+3)}}$ $= \frac{1}{\sqrt{(2r+3)}\sqrt{(2r+1)}\left[\sqrt{(2r+3)} + \sqrt{(2r+1)}\right]}$

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Qn	Solution
	$=\dfrac{\sqrt{(2r+3)}-\sqrt{(2r+1)}}{\sqrt{(2r+3)}\sqrt{(2r+1)}\ [2r+3-2r-1\ ]}$ $=\dfrac{\sqrt{2r+3}-\sqrt{2r+1}}{2\sqrt{2r+1}\sqrt{2r+3}}$ $=\dfrac{1}{2}\left(\dfrac{1}{\sqrt{2r+1}}-\dfrac{1}{\sqrt{2r+3}}\right)$ =RHS
8(ii)	$S_n=\sum_{r=1}^{n}\dfrac{1}{(2r+3)\sqrt{(2r+1)}+(2r+1)\sqrt{(2r+3)}}$ $=\dfrac{1}{2}\sum_{r=1}^{n}\left(\dfrac{1}{\sqrt{(2r+1)}}-\dfrac{1}{\sqrt{(2r+3)}}\right)$ ----------(1) $=\dfrac{1}{2}\begin{bmatrix}\dfrac{1}{\sqrt{3}}-\dfrac{1}{\sqrt{5}}\\ \dfrac{1}{\sqrt{5}}-\dfrac{1}{\sqrt{7}}\\ \cdots\\ \dfrac{1}{\sqrt{2n-1}}-\dfrac{1}{\sqrt{2n+1}}\\ \dfrac{1}{\sqrt{2n+1}}-\dfrac{1}{\sqrt{2n+3}}\end{bmatrix}$ $=\dfrac{1}{2}\left(\dfrac{1}{\sqrt{3}}-\dfrac{1}{\sqrt{2n+3}}\right)$
8(iii)	$n\to\infty,\quad \dfrac{1}{\sqrt{2n+3}}\to 0$ $\lim_{x\to\infty} S_n=\dfrac{1}{2\sqrt{3}}$ $\left\lvert S_n-\dfrac{1}{2\sqrt{3}}\right\rvert<0.05$ $\left\lvert\dfrac{1}{2}\left(\dfrac{1}{\sqrt{3}}-\dfrac{1}{\sqrt{2n+3}}\right)-\dfrac{1}{2\sqrt{3}}\right\rvert<0.05$ ----------(2) $\left\lvert\dfrac{1}{2\sqrt{2n+3}}\right\rvert<0.05$

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Qn	Solution
	$\frac{1}{2\sqrt{2n+3}} < 0.05$ $\sqrt{2n+3} > 10$ $2n+3 > 100$ $n > 48.5$ OR From GC, $n = 48$, $\frac{1}{2\sqrt{2n+3}} = 0.05025$ $n = 49$, $\frac{1}{2\sqrt{2n+3}} = 0.04975$ therefore the least value n is 49.
9(i)	$\frac{dx}{dt} = \cos^2 t - \sin^2 t$ $\frac{dy}{dt} = \frac{d}{dt}\left(\cos t \cos\frac{\pi}{4} - \sin t \sin\frac{\pi}{4}\right)$ $= \frac{\sqrt{2}}{2}\frac{d}{dt}(\cos t - \sin t)$ $= -\frac{\sqrt{2}}{2}(\sin t + \cos t)$ $\therefore \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$ $= \frac{-\frac{\sqrt{2}}{2}(\sin t + \cos t)}{\cos^2 t - \sin^2 t}$ $= -\frac{\sin t + \cos t}{\sqrt{2}(\cos t + \sin t)(\cos t - \sin t)}$ $= \frac{1}{\sqrt{2}(\sin t - \cos t)}$ (shown)
9(ii)	For tangent parallel to $y-$axis, $\frac{dy}{dx}$ is undefined $\Rightarrow$ $\sin t - \cos t = 0$ $\tan t = 1$ $t = \frac{\pi}{4}$

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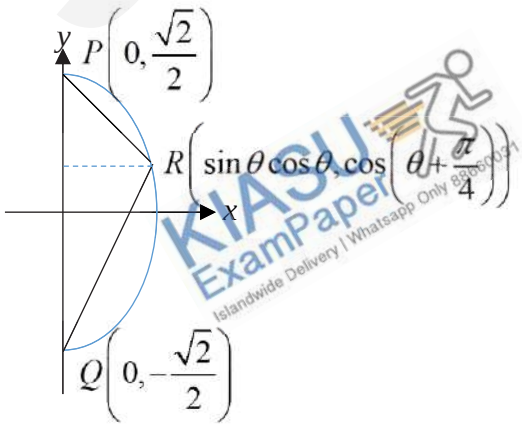
Qn	Solution
	$\Rightarrow$ Equation of tangent is: $x=\sin\frac{\pi}{4}\cos\frac{\pi}{4}=\left(\frac{\sqrt{2}}{2}\right)^2=\frac{1}{2}$ $\therefore x=\frac{1}{2}$
9(iii)	<u>Method 1</u> When $x=0$, $\sin t\cos t=0$ $2\sin t\cos t=0$ $\sin 2t=0$ $2t=0 \;\; or \;\; \pi$ $t=0\Rightarrow y=\cos\left(0+\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$ $or\ t=\frac{\pi}{2}\Rightarrow y=\cos\left(\frac{\pi}{2}+\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$ <u>Method 2</u> When $x=0$, $\sin t\cos t=0$ $\sin t=0\Rightarrow t=0\Rightarrow y=\cos\left(0+\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$ $\cos t=0\Rightarrow t=\frac{\pi}{2}\Rightarrow y=\cos\left(\frac{\pi}{2}+\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$ $\therefore P\left(0,\frac{\sqrt{2}}{2}\right), Q\left(0,-\frac{\sqrt{2}}{2}\right)$
9(iv)	 From the diagram, area of triangle PQR is given by

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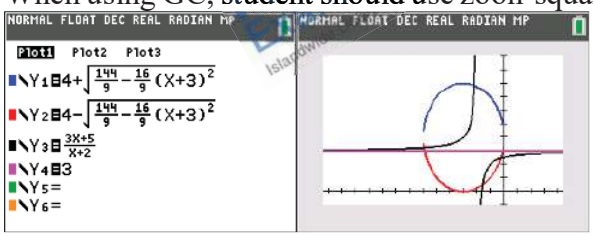
Qn	Solution
	$\frac{1}{2}\times 2\left(\frac{\sqrt{2}}{2}\right)\times \sin\theta\cos\theta$ $=\frac{\sqrt{2}}{2}\left(\frac{1}{2}\sin 2\theta\right)$ $=\frac{\sin 2\theta}{2\sqrt{2}}$ $=\frac{\sqrt{2}\sin 2\theta}{4}\,(\text{shown})$ Since $0\le\sin 2\theta\le 1$ for $0\le\theta\le\frac{\pi}{2}$, max area of triangle PQR is when $\sin 2\theta=1\Rightarrow\theta=\frac{\pi}{4}$
10 (i)	$y=\frac{3x+5}{x+2}=3-\frac{1}{x+2}$ <u>Method 1:</u> $\frac{1}{x}\xrightarrow{1}\frac{1}{x+2}\xrightarrow{2}-\frac{1}{x+2}\xrightarrow{3}3-\frac{1}{x+2}$ 1. Translate 2 units in the negative x-direction 2. Reflect in x-axis 3. Translate 3 units in the positive y-direction <u>Method 2:</u> $\frac{1}{x}\to-\frac{1}{x}\to-\frac{1}{x+2}\to 3-\frac{1}{x+2}$ 1. Reflect in y-axis 2. Translate 2 units in the negative x-direction 3. Translate 3 units in the positive y-direction Students should recognize the equation of the ellipse. $16(x+3)^2+9(y-4)^2=144$ $\frac{(x+3)^2}{3^2}+\frac{(y-4)^2}{4^2}=1$ When using GC, student should use zoon-square to ensure that the axis are both equal in scale. 

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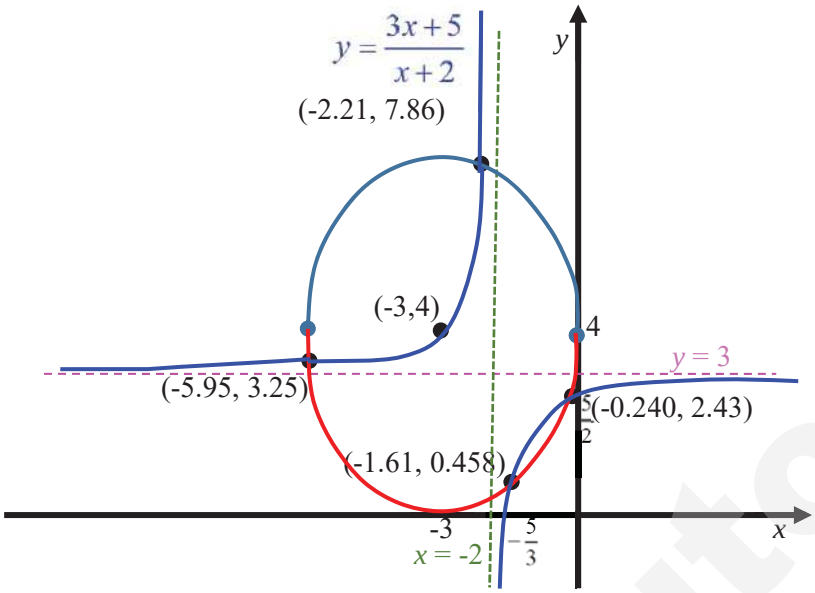
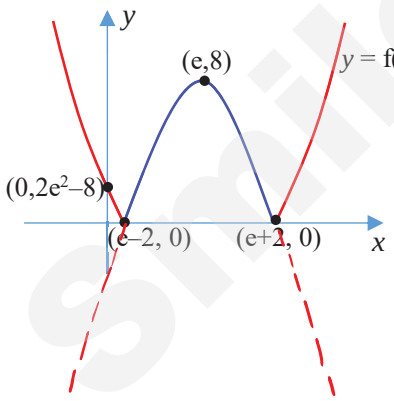
Qn	Solution
10 (ii)	
10 (iii)	From the graph, compare the curve with the lower half of the ellipse, $\{x \in \mathbb{R}: -5.95 < x < -2 \text{ or } -1.61 < x < -0.240\}$
11 (i)	 **To find exact *x*-coordinates of *x* intercepts:** $4-(x-e)^2=0$ $(x-e)^2=4$ $x-e=\pm 2$ $x=e\pm 2$ **To find exact *y*-coordinates of *y* intercept:** $2\left[-4+(0-e)^2\right]=y$ $y=2e^2-8$

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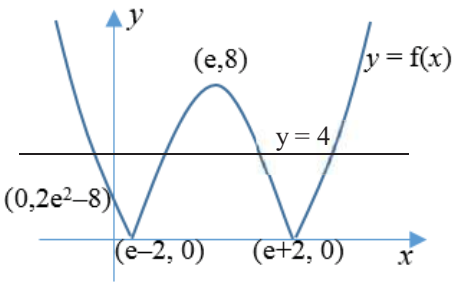
Qn	Solution
	NOTE: $2\left\|4-(x-\mathrm{e})^2\right\| = \begin{cases} 2\left[4-(x-\mathrm{e})^2\right] & ,\mathrm{e}-2\le x\le \mathrm{e}+2 \\ 2\left[-4+(x-\mathrm{e})^2\right] & ,x<\mathrm{e}-2 \text{ or } x>\mathrm{e}+2 \end{cases}$
11 (ii)	Since $\mathrm{f}(\mathrm{e}-2)=\mathrm{f}(\mathrm{e}+2)$ Therefore f is **not a one to one function**. Hence f^{-1} does not exist.  **Alternatively**, Since the **horizontal line** $y=4$ (give a specific counter-example) intersects $y=\mathrm{f}(x)$ **more than once**, therefore f is not a one to one function. Hence f^{-1} does not exist.
11 (iii)	The function f has an inverse if its domain is restricted to $a\le x<\mathrm{e}$. Smallest $a=\mathrm{e}-2$ Therefore $y=-2(x-\mathrm{e})^2+8$ $-\frac{y}{2}=(x-\mathrm{e})^2-4$ $(x-\mathrm{e})^2=4-\frac{y}{2}$ $x-\mathrm{e}=\pm\sqrt{4-\frac{y}{2}}$ $x=\mathrm{e}\pm\sqrt{4-\frac{y}{2}}$ Since $\mathrm{e}-2\le x<\mathrm{e}$ $x=\mathrm{e}-\sqrt{4-\frac{y}{2}}$ $\mathrm{f}^{-1}:x\mapsto \mathrm{e}-\sqrt{4-\frac{x}{2}},\quad 0\le x<8$
11 (iv)	From part **(iii)**, $\mathrm{e}-2\le x<\mathrm{e}$ $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{f}}=[\mathrm{e}-2,\mathrm{e})$ $\mathrm{D}_{\mathrm{g}}=\mathbb{R}\setminus\{-1\}$

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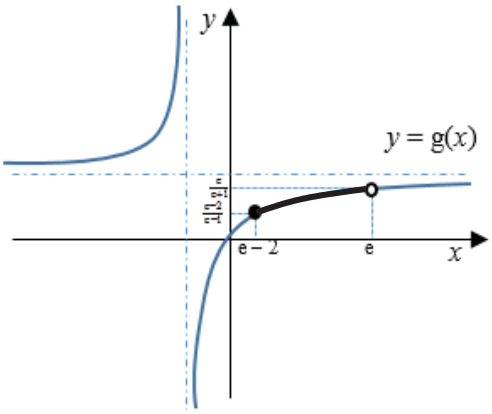
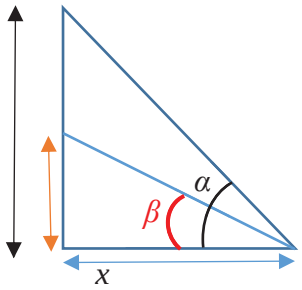
Qn	Solution
	Since $R_{f^{-1}} \subseteq D_g$, gf^{-1} exists  $R_{gf^{-1}} = \left[\frac{e-2}{e-1}, \frac{e}{e+1}\right)$
12 (i)	Let the angle α and β be such that $\tan\alpha = \frac{\frac{y}{2}-4.5+1.5}{x} = \frac{y-6}{2x}$ and $\tan\beta = \frac{\frac{y}{2}-4.5-1.5}{x} = \frac{y-12}{2x}$ $\tan\theta = \tan(\alpha-\beta)$ $= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$ $= \frac{\frac{y-6}{2x} - \frac{y-12}{2x}}{1+\left(\frac{y-6}{2x}\right)\left(\frac{y-12}{2x}\right)}$ $= \frac{12x}{4x^2+(y-6)(y-12)}$ $= \frac{12x}{4x^2+A}$ 
12 (ii)	Differentiate θ implicitly throughout with respect to x

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Qn | **Solution**

$$\sec^2\theta\left(\frac{d\theta}{dx}\right)=\frac{12\left(4x^2+A\right)-12x(8x)}{\left[4x^2+A\right]^2}$$

$$\sec^2\theta\left(\frac{d\theta}{dx}\right)=\frac{12\left[A-4x^2\right]}{\left[4x^2+A\right]^2}$$

$$\sec^2\theta\left(\frac{d\theta}{dx}\right)=\frac{12\left(\sqrt{A}-2x\right)\left(\sqrt{A}+2x\right)}{\left[4x^2+A\right]^2}$$

For $\frac{d\theta}{dx}=0$,

$x=\frac{\sqrt{A}}{2}\quad(x>0)$

Using 1st Derivative Test below,

x	$\frac{\sqrt{A}}{2}^-$	$\frac{\sqrt{A}}{2}$	$\frac{\sqrt{A}}{2}^+$
$\frac{d\theta}{dx}$	+ve	0	-ve
Shape of tangent			

Thus for maximum θ, the player should be $\frac{\sqrt{A}}{2}$ m away from the width of the pool

12 (iii)

$y=20$ or $A=112$

maximum θ, the player should be $\frac{\sqrt{112}}{2}=2\sqrt{7}$ m away from the width of the pool.

12 (iv)

Using Chain Rule

Since $x=15$, $\tan\theta=\frac{45}{253}$, $A=112$ and $\frac{dx}{dt}=-50$,

$$\frac{d\theta}{dt}=\frac{d\theta}{dx}\times\frac{dx}{dt}$$

$$=\frac{12(112)-48(15)^2}{\left[4(15)^2+112\right]^2\left[1+\left(\frac{45}{253}\right)^2\right]}\times-50$$

$$=0.447\text{ rad/min}\left(\text{to 3 sf}\right)$$

13 (i)

Since L lies on the lines of AA' and BB', we need to find the equation of these 2 lines.

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Qn	Solution
	$l_{AA'}: \boldsymbol{r} = \begin{pmatrix}2\\3\\3\end{pmatrix} + \lambda\begin{pmatrix}-3-2\\13-3\\8-3\end{pmatrix} = \begin{pmatrix}2\\3\\3\end{pmatrix} + \lambda\begin{pmatrix}-5\\10\\5\end{pmatrix}$ $l_{BB'}: \boldsymbol{r} = \begin{pmatrix}4\\3\\4\end{pmatrix} + \mu\begin{pmatrix}9-4\\13-3\\14-4\end{pmatrix} = \begin{pmatrix}4\\3\\4\end{pmatrix} + \mu\begin{pmatrix}5\\10\\10\end{pmatrix}$ $\begin{pmatrix}2\\3\\3\end{pmatrix} + \lambda\begin{pmatrix}-5\\10\\5\end{pmatrix} = \begin{pmatrix}4\\3\\4\end{pmatrix} + \mu\begin{pmatrix}5\\10\\10\end{pmatrix}$ By G.C., $\lambda = -\frac{1}{5}$ and $\mu = -\frac{1}{5}$ $\overrightarrow{OL} = \begin{pmatrix}2\\3\\3\end{pmatrix} - \frac{1}{5}\begin{pmatrix}-5\\10\\5\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix}$
13 (ii)	$p_1: \boldsymbol{r} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = k$ Since $\begin{pmatrix}-3\\13\\8\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = 13$, $k = 13$. $l_{LC}: \boldsymbol{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2-3\\2-1\\5-2\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\3\end{pmatrix}$ $\left(\begin{pmatrix}3\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\3\end{pmatrix}\right) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = 13$ $1 + \lambda = 13$ $\lambda = 12$ $\overrightarrow{OC'} = \begin{pmatrix}3\\1\\2\end{pmatrix} + 12\begin{pmatrix}-1\\1\\3\end{pmatrix} = \begin{pmatrix}-9\\13\\38\end{pmatrix}$
13 (iii)	Since $(0,1053,0)$ is on p_2, shortest distance is

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Qn	Solution
	$\left\lvert \begin{pmatrix} 0-3 \\ 1053-1 \\ 0-2 \end{pmatrix} \bullet \frac{1}{\sqrt{17}} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \right\rvert = \frac{1044}{\sqrt{17}} \approx 253 \text{ cm (3 s.f.)}$
13 (iv)	Solving $p_1 : y = 13$ and $p_2 : y + 4z = 1053$, $x = x$ $y = 13$. $z = 260$ So, $l : \boldsymbol{r} = \begin{pmatrix} 0 \\ 13 \\ 260 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, where $s \in \mathbb{R}$.

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Questions from 2019 NYJC Promos

1 removed (not in syllabus)

2 Referred to the origin O, three distinct points A, B and C are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OC}=\mathbf{c}$. If $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{BC}$ and $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{CA}$, show that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$. [4]

3 It is given that $\mathrm{f}(x)=ax^3+bx^2+cx+d$, where a, b, c and d are constants. The graph $y=\mathrm{f}(x)$ has a minimum point at $x=-1$ and a maximum point at $x=3$. The graphs $y=\mathrm{f}(|x|)$ and $y=|\mathrm{f}(x)|$ pass through $(-3,25)$ and $(-1,7)$ respectively. Find two possible expressions for $\mathrm{f}(x)$. [5]

4 **(i)** Using an algebraic method, solve the inequality $\dfrac{5}{4-2x-x^2}\le 1$. [5]

(ii) Hence, solve $\dfrac{5}{4-2|x|-x^2}\le 1$. [2]

5 removed (not in syllabus)

6 In this question, you may use expansions from the List of Formulae (MF 26).

(i) Find the Maclaurin expansion of $\ln(1+\sin 2x)$ in ascending powers of x, up to and including the term in x^4. State any value(s) of x in the interval $-\dfrac{\pi}{4}\le x\le\dfrac{\pi}{4}$ for which the expansion is **not** valid. [4]

(ii) It is given that the first three terms found in part **(i)** are equal to the first three terms in the series expansion of $ax(1+bx)^c$ for small x. Find the exact values of the constants a, b and c and use these values to find the coefficient of x^4 in the expansion of $ax(1+bx)^c$, giving your answer as a simplified rational number. [5]

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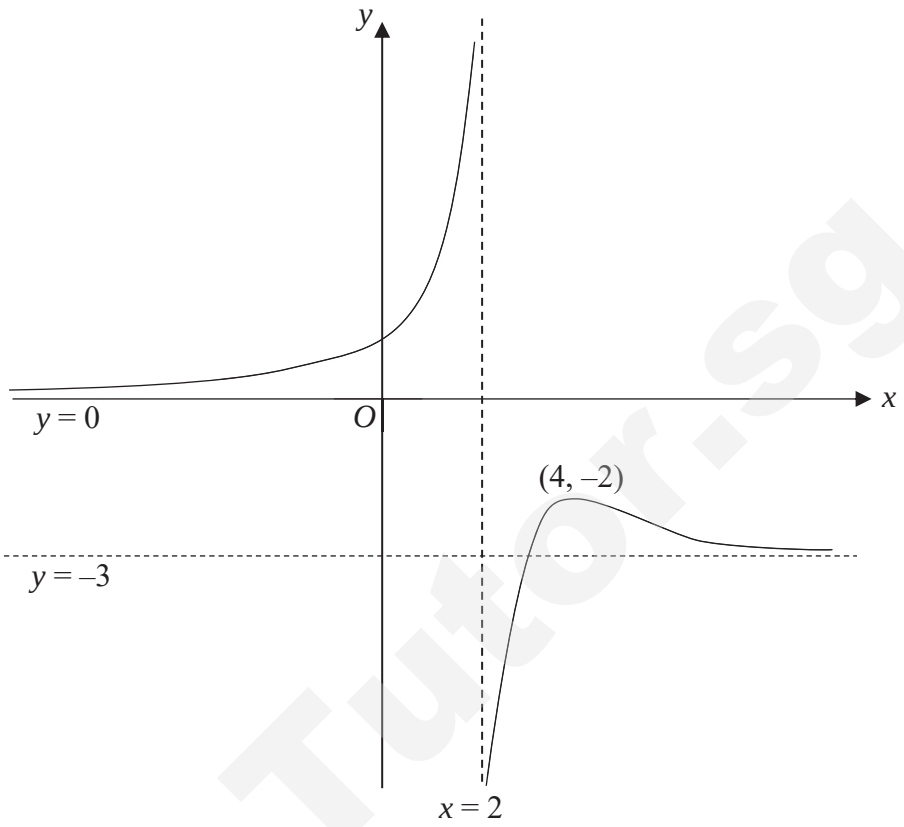
7 **(a)** The diagram below shows the graph with equation $y = \mathrm{f}(x)$. It has a maximum point at $(4, -2)$ and asymptotes $x = 2$, $y = -3$ and $y = 0$.



Sketch on separate diagrams, the graphs of

(i) $y = 3\mathrm{f}(2x+1)$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

(b) A curve $y = \mathrm{g}(x)$ undergoes, in succession, the following transformations to get $y = \dfrac{1}{x}$:

A: A scaling with scale factor $\dfrac{1}{3}$ parallel to the *y*-axis

B: A reflection in the *y*-axis

C: A translation of 1 unit in the direction of the *x*-axis.

Find $\mathrm{g}(x)$, showing your workings clearly. [3]

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8 The functions f, g and h are defined as follows.

$$\begin{aligned} &\mathrm{f}: x \mapsto |(x-1)(3-x)|, \quad x > k,\ k \in \mathbb{R}, \\ &\mathrm{g}: x \mapsto -mx+1, \quad x \in \mathbb{R},\ m > 1, \\ &\mathrm{h}: x \mapsto |(x-1)(3-x)|, \quad x \in \mathbb{R}. \end{aligned}$$

(i) State the least value of k such that f^{-1} exists. [1]

Using the value of k found in part **(i)**,

(ii) find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}, [4]

(iii) sketch on the same diagram the graphs of $y = \mathrm{f}(x), y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$. [3]

(iv) On a separate diagram, sketch the graph of $y = \mathrm{gh}\left(\frac{1}{2}x\right)$, indicating clearly the coordinates of the turning point. [2]

9 Planes p_1 and p_2 have equations $x+2y+z=4$ and $ax+3y+2z=1$ respectively, where a is a constant.

(i) Show that p_1 and p_2 have a common point (0, 7, –10). [1]

(ii) Find, in terms of a, the vector equation of the line l where p_1 and p_2 meet. [3]

Plane p_3 has equation $2x+y-2z=b$, where b is a constant and l has a distance of 5 units from p_3.

(iii) Show that $a = \frac{6}{5}$. [1]

(iv) Find the possible values of b. [3]

(v) Find the acute angle between p_1 and p_2. [2]

10 A curve D has parametric equations

$$x = 2t^3,\ y = -3t^2.$$

(i) Find the equation of the tangent to D at the point with parameter t. [2]

(ii) Points P and Q on D have non-zero parameters p and q respectively. The tangent at P meets the tangent at Q at the point R. Show that the x-coordinate of R is $pq(p+q)$, and find the y-coordinate of R in terms of p and q. Given that $pq = -1$, show that R lies on the curve E with equation $y = -x^2 - 1$. [5]

(iii) It is given that D and E meet at the point M where the x-coordinate of M is positive. Find the exact coordinates of M. [3]

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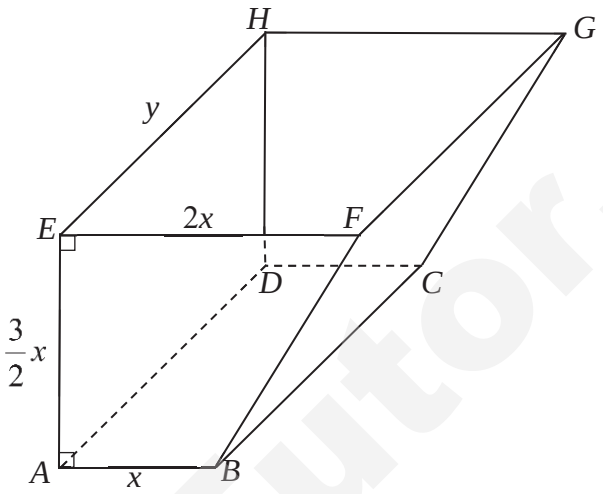
11 removed (not in syllabus)

12 The diagram shows an open tank made of metal with negligible thickness and a fixed capacity of 108 m^3. The open top *EFGH*, base *ABCD*, sides *ADHE* and *BCGF* are rectangles, while *ABFE* and *DCGH* are trapeziums with *AE* perpendicular to *AB* and *EF*. The dimensions of the tank are $AB = x$ m, $EF = 2x$ m, $AE = \frac{3}{2}x$ m and $EH = y$ m.



(i) The interior of the tank needs to be treated to prevent rusting. The cost of treating the interior base is \$3 per m^2 and the cost of treating the interior sides is \$4 per m^2. Show that the total cost of treatment is $\$\left(\frac{K}{x}+18x^2\right)$, where *K* is an exact constant to be determined. [3]

(ii) Find, by differentiation, the values of *x* and *y* such that the cost of treatment is minimum and show that it is less than \$420. [5]

(iii) The tank is mounted on a wall and filled with a liquid to maximum capacity. However, due to poor construction, the liquid begins to leak and spread in a circle of uniform thickness

2 mm on the floor. Assuming that the liquid is leaking from the tank at a constant rate of 8.4×10^{-7} m^3s^{-1} and the leak is discovered one minute after it started, find the rate at which the radius of the circle of liquid is increasing at this instant, giving your answer in ms^{-1}. [4]

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2	Vectors	
3	Equations & Inequalities	$\mathrm{f}(x)=-\frac{9}{16}x^3+\frac{27}{16}x^2+\frac{81}{16}x+\frac{157}{16}$ or $\mathrm{f}(x)=-x^3+3x^2+9x-2$
4	Equations & Inequalities	(i) $x<-1-\sqrt{5}$ or $x=-1$ or $x>-1+\sqrt{5}$ (ii) $x<1-\sqrt{5}$ or $x>-1+\sqrt{5}$
6	Maclaurin & Binomial Series	i) $2x-2x^2+\frac{4}{3}x^3-\frac{4}{3}x^4+\cdots;\ x\neq-\frac{\pi}{4}$ ii) $a=2$, $c=-3$, $b=\frac{1}{3}$; coefficient of $x^4=-\frac{20}{27}$
7	Graphs & Transformations	(iii) $\mathrm{g}(x)=\frac{3}{1-x}$
8	Functions	(i) $k=3$ (ii) $\mathrm{f}^{-1}(x)=2+\sqrt{x+1}$; $\mathrm{D}_{\mathrm{f}^{-1}}=(0,\infty)$
9	Vectors	(ii) $\mathbf{r}=\begin{pmatrix}0\\7\\-10\end{pmatrix}+\lambda\begin{pmatrix}-1\\2-a\\2a-3\end{pmatrix},\lambda\in\mathbb{R}$ (iv) $b=12$ or $b=42$ (v) $\theta=8.7°$ (to 1 dec. pl.)
10	Differentiation & Applications	(i) $y=-\frac{x}{t}-t^2$ (ii) $y_{\mathrm{R}}=-pq-q^2-p^2$ (iii) $\left(\sqrt{\frac{1}{2}},-\frac{3}{2}\right)$
12	Differentiation & Applications	(i) $K=48\left(9+2\sqrt{13}\right)$ (ii) $x\approx2.79\text{m}$; $y\approx6.19\text{m}$ (iii) $7.46\times10^{-4}\text{ms}^{-1}$.

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NATIONAL JUNIOR COLLEGE
SENIOR HIGH 1
Promotional Examinations

NAME

SUBJECT CLASS: 1ma2

REGISTRATION NUMBER

H2 MATHEMATICS 9758

02 October 2019

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)
Writing Paper

READ THESE INSTRUCTIONS FIRST

This paper constitutes 50% of your overall score for SH1 H2 Mathematics.

Write your name, class and registration number in the boxes above.
Please write clearly and use capital letters.

Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.
Up to 2 marks may be deducted for improper presentation.

The number of marks is given in the brackets [ ] at the end of each question or part question.

Question Number	Marks Possible	Marks Obtained
1	4	
2	4	
3	6	
4	7	
5	7	
6	8	
7	8	
8	8	
9	8	
10	14	
11	14	
12	12	
Presentation Deduction		– 1 / –2
TOTAL	100	

This document consists of **31** printed pages and **1** blank page.

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[Turn over

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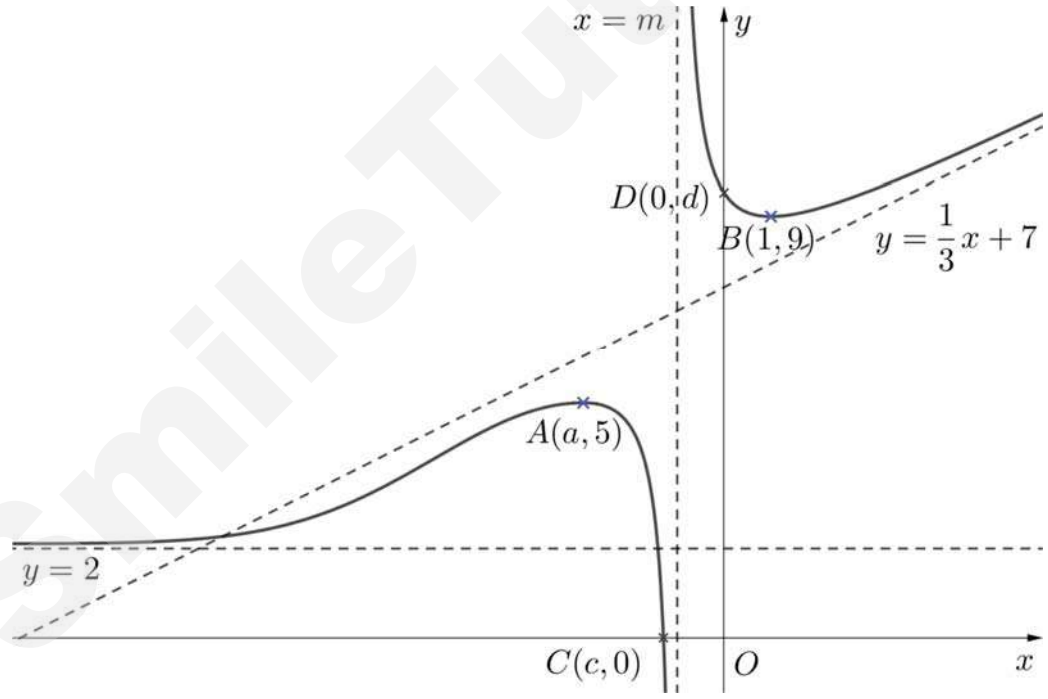
1 Given that $\mathrm{f}(x) = \tan\left(\frac{\pi}{4} - x\right)$, find $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. Hence write down the first three non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. [4]

2 The function $\mathrm{f}(x)$ is a quadratic polynomial such that $\mathrm{f}(4) = 3$ and $\int_0^4 \mathrm{f}(x)\,\mathrm{d}x = \frac{20}{3}$. If the curve with equation $y = \mathrm{f}(x)$ is transformed by a translation of 1 unit in the negative x-direction, the new curve has a y-intercept at $\frac{3}{2}$. Find $\mathrm{f}(x)$. [4]

3 The diagram below shows the curve with equation $y = \mathrm{f}(x)$. It has turning points $A(a, 5)$ and $B(1, 9)$ and asymptotes with equations $y = 2$, $x = m$ and $y = \frac{1}{3}x + 7$. The curve also crosses the axes at the points $C(c, 0)$ and $D(0, d)$. The gradient of the curve at D is -3.



Sketch, on separate diagrams, the following curves.

(a) $y = \frac{1}{\mathrm{f}(x)}$ [3]

(b) $y = \mathrm{f}'(x)$ [3]

Label the coordinates of the points corresponding to A, B, C, and D (where applicable), the points where the curve crosses the axes, and the equations of any asymptotes.

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4 A curve C has equation $y = e^x$.

(a) Find the exact volume of revolution when the region bounded by C, the line $y = 3$ and the y-axis is rotated completely about the y-axis. [5]

(b) Describe a pair of transformations which transforms the curve with equation $y = e^{1+2x}$ on to the curve C. [2]

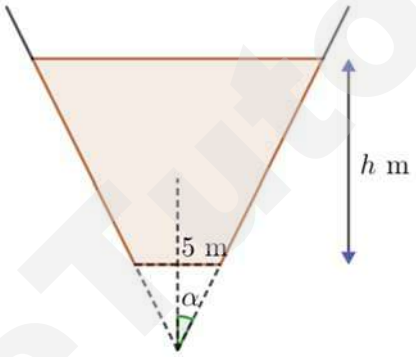
5 A curve C has parametric equations

$$x = \sin\theta, \qquad y = \sin 2\theta, \qquad \text{for } 0 \le \theta \le \pi.$$

(i) Sketch C, labelling the coordinates of any axial intercepts. [2]

(ii) Find the exact area of the region bounded by C. [5]

6 A vessel is formed by removing a smaller cone of radius 5 m from a bigger cone whose semi-vertical angle is α, where $\tan\alpha = 0.5$. Water flows out of the vessel at a rate of $k\sqrt{h}$ m^3 per minute, where k is a positive constant. At time t minutes, the height of the water surface from the hole is h m (see diagram).



(i) Show that the volume of the water V, in m^3, is given by $\frac{1}{12}\pi\left[(h+10)^3 - 1000\right]$. [4]

(ii) Find the rate of change of h, in terms of k, when $V = 120\pi$. [4]

7 Determine the constants A and B such that

$$\frac{1}{(3x^2+1)(x^2+3)} = \frac{A}{3x^2+1} + \frac{B}{x^2+3}.$$

Hence, find the exact value of p, where $p > -1$, such that

$$\int_0^1 \frac{8}{(3x^2+1)(x^2+3)}\,\mathrm{d}x = \int_{-1}^{p} \left|\frac{\sqrt{3}}{9}x\right|\,\mathrm{d}x,$$

giving your answer in terms of π. [8]

8 A curve C has equation $\ln(y+1) = 1 + \tan^{-1}x$, where $y > -1$.

(i) Explain why C has no tangent parallel to the y-axis. [2]

(ii) Without using a calculator, find the equation of the normal to C at the point where it crosses the y-axis. [3]

(iii) Determine the range of values of x for which C is concave downwards. [3]

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9 Non-zero and non-parallel vectors **a**, **b** and **c** are such that $\mathbf{b}\times 3\mathbf{c} = \mathbf{c}\times\mathbf{a}$.

(i) Determine the relationship between **c** and $\mathbf{a}+3\mathbf{b}$, justifying your answer. [2]

It is given that **a** and $3\mathbf{b}$ are unit vectors and that the angle between **a** and **b** is $60°$.

(ii) Evaluate $|\mathbf{a}+3\mathbf{b}|$. [3]

(iii) Given further that $\mathbf{a}+3\mathbf{b}$ makes an angle of $60°$, $120°$ and $135°$ with the positive x-, y- and z-axes respectively, show that **c** is parallel to $\mathbf{i}-\mathbf{j}-\sqrt{2}\mathbf{k}$. [3]

10 Functions f and g are defined by

$$\mathrm{f}: x \mapsto 2-x+\frac{8}{x+2}, \quad x\in\mathbb{R}, x\neq -2, x>k,$$
$$\mathrm{g}: x \mapsto x^2-6x+a, \quad x\in\mathbb{R}, x>0,$$

where a is a constant.

(i) State the least value of k for which the function f^{-1} exists. [1]

Using this value of k,

(ii) Without finding f^{-1}, sketch, on the same diagram, the graphs of $y=\mathrm{f}(x)$, $y=\mathrm{f}^{-1}(x)$ and $y=\mathrm{f}^{-1}\mathrm{f}(x)$, showing clearly their geometrical relationship. State the equations of any asymptotes. [4]

(iii) Find the smallest integer value of a for which the composite function fg exists and use this value to state the range of fg. [4]

(iv) Given instead that $a=10$, solve the inequality $\mathrm{fg}(x)+\mathrm{g}(x)\leq 4$ algebraically. [5]

11 The plane p passes through the points A, B and C with coordinates $(-3,-4,11)$, $(1,-2,0)$ and $(-5,2,-1)$ respectively. The point M has position vector given by $\mathbf{m}=3\mathbf{i}+6\mathbf{j}+15\mathbf{k}$.

(i) Show that $\overrightarrow{AM}$ is perpendicular to p. [3]

(ii) Find the coordinates of N which is the mirror image of M in p. [2]

(iii) Find a vector equation of the line which is a reflection of the line MB in p. [2]

The plane q has equation $11x+6y-z-4+k(-x-3y+z-4)=0$ for some constant k.

(iv) Given that q contains M, find a cartesian equation of q. [2]

(v) By finding the line of intersection between p and q, or otherwise, find a cartesian equation of the plane which is a reflection of q in p. [5]

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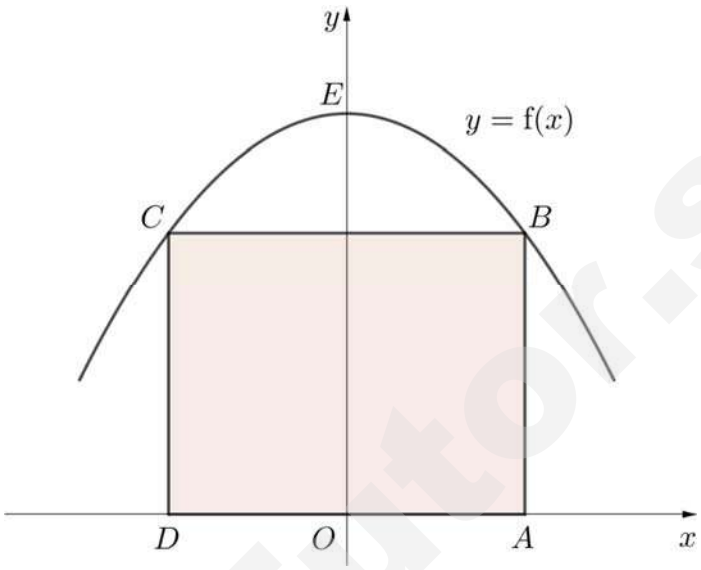
12 It is given that $\mathrm{f}(x) = 9 - \frac{x^2}{6}$. The diagram below shows a vertical cross section of a building. The cross section of the roof of the building can be modelled by the curve $y = \mathrm{f}(x)$, where $-6 \le x \le 6$ and x is in metres. The ground level is represented by the x-axis.



The cross section of the living space under the roof can be modelled by a rectangle *ABCD* with points $D(-a, 0)$ and $A(a, 0)$, where $0 < a \le 6$.

The P/A ratio is a measure of the thermal insulation of a space. The lower the P/A ratio, the smaller the amount of insulation is required, thus saving costs. The P/A ratio can be defined by the function $I(a) = \frac{P(a)}{A(a)}$ where $P(a)$ is the perimeter and $A(a)$ is the area of the rectangle *ABCD*.

(i) Show that $I(a) = \frac{12}{54 - a^2} + \frac{1}{a}$. [3]

(ii) Given that $a = a_1$ is the value of a which gives the minimum value of I, show that a_1 satisfies the equation $a^2 + \sqrt{24}a^{\frac{3}{2}} - 54 = 0$. [3]

(iii) Find a_1, correct to 3 decimal places, and show that it minimises I. [2]

The developer would prefer the living space *ABCD* to be at least 80% of the area *ABECD*, which is currently not satisfied. It is suggested to replace the curve $y = \mathrm{f}(x)$ to model the roof with part of the curve in part **(iv)**.

(iv) Sketch the curve with equation $(2y - 19)^2 - 2x^2 = 1$ for $y \le 9$ for **all** values of x, stating the equations of any asymptotes. [2]

(v) Based on the value of a_1 found in part **(iii)**, determine if the developer would accept the suggestion. [2]

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Qn	Suggested Solutions
1	$\mathrm{f}(x)=\tan\left(\frac{\pi}{4}-x\right) \quad \Rightarrow \quad \mathrm{f}(0)=1$ $\mathrm{f}'(x)=-\sec^2\left(\frac{\pi}{4}-x\right) \quad \Rightarrow \quad \mathrm{f}'(0)=-2$ $\mathrm{f}''(x)=2\sec^2\left(\frac{\pi}{4}-x\right)\tan\left(\frac{\pi}{4}-x\right) \quad \Rightarrow \quad \mathrm{f}''(0)=4$ $\mathrm{f}(x)=1+(-2)x+(4)\frac{x^2}{2!}+\ldots$ $=1-2x+2x^2+\ldots$
2	Let $\mathrm{f}(x)=ax^2+bx+c$ $\mathrm{f}(4)=3$ $a(4)^2+b(4)+c=3$ $16a+4b+c=3 \quad \cdots(1)$ $\int_0^4 ax^2+bx+c\,\mathrm{d}x=\frac{20}{3}$ $\left[\frac{a}{3}x^3+\frac{b}{2}x^2+cx\right]_0^4=\frac{20}{3}$ $\frac{64}{3}a+8b+4c=\frac{20}{3} \quad \cdots(2)$ When translated 1 unit in the direction of the negative x-direction, the equation of curve becomes $y=a(x+1)^2+b(x+1)+c$. At the y-intercept, $a+b+c=\frac{3}{2} \quad \cdots(3)$ Solving (1), (2) and (3), $a=\frac{1}{2}, \quad b=-2, \quad c=3$. Hence $\mathrm{f}(x)=\frac{1}{2}x^2-2x+3$

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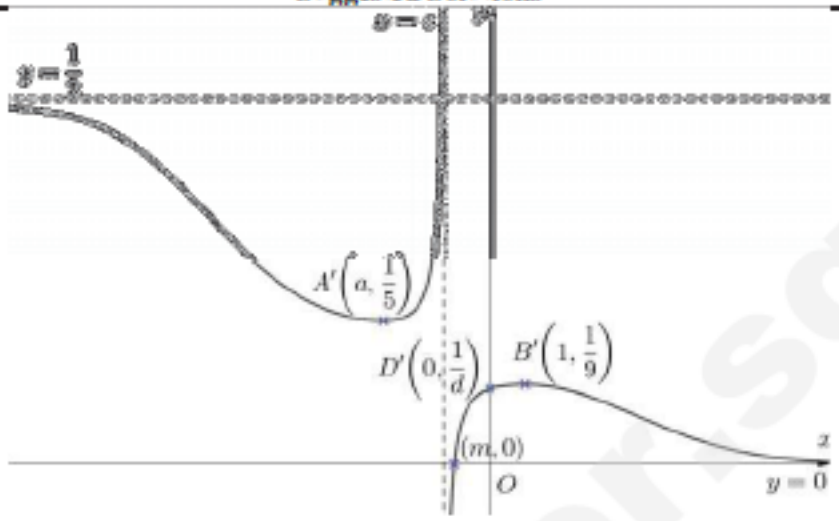
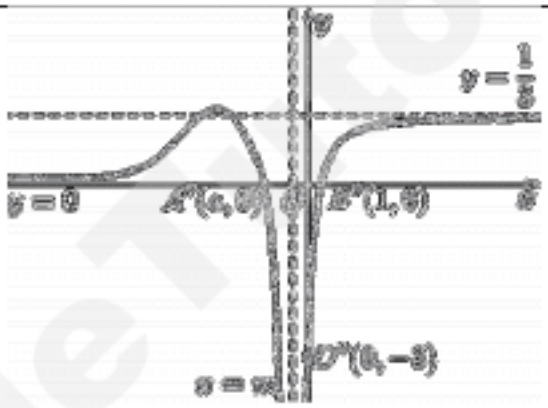
Qn	Suggested Solutions
3(a)	
3(b)	
4(a)	$y=e^x \Rightarrow x=\ln y$ $\int (\ln y)^2\,dy$ $= y(\ln y)^2 - \int y\left(\frac{2\ln y}{y}\right)dy$ $= y(\ln y)^2 - 2\int \ln y\,dy$ $= y(\ln y)^2 - 2\left[y\ln y - \int y\left(\frac{1}{y}\right)dy\right]$ $= y(\ln y)^2 - 2y\ln y + 2y + c$ volume of the solid formed $= \pi\int_1^3 (\ln y)^2\,dy$ $= \pi\left[y(\ln y)^2 - 2y\ln y + 2y\right]_1^3$ $= \pi\left[3(\ln 3)^2 - 6\ln 3 + 6 - 2\right]$ $= \pi\left[3(\ln 3)^2 - 6\ln 3 + 4\right]$ units^3

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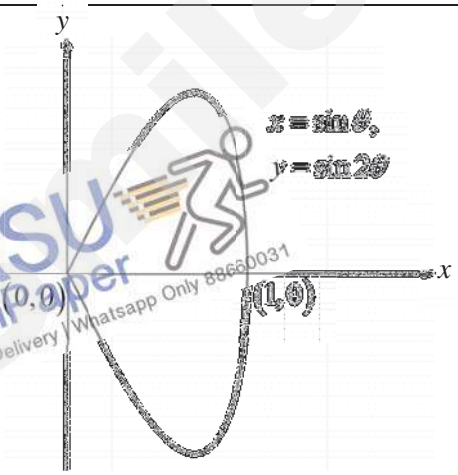
Qn	Suggested Solutions
4(b)	$y = \mathrm{e}^{1+2x}$ $\downarrow$ Replace x by $\frac{x}{2}$ $y = \mathrm{e}^{1+x}$ $\downarrow$ Replace x by $x-1$ $y = \mathrm{e}^{x}$ 1. Scaling by a factor of 2 parallel to the x-axis. 2. Translation of 1 unit in the positive x-direction.
	Alternatively $y = \mathrm{e}^{1+2x}$ $\downarrow$ Replace x by $x-\frac{1}{2}$ $y = \mathrm{e}^{2x}$ $\downarrow$ Replace x by $\frac{x}{2}$ $y = \mathrm{e}^{x}$ 1. Translation of $\frac{1}{2}$ unit in the positive x-direction. 2. Scaling by a factor of 2 parallel to the x-axis.
5(i)	

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Qn	Suggested Solutions
5(ii)	When $x=0$, $\sin\theta=0$, $\theta=1$. When $x=1$, $\sin\theta=1$, $\theta=\frac{\pi}{2}$.

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=\cos\theta.$$

Method 1: Using the form $\mathrm{f}'(x)\left[\mathrm{f}(x)\right]^n$.

Area of region bounded by C

$$=2\int_0^{\frac{\pi}{2}}\sin 2\theta\cos\theta\ \mathrm{d}\theta$$

$$=2\int_0^{\frac{\pi}{2}}2\sin\theta\cos^2\theta\ \mathrm{d}\theta$$

$$=-4\int_0^{\frac{\pi}{2}}(-\sin\theta)\cos^2\theta\ \mathrm{d}\theta$$

$$=-4\left[\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}}$$

$$=-4\left(0-\frac{1}{3}\right)$$

$$=\frac{4}{3}\ \text{units}^2$$

Method 2: Using the factor formula.

Area of region bounded by C

$$=2\int_0^{\frac{\pi}{2}}\sin 2\theta\cos\theta\ \mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{2}}\sin 3\theta+\sin\theta\ \mathrm{d}\theta$$

$$=\left[-\frac{\cos 3\theta}{3}-\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$=0-\left(-\frac{1}{3}-1\right)$$

$$=\frac{4}{3}\ \text{units}^2$$

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Qn	Suggested Solutions
6(i)	Let h' be the height of the smaller cone $\tan\alpha = 0.5 = \frac{5}{h'}$ $\Rightarrow h' = 10$ Let R and H be the radius and height of the bigger cone respectively. $\tan\alpha = 0.5 = \frac{R}{H}$ $\Rightarrow H = 2R$ Also, we have $H = h + 10$ $V = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi(5)^2(10)$ $= \frac{1}{3}\pi\left(R^2 H - 250\right)$ $= \frac{1}{3}\pi\left[\left(\frac{H}{2}\right)^2 H - 250\right]$ $= \frac{1}{3}\pi\left(\frac{H^3}{4} - 250\right)$ $= \frac{1}{12}\pi\left(H^3 - 1000\right)$ $= \frac{1}{12}\pi\left[(h+10)^3 - 1000\right]$
6(ii)	$\frac{dV}{dh} = \frac{1}{12}\pi\left[3(h+10)^2\right]$ $= \frac{\pi}{4}(h+10)^2$ $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $-k\sqrt{h} = \frac{1}{4}\pi(h+10)^2 \times \frac{dh}{dt}$ $\frac{dh}{dt} = \frac{-4k\sqrt{h}}{\pi(h+10)^2}$ When $V = 120\pi$, $\frac{1}{12}\pi\left[(h+10)^3 - 1000\right] = 120\pi$. $(h+10)^3 - 1000 = 1440$ $(h+10)^3 = 2440$ $h + 10 = \sqrt[3]{2440}$ $h = \sqrt[3]{2440} - 10$ When $h = \sqrt[3]{2440} - 10$, $\frac{dh}{dt} = \frac{-4k\sqrt{\sqrt[3]{2440} - 10}}{\pi\left(\sqrt[3]{2440}\right)^2}$ $= -0.0131k$ m/min (to 3 s.f.)

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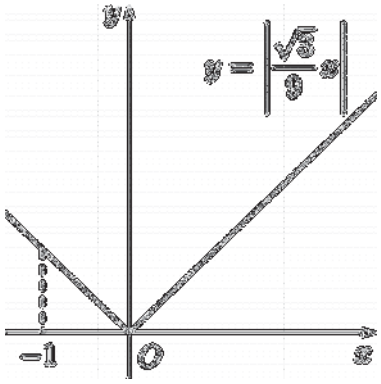
Qn	Suggested Solutions
7	$\dfrac{1}{(3x^2+1)(x^2+3)} = \dfrac{A(x^2+3)+B(3x^2+1)}{(3x^2+1)(x^2+3)}$ Comparing coefficients, $A+3B=0$ (1) and $3A+B=1$ (2) Solving (1) and (2), $A=\dfrac{3}{8}$ and $B=-\dfrac{1}{8}$ $\displaystyle\int_0^1 \frac{8}{(3x^2+1)(x^2+3)}\,\mathrm{d}x$ $\displaystyle= \int_0^1 \frac{3}{3x^2+1} - \frac{1}{x^2+3}\,\mathrm{d}x$ $\displaystyle= \int_0^1 \frac{1}{x^2+\frac{1}{3}}\,\mathrm{d}x - \int_0^1 \frac{1}{x^2+3}\,\mathrm{d}x$ $\displaystyle= \int_0^1 \frac{1}{x^2+\left(\frac{1}{\sqrt{3}}\right)^2}\,\mathrm{d}x - \int_0^1 \frac{1}{x^2+\left(\sqrt{3}\right)^2}\,\mathrm{d}x$ $\displaystyle= \left[\sqrt{3}\tan^{-1}\left(\sqrt{3}x\right)\right]_0^1 - \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$ $\displaystyle= \left[\sqrt{3}\tan^{-1}\left(\sqrt{3}\right)-0\right]_0^1 - \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-0\right]_0^1$ $\displaystyle= \frac{\pi\sqrt{3}}{3} - \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right)$ $\displaystyle= \frac{5\sqrt{3}\pi}{18}$ Consider the graph of $y=\left\lvert\dfrac{\sqrt{3}}{9}x\right\rvert$ 

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Qn	Suggested Solutions
	Observe that area of the triangle bounded by $y=-\frac{\sqrt{3}}{9}x$, the negative x-axis and $x=-1$ is $\frac{1}{2}(1)\left(\frac{\sqrt{3}}{9}\right)=\frac{\sqrt{3}}{18}<\frac{5\sqrt{3}\pi}{18}$. Thus $p>0$. Hence, $\frac{5\sqrt{3}\pi}{18}=\frac{\sqrt{3}}{18}+\frac{1}{2}(p)\left(\frac{\sqrt{3}}{9}p\right)$ $5\pi=1+p^2$ $p=\sqrt{5\pi-1}$
8(i)	Differentiating $\ln(y+1)=1+\tan^{-1}x$ w.r.t. x, we get $\frac{1}{y+1}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+x^2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y+1}{1+x^2}.$ Since $1+x^2>0$ for all real values of x, thus $\frac{\mathrm{d}y}{\mathrm{d}x}$ is defined for all values of x and y. Therefore, C has no tangent parallel to the y-axis.
8(ii)	At $x=0$, we have $\ln(y+1)=1+\tan^{-1}0$ $\Rightarrow \quad y=\mathrm{e}-1,$ and $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}-1+1}{1+0^2}=\mathrm{e}.$ Then the gradient of normal at point where $x=0$ is $-\mathrm{e}^{-1}$. Therefore the equation of normal at point where $x=0$ is $\Rightarrow y-(\mathrm{e}-1)=\left(-\mathrm{e}^{-1}\right)(x-0)$ $\Rightarrow y=-\mathrm{e}^{-1}x+\mathrm{e}-1$

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Qn	Suggested Solutions
8(iii)	Where C is concave downwards, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}<0$. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y+1}{1+x^2}$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}-\left(y+1\right)2x}{\left(1+x^2\right)^2}$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\left(y+1\right)-\left(y+1\right)2x}{\left(1+x^2\right)^2}=\frac{\left(y+1\right)\left(1-2x\right)}{\left(1+x^2\right)^2}<0$ $\Rightarrow \left(1-2x\right)<0$ $\Rightarrow x>\frac{1}{2}$
9(i)	$\mathbf{b}\times 3\mathbf{c}=\mathbf{c}\times\mathbf{a}$ $\Rightarrow \mathbf{b}\times 3\mathbf{c}-\mathbf{c}\times\mathbf{a}=\mathbf{0}$ $\Rightarrow 3\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{c}=\mathbf{0}$ $\Rightarrow \left(3\mathbf{b}+\mathbf{a}\right)\times\mathbf{c}=\mathbf{0}$ Hence $\mathbf{c}$ is parallel to $\mathbf{a}+3\mathbf{b}$.
9(ii)	$\left\|\mathbf{a}+3\mathbf{b}\right\|^2=\left(\mathbf{a}+3\mathbf{b}\right)\cdot\left(\mathbf{a}+3\mathbf{b}\right)$ $=\mathbf{a}\cdot\mathbf{a}+6\mathbf{a}\cdot\mathbf{b}+9\mathbf{b}\cdot\mathbf{b}$ $=\left\|\mathbf{a}\right\|^2+9\left\|\mathbf{b}\right\|^2+6\mathbf{a}\cdot\mathbf{b}$ $=\left\|\mathbf{a}\right\|^2+9\left\|\mathbf{b}\right\|^2+6\left(\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|\cos 60°\right)$ $=1+1+6\left(\left\|\mathbf{a}\right\|\frac{\left\|\mathbf{a}\right\|}{3}\frac{1}{2}\right)=3$ Thus $\left\|\mathbf{a}+3\mathbf{b}\right\|=\sqrt{3}$.
9(iii)	Since $\mathbf{a}+3\mathbf{b}$ makes an angle of $60°$, $120°$ and $135°$ with the x-, y- and z-axes respectively, then we have - $\left(\mathbf{a}+3\mathbf{b}\right)\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=\left\|\mathbf{a}+3\mathbf{b}\right\|\left\|1\right\|\cos 60°$ - $\left(\mathbf{a}+3\mathbf{b}\right)\bullet\begin{pmatrix}0\\1\\0\end{pmatrix}=\left\|\mathbf{a}+3\mathbf{b}\right\|\left\|1\right\|\cos 120°$ - $\left(\mathbf{a}+3\mathbf{b}\right)\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=\left\|\mathbf{a}+3\mathbf{b}\right\|\left\|1\right\|\cos 135°$

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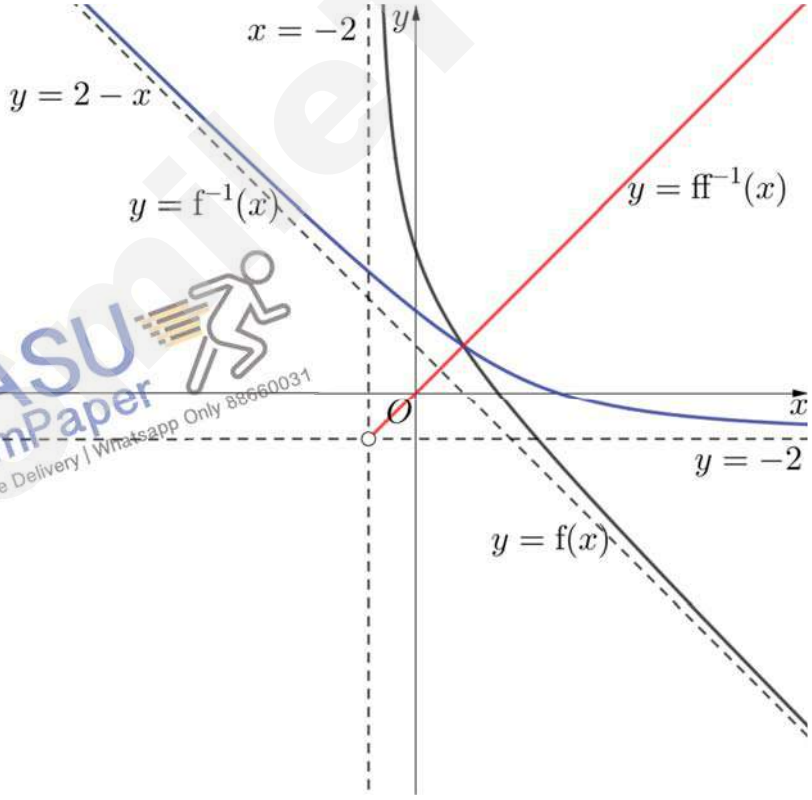
Qn	Suggested Solutions
	The LHS of all 3 expressions simplify to be the x-, y- and z-components of $\mathbf{a}+3\mathbf{b}$ respectively. With the information of part **(ii)**, the RHS of the expressions simplify to $\sqrt{3}\cos 60° = \frac{\sqrt{3}}{2}$, $\sqrt{3}\cos 120° = -\frac{\sqrt{3}}{2}$ and $\sqrt{3}\cos 135° = -\sqrt{\frac{3}{2}}$. Therefore $\mathbf{a}+3\mathbf{b} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \\ -\sqrt{\frac{3}{2}} \end{pmatrix} = \frac{\sqrt{3}}{2}\begin{pmatrix} 1 \\ -1 \\ -\sqrt{2} \end{pmatrix}$ Since $\mathbf{c}$ is parallel to $\mathbf{a}+3\mathbf{b}$ which can be expressed as $\frac{\sqrt{3}}{2}\begin{pmatrix} 1 \\ -1 \\ -\sqrt{2} \end{pmatrix}$, then $\mathbf{c}$ is parallel to $\mathbf{i}-\mathbf{j}-\sqrt{2}\mathbf{k}$.
10(i)	Minimum value of $k = -2$
10(ii)	

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Qn	Suggested Solutions
10(iii)	$\mathrm{g}(x) = x^2 - 6x + a$ $= (x-3)^2 + (a-9)$ $R_g = [a-9, \infty)$ $\quad D_f = (-2, \infty)$ For fg to exist, $R_g \subseteq D_f$ $a - 9 > -2$ $a > 7$ Thus, smallest integer $a = 8$ When $a = 8$, $R_g = [-1, \infty)$ $\mathrm{f}(-1) = 2 - (-1) + \dfrac{8}{-1+2} = 3 + 8 = 11$ Hence range of fg is $(-\infty, 11]$.
10(iv)	$\mathrm{fg}(x) + \mathrm{g}(x) \le 4$ $2 - \mathrm{g}(x) + \dfrac{8}{\mathrm{g}(x)+2} + \mathrm{g}(x) \le 4$ $2 - \dfrac{8}{\mathrm{g}(x)+2} \ge 0$ Since $\mathrm{g}(x) + 2 = x^2 - 6x + 12 = (x-3)^2 + 3 > 0$ for all real values of x, $2(x^2 - 6x + 12) - 8 \ge 0$ $2x^2 - 12x + 16 \ge 0$ $x^2 - 6x + 8 \ge 0$ $(x-2)(x-4) \ge 0$ $\therefore x \le 2 \quad \text{or} \quad x \ge 4$ In addition, the domain of g is $x > 0$. Thus, the range of x is $0 < x \le 2 \quad \text{or} \quad x \ge 4$
11(i)	$A(-3, -4, 11)$, $B(1, -2, 0)$, $C(-5, 2, -1)$, $M(3, 6, 15)$ $\overrightarrow{AB} = 4\mathbf{i} + 2\mathbf{j} - 11\mathbf{k}$ $\overrightarrow{BC} = -6\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ $\overrightarrow{AC} = -2\mathbf{i} + 6\mathbf{j} - 12\mathbf{k}$ $\overrightarrow{AM} = 6\mathbf{i} + 10\mathbf{j} + 4\mathbf{k}$ $\overrightarrow{AM} \bullet \overrightarrow{AB} = \begin{pmatrix} 6 \\ 10 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 2 \\ -11 \end{pmatrix} = 24 + 20 - 44 = 0$ $\overrightarrow{AM} \bullet \overrightarrow{BC} = \begin{pmatrix} 6 \\ 10 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} = -36 + 40 - 4 = 0$ Since $\overrightarrow{AM}$ is perpendicular to 2 of the vectors lying on p, then $\overrightarrow{AM}$ is perpendicular to p.

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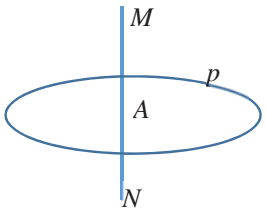
11(ii)

By ratio theorem,

$\overrightarrow{OA} = \frac{1}{2}\left(\overrightarrow{OM} + \overrightarrow{ON}\right)$ or $\overrightarrow{AN} = \overrightarrow{MA}$

$$\overrightarrow{ON} = 2\begin{pmatrix}-3\\-4\\11\end{pmatrix} - \begin{pmatrix}3\\6\\15\end{pmatrix} = \begin{pmatrix}-9\\-14\\7\end{pmatrix}$$

$N(-9, -14, 7)$

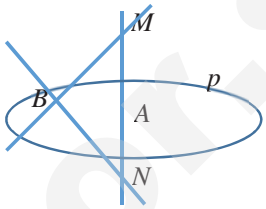


11(iii)

$$\overrightarrow{BN} = \begin{pmatrix}-9\\-14\\7\end{pmatrix} - \begin{pmatrix}1\\-2\\0\end{pmatrix} = \begin{pmatrix}-10\\-12\\7\end{pmatrix}$$

The equation of the reflected line is

$$\mathbf{r} = \begin{pmatrix}1\\-2\\0\end{pmatrix} + \lambda\begin{pmatrix}10\\12\\-7\end{pmatrix}, \lambda \in \mathbb{R}.$$



11(iv)

Since M lies on q, we can sub $x = 3,\ y = 6,\ z = 15$ into q

$$\begin{pmatrix}3\\6\\15\end{pmatrix} \bullet \begin{pmatrix}11-k\\6-3k\\-1+k\end{pmatrix} = 4k+4 \text{ or } 11x + 6y - z - 4 + k\left(-x - 3y + z - 4\right) = 0$$

$$33 + 36 - 15 - 4 + k\left(-3 - 18 + 15 - 4\right) = 0$$

$$50 - 10k = 0$$

$$k = 5$$

Sub $k = 5$ and the equation of q is $6x - 9y + 4z = 24$

11(v)

Since $\overrightarrow{AM}$ is perpendicular to p, the equation of p is

$$3x + 5y + 2z = -7$$

Using GC, we can find the line of intersection of p and q to be

$$\mathbf{r} = \begin{pmatrix}1\\-2\\0\end{pmatrix} + \alpha\begin{pmatrix}-\frac{2}{3}\\0\\1\end{pmatrix} \text{ or } \mathbf{r} = \begin{pmatrix}1\\-2\\0\end{pmatrix} + \alpha\begin{pmatrix}-2\\0\\3\end{pmatrix}.$$

Note: $B(1, -2, 0)$ is on line of intersection of p and q, so $\overrightarrow{NB} = \begin{pmatrix}-10\\-12\\7\end{pmatrix}$ and $N(-9, -14, 7)$

reflection of M is on q.

The vector equation of the reflected plane is

$$\mathbf{r} = \begin{pmatrix}1\\-2\\0\end{pmatrix} + \alpha\begin{pmatrix}-2\\0\\3\end{pmatrix} + \beta\begin{pmatrix}10\\12\\-7\end{pmatrix}$$

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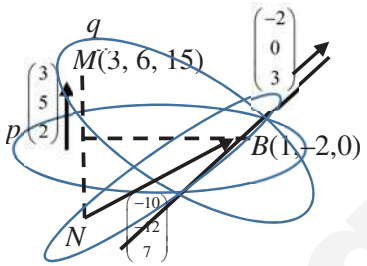
$$\begin{pmatrix}-2\\0\\3\end{pmatrix}\times\begin{pmatrix}10\\12\\-7\end{pmatrix}=\begin{pmatrix}0-36\\30-14\\-24-0\end{pmatrix}=\begin{pmatrix}-36\\16\\-24\end{pmatrix}=-4\begin{pmatrix}9\\-4\\6\end{pmatrix}$$

$$\mathbf{r}\bullet\begin{pmatrix}9\\-4\\6\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}\bullet\pm\begin{pmatrix}9\\-4\\6\end{pmatrix}=9+8+0$$

$$\mathbf{r}\bullet\begin{pmatrix}9\\-4\\6\end{pmatrix}=17$$

A cartesian equation of the reflected plane is $9x-4y+6z=17$.



12(i)

$$I(a)=\frac{2\left(2a+\mathrm{f}(a)\right)}{2a\mathrm{f}(a)}$$

$$=\frac{2a+\mathrm{f}(a)}{a\mathrm{f}(a)}$$

$$=\frac{2a}{a\mathrm{f}(a)}+\frac{\mathrm{f}(a)}{a\mathrm{f}(a)}$$

$$=\frac{2}{\mathrm{f}(a)}+\frac{1}{a}$$

$$=\frac{2}{9-\frac{a^2}{6}}+\frac{1}{a}$$

$$=\frac{12}{54-a^2}+\frac{1}{a}$$

12(ii)

$$\frac{\mathrm{d}I}{\mathrm{d}a}=\frac{(-1)(12)(-2a)}{\left(54-a^2\right)^2}-\frac{1}{a^2}$$

When $\frac{\mathrm{d}I}{\mathrm{d}a}=0$, we have

$$\frac{(-1)(12)(-2a)}{\left(54-a^2\right)^2}=\frac{1}{a^2}$$

$$24a^3=\left(54-a^2\right)^2$$

$$\sqrt{24}a^{\frac{3}{2}}=54-a^2 \qquad (54-a^2>0\because a^2\le 36)$$

$$a^2+\sqrt{24}a^{\frac{3}{2}}-54=0$$

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12(iii)

$a^2+\sqrt{24}a^{\frac{3}{2}}-54=0$

Using GC, $a=3.947$

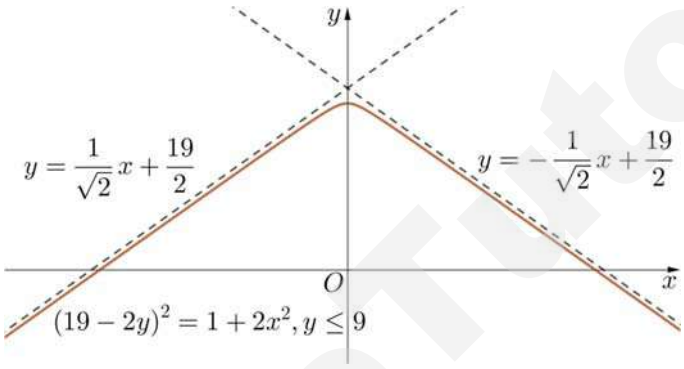
	3	3.947	4
$\frac{dI}{da}$	$-\frac{17}{225}$	0	$\frac{23}{5776}$
Shape	\	_	/

By the first derivative test, 3.947 gives the minimum value of I.

Alternatively,

By the second derivative test, $\left.\frac{d^2I}{da^2}\right|_{x=3.947}=0.0752>0$, thus 3.947 gives the minimum value of I.

12(iv)



12(v)

$$\frac{\text{area of } ABCD}{\text{area of } ABECD}=\frac{2\times 3.947\times\left(\frac{19-\sqrt{1+2(3.947)^2}}{2}\right)}{\int_{-3.947}^{3.947}\frac{19-\sqrt{1+2x^2}}{2}\,dx}$$

$$=\frac{52.61047582}{62.94592466}$$

$$=0.835804321$$

This implies that the living space takes up 83.6%, which satisfy the condition, thus the developer will take up the suggestion.

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2019 H2 Mathematics Promotion Examination: Solutions for Students

1 Find the equation of the tangent to the curve $y^3 - 2xy^2 + 3x^2 - 3 = 0$ at the point (2, 3). [3]

2 An arithmetic series has first term a and common difference d, where a and d are non-zero. The 21st and 53rd terms of the arithmetic series are 91 and 155 respectively. Given also that the sum of its first n terms is 6600. Find the values of a, d and n. [4]

3 The sequence $u_1, u_2, u_3, \ldots$ is given by $u_n = \tan(n+2)\tan(n+3)$ for $n \geq 1$.

(i) By considering $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, show that

$$u_n = \frac{\tan(n+3) - \tan(n+2)}{\tan 1} - 1.$$ [1]

(ii) Hence find $\sum_{r=2}^{n} u_r$ in terms of n. [3]

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4 It is given that $\mathrm{f}(x) = \ln(1+\mathrm{e}^{x})$. Without using a calculator, find the Maclaurin series for $\mathrm{f}(x)$, up to and including the term in x^{2}. [5]

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5 Given **a**, **b** and **c** are unit vectors such that $\mathbf{a}\cdot\mathbf{b} = 2\mathbf{a}\cdot\mathbf{c}$, what is the relationship between **a** and $\mathbf{b} - 2\mathbf{c}$? [2]

It is further given that **b** is perpendicular to $\mathbf{b} - 2\mathbf{c}$. Find the angle between **b** and **c**. [2]

Comment on the relationship between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} - 2\mathbf{c}$. [1]

6 An **open** tin box of negligible thickness is to be made. The design of the box and its horizontal base are shown below.

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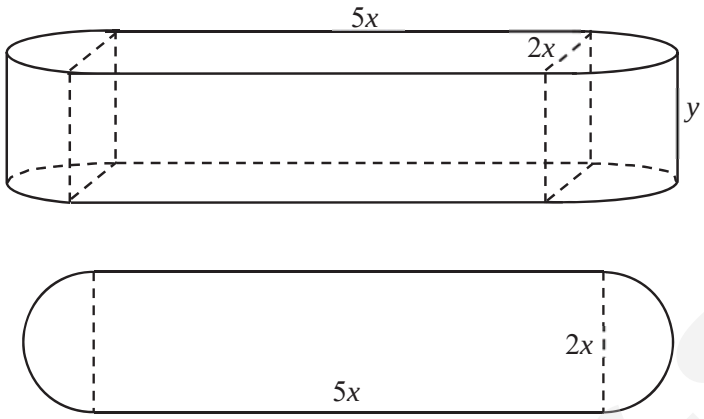

The middle portion of the horizontal base of the box is a rectangle of length $5x$ cm and width $2x$ cm while the two ends are semicircles of radius x cm. The box has a depth of y cm and its volume is 800 cm^3.

Show that the total external surface area, A in cm^2, of the box is given by

$$A = (\pi + 10)x^2 + \frac{1600(\pi + 5)}{(\pi + 10)x}.$$

Use differentiation to find the value of x which minimizes A. [6]

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7 A curve C has equation $y = \dfrac{x^2 + ax + b}{x + c}$, where a, b and c are constants. It is given that C has a stationary point at $(0, 2)$ and the line $x = 2$ is an asymptote to C.

(i) Find the values of a, b and c. [4]

(ii) Sketch the graph of C, stating the equations of any asymptotes and the coordinates of any axial intercepts and/or turning points. [3]

8 **(a)** Without using a calculator, solve the inequality $\dfrac{2x^2 - x}{x^2 + 3x - 4} > 1$. [4]

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(b) **(i)** On the same axes, sketch the graphs of $y = 2 + \frac{a}{x}$ and $y = 2 - |x|$, where a is a constant such that $1 < a < 2$. [3]

(ii) Hence, or otherwise, solve the inequality $2 + \frac{a}{x} < 2 - |x|$. [2]

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9 On 1 January 2018, Gary loans \$$A$ from a bank which charges compound interest at a rate of r% per month at the end of each month. Gary intends to fully repay the loan in n months with a fixed monthly instalment of \$$P_n$ which he pays on the first day of each subsequent month.

Show that $P_n = \frac{A(R-1)R^n}{R^n - 1}$ where $R = 1 + \frac{r}{100}$. [3]

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(i) Show that $\frac{P_{2n}}{P_n} = \frac{R^n}{R^n + 1}$. [1]

(ii) It is given that $r = 0.55$, find the least integral number of months, n, for which the ratio $\frac{P_{2n}}{P_n}$ is greater than $\frac{3}{5}$. [2]

(iii) Show that the ratio $\frac{P_{2n}}{P_n}$ is always greater than $\frac{1}{2}$, and explain what this statement means in the context of the question. [3]

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10 The complex numbers z and w are such that

$$z = -1 + \mathrm{i}a \quad \text{and} \quad w = b + \mathrm{i}$$

where a and b are real numbers.

(i) Given that $zz^* + w^2 = 4 - 2\mathrm{i}$, where z^* is the complex conjugate of z, find the exact values of a and b. [4]

(ii) Given instead that $a = -\frac{1}{\sqrt{3}}$ and $b = 1$, find $\left|\frac{z}{w}\right|$ and show that $\arg\left(\frac{z}{w}\right) = \frac{11\pi}{12}$. [4]

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Without using a calculator, find the smallest positive whole number n for which $\left(\frac{z}{w}\right)^n$ is purely imaginary. [2]

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11 The function f is defined by

$$\mathrm{f}: x \mapsto x^2 - mx, \qquad x \in \mathbb{R}, \ x \geq \frac{m}{2},$$

where m is a positive constant.

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. [3]

(ii) Sketch on the same diagram the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, showing clearly the graphical relationship between the two graphs. [2]

(iii) Find the value of m such that the curves in part **(ii)** intersect at the point where $x = 4$. [2]

In the rest of the question, the value of m is given to be 1.

The function g is defined by

$$\mathrm{g}: x \mapsto \ln x, \qquad x \in \mathbb{R}, \ x \geq \sqrt{\mathrm{e}}.$$

(iv) Find an expression for $\mathrm{fg}(x)$ and hence, or otherwise, find the exact value of $(\mathrm{fg})^{-1}(2)$. [3]

(v) Solve the inequality $\mathrm{fg}(x) > 5 - 0.1x$. [2]

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12	The Instrument Landing System (ILS) is used in many airports around the world. It is a precision runway approach aid based on two radio beams transmitters which together provide pilots with both the vertical and horizontal guidance during an approach to land. The ILS works with a localizer (LOC) that provides the guidance for horizontal planar movement of the aircraft, while a second transmitter known as the glideslope (GS) defines

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the correct vertical planar descent. The final approach of the aircraft onto the runway is guided by the trajectory defined by the intersection between the beams of LOC and GS.

The origin O is taken to be the base of the LOC and the ground is the xy-plane. The beams of LOC and GS are defined by planes with equations $\mathbf{r}\cdot\begin{pmatrix}1\\0\\-18\end{pmatrix}=1$ and $\mathbf{r}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=0$ respectively.

(i) Find a vector equation of the line l which the final approach trajectory lies on. [2]

(ii) Find the angle that l makes with the ground. [2]

Due to an emergency, aircraft A located at a point P with coordinates $\left(\frac{170}{9}, 1, 3\right)$ was redirected back to the airport. To determine if the aircraft is at a safe distance from the landing strip, the pilot needs to know how far the aircraft is from l.

(iii) Show that the coordinates of the point F on l such that F is closest to A is $(19, 0, 1)$. [3]

Aircraft B took off from the same airport but from a different runway. The flight path of B lies on the line l_B with equation $\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$.

(iv) Without using a calculator, show that the lines PF and l_B do not intersect each other. [3]

(v) The shortest distance between 2 skew lines l_1 and l_2 is given by the length of projection of $\overrightarrow{XY}$ onto $\mathbf{n}$ where X and Y are points on l_1 and l_2 respectively, and $\mathbf{n}$ is a vector perpendicular to both l_1 and l_2. Using this result, find the shortest distance between lines PF and l_B. [3]

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13 Taking O to be the origin of Cartesian coordinates, the end A of a thin inextensible string AB of length π units is fixed to a point $(-1, 0)$ on the circumference of a circle of unit radius and centre O. Initially, the end B is at $(-1, \pi)$ and the string is straight and tangent to the circle (see Fig. 1). The string remains taut and is then wrapped round the circle until the end B comes into contact with the circle.

By considering the coordinates of the point F and the length of FB (see Fig. 2) or otherwise, show that the path of B can be described by the curve C with parametric equations

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad \text{for} \quad 0 \leq t \leq \pi,$$

where t is the angle in radians as shown in Fig. 2. [4]

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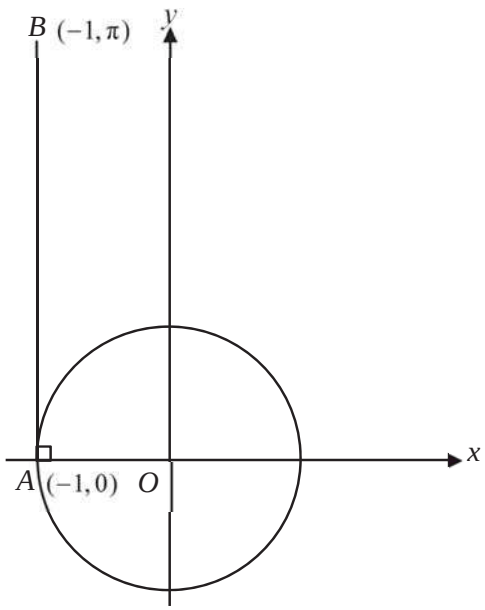

Fig. 1

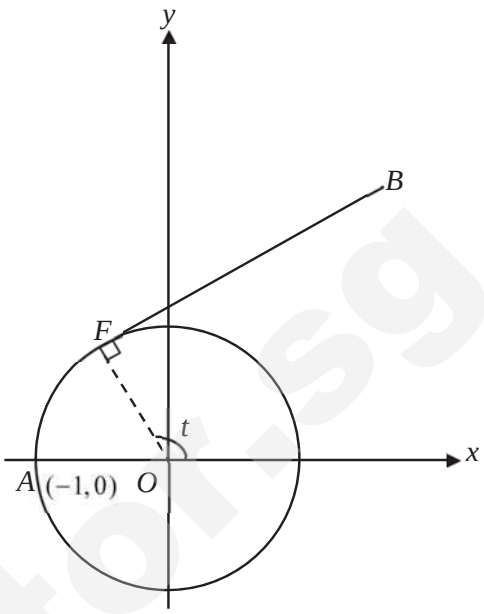

Fig. 2

(i) Find the exact value of t for which x takes its maximum value on C and sketch C. [5]

(ii) The point P on the curve C has parameter p where $0 < p < \pi$. The normal to C at the point P meets the y-axis at the point with coordinates $(0, k)$. Find k in terms of p and show that k is no less than one. [4]

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2019 H2 Mathematics Promotion Examination: Solutions for Students

1 Find the equation of the tangent to the curve $y^3 - 2xy^2 + 3x^2 - 3 = 0$ at the point (2, 3). [3]

1 [3]

$y^3 - 2xy^2 + 3x^2 - 3 = 0$

Differentiating implicitly wrt x,

$$3y^2\frac{dy}{dx} - 2\left(y^2 + 2xy\frac{dy}{dx}\right) + 6x = 0$$

Apply product rule and chain rule to differentiate xy^2

Apply chain rule to differentiate y^3

When $x = 2,\ y = 3$

$$3(3^2)\frac{dy}{dx} - 2\left(3^2 + 2(2)(3)\frac{dy}{dx}\right) + 6(2) = 0$$

$$3\frac{dy}{dx} = 6 \Rightarrow \frac{dy}{dx} = 2$$

The equation of the tangent is $y - 3 = 2(x - 2)$

i.e. $y = 2x - 1$

Read question carefully: find the equation of the tangent line, not find gradient at (2, 3) only.

2 An arithmetic series has first term a and common difference d, where a and d are non-zero. The 21st and 53rd terms of the arithmetic series are 91 and 155 respectively. Given also that the sum of its first n terms is 6600. Find the values of a, d and n. [4]

2 [4]

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = 6600$$

$n^2d + 2an - nd - 13200 = 0$ - (1)

$a + 20d = 91$ - (2)

$a + 52d = 155$ - (3)

Solving (2) and (3) using GC, $a = 51$ and $d = 2$.

Sub a and d into (1),

$2n^2 + 2n(51) - 2n - 13200 = 0$

$n^2 + 50n - 6600 = 0$

Using GC, $n = 60$ (–110 not accepted as $n > 0$).

$u_n = a + (n-1)d$

$S_n = \frac{n}{2}[2a + (n-1)d]$

No need to solve the equations by hand. Use GC to evaluate a, d and n.

3 The sequence $u_1, u_2, u_3, \ldots$ is given by $u_n = \tan(n+2)\tan(n+3)$ for $n \geq 1$.

(i) By considering $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, show that

$$u_n = \frac{\tan(n+3) - \tan(n+2)}{\tan 1} - 1.$$ [1]

(ii) Hence find $\sum_{r=2}^{n} u_r$ in terms of n. [3]

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3(i) **[1]**	$u_n = \tan(n+2)\tan(n+3)$ $\tan\left((n+3)-(n+2)\right) = \dfrac{\tan(n+3)-\tan(n+2)}{1+\tan(n+3)\tan(n+2)}$ $\tan(1) = \dfrac{\tan(n+3)-\tan(n+2)}{1+u_n}$ $(1+u_n)\tan(1) = \tan(n+3)-\tan(n+2)$ $u_n = \dfrac{\tan(n+3)-\tan(n+2)}{\tan 1} - 1$ (shown)	This is a SHOW question so detailed working should be shown. Show explicitly $\tan(A-B)$ $= \tan((n+2)-(n+3))$ $= \tan(-1) = -\tan 1$ or $\tan(A-B)$ $= \tan((n+3)-(n+2))$, $= \tan 1$ depending on was chosen for *A* and *B*.
(ii) **[3]**	$\sum_{r=2}^{n} u_r = \sum_{r=2}^{n}\left[\dfrac{\tan(r+3)-\tan(r+2)}{\tan 1} - 1\right]$ $= \dfrac{1}{\tan 1}\sum_{r=2}^{n}\left[\tan(r+3)-\tan(r+2)\right] - \sum_{r=2}^{n} 1$ $= \dfrac{1}{\tan 1}[\ \tan 5 - \tan 4$ $+ \tan 6 - \tan 5$ $+ \tan 7 - \tan 6$ $+ \ldots$ $+ \tan(n+2) - \tan(n+1)$ $+ \tan(n+3) - \tan(n+2)] - (n-1)$ $= \dfrac{\tan(n+3)-\tan 4}{\tan 1} + 1 - n$	Sufficient working needs to be shown (first 3 and last 2 rows). There is no need to evaluate tan 1 and tan 4. Note: the values are evaluated in radians instead of degree, so $\tan 4 = 1.15782$ $\tan 1 = 1.55741$ Alternative method : $\sum_{r=2}^{n} u_r$ $= \sum_{r=1}^{n} u_r - \left(\dfrac{\tan 4 - \tan 3}{\tan 1} - 1\right)$

4	It is given that $f(x) = \ln(1+e^x)$. Without using a calculator, find the Maclaurin series for $f(x)$, up to and including the term in x^2. [5]
4 **[5]**	$f(x) = \ln(1+e^x)$ $f'(x) = \dfrac{e^x}{1+e^x} = 1 - \dfrac{1}{1+e^x}$ $f''(x) = \dfrac{e^x}{(1+e^x)^2}$ When $x = 0$, $f(0) = \ln 2$, $f'(0) = \dfrac{1}{2}$, $f''(0) = \dfrac{1}{4}$. Thus the Maclaurin series for $f(x)$ is $\ln 2 + \dfrac{1}{2}x + \dfrac{\frac{1}{4}}{2!}x^2 + \ldots = \ln 2 + \dfrac{1}{2}x + \dfrac{1}{8}x^2 + \ldots$ *Alternative:*

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Let $y = \ln(1+e^x)$

$$e^y = 1 + e^x$$

$$e^y \frac{dy}{dx} = e^x$$

$$e^y \frac{d^2y}{dx^2} + e^y \left(\frac{dy}{dx}\right)^2 = e^x$$

When $x = 0,\ y = \ln 2,\ \frac{dy}{dx} = \frac{1}{2},\ \frac{d^2y}{dx^2} = \frac{1}{4}$.

Thus the Maclaurin series for $f(x)$ is $\ln 2 + \frac{1}{2}x + \frac{\frac{1}{4}}{2!}x^2 + \ldots = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$

Alternative:

Using the standard series expansions of e^x and $\ln(1+x)$,

$$f(x) = \ln(1+e^x)$$

$$= \ln\left(1 + \left(1 + x + \frac{x^2}{2} + \ldots\right)\right)$$

$$= \ln\left(2\left(1 + \frac{x}{2} + \frac{x^2}{4} + \ldots\right)\right)$$

$$= \ln 2 + \ln\left(1 + \frac{x}{2} + \frac{x^2}{4} + \ldots\right)$$

$$= \ln 2 + \left(\frac{x}{2} + \frac{x^2}{4} + \ldots\right) - \frac{1}{2}\left(\frac{x}{2} + \frac{x^2}{4} + \ldots\right)^2 + \ldots$$

$$= \ln 2 + \frac{x}{2} + \frac{x^2}{4} - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \ldots$$

$$= \ln 2 + \frac{x}{2} + \frac{x^2}{8} + \ldots$$

Note : When using the standard series, the following will ***not*** work.

$$\ln(1+e^x) = e^x - \frac{(e^x)^2}{2} + \frac{(e^x)^3}{3} + \ldots = \left(1 + x + \frac{x^2}{2!} + \ldots\right) - \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \ldots\right)^2 + \ldots$$

$$\ln(1+e^x) = \ln(1 + 1 + x + \frac{x^2}{2!} + \ldots) = \left(1 + x + \frac{x^2}{2!} + \ldots\right) - \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \ldots\right)^2 + \ldots$$

In each expansion, all the brackets have a constant term, x term, x^2 term etc., so we will not be able to collect all the respective terms totally.

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5	Given **a**, **b** and **c** are unit vectors such that **a·b = 2a·c**, what is the relationship between **a** and **b** – 2**c**? [2] It is further given that **b** is perpendicular to **b** – 2**c**. Find the angle between **b** and **c**. [2] Comment on the relationship between $\mathbf{a} \times \mathbf{b}$ and **b** – 2**c**. [1]	
5 **[2]**	$\mathbf{a} \cdot \mathbf{b} = 2\mathbf{a} \cdot \mathbf{c}$ $\mathbf{a} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{c} = 0$ $\mathbf{a} \cdot (\mathbf{b} - 2\mathbf{c}) = 0$ Therefore **a** is perpendicular to $\mathbf{b} - 2\mathbf{c}$ since $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq 2\mathbf{c}$.	Dot product is distributive, i.e., $\mathbf{a} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} - 2\mathbf{c})$ Note: $\mathbf{b} \neq 2\mathbf{c}$ as **b** and **c** are unit vectors.
[2]	Let θ be the angle between **b** and **c**. Given **b** is perpendicular to $\mathbf{b} - 2\mathbf{c}$, $\mathbf{b} \cdot (\mathbf{b} - 2\mathbf{c}) = 0$ $\mathbf{b} \cdot \mathbf{b} = 2\mathbf{b} \cdot \mathbf{c}$ $\lvert\mathbf{b}\rvert^2 = 2\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert\cos\theta$ $\cos\theta = \frac{1}{2}$ (since **b** and **c** are unit vectors) $\theta = \frac{\pi}{3}$ radians	Read question carefully to make use of $\lvert\mathbf{b}\rvert = \lvert\mathbf{c}\rvert = 1$.
[1]	Since **a** and **b** are perpendicular to **b** – 2**c**, and $\mathbf{a} \times \mathbf{b}$ is perpendicular to **a** and **b**, $\mathbf{a} \times \mathbf{b}$ is parallel to **b** – 2**c**. Some things to note : • $\mathbf{u} \perp \mathbf{w}$ and $\mathbf{v} \perp \mathbf{w}$ does ***not*** imply **u** and **v** are parallel (e.g., $\mathbf{i} \perp \mathbf{k}$ and $\mathbf{j} \perp \mathbf{k}$ but **i** and **j** are not parallel) • We do not have sufficient information to conclude whether the vectors are in the same direction, or in the opposite direction, or are equal. We can only conclude that the vectors are parallel (i.e. may be in the same or opposite direction, and may not be of the same magnitude).	

6	An **open** tin box of negligible thickness is to be made. The design of the box and its horizontal base are shown below.

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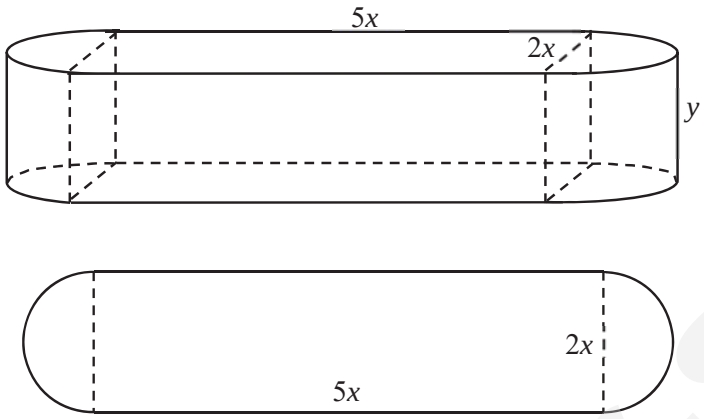

The middle portion of the horizontal base of the box is a rectangle of length $5x$ cm and width $2x$ cm while the two ends are semicircles of radius x cm. The box has a depth of y cm and its volume is 800 cm^3.

Show that the total external surface area, A in cm^2, of the box is given by

$$A = (\pi + 10)x^2 + \frac{1600(\pi + 5)}{(\pi + 10)x}.$$

Use differentiation to find the value of x which minimizes A. [6]

6 [6]

Since the volume of the box is 800 cm^3,

$$(\pi x^2 + 10x^2)y = 800 \Rightarrow y = \frac{800}{(\pi + 10)x^2}$$

Let the external surface area be A cm^2.

$$\begin{aligned} A &= 10x^2 + 10xy + \pi x^2 + 2\pi xy \\ &= (\pi + 10)x^2 + 2(\pi + 5)xy \\ &= (\pi + 10)x^2 + 2(\pi + 5)x\left[\frac{800}{(\pi + 10)x^2}\right] \\ &= (\pi + 10)x^2 + \frac{1600(\pi + 5)}{(\pi + 10)x} \end{aligned}$$

$$\frac{dA}{dx} = 2(\pi + 10)x - \frac{1600(\pi + 5)}{(\pi + 10)x^2} = 0$$

$$2(\pi + 10)x = \frac{1600(\pi + 5)}{(\pi + 10)x^2}$$

$$x^3 = \frac{800(\pi + 5)}{(\pi + 10)^2}$$

$$x = \left[\frac{800(\pi + 5)}{(\pi + 10)^2}\right]^{\frac{1}{3}} = 3.3535 = 3.35 \text{ (correct to 3 s.f.)}$$

To show that x minimizes A, it is recommended that the Second Derivative Test is used.
It is not easy to justify using the First Derivative Test because it is not easy to show whether the sign of

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	$\frac{d^2A}{dx^2} = 2(\pi+10) + \frac{3200(\pi+5)}{(\pi+10)x^3} > 0$ at $x^3 = \frac{800(\pi+5)}{(\pi+10)^2}$ Thus the external surface area is minimum at $x = 3.35$.	$\frac{dA}{dx} = 2(\pi+10)x - \frac{1600(\pi+5)}{(\pi+10)x^2}$ is positive or negative for 3.35^- or 3.35^+

7 A curve C has equation $y = \frac{x^2+ax+b}{x+c}$, where a, b and c are constants. It is given that C has a stationary point at $(0,2)$ and the line $x = 2$ is an asymptote to C.

(i) Find the values of a, b and c. [4]

(ii) Sketch the graph of C, stating the equations of any asymptotes and the coordinates of any axial intercepts and/or turning points. [3]

7(i) [4]

Since $x = 2$ is an asymptote to the curve, $c = -2$.

Also, C has a stationary point at $(0,2)$, $(0,2)$ lies on the curve and $\frac{dy}{dx} = 0$ at this point.

Sub $(0,2)$ into $y = \frac{x^2+ax+b}{x-2}$, $2 = \frac{b}{-2} \Rightarrow b = -4$

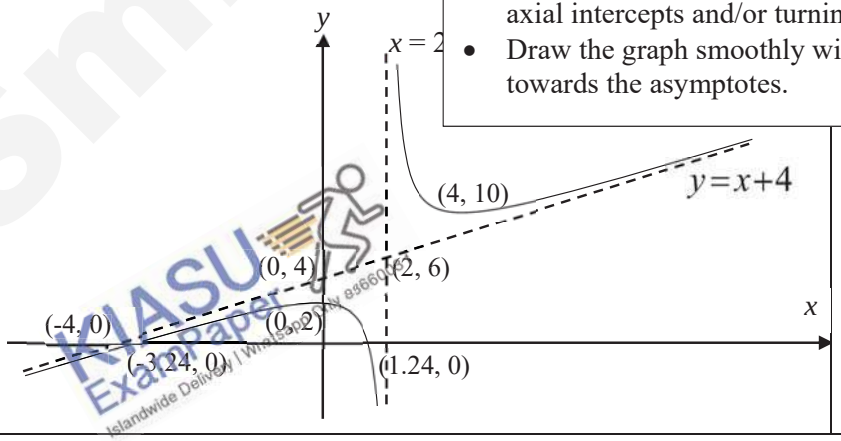
$y = \frac{x^2+ax-4}{x-2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{(2x+a)(x-2)-(x^2+ax-4)}{(x-2)^2} = \frac{x^2+4-2a}{(x-2)^2}$

$\frac{dy}{dx} = \frac{-2a+4}{(-2)^2} = 0 \Rightarrow a = 2$

(ii) [3]

$y = \frac{x^2+2x-4}{x-2} = x+4+\frac{4}{x-2}$

Oblique asymptote is $y = x+4$



Note :
- Be aware that rational functions should take up 2 'portions'. So you will need to adjust the window in your GC when you only see one portion.
- Label any asymptotes and the coordinates of any axial intercepts and/or turning points.
- Draw the graph smoothly with the end tending towards the asymptotes.

8 **(a)** Without using a calculator, solve the inequality $\frac{2x^2-x}{x^2+3x-4} > 1$. [4]

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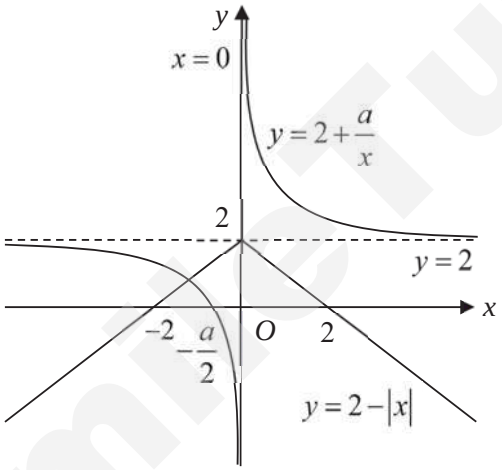
	(b) **(i)** On the same axes, sketch the graphs of $y = 2 + \frac{a}{x}$ and $y = 2 - \lvert x \rvert$, where a is a constant such that $1 < a < 2$. [3] **(ii)** Hence, or otherwise, solve the inequality $2 + \frac{a}{x} < 2 - \lvert x \rvert$. [2]	
8(a) **[4]**	$\frac{2x^2 - x}{x^2 + 3x - 4} > 1$ $\frac{2x^2 - x - (x^2 + 3x - 4)}{x^2 + 3x - 4} > 0$ $\frac{x^2 - 4x + 4}{x^2 + 3x - 4} > 0$ $\frac{(x-2)^2}{(x-1)(x+4)} > 0$ Since $(x-2)^2 > 0 \quad \forall\, x \in \mathbb{R} \setminus \{2\}$, $(x-1)(x+4) > 0, \quad x \neq 2$ $\therefore x < -4$ or $1 < x < 2$ or $x > 2$	Take note of inequality sign so as to exclude $x = 2$
(b) **(i)** **[3]**		The restriction $1 < a < 2$ should be taken into account when sketching the graph of $y = 2 + \frac{a}{x}$. Since the x-intercept is at $x = -\frac{a}{2}$ and $1 < a < 2 \Rightarrow -1 < -\frac{a}{2} < -\frac{1}{2}$, the curve should cut the x-axis at a point to the right of $x = -2$ (one of the x-intercepts of the other graph $y = 2 - \lvert x \rvert$).
(b) **(ii)** **[2]**	At point of intersection, $2 + \frac{a}{x} = 2 + x$ $x^2 = a$ $x = -\sqrt{a} \quad (\because x < 0)$ The solution to the inequality is $-\sqrt{a} < x < 0$.	Note that the graphs intersect at the point where $x < 0$ and so we have $\lvert x \rvert = -x$ at the intersection.

9	On 1 January 2018, Gary loans \$$A$ from a bank which charges compound interest at a rate of r% per month at the end of each month. Gary intends to fully repay the loan in n months with a fixed monthly instalment of \$$P_n$ which he pays on the first day of each subsequent month. Show that $P_n = \frac{A(R-1)R^n}{R^n - 1}$ where $R = 1 + \frac{r}{100}$. [3]

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(i) Show that $\frac{P_{2n}}{P_n} = \frac{R^n}{R^n + 1}$. [1]

(ii) It is given that $r = 0.55$, find the least integral number of months, n, for which the ratio $\frac{P_{2n}}{P_n}$ is greater than $\frac{3}{5}$. [2]

(iii) Show that the ratio $\frac{P_{2n}}{P_n}$ is always greater than $\frac{1}{2}$, and explain what this statement means in the context of the question. [3]

9 [3]

Month	Beginning of month	End of month
0 (Jan18)	A	AR
1 (Feb18)	$AR - P_n$	$AR^2 - P_nR$
2 (Mar18)	$AR^2 - P_nR - P_n$	$AR^3 - P_nR^2 - P_nR$
	…	…
n months later	$AR^n - P_nR^{n-1} - P_nR^{n-2} - ... - P_n$	

Given such word problems, it is best to create a table to identify the pattern.

Since loan is fully repaid in n months,

$$AR^n - P_nR^{n-1} - P_nR^{n-2} - ... - P_n = 0$$

$$AR^n - P_n\left(R^{n-1} + R^{n-2} + ... + 1\right) = 0$$

$$P_n\left(\frac{1\left(1-R^n\right)}{1-R}\right) = AR^n$$

$$P_n = \frac{AR^n\left(1-R\right)}{1-R^n}$$

$$P_n = \frac{AR^n\left(R-1\right)}{R^n-1} \quad \text{(shown)}$$

(i) [1]

$$\frac{P_{2n}}{P_n} = \frac{AR^{2n}\left(R-1\right)}{R^{2n}-1} \times \frac{R^n-1}{AR^n\left(R-1\right)}$$

$$= \frac{R^n\left(R^n-1\right)}{\left(R^{2n}-1\right)}$$

$$= \frac{R^n\left(R^n-1\right)}{\left(R^n-1\right)\left(R^n+1\right)}$$

$$= \frac{R^n}{R^n+1} \quad \text{(shown)}$$

Need to show explicitly how to factorise the denominator and cancel the terms as this is a SHOW question.

(ii) [2]

Given $r = 0.55$, to find n such that ratio is larger than $\frac{3}{5}$

$$R = 1 + \frac{r}{100} = 1.0055$$

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	$\frac{R^n}{R^n+1} > \frac{3}{5}$ $5R^n > 3R^n + 3 \quad (\text{since } R^n + 1 > 0)$ $R^n > \frac{3}{2}$ $n > \frac{\ln \frac{3}{2}}{\ln 1.0055} = 73.9$ Thus least n is 74. Alternatively, $\frac{R^n}{R^n+1} > \frac{3}{5}$ Using graphing calculator, When $n = 73$, $\frac{R^n}{R^n+1} = 0.59878 < \frac{3}{5}$ When $n = 74$, $\frac{R^n}{R^n+1} = 0.60010 > \frac{3}{5}$ Least $n = 74$	When using the GC to solve the inequality, there should be an attempt to show clearly why the least n is 74.
(iii) [3]	Since $R = 1 + \frac{r}{100} > 1 \quad (\text{since } r > 0)$ $R^n > 1$ $R^n + R^n > R^n + 1$ $\frac{2R^n}{R^n+1} > 1$ $\frac{R^n}{R^n+1} > \frac{1}{2}$ Hence $\frac{P_{2n}}{P_n}$ is always greater than $\frac{1}{2}$. If Gary intends to stretch his loan term to twice as long, the monthly instalment would be more than half of what he would need to pay if he stayed on with his initial loan term.	

10 The complex numbers z and w are such that

$$z = -1 + \mathrm{i}a \quad \text{and} \quad w = b + \mathrm{i}$$

where a and b are real numbers.

(i) Given that $zz^* + w^2 = 4 - 2\mathrm{i}$, where z^* is the complex conjugate of z, find the exact values of a and b. [4]

(ii) Given instead that $a = -\frac{1}{\sqrt{3}}$ and $b = 1$, find $\left|\frac{z}{w}\right|$ and show that $\arg\left(\frac{z}{w}\right) = \frac{11\pi}{12}$. [4]

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Without using a calculator, find the smallest positive whole number n for which $\left(\frac{z}{w}\right)^n$ is purely imaginary. [2]

10 (i) [4]

$zz^* + w^2 = 4 - 2\mathrm{i}$

$(-1+\mathrm{i}a)(-1-\mathrm{i}a) + (b+\mathrm{i})^2 = 4 - 2\mathrm{i}$

$1 + a^2 + b^2 + 2b\mathrm{i} - 1 = 4 - 2\mathrm{i}$

Comparing real and imaginary parts,

$a^2 + b^2 = 4$ and $2b = -2 \Rightarrow b = -1$

$a^2 = 3$

$a = \pm\sqrt{3}$

(ii) [4]

$$\frac{z}{w} = \frac{-1-\mathrm{i}\frac{1}{\sqrt{3}}}{1+\mathrm{i}} = \frac{\frac{2}{\sqrt{3}}\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}}{\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}\right)}}$$

$$= \frac{2}{\sqrt{6}}\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}-\frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{6}}{3}\mathrm{e}^{\mathrm{i}\left(-\frac{13\pi}{12}+2\pi\right)}$$

$$= \frac{\sqrt{6}}{3}\mathrm{e}^{\mathrm{i}\left(\frac{11\pi}{12}\right)}$$

Thus $\left|\frac{z}{w}\right| = \frac{\sqrt{6}}{3}$ and $\arg\left(\frac{z}{w}\right) = \frac{11\pi}{12}$ (shown).

Alternative Method:

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{\frac{2}{\sqrt{3}}}{\sqrt{2}} = \frac{\sqrt{6}}{3}$$

$$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w) + 2\pi$$

$$= -\frac{5\pi}{6} - \frac{\pi}{4} + 2\pi$$

$$= -\frac{13\pi}{12} + 2\pi = \frac{11\pi}{12} \text{ (shown)}$$

Note the presentation for the alternative method as it is mathematically incorrect to write

$$\arg\left(\frac{z}{w}\right) = -\frac{13\pi}{12} = \frac{11\pi}{12}$$

[2]

For $\left(\frac{z}{w}\right)^n$ to be purely imaginary, we require

$$\frac{11n\pi}{12} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$$

$$n = \frac{6}{11}, \frac{18}{11}, \frac{30}{11}, \frac{42}{11}, \frac{54}{11}, 6, \ldots$$

The smallest positive whole number n is 6.

$\left(\frac{z}{w}\right)^n$ purely imaginary implies that

$$\cos\left(\frac{11n\pi}{12}\right) = 0$$

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11 The function f is defined by

$$\mathrm{f}: x \mapsto x^2 - mx, \qquad x \in \mathbb{R}, \ x \geq \frac{m}{2},$$

where m is a positive constant.

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. [3]

(ii) Sketch on the same diagram the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, showing clearly the graphical relationship between the two graphs. [2]

(iii) Find the value of m such that the curves in part **(ii)** intersect at the point where $x = 4$. [2]

In the rest of the question, the value of m is given to be 1.

The function g is defined by

$$\mathrm{g}: x \mapsto \ln x, \qquad x \in \mathbb{R}, \ x \geq \sqrt{\mathrm{e}}.$$

(iv) Find an expression for $\mathrm{fg}(x)$ and hence, or otherwise, find the exact value of $(\mathrm{fg})^{-1}(2)$. [3]

(v) Solve the inequality $\mathrm{fg}(x) > 5 - 0.1x$. [2]

---

11 (i) [3]

Let $y = x^2 - mx = \left(x - \frac{m}{2}\right)^2 - \frac{m^2}{4}$

$$\left(x - \frac{m}{2}\right)^2 = y + \frac{m^2}{4}$$

$$x = \frac{m}{2} \pm \sqrt{y + \frac{m^2}{4}}$$

Since $x \geq \frac{m}{2}$, $\mathrm{f}^{-1}(x) = \frac{m}{2} + \sqrt{x + \frac{m^2}{4}}$

Domain of f^{-1} = Range of f = $[-\frac{m^2}{4}, \infty)$

To make x the subject, we can either apply 'complete the square' (as shown) or consider the quadratic equation of x, $x^2 - mx - y = 0$, and 'solve' for x using the quadratic formula.

Explanation should be given for the choice of $\mathrm{f}^{-1}(x)$.

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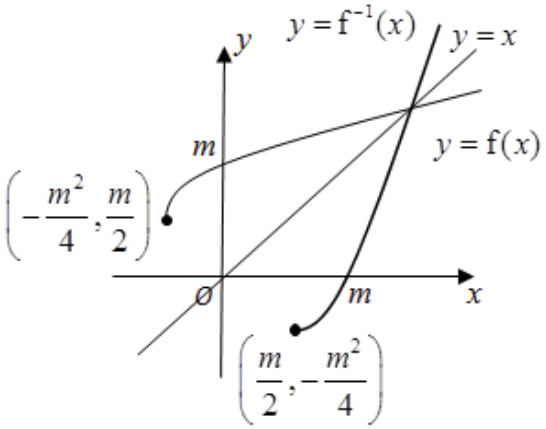
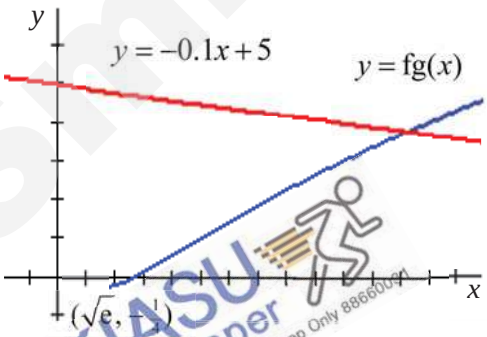
(ii) **[2]**		When drawing graph of f, the domain MUST be considered. The end-points should be indicated clearly.
(iii) **[2]**	Since the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect along the line $y = x$, the point of intersection is (4, 4). $f(4) = 4$ $4^2 - 4m = 4$ $m = 3$	
(iv) **[3]**	$fg(x) = (\ln x)^2 - \ln x$ Let $(fg)^{-1}(2) = k \Rightarrow fg(k) = 2$ for $k \geq \sqrt{e}$ $(\ln k)^2 - \ln k = 2$ $(\ln k - 2)(\ln k + 1) = 0$ $\ln k = 2$ or $\ln k = -1$ $k = e^2$ or $k = e^{-1}$ (N.A. $\because k \geq \sqrt{e}$) $\therefore (fg)^{-1}(2) = e^2$	Note : $(\ln x)^2 \neq 2\ln x$ $(fg)^{-1} \neq f^{-1}g^{-1}$
(v) **[2]**	$(\ln x)^2 - \ln x > 5 - 0.1x$  From GC, $x > 12.3$ (3 s.f.)	Note : $D_{fg} = D_g$, so consider only the graph of fg for $x \geq \sqrt{e}$ The GC should be used to solve the inequality. A rough sketch of the graph should be presented in the solution to support the answer from the GC. Writing down the answer without any working is a 'high-risk" strategy.

12	The Instrument Landing System (ILS) is used in many airports around the world. It is a precision runway approach aid based on two radio beams transmitters which together provide pilots with both the vertical and horizontal guidance during an approach to land. The ILS works with a localizer (LOC) that provides the guidance for horizontal planar movement of the aircraft, while a second transmitter known as the glideslope (GS) defines

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the correct vertical planar descent. The final approach of the aircraft onto the runway is guided by the trajectory defined by the intersection between the beams of LOC and GS.

The origin O is taken to be the base of the LOC and the ground is the xy-plane. The beams of LOC and GS are defined by planes with equations $\mathbf{r}\cdot\begin{pmatrix}1\\0\\-18\end{pmatrix}=1$ and $\mathbf{r}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=0$ respectively.

(i) Find a vector equation of the line l which the final approach trajectory lies on. [2]

(ii) Find the angle that l makes with the ground. [2]

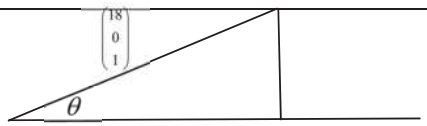
Due to an emergency, aircraft A located at a point P with coordinates $\left(\frac{170}{9}, 1, 3\right)$ was redirected back to the airport. To determine if the aircraft is at a safe distance from the landing strip, the pilot needs to know how far the aircraft is from l.

(iii) Show that the coordinates of the point F on l such that F is closest to A is $(19, 0, 1)$. [3]

Aircraft B took off from the same airport but from a different runway. The flight path of B lies on the line l_B with equation $\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$.

(iv) Without using a calculator, show that the lines PF and l_B do not intersect each other. [3]

(v) The shortest distance between 2 skew lines l_1 and l_2 is given by the length of projection of $\overrightarrow{XY}$ onto $\mathbf{n}$ where X and Y are points on l_1 and l_2 respectively, and $\mathbf{n}$ is a vector perpendicular to both l_1 and l_2. Using this result, find the shortest distance between lines PF and l_B. [3]

12 (i) [2]	$\mathbf{r}\cdot\begin{pmatrix}1\\0\\-18\end{pmatrix}=1$ and $\mathbf{r}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=0$ $x-18z=1 \quad ---(1)$ $y=0 \quad ---(2)$ Using the GC, Line of intersection is $l:\underset{\sim}{r}=\begin{pmatrix}1\\0\\0\end{pmatrix}+t\begin{pmatrix}18\\0\\1\end{pmatrix}$, $t\in\mathbb{R}$	Note the correct presentation for equation of a line. It is '$\underset{\sim}{r}=$' instead of '$l=$'.
(ii)		

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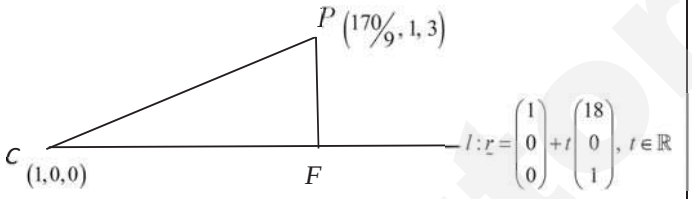
[2]	$\theta = \tan^{-1}\left(\frac{1}{18}\right) = 3.2^{\circ}$ Alternative Method: $\theta = \sin^{-1}\frac{\begin{pmatrix}18\\0\\1\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{18^2+1^2}} = \sin^{-1}\left(\frac{1}{\sqrt{325}}\right) = 3.2^{\circ}$	As the ground is taken to be the *xy* plane, a vector perpendicular to the ground is **k**.
(iii) **[3]**	 $l: \underset{\sim}{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}18\\0\\1\end{pmatrix},\ t \in \mathbb{R}$ Let *F* be point on line *l*. $\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP} = \begin{pmatrix}1+18t-170/9\\-1\\t-3\end{pmatrix}$ $\overrightarrow{PF} \perp l \Rightarrow \begin{pmatrix}18t-161/9\\-1\\t-3\end{pmatrix}\bullet\begin{pmatrix}18\\0\\1\end{pmatrix} = 0$ $324t - 322 + t - 3 = 0$ $t = 1$ Therefore, *F* is $(19,\ 0,\ 1)$	
(iv) **[3]**	$\overrightarrow{PF} = \begin{pmatrix}1/9\\-1\\-2\end{pmatrix}$ Equation of line PF is $\underset{\sim}{r} = \begin{pmatrix}19\\0\\1\end{pmatrix} + \mu\begin{pmatrix}1/9\\-1\\-2\end{pmatrix},\ \mu \in \mathbb{R}$ At point of intersection,	The $\underset{\sim}{r}$ in the equation of line PF represents the position vectors of all the points on the line while vector $\overrightarrow{PF}$ is just the vector itself. So line

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	$\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}19\\0\\1\end{pmatrix}+\mu\begin{pmatrix}1/9\\-1\\-2\end{pmatrix}$ $\lambda-\frac{1}{9}\mu=18 \quad -(1)$ $\mu=-1 \quad -(2)$ $\lambda+2\mu=0 \quad -(3)$ From (2), $\mu=-1$ (3)-(1), $\frac{19}{9}\mu=-18\Rightarrow\mu=-\frac{162}{19}$ Since there is no unique value of μ satisfying all 3 equations, the lines *PF* and l_B do not intersect each other.	PF and vector $\overrightarrow{PF}$ are NOT the same, so simply solving $\overrightarrow{PF}=\mathbf{i}+\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}+\mathbf{k})$ is incorrect. Since this is a SHOW question, the GC cannot be used to solve the equations.
(v) **[3]**	vector perpendicular to line *PF* and $l_B=\begin{pmatrix}1/9\\-1\\-2\end{pmatrix}\times\begin{pmatrix}1\\0\\1\end{pmatrix}$ $=\begin{pmatrix}-1\\-19/9\\1\end{pmatrix}=-\frac{1}{9}\begin{pmatrix}9\\19\\-9\end{pmatrix}$ Let $X(1,1,1)$ and $Y(19,0,1)$ be points on l_B and line *PF* respectively. $\overrightarrow{XY}=\begin{pmatrix}19-1\\0-1\\1-1\end{pmatrix}=\begin{pmatrix}18\\-1\\0\end{pmatrix}$ Shortest distance $=\frac{\left\|\begin{pmatrix}18\\-1\\0\end{pmatrix}\bullet\begin{pmatrix}9\\19\\-9\end{pmatrix}\right\|}{\sqrt{9^2+19^2+(-9)^2}}$ $=\frac{143}{\sqrt{523}}=6.25$ (correct to 3 s.f.)	

13 Taking *O* to be the origin of Cartesian coordinates, the end *A* of a thin inextensible string *AB* of length π units is fixed to a point $(-1,0)$ on the circumference of a circle of unit radius and centre *O*. Initially, the end *B* is at $(-1,\pi)$ and the string is straight and tangent to the circle (see Fig. 1). The string remains taut and is then wrapped round the circle until the end *B* comes into contact with the circle.

By considering the coordinates of the point *F* and the length of *FB* (see Fig. 2) or otherwise, show that the path of *B* can be described by the curve *C* with parametric equations

$$x=\cos t+t\sin t,\quad y=\sin t-t\cos t,\quad \text{for}\quad 0\le t\le\pi,$$

where *t* is the angle in radians as shown in Fig. 2. [4]

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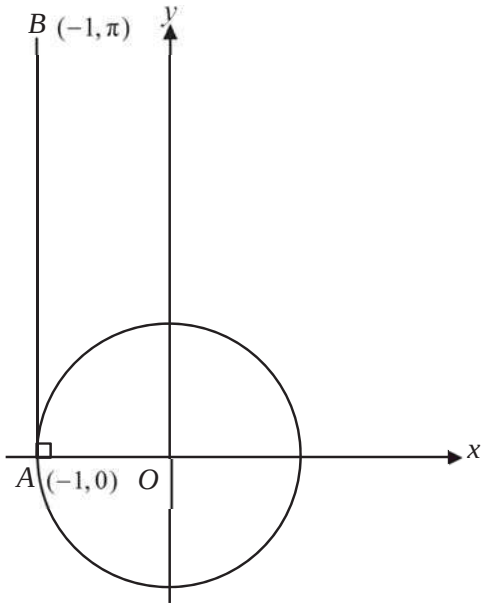


Fig. 1

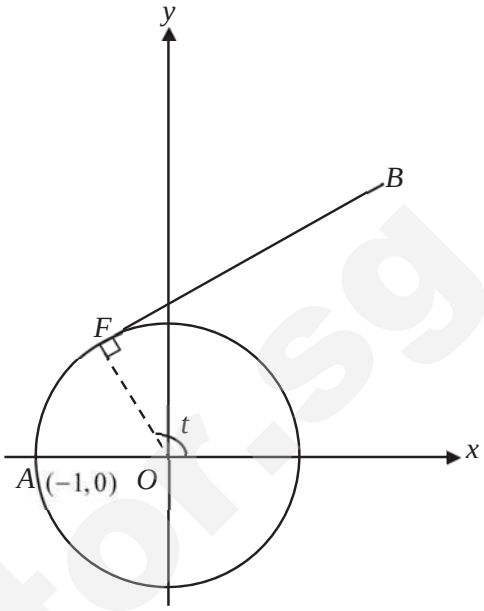
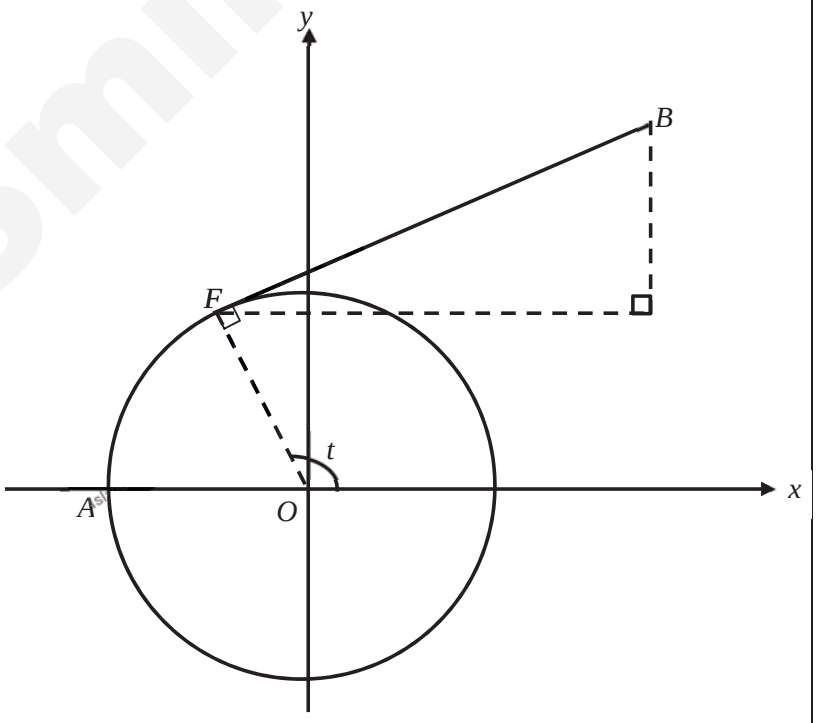


Fig. 2

(i) Find the exact value of t for which x takes its maximum value on C and sketch C. [5]

(ii) The point P on the curve C has parameter p where $0 < p < \pi$. The normal to C at the point P meets the y-axis at the point with coordinates $(0, k)$. Find k in terms of p and show that k is no less than one. [4]

13 **[4]**



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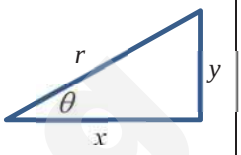
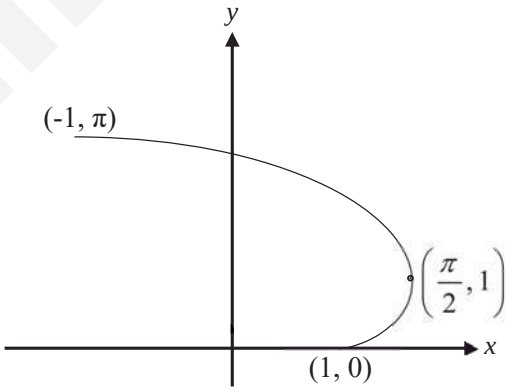
	Coordinates of point F are $(\cos t, \sin t)$. Also, angle $AOF = \pi - t$ and length of arc $AF = \pi - t$. Thus length $FB = t$ and the angle that FB makes with the horizontal is $t - \frac{\pi}{2}$. Hence the x-coordinate of B $= \cos t + t\cos\left(t - \frac{\pi}{2}\right) = \cos t + t\cos\left(\frac{\pi}{2} - t\right) = \cos t + t\sin t$ and the y-coordinate of B $= \sin t + t\sin\left(t - \frac{\pi}{2}\right) = \sin t - t\sin\left(\frac{\pi}{2} - t\right) = \sin t - t\cos t$	Here, make use of $x = r\cos\theta$ and $y = r\sin\theta$ in the triangle below. 
(i) **[5]**	$x = \cos t + t\sin t$ $\frac{dx}{dt} = -\sin t + \sin t + t\cos t = t\cos t$ Apply product rule to differentiate $t\sin t$ For maximum x, $\frac{dx}{dt} = t\cos t = 0$ $t = 0$ or $\cos t = 0 \Rightarrow t = \frac{\pi}{2}$ $\frac{d^2x}{dt^2} = \cos t - t\sin t$ When $t = 0$, $\frac{d^2x}{dt^2} = 1 > 0$. When $t = \frac{\pi}{2}$, $\frac{d^2x}{dt^2} = -\frac{\pi}{2} < 0$. Thus x is maximum when $t = \frac{\pi}{2}$. 	Need to solve $\frac{dx}{dt} = 0$ instead of $\frac{dy}{dx} = 0$ Label points clearly. Consider the required domain of t when drawing the curve (set domain in GC, under 'window').
(ii) **[4]**	$y = \sin t - t\cos t$ $\frac{dy}{dt} = \cos t - (\cos t - t\sin t) = t\sin t$ Thus, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t\sin t}{t\cos t} = \tan t$. At point $P(\cos p + p\sin p, \sin p - p\cos p)$,	

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gradient of normal $= -\cot p$

Equation of normal at P is
$y - (\sin p - p\cos p) = (-\cot p)(x - (\cos p + p\sin p))$

When $x = 0,\ y = k$

$$\begin{aligned} k &= \cot p\cos p + p\cos p + \sin p - p\cos p \\ &= \cot p\cos p + \sin p \\ &= \frac{\cos^2 p}{\sin p} + \sin p \\ &= \frac{\cos^2 p + \sin^2 p}{\sin p} \\ &= \frac{1}{\sin p} \end{aligned}$$

For $0 < p < \pi$, $0 < \sin p \le 1$ and thus $k = \frac{1}{\sin p} \ge 1$.

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2019 JC1 Promotional Examination
Higher 2

NAME	

CLASS						INDEX NUMBER		

MATHEMATICS 9758/01

Paper 1 **3 October 2019**

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your class, index number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
You are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
Up to **2 marks may be deducted** for poor presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
Total	

Calculator Model:

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1 u_n is a cubic polynomial in terms of n, where n is a positive integer. It is known that $u_0 = 0$, $u_1 = u_2 = 8$ and u_4 is four times of u_1. Find u_n in terms of n. [4]

2 Show that $\sin(n+1) - \sin(n-1) = k\cos n$, where k is an exact constant to be determined. [2]

Hence or otherwise, find $\cos(1) + \cos(2) + \cos(3) + ... + \cos N$ in terms of N, expressing your answer in the form $a(\sin(N+1) + \sin N) + b$, where a and b are constants to be determined. [4]

3 **(i)** By means of the substitution $x = \sqrt{12}\sin\theta$, show that

$$\int_0^{\sqrt{3}} \sqrt{12 - x^2}\, dx = \pi + \frac{3}{2}\sqrt{3}.$$ [4]

(ii) The region R is bounded by the curve $y = \sqrt[4]{12 - x^2}$, the lines $x = \sqrt{3}$ and $x = -\sqrt{3}$, and the x-axis. Using your result in part **(i)**, find the exact volume of the solid formed by rotating R through 2π radians about the x-axis. [3]

4 **(a)** Express $x + 3 = A(2x + 2) + B$, where A and B are constants to be determined.

Hence, find $\int \frac{x+3}{x^2 + 2x + 2}\, dx$. [3]

(b) The Mean Value Theorem for Integrals states that if f is continuous on a closed interval $[a, b]$, then there is at least one point $x = c$ in that interval such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx$, where f($c$) is known as the mean value of f on the interval $[a, b]$.

Let $f(x) = xe^x$. Given that $d > 2$, determine the exact value of d such that the mean value of f on the interval $[1, d]$ is f(2). [5]

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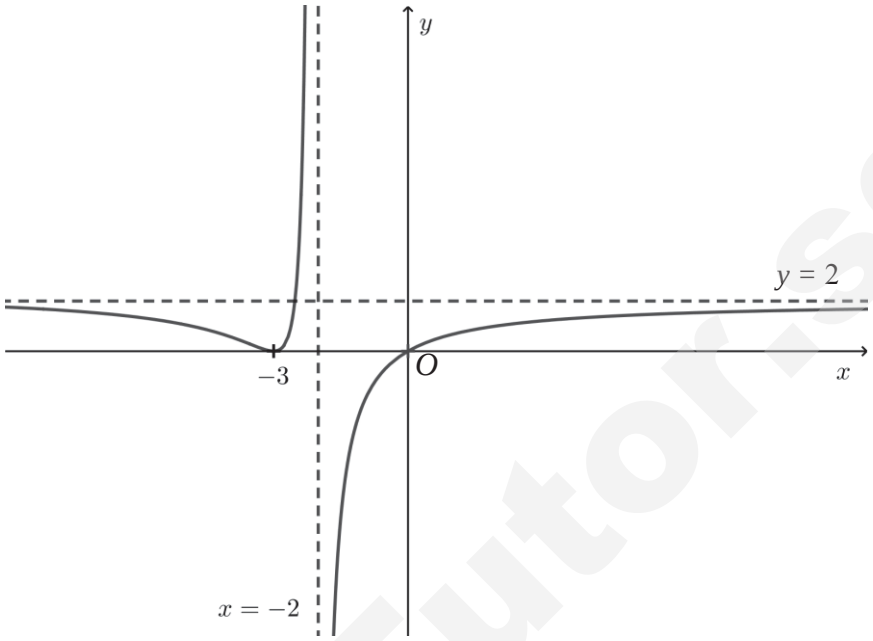
5 **(a)** The diagram shows the graph of $y = \mathrm{f}(x)$. The curve passes through the origin, has a minimum point at $(-3, 0)$ and its equations of asymptotes are $x = -2$ and $y = 2$.



Sketch the following graphs on separate axes.

(i) $y = \dfrac{1}{\mathrm{f}(x)}$ [3]

(ii) $y = \mathrm{f}(-|x|)$ [2]

(b) Describe a series of transformations that transforms the graph of $x^2 + (y+1)^2 = 1$ onto the graph of $(x-1)^2 + \dfrac{y^2}{9} = 1$. [3]

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6 The "Nephroid" (also known as a two-cusped epicycloid) is a kidney-shaped curve often seen on the surface of a cup of coffee in the sunshine – a crescent of light formed by sunlight reflecting off the inside of the cup onto the surface of the drink. More generally, it is the shape made by parallel rays of light reflecting from the inside of any semi-circle[1].

A particular Nephroid curve C has the following parametric equations:

$$x = 3\cos t - \cos 3t$$
$$y = 3\sin t - \sin 3t, \text{ where } 0 \le t < 2\pi.$$

(i) Sketch C for $0 \le t < 2\pi$, indicating clearly the axial intercepts. [1]

(ii) Find the exact equation of the normal to C at the point P where $t = \frac{\pi}{3}$. [4]

(iii) Hence, find the area of the region in the first quadrant bounded by C, the normal in part **(ii)** and the positive y-axis. [3]

7 Given that $xy = 1 - y^2$, where $y > 0$, show that

$$(x+2y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0. \quad [3]$$

Write down the Maclaurin series for y up to and including the term in x^2. [2]

Hence, by using the Maclaurin series for y found above and the standard series found in MF26, find the Maclaurin series for $\frac{1}{y}$ up to and including the term in x^2. [3]

8 The curve C has equation $y = \frac{x^2 + ax + 6}{x - 2}$, where a is a constant. It is given that the line $y = x + 5$ is an asymptote of C.

(i) Show that $a = 3$. [2]

(ii) Prove algebraically that y cannot lie between -1 and 15. [2]

[1] http://www.daviddarling.info/encyclopedia/N/nephroid.html

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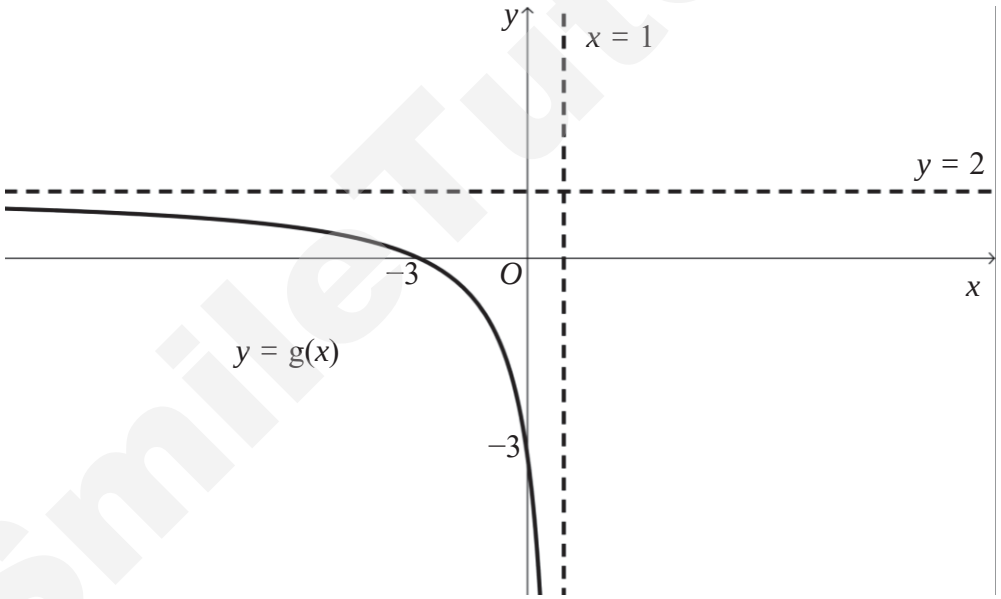
(iii) Sketch the curve C, stating clearly the coordinates of any points of intersection with the axes, the coordinates of any stationary points and the equations of asymptotes. [3]

(iv) Hence, solve the inequality $\dfrac{x^2+3x+6}{x-2} \geq -x-3$. [2]

9 The function f is defined as follows

$$\mathrm{f}: x \mapsto x-\sqrt{x^2+1}, \quad x \in \mathbb{R}.$$

The graph of function g with domain $(-\infty, 1)$ is given in the diagram below. It has asymptotes $x=1$ and $y=2$, and cuts the axes at $(0,-3)$ and $(-3, 0)$.



(a) **(i)** Show that f^{-1} exists. [2]

(ii) Show that gf exists and find the range of gf. [4]

(b) Sketch the graph of $y=\mathrm{g}(-x-1)$. [3]

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10 **(a)** Given that the sum of the first n terms of a sequence is n^2, show that the sequence is an arithmetic progression. [3]

(b) Given $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, find $\sum_{r=0}^{2n}\left(nr^2 + 2^{2r}\right)$, leaving your answer in terms of n. [3]

(c) Cauchy's Root Test states that a series of the form $\sum_{r=1}^{\infty} a_r$ converges when $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} < 1$ and diverges when $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} > 1$. When $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = 1$, the test is inconclusive. Using this test, explain why $\sum_{r=1}^{\infty} \frac{1}{\left(x^2+2\right)^r}$ converges for all real values of x and find the sum to infinity of this series. [3]

11 A tank contains 20 kg of salt dissolved in 5000 *l* of water. Salt solution that contains 0.03 kg of salt per litre of water is pumped into the tank at a rate of 25 *l* per minute. The solution in the tank is kept thoroughly mixed and is drained from the tank at a rate of 25 *l* per minute. Let x be the amount of salt, in kg, in the tank at time t minutes.

(i) Show that $\frac{dx}{dt} = \frac{150 - x}{200}$. [3]

(ii) Find the amount of salt in the tank at time t. [5]

(iii) Find the amount of salt in the tank eventually. [1]

(iv) Sketch the graph for the amount of salt in the tank over time. Comment on the suitability of the model. [3]

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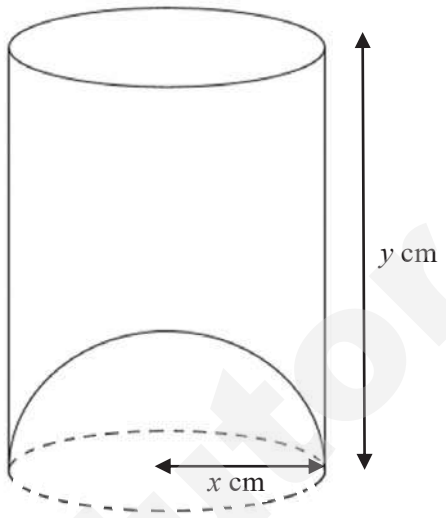
12 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

(a) An open metallic cup has a cylindrical body with height y cm and base radius x cm. The base of the can is shaped as a hemisphere, which shares the same diameter as the cylinder, as shown in the diagram.



The total area of metal sheet used to make the cup is $40\pi^2$ cm^2.

(i) Show that the volume V cm^3 of the cup is given by $V = \pi x\left(20\pi - \frac{5}{3}x^2\right)$. [3]

(ii) Find the exact value of x for which V has a maximum value. Hence, find the maximum volume, giving your answer correct to 2 decimal places. [4]

(iii) State an assumption made when carrying out your calculations. [1]

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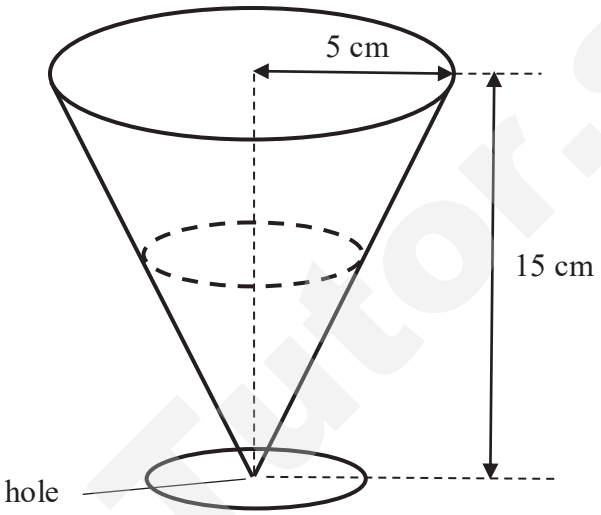
(b) Due to poor sales, the manufacturer decides to revamp the design of the metallic cup. The newly designed cup consists of an inverted right cone of radius 5 cm and height 15 cm, with its vertex fixed to a circular base. Before mass production, the manufacturer produced a sample cup and tested it by filling the cup completely with water. The sample cup was found to be defective as water leaked out of a hole at the vertex of the cone at a rate of 2 cm^3 per second. The diagram below shows the cup.



Find the rate at which the water level is decreasing when the water is 3 cm deep. [4]

- End of Paper -

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1	Solution [4]	
	Let $u_n = an^3 + bn^2 + cn + d$ $u_0 = 0 \Rightarrow d = 0$ $u_1 = a + b + c = 8$ ……(1) $u_2 = 8a + 4b + 2c = 8$ ……(2) $u_4 = 4u_1 \Rightarrow 64a + 16b + 4c = 32$ ……..(3) Using GC, $a = 2, b = -10, c = 16$ Hence, the cubic polynomial is: $u_n = 2n^3 - 10n^2 + 16n$	

2	Solution [6]	
	$\sin(n+1) - \sin(n-1)$ $= 2\cos\left(\frac{(n+1)+(n-1)}{2}\right)\sin\left(\frac{(n+1)-(n-1)}{2}\right)$ $= 2\sin(1)\cos n$ (shown, where $k = 2\sin(1)$) Alternatively, $\sin(n+1) - \sin(n-1)$ $= \sin n\cos(1) + \cos n\sin(1) - (\sin n\cos(1) - \cos n\sin(1))$ $= 2\sin(1)\cos n$ (shown, where $k = 2\sin(1)$)	

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$\cos(1)+\cos(2)+\cos(3)+\ldots+\cos N$

$=\sum_{n=1}^{N}\cos n$

$=\frac{1}{2\sin(1)}\sum_{n=1}^{N}\left[\sin(n+1)-\sin(n-1)\right]$

$=\frac{1}{2\sin(1)}\left[\cancel{\sin(2)}-\sin(0)\right.$

$+\cancel{\sin(3)}-\sin(1)$

$+\cancel{\sin(4)}-\cancel{\sin(2)}$

$+\cancel{\sin(5)}-\cancel{\sin(3)}$

$\vdots$

$\vdots$

$+\cancel{\sin(N-1)}-\cancel{\sin(N-3)}$

$+\ \sin(N)\ \ -\cancel{\sin(N-2)}$

$\left.+\sin(N+1)-\cancel{\sin(N-1)}\ \right]$

$=\frac{1}{2\sin(1)}\left[\sin(N+1)+\sin N-\sin(1)\right]$

$=\frac{1}{2\sin(1)}\left[\sin(N+1)+\sin N\right]-\frac{1}{2}$

$\therefore a=\frac{1}{2\sin(1)}$ (or 0.594 (to 3 s.f.)) , $b=-\frac{1}{2}$

3	Solution [7]	
(i)	$x=\sqrt{12}\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sqrt{12}\cos\theta$ When $x=0,\ \theta=0$ When $x=\sqrt{3}$, $\frac{1}{2}=\sin\theta\Rightarrow\theta=\frac{\pi}{6}$	

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	$\int_0^{\sqrt{3}} \sqrt{12-x^2}\ \mathrm{d}x$ $= \int_0^{\frac{\pi}{6}} \sqrt{12-12\sin^2\theta}\ \sqrt{12}\cos\theta\ \mathrm{d}\theta$ $= 12\int_0^{\frac{\pi}{6}} \cos^2\theta\ \mathrm{d}\theta$ $= 12\int_0^{\frac{\pi}{6}} \frac{\cos 2\theta + 1}{2}\ \mathrm{d}\theta$ $= 6\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$ $= 6\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$ $= \pi + \frac{3}{2}\sqrt{3}$ (shown)	
(ii)	Volume $= \pi\int_{-\sqrt{3}}^{\sqrt{3}} y^2\ \mathrm{d}x$ $= \pi\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{12-x^2}\ \mathrm{d}x$ $= 2\pi\int_0^{\sqrt{3}} \sqrt{12-x^2}\ \mathrm{d}x$ $= 2\pi\left(\pi + \frac{3}{2}\sqrt{3}\right)$ $= \pi\left(2\pi + 3\sqrt{3}\right)$ units^3	

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4	Solution [8]	
	(a) $x+3=A(2x+2)+B$ $A=\frac{1}{2},\ B=2$ $\int \frac{x+3}{x^2+2x+2}\,\mathrm{d}x$ $=\int \frac{\frac{1}{2}(2x+2)+2}{x^2+2x+2}\,\mathrm{d}x$ $=\frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,\mathrm{d}x+2\int \frac{1}{x^2+2x+2}\,\mathrm{d}x$ $=\frac{1}{2}\ln\left\|x^2+2x+2\right\|+2\int \frac{1}{1+(x+1)^2}\,\mathrm{d}x$ $=\frac{1}{2}\ln\left\|x^2+2x+2\right\|+2\tan^{-1}(x+1)+C$	
	(b) $\mathrm{f}(2)=\frac{1}{d-1}\int_1^d x\mathrm{e}^x\,\mathrm{d}x$ $\int_1^d x\mathrm{e}^x\,\mathrm{d}x=\left[x\mathrm{e}^x\right]_1^d-\int_1^d \mathrm{e}^x\,\mathrm{d}x$ $=\left[x\mathrm{e}^x-\mathrm{e}^x\right]_1^d$ $=\mathrm{e}^d(d-1)-(\mathrm{e}-\mathrm{e})$ $=\mathrm{e}^d(d-1)$ $\mathrm{f}(2)=\frac{1}{d-1}\left[\mathrm{e}^d(d-1)\right]=\mathrm{e}^d$ $2\mathrm{e}^2=\mathrm{e}^d$ $\ln \mathrm{e}^d=\ln\left(2\mathrm{e}^2\right)=\ln 2+2$ $d=\ln 2+2$	

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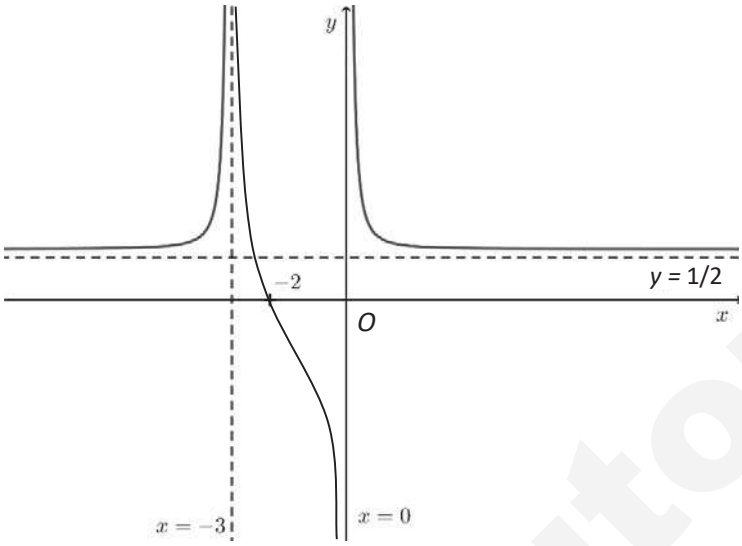
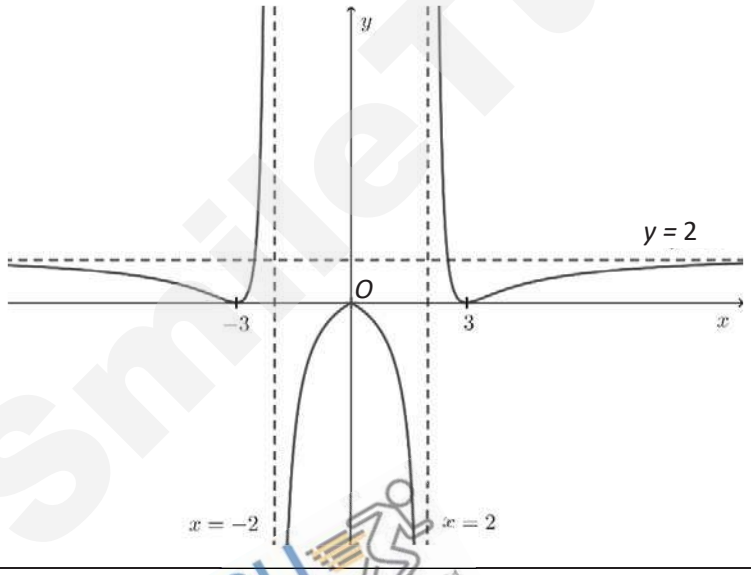
5	Solution [8]	
	(a)(i)  	
	(a)(ii)  	
	(b) Translate 1 unit in the positive x-axis Translate 1 unit in the positive y-axis Scale parallel to the y-axis by factor 3. OR	

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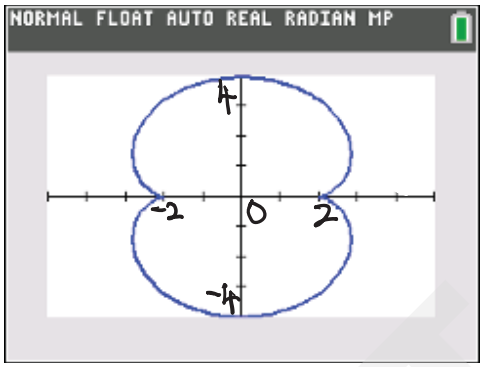
	Translate 1 unit in the positive x-axis Scale parallel to the y-axis by factor 3. Translate 3 units in the positive y-axis (Note: Transformation of x can be at any step)	

6	Solution [8]	
	(i) 	
	(ii) $\frac{dx}{dt} = -3\sin t + 3\sin 3t$ $\frac{dy}{dt} = 3\cos t - 3\cos 3t$ $\frac{dy}{dx} = \frac{3\cos t - 3\cos 3t}{-3\sin t + 3\sin 3t} = \frac{\cos t - \cos 3t}{-\sin t + \sin 3t}$ When $t = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{3\cos\frac{\pi}{3} - 3\cos\pi}{-3\sin\frac{\pi}{3} + 3\sin\pi} = \frac{\frac{3}{2} - 3(-1)}{-3\left(\frac{\sqrt{3}}{2}\right)} = -\sqrt{3}$ Gradient of normal $= \frac{1}{\sqrt{3}}$	

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$x = \frac{5}{2}$, $y = \frac{3\sqrt{3}}{2}$

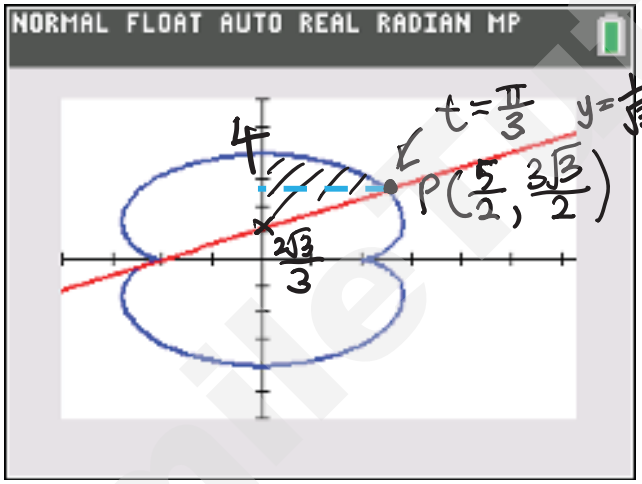
Thus equation of normal at P is

$$y - \frac{3\sqrt{3}}{2} = \frac{1}{\sqrt{3}}\left(x - \frac{5}{2}\right)$$

$$y = \frac{1}{\sqrt{3}}x - \frac{5\sqrt{3}}{6} + \frac{3\sqrt{3}}{2}$$

$$\therefore y = \frac{1}{\sqrt{3}}x + \frac{2\sqrt{3}}{3}$$

(iii)



Area

$$= \frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{3\sqrt{3}}{2} - \frac{2\sqrt{3}}{3}\right) + \int_{\frac{3\sqrt{3}}{2}}^{4} x \, dy$$

$$= \frac{25\sqrt{3}}{24} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (3\cos t - \cos 3t)(3\cos t - 3\cos 3t) \, dt$$

$= 4.30$ units^2

Alternatively,

Area

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	$= \int_0^{\frac{5}{2}} y\ dx - \int_0^{\frac{5}{2}} \frac{1}{\sqrt{3}}x + \frac{2\sqrt{3}}{3}\ dx$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} (3\sin t - \sin 3t)(-3\sin t + 3\sin 3t)\ dx - \int_0^{\frac{5}{2}} \frac{1}{\sqrt{3}}x + \frac{2\sqrt{3}}{3}\ dx$ $= 4.30$ units^2	

7	Solution [8]	
	$xy = 1 - y^2$ $y + x\frac{dy}{dx} = -2y\frac{dy}{dx}$ $y + \frac{dy}{dx}(x + 2y) = 0$ ---------- (1) $\frac{dy}{dx} + (x + 2y)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(1 + 2\frac{dy}{dx}\right) = 0$ $(x + 2y)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 = 0$ (shown) ---------- (2)	
	When $x = 0$, $y = 1$, $\frac{dy}{dx} = -\frac{1}{2}$ and $\frac{d^2y}{dx^2} = \frac{1}{4}$ $y = 1 - \frac{1}{2}x + \frac{1/4}{2!}x^2 + \ldots$ $\approx 1 - \frac{1}{2}x + \frac{1}{8}x^2$	
	$\frac{1}{y} \approx \left(1 - \frac{1}{2}x + \frac{1}{8}x^2\right)^{-1}$ $= \left(1 + \left(-\frac{1}{2}x + \frac{1}{8}x^2\right)\right)^{-1}$ $= 1 - \left(-\frac{1}{2}x + \frac{1}{8}x^2\right) + \left(-\frac{1}{2}x + \frac{1}{8}x^2\right)^2 + \ldots$ $= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{4}x^2 + \ldots$ $\approx 1 + \frac{1}{2}x + \frac{1}{8}x^2$	

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8	Solution [9]	
	(i) $y = \frac{x^2 + ax + 6}{x-2} = x + 5 + \frac{k}{x-2}$, for some $k \in \mathbb{R}$ $= \frac{x^2 + 3x + (k-10)}{x-2}$ $x^2 + ax + 6 = x^2 + 3x + (k-10)$ By comparing coefficients of x, $a = 3$ (shown) Alternatively, $y = \frac{x^2 + ax + 6}{x-2}$ $= x + (a+2) + \frac{2a+10}{x-2}$ (performing long division) Hence, $a + 2 = 5 \Rightarrow a = 3$ (shown)	
	(ii) $y = \frac{x^2 + 3x + 6}{x-2}$ $xy - 2y = x^2 + 3x + 6$ $x^2 + (3-y)x + (2y+6) = 0$ Want discriminant < 0 *(so that we get 'no real x')* i.e. $(3-y)^2 - 4(2y+6) < 0$ $\Rightarrow y^2 - 14y - 15 < 0$ $\Rightarrow (y+1)(y-15) < 0$ $\Rightarrow -1 < y < 15$ **(shown)**	

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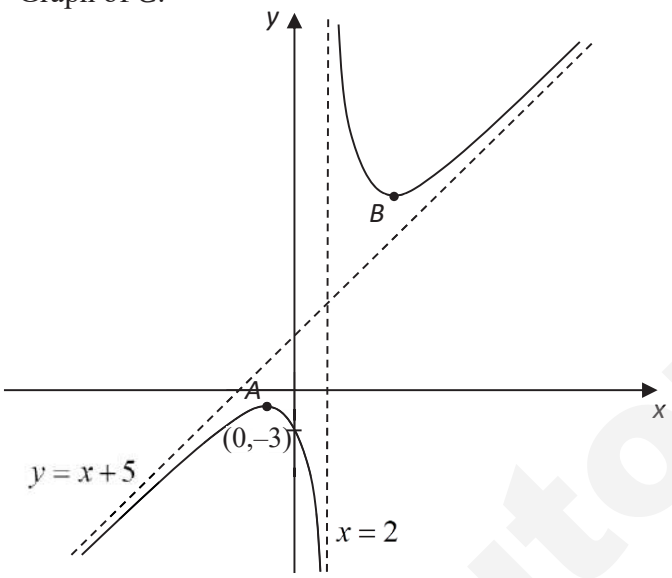
(iii) Graph of C:

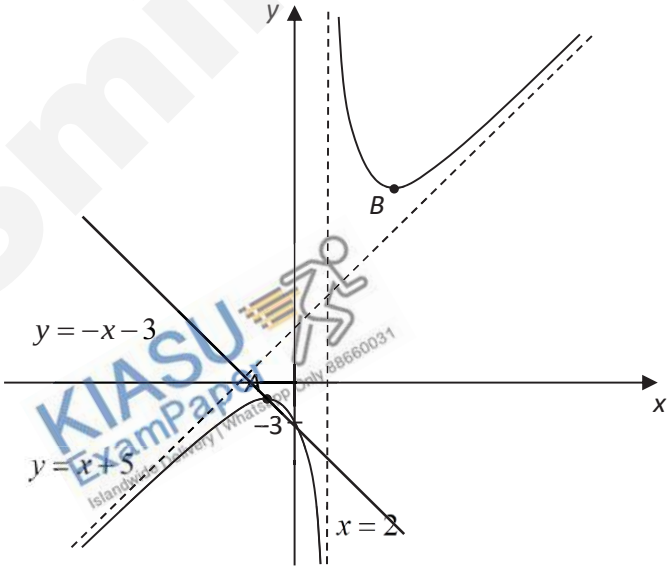


$A = (-2,-1)$, $B = (6,15)$

(iv)

$$\frac{x^2+3x+6}{x-2} \geq -x-3$$

Hence, sketch additionally the graph of $y = -x-3$:



The two graphs intersect at A ($x = -2$) and at $x = 0$.

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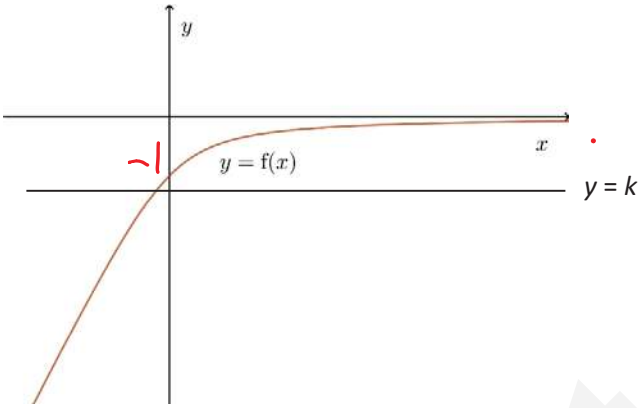
	$\therefore$ Answer: $-2 \le x \le 0$ or $x > 2$	
9	Solution [9]	
	(i)  Any horizontal line $y = k$ will intersect the graph of $y = \mathrm{f}(x)$ at most once. OR Any horizontal line $y = k$, $k < 0$ will intersect the graph of $y = \mathrm{f}(x)$ exactly once. Thus, f is one-one. Therefore f^{-1} exist.	
	(ii) $R_{\mathrm{f}} = (-\infty, 0) \subset (-\infty, 1) = D_{\mathrm{g}}$ Therefore gf exists. Using mapping method, $D_{\mathrm{f}} = \mathbb{R} \xrightarrow{\mathrm{f}} R_{\mathrm{f}} = (-\infty, 0) \xrightarrow{\mathrm{g}} (-3, 2) = R_{\mathrm{gf}}$ $\therefore R_{\mathrm{gf}} = (-3, 2)$	
	(b) $y = \mathrm{g}(x) \rightarrow y = \mathrm{g}(x-1) \rightarrow y = \mathrm{g}(-x-1)$	

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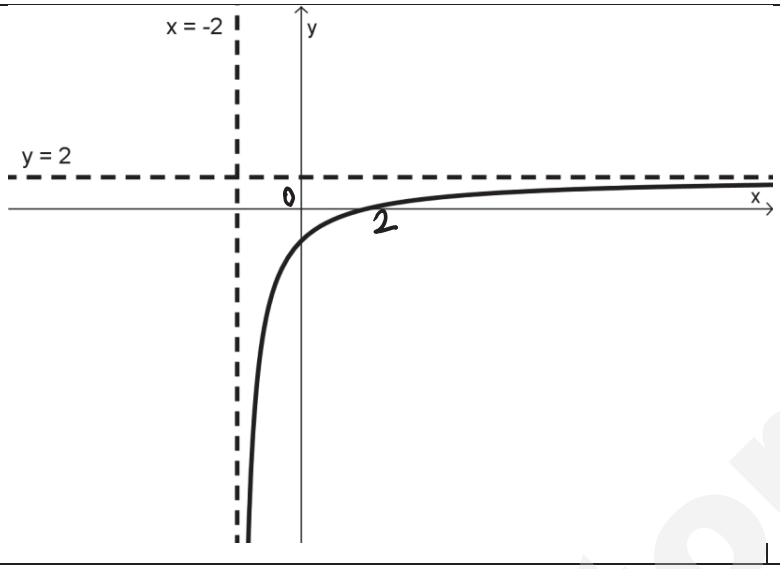

10	Solution [9]	
	(a) $u_n = S_n - S_{n-1}$ $= n^2 - (n-1)^2$ $= n^2 - (n^2 - 2n + 1)$ $= 2n - 1$ $u_n - u_{n-1}$ $= 2n - 1 - [2(n-1) - 1]$ $= 2$ (constant) Therefore, the series is a AP.	
	(b) $\sum_{r=0}^{2n} (nr^2 + 2^{2r})$ $= n\sum_{r=0}^{2n} r^2 + \sum_{r=0}^{2n} (2^2)^r$ $= n\sum_{r=1}^{2n} r^2 + \sum_{r=1}^{2n} (4^r) + 1$	

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$$= n\left[\frac{2n}{6}(2n+1)(4n+1)\right] + \frac{4\left(4^{2n}-1\right)}{4-1} + 1$$

$$= \frac{1}{3}n^2(2n+1)(4n+1) + \frac{1}{3}\left(4^{2n+1}-1\right)$$

$$= \frac{1}{3}\left(n^2(2n+1)(4n+1) + 4^{2n+1} - 1\right)$$

(c)

$$\lim_{n\to\infty}\left|\frac{1}{\left(x^2+2\right)^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\left(x^2+2\right)}$$

$$= \frac{1}{\left(x^2+2\right)} < 1 \text{ since } x^2+2>1 \text{ for all } x \in \mathbb{R}$$

$$\therefore \sum_{r=1}^{\infty}\frac{1}{\left(x^2+2\right)^r} \text{converges.}$$

$$\sum_{r=1}^{\infty}\frac{1}{\left(x^2+2\right)^r} = \frac{\frac{1}{x^2+2}}{1-\frac{1}{x^2+2}}$$

$$= \frac{1}{x^2+2} \div \frac{x^2+1}{x^2+2}$$

$$= \frac{1}{x^2+1}$$

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11	Solution [12]	
	(i) $\frac{\mathrm{d}x}{\mathrm{d}t} = \text{ rate in } - \text{ rate out}$ $= (0.03)(25) - \left(\frac{x}{5000}\right)(25)$ $= 0.75 - \frac{x}{200}$ $= \frac{150 - x}{200}$ (shown)	
	(ii) $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{150 - x}{200}$ $\frac{1}{150 - x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{200}$ $\int \frac{1}{150 - x}\,\mathrm{d}x = \frac{1}{200}\int 1\,\mathrm{d}t$ $-\ln\lvert 150 - x\rvert = \frac{1}{200}t + C$ $\lvert 150 - x\rvert = \mathrm{e}^{-\frac{1}{200}t - C}$ $150 - x = \pm\mathrm{e}^{-C}\mathrm{e}^{-\frac{1}{200}t}$ $150 - x = A\mathrm{e}^{-\frac{1}{200}t}$ $x = 150 - A\mathrm{e}^{-\frac{1}{200}t}$ When $t = 0,\ x = 20$ $150 - 20 = A\mathrm{e}^{-\frac{1}{200}(0)}$ $A = 130$ $\therefore 150 - x = 130\mathrm{e}^{-\frac{1}{200}t}$ $x = 150 - 130\mathrm{e}^{-\frac{1}{200}t}$	-

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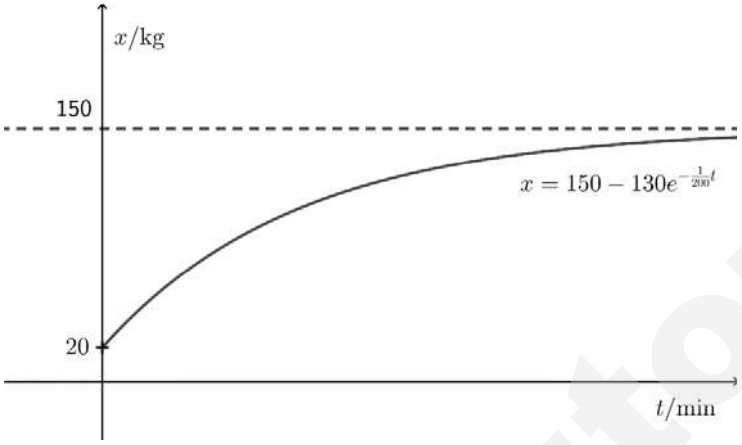
	(iii) As $t \to \infty$, $e^{-\frac{t}{200}} \to 0$. $\therefore x \to 150\,\text{kg}$	
	(iv)  Model is not suitable as the concentration of salt in the tank should be able to reach the same concentration as the salt solution flowing into the tank in a finite time.	

12	Solution [12]	
	(a) (i) Total surface area of metal sheet Total SA $= 2\pi xy + 2\pi x^2 = 40\pi^2$ $xy + x^2 = 20\pi$ $xy = 20\pi - x^2$ $y = \dfrac{20\pi - x^2}{x}$ $= \dfrac{20\pi}{x} - x$ Total Volume $= \pi x^2 y - \dfrac{2}{3}\pi x^3$ $= \pi x^2\left(\dfrac{20\pi}{x} - x\right) - \dfrac{2}{3}\pi x^3$	

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$= \pi x(20\pi - x^2) - \frac{2}{3}\pi x^3$

$= \pi x\left[20\pi - x^2 - \frac{2}{3}x^2\right]$

$= \pi x\left[20\pi - \frac{5}{3}x^2\right]$ (shown)

(a)(ii)

$\frac{dV}{dx} = \pi\left[20\pi - \frac{5}{3}x^2\right] + \pi x\left[-\frac{10}{3}x\right] = 0$

$20\pi - \frac{5}{3}x^2 - \frac{10}{3}x^2 = 0$

$20\pi - 5x^2 = 0$

$5x^2 = 20\pi$

$x^2 = 4\pi$

$x = \sqrt{4\pi}$ cm or $-\sqrt{4\pi}$ cm (Rejected since $x > 0$)

To check if V is maximum:

x	3.535	$\sqrt{4\pi} \approx 3.54$	3.545
$\frac{dV}{dx}$	$1.1018 > 0$	0	$-0.0103 < 0$

OR

$\frac{d^2V}{dx^2} = \pi\left[-\frac{10}{3}x\right] - \frac{20\pi}{3}x$

When $x = \sqrt{4\pi}$,

$\frac{d^2V}{dx^2} = -111.37 < 0$

Therefore, the volume is maximum at $x = \sqrt{4\pi}$

$V = \pi\sqrt{4\pi}\left(20\pi - \frac{5}{3}(4\pi)\right)$

$= 466.49 \text{ cm}^3$

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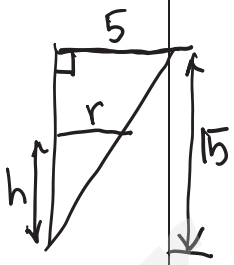
	(a)(iii) We assume negligible thickness for the metal sheet used.	
(b)	The water forms a smaller cone. Let its radius be r and height be h. From the diagram, by similar triangles, $\frac{r}{h} = \frac{5}{15}$ $r = \frac{h}{3}$  Let the volume of water in the cone be V_w. $V_w = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$ $= \frac{1}{3}\pi\left(\frac{h^2}{9}\right)h$ $= \frac{\pi}{27}h^3$ $V_w = \frac{\pi}{27}h^3$ $\frac{dV_w}{dh} = \frac{\pi}{9}h^2$ Given: $\frac{dV_w}{dt} = -2 \text{ cm}^3/\text{s}$ When $h = 3$, $\frac{dV_w}{dt} = \frac{dV_w}{dh} \times \frac{dh}{dt}$ $-2 = \frac{\pi}{9}(3)^2 \times \frac{dh}{dt}$ $\frac{dh}{dt} = -\frac{2}{\pi}$ or -0.637 cm/s (to 3 sf) The rate at which the water level is decreasing is 0.637 cm/s.	

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Questions from 2019 SAJC Promos

1 Solve the inequality $|3x+2| \geq 7|1-x|$. **[4]**

2 Given that a curve has the equation $x^{\tan^{-1}x} = y^{x+1}$ where $x > 0$ and $y > 0$, using a non-calculator method, find the exact gradient of the curve at the point where $x = 1$. **[5]**

3 removed (not in syllabus)

4 Relative to the origin O, the points A and B have position vectors **a** and **b** respectively. It is given that the magnitude of **a** is 4 and **b** is a unit vector perpendicular to **a**.

(i) Find the value of $(2\mathrm{a}+\mathrm{b})\bullet(3\mathrm{a}-5\mathrm{b})$. **[4]**

(ii) The point C is on AB such that $AC:CB = 3:1$. Write down the position vector of C, **c**, in terms of **a** and **b**. **[1]**

(iii) State the geometrical meaning of $|\mathrm{b}\times\mathrm{c}|$ and find its exact value. **[5]**

5 Functions f and g are defined by

$$\mathrm{f}: x \mapsto -\ln(x+a),\ x \in \mathbb{R},\ x > -a$$

$$\mathrm{g}: x \mapsto (x-\sqrt{a})^2,\ x \in \mathbb{R},\ x > 0$$

where a is a positive constant such that $1 < a \leq 3$.

(i) Sketch the graph of $y = \mathrm{f}(x)$ and show that f has an inverse. **[4]**

(ii) On the same diagram in part (i), sketch the graph of $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$. **[2]**

(iii) Find the range of the composite function fg. **[2]**

(iv) Without finding $\mathrm{f}^{-1}(x)$, find x given that $\mathrm{f}^{-1}\mathrm{g}(x) = \mathrm{e}^{-3} - a$. **[3]**

6 The curve C_1 has equation $y = \dfrac{x^2+ax+1}{x+b}$, where $x \in \mathbb{R}$, $x \neq -b$ and a and b are constants. The lines $x = -4$ and $y = x-6$ are asymptotes to C_1.

(i) Write down the value of b. Hence, show that $a = -2$. **[3]**

With the values of a and b found in **(i)**,

(ii) sketch C_1, stating the equations of any asymptotes, the coordinates of any turning points and any points of intersection with the axes, **[3]**

(iii) A second curve C_2 undergoes, in succession, the following transformations to get the resulting curve C_1:

A: Translation of 3 units in the negative x direction;
B: Scaling parallel to the x – axis by a scale factor of 2;
C: Reflection about the x – axis.

Find the equation of the curve C_2, showing your workings clearly. **[3]**

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7 removed (not in syllabus)

8 The curve C has parametric equations

$$x = \frac{t^3}{3}, \quad y = \left[\ln(t)\right]^2, \qquad \text{for } 0 < t \leq 3.$$

(i) Sketch the graph of C, giving the coordinates of its endpoint(s) and the point(s) where C meets the axes. State also the equation of the vertical asymptote. **[3]**

(ii) Find the equation of the tangent to the curve C at the point $\left(\frac{p^3}{3}, \left[\ln(p)\right]^2\right)$, simplifying your answer. **[5]**

(iii) Hence find the exact coordinates of the points Q and R where the tangent to the curve C when $t = \mathrm{e}$ meets the x-axis and y-axis respectively. **[3]**

(iv) Find the area of triangle OQR in exact form. **[2]**

9 removed (not in syllabus)

10 The plane Π_1 is defined by the equation $\mathbf{r} \bullet (-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 22$. A point P has coordinates $(3, -2, 1)$.

(i) Find the position vector of the foot of perpendicular, F, from the point P on Π_1. **[3]**

The line l has equation $\frac{x-3}{3} = \frac{y+2}{-3} = \frac{z-1}{2}$. The plane Π_2 contains the line l and is perpendicular to a plane with normal $\begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}$.

(ii) Find $\sin\theta$, where θ is the acute angle between the plane Π_2 and the line PF. **[4]**

(iii) A general point G has coordinates (x, y, z). Find the position vector of N, the midpoint of FG. **[1]**

(iv) Given that point N described in **(iii)** always lies in Π_2, find a cartesian equation that describes the set of points which G may take. Hence, describe the relationship between the set of points G and the plane Π_2. **[4]**

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11 [The volume of a cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$ and the arc length of a sector of radius r and angle θ radians is $r\theta$.]

Figure 1 shows a sector AOB of θ radians which is cut from a circular card of fixed radius a metres with centre O. A cup in the shape of an inverted right circular cone with radius r and height h is then formed by joining the two radii, OA and OB, of the sector together, without overlap (as shown in **Figure 2**).

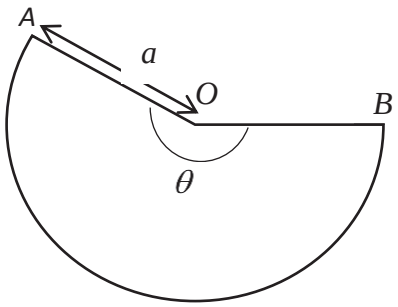

Figure 1

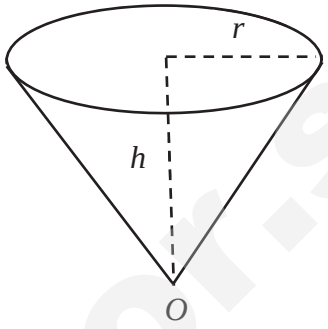

Figure 2

(i) Show that the volume of the cup in **Figure 2**, V cubic metres is given by

$$V = \frac{a^3}{24\pi^2}\theta^2\sqrt{4\pi^2 - \theta^2}\,. \qquad \textbf{[4]}$$

(ii) Use differentiation to find, in terms of a, the exact maximum volume of the cup as θ varies. You are not required to justify that the volume of the cup is a maximum. **[5]**

(iii) Hence, sketch the graph showing the volume of the cup, V as the angle of the sector AOB, θ varies. **[3]**

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1	Equations & Inequalities	$\frac{1}{2} \le x \le \frac{9}{4}$
2	Differentiation & Applications	$\frac{\pi}{8}$
4	Vectors	(i) 91 (ii) $\mathbf{c} = \frac{\mathbf{a}+3\mathbf{b}}{4}$ (iii) 1
5	Functions	(iii) $\mathrm{R}_{\mathrm{fg}} = (-\infty, -\ln a]$ (iv) $x = \sqrt{a} + \sqrt{3}$
6	Graphs & Transformations	(i) $y = x - 6 + \frac{25}{x+4}$ (iii) $y = \frac{-4x^2 + 28x - 49}{x - 22}$
8	Differentiation & Applications	(ii) $y = \left(\frac{2}{p^3}\ln p\right)x + [\ln p]^2 - \frac{2}{3}\ln p$ (iii) $(0, \frac{1}{3})$ (iv) $\frac{\mathrm{e}^3}{36}$ units^2
10	Vectors	(i) $\begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix}$ (ii) $\frac{4\sqrt{29}}{29}$ (iii) $\frac{1}{2}\begin{pmatrix} x \\ 4+y \\ 7+z \end{pmatrix}$ (iv) $2x + 4y + 3z = -35$
11	Differentiation & Applications	(ii) $\frac{2\sqrt{3}\pi a^3}{27}$ m^3

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TAMPINES MERIDIAN JUNIOR COLLEGE

JC1 YEAR-END EXAMINATION

CANDIDATE NAME	
CIVICS GROUP	

H2 MATHEMATICS **9758**

2 October 2019

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

For Examiners' Use	
1	
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Total	

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of 27 printed pages and 1 blank page.

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1 Using the substitution $x = 5\cos\theta$, find the exact value of $\int_{\frac{5}{\sqrt{2}}}^{5} \sqrt{1-\frac{x^2}{25}}\,\mathrm{d}x$. [6]

2 Functions f and g are defined by

$\mathrm{f}: x \mapsto \ln(x-a), \quad x \in \mathbb{R},\ a < x < a+1$, where a is a positive constant,

$\mathrm{g}: x \mapsto \frac{1}{x^2+1}, \quad x \in \mathbb{R},\ x \leq 2.$

(i) Show that the composite function gf exists. [2]

(ii) Find gf in a similar form. [2]

(iii) Find the range of gf, showing your working clearly. [2]

3 The curve C has equation $y = \frac{\sqrt{x}}{(16-x^2)^{\frac{1}{4}}},\ 0 \leq x < 4$.

(i) Sketch C. [2]

(ii) Find the exact volume of revolution when the region bounded by C, the line $x = 2\sqrt{3}$ and the x-axis, is rotated 2π radians about the x-axis. [4]

(iii) A horizontal line l intersects C at $x = 2\sqrt{3}$. Find the exact volume of revolution when the region bounded by C, the horizontal line l and the line $x = 0$, is rotated 2π radians about the x-axis. [2]

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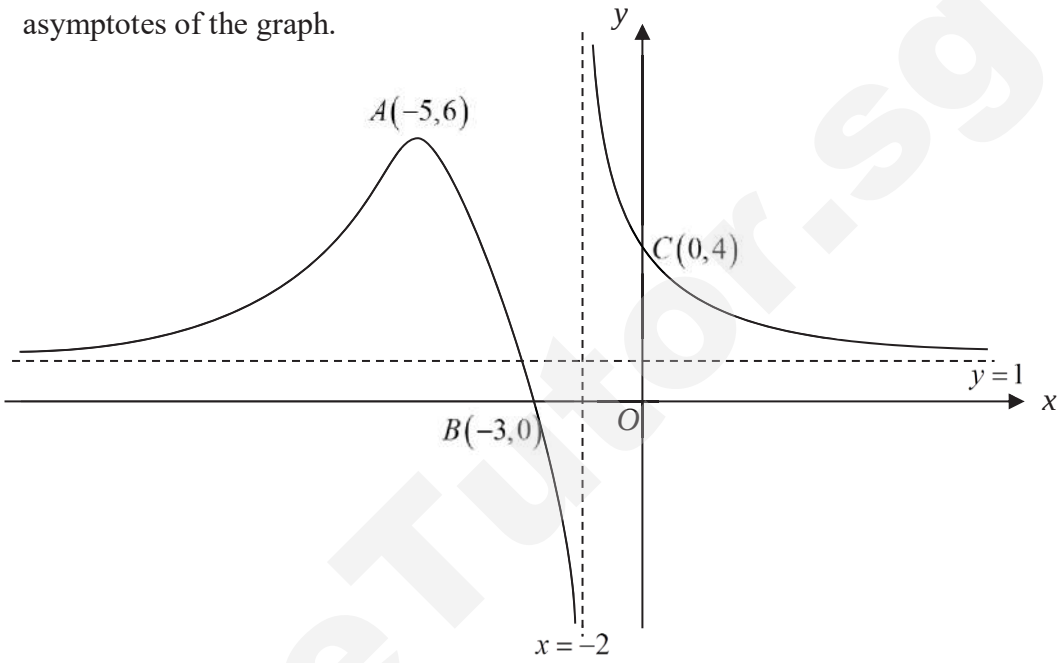
4 **(a)** State a sequence of transformations that will transform the curve with equation $y = x^2$ onto the curve with equation $y = 3 + \frac{1}{4}x^2$. [2]

(b) In the diagram, the graph of $y = \mathrm{f}(x)$ has a maximum turning point at $A(-5, 6)$ and axial intercepts at $B(-3, 0)$ and $C(0, 4)$. The lines $x = -2$ and $y = 1$ are the asymptotes of the graph.



Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}(3x + 5)$, [3]

(ii) $y = \mathrm{f}(-|x|)$, [3]

stating clearly, in each case, the equations of any asymptotes and the coordinates of the points corresponding to A, B and C.

5 Referred to the origin O, points A, B and C have position vectors $\mathbf{a} = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = 3\mathbf{j} - 3\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$ respectively.

(i) Show that A, B and C are collinear. [2]

(ii) Point D lies on AB such that $AD : DB = 1 : 2$. Find the position vector of D. [2]

(iii) Find the area of triangle OAB. Hence write down the area of triangle OBD. [4]

(iv) Evaluate $|\mathbf{c} \cdot \hat{\mathbf{a}}|$ and give a geometrical interpretation of $|\mathbf{c} \cdot \hat{\mathbf{a}}|$. [3]

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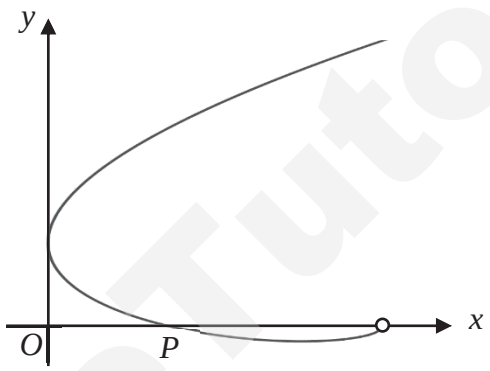
6 **(a)** Given that $y = \ln\left(2 - e^{2x}\right)$, where $x < \frac{1}{2}\ln 2$, show that

$$e^{y}\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = -4e^{2x}.$$

Hence, find the Maclaurin series of y, up to and including the term in x^3. [5]

(b) Find the expansion of $\frac{\sqrt{1-2x}}{2+3x}$ in ascending powers of x, up to and including the term in x^2. State the range of values of x for which this expansion is valid. [5]

7



The diagram shows the curve C with parametric equations

$$x = (t-3)^2, \quad y = t\ln t, \text{ where } t > 0.$$

(i) Find by differentiation, the value of t when the tangent to C is parallel to the y-axis. [3]

(ii) The point P on C has coordinates $(4, 0)$. The normal to C at point P is denoted by l. Find the equation of l. [3]

(iii) The line l intersects C again at point Q. Find the coordinates of point Q. [3]

(iv) Find the finite area bounded by C and the line l. [3]

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8 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{x^2 - 8x + 28}{4x - 32} \qquad \text{for } x \in \mathbb{R}, \quad x \neq 8.$$

(i) Find the exact x-coordinates of the turning points of $y = \mathrm{f}(x)$. [3]

(ii) Sketch the graph of $y = \mathrm{f}(x)$, labelling clearly the equations of the asymptotes and coordinates of axial intercepts and turning points. [3]

For the rest of the question, the domain of f is restricted to $8 < x \leq a$, $x \in \mathbb{R}$, where a is a positive constant such that the function f^{-1} exists.

(iii) State the exact greatest value of a. [1]

(iv) Using the value of a found in part **(iii)**, find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. [3]

(v) Sketch, on a single diagram, the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = x$, showing the relationship between the graphs. [3]

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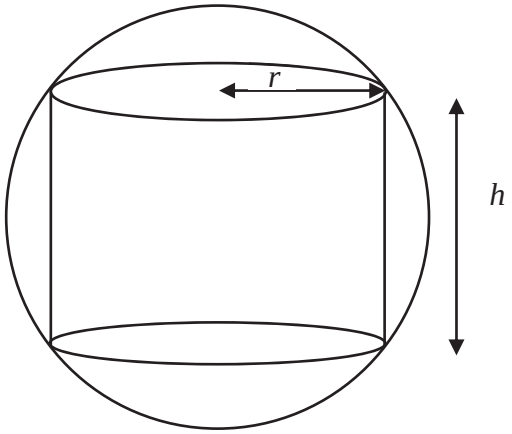
9 **(a)** [It is given that the volume of a cylinder of radius r and height h is $\pi r^2 h$.]

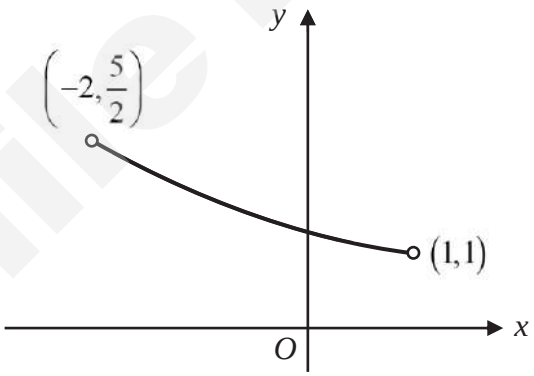


The diagram above shows a cylinder of radius r cm and height h cm inscribed in a spherical container with fixed radius 5 cm.

(i) Show that the volume, V cm^3, of the cylinder is given by $V = 2\pi r^2\sqrt{25-r^2}$. [2]

(ii) As r varies, use differentiation to find the exact value of r that gives a maximum V of the cylinder. Hence find the exact maximum value of V, showing that this value is a maximum. [6]

(b)



The diagram shows the graph of $y = \mathrm{f}(x)$, which is strictly decreasing on the interval $(-2,1)$. Given that $\mathrm{g}(x) = \frac{1}{2}\sin\left[2\mathrm{f}(x)\right]$, where $-2 < x < 1$, use differentiation to determine the number of stationary points on the graph of $y = \mathrm{g}(x)$. [4]

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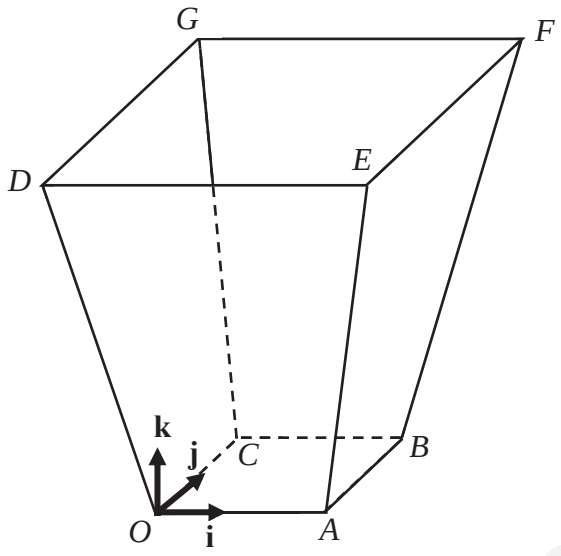
10



The diagram shows a structure with a horizontal square base *OABC* and a horizontal square top *DEFG*, where $OA = 2$ m and $DE = 6$ m. The vertical height of the structure is 8 m. Each sloping face of the structure is a trapezium, and the edges *AE*, *BF*, *CG* and *OD* are of equal length.

The point *O* is taken as the origin and perpendicular unit vectors **i**, **j**, **k** are such that **i** and **j** are parallel to *OA* and *OC* respectively. The coordinates of points *D*, *E*, *F* and *G* can be expressed as $(-p,\ -p,\ 8)$, $(2+p,\ -p,\ 8)$, $(2+p,\ 2+p,\ 8)$, and $(-p,\ 2+p,\ 8)$ respectively.

(i) Explain why $p = 2$. [1]

(ii) Find the Cartesian equation of plane *ABFE*. [3]

(iii) Given that the Cartesian equation of plane *OAED* is $4y + z = 0$, determine the obtuse angle between planes *OAED* and *ABFE*. [2]

A rod connects point *M*, the midpoint of edge *FG*, to a point *N* on the plane *OAED* such that *MN* is made as small as possible.

(iv) Find the coordinates of *N* and the exact minimum length of the rod. [5]

The structure sits on the ground and is secured by a straight cable connecting a point on the ground to the point *D*. This cable coincides with the line of reflection of the line segment *DM* in the plane *OAED*.

(v) Using your answer in part **(iv)** or otherwise, find a vector equation representing the straight cable. [3]

End of Paper

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Qn	Solution
1	**Integration by Substitution**

$x = 5\cos\theta \quad \Rightarrow \quad \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -5\sin\theta$

When $x = 5$, $5 = 5\cos\theta \quad \Rightarrow \cos\theta = 1 \quad \Rightarrow \theta = 0$

When $x = \dfrac{5}{\sqrt{2}}$, $\dfrac{5}{\sqrt{2}} = 5\cos\theta \quad \Rightarrow \cos\theta = \dfrac{1}{\sqrt{2}} \quad \Rightarrow \theta = \dfrac{\pi}{4}$

$$\int_{\frac{5}{\sqrt{2}}}^{5} \sqrt{1-\frac{x^2}{25}}\,\mathrm{d}x = \int_{\frac{\pi}{4}}^{0} \sqrt{1-\cos^2\theta}\,\left(-5\sin\theta\right)\,\mathrm{d}\theta$$

$$= \int_{\frac{\pi}{4}}^{0} -5\sin^2\theta\,\mathrm{d}\theta$$

$$= 5\int_{0}^{\frac{\pi}{4}} \frac{1}{2}\left(1-\cos 2\theta\right)\mathrm{d}\theta$$

$$= \frac{5}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{4}}$$

$$= \frac{5}{2}\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} - 0 - 0\right) = \frac{5\pi}{8} - \frac{5}{4}$$

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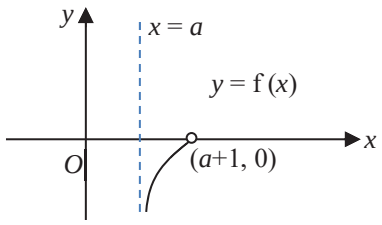
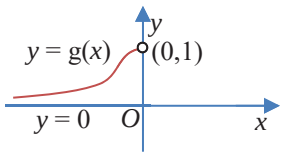
Qn	Solution
2	**Functions**
(i)	 $R_f = (-\infty, 0)$, $D_g = (-\infty, 2]$. Since $R_f \subseteq D_g \Rightarrow$ gf exists.
(ii)	$\text{gf}(x) = \text{g}(\ln(x-a))$ $= \dfrac{1}{[\ln(x-a)]^2 + 1}$ $\text{gf} : x \mapsto \dfrac{1}{[\ln(x-a)]^2 + 1}, x \in \mathbb{R},\ a < x < a+1$
(iii)	$D_f \xrightarrow{f} R_f = (-\infty, 0) \xrightarrow{g} (0,1) = R_{gf}$ $\therefore R_{gf} = (0,1)$ 

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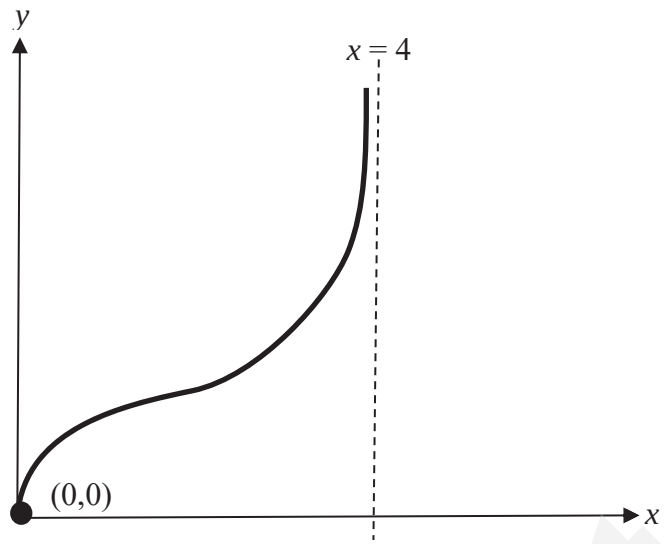
Qn	Solution
3	**Volume generated given a bounded region (Definite Integrals)**
(i)	$y = \dfrac{\sqrt{x}}{(16-x^2)^{\frac{1}{4}}}$ 
(ii)	Volume generated $= \pi\int_0^{2\sqrt{3}} \left[\dfrac{\sqrt{x}}{\left(16-x^2\right)^{\frac{1}{4}}}\right]^2 \mathrm{d}x$ $= -\dfrac{\pi}{2}\int_0^{2\sqrt{3}} \left[\dfrac{-2x}{\sqrt{16-x^2}}\right] \mathrm{d}x$ $= -\dfrac{\pi}{2}\left[\dfrac{\sqrt{16-x^2}}{\frac{1}{2}}\right]_0^{2\sqrt{3}}$ $= -\pi\left[\sqrt{16-\left(2\sqrt{3}\right)^2} - \sqrt{16}\right]$ $= -\pi(2-4)$ $= 2\pi$
(iii)	Volume generated $= \pi\left[\dfrac{\sqrt{2\sqrt{3}}}{\left(16-\left(2\sqrt{3}\right)^2\right)^{\frac{1}{4}}}\right]^2 \left(2\sqrt{3}\right) - 2\pi$ $= \pi\left(\dfrac{2\sqrt{3}}{2}\right)\left(2\sqrt{3}\right) - 2\pi$ $= 4\pi$

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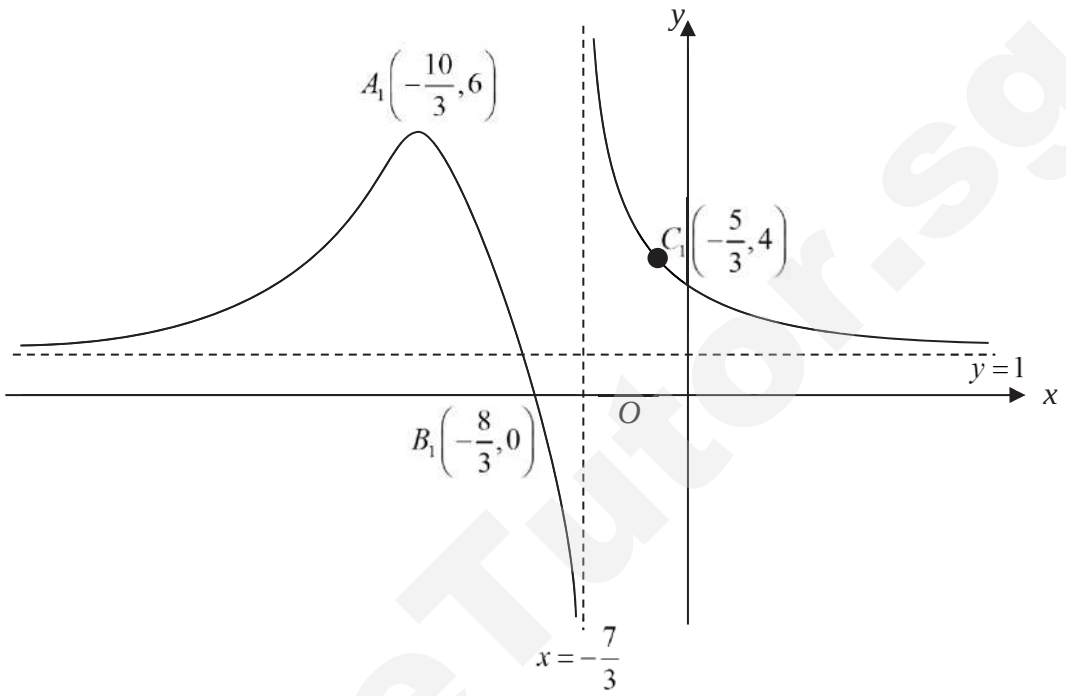
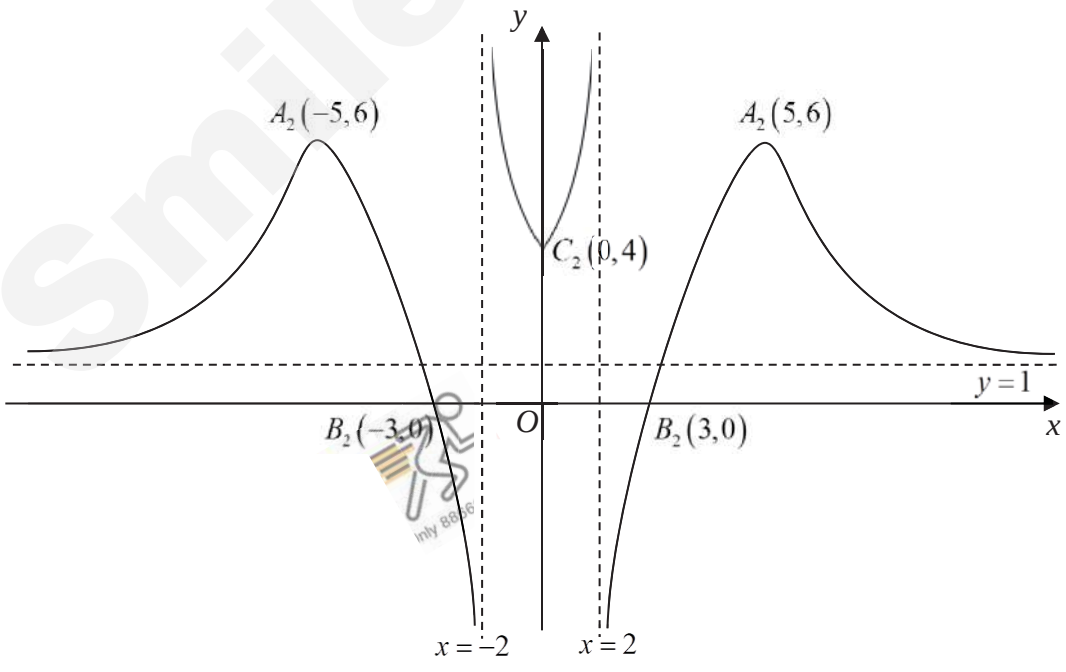
Qn	Solution
4	**Transformation of Curves**
(a)	(1) Stretch/Scaling of factor $\frac{1}{4}$ parallel to the y-axis. **OR** Stretch/Scaling of factor 2 parallel to the x-axis. (2) Translation of 3 units in the positive y-direction, *or (2),(1)
(bi)	
(bii)	

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Qn	Solution
5	**Vectors**
(i)	For A, B and C to be collinear, there exists $\lambda \in \mathbb{R}$ such that $\overrightarrow{AC} = \lambda \overrightarrow{AB}$ $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\therefore \overrightarrow{AC} = \frac{3}{2}\overrightarrow{AB}$ Hence A, B and C are collinear.
(ii)	Using ratio theorem and ΔOAB, $\overrightarrow{OD} = \frac{1}{3}\left[2\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix} -4 \\ 5 \\ -5 \end{pmatrix}$
(iii)	Area of triangle $OAB = \frac{1}{2}\lvert \mathbf{a} \times \mathbf{b} \rvert$ $= \frac{1}{2}\left\lvert \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} \right\rvert$ $= \frac{1}{2}\left\lvert \begin{pmatrix} 0 \\ -6 \\ -6 \end{pmatrix} \right\rvert$ $= 3\left\lvert \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \right\rvert = 3\sqrt{2}$ Area of triangle $OBD = 2\sqrt{2}$
(iv)	$\lvert \mathbf{c} \cdot \hat{\mathbf{a}} \rvert = \left\lvert \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix} \quad \frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \right\rvert$ $= \frac{1}{\sqrt{6}}\lvert -2 + 4 + 4 \rvert$ $= \sqrt{6}$ $\lvert \mathbf{c} \cdot \hat{\mathbf{a}} \rvert$ is the length of projection of $\boldsymbol{c}$ onto $\boldsymbol{a}$.

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Qn	Solution
6	**Maclaurin Series and Binomial Theorem**
(a)	$y = \ln\left(2 - e^{2x}\right)$ Differentiate with respect to x: $\frac{dy}{dx} = \frac{-2e^{2x}}{2 - e^{2x}}$ $= \frac{-2e^{2x}}{e^{y}}$ $e^{y}\frac{dy}{dx} = -2e^{2x}$ Differentiate with respect to x: $e^{y}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(e^{y}\frac{dy}{dx}\right) = -4e^{2x}$ $e^{y}\frac{d^2y}{dx^2} + e^{y}\left(\frac{dy}{dx}\right)^2 = -4e^{2x}$ $e^{y}\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = -4e^{2x}$ Differentiate with respect to x: $e^{y}\left(\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\frac{d^2y}{dx^2}\right) + \left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]\left(e^{y}\frac{dy}{dx}\right) = -8e^{2x}$ $e^{y}\left[\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\right] = -8e^{2x}$ When $x = 0$, $y = 0$, $\frac{dy}{dx} = -2$, $\frac{d^2y}{dx^2} = -8$, $\frac{d^3y}{dx^3} = -48$ Maclaurin series for y is $y = -2x - 8\left(\frac{x^2}{2!}\right) - 48\left(\frac{x^3}{3!}\right) + \ldots$ $y = -2x - 4x^2 - 8x^3 + \ldots$
(b)	$\frac{\sqrt{1-2x}}{(2+3x)}$ $= (1-2x)^{\frac{1}{2}}(2+3x)^{-1}$ $= \frac{1}{2}(1-2x)^{\frac{1}{2}}\left(1+\frac{3}{2}x\right)^{-1}$ $= \frac{1}{2}\left(1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(-2x)^2 + \ldots\right)\left(1 - \frac{3}{2}x + \left(\frac{3}{2}x\right)^2 + \ldots\right)$ $= \frac{1}{2}\left(1 - x - \frac{1}{2}x^2 + \ldots\right)\left(1 - \frac{3}{2}x + \frac{9}{4}x^2 + \ldots\right)$ $= \frac{1}{2}\left(1 - \frac{3}{2}x + \frac{9}{4}x^2 - x + \frac{3}{2}x^2 - \frac{1}{2}x^2 + \ldots\right)$ $= \frac{1}{2}\left(1 - \frac{5}{2}x + \frac{13}{4}x^2 + \ldots\right)$ $= \frac{1}{2} - \frac{5}{4}x + \frac{13}{8}x^2 + \ldots$

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	The expansion of $(1-2x)^{\frac{1}{2}}$ is valid for $\lvert 2x\rvert<1\Rightarrow\lvert x\rvert<\frac{1}{2}$ The expansion of $(2+3x)^{-1}$ is valid for $\left\lvert\frac{3x}{2}\right\rvert<1\Rightarrow\lvert x\rvert<\frac{2}{3}$ $-\frac{1}{2}<x<\frac{1}{2}$ and $-\frac{2}{3}<x<\frac{2}{3}$ $\Rightarrow-\frac{1}{2}<x<\frac{1}{2}$ Therefore, the expansion of $\frac{\sqrt{1-2x}}{(2+3x)}$ is valid for $-\frac{1}{2}<x<\frac{1}{2}$.

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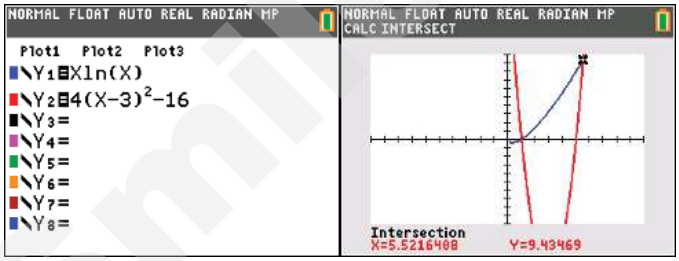
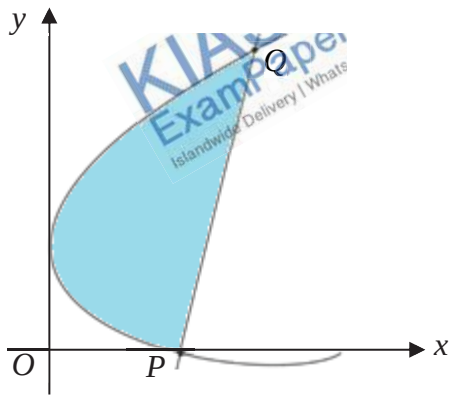
Qn	Solution
7	**Tangent & Normal, Integration**
(i)	$x=(t-3)^2 \Rightarrow \dfrac{dx}{dt}=2(t-3)$ $y=t\ln t \Rightarrow \dfrac{dy}{dt}=\ln t+\dfrac{t}{t}=\ln t+1$ $\dfrac{dy}{dx}=\dfrac{\ln t+1}{2(t-3)}$ Tangent parallel to the y-axis $\Rightarrow 2(t-3)=0$ $t=3$
(ii)	Gradient of normal $=\dfrac{-1}{\left(\frac{dy}{dx}\right)}=\dfrac{-2(t-3)}{\ln t+1}$ At (4, 0), $4=(t-3)^2 \Rightarrow t-3=\pm 2,\ t=1$ or $t=5$ (reject since $y=5\ln 5\neq 0$) OR $t\ln t=0 \Rightarrow t=0$ (reject $\because t>0$) or $t=1$ Equation of normal at point P $y-(1)\ln(1)=\dfrac{-2(1-3)}{\ln(1)+1}\left[x-(1-3)^2\right]$ $\Rightarrow y=4(x-4) \Rightarrow y=4x-16$
(iii)	$y=4x-16$ ----(1) Substi. $x=(t-3)^2$, $y=t\ln t$ into (1): $t\ln t=4(t-3)^2-16$  Using G.C., $t=5.52164$ When $t=5.52164$, $x=(5.52164-3)^2=6.35867=6.36$ $y=5.52164\ln(5.52164)=9.43469=9.43$ Coordinates of Q is (6.36, 9.43) (3 s.f.)
(iv)	 Required Area

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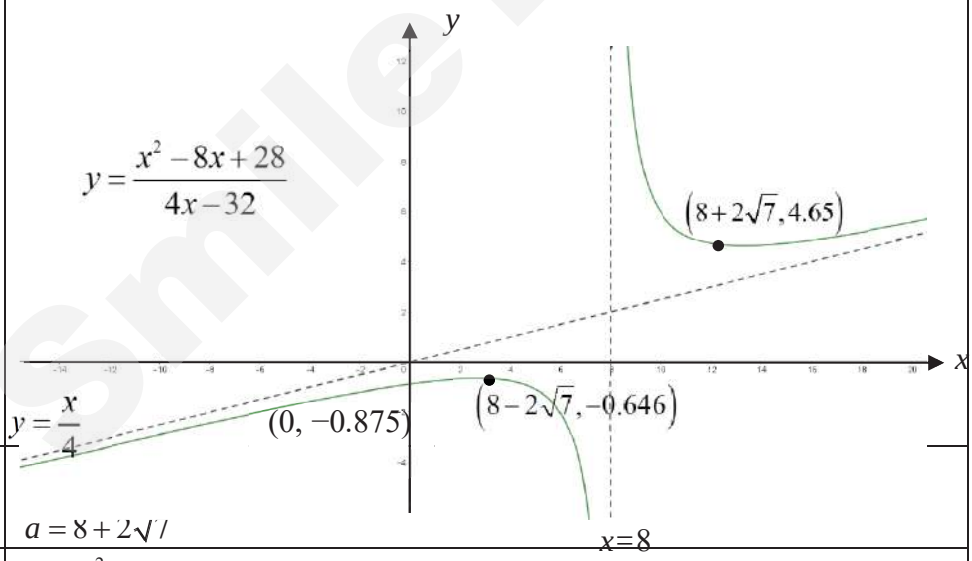
$$= \int_0^{9.43469} \left(\frac{y+16}{4} \right) \mathrm{d}y - \int_0^{9.43469} x_C \, \mathrm{d}y$$

$$= \int_0^{9.43469} \left(\frac{y+16}{4} \right) \mathrm{d}y - \int_1^{5.52164} (t-3)^2 \, (\ln t + 1) \, \mathrm{d}t$$

$$= 31.3995 = 31.4 \text{ units}^2 \quad (3 \text{ s.f.})$$

Qn	Solution
8	**Functions**
(i)	$y = \dfrac{x^2 - 8x + 28}{4x - 32}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(4x-32)(2x-8) - (x^2 - 8x + 28)4}{(4x-32)^2}$ At turning points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(4x-32)(2x-8) - (x^2 - 8x + 28)4}{(4x-32)^2} = 0$ $4x^2 - 64x + 144 = 0$ $x = \dfrac{-(-64) \pm \sqrt{(-64)^2 - 4(4)(144)}}{2(4)}$ $x = 8 \pm 2\sqrt{7}$
(ii)	
(iii)	$a = 8 + 2\sqrt{7}$
(iv)	$y = \dfrac{x^2 - 8x + 28}{4x - 32}$ $y(4x - 32) = x^2 - 8x + 28$ $x^2 + (-8 - 4y)x + (28 + 32y) = 0$ $x = \dfrac{8 + 4y \pm 4\sqrt{4 + 4y + y^2 - 7 - 8y}}{2}$ $= 4 + 2y \pm 2\sqrt{y^2 - 4y - 3}$ Since $8 < x \leq 8 + 2\sqrt{7}$, $x = 4 + 2y - 2\sqrt{y^2 - 4y - 3}$

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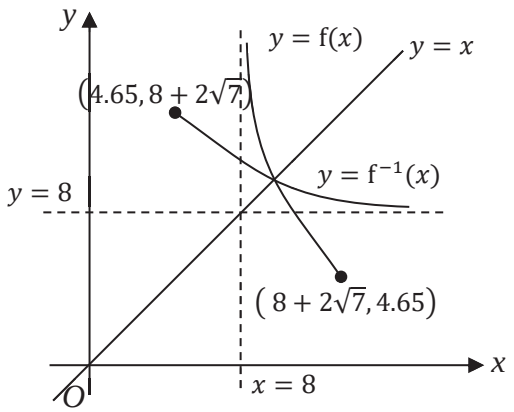
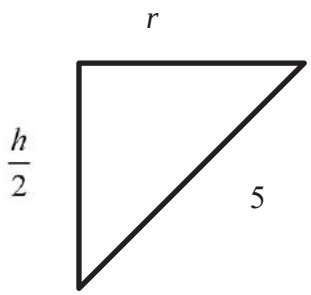
	I.e. $f^{-1}(x) = 4 + 2x - 2\sqrt{x^2 - 4x - 3}$ Domain of f^{-1} = Range of f = $[4.65, \infty)$
(v)	

Qn	Solution
9	**Application of Differentiation**
(i)	 $r^2 + \left(\frac{h}{2}\right)^2 = 5^2$ $r^2 + \frac{h^2}{4} = 25$ $4r^2 + h^2 = 100$ $\therefore$ since $h > 0$, $h = \sqrt{100 - 4r^2}$ $= \sqrt{4(25 - r^2)}$ $= 2\sqrt{25 - r^2}$ $V = \pi r^2 h$ $= \pi r^2 \left(2\sqrt{25 - r^2}\right)$ $= 2\pi r^2 \left(\sqrt{25 - r^2}\right)$ (shown) Alternatively, $r^2 + \left(\frac{h}{2}\right)^2 = 5^2$ since $h > 0$, $\frac{h}{2} = \sqrt{25 - r^2}$ $h = 2\sqrt{25 - r^2}$
(ii)	For maximum volume, $\frac{dV}{dr} = 0$

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$$\frac{dV}{dr} = 2\pi\left[2r\sqrt{25-r^2} + r^2\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{25-r^2}}\right)(-2r)\right]$$

$$2r\sqrt{25-r^2} - \frac{r^3}{\sqrt{25-r^2}} = 0$$

$$2r\sqrt{25-r^2} = \frac{r^3}{\sqrt{25-r^2}}$$

$$2r\left(25-r^2\right) = r^3$$

$$50r = 3r^3$$

$$r^2 = \frac{50}{3} \quad , \quad r \neq 0$$

since $r > 0$, $r = \sqrt{\frac{50}{3}} = \frac{5\sqrt{6}}{3}$

r	$\sqrt{\frac{50}{3}}^-$	$\sqrt{\frac{50}{3}}$	$\sqrt{\frac{50}{3}}^+$
$\frac{dV}{dr}$	/	—	\

$\therefore r = \sqrt{\frac{50}{3}}$ gives a maximum volume.

Method 2 (2nd derivative test)

$$\frac{d^2V}{dr^2} = -217.65 < 0$$

$\therefore r = \sqrt{\frac{50}{3}}$ gives a maximum volume.

Maximum volume, $V = 2\pi\left(\sqrt{\frac{50}{3}}\right)^2\sqrt{25-\left(\sqrt{\frac{50}{3}}\right)^2}$

$$= \frac{500}{3\sqrt{3}}\pi$$

$$= \frac{500\sqrt{3}}{9}\pi$$

(b)

$$g(x) = \frac{1}{2}\sin\left[2f(x)\right]$$

$$g'(x) = \frac{1}{2}(2)f'(x)\cos\left[2f(x)\right] = 0$$

Since f is strictly decreasing on $(-2,1)$, $f'(x) < 0$ and

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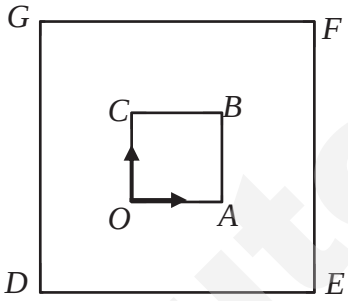
	for $\cos\left[2\mathrm{f}\left(x\right)\right]=0$, $2\mathrm{f}\left(x\right)=...,\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},...$ $\mathrm{f}\left(x\right)=...,\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},...$ Since $1<\mathrm{f}\left(x\right)<\frac{5}{2}$ for $-2<x<1$, $\therefore \mathrm{f}\left(x\right)=\frac{3\pi}{4}$ Hence there is one stationary point on the graph $y=\mathrm{g}\left(x\right)$.

Qn	Solution
10	**Vectors**
(i)	From the top view, $2p=DE-OA$ $p=\frac{6-2}{2}=2$ (shown) 
(ii)	$\overrightarrow{OA}=\begin{pmatrix}2\\0\\0\end{pmatrix},\overrightarrow{OB}-\begin{pmatrix}2\\2\\0\end{pmatrix},\overrightarrow{OE}-\begin{pmatrix}4\\-2\\8\end{pmatrix}$ $\overrightarrow{AB}=\begin{pmatrix}2\\2\\0\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix},\overrightarrow{AE}=\begin{pmatrix}4\\-2\\8\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}2\\-2\\8\end{pmatrix}$ $\overrightarrow{AB}\times\overrightarrow{AE}=\begin{pmatrix}0\\2\\0\end{pmatrix}\times\begin{pmatrix}2\\-2\\8\end{pmatrix}=\begin{pmatrix}16\\0\\-4\end{pmatrix}$ A normal vector to the plane is $\begin{pmatrix}4\\0\\-1\end{pmatrix}$. $\mathbf{r}\bullet\begin{pmatrix}4\\0\\-1\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}\bullet\begin{pmatrix}4\\0\\-1\end{pmatrix}=8$ $\therefore$ The Cartesian equation of plane $ABFE$ is $4x-z=8$.

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(iii)

$4y + z = 0 \Rightarrow \mathbf{r} \bullet \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} = 0$

$$\text{Angle between planes } OAED \text{ and } ABFE = \cos^{-1} \frac{\begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}}{\sqrt{17}\sqrt{17}}$$

$$= \cos^{-1}\left(-\frac{1}{17}\right)$$

$$= 93.4° \text{ (to 1 d.p.)}$$

(iv)

Coordinates of $M = (1, 4, 8)$

Equation of line MN: $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ **---(1)**

Equation of plane $OAED$: $\mathbf{r} \bullet \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} = 0$ **---(2)**

Solving (1) and (2),

$$\left[\begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}\right] \bullet \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow 24 + 17\lambda = 0$$

$$\Rightarrow \lambda = -\frac{24}{17}$$

$$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} - \frac{24}{17}\begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{28}{17} \\ \frac{112}{17} \end{pmatrix}$$

$\therefore$ Coordinates of point $N = \left(1, -\frac{28}{17}, \frac{112}{17}\right)$

$$\text{Length of cable} = \left| \begin{pmatrix} 1 \\ -\frac{28}{17} \\ \frac{112}{17} \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 \\ -\frac{96}{17} \\ -\frac{24}{17} \end{pmatrix} \right| = \frac{24}{\sqrt{17}}$$

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Alternatively,

$$\overrightarrow{DM} = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

$$\overrightarrow{NM} = \left(\frac{\begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}}{\sqrt{17}} \right) \frac{\begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}}{\sqrt{17}}$$

$$= \frac{24}{17} \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$$

$$\overrightarrow{OM} - \overrightarrow{ON} = \frac{24}{17} \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$$

$$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} - \frac{24}{17} \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{28}{17} \\ \frac{112}{17} \end{pmatrix}$$

$\therefore$ Coordinates of point $N = \left(1, -\frac{28}{17}, \frac{112}{17} \right)$

$$\text{Length of cable} = \frac{\left| \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \right|}{\sqrt{17}} = \frac{24}{\sqrt{17}}$$

(v)

By ratio theorem,

$$\overrightarrow{ON} = \frac{\overrightarrow{OM} + \overrightarrow{OM'}}{2}$$

$$\overrightarrow{OM'} = 2\overrightarrow{ON} - \overrightarrow{OM}$$

$$= 2 \begin{pmatrix} 1 \\ -\frac{28}{17} \\ \frac{112}{17} \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{124}{17} \\ \frac{88}{17} \end{pmatrix}$$

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$\overrightarrow{DM'} = \overrightarrow{OM'} - \overrightarrow{OD}$

$$= \begin{pmatrix} 1 \\ -\frac{124}{17} \\ \frac{88}{17} \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{90}{17} \\ -\frac{48}{17} \end{pmatrix} = -\frac{3}{17}\begin{pmatrix} -17 \\ 30 \\ 16 \end{pmatrix}$$

A vector equation of the reflection of line *DM* in the plane *OAED* is

$\mathbf{r} = \begin{pmatrix} -2 \\ -2 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -17 \\ 30 \\ 16 \end{pmatrix}, \mu \in \mathbb{R}$.

Also accept $\mathbf{r} = \begin{pmatrix} 1 \\ -\frac{124}{17} \\ \frac{88}{17} \end{pmatrix} + \mu \begin{pmatrix} -17 \\ 30 \\ 16 \end{pmatrix}, \mu \in \mathbb{R}$

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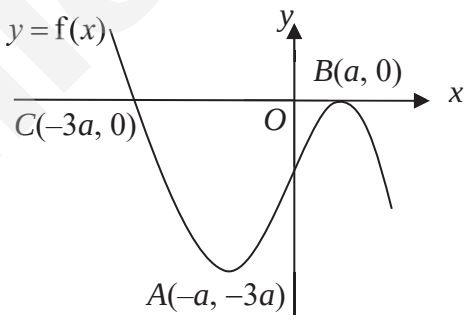
1 **(i)** Sketch the curve with equation $y = \left|\frac{\alpha x}{x+1}\right|$, where α is a positive constant, stating the equations of the asymptotes. On the same diagram, sketch the line with equation $y = \alpha x - 2$. [3]

(ii) Solve the inequality $\left|\frac{\alpha x}{x+1}\right| \geq \alpha x - 2$, giving your answers in term of α. [3]

2 Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + \mu\mathbf{m}$ where $\mathbf{a}$ and $\mathbf{m}$ are constant vectors and μ is a parameter. [2]

Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, such that $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. The point C lies on the line AB such that the area of the triangle OBC is 6 units^2. Given that $\mathbf{a}$ is a unit vector, $|\mathbf{b}| = 4$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 30°, find the possible position vectors of C in terms of $\mathbf{a}$ and $\mathbf{b}$. [6]

3 **(a)** The diagram shows the curve $y = \mathrm{f}(x)$, where a is a positive constant. The curve has a minimum point at $A(-a, -3a)$, a maximum point at $B(a, 0)$ and cuts the x-axis at the point $C(-3a, 0)$.



Sketch, labelling each graph clearly and showing the coordinates of the points corresponding to A, B and C whenever possible, the graphs of

(i) $y = 3\mathrm{f}(x-a)$, [2]

(ii) $y = \mathrm{f}\left(\left|\frac{x}{2}\right|\right)$, [2]

(iii) $y = \frac{1}{\mathrm{f}(x)}$. [2]

(b) The curve with equation $y = 2 + \mathrm{e}^{-x}$ is reflected in the line $y = 5$. Find the equation of the reflected curve. [2]

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4 It is given that

$$f(x) = \begin{cases} 4x+2 & \text{for } 0 \le x \le 1, \\ \dfrac{12}{x+\sqrt{x}} & \text{for } 1 < x \le 4, \end{cases}$$

and that $f(x+4) = f(x)$ for all real values of x.

(i) Sketch the graph of $y = f(x)$ for $-2 < x \le 6$. [3]

(ii) Use the substitution $u = \sqrt{x}$ to find the exact value of $\int_0^4 f(x)\,dx$. [5]

5 **(a)** Find $\int \sin x \cos 3x \, dx$. [2]

(b) Find $\int \dfrac{x-1}{\sqrt{1+2x-x^2}} \, dx$. Find the greatest integer value of b such that $\int_0^b \dfrac{x-1}{\sqrt{1+2x-x^2}} \, dx$ is defined. [5]

(c) Find $\int x \cos x \, dx$. Hence find the exact value of $\int_0^{2\pi} x|\cos x| \, dx$. [5]

6 **(a)** In an arithmetic progression, the 8th term is 20 and the 27th term is greater than the 15th term by 24. It is given that the sum of the first n terms is greater than the sum of the 8th to the 40th term by more than 1218. Find the smallest value of n. [6]

(b) An infinite geometric progression is such that the sum of all the terms after the nth term is equal to twice the nth term. Show that the sum to infinity of the progression is three times the first term. [3]

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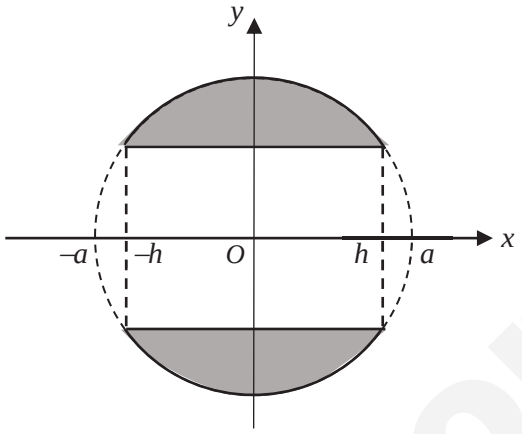
7 A napkin-holder is formed by boring a cylindrical hole, of length $2h$, through a wooden sphere of radius a, where a is a fixed constant. The axis of the hole passes through the centre O of the sphere. The diagram shows a cross-section through O, with x- and y-axes taken parallel and perpendicular to the axis of the hole respectively.



(i) Let S denote the napkin holder's total surface area, which is made up of its internal (cylindrical) area and its external (spherical) area. It is given that the external surface area is $4\pi ah$.

(a) Show that $S = 4\pi h\left(a + \sqrt{a^2 - h^2}\right)$. [2]

(b) Use differentiation to find, in terms of a, the exact maximum value of S as h varies. [You do not need to verify that this value of S is the maximum.] [4]

(ii) Let V denote the volume of the wood forming the napkin-holder. By considering the napkin-holder as a solid of revolution about the x-axis, find V in terms of h, verifying that it is independent of a. [4]

8 Functions f and g are defined by

$$\mathrm{f} : x \mapsto \frac{x^2 - 6x - 6}{x+1}, \quad x \in \mathbb{R}, \quad x \neq -1,$$
$$\mathrm{g} : x \mapsto \frac{ax+1}{x+b}, \qquad x \in \mathbb{R}, \qquad x \neq -b,$$

where a and b are constants.

(i) Sketch the graph of $y = \mathrm{f}(x)$, giving the coordinates of the turning points and the equation of the asymptotes. Write down the range of f. [3]

(ii) Find the value of a and the range of values of b such that both composite functions fg and gf exist. [4]

(iii) Find $\mathrm{g}^{-1}(x)$. Given that $\mathrm{g}^{-1}(x) = \mathrm{g}(x)$ for all real x, $x \neq -b$, find b in terms of a. [3]

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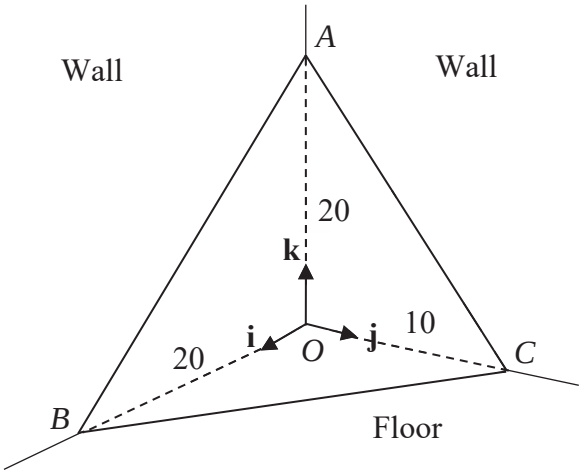

A shoe store owner plans to install a triangular mirror ABC with negligible thickness at one of the floor corner of his shop to allow his customers to view the fitting of their selected shoes. Points (x, y, z) are defined relative to the corner point at O where the two vertical walls, which are perpendicular to each other, and the horizontal floor meet. The z-axis points vertically upwards. The x-axis and y-axis are the intersections of the floor with the two walls. A, B and C lie on the z-axis, x-axis and y-axis and are 20 units, 20 units and 10 units from O respectively. The units of length are measured in inches.

(i) Find the cartesian equation of the face of the mirror ABC. Hence find the exact shortest distance from O to the face of the mirror ABC. [4]

(ii) Find the coordinates of the point N on the face of the mirror ABC which is nearest to O. [2]

(iii) Find the acute angle between the face of the mirror ABC and the floor. [2]

As a safety measure, a triangular plank OBR is installed to support the mirror, where R is a point between A and C such that $AR : RC = \mu : 1-\mu$. The face of the mirror ABC meets the plank OBR on l.

(iv) Given that N in **(iii)** lies on l, find the coordinates of R. [5]

10 A curve C has parametric equations

$$x = 2t + \sin 2t, \quad y = \cos 2t, \quad \text{for } 0 \le t \le \pi.$$

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\tan t$. What can be said about the tangent to C at the point where $t = \dfrac{\pi}{2}$. [4]

(ii) Find the exact x-coordinates, x_1 and x_2 where $x_1 < x_2$, of the two points where C cuts the x-axis. [2]

(iii) Sketch C, indicating the exact coordinates of the end-points. [2]

(iv) Find the exact area of the region bounded by C and the x-axis. [5]

(v) Find the value of t at which the tangent to C at the point $x = x_1$ intersects C again. [3]

End of Paper

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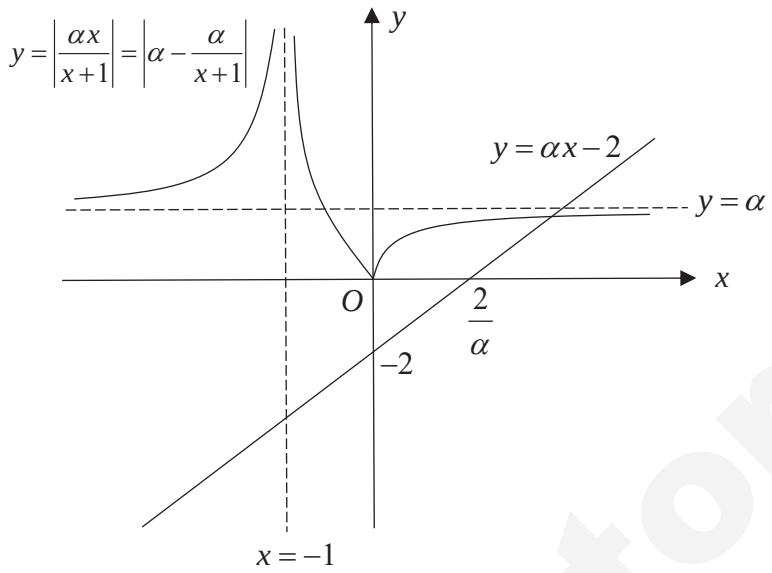
1 **[Solution]**

(i)



(ii)

$$\left|\frac{\alpha x}{x+1}\right| \geq \alpha x - 2$$

Solve $\dfrac{\alpha x}{x+1} = \alpha x - 2$

$$\alpha x^2 - 2x - 2 = 0$$

$$x = \frac{2 \pm \sqrt{4+8\alpha}}{2\alpha} = \frac{1 \pm \sqrt{2\alpha+1}}{\alpha}$$

$$x = \frac{1+\sqrt{2\alpha+1}}{\alpha} \quad \text{(Larger value of } x\text{)}$$

From graph, $x \leq \dfrac{1+\sqrt{2\alpha+1}}{\alpha}$, $x \neq -1$

(or equivalent form $x < -1$ or $-1 < x \leq \dfrac{1+\sqrt{2\alpha+1}}{\alpha}$)

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2 **[Solution]**

Vector equation $\mathbf{r} = \mathbf{a} + \mu\mathbf{m}$:
The equation gives the set of position vectors of the points on the line which passes through the point with position vector $\mathbf{a}$ and is parallel to $\mathbf{m}$.

Equation of line AB: $\mathbf{r} = \mathbf{a} + \mu(\mathbf{b} - \mathbf{a}) = (1 - \mu)\mathbf{a} + \mu\mathbf{b}$
Since C lies on line AB, $\mathbf{c} = (1 - \mu)\mathbf{a} + \mu\mathbf{b}$ for some μ

OR using ratio theorem, $\mathbf{c} = (1 - \mu)\mathbf{a} + \mu\mathbf{b}$ for some μ

Area of triangle OBC

$$= \frac{1}{2}|\mathbf{c} \times \mathbf{b}|$$

$$= \frac{1}{2}|[(1-\mu)\mathbf{a} + \mu\mathbf{b}] \times \mathbf{b}|$$

$$= \frac{1}{2}|(1-\mu)\mathbf{a} \times \mathbf{b} + \mu\mathbf{b} \times \mathbf{b}|$$

$$= \frac{1}{2}|(1-\mu)||\mathbf{a} \times \mathbf{b}| \quad \text{since } \mathbf{b} \times \mathbf{b} = 0$$

$$= \frac{1}{2}|(1-\mu)|(1)(4)\sin 30°$$

$$= |(1-\mu)|$$

$$|(1-\mu)| = 6$$

$1 - \mu = 6$ or $1 - \mu = -6$

$\mu = -5$ or 7

$\therefore \overrightarrow{OC} = 6\mathbf{a} - 5\mathbf{b}$ or $\overrightarrow{OC} = 7\mathbf{b} - 6\mathbf{a}$

Alternatively

Area of triangle $OAB = \frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin 30° = 1$

Since A, B, C are collinear and Area triangle $OBC = 6$,

$$\frac{\text{Area of } \Delta OBC}{\text{Area of } \Delta OAB} = \frac{6}{1}$$

$$\frac{\frac{1}{2}h|\overrightarrow{BC}|}{\frac{1}{2}h|\overrightarrow{AB}|} = \frac{6}{1} \Rightarrow \frac{|\overrightarrow{BC}|}{|\overrightarrow{AB}|} = \frac{6}{1}$$

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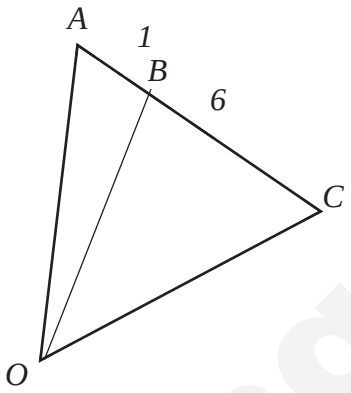
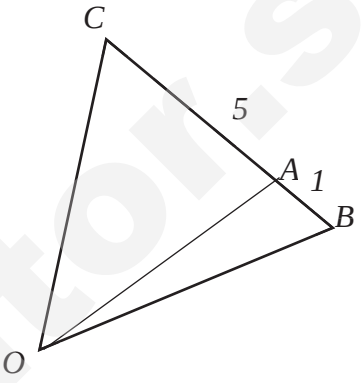
By ratio Theorem,
Case 1:

$$\overrightarrow{OB} = \frac{6\overrightarrow{OA} + \overrightarrow{OC}}{7} \Rightarrow \overrightarrow{OC} = 7\mathbf{b} - 6\mathbf{a}$$



Case 2:

$$\overrightarrow{OA} = \frac{5\overrightarrow{OB} + \overrightarrow{OC}}{6} \Rightarrow \overrightarrow{OC} = 6\mathbf{a} - 5\mathbf{b}$$



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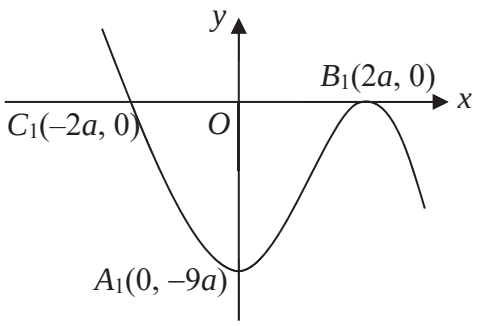
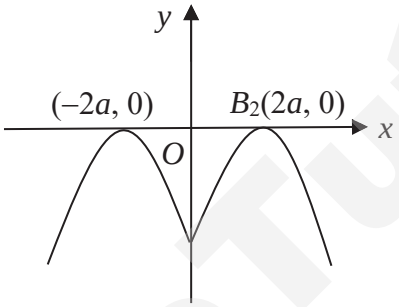
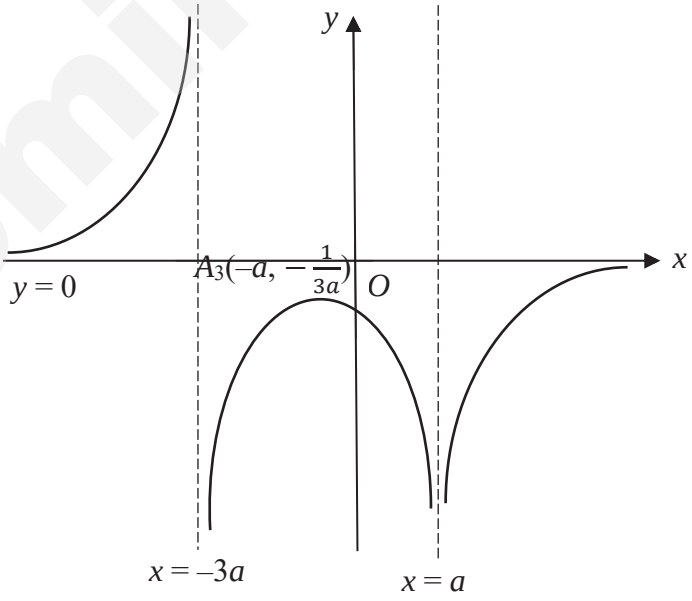
3	[Solution]
(a)(i)	$y = 3\mathrm{f}(x-a)$ 
(ii)	$y = \mathrm{f}\left(\left\|\frac{x}{2}\right\|\right)$ 
(iii)	$y = \frac{1}{\mathrm{f}(x)}$ 
(b)	$y = 2 + \mathrm{e}^{-x}$ $\rightarrow y = 2 + \mathrm{e}^{-x} - 5 = -3 + \mathrm{e}^{-x}$ $\rightarrow y = -(-3 + \mathrm{e}^{-x}) = 3 - \mathrm{e}^{-x}$ $\rightarrow y = 3 - \mathrm{e}^{-x} + 5 \Rightarrow y = 8 - \mathrm{e}^{-x}$

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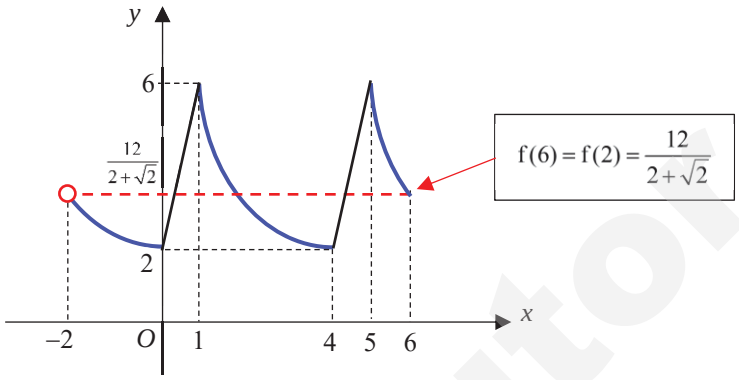
	Alternative method $y = 2 + e^{-x}$ $\rightarrow y = -(2 + e^{-x})$ $\rightarrow y = -(2 + e^{-x}) + 10 \Rightarrow y = 8 - e^{-x}$

4	[Solution]
(i)	
(ii)	$\int_0^4 f(x)\,dx = \frac{1}{2}(2+6)(1) + \int_1^4 \frac{12}{x+\sqrt{x}}\,dx$ Let $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$ (giving $dx = 2u\,du$) When $x = 1$, $u = 1$ When $x = 4$, $u = 2$ $\int_1^4 \frac{12}{x+\sqrt{x}}\,dx = 12\int_1^2 \frac{1}{u^2+u}\,2u\,du$ $= 24\int_1^2 \frac{1}{u+1}\,du$ $= 24\left[\ln(u+1)\right]_1^2$ $= 24\ln\left(\frac{3}{2}\right)$ $\therefore \int_0^4 f(x)\,dx = 4 + 24\ln\left(\frac{3}{2}\right)$

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5 **[Solution]**

(a)

$$\int \sin x \cos 3x \, dx = \int \cos 3x \sin x \, dx$$

$$= \frac{1}{2}\int \sin 4x - \sin 2x \, dx$$

$$= -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + c$$

(b)

$$\int \frac{x-1}{\sqrt{1+2x-x^2}} \, dx = -\frac{1}{2}\int \frac{2(1-x)}{\sqrt{1+2x-x^2}} \, dx$$

$$= -\frac{1}{2}\frac{\left(1+2x-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -\sqrt{1+2x-x^2} + c$$

For $\dfrac{x-1}{\sqrt{1+2x-x^2}}$ to be defined,

$$1+2x-x^2 > 0$$

$$(x-1)^2 - 2 < 0$$

$$\left(x-1+\sqrt{2}\right)\left(x-1-\sqrt{2}\right) < 0$$

$$\Rightarrow 1-\sqrt{2} < x < 1+\sqrt{2}$$

Greatest integer value of b is 2

(c)

$$\int x\cos x \, dx = x\sin x - \int \sin x \, dx$$

$$= x\sin x + \cos x + c$$

$$\int_0^{2\pi} x|\cos x| \, dx = \int_0^{\frac{\pi}{2}} x\cos x \, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\cos x \, dx + \int_{\frac{3\pi}{2}}^{2\pi} x\cos x \, dx$$

$$= \left[x\sin x + \cos x\right]_0^{\frac{\pi}{2}} - \left[x\sin x + \cos x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$+ \left[x\sin x + \cos x\right]_{\frac{3\pi}{2}}^{2\pi} \quad \text{(using above result)}$$

$$= \left(\frac{\pi}{2} - 1\right) - \left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) + \left(1 + \frac{3\pi}{2}\right)$$

$$= 4\pi$$

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6 **[Solution]**

(a)

$T_8 = a + 7d = 20$ --- (1)

$T_{27} - T_{15} = 24$

$(a + 26d) - (a + 14d) = 24$

$12d = 24$

$\Rightarrow\ d = 2$

Subst into (1): $a = 20 - 7(2) = 6$

$S_{40} = \frac{40}{2}[12 + 39(2)] = 1800$

$S_7 = \frac{7}{2}[12 + 6(2)] = 84$

Sum of T_8 to $T_{40} = S_{40} - S_7 = 1800 - 84 = 1716$

$\frac{n}{2}[12 + (n-1)(2)] - 1716 > 1218$ (more than 1218)

$n^2 + 5n - 2934 > 0$

Method 1

n	$n^2 + 5n - 2934$
51	$-78 < 0$
52	$30 > 0$

Using GC, the smallest value of n is 52

Method 2

$n^2 + 5n - 2934 > 0$

$n < -56.7$ or $n > 51.7$

$\therefore$ smallest value of n is 52

(b)

Let the first term be a and the common ratio be r.

Given $S_\infty - S_n = 2T_n$

$$\frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = 2ar^{n-1}$$

$$\frac{ar^n}{1-r} = 2ar^{n-1} \quad \Rightarrow \frac{r}{1-r} = 2 \Rightarrow r = \frac{2}{3}$$

$$S_\infty = \frac{a}{1-\left(\frac{2}{3}\right)} = 3a$$

Hence, sum to infinity is 3 times of the first term.

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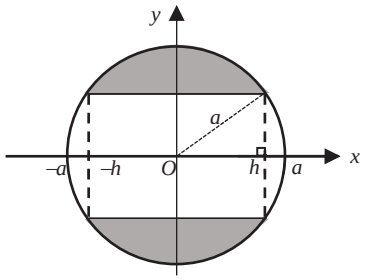
7	**[Solution]**
(i)(a)	 Radius of cylinder $= \sqrt{a^2-h^2}$ Internal cylindrical area $= 2\pi\sqrt{a^2-h^2}\,(2h) = 4\pi h\sqrt{a^2-h^2}$ $\therefore S = 4\pi ah + 4\pi h\sqrt{a^2-h^2}$ $= 4\pi h\left(a+\sqrt{a^2-h^2}\right)$ (shown)
(b)	$\frac{\mathrm{d}S}{\mathrm{d}h} = 4\pi\left(a+\sqrt{a^2-h^2}\right) + 4\pi h\cdot\frac{1}{2}\left(\frac{-2h}{\sqrt{a^2-h^2}}\right)$ $= 4\pi\left(a+\frac{a^2-h^2-h^2}{\sqrt{a^2-h^2}}\right)$ $= 4\pi\left(a+\frac{a^2-2h^2}{\sqrt{a^2-h^2}}\right)$ Let $\frac{\mathrm{d}S}{\mathrm{d}h} = 0$ $\frac{2h^2-a^2}{\sqrt{a^2-h^2}} = a$ $\left(2h^2-a^2\right)^2 = a^2\left(a^2-h^2\right)$ $4h^4 - 4h^2a^2 + a^4 = a^4 - a^2h^2$ $4h^4 - 3h^2a^2 = 0$ $h^2(4h^2-3a^2) = 0$ Since $h \neq 0,\ h^2 = \frac{3a^2}{4}$ $h = \frac{\sqrt{3}}{2}a \quad (\because h > 0)$ Max value of $S = 4\pi\left(\frac{\sqrt{3}}{2}a\right)\left(a+\sqrt{a^2-\frac{3}{4}a^2}\right)$ $= 2\sqrt{3}\pi a\left(a+\frac{1}{2}a\right)$ $= 3\sqrt{3}\pi a^2$

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(ii)

Equation of circle is $x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2$

$$V = 2\times\pi\int_0^h \left(a^2 - x^2\right)\,\mathrm{d}x \;-\; \pi\left(\sqrt{a^2 - h^2}\right)^2 (2h)$$

$$= 2\pi\left[a^2 x - \frac{x^3}{3}\right]_0^h - 2\pi h(a^2 - h^2)$$

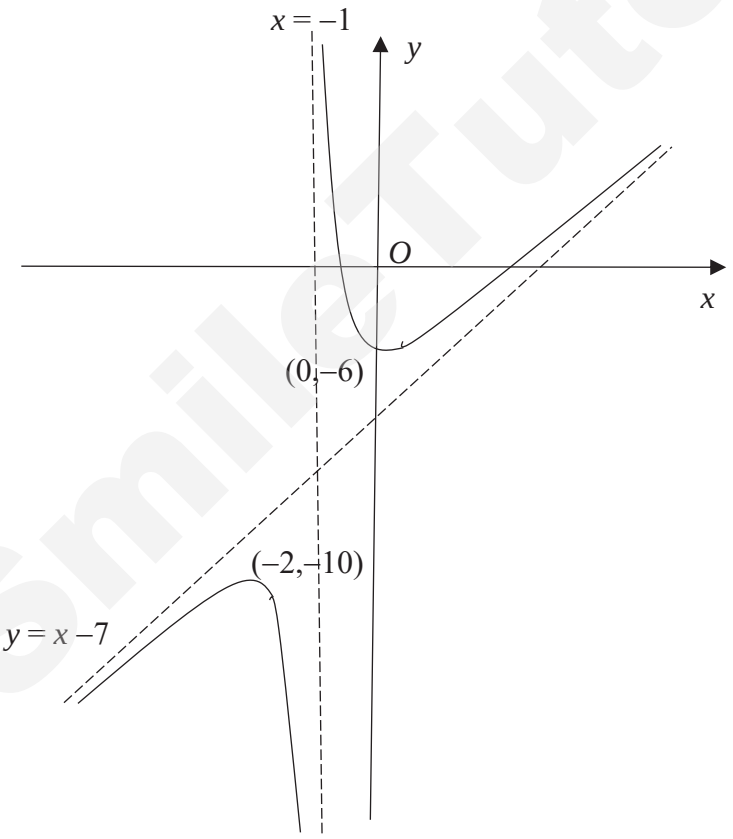
$$= 2\pi\left(a^2 h - \frac{h^3}{3}\right) - 2\pi h a^2 + 2\pi h^3$$

$$= \frac{4}{3}\pi h^3 \quad \text{which is independent of } a$$

8 **[Solution]**

(i)

$$y = \mathrm{f}(x) = x - 7 + \frac{1}{x+1}$$



$$\mathrm{R_f} = \left(-\infty, -10\right] \cup \left[-6, \infty\right)$$

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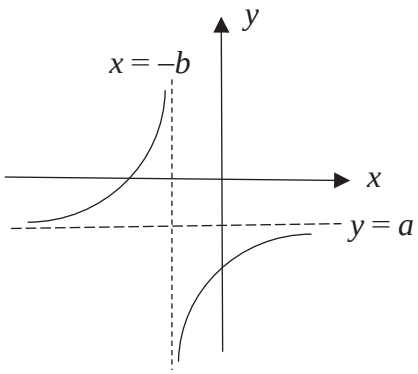
(ii)

$$g(x) = \frac{ax+1}{x+b} = a + \frac{1-ab}{x+b}$$



$R_g = \mathbb{R} \setminus \{a\}$

$D_f = \mathbb{R} \setminus \{-1\}$

For fg to exist, $R_g \subseteq D_f$

i.e $\mathbb{R} \setminus \{a\} \subseteq \mathbb{R} \setminus \{-1\}$

Hence, $a = -1$

$R_f = (-\infty, -10] \cup [-6, \infty)$

$D_g = \mathbb{R} \setminus \{-b\}$

For gf to exist, $R_f \subseteq D_g$

i.e $(-\infty, -10] \cup [-6, \infty) \subseteq \mathbb{R} \setminus \{-b\}$

Hence $-10 < -b < -6$

$\Rightarrow \quad 6 < b < 10$

(iii)

Let $y = \frac{ax+1}{x+b}$

$xy + by = ax + 1$

$x = \frac{1-by}{y-a}$

$\therefore \ g^{-1}(x) = \frac{1-bx}{x-a}$

Given $g^{-1}(x) = g(x)$ for all real values of x,

$$\frac{1-bx}{x-a} = \frac{ax+1}{x+b}$$

By comparison, $b = -a$

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9 **[Solution]**

(i)

$$\overrightarrow{OA}=\begin{pmatrix}0\\0\\20\end{pmatrix},\quad \overrightarrow{OB}=\begin{pmatrix}20\\0\\0\end{pmatrix},\quad \overrightarrow{OC}=\begin{pmatrix}0\\10\\0\end{pmatrix}$$

$$\overrightarrow{AB}-\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}20\\0\\-20\end{pmatrix},$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}0\\10\\-20\end{pmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}200\\400\\200\end{pmatrix}=200\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$\mathbf{r}\bullet\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}20\\0\\0\end{pmatrix}\bullet\begin{pmatrix}1\\2\\1\end{pmatrix}=20$$

$\therefore$ Equation of the face ABC is $x + 2y + z = 20$

Shortest distance from O to ABC

$$=\frac{20}{\sqrt{1^2+2^2+1^2}}=\frac{20}{\sqrt{6}}$$

(ii)

Method 1 (Using Unit vector)

Using **(i)**, $\overrightarrow{ON}=\frac{20}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}10\\20\\10\end{pmatrix}$

$\therefore$ Coordinates of N is $\left(\frac{10}{3},\ \frac{20}{3},\ \frac{10}{3}\right)$

Alternative Solution

Line $ON: \mathbf{r}=\lambda\begin{pmatrix}1\\2\\1\end{pmatrix},\ \lambda\in\mathbb{R}$

$$\overrightarrow{ON}\bullet\begin{pmatrix}1\\2\\1\end{pmatrix}=20\quad\Rightarrow\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}1\\2\\1\end{pmatrix}=20$$

$$\Rightarrow 6\lambda=10\Rightarrow\lambda=\frac{10}{3}$$

Coordinates of N is $\left(\frac{10}{3},\ \frac{20}{3},\ \frac{10}{3}\right)$

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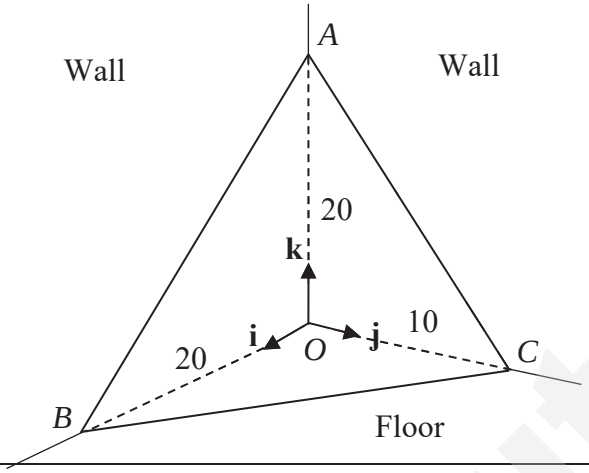
(iii)

Let θ be the required acute angle.

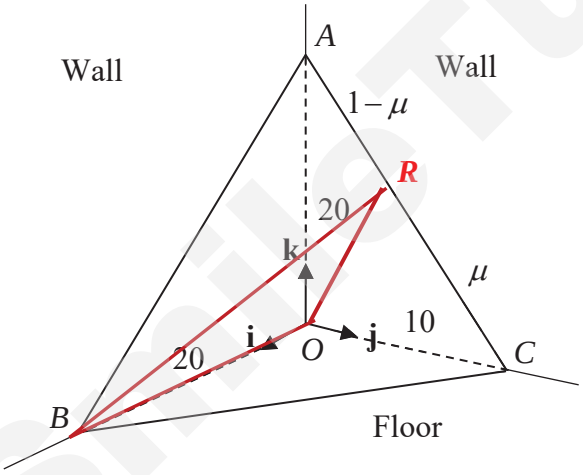
$$\cos\theta = \frac{\left|\begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{6}\sqrt{1}} = \frac{1}{\sqrt{6}}$$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$$

$\theta = 65.9°$ (1 d.p.)



(iv)



Method 1

(Use concept of point *R* is a point on plane *OBR*)

Using ratio theorem, $OR = \mu OC + (1-\mu)OA = \begin{pmatrix}0\\10\mu\\20(1-\mu)\end{pmatrix}$

Normal of plane $OBR = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\2\end{pmatrix}$

Equation of plane OBR is $\mathbf{r} \cdot \begin{pmatrix}0\\-1\\2\end{pmatrix} = 0$

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Since point R is a point on plane OBR,

$$\begin{pmatrix} 0 \\ 10\mu \\ 20(1-\mu) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = 0 \qquad \text{----- (*)}$$

$$-10\mu + 40 - 40\mu = 0 \Rightarrow \mu = \frac{4}{5}$$

$$\overrightarrow{OR} = \begin{pmatrix} 0 \\ 10\left(\frac{4}{5}\right) \\ 20\left(1-\frac{4}{5}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 4 \end{pmatrix}$$

Therefore, coordinates of R is (0, 8, 4)

Method 2 (Use concept of point *N* is a point on line *l*)

Using ratio theorem, $\overrightarrow{OR} = \mu\overrightarrow{OC} + (1-\mu)\overrightarrow{OA} = \begin{pmatrix} 0 \\ 10\mu \\ 20(1-\mu) \end{pmatrix}$

$$\overrightarrow{BR} = \overrightarrow{OR} - \overrightarrow{OB} = \begin{pmatrix} -20 \\ 10\mu \\ 20(1-\mu) \end{pmatrix}$$

Equation of line l: $\mathbf{r} = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ \mu \\ 2(1-\mu) \end{pmatrix}, \quad \mu \in \mathbb{R}$

Given N lies on line l,

$$\begin{pmatrix} \frac{10}{3} \\ \frac{20}{3} \\ \frac{10}{3} \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ \mu \\ 2(1-\mu) \end{pmatrix}$$

$$\frac{10}{3} = 20 - 2\alpha \qquad \Rightarrow \alpha = \frac{25}{3}$$

$$\frac{20}{3} = \alpha\mu \qquad \Rightarrow \mu = \frac{4}{5}$$

$$\frac{10}{3} = 2\alpha(1-\mu)$$

$$\overrightarrow{OR} = \begin{pmatrix} 0 \\ 10\left(\frac{4}{5}\right) \\ 20\left(1-\frac{4}{5}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 4 \end{pmatrix}$$

Therefore, coordinates of R is (0, 8, 4)

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10 **[Solution]**

(i)

$x = 2t + \sin 2t \quad \Rightarrow \quad \dfrac{dx}{dt} = 2 + 2\cos 2t$

$y = \cos 2t \quad \Rightarrow \quad \dfrac{dy}{dt} = -2\sin 2t$

$$\frac{dy}{dx} = \frac{-2\sin 2t}{2 + 2\cos 2t} = \frac{-\sin 2t}{1 + \cos 2t}$$

$$= -\frac{2\sin t \cos t}{1 + (2\cos^2 t - 1)}$$

$$= -\frac{\sin t}{\cos t}$$

$$= -\tan t \quad \text{(shown)}$$

As $t = \dfrac{\pi}{2}$, $\dfrac{dy}{dx} = -\tan\dfrac{\pi}{2}$ is undefined

The tangent to the curve is parallel to the y-axis.

(ii)

When $y = 0$, $\quad \cos 2t = 0$

For $0 \le t \le \pi$, $\quad 2t = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$

$t = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$

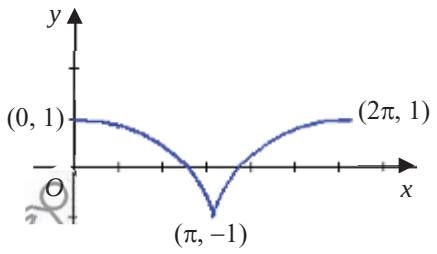
$\Rightarrow x_1 = \dfrac{\pi}{2} + 1$ or $x_2 = \dfrac{3\pi}{2} - 1$

(iii)

$t = 0,\ x = 0\ \&\ y = 1$

$t = \pi,\ x = 2\pi\ \&\ y = 1$

$t = \dfrac{\pi}{2},\ x = \pi\ \&\ y = -1$



(iv)

$$\text{Area} = -\int_{x_1}^{x_2} y \, dx$$

$$= -\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos 2t \times (2 + 2\cos 2t) \, dt$$

$$= -\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2\cos 2t + 2\cos^2 2t \, dt$$

$$= -\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2\cos 2t + (1 + \cos 4t) \, dt$$

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	$= -\left\{\left[\sin 2t\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \left[t + \frac{\sin 4t}{4}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\right\}$ $= -\left\{\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) + \left(\left(\frac{3\pi}{4} + \frac{\sin 3\pi}{4}\right) - \left(\frac{\pi}{4} + \frac{\sin\pi}{4}\right)\right)\right\}$ $= -\left(-2 + \frac{\pi}{2}\right)$ $= 2 - \frac{\pi}{2}$
(v)	From **(ii)**, $x_1 = \frac{\pi}{2} + 1,\ t = \frac{\pi}{4},\ \ \frac{dy}{dx} = -\tan\frac{\pi}{4} = -1$ Equation of tangent: $y = -\left(x - \left(\frac{\pi}{2} + 1\right)\right)$ When the tangent intersects C again, $\cos 2t = -\left(2t + \sin 2t - \left(\frac{\pi}{2} + 1\right)\right)$ i.e. $\cos 2t + \sin 2t = -2t + \frac{\pi}{2} + 1$ Using GC, $t = 1.99$ (3 s.f) , $t = 0.785$ (i.e. $\pi/4$) Thus the tangent to C intersects C again at $t = 1.99$ (3 s.f).

End of Paper

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1 **(i)** Sketch the curve with equation $y = 3\ln x + 3$, giving the equation of the asymptote and the coordinates of any points of intersection with the axes. On the same diagram, sketch the curve with equation $y = x^{\frac{5}{3}}$. [3]

(ii) Solve the inequality $3\ln x + 3 \geqslant x^{\frac{5}{3}}$. [2]

2 Given that $\left(3x^2 - y^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2xy$, and that $y = 1$ when $x = 0$, find the values of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = 0$.

Hence write down the first two non-zero terms in the Maclaurin series for y. [4]

3 **(i)** Using integration, show that $\int \mathrm{e}^{2x}\sin x\,\mathrm{d}x = \dfrac{1}{5}\mathrm{e}^{2x}\left(2\sin x - \cos x\right) + C$, where C is an arbitrary constant. [4]

(ii) Hence find the gradient of the curve

$$y = \mathrm{e}^{2x+2}\left[2\sin\left(x+1\right) - \cos\left(x+1\right)\right]$$

at $x = \dfrac{\pi}{2} - 1$, leaving your answer in an exact form. [2]

4 The curve C has equation $\dfrac{\left(y-2\right)^2}{9} - \dfrac{\left(x-3\right)^2}{4} = 1$.

(i) Sketch C, giving the equations of its asymptotes and the coordinates of any turning points. [4]

The curve D has equation $12y^2 - 48y + 48 + ax^2 - 6ax - 3a = 0$, where a is a positive constant.

(ii) Find the set of values of a for which C and D do not intersect. [2]

5 **(i)** Find the binomial expansion for $\dfrac{1+x^2}{2-x^2}$, up to and including the term in x^4. Give the coefficients as exact fractions in their simplest form. [3]

(ii) Find the set of values of x for which this expansion is valid. [2]

(iii) Using your answer in part **(i)**, find the series expansion of $\left(2-x^2\right)^{-2}$, up to and including the term in x^2. [3]

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6 Functions f and g are defined by

$$\mathrm{f}: x \mapsto 1-\lambda x^2, \quad x \in \mathbb{R}, x<-1,$$
$$\mathrm{g}: x \mapsto 2-\mathrm{e}^{1-x}, \quad x \in \mathbb{R}, x \leqslant 1,$$

where λ is a positive constant.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Show that gf exists and find the range of gf, giving your answer in terms of λ. [4]

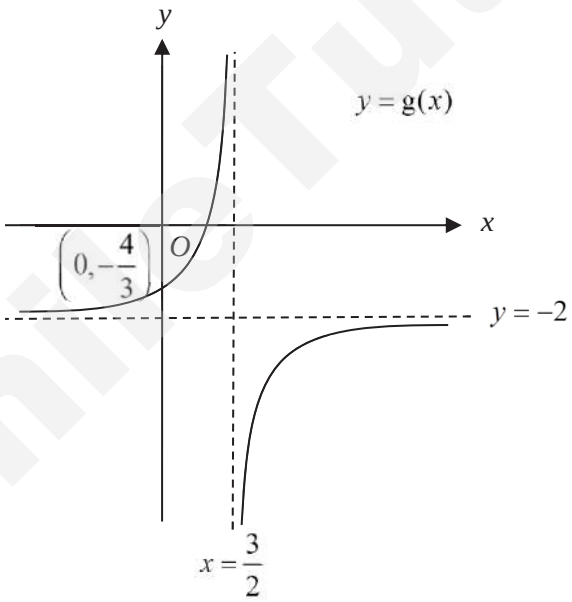
The function h is defined by

$$\mathrm{h}: x \mapsto 1-\lambda x^2, \quad x \in \mathbb{R}, x<k,$$

where k is a constant.

Determine the set of values of k for which the range of gh is the same as the range of g. [1]

7 The diagram below shows the graph of $y=\mathrm{g}(x)$, where $\mathrm{g}(x)=\dfrac{ax+b}{2x+c}$.



Determine the values of a, b and c. [3]

It is also given that $\mathrm{g}(x)=\mathrm{f}\left(\dfrac{1}{2}x-1\right)$. State a sequence of 2 transformations that will map the graph of $y=\mathrm{g}(x)$ to the graph of $y=\mathrm{f}(x)$.

Find $\mathrm{f}(x)$. [5]

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8 **(a)** Find $\int \sin 2x \cos^6 2x \, \mathrm{d}x$. [2]

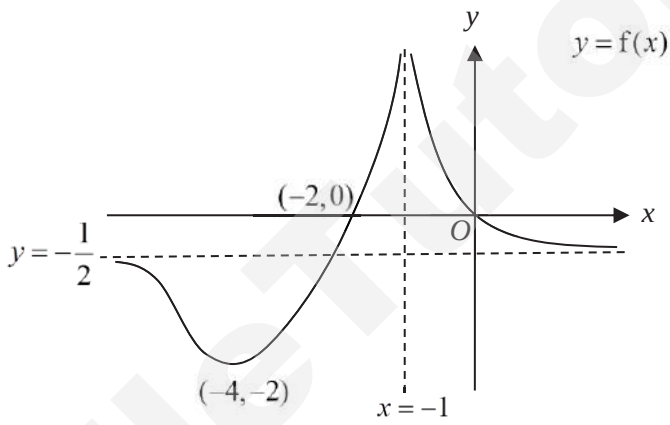
(b) **(i)** Find $\int \frac{3x}{x^2+2} \, \mathrm{d}x$. [2]

(ii) Show that $\frac{2}{x-3} + \frac{3x+1}{x^2+2} = \frac{Ax^2+Bx+1}{(x-3)(x^2+2)}$, where A and B are constants to be found. [2]

(iii) Using your answers to parts **(i)** and **(ii)**, find $\int_0^2 \frac{10x^2-16x+2}{(x-3)(x^2+2)} \, \mathrm{d}x$.

Give your answer in the form $a\tan^{-1}b - \ln c$, where a, b and c are constants to be determined. [4]

9



The diagram above shows the graph of $y = \mathrm{f}(x)$. The curve passes through (0,0) and (– 2,0), and has a minimum point at (– 4, –2). The curve has asymptotes $x = -1$ and $y = -\frac{1}{2}$.

(a) State the coordinates of the turning point of the curve $y = 1 - 2\mathrm{f}(x)$. [1]

(b) On separate diagrams, sketch the graphs of

(i) $y = \mathrm{f}(|x|)$, [2]

(ii) $y = \frac{1}{\mathrm{f}(x)}$, [3]

(iii) $y = \mathrm{f}'(x)$. [2]

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10 A curve C has equation $y = \dfrac{\alpha x^2 + x + 1}{x + 2}$, where α is a real, non-zero constant.

Show that if C has 2 stationary points, then $\alpha < 0$ or $\alpha > k$, where k is a constant to be determined. [4]

Sketch the curve C for $\alpha = -1$, giving the equations of asymptotes, the coordinates of stationary points and points of intersection with the axes. [4]

By considering C and an appropriate line, find the range of values of m such that the equation

$$-x^2 + x + 1 = mx^2 + (2m + 3)x + 6$$

has exactly 1 positive root and 1 negative root. [3]

[Questions 11 & 12 are printed on the next page]

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11 During test drives, a sensor is used to record the number of revolutions per minute made by a particular wheel of a vehicle.

(a) In a test drive, a car is initially travelling at constant speed but starts to slow down due to engine malfunction. The total number of revolutions is recorded on every 1-minute interval after malfunction. In the first n minutes, the total number of revolutions recorded, S_n, is given by $S_n = 54n(29-n)$.

(i) Show that the number of revolutions recorded in each minute after the malfunction occurs, before the car comes to a complete stop, follows an arithmetic progression. [3]

(ii) The diameter of each wheel of the car is measured to be 61 cm. Show that the car travels a distance of 21.7 km, correct to 3 significant figures, from the time the malfunction occurs until it comes to a complete stop. [3]

(b) In another test drive, a truck was travelling at constant speed before it entered the roughterrain. Before entering the rough terrain, the wheel was rotating at 486 revolutions per minute (rpm). After entering the rough terrain, engine power increases the rate of rotation by 20 rpm almost immediately at the beginning of each minute. However, at the end of each minute, friction slows the truck down such that the rate of rotation is $\frac{2}{3}$ of that recorded at the beginning of that minute. The rate of rotation of the wheel at the end of the the nth minute after entering the rough terrain is denoted by v_n rpm.

(i) Show that $v_n = 446\left(\frac{2}{3}\right)^n + 40$. [4]

(ii) Explain why the wheel always rotates at a rate of more than 40 rpm. [2]

(iii) Given that the rate of rotation of the wheel was less than 45 rpm at the end of m minutes, find the least integer value of m. [2]

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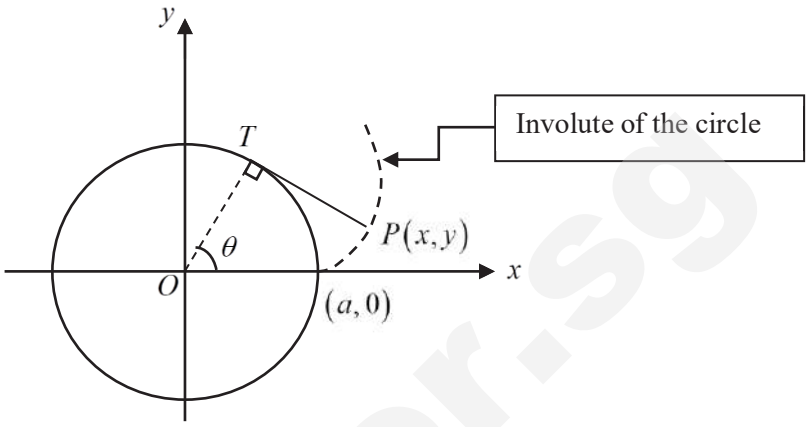
12 The diagram shows a string that is unwound from a circle while being held taut. The curve traced by the end point *P* of the string is called the *involute* of the circle. One of the major applications of involute of circle is in designing of gears for revolving parts where gear tooth follow the shape of involute.



A circle has fixed radius *a* units and centre *O* and the initial position of *P* is at $(a, 0)$.

The parameter θ, $\left(0 \leqslant \theta \leqslant \frac{\pi}{2}\right)$, is the angle measured from the positive *x*-axis to OT in the anti-clockwise direction, where *T* is the point on the circle such that *PT* is tangential to the circle.

Show that the involute has parametric equations

$$x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - \theta\cos\theta), \quad \text{for} \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}. \qquad [3]$$

The point *W* on the involute has parameter $\theta = \frac{\pi}{3}$.

(i) Show that the equation of the normal to the involute at *W* is

$$\sqrt{3}y = 2a - x. \qquad [5]$$

(ii) At *W*, *x* increases at a rate of 0.3 units per second. Given that $z = xy$, determine, in terms of *a*, the rate of change of z at *W*. [4]

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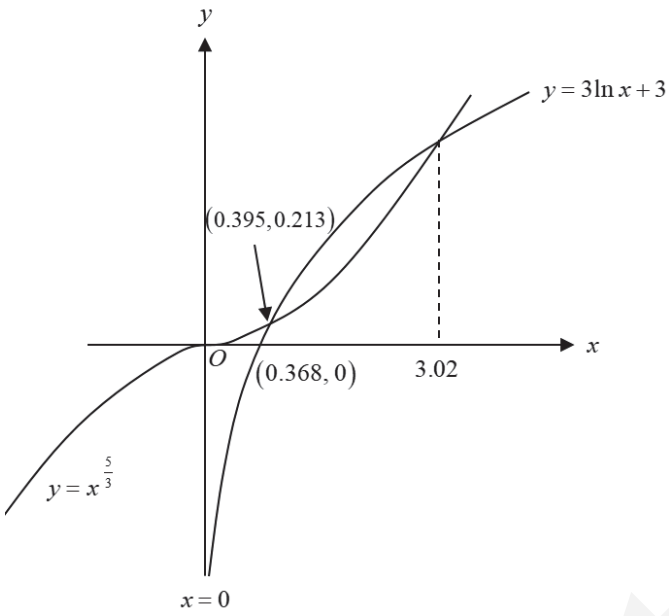
Qn	Solution
1i	
ii	Graphs of $y=3\ln x+3$ and $y=x^{\frac{5}{3}}$ intersect at $x=0.395$ and $x=3.02$. $3\ln x+3 \geq x^{\frac{5}{3}}$ $\therefore 0.395 \leq x \leq 3.02$
2	$\left(3x^2-y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2xy \quad ---(1)$ Differentiating w.r.t. x $\left(3x^2-y^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(6x-2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2y+2x\frac{\mathrm{d}y}{\mathrm{d}x}$ When $x=0$, $y=1$ $(0-1)\frac{\mathrm{d}y}{\mathrm{d}x}=2(0)(1)\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $(0-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(0-2(0))(0)=2+2(0)(0)\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2$ $\therefore$ the Maclaurin's series for y is $y=1+(0)x+\frac{-2}{2!}x^2+\ldots$ $=1-x^2+\ldots$

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Qn	Solution
3i	$\int e^{2x}\sin x\,dx$ $=\frac{1}{2}e^{2x}\sin x-\int\frac{1}{2}e^{2x}\cos x\,dx$ $=\frac{1}{2}e^{2x}\sin x-\frac{1}{2}\left[\left(\frac{1}{2}e^{2x}\cos x\right)-\int\frac{1}{2}e^{2x}(-\sin x)\,dx\right]$ $=\frac{1}{2}e^{2x}\sin x-\frac{1}{4}e^{2x}\cos x-\frac{1}{4}\int e^{2x}\sin x\,dx$ $\frac{5}{4}\int e^{2x}\sin x\,dx=\frac{1}{2}e^{2x}\sin x-\frac{1}{4}e^{2x}\cos x+D$ $\int e^{2x}\sin x\,dx=\frac{2}{5}e^{2x}\sin x-\frac{1}{5}e^{2x}\cos x+C$ $=\frac{1}{5}e^{2x}(2\sin x-\cos x)+C$
ii	Let $f(x)=e^{2x}(2\sin x-\cos x)$. Observe that $y=e^{2x+2}(2\sin(x+1)-\cos(x+1))=f(x+1)$ is a translation of $y=f(x)$ by 1 unit in the negative x-direction. Hence, the gradient of the curve $y=f(x+1)$ at $x=\frac{\pi}{2}-1$ is the gradient of the curve $y=f(x)$ at $x=\frac{\pi}{2}$, which is given by $y=f'\left(\frac{\pi}{2}\right)$. From part (i), $f'(x)=5e^{2x}\sin x$. Hence, the required gradient is $f'\left(\frac{\pi}{2}\right)=5e^{\pi}$.
4i	Asymptotes: $\frac{(y-2)^2}{9}=\frac{(x-3)^2}{4}$ $\frac{y-2}{3}=\pm\frac{x-3}{2}$ $y=\pm\frac{3(x-3)}{2}+2$ $y=\frac{3}{2}x-\frac{5}{2}$ or $y=-\frac{3}{2}x+\frac{13}{2}$

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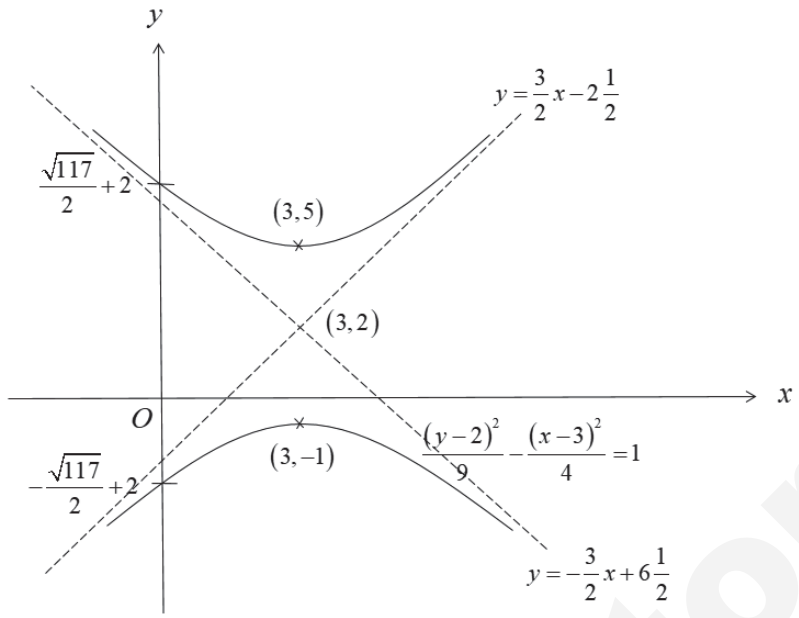
Qn	Solution
	
ii	$12y^2-48y+48+ax^2-6ax-3a=0$ $12\left(y^2-4y\right)+a\left(x^2-6x\right)-3a+48=0$ $12\left(y^2-4y+4\right)-48+a\left(x^2-6x+9\right)-9a-3a+48=0$ $12(y-2)^2+a(x-3)^2=12a$ $\frac{(y-2)^2}{a}+\frac{(x-3)^2}{12}=1$ which is an ellipse with centre $(3,2)$ and vertices at $\left(3,2+\sqrt{a}\right)$ and $\left(3,2-\sqrt{a}\right)$ For curve C and D to not intersect, $\sqrt{a}<3\Rightarrow a<9$ Since a is a positive constant, $\therefore\{a\in\mathbb{R}:0<a<9\}$
5i	$\frac{1+x^2}{2-x^2}=\left(1+x^2\right)\left(2-x^2\right)^{-1}$ $=\left(1+x^2\right)2^{-1}\left(1-\frac{x^2}{2}\right)^{-1}$ $=\frac{1}{2}\left(1+x^2\right)\left[1+(-1)\left(-\frac{x^2}{2}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x^2}{2}\right)^2+\ldots\right]$ $=\frac{1}{2}\left(1+x^2\right)\left(1+\frac{x^2}{2}+\frac{x^4}{4}+\ldots\right)$ $=\frac{1}{2}\left(1+x^2+\frac{x^2}{2}+\frac{x^4}{2}+\frac{x^4}{4}+\ldots\right)$ $=\frac{1}{2}+\frac{3x^2}{4}+\frac{3x^4}{8}+\ldots$

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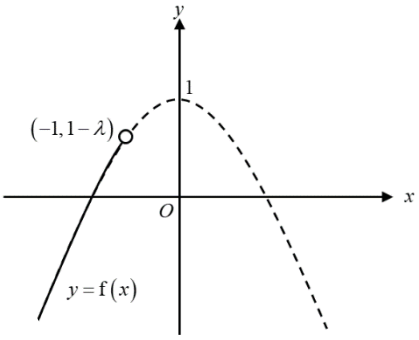
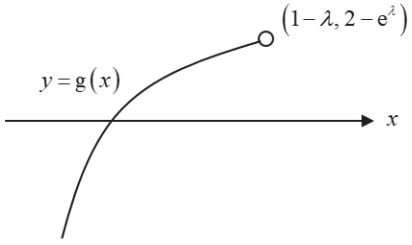
Qn	Solution
ii	For expansion to be valid, $\left\|-\frac{x^2}{2}\right\|<1 \Rightarrow \left\|\frac{x^2}{2}\right\|<1$ $\left\|x^2\right\|<2$ $x^2<2 \qquad \left(\left\|x^2\right\|=x^2 \text{ since } x^2\geq 0\right)$ $\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)<0$ $\left\{x\in\mathbb{R}:-\sqrt{2}<x<\sqrt{2}\right\}$
iii	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+x^2}{2-x^2}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}+\frac{3x^2}{4}+\frac{3x^4}{8}+\ldots\right)$ $\frac{\left(2-x^2\right)(2x)-\left(1+x^2\right)(-2x)}{\left(2-x^2\right)^2}=\frac{6x}{4}+\frac{12x^3}{8}+\ldots$ $\frac{4x-2x^3+2x+2x^3}{\left(2-x^2\right)^2}=\frac{6x}{4}+\frac{12x^3}{8}+\ldots$ $6x\left(2-x^2\right)^{-2}=\frac{6x}{4}+\frac{12x^3}{8}+\ldots$ $\left(2-x^2\right)^{-2}=\frac{1}{4}+\frac{x^2}{4}+\ldots$
6i	Let $y=\mathrm{f}(x)=1-\lambda x^2, \quad \lambda>0$ $x^2=\frac{1-y}{\lambda}$ $x=\pm\sqrt{\frac{1-y}{\lambda}}$ $=-\sqrt{\frac{1-y}{\lambda}} \quad (\because x<-1)$ $\mathrm{f}^{-1}(x)=-\sqrt{\frac{1-x}{\lambda}}$ $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}$ $=(-\infty,1-\lambda)$ 
6ii	$\mathrm{R}_{\mathrm{f}}=(-\infty,1-\lambda)$ Since $\lambda>0 \Rightarrow 1-\lambda<1$, hence $\mathrm{R}_{\mathrm{f}}\subseteq \mathrm{D}_{\mathrm{g}}=(-\infty,1]$. $\therefore$ gf exists $\mathrm{R}_{\mathrm{gf}}=\left(-\infty,2-\mathrm{e}^{\lambda}\right)$ 

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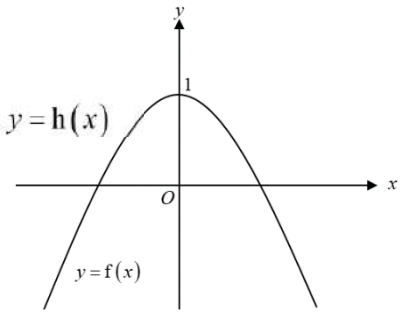
Qn	Solution
6	**Since g is a one –one function,** then if $R_{gh} = R_g$, we will have $R_h = D_g$ $R_h = D_g = (-\infty, 1]$ $\therefore \{k \in \mathbb{R} : k > 0\}$ 
7	$y = g(x) = \dfrac{ax+b}{2x+c}$ Vertical asym : $x = -\dfrac{c}{2} = \dfrac{3}{2} \Rightarrow c = -3$ Horizontal asym : $y = \dfrac{a}{2} = -2 \Rightarrow a = -4$ y-intercept : $y = \dfrac{b}{c} = -\dfrac{4}{3} \Rightarrow b = 4$ $\therefore y = g(x) = \dfrac{-4x+4}{2x-3}$ $y = f\left(\dfrac{1}{2}x - 1\right)$ $\downarrow$ Replace x by $x+2$ $y = f\left(\dfrac{1}{2}(x+2) - 1\right) = f\left(\dfrac{1}{2}x\right)$ $\downarrow$ Replace x by $2x$ $y = f\left(\dfrac{1}{2}(2x)\right) = f(x)$ The graph of $y = f\left(\dfrac{1}{2}x - 1\right)$ is translated 2 units in the negative x-direction and then stretched parallel to the x-axis by factor $\dfrac{1}{2}$ with y-axis invariant.

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Qn	Solution
	$\mathrm{f}\left(\frac{1}{2}x-1\right)=\frac{-4x+4}{2x-3}$ $\mathrm{f}\left(\frac{1}{2}x\right)=\frac{-4(x+2)+4}{2(x+2)-3}$ $=\frac{-4x-4}{2x+1}$ $\mathrm{f}(x)=\frac{-4(2x)-4}{2(2x)+1}$ $=\frac{-8x-4}{4x+1}$ **<u>Alternatively,</u>** $y=\mathrm{f}\left(\frac{1}{2}x-1\right)$ $\downarrow$ Replace x by $2x$ $y=\mathrm{f}\left(\frac{1}{2}(2x)-1\right)=\mathrm{f}(x-1)$ $\downarrow$ Replace x by $x+1$ $y=\mathrm{f}((x+1)-1)=\mathrm{f}(x)$ The graph of $y=\mathrm{f}\left(\frac{1}{2}x-1\right)$ is stretched parallel to the x-axis by factor $\frac{1}{2}$ with y-axis invariant and then translated 1 unit in the negative x-direction. $\mathrm{f}\left(\frac{1}{2}x-1\right)=\frac{-4x+4}{2x-3}$ $\mathrm{f}(x-1)=\frac{-4(2x)+4}{2(2x)-3}$ $=\frac{-8x+4}{4x-3}$ $\mathrm{f}(x)=\frac{-8(x+1)+4}{4(x+1)-3}$ $=\frac{-8x-4}{4x+1}$
8a	$\int \sin 2x\cos^6 2x\,\mathrm{d}x=-\frac{1}{2}\int -2\sin 2x\cos^6 2x\,\mathrm{d}x$ $=-\frac{1}{2}\left(\frac{\cos^7 2x}{7}\right)+C$ $=-\frac{\cos^7 2x}{14}+C$

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Qn	Solution
8bi	$\int \frac{3x}{x^2+2}\,dx = \frac{3}{2}\int \frac{2x}{x^2+2}\,dx$ $= \frac{3}{2}\ln\left\|x^2+2\right\| + C$ $= \frac{3}{2}\ln\left(x^2+2\right) + C \ (\because x^2+2>0)$
8bii	$\frac{2}{x-3}+\frac{3x+1}{x^2+2} = \frac{2\left(x^2+2\right)+(3x+1)(x-3)}{(x-3)\left(x^2+2\right)}$ $= \frac{2x^2+4+3x^2-8x-3}{(x-3)\left(x^2+2\right)}$ $= \frac{5x^2-8x+1}{(x-3)\left(x^2+2\right)}$
biii	$\int_0^2 \frac{10x^2-16x+2}{(x-3)\left(x^2+2\right)}\,dx$ $= 2\int_0^2 \frac{5x^2-8x+1}{(x-3)\left(x^2+2\right)}\,dx$ $= 2\int_0^2 \left(\frac{2}{x-3}+\frac{3x+1}{x^2+2}\right)dx$ $= 2\int_0^2 \left(\frac{2}{x-3}+\frac{3x}{x^2+2}+\frac{1}{x^2+2}\right)dx$ $= 2\left[2\ln\left\|x-3\right\| + \frac{3}{2}\ln\left(x^2+2\right) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^2$ $= 2\left[2\ln 1 + \frac{3}{2}\ln(6) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{2}{\sqrt{2}}\right)\right]$ $-2\left[2\ln 3 + \frac{3}{2}\ln(2) + \frac{1}{\sqrt{2}}\tan^{-1}(0)\right]$ $= 3\ln 6 + \sqrt{2}\tan^{-1}\left(\sqrt{2}\right) - 4\ln 3 - 3\ln 2$ $= \sqrt{2}\tan^{-1}\left(\sqrt{2}\right) - \ln 3$
9a	Effects on turning point $(-4,-2)$ on $y=f(x)$ ↓ Reflection in x – axis, and stretch // to y-axis, factor 2, x - axis invariant $(-4,4)$ on $-\frac{y}{2}=f(x)$ $\left[\text{i.e. } y=-2f(x)\right]$ ↓ Translate 1 unit in the positive y - direction

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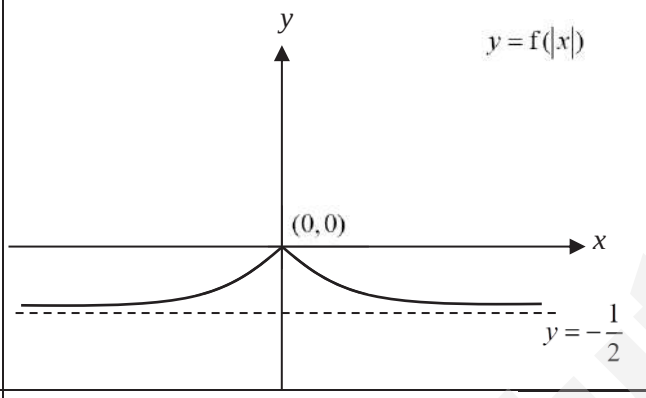
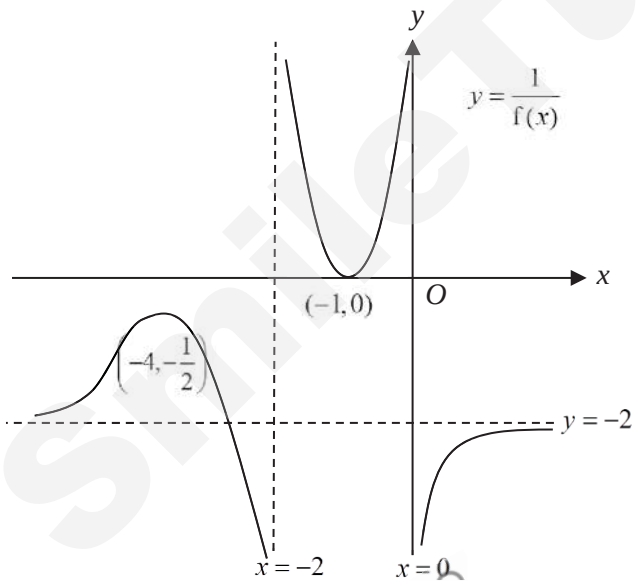
Qn	Solution
	$(-4,5)$ on $y=1-2\mathrm{f}(x)$ $(-4,5)$ is the turning point of the curve $y=1-2\mathrm{f}(x)$ **<u>Alternatively,</u>** Sub $x=-4$ $y=1-2\mathrm{f}(-4)=1-2(-2)=5$ $(-4,5)$ is the turning point of the curve $y=1-2\mathrm{f}(x)$
bi	
bii	

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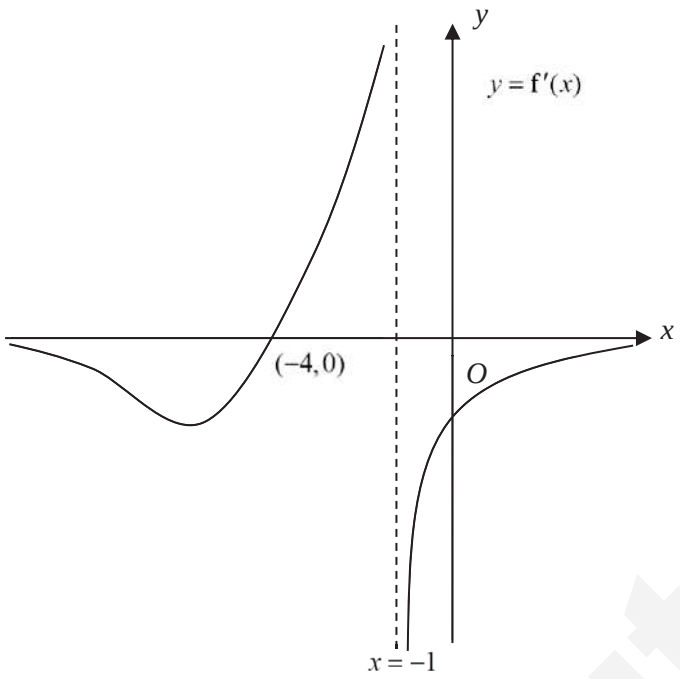
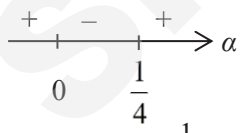
Qn	Solution
b iii	
10	$y=\dfrac{\alpha x^2+x+1}{x+2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x+2)(2\alpha x+1)-(\alpha x^2+x+1)}{(x+2)^2}=\dfrac{\alpha x^2+4\alpha x+1}{(x+2)^2}$ For C to have 2 stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ has 2 real roots. For $\alpha x^2+4\alpha x+1=0$ to have 2 real roots, Discrimant > 0 $(4\alpha)^2-4\alpha>0$ $4\alpha(4\alpha-1)>0$  $\alpha<0$ or $\alpha>\dfrac{1}{4}$ $\therefore k=\dfrac{1}{4}$ **Alternatively,** $y=\dfrac{\alpha x^2+x+1}{x+2}=\alpha x+1-2\alpha+\dfrac{4\alpha-1}{x+2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\alpha-\dfrac{4\alpha-1}{(x+2)^2}$

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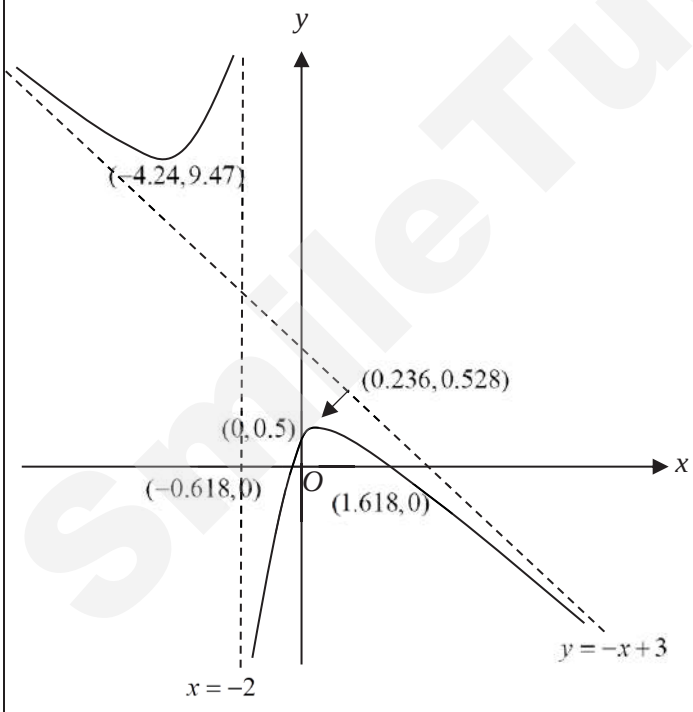
Qn	Solution
	For C to have 2 stationary points, $\frac{dy}{dx}=0$ has 2 real roots. $\frac{dy}{dx}=0 \Rightarrow \alpha-\frac{4\alpha-1}{(x+2)^2}=0$ $\Rightarrow (x+2)^2=\frac{4\alpha-1}{\alpha}$ for equation to have 2 real roots, $\frac{4\alpha-1}{\alpha}>0$ + – + α 0 $\frac{1}{4}$ $\alpha<0$ or $\alpha>\frac{1}{4}$ $\therefore k=\frac{1}{4}$  $-x^2+x+1=mx^2+(2m+3)x+6$ $=mx(x+2)+3(x+2)$ $\frac{-x^2+x+1}{x+2}=mx+3$ Add line $y=mx+3$ such that there are exactly 2 points of intersection with C, one with $x<0$ and one with $x>0$. For the equation to have 1 positive root and 1 negative root, $m<-1$

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Qn	Solution
11 ai	No. of revolutions in n^{th} minute, $u_n = S_n - S_{n-1}$ $= 54n(29-n) - 54(n-1)(29-n+1)$ $= 1620 - 108n$ $u_n - u_{n-1} = 1620 - 108n - [1620 - 108(n-1)]$ $= -108$ Since $u_n - u_{n-1} = -108$ is a constant (independent of n), the number of revolutions made in each minute follows an arithmetic progression.
11 aii	For $u_n = 1620 - 108n \le 0$, $n \ge 15$. Total number of revolutions, $S_{15} = 54(15)(29-15) = 11340$ Distance travelled $= 11340 \times \pi \times 61$ cm $= 21732165$ cm $= 21.7$ km (to 3 s.f.) (shown)
bi	$v_1 = (486+20)\left(\frac{2}{3}\right)$ $v_2 = \left[(486+20)\left(\frac{2}{3}\right)+20\right]\left(\frac{2}{3}\right)$ $= 486\left(\frac{2}{3}\right)^2 + 20\left(\frac{2}{3}\right)^2 + 20\left(\frac{2}{3}\right)$ $\vdots$ $v_n = 486\left(\frac{2}{3}\right)^n + 20\left(\frac{2}{3}\right)^n + 20\left(\frac{2}{3}\right)^{n-1} + \ldots + 20\left(\frac{2}{3}\right)$ $= 486\left(\frac{2}{3}\right)^n + 20\left[\frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\left(\frac{2}{3}\right)}\right]$ $= 486\left(\frac{2}{3}\right)^n + 40\left(1-\left(\frac{2}{3}\right)^n\right)$ $= 446\left(\frac{2}{3}\right)^n + 40$ **(shown)**
bii	Since $\left(\frac{2}{3}\right)^n > 0$ for all $n > 0$, $v_n = 446\left(\frac{2}{3}\right)^n + 40 > 40$. Thus, the wheel always rotates at a rate of more than 40 rpm.

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biii

$$446\left(\frac{2}{3}\right)^m + 40 < 45$$

$$\left(\frac{2}{3}\right)^m < \frac{5}{446}$$

$$m > \ln\frac{5}{446} \div \ln\frac{2}{3}$$

$$m > 11.1 \quad \text{(to 3 s.f.)}$$

Least $m = 12$

Alternative method

$$446\left(\frac{2}{3}\right)^m + 40 < 45$$

m	$446\left(\frac{2}{3}\right)^m + 40$
11	$45.156 > 45$
12	$43.437 < 45$
13	$42.292 < 45$

From the GC,

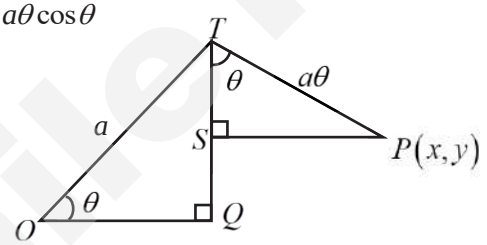
Least $m = 12$

12

$OQ = a\cos\theta; \quad TQ = a\sin\theta$

$TP = a\theta$ (arc length of unit circle)

$SP = a\theta\sin\theta; \quad TS = a\theta\cos\theta$



$x = OQ + SP$

$= a\cos\theta + a\theta\sin\theta$ (shown)

$y = TQ - TS$

$= a\sin\theta - a\theta\cos\theta$ (shown)

i

$$\frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

When $\theta = \frac{\pi}{3}$

$$x = a\left(\cos\frac{\pi}{3} + \frac{\pi}{3}\sin\frac{\pi}{3}\right) = a\left(\frac{1}{2} + \frac{\pi\sqrt{3}}{6}\right)$$

$$y = a\left(\sin\frac{\pi}{3} - \frac{\pi}{3}\cos\frac{\pi}{3}\right) = a\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$$

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Qn	Solution
	Gradient of normal $= -\dfrac{1}{\tan\frac{\pi}{3}} = -\dfrac{1}{\sqrt{3}}$ Equation of normal at W is $y - a\left(\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}\right) = -\dfrac{1}{\sqrt{3}}\left(x - \dfrac{a}{2} - \dfrac{\pi a\sqrt{3}}{6}\right)$ $\sqrt{3}y - \sqrt{3}a\left(\dfrac{\sqrt{3}}{2}\right) + \dfrac{\pi\sqrt{3}a}{6} = -x + \dfrac{a}{2} + \dfrac{\pi\sqrt{3}a}{6}$ $\sqrt{3}y = \dfrac{a}{2} + \dfrac{3a}{2} - x$ $\sqrt{3}y = 2a - x$ (shown)
12ii	At $\theta = \dfrac{\pi}{3}$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0.3$ $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}y}{\mathrm{d}x} \times \dfrac{\mathrm{d}x}{\mathrm{d}t} = \left(\tan\dfrac{\pi}{3}\right)(0.3) = \dfrac{3\sqrt{3}}{10}$ $z = xy$ -----(1) Differentiate (1) w.r.t. t $\dfrac{\mathrm{d}z}{\mathrm{d}t} = x\dfrac{\mathrm{d}y}{\mathrm{d}t} + y\dfrac{\mathrm{d}x}{\mathrm{d}t}$ $= a\left(\dfrac{1}{2} + \dfrac{\pi\sqrt{3}}{6}\right)\left(\dfrac{3\sqrt{3}}{10}\right) + a\left(\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}\right)(0.3)$ $= 0.834a$ (3sf) **<u>Alternatively,</u>** $\dfrac{\mathrm{d}z}{\mathrm{d}x} = y + x\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $= a\left(\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}\right) + \left(\sqrt{3}\right)a\left(\dfrac{1}{2} + \dfrac{\pi\sqrt{3}}{6}\right)$ $= 2.7792a$ $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{\mathrm{d}z}{\mathrm{d}x} \times \dfrac{\mathrm{d}x}{\mathrm{d}t} = (2.7792a)(0.3) = 0.834a$ (3sf)

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Questions from 2019 YIJC Promos

1 Without using a calculator, solve the inequality

$$\frac{x^2-4x+7}{2x^2+9x-5}<0\,.$$ [4]

2 Removed (not in syllabus)

3 Differentiate $2xe^{\cos 3x}$ with respect to x. Hence find $\int \frac{e^{\cos 3x}(1-3x\sin 3x)}{\sqrt{1+2xe^{\cos 3x}}}\,dx$. [3]

4 **(a)** Find $\int x(\ln x)^2\,dx$. [4]

(b) The region R is bounded by the curves $y=\tan x$, $y=\cos 2x$ and the y-axis. Find the numerical value of the volume of the solid obtained when R is rotated through 2π radians about the y-axis. [3]

5 A curve $y=f(x)$ undergoes, in succession, the following transformations.

A: A translation of 1 unit in the negative x-direction.

B: A reflection about the y-axis.

C: A scaling parallel to the y-axis with scale factor of 2.

The equation of the resulting curve is $y=g(x)$, where $g(x)=\frac{4x-1}{x^2+3}$.

(i) Find $f(x)$. [3]

(ii) Find the exact area of the region bounded by the graph of $y=g(x)$, the x-axis and the lines $x=1$ and $x=3$. [4]

6 The function f is defined as follows.

$$f: x \mapsto x^2-3x+2, \qquad x\in\mathbb{R},\ x\geq\frac{3}{2}\,.$$

(i) Find $f^{-1}(x)$ and write down the domain and range of f^{-1}. [4]

The function g is defined as follows.

$$g: x \mapsto \ln x, \qquad x\in\mathbb{R},\ x>0\,.$$

(ii) Explain why the composite function gf^{-1} exists. [1]

(iii) Find $gf^{-1}(x)$, stating the domain and range of gf^{-1}. [3]

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7 The curve C has equation

$$(2x-y)^2 = x+2y\,.$$

It is given that C has only one turning point.

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2 = \dfrac{-5}{4x-2y+2}$. [3]

(ii) Hence, or otherwise, show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = k\left(\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2\right)^3$, where k is a constant to be determined. [3]

(iii) Hence state, with a reason, whether the turning point is a maximum or a minimum. [2]

8 Removed (not in syllabus)

9 **(a)** The function f is given by

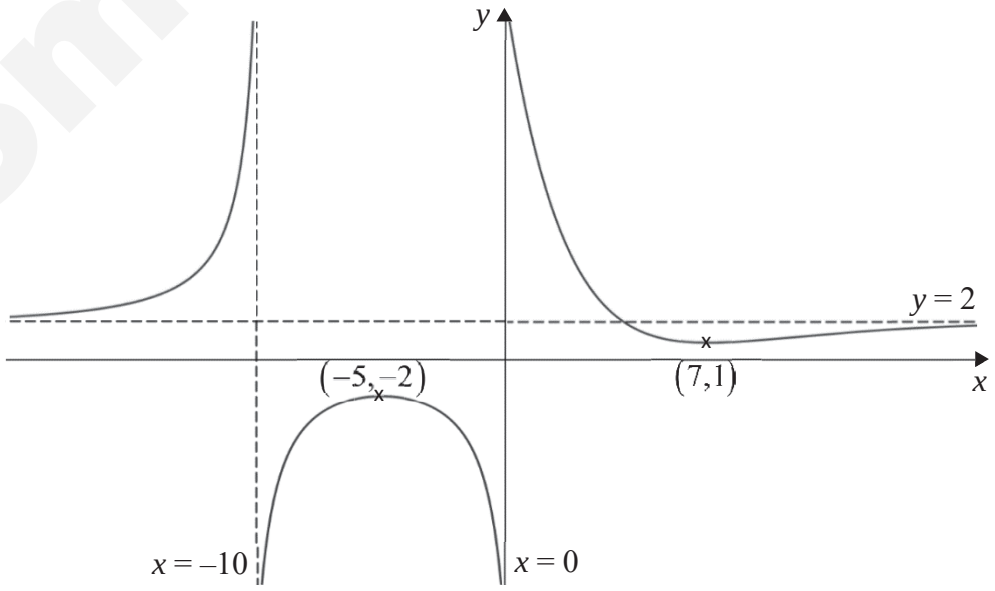
$$\mathrm{f}(x) = \begin{cases} x+1, & 0 \le x < 1 \\ -x+3, & 1 \le x < 3 \\ 0, & 3 \le x < 4 \end{cases}$$

and $\mathrm{f}(x) = \mathrm{f}(x+4)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-2 \le x \le 7$. [3]

(ii) Hence find $\int_{-2}^{7} \mathrm{f}(x)\,\mathrm{d}x$. [2]

(b) The diagram shows the curve $y = \mathrm{g}(x)$ with asymptotes $y = 2$, $x = -10$ and $x = 0$. The maximum point and the minimum point of the curve are $(-5,-2)$ and $(7,1)$ respectively.



On separate diagrams, sketch the graphs of

(i) $y = \dfrac{1}{\mathrm{g}(x)}$, [3]

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(ii) $y = \mathrm{g}'(x)$, [3]

stating clearly the equations of any asymptotes, the coordinates of any turning points and any points of intersection with the x- and y-axes.

10 A curve C has parametric equations

$$x = 2\sin^3 t\ ,\ y = 5\cos t \ \text{ for } 0 < t \leq \frac{\pi}{2}.$$

(i) Sketch the curve C, stating clearly the coordinates of the points of intersection with the axes. [2]

(ii) The tangent to the curve at the point P where $t = \frac{\pi}{4}$ is denoted by l. Show that the equation of l is $3y + 5x = 10\sqrt{2}$. [4]

(iii) The tangent l meets the x-axis at Q. Given that the point R has coordinates $\left(\frac{1}{\sqrt{2}}, 0\right)$, find the exact area of triangle PQR. [2]

(iv) Show that the area of the region bounded by the curve C, the line $x = \frac{1}{\sqrt{2}}$ and the x-axis is given by $\frac{15}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 2t \, \mathrm{d}t$. Hence find the exact area of the region bounded by the curve C, the tangent l and the x-axis. [6]

11 [It is given that a cone with base radius r and height h has volume $\frac{1}{3}\pi r^2 h$ and external curved surface area, also known as lateral area, $\pi r\sqrt{r^2 + h^2}$.]

A conical paper cup is to hold a fixed volume, k cm^3 of water. It is given that the cup has height h cm, base radius r cm and lateral area S cm^2.

(i) Show that $S^2 = \pi^2 r^4 + \frac{9k^2}{r^2}$. [2]

(ii) Given that the lateral area is a minimum, use differentiation to find the values of r and h in terms of k. Simplify your answers. [5]

It is now given that the volume of the cup is 125 cm^3.

(iii) Sketch the graph of S against r, indicating clearly the coordinates of the minimum point. [3]

(iv) It is also given that the lateral area is 120 cm^2 and the base radius is less than 4 cm. Find the values of r and h. [2]

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1	Equations & Inequalities	$-5 < x < \frac{1}{2}$
3	Integration & Applications	$2e^{\cos 3x}(1-3x\sin 3x)$; $\left(1+2xe^{\cos 3x}\right)^{\frac{1}{2}}+C$
4	Integration & Applications	$\frac{x^2}{2}(\ln x)^2-\frac{x^2}{2}\ln x+\frac{x^2}{4}+C$ (a) $\left(\text{or } \frac{x^2}{4}\left[2(\ln x)^2-2\ln x+1\right]+C\right)$ (b) 0.324 (to 3 s.f.)
5	Graphs & Transformations	(i) $f(x)=\frac{3-4x}{2(x-1)^2+6}$ or $\frac{3-4x}{2x^2-4x+8}$ (ii) $2\ln 3-\frac{\pi}{6\sqrt{3}}$
6	Functions	(i) $f^{-1}(x)=\frac{3}{2}+\sqrt{x+\frac{1}{4}}$, $D_{f^{-1}}=\left[-\frac{1}{4},\infty\right)$, $R_{f^{-1}}=\left[\frac{3}{2},\infty\right)$ (iii) $gf^{-1}(x)=\ln\left(\frac{3}{2}+\sqrt{x+\frac{1}{4}}\right)$, $D_{gf^{-1}}=\left[-\frac{1}{4},\infty\right)$, $R_{gf^{-1}}=\left[\ln\left(\frac{3}{2}\right),\infty\right)$ or $[0.405,\infty)$
7	Differentiation & Applications	(ii) $k=-\frac{2}{5}$
9	Graphs & Transformations	(a)(ii) $\frac{15}{2}$
10	Integration & Applications	(iii) $\frac{15}{4}$ (iv) $\frac{15}{4}-\frac{15}{16}\pi$
11	Differentiation & Applications	(ii) $r=\left(\frac{9k^2}{2\pi^2}\right)^{\frac{1}{6}}$ or $\left(\frac{9}{2}\right)^{\frac{1}{6}}\left(\frac{k}{\pi}\right)^{\frac{1}{3}}$, $h=\left(\frac{6k}{\pi}\right)^{\frac{1}{3}}$ (iv) $r=3.25$ (3 s.f.), $h=11.3$ (3 s.f.)

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