2018

H2 Math

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ANDERSON JUNIOR COLLEGE
2018 PRELIMINARY EXAMINATION
H2 MATHEMATICS 9758/01

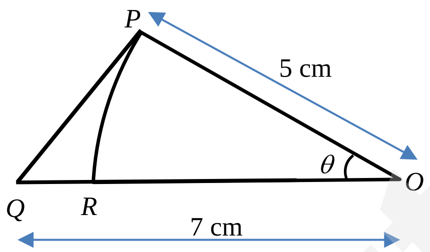
PAPER 1

Duration: 3 hours

Answer ALL questions.

1 Differentiate e^{x^2+1} with respect to x. Hence, find $\int x^3 e^{x^2+1}\, dx$. [3]

2



The diagram shows a triangle OPQ with $OP = 5$ cm, $OQ = 7$ cm and $\angle POQ = \theta$ radians. Given that OP and OR are radii of a circle with centre O and θ is a sufficiently small angle, show that the perimeter of PQR can be approximated by $a + b\theta + c\theta^2$ where a, b and c are constants to be determined. [5]

3 It is given that $y = \dfrac{ax+1}{x-b}$, where a, b are positive integers.

(i) Show that the equation of $y = \dfrac{ax+1}{x-b}$ can be written as $y = p + \dfrac{q}{x-b}$, where p and q are constants to be found in terms of a and b. [1]

(ii) Hence, describe a sequence of three transformations which transforms the graph of $y = \dfrac{ax+1}{x-b}$ on to the graph of $y = \dfrac{1}{x}$. [3]

4 **(i)** Express $\dfrac{1}{(r-1)r(r+1)}$ in partial fractions. [2]

(ii) Hence find $\displaystyle\sum_{r=2}^{n} \frac{4}{(r-1)r(r+1)}$ in terms of n. [4]

(iii) State the sum to infinity of the series in part **(ii)**. Hence, find the smallest value of n for which $\displaystyle\sum_{r=2}^{n} \frac{4}{(r-1)r(r+1)}$ is within 10^{-5} of the sum to infinity. [3]

5 It is given that $\mathrm{f}(x) = \ln(\sin x + \cos x)$ for $-\frac{\pi}{4} < x \le \frac{\pi}{4}$.

(i) Show that $\mathrm{f}''(x) + [\mathrm{f}'(x)]^2 + 1 = 0$. [2]

(ii) By further differentiation of the result in part **(i)**, find the Maclaurin series for $\mathrm{f}(x)$ up to and including the term in x^3. [4]

(iii) Using the expansion obtained in part **(ii)** and the standard series from the List of Formulae (MF26), find the series expansion of $\mathrm{g}(x) = \ln(1 + \tan x)$ for $-\frac{\pi}{4} < x \le \frac{\pi}{4}$ as far as the term in x^3. [4]

6 A curve C has parametric equations

$$x = \frac{2}{1+t^2}, \quad y = \ln(2-t), \quad \text{where } -5 < t < 2 .$$

(i) Write down the equation of the vertical asymptote of C. [1]

(ii) Find the value of t, where $t < -0.5$, at the point on C where the tangent has gradient $-\frac{1}{3}$. [4]

(iii) Hence find the equation of this tangent. [2]

(iv) Find the point of intersection of C and the vertical asymptote found in part **(i)**. [2]

7 **Do not use a calculator in answering this question.**

(a) Given that $z = -2 + 3\mathrm{i}$ is a root of the equation $2z^2 + (-1+4\mathrm{i})z + c = 0$, find the complex number c and the other root. [4]

(b) The complex number u is given by $u = \cos\theta + \mathrm{i}\sin\theta$, where $0 < \theta < \pi$.

(i) Express $u - 1$ in terms of $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$. Hence, find $|u-1|$ and show $\arg(u-1) = \frac{\pi}{2} + \frac{\theta}{2}$. [4]

(ii) Given that $v = -\sqrt{3} + \mathrm{i}$ and $(v^*u)^8$ is real and negative, find the smallest value of θ in terms of π. [4]

8 Referred to the origin O, a non-zero vector, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, makes an angle of α, β and γ with the positive x, y and z-axis respectively.

(i) Explain with clear workings, if $\alpha = \beta = \gamma = 45°$ is possible. [2]

It is now given that $\alpha = \beta = 45°$ and $\gamma = 90°$.

(ii) Find $\mathbf{r}$ in terms of x. [2]

(iii) $OABC$ is a tetrahedron. $\overrightarrow{OA}$ is parallel to $\mathbf{r}$ and the point B is the foot of the perpendicular from the point A to the y-axis. $\overrightarrow{AC}$ is parallel to the z-axis and $AC = AB$.

Given that the volume of tetrahedron $OABC$ is 36 unit^3, by considering the area of triangle OAB in terms of x, find the coordinates of a possible point C. [4]

[Volume of tetrahedron = $\frac{1}{3}(\text{base area})(\text{height})$]

9 The functions f and g are defined by

$$\mathrm{f} : x \mapsto (x+3)|x-2|, \quad x \in \mathbb{R},$$
$$\mathrm{g} : x \mapsto \frac{24}{x^2+3} - 1, \quad x \in \mathbb{R},\ x > 0.$$

(i) Explain why the function f^{-1} does not exist. [1]

In the rest of the question, the domain of f is restricted to $[k, 2]$ where k is the least value such that f^{-1} exists.

(ii) Write down the value of k. Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

(iii) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram, showing clearly the relationship between the two graphs. Hence, find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [4]

(iv) Determine whether $\mathrm{g}^{-1}\mathrm{f}$ exists. [2]

(v) Show that g is a strictly decreasing function. Hence, without finding g^{-1}, find the range of values of x for which $\mathrm{g}^{-1}(\mathrm{f}(x)) < 1$, giving your answer in exact form. [3]

10 The human body carries out one of its main functions by consuming food and turning it into usable energy. A person's mass depends both on the rate of calories intake and on the rate of calories used. A health report shows:

> ……. the calories usage of a person per day can be estimated by……… Calories usage per day = 38.5 × (mass (in kg) of the person) …..

A man with mass m kg follows a dietary intake of u calories per day. Taking $\frac{\mathrm{d}m}{\mathrm{d}t}$ to represent the rate of change of his mass with respect to time t days, and assuming that the rate of mass change is proportional to the net excess (or deficit) in the number of calories per day, show that $\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{u}{7700} - \frac{m}{200}$. [2]

(i) Assuming that the rate of calories intake is a constant, find m in terms of u and t. [4]

(ii) A person with initial mass 80kg adopts a diet of 2500 calories per day. Write down the mass m in terms of t for this individual. [2]

A weight management company applies another model of mass function M with a differential equation given by $38.5\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{77(77M+154t)+76}{77M+154t+1}$.

Using the substitution $z = 38.5M + 77t$, find the general solution of M in terms of t. [5]

11 Mr and Mrs Lai have decided to buy an education investment fund for their daughter, Adeline, on the day she turns one-year old to provide financial security for her future education.

An insurance company offers two different types of education investment funds as shown below.

SupremeEdu Fund: There is a fixed annual contribution of \$1040 to the fund. The last contribution will be on the child's seventeenth birthday. This fund will earn 3.5% interest per annum.

UltimateEdu Fund: The first annual contribution to the fund is \$600. The subsequent contributions will increase \$80 per annum such that the second contribution will be \$680 and so on, until the last contribution on the child's seventeenth birthday. The fund will earn an annual interest of \$200.

For both funds, the contributions will be deposited into the account on the child's birthday. The interest will be credited into the account annually a day before the child's next birthday. On the child's eighteenth birthday, the total sum of money in the account will be paid out to the child.

Leave all your answers to the nearest dollar.

(i) If Mr and Mrs Lai were to choose the UltimateEdu Fund, they need to know if they are able to afford to pay for the annual contributions. Show that the last contribution to the fund is \$1880. [2]

(ii) Mr and Mrs Lai wish to invest in one of these two funds. Find the amount that Adeline would get on her eighteenth birthday from each fund. [7]

(iii) Determine, with a clear explanation, which investment fund Mr and Mrs Lai should choose. [2]

(iv) Based on the selected fund, what should be the least amount of first contribution to the fund so that Adeline will be able to receive \$50000 on her eighteenth birthday? [2]

END OF PAPER

H2 Mathematics 9758/01 (JC2) 2018
Paper 1 Solutions

Qn	Solutions
1	$\frac{\mathrm{d}}{\mathrm{d}x} e^{x^2+1} = 2x\, e^{x^2+1}$ $\int x^3 e^{x^2+1}\,\mathrm{d}x = \frac{1}{2}\int x^2\left(2x e^{x^2+1}\right)\mathrm{d}x$ $= \frac{1}{2}\left[x^2 e^{x^2+1} - \int (2x) e^{x^2+1}\,\mathrm{d}x\right]$ $= \frac{1}{2}\left[x^2 e^{x^2+1} - e^{x^2+1}\right] + c$ $= \frac{1}{2}\left[x^2 - 1\right] e^{x^2+1} + c$
2	$PQ^2 = 5^2 + 7^2 - 2(5)(7)\cos\theta$ $\Rightarrow PQ = \sqrt{74 - 70\cos\theta}$ Perimeter of PQR $= PQ + QR +$ arc length PR $= \sqrt{74-70\cos\theta} + 2 + 5\theta \approx \left(74 - 70\left(1 - \frac{\theta^2}{2}\right)\right)^{\frac{1}{2}} + 2 + 5\theta$ $= \left(4 + 35\theta^2\right)^{\frac{1}{2}} + 2 + 5\theta$ $= 2\left(1 + \frac{35}{4}\theta^2\right)^{\frac{1}{2}} + 2 + 5\theta$ $\approx 2\left(1 + \frac{35}{8}\theta^2\right) + 2 + 5\theta = 4 + 5\theta + \frac{35}{4}\theta^2$ $\therefore\ a = 4,\quad b = 5,\quad c = \frac{35}{4}$
3	**(i)** $y = \frac{ax+1}{x-b}$ $= \frac{a(x-b)+1+ab}{x-b} = a + \frac{1+ab}{x-b}$ **(ii)** The sequence of 3 transformations are: (a) translation of b units in the negative x-direction, followed by (b) translation of a units in the negative y-direction, followed by (c) scaling parallel to the y-axis with scale factor $\frac{1}{1+ab}$.
4	**(i)** $\frac{1}{(r-1)r(r+1)} = \frac{A}{r-1} + \frac{B}{r} + \frac{C}{r+1}$

By Cover Up rule or any other methods,

$A = \dfrac{1}{2}$, $B = -1$ and $C = \dfrac{1}{2}$

$$\therefore \frac{1}{(r-1)r(r+1)} = \frac{1}{2(r-1)} - \frac{1}{r} + \frac{1}{2(r+1)}$$

(ii)

$$\sum_{r=2}^{n} \frac{4}{(r-1)r(r+1)}$$

$$= 4\sum_{r=2}^{n}\left(\frac{1}{2(r-1)} - \frac{1}{r} + \frac{1}{2(r+1)}\right)$$

$$= 4\left(\frac{1}{2(1)} - \frac{1}{2} + \frac{1}{2(3)}\right.$$
$$+\frac{1}{2(2)} - \frac{1}{3} + \frac{1}{2(4)}$$
$$+\frac{1}{2(3)} - \frac{1}{4} + \frac{1}{2(5)}$$
$$\vdots$$
$$+\frac{1}{2(n-3)} - \frac{1}{n-2} + \frac{1}{2(n-1)}$$
$$+\frac{1}{2(n-2)} - \frac{1}{n-1} + \frac{1}{2(n)}$$
$$\left.+\frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}\right)$$

$$= 4\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2n} - \frac{1}{n} + \frac{1}{2(n+1)}\right)$$

$$= 1 - \frac{2}{n(n+1)} \quad \text{or} \quad 1 - \frac{2}{n} + \frac{2}{n+1}$$

(iii) $\displaystyle\sum_{r=2}^{n} \frac{4}{(r-1)r(r+1)} = 1 - \frac{2}{n(n+1)}$

As $n \to \infty, \dfrac{2}{n(n+1)} \to 0$, Sum to infinity = 1

$$1 - \left(1 - \frac{2}{n(n+1)}\right) < 10^{-5}$$

$$\frac{2}{n(n+1)} < 10^{-5}$$

$$n(n+1) > 2(10^5)$$

Using G.C, $n < -447.71$ (rejected, $n > 0$) or $n > 446.71$

Least $n = 447$

5

(i) $\mathrm{f}(x) = \ln(\sin x + \cos x)$

$\Rightarrow e^{\mathrm{f}(x)} = \sin x + \cos x$

Differentiating wrt x :

$e^{\mathrm{f}(x)}\,\mathrm{f}'(x) = \cos x - \sin x$

$e^{\mathrm{f}(x)}\,\mathrm{f}''(x) + e^{\mathrm{f}(x)}\,\mathrm{f}'(x) \times \mathrm{f}'(x) = -\sin x - \cos x = -e^{\mathrm{f}(x)}$

$\Rightarrow \mathrm{f}''(x) + \left[\mathrm{f}'(x)\right]^2 = -1$ since $e^{\mathrm{f}(x)} \neq 0$

$\Rightarrow \mathrm{f}''(x) + \left[\mathrm{f}'(x)\right]^2 + 1 = 0$ (shown)

<u>Alternative Method 1</u>

$\mathrm{f}(x) = \ln(\sin x + \cos x)$

$\mathrm{f}'(x) = \dfrac{\cos x - \sin x}{\sin x + \cos x}$

$$f''(x)=\frac{(\sin x+\cos x)(-\sin x-\cos x)-(\cos x-\sin x)(\cos x-\sin x)}{(\sin x+\cos x)^2}$$

$$=-1-\frac{(\cos x-\sin x)^2}{(\sin x+\cos x)^2}\ =-1-\left[f'(x)\right]^2$$

$$\Rightarrow\quad f''(x)+\left[f'(x)\right]^2+1=0\quad \text{(shown)}$$

Alternative Method 2

Let $f(x)=\ln(\sin x+\cos x)$

$$f'(x)=\frac{\cos x-\sin x}{\sin x+\cos x}$$

$$(\sin x+\cos x)f'(x)=\cos x-\sin x$$

$$(\sin x+\cos x)f''(x)+(\cos x-\sin x)f'(x)=-\sin x-\cos x$$

$$f''(x)+\frac{\cos x-\sin x}{\sin x+\cos x}\left[f'(x)\right]=-1$$

$$f''(x)+\left[f'(x)\right]^2+1=0\ \text{(shown)}$$

(ii) From $f''(x)+\left[f'(x)\right]^2+1=0$

Differentiating wrt x,

$$f'''(x)+2\left[f'(x)\right]\left[f''(x)\right]=0$$

When $x=0$, $f(0)=0$, $f'(0)=1$, $f''(0)=-2$, $f'''(0)=4$

By Maclaurin's Theorem,

$$y=0+x+\frac{(-2)}{2!}x^2+\frac{4}{3!}x^3+\ldots$$

$$=x-x^2+\frac{2}{3}x^3\quad \text{(up to term in } x^3\text{)}$$

(iii) $g(x)=\ln(1+\tan x)$

$$=\ln\left(\frac{\cos x+\sin x}{\cos x}\right)$$

$$=\ln(\cos x+\sin x)-\ln(\cos x)$$

$$\approx\left(x-x^2+\frac{2}{3}x^3\right)-\ln\left(1-\frac{x^2}{2}\right)$$

$$=x-x^2+\frac{2}{3}x^3-\left[\left(-\frac{x^2}{2}\right)-\ldots\right]$$

$$=x-\frac{1}{2}x^2+\frac{2}{3}x^3\ \text{(up to term in } x^3\text{)}$$

6

(i) when $t \to 2$; $y \to -\infty$ and $x \to \dfrac{2}{5}$,

Equation of the vertical asymptote is $x = \dfrac{2}{5}$.

(ii) $x = \dfrac{2}{1+t^2} \Rightarrow \dfrac{dx}{dt} = \dfrac{-2(2t)}{(1+t^2)^2}$; $y = \ln(2-t) \Rightarrow \dfrac{dy}{dt} = \dfrac{-1}{2-t}$,

hence $\dfrac{dy}{dx} = \dfrac{(1+t^2)^2}{4t(2-t)}$

$$\frac{(1+t^2)^2}{4t(2-t)} = -\frac{1}{3} \Rightarrow 3t^4 + 2t^2 + 8t + 3 = 0,$$

By GC $t = -1$ or $t = -0.436114$ (rej. as $t < -0.5$)

(iii) $\dfrac{dy}{dx} = \dfrac{-1}{3}$ when $t = -1$, $x = 1$, $y = \ln(3)$

Equation of tangent is $y - \ln(3) = \dfrac{-1}{3}(x-1)$ i.e. $y = \dfrac{-x}{3} + \dfrac{1}{3} + \ln 3$.

(iv) Solving $x = \dfrac{2}{5}$ and $x = \dfrac{2}{1+t^2}$, $y = \ln(2-t)$

$\dfrac{2}{5} = \dfrac{2}{1+t^2} \Rightarrow t = -2$ hence $y = \ln(2-(-2)) = \ln(4)$

Point of intersection is $\left(\dfrac{2}{5}, \ln 4\right)$.

7

(a) Since $z = -2 + 3i$ is a root of the equation $2z^2 + (-1+4i)z + c = 0$,

$2(-2+3i)^2 + (-1+4i)(-2+3i) + c = 0$

$2(4-9-12i) + (2-8i-3i-12) + c = 0$

$c = 20 + 35i$

$\therefore \quad 2z^2 + (-1+4i)z + 20 + 35i$

$= \left(z - (-2+3i)\right)\left(2z - (a+ib)\right)$

$= 2z^2 + (4 - 6i - a - bi)z + (-2a - 3b - 2bi + 3ai)$

Comparing real and imaginary parts of coefficient of z:

$-1 = 4 - a$ and $4 = -6 - b$

$\Rightarrow a = 5$ and $b = -10$

The other root is $\frac{1}{2}(5 - 10i) = \frac{5}{2} - 5i$.

OR

$\therefore \quad 2z^2+(-1+4\text{i})z+20+35\text{i}$
$=\left(z-(-2+3\text{i})\right)\left(2z-(a+\text{i}b)\right)$

Compare the coefficient of z^0,
$20+35\text{i}=(-2+3\text{i})(a+\text{i}b)$

$$(a+\text{i}b)=\frac{20+35\text{i}}{-2+3\text{i}}$$
$$=\frac{20+35\text{i}\left(-2-3\text{i}\right)}{-2+3\text{i}\left(-2-3\text{i}\right)}$$
$$=5+10\text{i}$$

The other root is $\frac{1}{2}(5-10\text{i})=\frac{5}{2}-5\text{i}$.

OR

Let c be p + iq, where p and q are real numbers.
$2z^2+(-1+4\text{i})z+p+\text{i}q$
$=(z-(-2+3\text{i}))(2z-(a+\text{i}b))$
$=2z^2+(-a-\text{i}b+4-6\text{i})z+(-2+3\text{i})(a+\text{i}b)$
$=2z^2+(-a+4-\text{i}\,(b+6))z+(-2a-3b+\text{i}\,(3a-2b))$

Compare real and imaginary parts of the coefficient of z:
$-a+4=-1$ and $-b-6=4$
$\Rightarrow a=5$ and $b=-10$
$\therefore$ The other root is $\frac{1}{2}(5-10\text{i})=\frac{5}{2}-5\text{i}$

Compare real and imaginary parts of the coefficient of z^0:
$p=-2a-3b$ and $q=3a-2b$
$\Rightarrow p=20$ and $q=35$
$\Rightarrow c=20+35\text{i}$

(b)(i) $u-1=\cos\theta-1+\text{i}\sin\theta$
$$=-2\sin^2\frac{\theta}{2}+2\text{i}\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$=2\sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2}+\text{i}\cos\frac{\theta}{2}\right)$$

$$|u-1|=2\sin\frac{\theta}{2}$$

$\arg(u-1)=\pi-\tan^{-1}\left(\cot\frac{\theta}{2}\right)$ since $u-1$ is in 2nd quadrant.
$$=\pi-\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)$$
$$=\pi-\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$
$$=\frac{\pi}{2}+\frac{\theta}{2}$$

	OR

$$u-1=\cos\theta-1+\mathrm{i}\sin\theta$$
$$=-2\sin^2\frac{\theta}{2}+2\mathrm{i}\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$=2\sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2}+\mathrm{i}\cos\frac{\theta}{2}\right)$$
$$=2\mathrm{i}\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)$$

$$|u-1|=\left|2\mathrm{i}\sin\frac{\theta}{2}\right|=2\sin\frac{\theta}{2}$$

$$\arg(u-1)=\arg(\mathrm{i})+\frac{\theta}{2}=\frac{\pi}{2}+\frac{\theta}{2}$$

(b)(ii) $v^*=-\sqrt{3}-\mathrm{i}=2\mathrm{e}^{-\mathrm{i}\frac{5\pi}{6}}$

$$(v^*u)^8=\left(2\mathrm{e}^{-\mathrm{i}\frac{5\pi}{6}}\mathrm{e}^{\theta}\right)^8=2^8\mathrm{e}^{\mathrm{i}\left(-\frac{40\pi}{6}+8\theta\right)}$$

Given $(w^*u)^8$ is real and negative ,

$$-\frac{20}{3}\pi+8\theta=\ldots\ ,-\pi,\ -3\pi,\ -5\pi,\ -7\pi,\ \ldots$$

$$8\theta=\ \ldots\ ,\ -3\pi+\frac{20}{3}\pi,\ -5\pi+\frac{20}{3}\pi,\ -7\pi+\frac{20}{3}\pi,\ \ldots$$

Smallest $\theta=\frac{5\pi}{24}$

8 **(i)** Using direction cosines,

$\cos 45^\circ=\frac{x}{\sqrt{x^2+y^2+z^2}}$	$\cos 45^\circ=\frac{y}{\sqrt{x^2+y^2+z^2}}$	$\cos 45^\circ=\frac{z}{\sqrt{x^2+y^2+z^2}}$
$\frac{1}{\sqrt{2}}=\frac{x}{\sqrt{x^2+y^2+z^2}}$	$\frac{1}{\sqrt{2}}=\frac{y}{\sqrt{x^2+y^2+z^2}}$	$\frac{1}{\sqrt{2}}=\frac{z}{\sqrt{x^2+y^2+z^2}}$
$-x^2+y^2+z^2=0$	$x^2-y^2+z^2=0$	$x^2+y^2-z^2=0$

From G.C, solving the 3 equations, $x=y=z=0$

Since $\mathbf{r}$ is a <u>non-zero vector</u>, $\alpha=\beta=\gamma=45^\circ$ is not possible.

Alternatively,

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2=\frac{3}{2}\neq 1$$

$\alpha=\beta=\gamma=45^\circ$ is not possible.

(ii) When $\alpha=\beta=45^\circ$ and $\gamma=90^\circ$, $x=y$ and $z=0$ $\quad \mathbf{r}=\begin{pmatrix}x\\y\\0\end{pmatrix}=\begin{pmatrix}x\\x\\0\end{pmatrix}$

(iii) Let $\mathbf{a}=\begin{pmatrix}x\\x\\0\end{pmatrix}$. Since point B is the foot of perpendicular from point A to the y-axis,

$\Rightarrow \mathbf{b}=\begin{pmatrix}0\\x\\0\end{pmatrix}$.

Since $AC = AB$, and $\overrightarrow{AC}$ is parallel to the z-axis, $\mathbf{c}=\begin{pmatrix}x\\x\\x\end{pmatrix}$ or $\begin{pmatrix}x\\x\\-x\end{pmatrix}$.

Volume of tetrahedron = $\frac{1}{3}(\text{base area})(\text{height})$

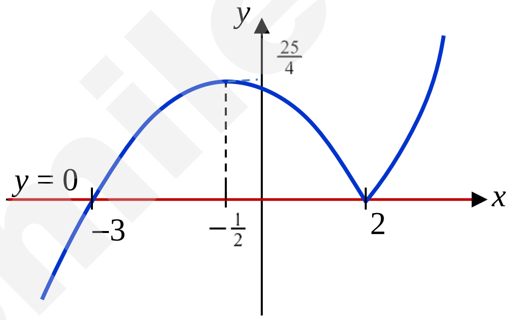
$$36=\frac{1}{3}\left(\frac{1}{2}(x)^2\right)(x)$$
$$x=6$$

Possible coordinates of point C are $(6,6,6)$ or $(6,6,-6)$.

9 **(i)** Since f(–3) = f(2) = 0 where $-3, 2 \in D_f$, f is not one-one.

Thus f^{-1} does not exist.

OR **(i)** Since the horizontal line
y = 0, cuts the graph of
y = f(x) at more than 1 point,
f is not one to on, thus f^{-1} does not exist.



(ii) Since $k \le x \le 2$, f(x) = –(x – 2)(x + 3). Least $k = -\frac{1}{2}$.

Let $y=\mathrm{f}(x)$, where $-\frac{1}{2}\le x\le 2$.

$$y=-(x+3)(x-2)=-(x^2+x-6)=-\left(x+\frac{1}{2}\right)^2+\frac{25}{4}$$

$$\left(x+\frac{1}{2}\right)^2=\frac{25}{4}-y \quad \Rightarrow x=-\frac{1}{2}\pm\sqrt{\frac{25-4y}{4}}$$

Since $-\frac{1}{2}\le x\le 2$, $\quad x=-\frac{1}{2}+\frac{1}{2}\sqrt{25-4y}$

$\therefore\ \mathrm{f}^{-1}:\ x\mapsto -\frac{1}{2}+\frac{1}{2}\sqrt{25-4x},\quad x\in\mathbb{R},\ \ 0\le x\le\frac{25}{4}.$

9

Alternative Method

Let $y = \mathrm{f}(x)$, where $-\frac{1}{2} \le x \le 2$.

$y = -(x+3)(x-2) = -x^2 - x + 6$

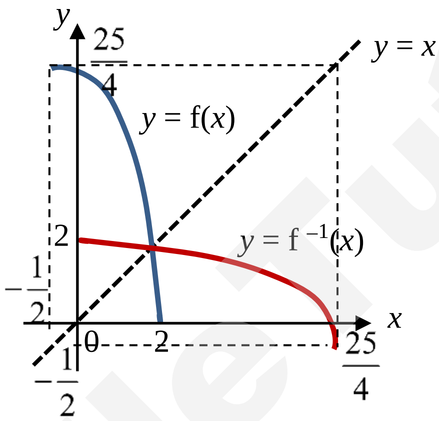
$x^2 + x + y - 6 = 0$

$x = \frac{-1 \pm \sqrt{1-4(y-6)}}{2} = \frac{-1 \pm \sqrt{25-4y}}{2}$

Since $-\frac{1}{2} \le x \le 2$, $\quad x = \frac{-1+\sqrt{25-4y}}{2}$

$\therefore \ \mathrm{f}^{-1}: x \mapsto \frac{-1+\sqrt{25-4x}}{2}, \quad x \in \mathbb{R}, \ 0 \le x \le \frac{25}{4}.$

(iii)



$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$

$\mathrm{f}(x) = x$

$-x^2 - x + 6 = x$

$x^2 + 2x - 6 = 0$

$x = \frac{-2 \pm \sqrt{28}}{2} = -1 \pm \sqrt{7}$

From the diagram, $x \ge 0$, $\quad \therefore \ x = -1+\sqrt{7}$

9

(iv) Range of f $= \left[0, \frac{25}{4}\right]$, Domain of g^{-1} = Range of g $= (-1, 7)$

Since $R_{\mathrm{f}} \subseteq D_{\mathrm{g}^{-1}}$, $\therefore$ $\mathrm{g}^{-1}\mathrm{f}$ exists.

(v) $\mathrm{g}(x) = \frac{24}{x^2+3} - 1, \ x \in \mathbb{R}, \ x > 0$

$\mathrm{g}'(x) = -\frac{24(2x)}{(x^2+3)^2} < 0$ for $x > 0$ since $(x^2+3)^2 > 0$

Hence, g is a strictly decreasing function.

	$g^{-1}(f(x))<1 \Rightarrow f(x)>g(1)$ (since g is a strictly decreasing fn) $\Rightarrow f(x)>5$ Since $f^{-1}(5)=-\frac{1}{2}+\frac{1}{2}\sqrt{25-4(5)}=\frac{\sqrt{5}-1}{2}$, from the graph of $y = f(x)$, $-\frac{1}{2}\le x<\frac{\sqrt{5}-1}{2}$
9	OR $g^{-1}(f(x))<1 \Rightarrow f(x)>g(1)$ (since g is a decreasing function) $\Rightarrow f(x)>5$ $\Rightarrow -x^2-x+6>5$ $\Rightarrow x^2+x-1<0$ $\Rightarrow \frac{-1-\sqrt{5}}{2}<x<\frac{-1+\sqrt{5}}{2}$ Since $-\frac{1}{2}\le x\le 2$, $\therefore -\frac{1}{2}\le x<\frac{-1+\sqrt{5}}{2}$
10	**(i)** $\frac{dm}{dt}\propto(u-38.5m)$. Let $\frac{dm}{dt}=k(u-38.5m)$ When $(u-38.5m)=-7700$, $\frac{dm}{dt}=-1 \Rightarrow k=\frac{1}{7700}$ $\Rightarrow \frac{dm}{dt}=\frac{1}{7700}(u-38.5m)=\frac{u}{7700}-\frac{m}{200}$ **(ii)** Solve DE: $\frac{dm}{dt}=\frac{u}{7700}-\frac{m}{200}=\frac{1}{200}\left(\frac{2u}{77}-m\right)$ $\int \frac{1}{\frac{2u}{77}-m}\,dm=\int \frac{1}{200}\,dt$ $-\ln\left\lvert\frac{2u}{77}-m\right\rvert=\frac{t}{200}+c$ $\frac{2u}{77}-m=\pm e^{-c}.e^{-\frac{t}{200}}=Be^{-\frac{t}{200}}$ $m=\frac{2u}{77}-Be^{-\frac{t}{200}}$ **(iii)** Given: $u=2500$ and $t=0, m=80$, $B=\frac{5000}{77}-80=-\frac{1160}{77}$

	$m=\frac{5000}{77}+\frac{1160}{77}e^{-\frac{t}{200}}$
10	**(iv)** New DE: $38.5\frac{\mathrm{d}M}{\mathrm{d}t}=-\frac{77(77M+154t)+76}{77M+154t+1}$ By substitution: $z=38.5M+77t \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}t}=38.5\frac{\mathrm{d}M}{\mathrm{d}t}+77$ $\frac{\mathrm{d}z}{\mathrm{d}t}-77=-\frac{77(2z)+76}{2z+1}=-77+\frac{1}{2z+1}$ $\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{1}{2z+1}$ Integrate w.r.t. t, $\int (2z+1)\,\mathrm{d}z=\int 1\,\mathrm{d}t$ $z^2+z=t+C$ $\left(z+\frac{1}{2}\right)^2-\frac{1}{4}=t+C \Rightarrow \left(z+\frac{1}{2}\right)^2=t+D$ $\Rightarrow \left(38.5M+77t+\frac{1}{2}\right)^2=t+D$ $\Rightarrow 38.5M+77t+\frac{1}{2}=\sqrt{t+D}$ (-ve square root rejected) $\Rightarrow M=\frac{1}{38.5}\left[\sqrt{t+D}-77t-\frac{1}{2}\right]$ where D is an arbitrary constant

11	**(i)** A.P with a = 600, d = 80

Last contribution is when n = 17

$T_{17} = 600 + (17-1)(80) = 1880$

The last contribution is $1880.

(ii)

nth contribution	Amount of money in SupremeEdu Fund
1 (1 year old)	1040
2	1040(1.035)+1040
3	(1040(1.035)+1040)(1.035) + 1040 $= 1040(1.035)^2 + 1040(1.035) + 1040$
…	…
17	$1040\left[(1.035)^{16} + (1.035)^{15} + \ldots + (1.035) + 1\right]$
18 *no contribution	$1.035 \times 1040\left[(1.035)^{16} + (1.035)^{15} + \ldots + (1.035) + 1\right]$

G.P with $a = 1040(1.035)$, $r = 1.035$

Total amount = $\$\dfrac{1040(1.035)\left[(1.035)^{17} - 1\right]}{1.035 - 1}$ = $24439.67

= $24440 (nearest dollar)

nth contribution	Amount of money in UltimateEdu Fund
1 (1 year old)	600
2 (2 year old)	600+(600+80)+(200)
3	600+(600+80)+(600+80+80)+(200+200)
…	…

Total contribution is an A.P with a = 600, d = 80, n = 17

$S_{17} = \dfrac{17}{2}\left(2(600) + 16(80)\right) = \$21080$ OR $S_{17} = \dfrac{17}{2}\left(600 + 1880\right) = \$21080$

Total interest earn = 17 x $200 = $3400

Total amount on eighteenth birthday = $21080 + $3400 = $24480

OR $\dfrac{17}{2}\left(2(800) + (17-1)(80)\right) = \$24480$

11	**(iii)**

	Invested amount	Amt received on 18^{th} Birthday	Interest earned
SupremeEdu	\$1040 x 17 = \$17680	\$24440	\$24440 - \$1040 x 17 = \$6760
UltimateEdu	\$21080	\$24480	17 x \$200 = \$3400

Since SupremeEdu earns a higher interest with a lower invested amount, Mr and Mrs Lai should get SupremeEdu instead.

(iv) IF chosen SupremeEdu
Let the first investment amount be \$*a*.

$$\text{Total amount} = \frac{a(1.035)\left[(1.035)^{17}-1\right]}{1.035-1} \geq 50000$$

$$a \geq 2127.69$$

$$a = \$2128 \text{ (nearest dollar)}$$

IF chosen UltimateEdu

$$\frac{17}{2}\left(2(a)+(17-1)(80)\right)+3400 \geq 50000$$

$$a \geq 2101.18$$

$$a = \$2102 \text{ (nearest dollar)}$$

ANDERSON JUNIOR COLLEGE
2018 PRELIMINARY EXAMINATION
H2 Mathematics 9758/02

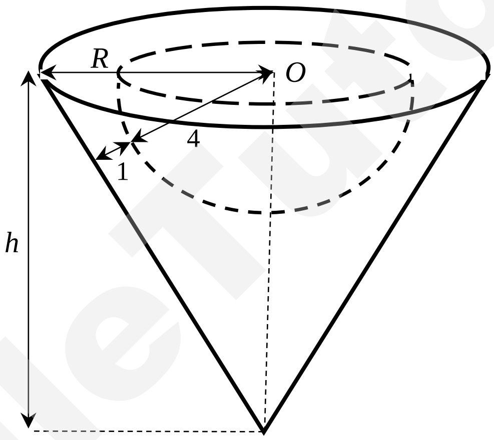
PAPER 2

Duration: 3 hours

Answer ALL questions.

Section A: Pure Mathematics [40 marks]

1 [It is given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



Michelle wants to design a décor ornament for Christmas. The ornament is to be made up of a right circular cone of radius R cm and height h cm, with a hemisphere of fixed radius 4 cm being carved out from the cone as shown in the diagram above. O is the center of the circular top of both the cone and the hemisphere. The hemisphere is carved out such that the shortest distance between the curved surface of the cone and the hemisphere is 1 cm (see diagram). Show that the volume V of the ornament is given by

$$V = \frac{\pi}{3}\left(\frac{25h^3}{h^2 - 25}\right) - \frac{128\pi}{3}.$$

Michelle wishes to minimise the volume of the ornament so that it is light in weight. Find the exact value of h that corresponds to the minimum volume, and find the minimum volume. [8]

2 **(i)** Given that **a** and **b** are non-zero constant vectors, show that the points with position vector $\mathbf{r}$ and satisfying the equation $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$ lie on a line. [3]

The line L_1 has equation $\left[\mathbf{r}-\begin{pmatrix}2\\1\\-1\end{pmatrix}\right]\times\begin{pmatrix}0\\1\\1\end{pmatrix}=\mathbf{0}$, and the line L_2 has equation $\mathbf{r}=\begin{pmatrix}-1\\2\\3\end{pmatrix}+\mu\begin{pmatrix}m\\0\\1\end{pmatrix}$, where m is a constant and μ is a real parameter.

(ii) Find the vector equations of the parallel planes p_1 and p_2, in scalar product form, such that p_1 contains L_1 and p_2 contains L_2, leaving your answers in terms of m. [3]

(iii) Find the acute angle between the plane p_1 and the line which contains both the points $(2,1,-1)$ and $(-1,2,3)$, leaving your answer in terms of m. [2]

(iv) It is now given that L_1 and L_2 is contained in a common plane p_3. Using your answer in part **(iii)**, or otherwise, find the cartesian equation of the plane p_3. [1]

3 The curve C has equation $y=ax+b+2a+\dfrac{2(b+2a)}{x-2}$, where a and b are real constants such that $a>0$, $b\neq -2a$ and $x\neq 2$.

(i) By using differentiation, find the range of values of b in terms of a such that C has no stationary points. [4]

(ii) Given that $b=-3a$, sketch C, stating the equations of any asymptotes and the coordinates of the points where C crosses the axes. [3]

(iii) On the same diagram as C, sketch the graph of $\left(\dfrac{y}{a}\right)^2-(x-1)^2=4$, showing clearly the equations of the asymptotes and turning points. Hence solve the inequality $x-1-\dfrac{2}{x-2}\geq\sqrt{(x-1)^2+4}$. [4]

4 It is given that $\mathrm{f}(x)=\dfrac{x}{\sqrt{3+2x-x^2}}$ for $-1<x<3$.

(i) Find (a) $\int \mathrm{f}(x)\,\mathrm{d}x$. [4]

(b) $\int [\mathrm{f}(x)]^2\,\mathrm{d}x$ by using partial fractions. [3]

(ii) Find the exact area of the region bounded by the curve $y=\mathrm{f}(x)$, the x-axis and the lines $x=1-\sqrt{3}$ and $x=2$. [3]

(iii) Find the exact volume of revolution when the region bounded by the curve $y=\mathrm{f}(x)$, the x-axis and the lines $x=1$ and $x=2$ is rotated completely about the x-axis. [2]

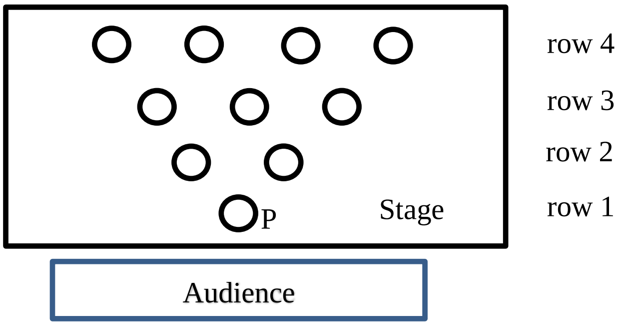
Section B: Probability and Statistics [60 marks]

5 There are 18 participants in an Idol Survival Competition. In the first round of competition, these participants are grouped into 3 groups of equal size. James and Michael are two participants in this competition.

(i) Show clearly that the number of ways the participants can be grouped is 2858856. [1]

(ii) Find the probability that James and Michael are in the same group. [2]

After many rounds of elimination, the top 10 participants (including James and Michael) remain in the competition. During their final performance, the participants stand in a triangular formation as shown in the diagram below.



Given that position P must be occupied by Michael, and that James and 2 particular participants would like to stand next to one another in the same row, find the number of ways the participants can arrange themselves. [3]

6 A farm produces a large number of eggs every day. The eggs are randomly packed into trays of 30. A small proportion p of these eggs are known to have weights which are substandard.

A check is carried out each day by taking a random sample of 50 trays and examining their weights. If more than 1 egg in a tray is substandard, the entire tray of eggs is rejected.

If 4 or more trays are rejected in the sample, the day's production is rated as 'poor'. If none of the trays is rejected in the sample, the day's production is rated as 'excellent'. Otherwise, the day's production is rated as 'fair'.

(i) State, in this context, two conditions that must be met for the number of eggs in a tray that are substandard to be well modelled by a binomial distribution. [2]

Assume now that these conditions are met.

(ii) Show that the probability G that a tray is not rejected is given by

$$G = (1-p)^{29}(1+29p).$$ [2]

Given that $G = 0.96$,

(iii) find the value of p correct to four decimal places, [1]

(iv) find the probability that two trays are rejected if the day's production is rated as 'fair'. [3]

7 Soya bean drink is sold in cups of two sizes – small and large. For each size, the amount of content, in ml, of a randomly chosen cup is normally distributed with mean and variance as given in the table. The selling prices are also given in the table.

	Mean (ml)	Variance (ml^2)	Selling Price ($)
Small	202	21	1.10
Large	405	74	2.20

The amount of content in any cup may be assumed to be independent of the amount of content in any other cup.

(i) A small cup and two large cups are selected at random. Find the probability that the total amount of content in the two large cups is less than four times the amount of content in the small cup. [2]

(ii) A boy needs at least 600 ml of soya bean drink but he only has $3.80. In what way should he make his purchase so that he has the highest probability of getting at least 600 ml? Support your answer with clear workings. [3]

(iii) A random sample of 20 small cups and n large cups of soya bean drink is taken. Find the least value of n such that there is a probability of more than 0.8 that the mean amount of content in these cups is more than 350 ml. [4]

8 To promote the sale of its products on Valentine's Day, a shop owner gives free vouchers to its customers. To get the vouchers, customers must first participate in a game. Each customer is allowed to play the game only once.

There are four rounds in the game. In each round, four cards are used. The message printed on one of the cards is 'Congratulations' and the message printed on the other three cards is 'Thank you'. These cards are placed facedown, and the customer would choose a card to flip in that round. Two points are scored if the card with the message 'Congratulations' is flipped. Otherwise, one point is deducted. Each customer is equally likely to choose any of the cards to flip.

At the end of the game, the customer's final score is X, where X is the sum of the points scored in the four rounds.
The shop will then reward the customer with a voucher worth $\$\ (8+2X)$.

(i) Show that $\mathrm{P}(X=2)=\dfrac{27}{128}$. [1]

(ii) Tabulate the probability distribution of X. [3]

(iii) Find the mean and variance of the value of the voucher a customer receives from the shop. [4]

If 100 customers visit the shop and play the game, what is the total value of the vouchers the shopkeeper is expected to give away? Estimate the probability that the total value of the vouchers given away would be more than $650. [3]

9 **(a)** The following set of bivariate data was obtained for x and y:

$$n=8,\quad \sum(x-150)=47,\quad \sum(x-150)^2=483,$$

$$\sum(y-170)=87,\quad \sum(y-170)^2=1101,\quad \sum(x-150)(y-170)=677.$$

Find the

(i) equation of the least-squares regression line of x on y,

(ii) product moment correlation coefficient. [4]

(b) To understand the correlation between the level of happiness and the time spent at work, the human resource department of an organisation polls nine of its officers. The number of work hours per week (t) of each officer and the happiness index (h) are recorded, where a higher value of h indicates a higher level of happiness. The data are shown in the table below:

t	20.1	22.0	24.4	25.3	28.8	36.5	40.6	46.0	55.1
h	24.5	16.3	18.6	12.5	5.2	4.7	1.4	1.8	0.8

(i) Draw a scatter diagram of h against t for the data and comment whether a linear model would be appropriate. [3]

It is thought that the relationship between the happiness index h and the number of work hours per week t can be modelled by one of the following formulae

$$\textbf{(A)}\quad h=a+\frac{b}{t} \qquad\qquad \textbf{(B)}\quad h=c+d\ln(t)$$

where a, b, c and d are constants.

(ii) Find, correct to 5 decimal places, the value of the product moment correlation coefficient for each model. Explain which is the better model. [3]

(iii) Use the model you have chosen in part **(ii)** to predict the happiness index for an officer who works 15.0 hours per week. Comment on the reliability of your prediction. [3]

10 The public relations officer of In-star-gram stated in an education report that a student spends an average of 2.49 hours a day on his/her social media platform. A survey is done to collect the usage time, x hours, of 100 students in the age group 13 to 16 years old, and the data is summarised as follows:

$$\sum x = 250.5 \text{ and } \sum x^2 = 628.45.$$

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) A hypothesis test is carried out at the $\alpha\%$ level of significance, and it was found that the public relations officer had underestimated the usage hours. Find the set of values of α and state any necessary assumption(s). [6]

(iii) Explain, in context, the meaning of the p-value found in part **(ii)**. [1]

In another survey done to collect the usage hours of 100 students in the age group 17 to 20 years old, it is found that the average usage is k hours. Assuming now that the population standard deviation is 1.5 hours, find the range of values of k such that there is sufficient evidence to conclude that the average usage hours of In-star-gram is not 2.49 hours at the 5% level of significance. [4]

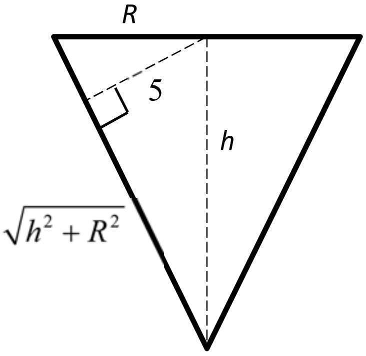
END OF PAPER

1

Volume of the ornament (V)

$$=\frac{1}{3}\pi R^2 h-\frac{2}{3}\pi(4)^3$$

$$=\frac{1}{3}\pi R^2 h-\frac{128}{3}\pi$$



By similar triangle,

$$\frac{R}{\sqrt{h^2+R^2}}=\frac{5}{h}$$

$$h^2R^2=25\left(h^2+R^2\right)$$

$$R^2=\frac{25h^2}{h^2-25}$$

$$\therefore V=\frac{1}{3}\pi\left(\frac{25h^2}{h^2-25}\right)h-\frac{128}{3}\pi=\frac{\pi}{3}\left(\frac{25h^3}{h^2-25}\right)-\frac{128\pi}{3}$$

$$\frac{dV}{dh}=\frac{25\pi}{3}\left(\frac{\left(h^2-25\right)\left(3h^2\right)-h^3(2h)}{\left(h^2-25\right)^2}\right)=\frac{25\pi}{3}\left(\frac{h^2\left(h^2-75\right)}{\left(h^2-25\right)^2}\right)$$

When $\frac{dV}{dh}=0$, $h^2\left(h^2-75\right)=0$

Since $h\neq 0$, $h^2-75=0$

$\Rightarrow h=\sqrt{75}=5\sqrt{3}$ (Since $h>0$)

<u>1st derivative test</u>

	$h=\left(\sqrt{75}\right)^-$	$h=\sqrt{75}$	$h=\left(\sqrt{75}\right)^+$
$\frac{dV}{dh}=\frac{25\pi}{3}\left(\frac{h^2\left(h^2-75\right)}{\left(h^2-25\right)^2}\right)$	$h^2<75$ $h^2-75<0$ $\frac{dV}{dh}<0$	$h^2=75$ $\frac{dV}{dh}=0$	$h^2>75$ $h^2-75>0$ $\frac{dV}{dh}>0$
gradient			

2nd derivative test

$$\frac{d^2V}{dh^2}=\frac{\left(h^2-25\right)\left(4h^3-150h\right)-2\left(h^2-25\right)(2h)\left(h^2\right)\left(h^2-75\right)}{\left(h^2-25\right)^4}$$

When $h=\sqrt{75}$, $\frac{d^2V}{dh^2}=0.51962>0$, $V=\frac{\pi}{3}\left(\frac{25\left(5\sqrt{3}\right)^3}{\left(5\sqrt{3}\right)^2-25}\right)-\frac{128\pi}{3}=206.046$ or

$$\frac{\pi}{6}\left(375\sqrt{3}-256\right)$$

$\therefore$When $h=5\sqrt{3}$, V is minimum. & $V_{min}=\approx 206\text{ cm}^3$ (3s.f)

2	(i) $(\underset{\sim}{r}-\underset{\sim}{a})\times\underset{\sim}{b}=\underset{\sim}{0}$ Either $\underset{\sim}{b}=\underset{\sim}{0}$ (reject since $\underset{\sim}{b}$ is a non-zero vector) or $(\underset{\sim}{r}-\underset{\sim}{a})=\underset{\sim}{0}$ or $(\underset{\sim}{r}-\underset{\sim}{a})//\underset{\sim}{b}$ $\Rightarrow \underset{\sim}{r}=\underset{\sim}{a}$ OR $(\underset{\sim}{r}-\underset{\sim}{a})=k\underset{\sim}{b}$ for $k\in\mathbb{R}\setminus\{0\}$ $\Rightarrow \underset{\sim}{r}=\underset{\sim}{a}+k\underset{\sim}{b}$ for $k\in\mathbb{R}$ (Note that $\underset{\sim}{r}=\underset{\sim}{a}$ is included in these solutions) This is the equation of a line passing through point with position vector $\underset{\sim}{a}$ and parallel to the vector $\underset{\sim}{b}$. (ii) $\underset{\sim}{n}=\begin{pmatrix}0\\1\\1\end{pmatrix}\times\begin{pmatrix}m\\0\\1\end{pmatrix}=\begin{pmatrix}1\\m\\-m\end{pmatrix}$ p1: $\underset{\sim}{r}\bullet\begin{pmatrix}1\\m\\-m\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}\bullet\begin{pmatrix}1\\m\\-m\end{pmatrix}=2+2m$ p2: $\underset{\sim}{r}\bullet\begin{pmatrix}1\\m\\-m\end{pmatrix}=\begin{pmatrix}-1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}1\\m\\-m\end{pmatrix}=-1-m$ (iii) Let θ be the acute angle between the plane and the line. $\sin\theta=\left\|\frac{\underset{\sim}{d}\bullet\underset{\sim}{n}}{\|\underset{\sim}{d}\|\|\underset{\sim}{n}\|}\right\|=\frac{\left\|\begin{pmatrix}-3\\1\\4\end{pmatrix}\bullet\begin{pmatrix}1\\m\\-m\end{pmatrix}\right\|}{\sqrt{26}\sqrt{1+2m^2}}=\frac{\|-3-3m\|}{\sqrt{26+52m^2}}$ $\theta=\sin^{-1}\left\|\frac{3+3m}{\sqrt{26+52m^2}}\right\|$ (iv) For p_3 to contain both L_1 and L_2, $\theta=0$. $\Rightarrow 3+3m=0$ $\Rightarrow m=-1$ Substituting $m=-1$ into either plane equation above, $\underset{\sim}{r}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$ $x-y+z=0$ is the Cartesian equation of p_3.

3

$$y = ax + b + 2a + \frac{2(b+2a)}{x-2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a - \frac{2(b+2a)}{(x-2)^2}$$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad \frac{2(b+2a)}{(x-2)^2} = a$

$$a(x-2)^2 = 2(b+2a)$$

For C to have no stationary points,

$2(b+2a) < 0$ (since $a > 0$)

$$b < -2a$$

Alternatively,

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad \frac{2(b+2a)}{(x-2)^2} = a$

$$a(x-2)^2 = 2(b+2a)$$

$$ax^2 - 4ax - 2b = 0$$

For C to have no stationary points,

Discriminant $= 16a^2 - 4a(-2b) = 8a(2a+b) < 0$

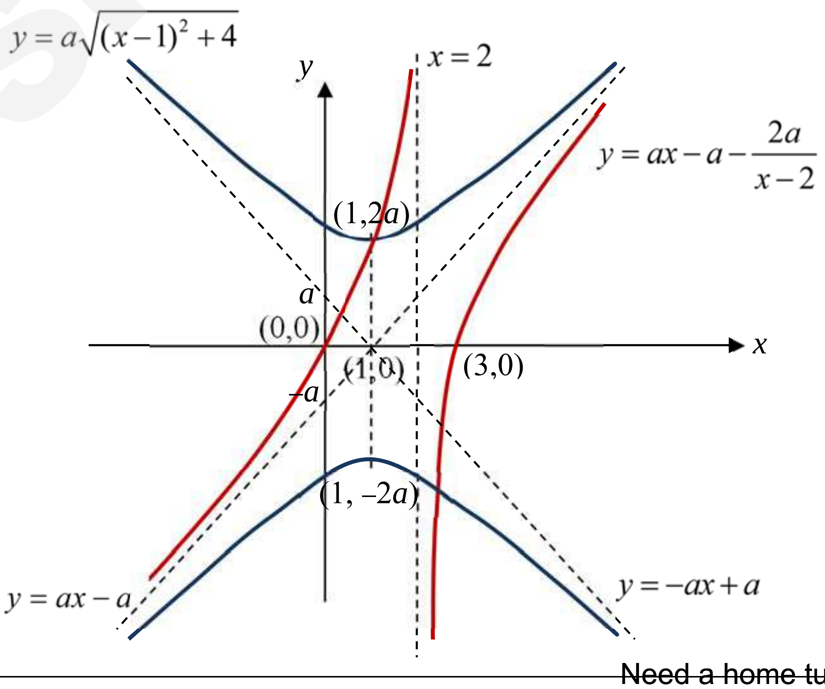
Since $a > 0$, $\quad 2a + b < 0 \Rightarrow b < -2a$

For $b = -3a$, $\quad C: \; y = ax - a + \frac{-2a}{x-2}$

Equations of asymptotes: $\quad y = ax - a, \; x = 2.$

$$\left(\frac{y}{a}\right)^2 - (x-1)^2 = 4 \Rightarrow \left(\frac{y}{2a}\right)^2 - \left(\frac{x-1}{2}\right)^2 = 1$$

Asymptotes: $\frac{y}{2a} = \pm\left(\frac{x-1}{2}\right) \quad \Rightarrow y = ax - a \quad$ or $\quad y = -ax + a$



	$x-1-\dfrac{2}{x-2} \geq \sqrt{(x-1)^2+4}$ From the graphs of $y = ax - a - \dfrac{2a}{x-2}$ and $y = a\sqrt{(x-1)^2+4}$ (*i.e.* $y > 0$), $1 \leq x < 2$.
4	(i) (a) $\displaystyle\int \frac{x}{\sqrt{3+2x-x^2}}\,dx = \int \frac{\frac{-1}{2}(2-2x)+1}{\sqrt{3+2x-x^2}}\,dx$ $\displaystyle = \frac{-1}{2}\int \frac{(2-2x)}{\sqrt{3+2x-x^2}}\,dx + \int \frac{1}{\sqrt{3+2x-x^2}}\,dx$ $\displaystyle = \frac{-1}{2}.2\sqrt{3+2x-x^2} + \int \frac{1}{\sqrt{4-(x-1)^2}}\,dx$ $\displaystyle = \sin^{-1}\left(\frac{x-1}{2}\right) - \sqrt{3+2x-x^2} + c$ (i) (b) $\displaystyle\int \frac{x^2}{3+2x-x^2}\,dx = \int \left(-1 + \frac{9}{4(3-x)} + \frac{1}{4(1+x)}\right)dx$ $\displaystyle = -x - \frac{9}{4}\ln(3-x) + \frac{1}{4}\ln(1+x) + c$ (ii) Area = $\displaystyle\int_0^2 \frac{x}{\sqrt{3+2x-x^2}}\,dx - \int_{1-\sqrt{3}}^0 \frac{x}{\sqrt{3+2x-x^2}}\,dx$ $\displaystyle = \sin^{-1}\left(\frac{1}{2}\right) - \sqrt{3} - \left[\sin^{-1}\left(\frac{-1}{2}\right) - \sqrt{3}\right]$ $\displaystyle - \left\{\sin^{-1}\left(\frac{-1}{2}\right) - \sqrt{3} - \left[\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) - \sqrt{1}\right]\right\}$ where $\sqrt{3+2(1-\sqrt{3})-(1-\sqrt{3})^2} = 1$ $\displaystyle = \frac{\pi}{6} + \frac{\pi}{6} - \left[-\frac{\pi}{6} - \sqrt{3} - \left(-\frac{\pi}{3}\right) + 1\right] = \frac{\pi}{6} + \sqrt{3} - 1$ (iii) Volume = $\displaystyle\pi\int_1^2 \frac{x^2}{3+2x-x^2}\,dx$ $\displaystyle = \pi\left\{-2 - \frac{9}{4}\ln(1) + \frac{1}{4}\ln(3) - \left[-1 - \frac{9}{4}\ln(2) + \frac{1}{4}\ln(2)\right]\right\}$ $\displaystyle = \pi\left\{\frac{1}{4}\ln(3) + 2\ln(2) - 1\right\}$

5	(i) Number of ways $= \dfrac{{}^{18}C_6 \times {}^{12}C_6 \times {}^{6}C_6}{3!} = 2858856$ (Shown) (ii) Number of ways M and J in the same group $= \dfrac{{}^{16}C_4 \times {}^{12}C_6 \times {}^{6}C_6}{2!} = 840840$ Required probability $= \dfrac{840840}{2858856} = 0.294$ (3 s.f.) **Case 1** : 3 particular participants stand in row 3 Number of ways $= 3! \times 6! = 4320$ **Case 2 :** 3 participants stand in row 4 Number of ways $= 2 \times 3! \times 6! = 8640$ Total number of ways = 12960
6	(i) Assumptions: The probability that the eggs are lighter than standard weight is constant. The weight of an (randomly chosen) egg is independent of the weight of another (randomly chosen) egg. (ii) Let X be the number of eggs that are lighter than standard weight in a tray of 30 eggs. $X \sim \mathrm{B}(30, p)$ P (tray is not rejected) $= \mathrm{P}(X = 0) + \mathrm{P}(X = 1)$ $= \binom{30}{0} p^0 (1-p)^{30} + \binom{30}{1} p^1 (1-p)^{29}$ $G = (1-p)^{30} + 30p(1-p)^{29} = (1-p)^{29}(1+29p)$ (shown) (iii) $(1-p)^{29}(1+29p) = 0.96$ Using GC, $p = 0.1057495\ldots = 0.1057$ (4 dec pl) (iv) Let Y be the number of trays that are rejected in a day. $Y \sim \mathrm{B}(50, 1-0.96)$ $Y \sim \mathrm{B}(50, 0.04)$ $\mathrm{P}(1 \le Y \le 3) = \mathrm{P}(Y \le 3) - \mathrm{P}(Y = 0) = 0.731$ Probability that two trays are rejected if the day's production is rated as 'fair' $= \mathrm{P}(2 \text{ trays are rejected} \mid \text{the day's production is 'fair'})$ $= \dfrac{\mathrm{P}(Y = 2 \text{ and } 1 \le Y \le 3)}{\mathrm{P}(1 \le Y \le 3)} = \dfrac{\mathrm{P}(Y = 2)}{\mathrm{P}(1 \le Y \le 3)}$ $= \dfrac{0.2762328}{0.7309834} = 0.378$

7	(i) Let L be the amount of drink in a large cup and S be the amount of drink in a small cup. $L \sim \mathrm{N}(405, 74)$ $S \sim \mathrm{N}(202, 21)$ $L_1 + L_2 - 4S \sim \mathrm{N}(2, 484)$ $\mathrm{P}(L_1 + L_2 - 4S < 0) = 0.464$ (ii) With \$3.80, he can buy 3 small cups OR 1 large cup and 1 small cup of drinks. 3 small cups of drinks : $S_1 + S_2 + S_2 \sim \mathrm{N}(606, 63)$ $\mathrm{P}(S_1 + S_2 + S_2 > 600) = 0.775$ 1 large cup and 1 small cup of drinks : $L + S \sim \mathrm{N}(607, 95)$ $\mathrm{P}(L + S > 600) = 0.764$ Hence he should buy 3 small cups. (Note: the amount he needs to spend is the same for both cases) (iii) Let $T = \dfrac{S_1 + S_2 + \ldots + S_{20} + L_1 + L_2 + \ldots + L_n}{20 + n}$ $T \sim N\left(\dfrac{20(202) + n(405)}{20 + n}, \dfrac{20(21) + n(74)}{(20 + n)^2}\right)$ $T \sim N\left(\dfrac{4040 + 405n}{20 + n}, \dfrac{420 + 74n}{(20 + n)^2}\right)$ $\mathrm{P}(T > 350) > 0.8$ Using GC, when $n = 54$, $\mathrm{P}(T > 350) = 0.5598 < 0.8$ when $n = 55$, $\mathrm{P}(T > 350) = 0.834 > 0.8$ Least value of $n = 55$ **<u>Alternative method (using standardization)</u>** $\mathrm{P}\left(Z > \dfrac{350 - \dfrac{4040 + 405n}{20 + n}}{\sqrt{\dfrac{420 + 74n}{(20 + n)^2}}}\right) > 0.8$ $\dfrac{350 - \dfrac{4040 + 405n}{20 + n}}{\sqrt{\dfrac{420 + 74n}{(20 + n)^2}}} < -0.8416212$ Using GC, least value of $n = 55$

8

(i) $P(X=2)=\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2\,{}^4C_2=\frac{27}{128}$

(ii) C = ‘Congratulations’ (+2 points), T = ‘Thank You’ (-1 points)

X	cards	Prob	
-4	0C4T	$\left(\frac{3}{4}\right)^4=\frac{81}{256}$	$\frac{81}{256}$
-1	1C3T	$\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3\,{}^4C_1=\frac{27}{64}$	$\frac{108}{256}$
2	2C2T	$\frac{27}{128}$	$\frac{54}{256}$
5	3C1T	$\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1\,{}^4C_3=\frac{3}{64}$	$\frac{12}{256}$
8	4C0T	$\left(\frac{1}{4}\right)^4=\frac{1}{256}$	$\frac{1}{256}$

(iii) E (X) $=-4\left(\frac{81}{256}\right)-1\left(\frac{27}{64}\right)+2\left(\frac{27}{64}\right)+5\left(\frac{3}{64}\right)+8\left(\frac{1}{256}\right)=-1$

Mean amount = E (8+ 2X) = (8+2 (–1)) = 6

$$\text{Var}(X)=E(X^2)-[E(X)]^2$$
$$=\left[(-4)^2\left(\frac{81}{256}\right)+(-1)^2\left(\frac{27}{64}\right)+2^2\left(\frac{27}{64}\right)+5^2\left(\frac{3}{64}\right)+8^2\left(\frac{1}{256}\right)\right]-(-1)^2$$
$$=\frac{31}{4}-1=\frac{27}{4}$$

Variance amount = Var (8+ 2X) = 4 Var (X) = $4\left(\frac{27}{4}\right)=27$

The total value of the vouchers the shop keeper is expected to give away
= 100 (6) = \$600

Let T be the total value of the vouchers given away to 100 customers
As sample size 100 is large, by Central Limit Theorem,
$T\sim N(6(100),27(100))$
$T\sim N(600,2700)$ approximately

Required probability = $P(T>650)\approx 0.168$

9

(a)(i) Let $v = x - 150, \quad w = y - 170$

$$d = \frac{\sum vw - \frac{\sum v \sum w}{n}}{\sum w^2 - \frac{\left(\sum w\right)^2}{n}} = \frac{677 - \frac{47 \times 87}{8}}{1101 - \frac{87^2}{8}} = 1.0710$$

$$\bar{x} = \bar{v} + 150 = \frac{47}{8} + 150 = 155.875$$

$$\bar{y} = \bar{w} + 170 = \frac{87}{8} + 170 = 180.875$$

Regression line of x on y :

$$x - \bar{x} = d\left(y - \bar{y}\right)$$

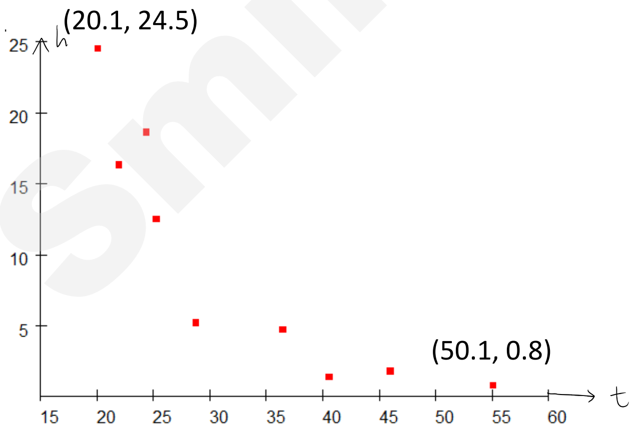
$$\Rightarrow x - 155.875 = 1.0710\left(y - 180.875\right)$$

$$\Rightarrow x = 1.07y - 37.8 \text{ (3sf)}$$

(a)(ii) $$r_{xy} = r_{vw} = \frac{\sum vw - \frac{\sum v \sum w}{n}}{\sqrt{\left[\sum v^2 - \frac{\left(\sum v\right)^2}{n}\right]\left[\sum w^2 - \frac{\left(\sum w\right)^2}{n}\right]}}$$

$$= \frac{677 - \frac{47 \times 87}{8}}{\sqrt{\left(483 - \frac{47^2}{8}\right)\left(1101 - \frac{87^2}{8}\right)}} = 0.92668 \approx 0.927$$

(b)(i)



A linear model is not suitable, from the scatter diagram as t increases, h decrease at a decreasing rate, whereas a linear model predicts a constant rate of decrease.

(b)(ii)

Model (A) between h and $\frac{1}{t}$, $r = 0.94438$

Model (B) between h and $\ln(t)$, $r = -0.90997$

Model A has a better fit since $|r| = 0.94438 > |r| = 0.90997$

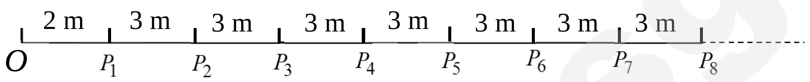
(or) Model A has a better fit since $|r| = 0.94438$ is closer to 1.

9	(b)(iii) From GC, regression line $h = -15.264 + \frac{738.12}{t}$ $h = -15.264 + \frac{738.12}{15} = 33.944 \approx 33.9$ (3 s.f.) $t = 15$ lies outside the data range of t (extrapolation), thus the linear relationship between h and $\frac{1}{t}$ may not be valid beyond the data range, hence estimate is not likely to be reliable.
10	(i) Unbiased estimate of population mean = $\bar{x} = \frac{250.5}{100} = 2505$ Unbiased estimate of population variance = $s^2 = \frac{1}{99}\left(628.45 - \frac{250.5^2}{100}\right)$ $= 0.0095707 = 0.00957$ (3 sf)
	(ii) Let X be the no. of hours spent on In-star-gram and μ be the mean usage hours. H_0: $\mu = 2.49$ H_1: $\mu > 2.49$ Under H_0, since sample size $n = 100$ is large, by Central Limit Theorem, The test statistic is $Z = \frac{\bar{X} - 2.49}{\sqrt{\frac{0.0095707}{100}}} \sim N(0,1)$ approximately. Given $\bar{x} = 2.505$, From GC, $p = 0.0626$ Since H_0 is rejected under α % significance level p-value < α % $\alpha > 6.26$ Assumption: The sample is randomly obtained.
	(iii) The p – value is the probability of getting a mean usage time of 2.505 or more from any sample of size 100 if the population mean is indeed 2.49. (i.e. There is 6.26% chance of getting a sample mean of 2.505 or more if the population mean is 2.49.)
	(iv) H_0: $\mu = 2.49$ H_1: $\mu \neq 2.49$ Given level of significance = 5%, σ = 1.5 and n = 100 Under H_0, since the sample size n = 100 is large, by Central Limit Theorem, the test statistic: $Z = \frac{\bar{X} - 2.49}{\left(\frac{1.5}{\sqrt{100}}\right)} \sim N(0,1)$ approximately. At 5% level of significance, critical region is $z \leq -1.95996$ or $z \geq 1.95996$

	Since there is sufficient evidence at 5% level of significance to conclude that average usage hours of Instar-gram is not 2.49 hours, $\mathrm{H_0}$ is rejected. $\dfrac{k-2.49}{1.5/\sqrt{100}} \le -1.95996$ or $\dfrac{k-2.49}{1.5/\sqrt{100}} \ge 1.95996$ $\Rightarrow 0 < k \le 2.20$ or $k \ge 2.78$ (3s.f.)

2018 ACJC H2 Prelim Paper 1 Question

1 Points $P_1, P_2, P_3, ..., P_n$ are marked on a straight line increasingly far away from the origin O such that distance $OP_1 = 2$ m and the distances between subsequent adjacent points are all 3 m as shown in the diagram below.



(i) Express the distances OP_n and P_1P_n in terms of n. [2]

(ii) Hence, or otherwise, show that the total distance $OP_n + P_1P_n + P_2P_n + ... + P_{n-1}P_n$ is $\frac{3}{2}n^2 + \frac{3}{2}n - 1$. [3]

2 Two series S_n and T_{n-1} are given by

$$S_n = \sum_{r=1}^{n}\left(\frac{r}{2^{r-1}}\right) \text{ and } T_{n-1} = \sum_{r=1}^{n-1}\left(\frac{r}{2^r}\right).$$

By listing the terms of S_n and T_{n-1}, write $S_n - T_{n-1}$ as a geometric series and find $S_n - T_{n-1}$, leaving your answer in terms of n. [3]

Hence find the value of $S_n - T_{n-1}$ as $n \to \infty$. [1]

3 **(a)** The quadrilateral *ABCD* is such that *P*, *Q*, *R* and *S* are the midpoints of *AB*, *BC*, *CD* and *DA* respectively. Prove that *PQRS* is a parallelogram. [3]

(b) Referred to the origin *O*, points *A*, *B* and *C* have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where $\mathbf{a}$ is a unit vector, $|\mathbf{b}| = 3$, $|\mathbf{c}| = \sqrt{3}$ and angle AOC is $\frac{\pi}{6}$ radians. Given that $3\mathbf{a} + \mathbf{c} = k\mathbf{b}$ where $k \neq 0$, by considering $(3\mathbf{a} + \mathbf{c}) \cdot (3\mathbf{a} + \mathbf{c})$, find the exact values of k. [4]

4 In the triangle *ABC*, *AB* = 2, *BC* = 3 and angle *ABC* = $\frac{\pi}{3} - \theta$ radians. Given that θ is a sufficiently small angle, show that

$$AC \approx \left(7 - 6\sqrt{3}\,\theta + 3\theta^2\right)^{\frac{1}{2}} \approx a + b\theta + c\theta^2,$$

where constants a, b and c are to be determined in exact form. [7]

5 A function h is said to self-inverse if $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$ for all x in the domain of h.

Functions f and g are defined by

$$\mathrm{f} : x \mapsto \frac{5x-3}{x-5},\ x \in \mathbb{R}, x \neq a, \text{ where } a \text{ is a constant,}$$

$$\mathrm{g} : x \mapsto \ln x,\ \ x \in \mathbb{R}, x \geq \mathrm{e}^{10}.$$

(i) State the value of a and explain why this value has to be excluded from the domain of f. [2]

(ii) Show that f is self-inverse. [2]

(iii) Find the exact values of b such that $\mathrm{f}^4(b) - 2 = \mathrm{f}^{-1}(b)$. [2]

(iv) Find the exact range of fg. [2]

6 Find

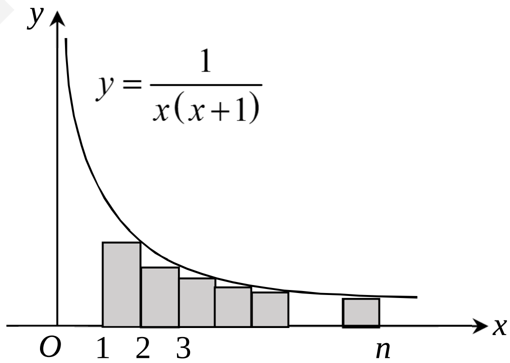
(a) $\displaystyle\int \left(\sin^{-1} 2x\right)\frac{x}{\sqrt{1-4x^2}}\ \mathrm{d}x$. [4]

(b) $\displaystyle\int \frac{x-1}{x^2+2x+6}\ \mathrm{d}x$. [4]

7

(i) Express $\dfrac{1}{x(x+1)}$ in partial fractions. [1]

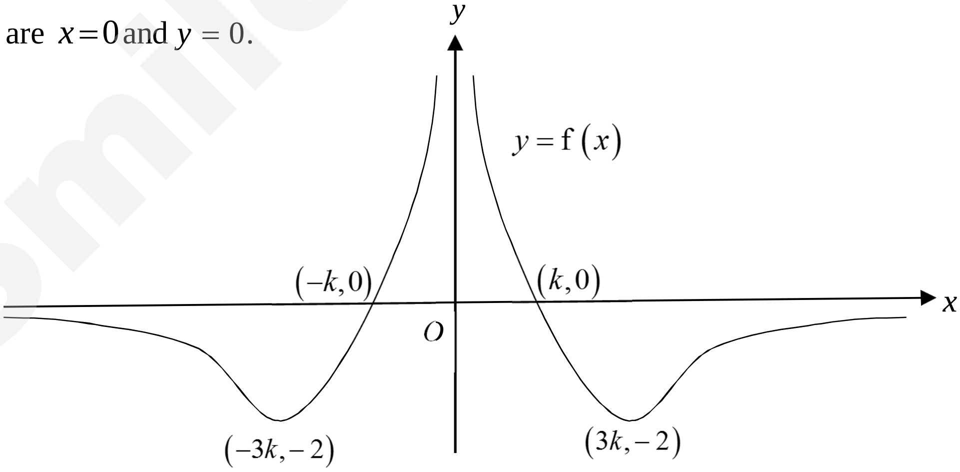
(ii)



The graph of $y = \dfrac{1}{x(x+1)}$, for $0 \leq x \leq n$, is shown in the diagram. Rectangles, each of width 1 unit, are drawn below the curve from $x = 1$ to $x = n$, where $n \geq 3$.

By considering $\displaystyle\sum_{x=a}^{b} \frac{1}{x(x+1)}$ where a and b are constants to be found, find the total area of the $n - 1$ rectangles in terms of n. [3]

Find the actual area bounded by the curve $y = \dfrac{1}{x(x+1)}$, the x-axis and the lines $x = 1$ and $x = n$. [2]

	Hence show that $\frac{1}{2}-\ln 2<\frac{1}{n+1}+\ln\left(1-\frac{1}{n+1}\right)$ for all $n \geq 3$. [1] Using a standard series from the List of Formulae (MF26), show that, for all $n \geq 3$, $$\frac{1}{2}-\ln 2<\sum_{r=2}^{\infty}\frac{-1}{r(n+1)^{r}}.$$ [2]
8	**(a)** Without using a calculator, solve the inequality $\frac{x+5}{x-1}>\frac{9}{x}$. [3] Hence find the set of values of x which satisfies $\frac{e^{x}+5}{e^{x}-1}>9e^{-x}$. [2] **(b)** It is given that $y=\frac{1+kx}{x^{2}-1}$, $x\in\mathbb{R}$, $x\neq\pm1$, and $-1\leq k\leq 1$. Find, in terms of k, the set of values that y can take. [4]
9	**(a)** The diagram shows the sketch of $y=\mathrm{f}(x)$, where k is a positive constant. The curve passes through the points with coordinates $(-k,0)$ and $(k,0)$, and has two minimum points with coordinates $(-3k,-2)$ and $(3k,-2)$. The asymptotes are $x=0$ and $y=0$.  Sketch on separate diagrams, the graphs of **(i)** $y=\mathrm{f}(2x-2k)$, [3] **(ii)** $y=\mathrm{f}'(x)$, [3] **(iii)** $y=\frac{1}{\mathrm{f}(x)}$, [3]

		showing clearly, in terms of k, the equations of any asymptote(s), the coordinates of any turning point(s) and any points where the curve crosses the x- and y-axes, where possible.
	(b)	The curve C has equation $y=\dfrac{3x^2+9x+7}{x+2}$.
		(i) By expressing the equation of C in the form $y=3x+a+\dfrac{b}{x+2}$, where a and b are non-zero constants, write down the equations of the asymptotes of C. [2]
		(ii) Describe a pair of transformations that will transform C to the graph of $y=3x+\dfrac{1}{x}$. [2]
10	**(a)**	Lion City Hospital is situated on the island of Jali. At the hospital, there are 3 classes of wards: A, B and C. The table below shows the number of patients admitted to each class of wards over 4 days, as well as the total amount collected by the hospital for bed charges.

	Day 1	***Day 2***	***Day 3***	***Day 4***
Class A	20	25	30	30
Class B	35	37	k	62
Class C	38	40	61	65
Total bed charges ($)	14150	16300	20600	23050

Assuming that all the patients admitted to a ward of the same class pay the same bed charge, find the value of k. [4]

(b) An epidemic occurred in the island of Jali and the number of people, P, who have caught the disease t weeks after the epidemic has started is given by the logistic growth equation proposed by Verhulst shown below.

$$\frac{dP}{dt}=0.9P\left(1-\frac{P}{300}\right)$$

It is given that when the epidemic started, 5 people had the disease.

(i) Show that $P=\dfrac{A}{Be^{-0.9t}+1}$, where A and B are constants. [5]

(ii) Sketch a graph of P against t. [2]

(iii) Given that Lion City Hospital has 350 beds, explain clearly whether this hospital can be used solely as a quarantine centre for all the people who have caught the disease during the epidemic. [1]

(iv) Once the epidemic reaches its peak, the spread of the disease will begin to slow down. Find the number of people who had the disease when the disease was spreading most rapidly. [2]

11

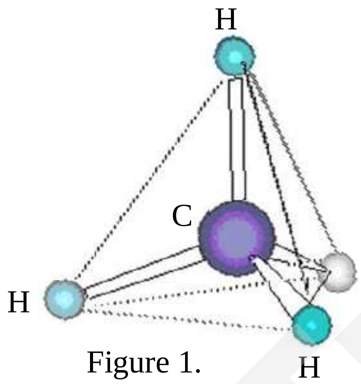


Figure 1.

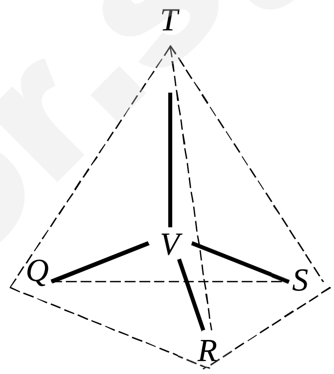


Figure 2.

Figure 1 shows a methane molecule consisting of a carbon atom with four hydrogen atoms symmetrically placed around it. Figure 2 shows the tetrahedron structure of the methane molecule with the centres of the hydrogen atoms represented by points Q, R, S and T and the centre of the carbon atom represented by point V.

The points Q, R and S has coordinates (8, 1, 8), (8, 7, 2) and (2, 1, 2) respectively and form an equilateral triangle.

(i) Find a cartesian equation of the plane p which passes through the points Q, R and S. [4]

(ii) Find a cartesian equation of the plane p_1 which passes through the midpoint of QR and is perpendicular to QR. [2]

Plane p_2 which passes through the midpoint of RS and is perpendicular to RS has equation $x+y=9$.

(iii) Find a vector equation of the line l where p_1 and p_2 meet. [1]

(iv) The point T is on the line l such that $QRST$ is a regular tetrahedron with QR = QT.

Show that the possible coordinates for point T is (2, 7, 8). Hence, or otherwise, find the coordinates of a point on plane p that is closest to point T. [5]

	(v) Given that *TV* is $\frac{3}{4}$ of the vertical height of tetrahedron *QRST*. Find the coordinates of point *V* and hence show that the bonding angle *TVQ* of the methane molecule is $109.5°$ (correct to 1 decimal place). [4]

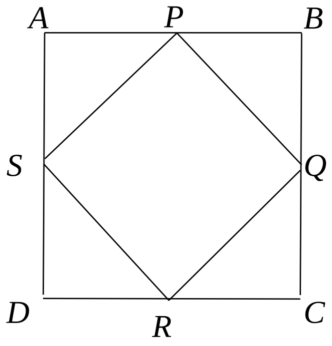
Anglo-Chinese Junior College
2018 H2 Mathematics Prelim Paper 1 Marker's Report

Qn	Solution	Remarks
1	(i) $OP_n = 2+3(n-1) = 3n-1$ $P_1P_n = 3(n-1) = 3n-3$ Total distance $= OP_n + P_1P_n + P_2P_n + P_3P_n + ... + P_{n-1}P_n$ $= (3n-1)+(3n-3)+(3n-6)$ $+(3n-9)+...+3$ $= (3n-1)+\frac{n-1}{2}((3n-3)+3)$ $= (3n-1)+\frac{(n-1)(3n)}{2}$ $= 3n-1+\frac{3}{2}n^2-\frac{3}{2}n$ $= \frac{3}{2}n^2+\frac{3}{2}n-1$ (*shown*)	Some students left the answer as $OP_n = 2+3(n-1)$, without simplifying the answer. Since the question require you to express in terms of *n*, answers must be given in simplified form. When you are using the sum of AP formula it is a good habit to state or list out the AP first, especially since this is a proving question. Note: The value of *n* in the Sum of *n* terms formula for an AP is the **number of terms** that are summed. Some students are not careful about this.
2	(i) $S_n = \sum_{r=1}^{n}\left(\frac{r}{2^{r-1}}\right) = \frac{1}{2^0}+\frac{2}{2^1}+\frac{3}{2^2}+...+\frac{n-1}{2^{n-2}}+\frac{n}{2^{n-1}}$ $T_{n-1} = \sum_{r=1}^{n-1}\left(\frac{r}{2^r}\right) = \frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+...+\frac{n-1}{2^{n-1}}$ $\therefore\ S_n - T_{n-1} = \frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^{n-1}}$ $S_n - T_{n-1} = \frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^{n-1}}$ $= \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$ $= 2\left[1-\frac{1}{2^n}\right] = 2-\left(\frac{1}{2}\right)^{n-1}$ (ii) As $n\to\infty$, $\left(\frac{1}{2}\right)^{n-1}\to 0$, $\therefore S_n - T_{n-1} \to 2$	Some students try to find the answer without listing the terms, usually could not get the answer. Many are able to list the terms but unable to see the geometric series.

3

(a)

$$\overrightarrow{OP}=\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$$



$$\overrightarrow{OQ}=\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}$$

$$\overrightarrow{OR}=\frac{\overrightarrow{OC}+\overrightarrow{OD}}{2}$$

$$\overrightarrow{OS}=\frac{\overrightarrow{OD}+\overrightarrow{OA}}{2}$$

$$\overrightarrow{PQ}=\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}-\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}=\frac{\overrightarrow{OC}-\overrightarrow{OA}}{2}$$

$$\overrightarrow{SR}=\frac{\overrightarrow{OC}+\overrightarrow{OD}}{2}-\frac{\overrightarrow{OD}+\overrightarrow{OA}}{2}=\frac{\overrightarrow{OC}-\overrightarrow{OA}}{2}=\overrightarrow{PQ}$$

Since $\overrightarrow{SR}=\overrightarrow{PQ}$, $\therefore$ $PQRS$ is a parallelogram.

Alternatively,

$$\overrightarrow{SP}=\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}-\frac{\overrightarrow{OD}+\overrightarrow{OA}}{2}=\frac{\overrightarrow{OB}-\overrightarrow{OD}}{2}$$

$$\overrightarrow{RQ}=\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}-\frac{\overrightarrow{OC}+\overrightarrow{OD}}{2}=\frac{\overrightarrow{OB}-\overrightarrow{OD}}{2}=\overrightarrow{SP}$$

Since $\overrightarrow{RQ}=\overrightarrow{SP}$, $\therefore$ $PQRS$ is a parallelogram.

Common mistakes:
Starting the proof by assuming a $ABCD$ is a parallelogram.

The use of ratio theorem is not common, and some students made the mistake by writing $\overrightarrow{OP}=\frac{1}{2}\overrightarrow{AB}$ instead.

(b)

$$(3\mathbf{a}+\mathbf{c})\bullet(3\mathbf{a}+\mathbf{c})=9\mathbf{a}\bullet\mathbf{a}+3\mathbf{a}\bullet\mathbf{c}+3\mathbf{c}\bullet\mathbf{a}+\mathbf{c}\bullet\mathbf{c}$$

$$(3\mathbf{a}+\mathbf{c})\bullet(3\mathbf{a}+\mathbf{c})=9\mathbf{a}\bullet\mathbf{a}+6\mathbf{a}\bullet\mathbf{c}+\mathbf{c}\bullet\mathbf{c}$$

$$k\mathbf{b}\bullet k\mathbf{b}=9\mathbf{a}\bullet\mathbf{a}+6\mathbf{a}\bullet\mathbf{c}+\mathbf{c}\bullet\mathbf{c}$$

$$k^2|\mathbf{b}|^2=9|\mathbf{a}|^2+6|\mathbf{a}||\mathbf{c}|\cos\frac{\pi}{6}+|\mathbf{c}|^2$$

$$k^2|\mathbf{b}|^2=9(1)+6(1)(\sqrt{3})\cos\frac{\pi}{6}+(\sqrt{3})^2$$

$$k^2(3)^2=21$$

$$k=\pm\frac{\sqrt{21}}{3}$$

A lot of careless mistakes in the expansion. Many students did not know how to proceed to find k.

4

$$AC^2 = AB^2 + BC^2 - 2AB\times BC\cos\left(\frac{\pi}{3}-\theta\right)$$
$$= 2^2+3^2-2\times2\times3\cos\left(\frac{\pi}{3}-\theta\right)$$
$$= 4+9-12\left(\cos\frac{\pi}{3}\cos\theta+\sin\frac{\pi}{3}\sin\theta\right)$$
$$= 13-12\left(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)$$
$$\approx 13-6\left(1-\frac{1}{2}\theta^2+\sqrt{3}\,\theta\right)$$
$$= 7-6\sqrt{3}\,\theta+3\theta^2$$
$$AC \approx \left(7-6\sqrt{3}\,\theta+3\theta^2\right)^{\frac{1}{2}} \qquad (shown)$$

$$\left(7-6\sqrt{3}\,\theta+3\theta^2\right)^{\frac{1}{2}}$$
$$= \sqrt{7}\left(1-\frac{6\sqrt{3}}{7}\theta+\frac{3}{7}\theta^2\right)^{\frac{1}{2}}$$
$$\approx \sqrt{7}\left(1+\frac{1}{2}\left(-\frac{6\sqrt{3}}{7}\theta+\frac{3}{7}\theta^2\right)+\frac{\frac{1}{2}.-\frac{1}{2}}{\frac{1}{2}}\left(-\frac{6\sqrt{3}}{7}\theta+\frac{3}{7}\theta^2\right)^2\right)$$
$$= \sqrt{7}\left(1-\frac{3\sqrt{3}}{7}\theta+\frac{3}{14}\theta^2-\frac{1}{8}\left(\frac{36\times3}{49}\theta^2\right)\right)$$
$$= \sqrt{7}\left(1-\frac{3\sqrt{3}}{7}\theta+\frac{3}{14}\theta^2-\frac{27}{98}\theta^2\right)$$
$$= \sqrt{7}\left(1-\frac{3\sqrt{3}}{7}\theta-\frac{6}{98}\theta^2\right) = \sqrt{7}\left(1-\frac{3\sqrt{3}}{7}\theta-\frac{3}{49}\theta^2\right)$$
$$= \sqrt{7}-\frac{3\sqrt{21}}{7}\theta-\frac{3\sqrt{7}}{49}\theta^2$$
$$\text{or} \quad \sqrt{7}-\frac{3\sqrt{3}\sqrt{7}}{7}\theta-\frac{3\sqrt{7}}{49}\theta^2$$

$$\therefore\ AC \approx \sqrt{7}-\frac{3\sqrt{21}}{7}\theta-\frac{3\sqrt{7}}{49}\theta^2$$
$$\text{or} \quad \sqrt{7}-\frac{3\sqrt{3}\sqrt{7}}{7}\theta-\frac{3\sqrt{7}}{49}\theta^2$$

$$a=\sqrt{7},\ b=-\frac{3\sqrt{21}}{7},\ c=-\frac{3\sqrt{7}}{49}$$

This question tested students' knowledge on the standard Maclaurin series given in MF26.

First Part

Among the errors,
Very common:
1. Forgetting **cosine rule**.
2. Assuming $\left(\frac{\pi}{3}-\theta\right)$ is a small angle.
3. Carelessness with the signs.

Less common:
1. Applied **cosine addition formula** wrongly.
2. Not using cosine rule.

Second Part

Quite a number used the method of repeated differentiation, losing time and making careless mistakes.

Among the errors,
Very common:
1. Trying to use the binomial series with a variable term that will not allow them to collect all the terms they require.
2. Carelessness with the signs.
3. Expanding the $\sqrt{7}$ back.

Less common:
1. Factoring 7 or $1/\sqrt{7}$ out.
2. Forgetting to factor the 7 out from the $-6\sqrt{3}\,\theta+3\theta^2$ expression.

5	(i) $a = 5$ 5 has to be excluded from the domain of f as it does not have an image under f, which will then mean that f is not a function.	Value of *a* well-done. Explanation of why 5 had to be excluded from the domain of f was not well done. Students need to explain that for f to be a function, **every element in its domain must have a unique image**. Many students just wrote that *x*=5 is a vertical asymptote, or that f is undefined when *x*=5.
	ii) Let $y = \frac{5x-3}{x-5}$. $y = \frac{5x-3}{x-5}$ $y(x-5) = 5x-3$ $yx - 5y = 5x - 3$ $yx - 5x = 5y - 3$ $x = \frac{5y-3}{y-5}$ $\mathrm{f}^{-1}(x) = \frac{5x-3}{x-5} = \mathrm{f}(x)$ $\therefore$ f is self-inverse.	Supposed to be easy to show but many students just sketched the graph of f(*x*) and showed that it is one-one, therefore it is self-inverse, which is wrong. A one-one function <u>does not mean</u> it is self-inverse.
	(iii) Since f is self-inverse, $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. $\therefore \mathrm{f}^4(x) = \mathrm{ff}\left[\mathrm{f}^2(x)\right]$ $= \mathrm{ff}\left[\mathrm{ff}^{-1}(x)\right]$ $= \mathrm{ff}[x]$ $= x$ $\mathrm{f}^4(b) - 2 = \mathrm{f}^{-1}(b)$ $b - 2 = \mathrm{f}(b)$ $b - 2 = \frac{5b-3}{b-5}$ $b^2 - 7b + 10 = 5b - 3$ $b^2 - 12b + 13 = 0$ $b = 6 + \sqrt{23}$ or $b = 6 - \sqrt{23}$	Only about half the cohort could do this. Many did not realise $\mathrm{f}^4(x) = x$. Of those who only got 1 mark for this part, many used the G.C to calculate the values of *b* when the question clearly asked for exact values. <u>WRONG CONCEPTS</u> that were seen: $\mathrm{f}^4(b) = \left[\mathrm{f}(b)\right]^4$; $\mathrm{f}(a-b) = \mathrm{f}(a) - \mathrm{f}(b)$ These are both WRONG

	(iv) Take the $R_g = [10, \infty)$ as the restricted domain of f and read off the corresponding range. $[e^{10}, \infty) \xrightarrow{g} [10, \infty) \xrightarrow{f} (5, \frac{47}{5}]$ $R_{fg} = (5, \frac{47}{5}]$	Many students also did not realise $\sqrt{93} = 2\sqrt{23}$, and so did not simplify their answers, but credit was given anyway. Usual careless algebraic errors were seen in simplifying the quadratic equation. Most common error for students who could find the two end points of the range: [47/5, 5), which does not make sense.
6	(a) Let $u = \sin^{-1} 2x$, $\frac{dv}{dx} = \frac{x}{\sqrt{1-4x^2}} = x\left(1-4x^2\right)^{-\frac{1}{2}}$ $\frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}$, $v = \int x\left(1-4x^2\right)^{-\frac{1}{2}} = -\frac{1}{8}\int -8x\left(1-4x^2\right)^{-\frac{1}{2}} dx$ $= -\frac{1}{4}\sqrt{1-4x^2} + C$ $\int \sin^{-1} 2x \frac{x}{\sqrt{1-4x^2}}\, \mathrm{d}x$ $= \left[\left(\sin^{-1} 2x\right)\left(-\frac{1}{4}\sqrt{1-4x^2}\right)\right] - \int \left(-\frac{1}{4}\sqrt{1-4x^2}\right)\left(\frac{2}{\sqrt{1-4x^2}}\right) \mathrm{d}x$ $= \left[-\frac{1}{4}\left(\sin^{-1} 2x\right)\sqrt{1-4x^2}\right] + \int \frac{1}{2} \mathrm{d}x$ $= \left[-\frac{1}{4}\left(\sin^{-1} 2x\right)\sqrt{1-4x^2}\right] + \frac{1}{2}x + C$	Students did better in 6(b) than 6(a). 6(a) **common mistakes**: 1) Can't identify u & $\frac{dv}{dx}$ correctly. 2) $-\frac{1}{8}\int (\frac{(-8x)}{\sqrt{1-4x^2}} dx$ $= -\frac{1}{8}\left[\frac{\left(1-4x^2\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]$ 3) $-\frac{1}{8}\left[\frac{\left(1-4x^2\right)^{\frac{1}{2}}}{\frac{1}{2}}\right]$ $= -\frac{1}{16}\left(1-4x^2\right)^{\frac{1}{2}}$ 4) $\mathrm{f}(x) = \sin^{-1} 2x$ but $\mathrm{f}'(x) = \frac{x}{\sqrt{1-4x^2}}$ A few solved using substitution $\theta = \sin^{-1} 2x$ correctly

	(b) $\int \frac{x-1}{x^2+2x+6}\ \mathrm{d}x$ $= \frac{1}{2}\int \frac{2x+2}{x^2+2x+6}\ \mathrm{d}x - \int \frac{2}{(x+1)^2+5}\ \mathrm{d}x$ $= \frac{1}{2}\ln\left\|x^2+2x+6\right\| - \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + C$	6(b) A few **common mistakes** 1) Split/ present $\int \frac{x-1}{x^2+2x+6}dx$ wrongly due to bracket/ without bracket 2) Careless mistake in complete square x^2+2x+6 3) $\int \frac{1}{(x+1)^2+5}dx$ $= \ln\left[(x+1)^2+5\right]+c$
7	(i) $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ (ii) Total area of rectangles $= \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + ... + \frac{1}{n(n+1)}$ $= \sum_{x=2}^{n} \frac{1}{x(x+1)}$ so $a = 2$, $b = n$ $= \sum_{x=2}^{n}\left(\frac{1}{x} - \frac{1}{x+1}\right)$ $= \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + ... + \left(\frac{1}{n}-\frac{1}{n+1}\right)$ $= \frac{1}{2} - \frac{1}{n+1}$	(i) Most students were able to obtain the correct partial fractions. (ii) **Area of rectangles:** Many students provided incorrect values for *a* and *b*, the most common pairs being $a=1$, $b=n-1$ (because of the phrase "$n-1$ rectangles") and $a=1$, $b=n$ (because of familiarity with these values). A good starting point for this question would have been to write down the areas of each rectangle according to the diagram. The method of difference was generally done correctly, but some students preferred to begin the series from $x=1$ and subtract $\frac{1}{2}$ afterwards. This, along with the previous observation about *a*, shows that students prefer to handle series that begin from $r=1$ and not any other values. This may be problematic when extra algebraic manipulation is then required to simply the answer. Some other minor presentation points are: • ensuring the running

Actual area $= \int_1^n \frac{1}{x(x+1)}\,\mathrm{d}x = \int_1^n \frac{1}{x} - \frac{1}{(x+1)}\,\mathrm{d}x$

$$= \left[\ln x - \ln(x+1)\right]_1^n$$
$$= \ln n - \ln(n+1) - \ln 1 + \ln 2$$
$$= \ln n - \ln(n+1) + \ln 2$$

Area of rectangles < actual area

$$\therefore \quad \frac{1}{2} - \frac{1}{n+1} < \ln n - \ln(n+1) + \ln 2$$
$$\frac{1}{2} - \ln 2 < \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right)$$
$$\frac{1}{2} - \ln 2 < \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1}\right) \quad (shown)$$

Using MF26,

$$\ln\left(1 - \frac{1}{n+1}\right) = -\frac{1}{n+1} - \frac{1}{2}\left(\frac{1}{n+1}\right)^2 - \ldots - \frac{1}{r}\left(\frac{1}{n+1}\right)^r - \ldots$$

variable is consistent in the summation notation;
- checking the signs of the terms in the method of difference very carefully.

Actual area

The integration was generally done correctly, with students either using the same partial fractions from part (i) or completing the square in the denominator to use an MF26 formula. A handful of students carelessly missed out the ln 2 term. There were also students who swapped the order of the limits of integration, or worked out $\mathrm{F}(1) - \mathrm{F}(n)$ instead of $\mathrm{F}(n) - \mathrm{F}(1)$.

Comparing areas

Students who were able to find the 2 areas correctly were able to complete this proof. The proof would be more complete if students wrote down the inequality about the areas in words, instead of writing down the expressions immediately. Some who had gotten one of the areas wrong were able to use this part to realise their earlier error and correct it; this shows good awareness and ability to review their own work.

Some students had written the reason for the inequality as "the area of rectangles is an approximation of the actual area". This wasn't marked down, but students should be aware that this is not the direct reason for the direction of the inequality. The inequality would have gone the other way if the rectangles were drawn to be an overestimation of the actual area.

Expansion of $\ln(1+x)$

This part was poorly done. Many students attempted to begin their working from the summation in the end result, $\sum_{r=2}^{\infty} \frac{-1}{r(n+1)^r}$, which is already wrong in terms of proving

	$\therefore\ \frac{1}{2}-\ln 2<\frac{1}{n+1}-\frac{1}{n+1}-\frac{1}{2}\left(\frac{1}{n+1}\right)^2-\ldots-\frac{1}{r}\left(\frac{1}{n+1}\right)^r-\ldots$ $\therefore\ \frac{1}{2}-\ln 2<-\frac{1}{2}\frac{1}{(n+1)^2}-\ldots-\frac{1}{r}\frac{1}{(n+1)^r}-\ldots$ $\Rightarrow \frac{1}{2}-\ln 2<\sum_{r=2}^{\infty}\frac{-1}{r(n+1)^r}\quad (shown)$	techniques. For those who started correctly from the previous inequality, some tried to apply the expansion of $\ln(1+x)$ to $\ln 2$, not realising that the LHS of the inequalities are unchanged. Some carelessly let $x=\frac{1}{n+1}$ instead of $x=-\frac{1}{n+1}$, and had difficulties proving the -1 in the final expression. Several students also got stuck because they also expanded $(1+n)^{-1}$, unable to see that the term will cancel out the first term in the expansion of $\ln\left(1-\frac{1}{n+1}\right)$.
8	(a) $\frac{x+5}{x-1}-\frac{9}{x}>0$ $\frac{x^2-4x+9}{(x-1)x}>0$ $\frac{(x-2)^2+5}{(x-1)x}>0$ Since $(x-2)^2+5>0$ for all real values of *x*, $(x-1)x>0$ $x<0$ or $x>1$ Using previous result, replace *x* by e^x. $e^x<0$ (rejected) or $e^x>1$ $\therefore x>0$	A significant number of students did not obtain full marks for (a) although this is an easy inequalities question. **Common Mistakes**: 1. Some students simply cross-multiply. 2. Some tried to factorise x^2-4x+9 into $(x-(2+\sqrt{5}i))(x-(2-\sqrt{5}i))$ and then marked $2+\sqrt{5}i$ and $2-\sqrt{5}i$ on the number line. 3. For those who tried to show that $x^2-4x+9>0$ by showing discriminant < 0, many did NOT show that coefficient of $x^2>0$. Most students recognised that *x* is to be replaced by e^x. Only a small number replaced *x* is by e^{-x} instead. Some students left the answer as *x* > ln 1 without evaluating ln 1.

(b) Method 1:

$$y=\frac{1+kx}{x^2-1}$$

$$yx^2-y=1+kx$$

$$yx^2-kx-(y+1)=0$$

For discriminant ≥ 0

$$(-k)^2-4y(-y-1)\geq 0$$

$$4y^2+4y+k^2\geq 0$$

Consider $4y^2+4y+k^2=0$,

$$y=\frac{-1\pm\sqrt{1-k^2}}{2}$$

$$\therefore\ y\leq\frac{-1-\sqrt{1-k^2}}{2}\ \text{ or } y\geq\frac{-1+\sqrt{1-k^2}}{2}$$

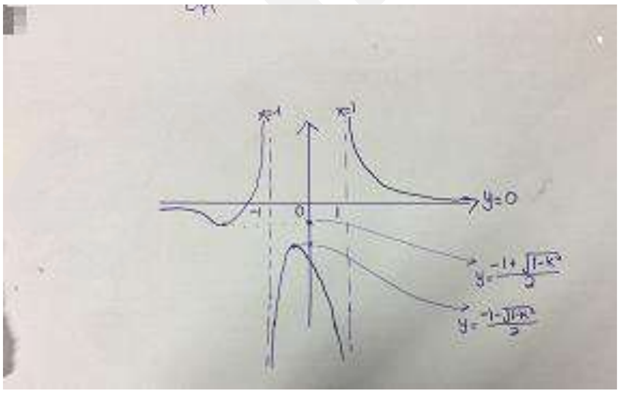
Method 2:

$$\frac{dy}{dx}=0$$

$$kx^2+2x+k=0$$

$$x=\frac{-1\pm\sqrt{1-k^2}}{k}$$

stationary points are: $\left(\frac{-1-\sqrt{1-k^2}}{k},\frac{-1+\sqrt{1-k^2}}{2}\right),\left(\frac{-1+\sqrt{1-k^2}}{k},\frac{-1-\sqrt{1-k^2}}{2}\right)$

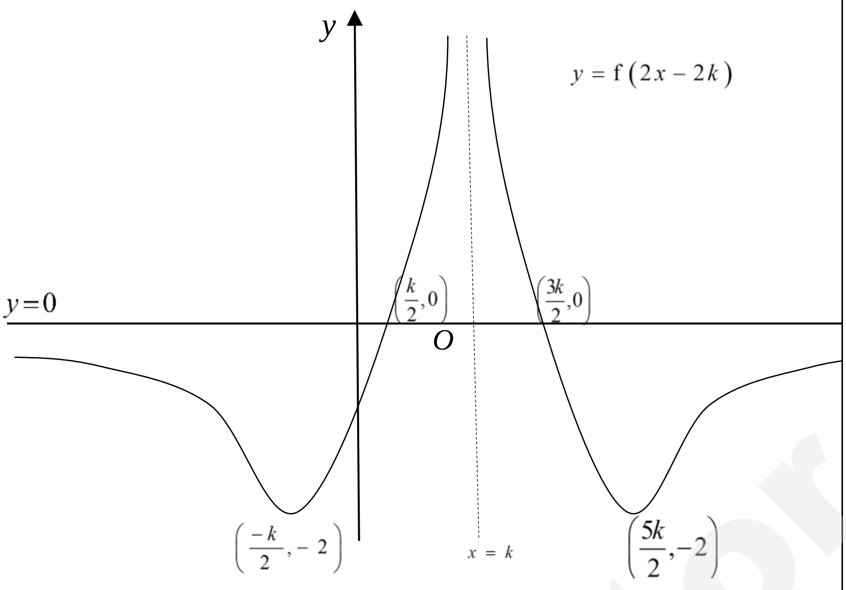
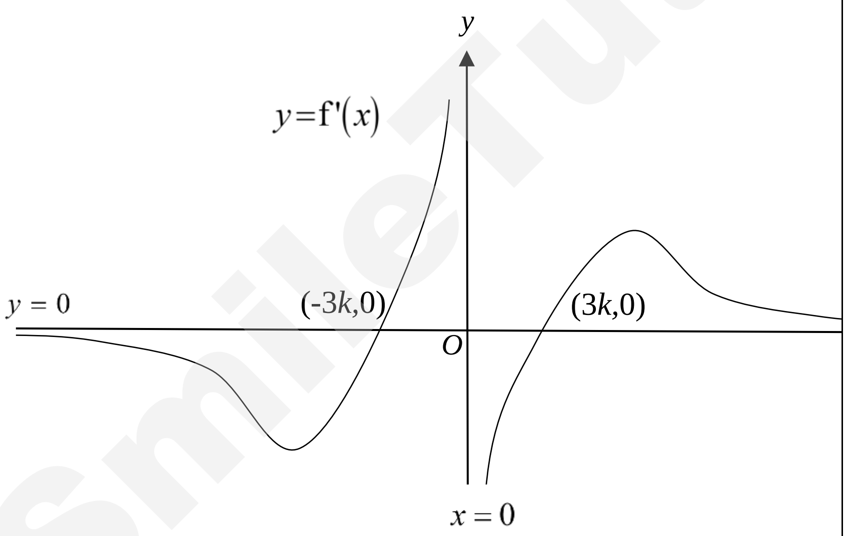


$$\therefore\ y\leq\frac{-1-\sqrt{1-k^2}}{2}\ \text{ or } y\geq\frac{-1+\sqrt{1-k^2}}{2}$$

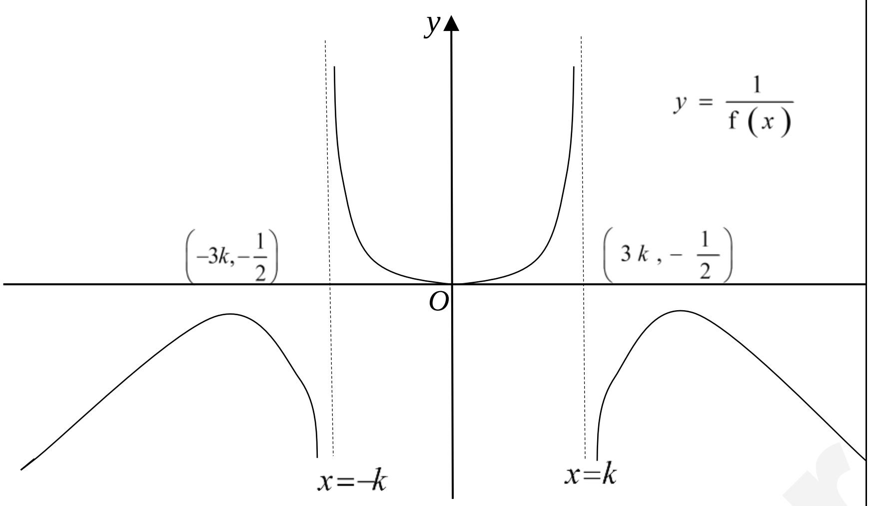
Most students did not know how to approach (b) and started by considering different values of *k* or tried to express the expression in partial fractions.

For those who were able to obtain the quadratic equation in *x*, a significant number consider discriminant > 0 instead of discriminant ≥ 0. Some made careless mistakes while trying to solve for the roots of the quadratic equation $4y^2+4y+k^2=0$.

For those who tried to use the graphical approach, many failed to show the sketch of the graph. Many also failed to recognise that there is a minimum point as the GC did not show clearly the presence of minimum point.

9	(a)(i)  (a)(ii)  	9(ai)Shape of the curve was done well but wrong *x*-intercepts, turning points and asymptotes. Quite a number did $y = f(2x) = g(x)$ then $y = g(x-2k)$ and gave answers like (5k/2,0) instead of (3k/2,0) or asymptote $x = 2k$ instead of $x = k$ Some did not include $y = 0$ as one of the asymptote. (9)(a)(ii) Many candidates know that (3k,0) and (-3k,0) are points on the curve. Common errors (a) label *x*-axis as $x = 0$ and *y* –axis as $y = 0$ as asymptotes. The shape of curve is not very well drawn. Quite a number drew curves which is similar in shape to $y = f(x)$.

(a)(iii)



(9)(a)(iii) Many has problem drawing the curve as $x \to \pm\infty$. Many answers with $y = 0$ is an asymptote seen. Produce curves such that (3k, -1/2) and (-3k, -½) are not max pts or they include two more minimum points so that the x-axis is an asymptote. Some drew the correct shape but include $y = 0$ as one of the asymptotes which is incorrect.

(b)(i)

$$y = \frac{3x^2+9x+7}{x+2}$$

$$= 3x+3+\frac{1}{x+2}$$

$$\begin{array}{r} 3x+3 \\ x+2\overline{)3x^2+9x+7} \\ \underline{3x^2+6x} \\ 3x+7 \\ \underline{3x+6} \\ 1 \end{array}$$

$\therefore a = 3, \quad b = 1$

The equations of the asymptotes of C are $x = -2$ and $y = 3x+3$.

(9)(b)(i) For oblique asymptote some wrote y =3x +3 and not y = 3x + 3.

(b)(ii)

<u>Method 1</u>

Translation of 2 units in the positive x-direction

Then translation of 3 units in the positive y-direction.

<u>Method 2</u>

Translation of 3 units in the positive y-direction.

Then translation of 2 units in the positive x-direction

9)(b)(ii) Use **wrong words** to describe the transformation such as shift, move, transform or right, up was often seen. Some wrote translate factor 2.

Students who do translation parallel to y-axis first tend to say translate 3 units in the negative y direction, which is incorrect. Many students wrote translation of 2 units in the positive "x-axis" instead of translation of 2 units in the positive ***x*-direction**

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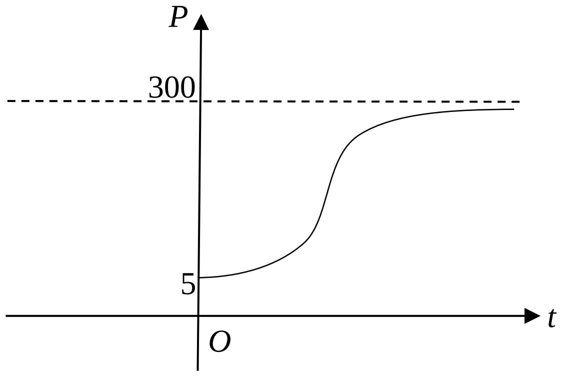
10	(a) Let a, b and c be the number of patients admitted to wards A, B and C respectively. $20a+35b+38c=14150$ $25a+37b+40c=16300$ $30a+62b+65c=23050$ By GC, $a=350,\ b=150,\ c=50$ $k=\dfrac{20600-30a-61c}{b}=47$ (b)(i) $\dfrac{\mathrm{d}P}{\mathrm{d}t}=0.9P\left(1-\dfrac{P}{300}\right)$ $\dfrac{\mathrm{d}P}{\mathrm{d}t}=0.003P(300-P)$ $\displaystyle\int\frac{1}{P(300-P)}\ \mathrm{d}P=\int 0.003\ \mathrm{d}t$ $\displaystyle\int\frac{1}{P}+\frac{1}{(300-P)}\ \mathrm{d}P=\int 0.9\ \mathrm{d}t$ $\ln\left\|\dfrac{P}{300-P}\right\|=0.9t+C$ $\dfrac{P}{300-P}=Ae^{0.9t}$, where $A=\pm e^{C}$ $P=300Ae^{0.9t}-PAe^{0.9t}$ $P+PAe^{0.9t}=300Ae^{0.9t}$ $P\left(1+Ae^{0.9t}\right)=300Ae^{0.9t}$ $P=\dfrac{300Ae^{0.9t}}{1+Ae^{0.9t}}$ $P=\dfrac{300}{Be^{-0.9t}+1}$, where $B=\dfrac{1}{A}$ when $t=0,\ P=5,$ $5=\dfrac{300}{Be^{0}+1}$ $B=\dfrac{300-5}{5}$ $B=59$ $\therefore\ P=\dfrac{300}{59e^{-0.9t}+1}$	The first part was very well done. Some students did not introduce the modulus sign when they did the integration. The $\pm$ sign was also missing. Some students still are not using the correct method to deduce the correct sign. As shown in the answer it is always better to handle the $\pm$ sign by expressing in exponential form before substituting the initial values.

10	(b)(ii) 	(ii) Graph should be for *t* >0 and the shape should be such that the graph tends to an asymptote and not touch the asymptote in a straight line as seen on the calculator screen at times. Students should explore the graphs they see on their calculator further and not just copy what they see.
	(b)(iii) Since *P* does not exceed 300 in the long run, Lion City Hospital can be used solely as the quarantine centre as it has 350 beds.	(iii) Students should answer the question that is asked with a reason. Some just said that the population tends to 300 without answering the question. Many students used the word 'maximum' or the 'graph peaks at' both of which are incorrect. The words 'does not exceed' are good to be used when describing an asymptote.
	(b)(iv) Let $w = 0.9P\left(1-\frac{P}{300}\right)$ Solve $\frac{\mathrm{d}w}{\mathrm{d}P} = 0$ $0.9\left(1-\frac{P}{300}\right)+0.9P\left(-\frac{1}{300}\right)=0$ $P = 150$	(iv) Many made the mistake of making the rate of growth of *P* to be 0 i.e *P* is max, when it should be the rate at which the disease is spreading maximum.

11	(i) $\overrightarrow{OQ}=\begin{pmatrix}8\\1\\8\end{pmatrix},\quad \overrightarrow{OR}=\begin{pmatrix}8\\7\\2\end{pmatrix},\quad \overrightarrow{OS}=\begin{pmatrix}2\\1\\2\end{pmatrix}$ $\overrightarrow{QR}=\overrightarrow{OR}-\overrightarrow{OQ}=\begin{pmatrix}8\\7\\2\end{pmatrix}-\begin{pmatrix}8\\1\\8\end{pmatrix}=\begin{pmatrix}0\\6\\-6\end{pmatrix}=6\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\overrightarrow{QS}=\overrightarrow{OS}-\overrightarrow{OQ}=\begin{pmatrix}2\\1\\2\end{pmatrix}-\begin{pmatrix}8\\1\\8\end{pmatrix}=\begin{pmatrix}-6\\0\\-6\end{pmatrix}=6\begin{pmatrix}-1\\0\\-1\end{pmatrix}$ $n=\begin{pmatrix}0\\1\\-1\end{pmatrix}\times\begin{pmatrix}-1\\0\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$ $r\bullet\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}\bullet\begin{pmatrix}-1\\1\\1\end{pmatrix}=1$ The cartesian equation of the plane p is $-x+y+z=1$ (ii) Midpoint of $\overrightarrow{QR}=\dfrac{\overrightarrow{OR}+\overrightarrow{OQ}}{2}=\dfrac{\begin{pmatrix}8\\7\\2\end{pmatrix}+\begin{pmatrix}8\\1\\8\end{pmatrix}}{2}=\begin{pmatrix}8\\4\\5\end{pmatrix}$ $r\bullet\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}8\\4\\5\end{pmatrix}\bullet\begin{pmatrix}0\\1\\-1\end{pmatrix}=-1$ The cartesian equation of the plane p_1 is $y-z=-1$ (iii) The vector equation of the line l is $r=\begin{pmatrix}10\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$ where $\lambda\in\mathbb{R}$	Common mistake: $n=\overrightarrow{OQ}\times\overrightarrow{OR}$ is **<u>not</u>** the normal. These are **<u>NOT</u>** the cartesian equation of the plane. $r=\begin{pmatrix}8\\1\\8\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\-1\end{pmatrix}+\beta\begin{pmatrix}-1\\0\\-1\end{pmatrix}$ $r\bullet\begin{pmatrix}-1\\1\\1\end{pmatrix}=1$ **Common mistake**: Midpoint of $\overrightarrow{QR}=\frac{1}{2}\overrightarrow{QR}$ Some students did not use GC to solve and wasted time working alegbraically for the answer. Student should note that this question is worth only 1 mark and hence the workings should not be long.

11 (iv)

Since point T is on the line l,

$$\overrightarrow{OT}=\begin{pmatrix}10-\lambda\\-1+\lambda\\\lambda\end{pmatrix}$$

$$\left|\overrightarrow{QR}\right|=\left|\overrightarrow{QT}\right|$$

$$\left|\begin{pmatrix}0\\6\\-6\end{pmatrix}\right|=\left|\begin{pmatrix}10-\lambda\\-1+\lambda\\\lambda\end{pmatrix}-\begin{pmatrix}8\\1\\8\end{pmatrix}\right|$$

$$\left|\begin{pmatrix}0\\6\\-6\end{pmatrix}\right|=\left|\begin{pmatrix}2-\lambda\\-2+\lambda\\\lambda-8\end{pmatrix}\right|$$

$$\sqrt{72}=\sqrt{(2-\lambda)^2+(-2+\lambda)^2+(\lambda+8)^2}$$

$$72=3\lambda^2-24\lambda+72$$

$$\lambda=0 \quad \text{or} \quad \lambda=8$$

When $\lambda=8$,

$$\overrightarrow{OT}=\begin{pmatrix}10-8\\-1+8\\8\end{pmatrix}=\begin{pmatrix}2\\7\\8\end{pmatrix}$$

When $\lambda=0$,

$$\overrightarrow{OT}=\begin{pmatrix}10-0\\-1+0\\0\end{pmatrix}=\begin{pmatrix}10\\-1\\0\end{pmatrix}$$

Hence, a possible coordinates for point T is (2, 7, 8). (Shown)

Method 1

$$\frac{\begin{pmatrix}2\\7\\8\end{pmatrix}+\begin{pmatrix}10\\-1\\0\end{pmatrix}}{2}=\begin{pmatrix}6\\3\\4\end{pmatrix}$$

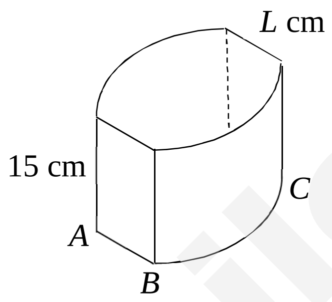
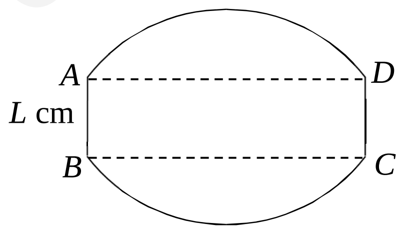
The coordinates of the point on the plane p that is closest to point T is (6, 3, 4).

Common mistake

Some students use the point (2, 7, 8) to verify that it lies on line l. This is not allowed as this is a show question.

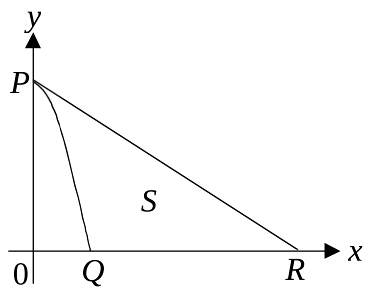
	Method 2 The vector equation of the line l through point T perpendicular to plane p is $r=\begin{pmatrix}2\\7\\8\end{pmatrix}+\mu\begin{pmatrix}-1\\1\\1\end{pmatrix}$ where $\mu\in\mathbb{R}$ $\begin{pmatrix}2-\mu\\7+\mu\\8+\mu\end{pmatrix}\bullet\begin{pmatrix}-1\\1\\1\end{pmatrix}=1$ $-2+\mu+7+\mu+8+\mu=1$ $3\mu=-12$ $\mu=-4$ When $\mu=-4$ $r=\begin{pmatrix}2\\7\\8\end{pmatrix}+(-4)\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}6\\3\\4\end{pmatrix}$ The coordinates of the point on the plane p that is closest to point T is (6, 3, 4). **(**v) Using ratio theorem, $\overrightarrow{OV}=\dfrac{3\begin{pmatrix}6\\3\\4\end{pmatrix}+\begin{pmatrix}2\\7\\8\end{pmatrix}}{4}=\begin{pmatrix}5\\4\\5\end{pmatrix}$ The coordinates of V is (5, 4, 5). $\overrightarrow{OQ}=\begin{pmatrix}8\\1\\8\end{pmatrix}$, $\overrightarrow{OV}=\begin{pmatrix}5\\4\\5\end{pmatrix}$, $\overrightarrow{OT}=\begin{pmatrix}2\\7\\8\end{pmatrix}$ $\overrightarrow{VQ}=\overrightarrow{OQ}-\overrightarrow{OV}=\begin{pmatrix}8\\1\\8\end{pmatrix}-\begin{pmatrix}5\\4\\5\end{pmatrix}=\begin{pmatrix}3\\-3\\3\end{pmatrix}$ $\overrightarrow{VT}=\overrightarrow{OV}-\overrightarrow{OT}=\begin{pmatrix}2\\7\\8\end{pmatrix}-\begin{pmatrix}5\\4\\5\end{pmatrix}=\begin{pmatrix}-3\\3\\3\end{pmatrix}$	Most students use method 2 to solve. Common mistake: Coordinates is **not** $\begin{pmatrix}6\\3\\4\end{pmatrix}$ but (6, 3, 4) Common mistake: Coordinates is **not** $\begin{pmatrix}5\\4\\5\end{pmatrix}$ but (5, 4, 5).

	$\angle TVQ = \cos^{-1}\left[\dfrac{\begin{pmatrix}3\\-3\\3\end{pmatrix}\cdot\begin{pmatrix}-3\\3\\3\end{pmatrix}}{\sqrt{3^2+(-3)^2+3^2}\sqrt{(-3)^2+3^2+3^2}}\right]$ $= \cos^{-1}\left(\dfrac{-9-9+9}{27}\right) = \cos^{-1}\left(\dfrac{-9}{27}\right) = 109.471° = 109.5°$ (1 decimal place)	Careless mistake: $\sqrt{27}\sqrt{27} = 2\sqrt{27}$ instead of $\sqrt{27}\sqrt{27} = 27$

	2018 ACJC H2 Prelim Paper 2 Questions
	Section A: Pure Mathematics [40 marks]
1	The curve C has equation $\frac{(x-1)^2}{2^2}+y^2=1$. **(i)** Sketch C, giving the exact coordinates of any points of intersection with the axes. [2] **(ii)** Use the substitution $x=\sin\theta$, where $0\le\theta\le\frac{\pi}{2}$, to find the exact value of $\int_0^{\frac{\sqrt{3}}{2}}\sqrt{1-x^2}\ \mathrm{d}x$. [4] **(iii)** The region bounded by C for $x<0$ and the line $y=\frac{\sqrt{3}}{2}(x+1)$ is rotated through 2π radians about the y-axis. Using the result in **(ii)**, find the exact volume of revolution formed. [4]
2	Fig. 1 Fig. 2 Fig. 1 shows an open container with vertical sides and a height of 15 cm. Fig. 2 shows the base $ABCD$ of the container which is made up of • a rectangle, where $AB = DC = L$ cm, • and two segments of a circle of radius r, where each of the arc AD and arc BC is $\frac{1}{3}$ of the circumference of a circle with radius r. Both r and L are variables. Assume the sides and base of the container are of negligible thickness. Show that the area of the base $ABCD$ is $\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)r^2+\sqrt{3}\,Lr$. [2] It is given that the volume of the container is 1000 cm^3, and the external surface area of the four vertical sides and the base is to be the least possible value. Use

differentiation to find the value of r that would give the least possible external surface area, and justify why it is the least possible. [6]

3



A curve C has parametric equations

$$x = k\cos 3t,\quad y = k\sin t,\quad \text{where } 0 \le t \le \frac{\pi}{6} \text{ and } k \text{ is a positive constant.}$$

C meets the x-axis at Q and y-axis at P. The tangent to C at the point P meets the x-axis at R.

The region bounded by the curve C, the line PR and the x-axis is denoted by S (see diagram).

(i) Find the exact value of $\int_0^{\frac{\pi}{6}} \sin t \sin 3t \; \mathrm{d}t$. [3]

(ii) Find the equation of the line PR, in terms of k, simplifying your answer. [3]

(iii) Hence find the exact area of S, giving your answer in terms of k. [4]

4 **(a)** The complex numbers z and w is such that $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, where x, y, u and v are real numbers

(i) Express $\dfrac{w}{z}$ in cartesian form $a + \mathrm{i}b$. [2]

(ii) If $\mathrm{Re}\left(\dfrac{w}{z}\right) = \dfrac{\mathrm{Re}(w)}{\mathrm{Re}(z)}$ where $\mathrm{Re}(z) \neq 0$. Show that either z is real or $\dfrac{w}{z}$ is real. [3]

(b) Given $\left(\mathrm{i}e^{\frac{\pi}{3}}z\right)^* = \dfrac{(1+\mathrm{i})^6}{\left(-1+\mathrm{i}\sqrt{3}\right)^8}$ where z^* is the conjugate of a complex number z.

Without using a calculator, find $|z|$ and $\arg z$ exactly, where $-\pi < \arg z \le \pi$. [5]

	(c) The roots of the equation $x^4 - 2x^3 + ax^2 - x + b = 0$, where a and b are non-zero real numbers, are z_1, z_2, z_3 and z_4. It is given that $z_1^{\,2} + z_2^{\,2} + z_3^{\,2} + z_4^{\,2} < 0$. Explain why at most two of z_1, z_2, z_3 and z_4 are real. [2]

	Section B : Probability and Statistics [60 marks]
5	The management of a supermarket wishes to analyse the effectiveness of an advertising campaign. Before the campaign, the daily takings $\$X$ is normally distributed with mean \$72 300 and standard deviation \$4410. Immediately after the campaign, a sample of 30 shopping days is taken and the mean daily takings was found to be $\$\bar{x}$. A test is carried out, at the 5% significance level, to determine whether the campaign is effective, assuming that there is no change in the standard deviation of the daily takings after the campaign. **(i)** State appropriate hypotheses for the test. [1] **(ii)** By stating a necessary assumption, find the set of values of $\bar{x}$, to the nearest dollar, for which the result of the test would be to reject the null hypothesis at 5% level of significance. [4] **(iii)** The 30 readings taken on X immediately after the campaign is summarised as follows. $$\sum (x - 70\ 000) = 129\ 000$$ By finding $\bar{x}$, state the conclusion of the test, at the 5% significance level, whether the campaign is effective. [2]
6	On average, 40% of Christmas tree light bulbs manufactured by a company are red and the rest are blue. The lights are sold in boxes of 20. Assume that the number of red light bulbs in a box has a binomial distribution. **(i)** Find the probability that a box of 20 light bulbs contains fewer red light bulbs than blue light bulbs. [1] **(ii)** In n randomly chosen boxes, the probability that there will be at least 2 boxes with fewer red light bulbs than blue is at most 0.999. Find the greatest value of n. [3] **(iii)** A random sample of 50 boxes is chosen. Using an approximate distribution, find the probability that the mean number of red light bulbs in 1 box will not exceed 7.5. [3] **(iv)** A customer selects boxes of light bulbs at random from a large consignment until she finds a box with fewer red lights than blue. Give a reason why a binomial distribution is not an appropriate model for the number of boxes selected. [1]

7	Ozone, is a colourless gas that is always found in the air that we breathe. It is also the main ingredient of smog, which presents a serious air quality problem in many cities. A research on a city's air quality found that the wind speed, s, will result in a change in the ozone level, y, given below, where s and y are measured in suitable units.

s	7.0	8.0	9.2	11.1	12.7	14.4	19.6	27.7	35.0	39.
y	49	39	36	34	31	27	24	22	41	20

(i) Draw a scatter diagram for these values. [1]

(ii) Circle the point on the scatter diagram that does not seem to follow the trend and label it as P. Suggest a possible reason for it. [2]

For the remaining parts of this question, you should **omit** point P.

(iii) Suppose that the relationship between s and y are modelled by an equation of the form $y = a - b\ln s$, where a and b are positive constants.

(a) State the product moment correlation coefficient for this model. [1]

(b) Explain, in context, why $y = a - b\ln s$ may not be a good model. [1]

(iv) Use the model $y = c + \frac{d}{s}$, to predict the value of y when $s = 50$. Comment on the reliability of your prediction. [2]

(v) Find the intersection point of the least squares regression line in part **(iv)** with the least squares regression line of $\frac{1}{s}$ on y. [1]

(vi) Explain why the regression line of $\frac{1}{s}$ on y should not be used. [1]

8	An electronic device contains 5 components, two of which are faulty. To isolate the faults, the components are tested one by one in a random order until both the faulty components are identified. The random variable X denotes the number of tests required to locate both the faulty components. Show that $\mathrm{P}(X = 3) = \frac{1}{5}$. [2]

(i) Find the probability distribution of X. [2]

(ii) Find $\mathrm{P}(X_1 = 3 \mid X_1 + X_2 \le 5)$ if X_1 and X_2 are two independent observations of X. [3]

	The cost C (in dollars) depends in part on the number of tests required and is given by the formula $C = 10 + 3X$. **(iii)** Find Var(C). [3]
9	A shop sells two brands of car battery, A and B. The battery life, in months, of each brand of car battery have independent normal distributions. The means and standard deviations of these distributions, are shown in the following table.

	Mean battery life	Standard deviation
Brand A	30	8
Brand B	25	σ

(i) Explain why the battery life of a randomly chosen Brand B battery cannot be normally distributed if $\sigma = 22$. [1]

(ii) The probability that the battery life of a randomly chosen Brand B battery is within 5 months of the mean battery life of Brand B batteries is 0.8. Find the variance of the distribution of Brand B battery life. [3]

Use $\sigma = 4$ for the rest of the question.

(iii) Sketch the distributions of the battery life of the Brand A battery and the Brand B battery on a single diagram. [1]

(iv) Find the probability that the battery life of a randomly chosen Brand A car battery is exactly 26 months. [1]

(v) Find the probability that the difference between the mean battery life of 3 randomly chosen Brand B batteries and 75% of the battery life of a randomly chosen Brand A battery is less than 3 months. State the parameters of any distribution you use. [4]

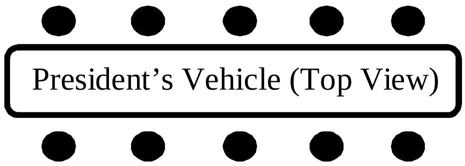
(vi) The manufacturer of Brand B battery replaces for free all batteries that fail within the warranty period of k months. If they are willing to replace for free less than 1% of all batteries sold, find the longest warranty period (to the nearest integer), that the manufacturer can offer. [2]

10	The President's bodyguard unit consists of 40 men. There are 10 experts in firearms, 10 experts in unarmed combat, 10 experts in tactical driving and 10 experts in first aid. Each man is an expert in only one area. In preparation for a historical summit, a team of 10 bodyguards is to be chosen from the 40 men.

(i) Find the number of ways in which the team can be chosen, if there are at least 2 experts chosen from each area. [3]

The selected team comprises of 3 experts in firearms, 3 experts in unarmed combat, 2 experts in tactical driving and 2 experts in first aid.

(ii) These 10 bodyguards are to walk alongside the president's vehicle in a tactical formation as shown below.



Find the number of ways the 10 bodyguards can stand, if the 2 experts in first aid must not be standing next to each other on the same side of the vehicle. [3]

(iii) The ten bodyguards are to stand at random in a circular formation surrounding the president as he walks. Find the probability that the 3 experts in firearms are separated from each other. [3]

Each of the bodyguard is wearing a surveillance earpiece whose electrical circuit is controlled by two relay switches, A and B. The probability that switch A fails is 0.1 and the probability that switch B fails is 0.23. The probability that both switches do not fail is 0.7.

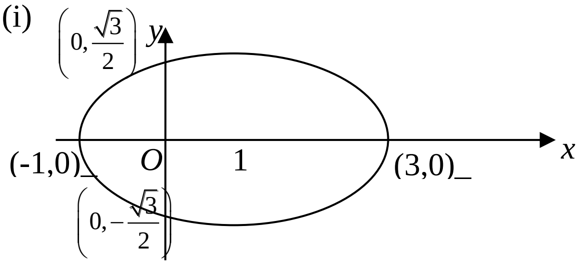
(iv) Find the probability that only switch B fails. [2]

A third relay switch C is added to the electrical circuit. The event that switch C fails is independent of the event that switch A fails. The probability that switch C fails given that switch A has failed is 0.15.

(v) Find the probability that switch C fails but not A. [1]

(vi) Hence find the maximum probability that, switches B and C fail but not A. [2]

Anglo-Chinese Junior College
2018 H2 Mathematics Prelim Paper 2 Markers Report

Qn	Solution	Remarks
1	(i)  (ii) $\int_0^{\frac{\sqrt{3}}{2}} \boxed{\sqrt{1-x^2}}\,dx$ $x=\sin\theta$ $\frac{dx}{d\theta}=\cos\theta$ $=\int_0^{\frac{\pi}{3}} \boxed{\sqrt{1-\sin\theta^2}}\cos\theta\,d\theta$ $=\int_0^{\frac{\pi}{3}} \cos^2\theta\,d\theta$ $=\int_0^{\frac{\pi}{3}} \frac{\cos 2\theta+1}{2}\,d\theta$ $=\frac{\sin\frac{2\pi}{3}}{4}+\frac{\pi}{6}$ $=\frac{\sqrt{3}}{8}+\frac{\pi}{6}$ Using $x=\sin\theta$, when $x=0$, $\sin\theta=0\Rightarrow\theta=0$ when $x=\frac{\sqrt{3}}{2}$, $\sin\theta=\frac{\sqrt{3}}{2}\Rightarrow\theta=\frac{\pi}{3}$ (since $0\le\theta\le\frac{\pi}{3}$) (iii) Required volume $=\pi\int_0^{\frac{\sqrt{3}}{2}} x^2\,dy-\left(\frac{1}{3}\right)(\pi)(1)^2\left(\frac{\sqrt{3}}{2}\right)$ $=\pi\int_0^{\frac{\sqrt{3}}{2}} x^2\,dy-\frac{\pi\sqrt{3}}{6}$ $=\pi\int_0^{\frac{\sqrt{3}}{2}} \left(1-\sqrt{4\left(1-y^2\right)}\right)^2 dy-\frac{\pi\sqrt{3}}{6}$ $=\pi\int_0^{\frac{\sqrt{3}}{2}} 1-2\sqrt{4\left(1-y^2\right)}+4\left(1-y^2\right)dy-\frac{\pi\sqrt{3}}{6}$ $=\pi\int_0^{\frac{\sqrt{3}}{2}} 5-4y^2-4\sqrt{\left(1-y^2\right)}dy-\frac{\pi\sqrt{3}}{6}$ $=\pi\left[5y-\frac{4y^3}{3}\right]_0^{\frac{\sqrt{3}}{2}}-4\pi\left[\frac{\sqrt{3}}{8}+\frac{\pi}{6}\right]-\frac{\pi\sqrt{3}}{6}$ $=\pi\left[5\left(\frac{\sqrt{3}}{2}\right)-\frac{4\left(\frac{\sqrt{3}}{2}\right)^3}{3}\right]-\left[\frac{\sqrt{3}}{2}\pi+\frac{2\pi^2}{3}\right]-\frac{\pi\sqrt{3}}{6}$ $=\frac{5\sqrt{3}}{2}\pi-\frac{\sqrt{3}}{2}\pi-\frac{\sqrt{3}}{2}\pi-\frac{2\pi^2}{3}-\frac{\pi\sqrt{3}}{6}$ $=\frac{14\sqrt{3}}{6}\pi-\pi\sqrt{3}-\frac{2\pi^2}{3}$ $=\frac{4\pi\sqrt{3}}{3}-\frac{2\pi^2}{3}$	(i) Many fail to READ the question carefully and hence did not write the axial intercepts in **coordinates** form. (ii) Students must remember to change all the 3 fields highlighted in the solution concurrently in one step. Some did partial changes in different steps. (iii) Many have identified the region wrongly. Many used $\pi\int y^2\,dx$ although the question ask for volume of revolution about the y-axis. Students should learn to use the formula for volume of cone instead of finding the cone volume by integration. Many students chose the incorrect expression for x in terms of y not realising that $x<0$.

2	Area between arc AD and line AD $= \frac{1}{3}\pi r^2 - \frac{1}{2}r^2 \sin 120° = \frac{1}{3}\pi r^2 - \frac{1}{2}r^2\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{3}\pi r^2 - \frac{\sqrt{3}}{4}r^2$ Length of line $AD = 2\times r\sin 60° = \sqrt{3}\,r$ Area of base $ABCD = 2\left(\frac{1}{3}\pi r^2 - \frac{\sqrt{3}}{4}r^2\right) + \sqrt{3}\,rL$ $= \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)r^2 + \sqrt{3}\,Lr$ (shown) Volume $= 15\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)r^2 + 15\sqrt{3}\,Lr = 1000$ $15\sqrt{3}\,Lr = 1000 - 15\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)r^2$ $\sqrt{3}\,Lr = \frac{200}{3} - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)r^2$ *or* $L = \frac{200}{3\sqrt{3}\,r} - \left(\frac{2\pi}{3\sqrt{3}} - \frac{1}{2}\right)r$ Perimeter of base $ABCD$ = 2 × arc AD + 2 AB $= 2\left(\frac{1}{3}\times 2\pi r\right) + 2L = \frac{4\pi}{3}r + 2L$ Area of 4 vertical sides $= \left(\frac{4\pi}{3}r + 2L\right)\times 15 = 20\pi r + 30L$ Required area $= A = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)r^2 + \sqrt{3}\,Lr + \boxed{(20\pi r + 30L)}$ $= \left(\frac{2\pi}{3}r^2 - \frac{\sqrt{3}}{2}r^2\right) + \left(\frac{200}{3} - \frac{2\pi}{3}r^2 + \frac{\sqrt{3}}{2}r^2\right)$ $+20\pi r + \frac{30}{\sqrt{3}\,r}\left(\frac{200}{3} - \frac{2\pi}{3}r^2 + \frac{\sqrt{3}}{2}r^2\right)$ $= \frac{200}{3} + 20\pi r + \frac{2000}{\sqrt{3}\,r} - \frac{20\pi}{\sqrt{3}}r + 15r$ $\frac{dA}{dr} = 20\pi - \frac{2000}{\sqrt{3}\,r^2} - \frac{20\pi}{\sqrt{3}} + 15 = 0$ $\frac{2000}{\sqrt{3}\,r^2} = 20\pi - \frac{20\pi}{\sqrt{3}} + 15$ $r^2 = \frac{2000}{\sqrt{3}\left(20\pi - \frac{20\pi}{\sqrt{3}} + 15\right)}$ r = 5.271 3091 = 5.27 (3 sf)	Many students cannot recall formula for area of a SECTOR. Some didn't read question carefully and did not realise that the height is 15. Some left out area of vertical sides in their calculation for the total area. Some did not sub L in terms of r BEFORE differentiation.

	Note: L = 3.56 (3 sf) and A = area = 504.77 (5 sf) Using GC, $\frac{\mathrm{d}^2A}{\mathrm{d}r^2}=15.8>0$. This area is the least possible. Or, $\frac{\mathrm{d}^2A}{\mathrm{d}r^2}=\frac{4000}{\sqrt{3}\,r^3}=15.8>0$. This area is the least possible. Or, r 5.2 5.27 5.3 $\frac{\mathrm{d}A}{\mathrm{d}r}$ Thus this area is the least possible.	Some didn't justify for why the area is the least.
3	(i) $\int_0^{\frac{\pi}{6}}(\sin t)(\sin 3t)\,\mathrm{d}t$ $=\boxed{-\frac{1}{2}\int_0^{\frac{\pi}{6}}\cos 4t-\cos 2t\ \mathrm{d}t}=-\frac{1}{2}\left[\frac{\sin 4t}{4}-\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{6}}$ $=-\frac{1}{2}\left[\frac{\sin\left(\frac{2\pi}{3}\right)}{4}-\frac{\sin\left(\frac{\pi}{3}\right)}{2}-0\right]=-\frac{1}{2}\left[\frac{\left(\frac{\sqrt{3}}{2}\right)}{4}-\frac{\left(\frac{\sqrt{3}}{2}\right)}{2}\right]=\frac{\sqrt{3}}{16}$ (ii) $x=k\cos 3t$ $y=k\sin t$ At P, $x=0\Rightarrow t=\frac{\pi}{6}\Rightarrow y=\frac{k}{2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{k\cos t}{-3k\sin 3t}=\frac{\cos t}{-3\sin 3t}$ $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{t=\frac{\pi}{6}}=\frac{\sqrt{3}}{-6}$ $\Rightarrow$ equation of tangent at P: $y-\frac{k}{2}=\frac{\sqrt{3}}{-6}(x-0)$ $y=-\frac{\sqrt{3}}{6}x+\frac{k}{2}$	Many used integration by parts twice which led to tedious computation. Students should be able to arrive at this step easily using the factor formulae in MF 26. Some took a few steps to arrive at this. (ii) Many students did not evaluate $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{t=\frac{\pi}{6}}=\frac{\sqrt{3}}{-6}$. The gradient of tangent at P shouldn't be in terms of t since t is found for point P.

	(iii) Area of S $= \left(\frac{1}{2}\right)\left(k\sqrt{3}\right)\left(\frac{k}{2}\right) - \int_0^{k} \boxed{y}\, dx$ $= \frac{k^2\sqrt{3}}{4} - \int_{\frac{\pi}{6}}^{0} \boxed{k\sin t}\left(-3k\sin 3t\right) dt$ $= \; = \frac{k^2\sqrt{3}}{4} - 3k^2\int_0^{\frac{\pi}{6}} \left(\sin t\right)\left(\sin 3t\right) dt$ $= \frac{k^2\sqrt{3}}{4} - 3k^2\left(\frac{\sqrt{3}}{16}\right)$ $= \frac{k^2\sqrt{3}}{16}$	(iii) Students should use the formula for area of triangle instead of through integration. Many assumed that (i) is the area under curve PQ which is not the case. They should begin with this and replace all fields by t. Many **got the wrong idea** that the the lower limits needs to be smaller than the upper limit and hence did not indicate this. Point R should be found by substituting y=0 into the equation of tangent. However, some substituted $y = k\sin t$ =0 which is actually finding the value of t at point Q.
4	(a) (i) $\frac{w}{z} = \frac{u+iv}{x+iy} = \frac{u+iv}{x+iy} \times \frac{x-iy}{x-iy}$ $= \frac{(ux+vy)+i(vx-uy)}{x^2+y^2}$ $= \frac{(ux+vy)}{x^2+y^2} + i\frac{(vx-uy)}{x^2+y^2}$ (a) (ii) $\text{Re}\left(\frac{w}{z}\right) = \frac{\text{Re}(w)}{\text{Re}(z)}$ Hence $\frac{ux+vy}{x^2+y^2} = \frac{u}{x}$ $ux^2 + vxy = ux^2 + uy^2$ $\Rightarrow y(vx-uy) = 0$ $\Rightarrow y = 0$ or $vx - uy = 0$	Most students can rationalize the fraction. Some students didn't FACTORISE but divided away the variable and lose one set of solution.

	$\Rightarrow$ Im(z) = 0 or Im$\left(\dfrac{w}{z}\right)$ = 0 $\Rightarrow$ z is real or $\dfrac{w}{z}$ is real (b) **Method 1** $\lvert 1+i\rvert = \sqrt{2}$, arg $(1+i) = \dfrac{\pi}{4}$ $\lvert -1+i\sqrt{3}\rvert = 2$, $\arg(-1+i\sqrt{3}) = \pi - \tan^{-1}\sqrt{3} = \dfrac{2\pi}{3}$ $\left\lvert \dfrac{(1+i)^6}{(-1+i\sqrt{3})^8}\right\rvert = \dfrac{\lvert 1+i\rvert^6}{\lvert -1+i\sqrt{3}\rvert^8} = \dfrac{(\sqrt{2})^6}{2^8} = \dfrac{1}{2^5}$ $\text{Arg}\left(\dfrac{(1+i)^6}{(-1+i\sqrt{3})^8}\right) = 6\arg(1+i) - 8\arg(-1+i\sqrt{3})$ $= 6\left(\dfrac{\pi}{4}\right) - 8\left(\dfrac{2\pi}{3}\right)$ $= -\dfrac{23\pi}{6} = \dfrac{\pi}{6}$ (adjusted) $\left(ie^{\frac{\pi}{3}}z\right)^* = -ie^{\frac{\pi}{3}}z^*$ $\left\lvert -ie^{\frac{\pi}{3}}z^*\right\rvert = e^{\frac{\pi}{3}}\lvert z\rvert$ Since $\left\lvert -ie^{\frac{\pi}{3}}z^*\right\rvert = \left\lvert \dfrac{(1+i)^6}{(-1+i\sqrt{3})^8}\right\rvert$, $e^{\frac{\pi}{3}}\lvert z\rvert = \dfrac{1}{2^5}$ $\therefore \lvert z\rvert = \dfrac{1}{e^{\frac{\pi}{3}}2^5}$ $\arg(-ie^{\frac{\pi}{3}}z^*) = -\dfrac{\pi}{2} + \arg(z^*) = -\dfrac{\pi}{2} - \arg(z)$ Since $\arg(-ie^{\frac{\pi}{3}}z^*) = \text{Arg}\left(\dfrac{(1+i)^6}{(-1+i\sqrt{3})^8}\right)$, $\therefore -\dfrac{\pi}{2} - \arg(z) = \dfrac{\pi}{6}$ Hence arg $z = -\dfrac{2\pi}{3}$	Some students didn't change $-\dfrac{23\pi}{6}$ to $\dfrac{\pi}{6}$ so that it is within principal range. Please note that $e^{\frac{\pi}{3}} \neq e^{\frac{\pi}{3}i}$ and $\left\lvert ie^{\frac{\pi}{3}}\right\rvert = e^{\frac{\pi}{3}}$

(b)

Method 2

$|1+i| = \sqrt{2}$, $\quad$ arg $(1+i) = \dfrac{\pi}{4}$

$\left|-1+i\sqrt{3}\right| = \sqrt{1+3} = 2$, $\quad \arg(-1+i\sqrt{3}) = \pi - \tan^{-1}\sqrt{3} = \dfrac{2\pi}{3}$

$1+i = \sqrt{2}e^{i\frac{\pi}{4}}$

$-1+i\sqrt{3} = 2e^{i\frac{2\pi}{3}}$

$$\frac{(1+i)^6}{\left(-1+i\sqrt{3}\right)^8} = \frac{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^6}{\left(2e^{i\frac{2\pi}{3}}\right)^8} = \frac{2^3 e^{i\frac{3\pi}{2}}}{2^8 e^{i\frac{16\pi}{3}}} = \frac{1}{2^5}e^{-i\frac{23\pi}{6}} = \frac{1}{2^5}e^{i\frac{\pi}{6}}$$

$$\left(ie^{\frac{\pi}{3}}z\right)^* = -ie^{\frac{\pi}{3}}z^*$$

Since $\left(\mathrm{i}e^{\frac{\pi}{3}}z\right)^* = \dfrac{(1+\mathrm{i})^6}{\left(-1+\mathrm{i}\sqrt{3}\right)^8}$,

$$-ie^{\frac{\pi}{3}}z^* = \frac{1}{2^5}e^{i\frac{\pi}{6}}$$

$$\Rightarrow z^* = \frac{1}{-ie^{\frac{\pi}{3}}2^5}e^{i\frac{\pi}{6}} = \frac{i}{e^{\frac{\pi}{3}}2^5}e^{i\frac{\pi}{6}} = \frac{e^{i\frac{\pi}{2}}}{e^{\frac{\pi}{3}}2^5}e^{i\frac{\pi}{6}} = \frac{e^{i\frac{\pi}{2}}e^{i\frac{\pi}{6}}}{e^{\frac{\pi}{3}}2^5} = \frac{e^{i\frac{2\pi}{3}}}{e^{\frac{\pi}{3}}2^5}$$

Hence $z = \dfrac{e^{-i\frac{2\pi}{3}}}{e^{\frac{\pi}{3}}2^5}$

$|z| = \dfrac{1}{e^{\frac{\pi}{3}}2^5}$ and arg $z = -\dfrac{2\pi}{3}$

(c)

Since $z_1^2 + z_2^2 + z_3^2 + z_4^2 < 0$, hence equation has complex roots. Also as coefficients of x of equation are all real, hence the complex roots occur in conjugate pairs. Thus at most two of z_1, z_2, z_3 and z_4 are real.

Many were unable to explain WHY there were complex roots in the first place before citing the 'real coefficients...... and therefore conjugate pair of complex roots'

5

(i) To test $H_0: \mu = 72300$

against $H_1: \mu > 72300$ at 5 % level of significance

(ii) Assume that the sample of 30 daily takings after the advertising campaign is a random sample

Under H_o, $\overline{X} \sim N(72300, \dfrac{4410^2}{30})$

<u>Method 1</u>

(i)A number of students do not seem to know what is the meaning of hypothesis for the test

(ii)It is given in the question that the distribution is normal but answers like "assume normally distributed" is commonly seen. Use

	Using GC P($\overline{X}$ > 73624) = 0.05 Reject H_0 at 5% level of significance if $\bar{x} > 73624$ **Method 2** $Z = \dfrac{\overline{X} - 72300}{\frac{4410}{\sqrt{30}}} \sim N(0,1)$ And P(Z > 1.64485) = 0.05 Reject H_0 at 5% level of significance if $\dfrac{\bar{x} - 72300}{\frac{4410}{\sqrt{30}}} > 1.66485$ i.e $\bar{x} > 73624$ (iii) $\bar{x} = \dfrac{\sum(x-70000)}{30} + 70000 = \dfrac{129000}{30} + 70000 = 74300$ Using previous result in (ii), since 74300 > 73624, we reject H_0 . There is sufficient evidence at 5% level of significance that the advertising campaign is effective	of central limit theorem because n = 30 is also large frequently seen. This shows that the candidate is unsure of when to apply central limit theorem. **Wrong** H_1 written, eg: H_1 : $\mu < 72300$ or $\mu \neq 72300$ or Other frequently seen strange hypothesis like H_0: $\mu = \bar{x}$, H_1: $\mu > \bar{x}$ H_0: $\bar{x} = 72300$ H_1: $\bar{x} > 72300$ Common **error**: H_1: $\mu_1 > 72300$ H_0 = 72300, H_1 > 72300 Test statistic Z= $\dfrac{72300 - \overline{X}}{\frac{4410}{\sqrt{30}}}$ frequently seen. Using $s^2 = \dfrac{(4410)(30)}{29}$ seen or $\overline{X} \sim N(\bar{x}, \dfrac{4410^2}{30})$ Some students did not express answer to the nearest integer for the range of $\bar{x}$ (iii) Quite a number used *p*-value instead of range found in (ii). Wrong use of GC like Use $\mu = 74300$ to find the *p*-value = 0.9930 Wrong reason for the conclusion stated such as: "Campaign is effective since 74300 > 72300 frequently

<table>
<tr><td></td><td></td><td>seen.”
Some also did not state in the conclusion the 5% level of sig or did not write “sufficient evidence”.</td></tr>
<tr><td>6</td><td>(i) Let random variable X be the number of red light bulbs
$X \sim B(20, 0.4)$
$P(X<20-X) = P(X<10) = P(X \le 9) = 0.755337 = 0.755$ (to 3 s.f)

(ii) Let random variable A be the number of boxes with fewer red light bulbs than blue
$A \sim B(n, 0.755337)$
$P(A \ge 2) \le 0.999$
$1 - P(A \le 1) \le 0.999$
$P(A \le 1) \ge 0.001$

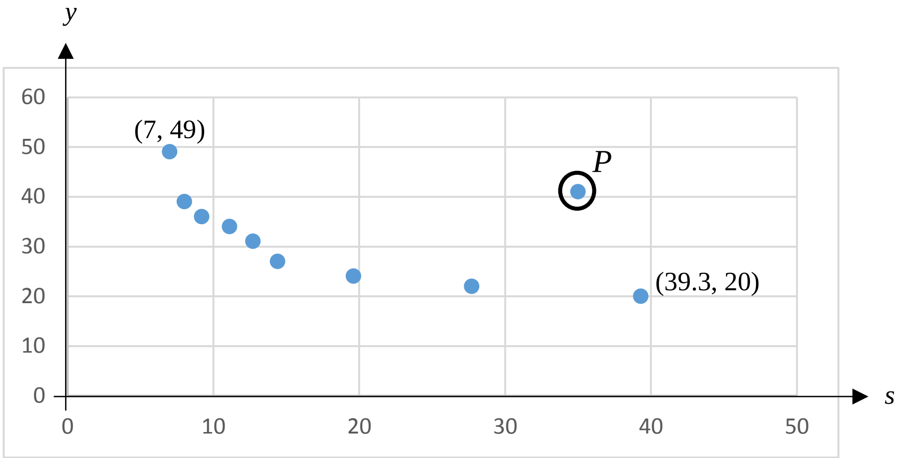
From GC,
<table><tr><td>n</td><td>$P(A \le 1)$</td></tr><tr><td>6</td><td>0.00419</td></tr><tr><td>7</td><td>0.00119</td></tr><tr><td>8</td><td>0.00033</td></tr></table>
Least possible value of $n = 7$

(iii)</td><td>(i) Many students could not figure out that the answer is $P(X \le 9)$ just by mere counting. Those who attempted to figure out by $P(X<Y)$ ended up using normal distribution. Another group of students wrote $P(X<10)$ which is correct theoretically but they computed $P(X \le 10)$ instead with the GC.

(ii) Many could get the first 2 steps correct:
$A \sim B(n, 0.755337)$
$P(A \ge 2) \le 0.999$.
However, those who continued further ended up with
$P(A \le 1) \le 0.001$
which is wrong.
Other common errors were:
$P(A \ge 2) \le 0.99$
$P(A \le 1) \le 0.01$
$P(A \ge 2) = 0.999$
$P(A \le 1) = 0.001$
or a mixture of the above. Even if the first 2 steps were correct, the wrong value of n is chosen because the computation of the values of $P(A \le 1)$ were wrong or the decimal place read wrongly.

(iii) Many did not</td></tr>
</table>

		$E(X) = 20(0.4) = 8$ $\text{Var}(X) = 20(0.4)(0.6) = 4.8$ $n = 50$ is large. From central limit theorem $\overline{X} \sim N(8, \frac{4.8}{50})$ approximately $P(\overline{X} \le 7.5) \approx 0.05329158 \approx 0.0533$ (3 s.f) (iv) In the context of the question, the number of boxes that the customer can select is not fixed (sample size is not fixed). Hence a binomial distribution is not an appropriate model as it requires a fixed number of trials.	understand Central Limit Theorem nor understand that for any binomial random variable, E(X)=np and Var(X)=npq. They thought that when they used these results, X changes from binomial to normal distribution. Some stated Central Limit Theorem gives you X~N(np, npq). In finding Var(X)=npq, many used n=50 instead of n=20. Many left out the word 'approx' and most students wrote $P(\overline{X} \le 7.5) = 0.05329158$ instead of $P(\overline{X} \le 7.5) \approx 0.05329158$. (iv) Most students gave the reason that the probability that there are fewer red light bulbs is not a constant or each trial is not independent of each other. On one hand this is true but the condition that these light bulbs are from a large consignment is what allows us to assume that question (i) to (iii) are binomially distributed.
7	(i) & (ii)		(i)

	 Point P could be the result of a recording error or an instrument error. (iii) (a) $r = -0.9258429505 = -0.926$ (3s.f) (b) $y = a - b\ln s$ may not be a good model as it seems to suggest that as wind speed (x) increases, ozone (y) will decrease and eventually be negative in value. This is inappropriate as ozone level will never be negative in value. There is always ozone present in the air. (iv) Least square regression line of y on $\frac{1}{s}$:	Most students are able to draw the scatter diagram but lost marks due to some of the following common mistakes: - Not labelling the axis. - Did not indicate in diagram the data range for y and s. - Poor relative positions of data points, particularly point P being marked lower than the point (8,39). (ii) Quite a number of student are able to come out with a logical reason for the presence of the outlier point. (iii)(a) Most students know how to calculate the product moment correlation coefficient. Many lost the mark as they did not leave their answer in 3.s.f. (iii)(b) Many students are unable to give a reason why $y = a - b\ln s$ is not a good model. Quite a number of students got the **wrong** idea that $y = a - b\ln s$ is a "linear model" which they then claim that since it is a "linear model", this makes it not suitable as the scatter diagram in (i) shows a curvilinear relationship. (iv)

	$y=12.93341953+\dfrac{227.6608861}{s}$ When $s=50$, $y=17.486637=17.5$ (3s.f) The prediction is not reliable as $s=50$ does not lies within the data range of the wind speed (s). Hence, we are extrapolating. (v) Using G.C, mean of $\dfrac{1}{s}$ = 0.0808 (3s.f) mean of y = 31.3 (3s.f). The least squares regression line of y on $\dfrac{1}{s}$ intersects the least squares regression line of $\dfrac{1}{s}$ on y at (0.0808, 31.3). (vi) The regression line of $\dfrac{1}{s}$ on y should not be used as y is not the independent variable. It is the dependent variable.	This part is well done as most students know how to find the regression line and predict the value of *y*. However, many did not get the mark as they fail to read the question carefully and DID NOT OMIT point *P* and hence got the wrong equation for the regression line and hence the wrong predicted value. However, most students are able to give the correct reason for why their prediction is not reliable. (v) Few students manage to give the correct intersection points. Many did the tedious method of finding the equation of the least square regression line of of $\dfrac{1}{s}$ on *y* and the equation of *y* on $\dfrac{1}{s}$. Many also gave the **wrong** intersection point as $(\bar{y},\bar{s})$ or $(\bar{y},\frac{1}{\bar{s}})$. Many also did not get the answer as they **did not omit** point *P* in their calculation. (vi) The last part was very well done.
8		(i)

F is the event that the component is faulty
N is the event that the component is not faulty
Outcomes: NFF or FNF

P(X = 3) = $\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}+\frac{2}{5}\times\frac{3}{4}\times\frac{1}{3}=\frac{1}{10}+\frac{1}{10}=\frac{1}{5}$

(i)

x	2	3	4	5
outcomes	FF		NNFF, NFNF, FNNF	NNNFF NNFNF NFNNF FNNNF
P($X = x$)	$\frac{2}{5}\times\frac{1}{4}=\frac{1}{10}$	$\frac{1}{5}$	$3\times\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}\times\frac{1}{2}=\frac{3}{10}$	$4\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\times\frac{2}{2}\times\frac{1}{1}=\frac{2}{5}$ or $1-\frac{1}{10}-\frac{1}{5}-\frac{3}{10}=\frac{2}{5}$

(ii) $P(X_1 = 3 \mid X_1 + X_2 \le 5) = \dfrac{P([X_1=3]\cap[X_1+X_2\le 5])}{P(X_1+X_2\le 5)} =$

$$\frac{P(X_1=3\,,\,X_2=2)}{2P(X_1=3,X_2=2)+P(X_1=2,X_2=2)}=$$

$$\frac{P(X_1=3)P(X_2=2)}{2P(X_1=3)P(X_2=2)+P(X_1=2)P(X_2=2)}$$

$$=\frac{\left(\frac{1}{5}\right)\left(\frac{1}{10}\right)}{\left(\frac{1}{10}\right)^2+2\left(\frac{1}{5}\right)\left(\frac{1}{10}\right)}=\frac{2}{5}=0.4\ .$$

(iii) $E(X) = \sum_{all\,x} xP(X=x) = 2\left(\frac{1}{10}\right)+3\left(\frac{1}{5}\right)+4\left(\frac{3}{10}\right)+5\left(\frac{2}{5}\right)=4$

$E(X^2) = \sum_{all\,x} x^2P(X=x) = 2^2\left(\frac{1}{10}\right)+3^2\left(\frac{1}{5}\right)+4^2\left(\frac{3}{10}\right)+5^2\left(\frac{2}{5}\right)=17$

$\text{Var}(X) = 17-\left(4^2\right)=1$

$\text{Var}(C) = 9\text{Var}(X) = 9(1) = 9$

Most students can show that $P(X=3)=\frac{1}{5}$.

For the probability distribution of X, most can list out correctly the various outcomes of X. However, some gave the impossible outcomes of X=0 and X=1. Some did not realise that X=5 is possible. Quite a number of students have problem find P(X=4) or P(X=5).

(ii)
For the numerator, many students wrote $P(X_1=3)\times P(X_1+X_2 \le 5)$. This is **incorrect** as $(X_1=3)$ and $(X_1+X_2 \le 5)$ are NOT independent events.

(iii)
The last part is very well done. A minority of the students lost marks as they recalled the formulae for Var(X) incorrectly. Note that Var(X)$\neq$ E(X^2).

9 (i) Let random variable X be the battery life of a randomly selected Brand B car battery

(i) Most merely said that the standard

	$X \sim N(25, 22^2)$ $P(X < 0) = 0.128$ A randomly chosen Brand B battery cannot be normally distributed as this will result in 12.8% of Brand B batteries having a negative battery life which is not a negligible value and does not make sense.	deviation is too large and some try to give some consequence of having a large standard deviation except stating that the battery life could be negative as a result. Some wrongly stated that the mean battery life may be negative when the mean battery life has already been given.
	(ii) $P(\lvert X-25\rvert<5)=0.8$ $P(-5<X-25<5)=0.8$ $P(20<X<30)=0.8$ $P\left(\frac{20-25}{\sigma}<Z<\frac{30-25}{\sigma}\right)=0.8$ $P\left(\frac{-5}{\sigma}<Z<\frac{5}{\sigma}\right)=0.8$ $P\left(Z<\frac{-5}{\sigma}\right)=0.1$ Using G.C, $\frac{-5}{\sigma}=-1.281551567$ $\sigma^2=15.221864=15.2$ (3s.f)	(ii) Many did not understand the phrase "within 5 month of the mean battery life". Many interpreted that to be $P\left(\lvert X-\overline{X}\rvert<5\right)=0.8$. In fact, this is an indication that many do not know the difference between sample mean and population mean. Many also do not know how to standardize to Z or at least present the working of standardization properly. They also do not know how to use the symmetrical properties to arrive with the result $P\left(Z<\frac{-5}{\sigma}\right)=0.1$. Some managed to use the new GC feature for invnorm using Tail: CENTER to get $\frac{-5}{\sigma}=-1.281551567$.
	(iii)	(iii) Most drew the 2 curves with the same height while a few one

			curve above another with the same mean. Some others seemed to have drawn the right curve to be lower but not much lower with at the most 2mm lower that is difficult to tell did they mean to draw the curve lower.
	(iv)	Let random variable A be the battery life of a randomly selected Brand A car battery P(A = 26) = 0	(iv) Most equate the probability to the y-coordinate, i.e. the value obtained from normalpdf. Some others compute by using P(25<A<27).
	(v)	A ~ N(30, 8^2) $\bar{X} \sim N(25, \frac{4^2}{3})$ 0.75A ~ N(30(0.75), $8^2(0.75)^2$) i.e 0.75A ~ N(22.5, 36) $\bar{X} - 0.75A \sim N(25 - 22.5, \frac{4^2}{3} + 36)$ $\bar{X} - 0.75A \sim N(2.5, \frac{124}{3})$ P($\left\| \bar{X} - 0.75A \right\| < 3$) = P($-3 < \bar{X} - 0.75A < 3$) =0.33485 = 0.335 (3 s.f)	(v) It is strange that majority of the students attempted to find $P(\|(X_1 + X_2 + X_3) - 0.75A\|) <$ Otherwise, we have $P(\bar{X} - 0.75A < 3)$. Many did not give clear working on how they arrive with the variance.
	(vi)	P(X < k) < 0.01 From GC, k < 15.695 Largest integer k = 15 ${}^{10}C_2\,{}^{10}C_2\,{}^{10}C_2\,{}^{10}C_2\,{}^{32}C_2$	(vi) Common errors: k < 15.695 $\Rightarrow$ k=16 (nearest integer) k = 15.695 $\Rightarrow$ k = 15 P(B < k) = 0.01 $\Rightarrow$ k < 15.6 or a mixture of '=', '<', '>' .

10

(i)

Case 1:

Team comprises of 2+2+3+3 (selected from each of the expert area).

$$ {}^{10}C_2 \times {}^{10}C_2 \times {}^{10}C_3 \times {}^{10}C_3 \times \frac{4!}{2!2!} = 174960000 $$

Case 2:

Team comprises of 2+2+2+4 (selected from each of the expert area).

$$ {}^{10}C_2 \times {}^{10}C_2 \times {}^{10}C_2 \times {}^{10}C_4 \times \frac{4!}{3!} = 76545000 $$

Total ways = $174960000 + 76545000 = 251505000$.

Poorly attempted. Very few students get the full credit.

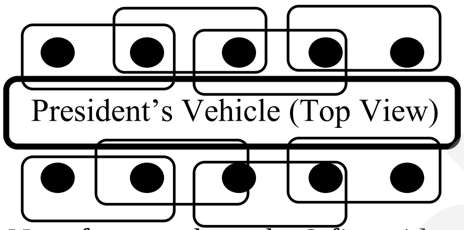
About half of the cohort wrote ${}^{10}C_2\,{}^{10}C_2\,{}^{10}C_2\,{}^{10}C_2\,{}^{32}C_2$, a misconception of using the "remaining" people.

About 25% of the cohort tried to use complement method, but there are too many cases to consider, thus no one gets correct.

(ii)

Method 1 (complement Method):

No. of ways of standing without restrictions = 10!



No. of ways where the 2 first aid experts are together = (8)(2!)(8!)
=645120

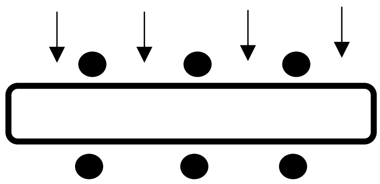
No of required ways = 10! − 645120 = 2983680

Many students tried using the complement method, but most of them didn't get the full credit.

Method 2:

Case 1: The two first aiders are on the same side of car and separated.

No. of ways= $\left({}^{4}C_2 \times 2! \times 8!\right) \times 2 = 967680$

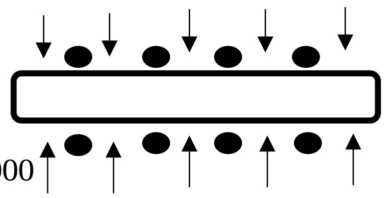


Case 2: The two first aiders are on the opposite sides of car.

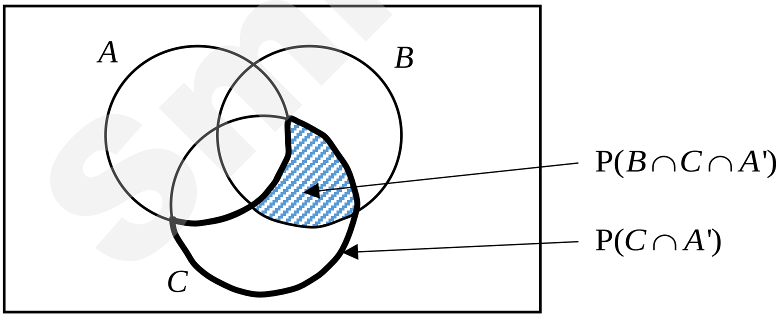
No. of ways= $\left({}^{5}C_1 \times {}^{5}C_1 \times 2! \times 8!\right) = 2016000$

OR

No. of ways= $\left({}^{8}C_4 \times {}^{4}C_4 \times 5! \times 5!\right) \times 2 = 2016000$



Total no. of ways = $967680 + 2016000 = 2983680$.

	(iii) **Method 1:** Required probability $=\dfrac{(7-1)!\times {}^{7}C_{3}\times 3!}{(10-1)!}=\dfrac{5}{12}$ **Method 2:** Required probability $=\dfrac{(10-1)!\ -{}^{3}C_{2}\times(9-1)!\times 2!+(8-1)!\times 3!}{(10-1)!}$	Most students got the method correct but did not realize that the question is asking for **PROBABILITY.** For students who tried the complement method, only a handful of students realized that it should be a "+" in the numerator.
	(iv) Let event A be switch A fails and event B be switch B fails. $\mathrm{P}(A)=0.1$ $\mathrm{P}(B)=0.23$ $\mathrm{P}(A'\cap B')=0.7$ $\mathrm{P}(B\cap A')=1-\mathrm{P}(A'\cap B')-\mathrm{P}(A)$ $=1-0.7-0.1$ $=0.2$	Most students wrote $\mathrm{P}(B\cap A')=\mathrm{P}(B)\times\mathrm{P}(A')$ which is **WRONG.** They simply assume that events A and B are independent.
	(v) Let event C be switch C fails. Since A and C are independent events, $\mathrm{P}(C\mid A)=\mathrm{P}(C)=0.15$. $\mathrm{P}(C\cap A')=\mathrm{P}(C)\times\mathrm{P}(A')$ $=0.15\times[1-0.1]$ $=0.135$	Only half the cohort answered this correctly. Students did not understand the question.
	(vi)  $0\le\mathrm{P}(B\cap C\cap A')\le 0.135$ $\therefore$ Maximum $\mathrm{P}(B\cap C\cap A')=0.135$	Almost whole cohort got this wrong.

CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC2 Preliminary Examination

MATHEMATICS

9758/01

Paper 1

21 Aug 2018

3 hours

Additional Materials: List of Formulae (MF26)
Answer paper

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

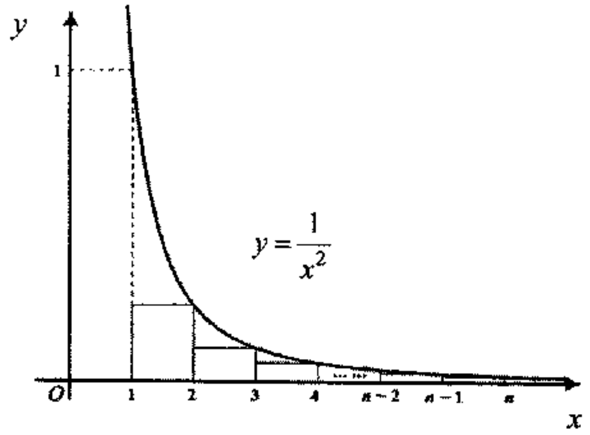
Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 6 printed pages, including the cover page.

1



The diagram above shows the graph of $y=\frac{1}{x^2}$ for $x>0$, together with a set of $(n-1)$ rectangles of unit width, starting at $x=1$.

(i) By considering the areas of these rectangles, explain why

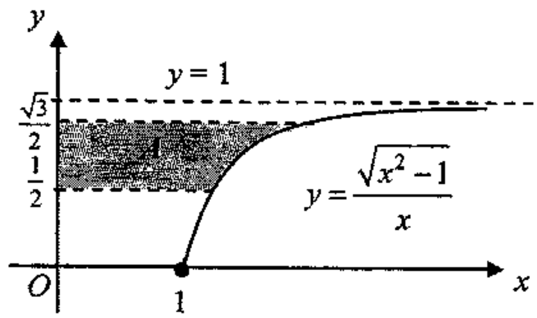
$$\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2} \ < \ \int_1^n \frac{1}{x^2}\,\mathrm{d}x \ ,$$

where n is an integer greater than 1. [1]

(ii) Find $\int_1^n \frac{1}{x^2}\,\mathrm{d}x$ in terms of n. Deduce that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+... \ < \ 2$. [2]

2 It is given that $\mathrm{f}(x)=x^3\ln a+bx^2+cx+d$, where a, b, c and d are constants. The curve with equation $y=\mathrm{f}(x)$ has a minimum point with coordinates $\left(\frac{5}{3},\ \frac{320}{27}\right)$. When this curve is translated 1 unit in the negative x-direction, it has a maximum point with coordinates $(0,\ 12)$. Find the values of a, b, c, and d. [4]

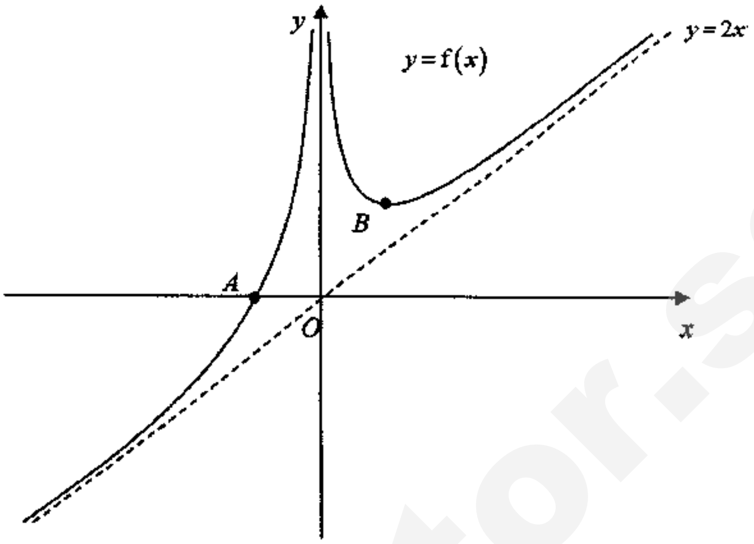
3 Find the exact area of the region A which is bounded by the curve $y=\frac{\sqrt{x^2-1}}{x}$, the horizontal lines $y=\frac{1}{2}$, $y=\frac{\sqrt{3}}{2}$, and the y-axis as shown in the diagram.



[4]

4 **(a)** State a sequence of transformations which transform the curve with equation $\frac{(x-3)^2}{4^2}+\frac{(y+1)^2}{5^2}=1$ to the curve with equation $\frac{(x-6)^2}{8^2}+\frac{(y+3)^2}{5^2}=1$. [2]

(b)



The diagram shows the graph of $y=\mathrm{f}(x)$. The curve crosses the x-axis at A, and has a minimum point at B. The coordinates of A and B are $(a,\ 0)$ and $(b,\ 2)$ respectively, where $a<0$ and $b>0$. The line $y=2x$ and y-axis are the asymptotes to the curve.

Sketch on separate diagrams, the graphs of

(i) $y=\mathrm{f}'(x)$, [3]

(ii) $y=\frac{1}{\mathrm{f}(x)}$, [3]

showing clearly, in terms of a and b where possible, the equations of any asymptote(s), the coordinates of any turning point(s) and any point(s) where the curve crosses the x- and y-axes.

5 **(i)** Let $y=\ln(\mathrm{e}^x+1)$.

Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$. [2]

(ii) By further differentiation of the result in part (i), find the first four non-zero terms in the Maclaurin series for y. [5]

(iii) Hence, expand $\frac{\ln(\mathrm{e}^x+1)}{4-x^2}$ in ascending powers of x up to and including the term in x^3. Leave the coefficients of the series in exact form. [3]

6 (a) Find $\int \frac{2-x}{4+x^2}\,dx$. [3]

(b) Use the substitution $x = \tan y$ to find the exact value of $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$. [4]

(c) Write down $\int x^2 e^{x^3}\,dx$. Hence find $\int x^5 e^{x^3}\,dx$. [4]

7 The function f is defined by

$$f : x \mapsto x^2 + 4x - 5, \quad \text{for } x \le k, k \in \mathbb{R}.$$

(i) Find the largest exact value of k such that f^{-1} exists. For this value of k, define f^{-1} in a similar form. [4]

Another function g is defined by

$$g : x \mapsto \begin{cases} 4 - x^2, & \text{for } 0 < x \le 2 \\ 2x - 4, & \text{for } 2 < x \le 4 \end{cases}$$

and that $g(x) = g(x+4)$ for all real values of x.

(ii) Sketch the graph of $y = g(x)$ for $-1 < x \le 7$. [3]

(iii) Using the results in part (i) and (ii), explain why composite function $f^{-1}g$ exists and find the exact value of $f^{-1}g(6)$. [4]

8 The logistic model for population growth states that the rate of growth of the population can be written as:

$$\frac{dP}{dt} = cP\left(1 - \frac{P}{K}\right),$$

where c is the proportionality constant, P is the size of population (in billions) at time t (in years after 2010), K is the carrying capacity (in billions). The carrying capacity is the maximum population that the environment is capable of sustaining in the long run.

At the start of 2010, the population of the world was about 7 billion. Many scientists estimated the Earth has a maximum capacity of 10 billion people, based on the calculation of the earth's available resources. It is assumed that when the population of the world is 9 billion, the rate of growth of the population is $\frac{9}{1750}$.

(i) Show that $\frac{dP}{dt} = \frac{1}{1750}P(10 - P)$. [2]

(ii) Use the logistic model to predict the world population at the start of 2020. [6]

(iii) After how many complete years will the world population first exceed 8.5 billion? [3]

(iv) Sketch a graph of P against t, where $t \ge 0$. [2]

9 The curve C has equation $y=\dfrac{x^2-4x+1}{2x+7}, x\in\mathbb{R}, x\neq -\dfrac{7}{2}$.

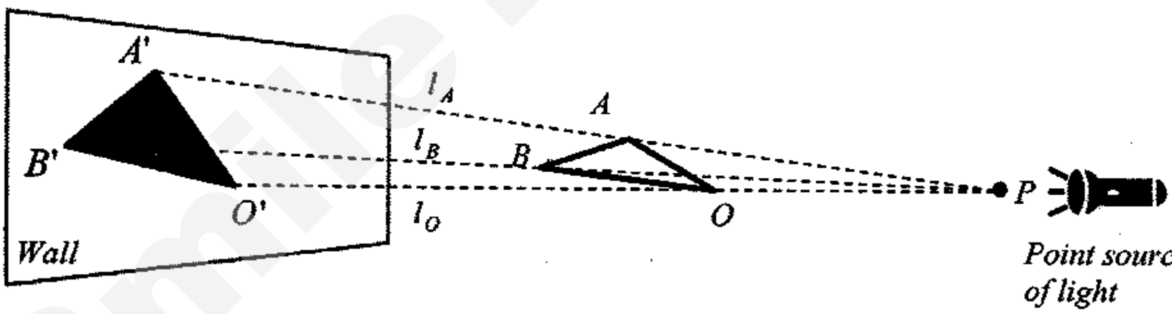
(i) Without using a calculator, find the set of values of y that C can take. [3]

(ii) Sketch C, stating the equations of any asymptotes, the coordinates of the points where the curve crosses the axes and the coordinates of the turning points. [4]

(iii) By drawing a suitable graph on the same diagram, solve the inequality $\dfrac{4x-x^2-1}{2x+7}>\dfrac{1}{(x+1)^2}$. [3]

10 In order to render the position of a shadow cast by an object on a wall in virtual reality, the concept of vector projection is used. When an object is placed between a point source of light and a wall, its shadow is projected onto the wall as shown in the diagram (not drawn to scale) below.

A triangular object OAB has O as the origin, $A(-23, 16, 10)$ and $B(-9.5, 6.5, 10.5)$ on the same plane where it is placed between the point source of light, $P(11, -22, -10)$, and the wall. Light rays l_A, l_B and l_O start at point P, passing through points A, B and O respectively and projecting their respective images A', B' and O' onto the wall.



Given that the coordinates of A' is $(-40, 35, 20)$ and the line that passes through A' and B' is parallel to the vector $10\mathbf{i}+11\mathbf{k}$,

(i) Find the coordinates of the point B'. [4]

It is given that the light ray l_O is perpendicular to the wall.

(ii) Find the equation of the plane $O'A'B'$ in scalar product form. Show that coordinates of O' is given by $(-22, 44, 20)$. [4]

(iii) Hence or otherwise, find the exact distance between point P and the wall. [2]

(iv) Are the planes OAB and $O'A'B'$ parallel? Justify your answer. [2]

11 It is given that $z=-\frac{1}{2}$ is a root of the equation

$$8z^3+\left(4-4\sqrt{2}\right)z^2+\left(2-2\sqrt{2}\right)z+1=0.$$

The roots of the equation are denoted by z_1, z_2, z_3, where $\arg(z_1)<\arg(z_2)<\arg(z_3)$.

(i) Find z_1, z_2 and z_3 in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [6]

The complex number w has modulus $\sqrt{2}$ and argument $\frac{\pi}{24}$.

(ii) Find the modulus and argument of z_4, where $z_4=\frac{w^2}{z_1}$. [3]

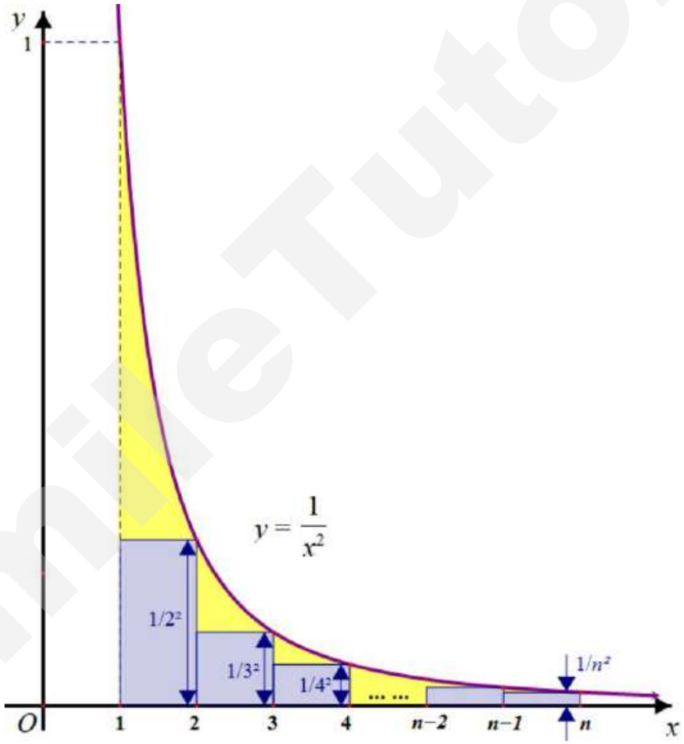
The complex numbers z_2, z_3 and z_4 are represented by the points Z_2, Z_3 and Z_4 respectively in an Argand diagram with origin O.

(iii) Mark, on an Argand diagram, the points Z_2, Z_3 and Z_4. [2]

(iv) By considering $\sin(A-B)$ with suitable values of A and B, show that $\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)$. [1]

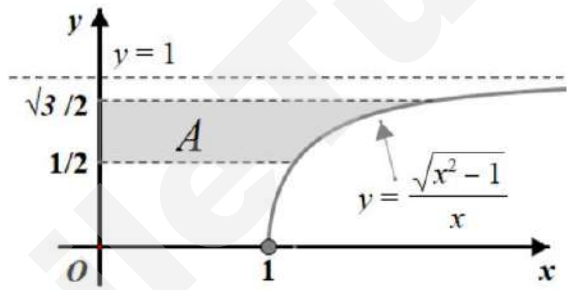
Hence or otherwise, find the exact area of the quadrilateral $OZ_3Z_4Z_2$. [2]

— **END Of PAPER** —

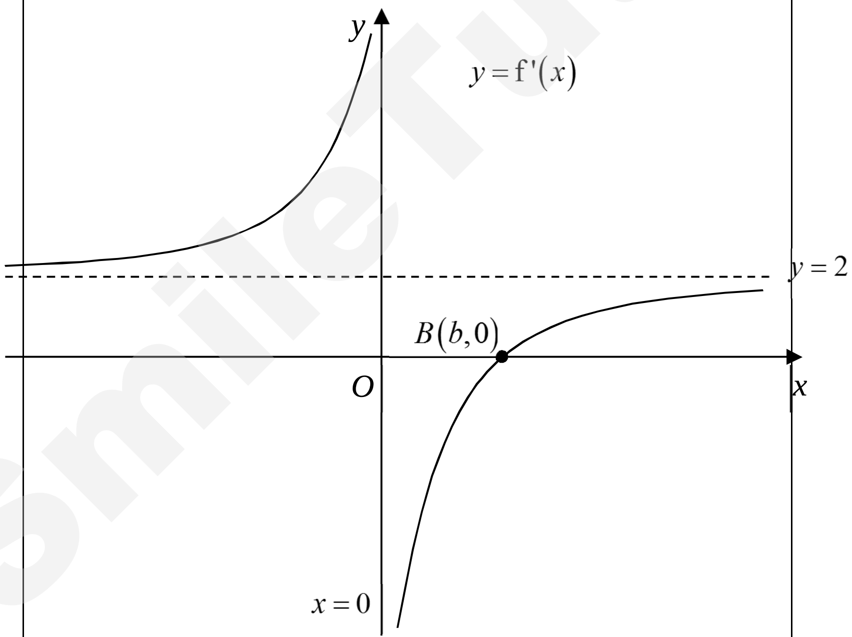
1	Topic: Definite Integrals, Integration Techniques	
Skill / Concept Assessed	**Solution**	**Feedback**
Identify a definite integral $\int_a^b \mathrm{f}(x)\ \mathrm{d}x$ as the area of a region under the curve $y=\mathrm{f}(x)$, bounded by the x-axis, and vertical lines $x=a$, $x=b$. Approximate / estimate the area of the region under the curve $y=\mathrm{f}(x)$ with the total area of an array of equally-wide, vertically-aligned rectangles between the curve and the x-axis distributed over the interval from $x=a$ to $x=b$. Using the graph to draw suitable conclusion about the under-estimation, and formulate a relevant inequality.	**1(i)** Use given sketch of the graph of $y=\frac{1}{x^2}$, together with an array of vertically-aligned rectangles, each of width 1, inscribed beneath the curve :  From the diagram, Area of first rectangle = length x breath $=\frac{1}{2^2}\times 1$	Most students know that the LHS of the expression refers to the area of the rectangles. However, very few could explain clearly how the individual components is the area of the rectangles, and only gave vague explanations based on the diagram. Most just merely used words to say there is a gap between the rectangles and the curve which is already obvious from the diagram but does not adequately link the LHS expression to area of the rectangles which are under the curve. Many also listed the rectangles up to the nth rectangle. As the rectangles are clearly drawn to be below the curve, to earn the mark, students need to explain how the areas of rectangles, especially the first and last rectangles, are found and summed to form the LHS expression.

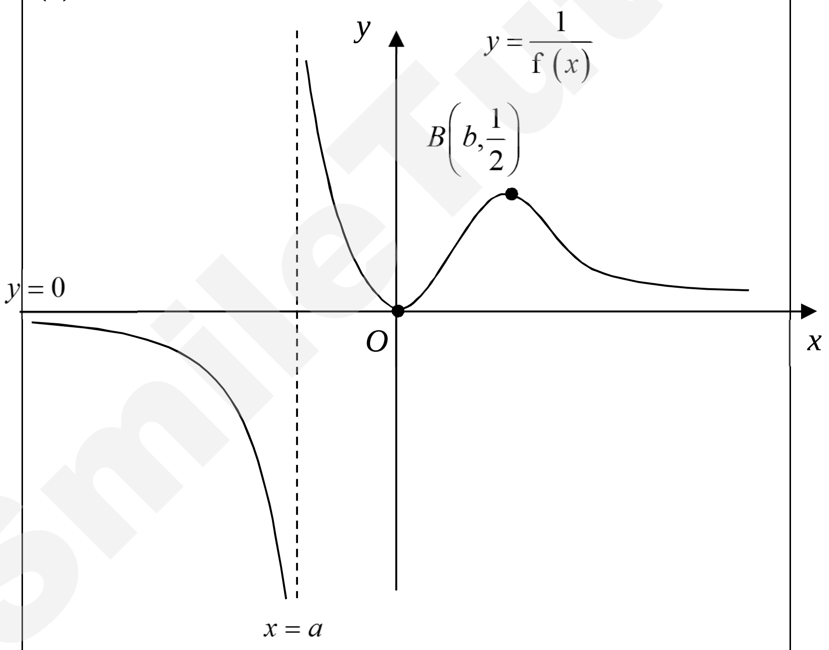
		Area of second rectangle $=\frac{1}{3^2}\times 1$ … Area of $(n-1)$th rectangle $=\frac{1}{n^2}\times 1$ Total area of $(n-1)$ rectangles $<$ $\begin{pmatrix}\text{Area under the curve } y=\frac{1}{x^2}\\ \text{over the interval } 1\le x\le n\end{pmatrix}$ $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}<\int_1^n \frac{1}{x^2}\,dx$		
	Evaluate the value of a definite integral $\int_a^b f(x)\,dx$. Take limits on both sides of an inequality.	**(ii)** $\int_1^n \frac{1}{x^2}\,dx=\left[\frac{x^{-1}}{-1}\right]_1^n$ $=\left[-\frac{1}{x}\right]_1^n$ $=\left(-\frac{1}{n}\right)-\left(-\frac{1}{1}\right)$ $=1-\frac{1}{n}$ Since $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\ <\int_1^n \frac{1}{x^2}\,dx=1-\frac{1}{n}$ Adding 1 to both sides, $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\ <\ 1+\left(1-\frac{1}{n}\right)$ As the difference between both sides of this inequality increases with n, as $n\to\infty$, $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots<\lim_{n\to\infty}\left(1+\left(1-\frac{1}{n}\right)\right)$ $=1+(1-0)=2$ (shown)		Surprisingly, many students could not integrate this basic definite integral correctly. Many seemed confused by the negative power and the unknown upper limit n. Common mistakes are: - reduce the power further to -3 - use ln formula - substitute n into the power instead of x - did not divide by – 1 - end up with $-\frac{1}{n}-1$ Few could present the last part correctly, often unable to decide when to change $\frac{1}{n}$ to $+\ldots$ as $n\to\infty$. Some did not link it to $n\to\infty$ but reasoned it as $1-\frac{1}{n}<1<2$.

2.	Topic – SOLE [Setter: YJX]			
	Skill / Concept Assessed	**Solution**		**Feedback**
	Able to form 4 equations with 4 unknowns using information provided: stationary point pass through the graph and gradient is zero. Able to interpret the graph after transformation	Marking Scheme: $\mathrm{f}(x)=x^3\ln a+bx^2+cx+d$ $\mathrm{f}'(x)=3x^2\ln a+2xb+c$ At $\left(\frac{5}{3},\ \frac{320}{27}\right)$, $\left(\frac{5}{3}\right)^3\ln a+b\left(\frac{5}{3}\right)^2+\frac{5}{3}c+d=\frac{320}{27}$ $125\ln a+75b+45c+27d=320 \qquad -(1)$ $3\left(\frac{5}{3}\right)^2\ln a+2\left(\frac{5}{3}\right)b+c=0$ $25\ln a+10b+3c=0 \qquad -(2)$ Let $\mathrm{g}(x)=\mathrm{f}(x+1)=(x+1)^3\ln a+b(x+1)^2+c(x+1)+d$ $\mathrm{g}'(x)=3(x+1)^2\ln a+2(x+1)b+c$ At $(0,\ 12)$, $\ln a+b+c+d=12 \qquad -(3)$ $3\ln a+2b+c=0 \qquad -(4)$ Using GC and solve, $\ln a=1, b=-4, c=5, d=10$ $\therefore a=\mathrm{e}, b=-4, c=5, d=10$		This question is surprisingly poorly done on the whole. Many students seem not to handle questions which combine curve transformations with SOLE. Some mistook ln*a* as a variable, when it is a constant. Many could not use the information on stationary points and formulate an equation out of it. Quite a number of students could not recognize that this is a SOLE question and wasted a lot of time solving it algebraically.

3.	**Topic: Definite Integrals, Integration Techniques**		
	Skill / Concept Assessed	**Solution**	**Feedback**
	Evaluate the exact value of a specified region bounded by a curve, and horizontal lines / axis, through forming suitable definite integral expression.	To obtain area of region A, consider integrating w.r.t. y . $y=\dfrac{\sqrt{x^2-1}}{x}$ where $x, y>0$ $\Leftrightarrow xy=\sqrt{x^2-1}$ $\Leftrightarrow x^2y^2=x^2-1$ $\Leftrightarrow 1=x^2(1-y^2)$ $\Leftrightarrow x^2=\dfrac{1}{1-y^2}$ $\Leftrightarrow x=\dfrac{1}{\sqrt{1-y^2}}$  $\therefore$ Area of $A=\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{\sqrt{1-y^2}}\,dy$ $=\left[\sin^{-1}y\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$ $=\sin^{-1}\frac{\sqrt{3}}{2}-\sin^{-1}\frac{1}{2}$ $=\frac{\pi}{3}-\frac{\pi}{6} \quad =\frac{\pi}{6}$	Since the diagram is provided, this question should have been well done but there is a significant number who did not recognize the straightforward method of using $\int xdy$. A few did not notice that exact value is required and lost unnecessary marks using GC instead. A few also did not evaluate $\sin^{-1}\frac{\sqrt{3}}{2}-\sin^{-1}\frac{1}{2}$ and left it as the final answer.

4.	Topic – Transformations		
	Skill / Concept Assessed	**Solution**	**Feedback**
	Able to state sequence of transformations for a conics equation.	**(a)** $\frac{(x-3)^2}{4^2}+\frac{(y+1)^2}{5^2}=1$ $\xrightarrow{\text{Replace } x \text{ with } \frac{x}{2}} \frac{\left(\left(\frac{x}{2}\right)-3\right)^2}{4^2}+\frac{(y+1)^2}{5^2}=1$ $\xrightarrow{\text{Replace } y \text{ with } y+2} \frac{(x-6)^2}{2^2 4^2}+\frac{((y+2)+1)^2}{5^2}=1$ Stage 1: Scale parallel to the x-axis by factor 2. Stage 2: Translate 2 units in the negative y-direction. (Order does not matter in this case) **Alternative solutions:** $\frac{(x-3)^2}{4^2}+\frac{(y+1)^2}{5^2}=1$ $\xrightarrow{\text{Replace } x \text{ with } x+3} \frac{x^2}{4^2}+\frac{(y+1)^2}{5^2}=1$ $\xrightarrow{\text{Replace } x \text{ with } \frac{x}{2}} \frac{\left(\frac{x}{2}\right)^2}{4^2}+\frac{(y+1)^2}{5^2}=1 \Rightarrow \frac{x^2}{8^2}+\frac{(y+1)^2}{5^2}=1$ $\xrightarrow{\text{Replace } x \text{ with } x-6} \frac{(x-6)^2}{8^2}+\frac{(y+1)^2}{5^2}=1$ $\xrightarrow{\text{Replace } y \text{ with } y+2} \frac{(x-6)^2}{8^2}+\frac{(y+3)^2}{5^2}=1$ Stage 1: Translate 3 units in the negative x-direction. Stage 2: Scale parallel to x-axis by factor 2. Stage 3: Translate 6 units in the positive x-direction.	This question is meant to access how well students know about the concepts of transformation. Students have to understand that translating and scaling is not about adding/multiplying constants to the bracket, it is about replacing x or y with a transformed variable. Please refer to Summary in MF26 for the replacement variables for each different transformations Most common mistakes given by students are: 1. Stage 1: Translate 3 units in the positive x-direction Stage 2: Scale parallel to the x-axis by factor 2. Stage 3: Translate 2 units in the negative y-direction. This will yield: $\frac{(x-3)^2}{4^2}+\frac{(y+1)^2}{5^2}=1$ $\xrightarrow{\text{Replace } x \text{ with } x-3} \frac{(x-3-3)^2}{4^2}+\frac{(y+1)^2}{5^2}=1$ $\xrightarrow{\text{Replace } x \text{ with } \frac{x}{2}} \frac{\left(\left(\frac{x}{2}\right)-6\right)^2}{4^2}+\frac{(y+1)^2}{5^2}=1$ $\xrightarrow{\text{Replace } y \text{ with } y+2} \frac{(x-12)^2}{2^2 4^2}+\frac{((y+2)+1)^2}{5^2}=1$ 2. Order does not matter in this case for x and y. However, many wrote "Translate 2 units in the positive y-direction." Instead of negative.

		Stage 4: Translate 2 units in the negative y-direction.		Where translation is concerned, please state "translate", "number of units translated", "direction", "axis". 3. Some candidates did not phrase the sequence of transformations clearly. "Shift" and "Transform" are words that cannot be accepted. "Scale x by 2" is a sloppy statement. 4. Candidates are strongly advised to spell out each word instead of writing short hand like –ve, 3u.
	Able to sketch a derivative graph	**(b)(i)**  		Similar mistakes from previous exam applies: 1. Candidates failed to read the question properly and did not label coordinates $(b,0)$ but rather B. This point lies on the x-axis so it should not still be $(b,2)$ 2. All asymptotes must be labelled correctly with their equations even with x- or y- axis are the asymptotes. However, most of the candidates are still confused with the equation of horizontal and vertical lines. To emphasize once more: Vertical lines are represented by $x=$ __ while horizontal lines are represented by $y=$ __ 3. Candidates do not need to label point A as there is no information

				about it for $y = \mathrm{f}'(x)$ graph. There will be no penalty. 4. Oblique asymptote $y = 2x$ will become horizontal asymptote $y = 2$. Think of differentiating *y* in terms of *x*. Majority of the candidates are still not sure of how to draw $y = \mathrm{f}'(x)$ and you are strongly advised to revise this transformation.
	Able to sketch $y = \dfrac{1}{\mathrm{f}(x)}$ graph	**(ii)** 		Since $x = 0$ is the original vertical asymptote, it will be transformed as $(0,0)$ point. The graph of $y = \dfrac{1}{\mathrm{f}(x)}$ will cut through the origin. Graphs that have a hollow circle (Excluding origin) but follow the correct shape will definitely be accepted as well. Common wrong coordinates of *B* include: $(b,2), \left(\dfrac{1}{b}, 2\right)$ Many of them sketch graphs where *y* is negative and increasing between $x = a$ and $x = 0$, this is incorrect as the original graph shows increasing *y* and positive. Therefore, graph of $y = \dfrac{1}{\mathrm{f}(x)}$ should show *y* positive (no change to sign) and decreasing (opposite from original).

5. Topic: Maclaurin Series

Skill / Concept Assessed			Feedback
Obtain a differential equation, through differentiation / implicit differentiation, using suitable differentiation formulae, rules and techniques.	**(i)** $y=\ln(\mathrm{e}^x+1)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{e}^x}{\mathrm{e}^x+1}$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{\mathrm{e}^x(\mathrm{e}^x+1)-\mathrm{e}^x(\mathrm{e}^x)}{(\mathrm{e}^x+1)^2}$ $=\dfrac{\mathrm{e}^x}{(\mathrm{e}^x+1)^2}$ The expression $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-\dfrac{\mathrm{d}y}{\mathrm{d}x}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\dfrac{\mathrm{e}^x}{(\mathrm{e}^x+1)^2}-\dfrac{\mathrm{e}^x}{\mathrm{e}^x+1}+\left(\dfrac{\mathrm{e}^x}{\mathrm{e}^x+1}\right)^2$ $=\dfrac{\mathrm{e}^x-\mathrm{e}^x(\mathrm{e}^x+1)+(\mathrm{e}^x)^2}{(\mathrm{e}^x+1)^2}$ $=\dfrac{\mathrm{e}^x-(\mathrm{e}^x)^2-\mathrm{e}^x+(\mathrm{e}^x)^2}{(\mathrm{e}^x+1)^2}$ $=0$ (shown) --- <u>Alternative Method 1 (simpler)</u> $y=\ln(\mathrm{e}^x+1)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{e}^x}{\mathrm{e}^x+1}$		This part is generally well answered. Approx. 75 – 85% of the candidates secured the full credit for this part. --- Notable source(s) of error include : • Not properly applying the chain-rule of differentiation e.g. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\mathrm{e}^x+1}$ ✗ • Assuming the intended result to be shown (poor presentation) e.g. $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-\dfrac{\mathrm{d}y}{\mathrm{d}x}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$ $\dfrac{\mathrm{e}^x}{(\mathrm{e}^x+1)^2}-\dfrac{\mathrm{e}^x}{\mathrm{e}^x+1}+\left(\dfrac{\mathrm{e}^x}{\mathrm{e}^x+1}\right)^2=0$ ✗ $\dfrac{\mathrm{e}^x-\mathrm{e}^x(\mathrm{e}^x+1)+(\mathrm{e}^x)^2}{(\mathrm{e}^x+1)^2}=0$ $\dfrac{\mathrm{e}^x-(\mathrm{e}^x)^2-\mathrm{e}^x+(\mathrm{e}^x)^2}{(\mathrm{e}^x+1)^2}=0$ (shown)

$$\frac{d^2y}{dx^2}=\frac{e^x(e^x+1)-e^x(e^x)}{(e^x+1)^2}$$

$$=\frac{e^x}{e^x+1}-\frac{(e^x)^2}{(e^x+1)^2}$$

$$=\frac{dy}{dx}-\left(\frac{dy}{dx}\right)^2$$

$$\therefore \quad \frac{d^2y}{dx^2}-\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2=0 \quad \text{(shown)}$$

Alternative Method 2 (via implicit differentiation)

$y=\ln(e^x+1)$

$e^y=e^x+1$

Differentiating implicitly w.r.t. x, $e^y\frac{dy}{dx}=e^x$

Differentiating implicitly w.r.t. x,

$$\left(e^y\frac{dy}{dx}\right)\frac{dy}{dx}+e^y\left(\frac{d^2y}{dx^2}\right)=e^x$$

$$e^y\left(\frac{dy}{dx}\right)^2+e^y\left(\frac{d^2y}{dx^2}\right)=e^y\frac{dy}{dx}$$

Multiplying throughout by e^{-y} produces

$$\left(\frac{dy}{dx}\right)^2+\frac{d^2y}{dx^2}=\frac{dy}{dx}$$

$$\left(\frac{dy}{dx}\right)^2-\frac{dy}{dx}+\frac{d^2y}{dx^2}=0 \quad \text{(shown)}$$

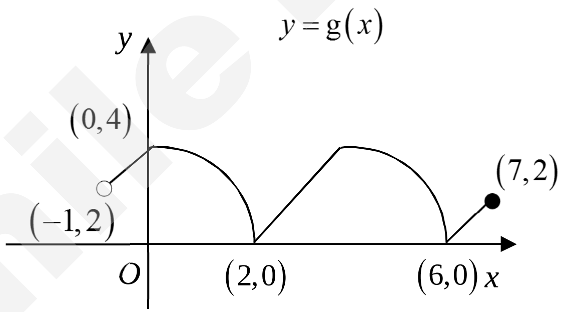
	Construct the Maclaurin series / power series representation for a function from its derivatives to required level of accuracy. Find derivative and successive higher derivatives of an explicitly defined function, involving implicit differentiation of a differential equation using suitable differentiation formulae, rules and techniques.	**(ii)** When $x = 0$, $y = \ln(e^x + 1)$; $y = \ln(e^0 + 1) = \ln 2$ $\frac{dy}{dx} = \frac{e^x}{e^x + 1}$; $\frac{dy}{dx} = \frac{e^0}{e^0 + 1} = \frac{1}{2}$ $\frac{d^2y}{dx^2} = \frac{e^x}{(e^x + 1)^2}$; $\frac{d^2y}{dx^2} = \frac{e^0}{(e^0 + 1)^2} = \frac{1}{4}$ $\because \frac{d^2y}{dx^2} - \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = 0$, Differentiating this implicitly w.r.t. x produces $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 0$ When $x = 0$, $\frac{d^3y}{dx^3} - \frac{1}{4} + 2\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = 0$ $\Rightarrow \frac{d^3y}{dx^3} = 0$ Differentiating the above once more implicitly w.r.t. x, $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} + 2\left[\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right)\right] = 0$ When $x = 0$, $\frac{d^4y}{dx^4} - 0 + 2\left[\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)(0)\right] = 0$ $\Rightarrow \frac{d^4y}{dx^4} = -\frac{1}{8}$ $\therefore$ The Maclaurin series for y		Most candidates (70 – 80%) secured partial credit for this part, while only 20 – 30% of responses could secure the full credit. Significant source(s) of error include : • Not properly applying the chain-rule of differentiation e.g. $\frac{d^2y}{dx^2} - \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = 0$ $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ ✗ OR $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{dy}{dx} = 0$ ✗ $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 0$ ✓. • Algebraic / sign error • Incorrect application of the Maclaurin theorem / stopping short of the first 4 non-zero terms Meanwhile, a handful of candidates proceeded with further differentiation from intermediate results involving $\frac{d^2y}{dx^2}$, producing more tedious working, instead of implicitly differentiating the result shown in (i), which is simpler/more elegant.

		$= y\big\vert_{x=0} + \left(\frac{dy}{dx}\big\vert_{x=0}\right)x + \left(\frac{d^2y}{dx^2}\big\vert_{x=0}\right)\frac{x^2}{2!} + \left(\frac{d^3y}{dx^3}\big\vert_{x=0}\right)\frac{x^3}{3!} + \left(\frac{d^4y}{dx^4}\big\vert_{x=0}\right)\frac{x^4}{4!} + \ldots$ $= \ln 2 + (\frac{1}{2})x + (\frac{1}{4})\frac{x^2}{2!} + (0)\frac{x^3}{3!} + (-\frac{1}{8})\frac{x^4}{4!} + \ldots$ $= \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \ldots$		
	Obtain the Maclaurin series / power series representation for a suitable function, through the use of <u>standard power series</u>, in particular the Binomial Theorem, in List MF26.	**(iii)** $\frac{\ln(e^x+1)}{4-x^2} = \ln(e^x+1)\ (4-x^2)^{-1}$ $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots$, where $-1<x<1$. $(4-x^2)^{-1} = 4^{-1}\left(1+(-\frac{x^2}{4})\right)^{-1}$ $= \frac{1}{4}\left(1+(-1)(-\frac{x^2}{4})+\frac{(-1)(-1-1)}{2!}(-\frac{x^2}{4})^2+\ldots\right)$ $= \frac{1}{4}\left(1+\frac{1}{4}x^2+\ldots\right)$ $= \frac{1}{4}+\frac{1}{16}x^2\ldots$ $\frac{\ln(e^x+1)}{4-x^2} = \ln(e^x+1)\ (4-x^2)^{-1}$ $= \left(\ln 2+\frac{1}{2}x+\frac{1}{8}x^2\ldots\right)\left(\frac{1}{4}+\frac{1}{16}x^2\ldots\right)$ $= (\ln 2)\cdot\frac{1}{4}+\left(\frac{1}{2}\cdot\frac{1}{4}\right)x+\left(\frac{1}{8}\cdot\frac{1}{4}+\frac{1}{16}\ln 2\right)x^2+\left(\frac{1}{2}\cdot\frac{1}{16}\right)x^3\ldots$ $= \frac{1}{4}\ln 2+\frac{1}{8}x+\left(\frac{1}{32}+\frac{1}{16}\ln 2\right)x^2+\frac{1}{32}x^3\ldots$		Only approx. half of the candidates could partial secure credit for this part. A significant portion (abt. 20 – 25%) left this part un-attempted. Notable errors include : • Algebraic error e.g. $(4-x^2)^{-1} = 4\left(1+(-\frac{x^2}{4})\right)^{-1}$ ✗ • Erroneous application of the binomial theorem e.g. $(4-x^2)^{-1} = 4^{-1}\left(1-\frac{x^2}{4}\right)^{-1}$ $= \frac{1}{4}\left(1+(-1)(\frac{x^2}{4})+\frac{(-1)(-1-1)}{2!}(\frac{x^2}{4})^2+\ldots\right)$ $= \frac{1}{4}\left(1-\frac{1}{4}x^2+\ldots\right)$ Meanwhile, a significant portion of responses factorized the denominator into a product of two linear factors before applying the binomial theorem :

				$(4-x^2)^{-1}=(2+x)^{-1}(2-x)^{-1}$ $=\frac{1}{2}(1+\frac{x}{2})^{-1}\cdot\frac{1}{2}(1-\frac{x}{2})^{-1}$', producing working that involve longer/more tedious steps, instead of writing $(4-x^2)^{-1}=\ \frac{1}{4}\left(1+(-\frac{x^2}{4})\right)^{-1}$, which is much simpler to expand into ascending powers of x via the binomial theorem.

6.	Topic – Integration Techniques			
	Skill / Concept Assessed	**Solution**		**Feedback**
	Evaluate an indefinite integral, by : - Splitting the numerator to break down the integration problem into simpler parts - Using integration formula for $\frac{1}{a^2+x^2}$ - Matching an integrand expression to the general form $\frac{f'(x)}{f(x)}$.	**(a)** $\int \frac{2-x}{4+x^2}\,dx$ $= \int \frac{2}{4+x^2}\,dx + \int \frac{-x}{4+x^2}\,dx$ $= 2\int \frac{1}{2^2+x^2}\,dx + \left(-\frac{1}{2}\right)\int \frac{2x}{4+x^2}\,dx$ $= 2\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right) - \frac{1}{2}\ln\left\|4+x^2\right\| + C$ $= \tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\ln\left\|4+x^2\right\| + C$, where C is an arbitrary constant.		This part was well answered by most of the candidates. Those who did wrongly mainly due to copied the expression wrongly, most commonly seen $\int \frac{2-x}{4-x^2}\,dx$ and resulted in getting the wrong answer. Most of the candidates knew that they needed to split the numerators but some used the wrong formula. A handful of candidates omitted the constant of integration and they were penalized if they repeated the same mistake for other part.
	Able to integrate by substitution	**(b)** $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2 y}}\sec^2 y\,dy$ $= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\sec^2 y}}\sec^2 y\,dy$ $= \int_0^{\frac{\pi}{4}} \sec y\,dy$ $= \left[\ln\left\|\sec y+\tan y\right\|\right]_0^{\frac{\pi}{4}}$ $= \ln\left\|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right\| - \ln\left\|\sec 0+\tan 0\right\|$ $= \ln\left(\sqrt{2}+1\right)$		Most of the candidates attempted to substitute the expression given. However, quite a number of candidates confused with the expression to be replaced for the term dy. For instance, many used this integral $\int_{y_1}^{y_2} \frac{1}{\sqrt{1+g(y)^2}}\frac{dy}{dx}dx$ instead of $\int_{y_1}^{y_2} \frac{1}{\sqrt{1+g(y)^2}}\frac{dx}{dy}dy$. Another common mistake was confusion between integration and differentiation. $\int_0^{\frac{\pi}{4}} \sec y\,dy = \left[\sec y\tan y\right]_0^{\frac{\pi}{4}}$. Many of candidates did not have the habit of

				putting a modulus sign for the logarithm term obtained. But they were not penalized because the expression for the range given was always positive, which may not be always the case. Lastly, many of them were not sensitive in variable used, they used x and y interchangeably. $\int_0^{\frac{\pi}{4}} \sec y \, \mathrm{d}y = \left[\ln\lvert\sec x + \tan x\rvert\right]_0^{\frac{\pi}{4}}$.
	Able to integrate by parts Able to integrate exponential form	**(c)** $\int x^2 \mathrm{e}^{x^3}\ \mathrm{d}x = \frac{1}{3}\int 3x^2\mathrm{e}^{x^3}\ \mathrm{d}x = \frac{1}{3}\mathrm{e}^{x^3} + C$ $\int x^5\mathrm{e}^{x^3}\ \mathrm{d}x = \int x^3\left(x^2\mathrm{e}^{x^3}\right)\mathrm{d}x$ $= x^3\left(\frac{\mathrm{e}^{x^3}}{3}\right) - \int\left(x^2\mathrm{e}^{x^3}\right)\mathrm{d}x$ $= \frac{1}{3}x^3\mathrm{e}^{x^3} - \frac{1}{3}\mathrm{e}^{x^3} + C$		This part proved to be more challenging as compared to previous two parts. Many failed to recognize the standard form $\int \mathrm{f}\,'(x)\mathrm{e}^{\mathrm{f}(x)}\ \mathrm{d}x$ for this integral. Such students applied integration by parts directly when they saw two terms in the integral. For those who successfully solved the first part, many of them made algebraic slips in second part and hence did not get full mark for this part.

7.	**Topic - Functions**			
	Skill / Concept Assessed	**Solution**		**Feedback**
	Able to analyze modulus function and find the largest domain for inverse function to exist. Able to define the rule and domain of inverse function.	**(i)** Largest value of $k = -2$ Let $y = \mathrm{f}(x)$ $y = x^2 + 4x - 5$ $= (x+2)^2 - 9$ $x + 2 = \pm\sqrt{y+9}$ $x = -2 \pm \sqrt{y+9}$ Since $x \le -2$, $x = -2 + \sqrt{y+9}$ is rejected. $\therefore x = -2 - \sqrt{x+9}$ Since $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_\mathrm{f} = [-9, \infty)$, $\mathrm{f}^{-1} : x \mapsto -2 - \sqrt{x+9}$, for $x \ge -9$		Well done. But a few did not answer to the question and wrote responses like: $k \le -2$; $x \le -2$ Many students did not put $\pm$ sign after taking square root which is not an acceptable mistake. So they subsequently lost marks as they would not have rejected one of the possible values. Many students still are not careful of the need to express the final answer in similar form.
	Able to sketch the graph of a piecewise function	**(ii)** 		About 60% of the candidates could sketch the piece-wise function completely. Common mistakes include not drawing to the required domain and not labelling the diagram adequately.
	Able to explain why composite function exists. Find value of composite function given a value of *x*.	**(iii)** $\mathrm{R}_\mathrm{g} = [0, 4]$ $\mathrm{D}_{\mathrm{f}^{-1}} = [-9, \infty)$ from part (i) Since $\mathrm{R}_\mathrm{g} \subseteq \mathrm{D}_{\mathrm{f}^{-1}}$, $\mathrm{f}^{-1}\mathrm{g}$ exists.		Quite well done in general if they got the earlier parts right.

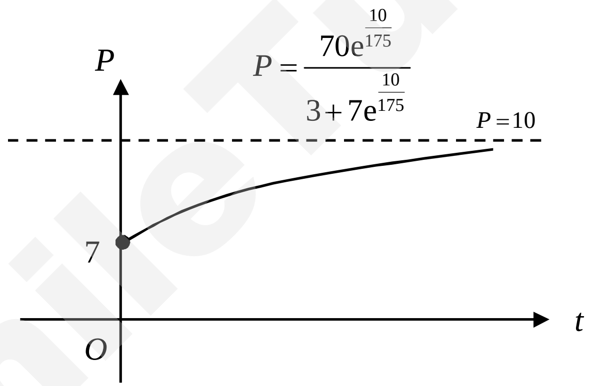
		$\mathrm{f}^{-1}\mathrm{g}(6)=\mathrm{f}^{-1}(0)$ $=-2-\sqrt{9}$ $=-5$		

8	**Differential Equations**		
	Assessment Objectives	**Solution**	**Feedback**
	Able to formulate a differential equation.	**(i)** Given carrying capacity, $K = 10$ When $P = 9$, $\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{9}{1750}$ Substituting into d.e. $\frac{\mathrm{d}P}{\mathrm{d}t}=cP(1-\frac{P}{K})$ $\frac{9}{1750}=c9\left(1-\frac{9}{10}\right)$ $\frac{1}{1750}=c\left(\frac{1}{10}\right)$ $c=\frac{1}{175}$ $\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{1}{175}P\left(1-\frac{P}{10}\right)$ $\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{1}{1750}P(10-P)$	On the whole, those who studied this topic could easily proceed with the question and get the required results. However, those who did not revise for this topic could not proceed beyond (i) as they did not know how to solve the d.e.. Many could get (i) correct. Those who did not generally did not take time to understand the question. A small minority went to solve for K when it is already given in paragraph 2. A small minority also could not factor out $\frac{1}{10}$ correctly.
	Able to solve a differential equation by variable separable.	**(ii)** $\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{1}{1750}P(10-P)$ $\int\frac{1}{P(10-P)}\frac{\mathrm{d}P}{\mathrm{d}t}\mathrm{d}t=\int\frac{1}{1750}\mathrm{d}t$ $\int\frac{1}{P(10-P)}\mathrm{d}P=\frac{1}{1750}t+C$ <u>Method 1: By Partial Fractions</u>	About 40% got this part correct. Common problem is unable to solve the d.e. by separating the variables. Common mistakes: - treating P as a constant and integrating with respect to t and forming a polynomial in P. - unable to complete the square for the denominator correctly

		Let $\dfrac{1}{P(10-P)}=\dfrac{A}{P}+\dfrac{B}{10-P}$ So $A=\dfrac{1}{10}$ and $B=\dfrac{1}{10}$ $\dfrac{1}{10}\int \dfrac{1}{P}+\dfrac{1}{10-P}\,\mathrm{d}P=\dfrac{1}{1750}t+C$ $\ln(P)-\ln(10-P)=\dfrac{1}{175}t+C$ since $0 < P < 10$ $\ln\left(\dfrac{P}{10-P}\right)=\dfrac{1}{175}t+C$ $\dfrac{P}{10-P}=\mathrm{e}^{c}\mathrm{e}^{\frac{1}{175}t}$ $\dfrac{P}{10-P}=B\mathrm{e}^{\frac{1}{175}t}$ where $B=\mathrm{e}^{c}$ $P=\dfrac{10B\mathrm{e}^{\frac{1}{175}t}}{1+B\mathrm{e}^{\frac{1}{175}t}}$		- using the integration formula for ln incorrectly. - substituting t = 20 when using t = 0 to find the arbitrary constant. - did not include arbitrary constant after integration and hence obtained wrong value for P - a small minority did not understand that P is expressed in billions and when on to use 7 000 000 000 for P and then complicate their working
		Method 2: By Formula $\int \dfrac{1}{10P-P^2}\,\mathrm{d}P=\dfrac{1}{1750}t+C$ $\int \dfrac{1}{5^2-(P-5)^2}\,\mathrm{d}P=\dfrac{1}{1750}t+C$ $\dfrac{1}{2(5)}\ln\left(\dfrac{5+(P-5)}{5-(P-5)}\right)=\dfrac{1}{1750}t+C$ since $0 < P < 10$ $\ln\left(\dfrac{P}{10-P}\right)=\dfrac{1}{175}t+10C$ $\dfrac{P}{10-P}=\mathrm{e}^{10c}\mathrm{e}^{\frac{1}{175}t}$		

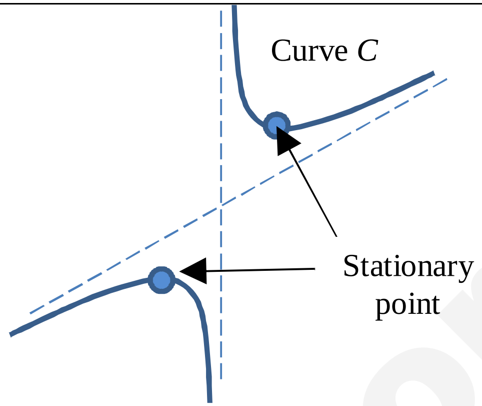
		$\frac{P}{10-P}=Be^{\frac{1}{175}t}$ where $B=e^{10c}$ $P=\frac{10Be^{\frac{1}{175}t}}{1+Be^{\frac{1}{175}t}}$		
		<u>Method 3: By Formula</u> $\int \frac{1}{10P-P^2}\,dP=\frac{1}{1750}t+C$ $-\int \frac{1}{(P-5)^2-5^2}\,dP=\frac{1}{1750}t+C$ $-\frac{1}{2(5)}\ln\left\|\frac{(P-5)-5}{(P-5)+5}\right\|=\frac{1}{1750}t+C$ since $0<P<10$ $\ln\left\|\frac{P-10}{P}\right\|=-\frac{1}{175}t-10C$ $\frac{P-10}{P}=\pm e^{-10c}e^{-\frac{1}{175}t}$ $\frac{P-10}{P}=Ae^{-\frac{1}{175}t}$ where $A=\pm e^{-10c}$ $P-10=PAe^{-\frac{1}{175}t}$ $P-PAe^{-\frac{1}{175}t}=10$ $P=\frac{10}{1+(-A)e^{-\frac{1}{175}t}}$ $P=\frac{10(-A)^{-1}e^{\frac{1}{175}t}}{(-A)^{-1}e^{\frac{1}{175}t}+1}$		

		$P = \dfrac{10Be^{\frac{1}{175}t}}{1+Be^{\frac{1}{175}t}}$ where $B = (-A)^{-1}$		
		At start of 2010, let $t=0$, then $P=7$ $7 = \dfrac{10B}{1+B}$ $7+7B=10B$ $B=\dfrac{7}{3}$ Therefore, $P = \dfrac{\frac{70}{3}e^{\frac{1}{175}t}}{1+\frac{7}{3}e^{\frac{1}{175}t}} = \dfrac{70e^{\frac{1}{175}t}}{3+7e^{\frac{1}{175}t}}$ At the start of 2020, t = 10. $P = \dfrac{70e^{\frac{10}{175}}}{3+7e^{\frac{10}{175}}}$ = 7.1186 Hence population at start of 2010 is 7.12 billion. **Note:** Can also use initial t = 2010 with P = 7 and then find P when t = 2020 or initial t = 10 with P = 7 and then find P when t = 20. However, this will give a corresponding different values for B. Final value for P will be the same.		
	Able to manipulate the equation and make the equation in terms of P.	**(iii)** When P > 8.5, Using GC on equation $P = \dfrac{70e^{\frac{1}{175}t}}{3+7e^{\frac{1}{175}t}} > 8.5$		Those who could do (ii) found no problems with (iii). A minority did not consider the inequality and round down the final answer to 155 years.

		Or using expression from <u>Method 1</u> Since at t = 0, P = 7, $\ln\left(\dfrac{P}{10-P}\right)=\dfrac{1}{175}t+\ln\dfrac{7}{3}$ $\ln\left(\dfrac{8.5}{10-8.5}\right)=\dfrac{1}{175}t+\ln\dfrac{7}{3}$ $t=175\left[\ln\left(\dfrac{8.5}{1.5}\right)-\ln\dfrac{7}{3}\right]$ $=155.2780591$ Hence, the number of complete years needed is 156		
	Able to sketch a graph in context	**(iv)** 		Those who could complete (ii) normally have no problems with (iii). Most common missing detail is the horizontal asymptote.

9	**Curve Sketching**			
	Assessment Objectives	**Solution**		**Feedback**
	Able to use discriminant inequality to determine the range of a curve.	**(i)** <u>Discriminant method</u> (recommended) : (x, y) is on curve C. $y=\dfrac{x^2-4x+1}{2x+7}$ for some $x\in\mathbb{R}$ $y(2x+7)=x^2-4x+1$ $x^2-4x-2xy+1-7y=0$ $x^2-(4+2y)x+(1-7y)=0$ for some $x\in\mathbb{R}$ $\because$ Real roots exist for this quadratic eqn. in x, $b^2-4ac\geq 0$ $(4+2y)^2-4(1)(1-7y)\geq 0$ $16+16y+4y^2-4+28y\geq 0$ $4y^2+44y+12\geq 0$ $y^2+11y+3\geq 0$ $\left(y+\dfrac{11}{2}\right)^2+3-\left(\dfrac{11}{2}\right)^2\geq 0$ $\left(y+\dfrac{11}{2}\right)^2-\left(\dfrac{\sqrt{109}}{2}\right)^2\geq 0$ $\left(y+\dfrac{11}{2}-\dfrac{\sqrt{109}}{2}\right)\left(y+\dfrac{11}{2}+\dfrac{\sqrt{109}}{2}\right)\geq 0$ $y\leq-\dfrac{\sqrt{109}}{2}-\dfrac{11}{2}$ or $y\geq\dfrac{\sqrt{109}}{2}-\dfrac{11}{2}$ Set of values of y that C can take		$\leq$ 25% of the candidates could obtain at least 1 mark of partial credit for this part. Close to half of the responses that did not obtain any credit actually left this part un-attempted. Amongst responses given partial credit, a significant portion considered the method of finding stationary points on curve C via differentiation, which requires more tedious steps subsequently w/o a calculator. Few candidates could link this portion to the use of the discriminant method to determine the range of values of y that curve C can take, which would be simpler.

		$=\left\{y\in\mathbb{R}: y\le -\frac{\sqrt{109}}{2}-\frac{11}{2} \text{ or } y\ge \frac{\sqrt{109}}{2}-\frac{11}{2}\right\}$ Alternative method : Finding stationary pts. on C via differentiation : (**Not** recommended — steps more tedious) $y=\frac{x^2-4x+1}{2x+7}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(2x-4)(2x+7)-(x^2-4x+1)(2)}{(2x+7)^2}$ $=\frac{4x^2+6x-28-(2x^2-8x+2)}{(2x+7)^2}$ $=\frac{2x^2+14x-30}{(2x+7)^2}$ If y is stationary, then $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $2x^2+14x-30=0$ $x^2+7x-15=0$ $x=\frac{-7\pm\sqrt{7^2-4(1)(-15)}}{2(1)}=\frac{-7\pm\sqrt{109}}{2}$ Since curve C is the graph of a rational function of the form $y=\frac{ax^2+bx+c}{dx+e}=Px+Q+\frac{R}{dx+e}$, with stationary points present, curve C would assume a general shape :		



At $x = \dfrac{-7 \pm \sqrt{109}}{2}$,

$$y = \frac{x^2 - 4x + 1}{2x + 7}$$

$$= \frac{\left(\frac{-7 \pm \sqrt{109}}{2}\right)^2 - 4\left(\frac{-7 \pm \sqrt{109}}{2}\right) + 1}{\cancel{2}\left(\frac{-7 \pm \sqrt{109}}{\cancel{2}}\right) + 7}$$

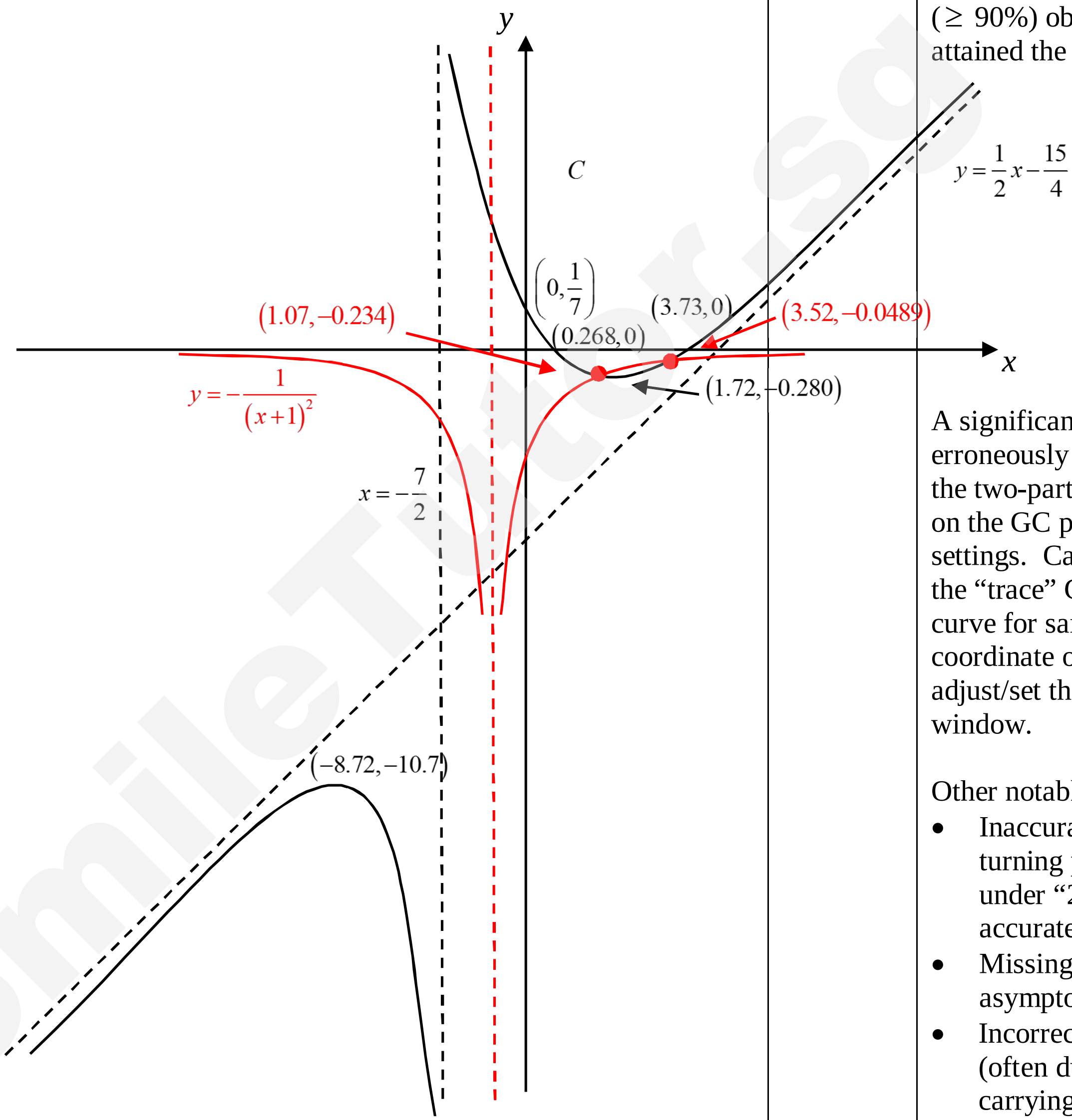
$$= \frac{\frac{1}{4}(49 \mp 14\sqrt{109} + 109) - 2(-7 \pm \sqrt{109}) + 1}{\pm\sqrt{109}}$$

$$= \frac{\frac{1}{2}(79) + 15 \ \mp \frac{7}{2}\sqrt{109} \mp 2\sqrt{109}}{\pm\sqrt{109}}$$

$$= \frac{\frac{109}{2} \mp \frac{11}{2}\sqrt{109}}{\pm\sqrt{109}} \quad = \pm\frac{\sqrt{109}}{2} - \frac{11}{2}$$

Set of values of y that C can take

$$= \left\{ y \in \mathbb{R} : y \le -\frac{\sqrt{109}}{2} - \frac{11}{2} \quad \text{or} \quad y \ge \frac{\sqrt{109}}{2} - \frac{11}{2} \right\}$$

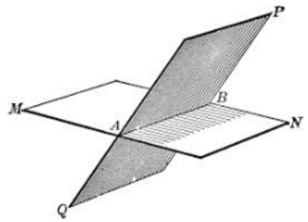
	Able to sketch graph with critical features such as asymptotes, turning points and axial intercepts.	**(ii)** 		Majority of the responses to this part ($\geq$ 90%) obtained partial credit, but few attained the full credit. A significant portion of sketches erroneously omitted the bottom half of the two-part curve, which is not visible on the GC plot under default window settings. Candidates are advised to use the "trace" GC function along the plotted curve for sampling typical value of the *y*-coordinate on the curve, and suitably adjust/set the range of values of *y* for the window. Other notable common mistakes include : • Inaccurate coordinates obtained for turning points on *C* (use GC function under "2nd"/"calc" to obtain the accurate coordinates) • Missing or incorrect oblique asymptote (O.A.) • Incorrect equation of O.A. quoted (often due to mistakes made in carrying out the long division for finding the O.A.) • Intercept with *x* / *y*-axis not indicated

	Able to sketch a suitable graph on existing diagram to solve for an inequality.	**(iii)** $\dfrac{x^2-4x+1}{2x+7}<-\dfrac{1}{(x+1)^2}$ Draw $y=-\dfrac{1}{(x+1)^2}$ Points of intersections $(1.07,-0.234)$ and $(3.52,-0.0489)$ From the diagram, $x<-\dfrac{7}{2}$ or $1.07<x<3.52$.		Only about half of the candidates correctly identified the suitable graph to insert/draw onto the same diagram as *C*. A significant error is the omission of minus sign when changing the direction of the inequality sign : • $\dfrac{4x-x^2-1}{2x+7}>\dfrac{1}{(x+1)^2}$ $\Rightarrow \dfrac{x^2-4x+1}{2x+7}<\dfrac{1}{(x+1)^2}$ ✗, resulting in the insertion of an inappropriate curve. Amongst candidates who inserted the correct graph, many could also correctly proceed to find the points of intersection, but many of these erroneously omitted the range $x<-\dfrac{7}{2}$, in extracting the set of values of x for which curve *C* is below the inserted curve.

10	**Topic – Vectors (Lines & Planes Contextual)**			
	Skill / Concept Assessed	**Solution**		**Feedback**
	Observing point of intersection between two lines in context. Finding point of intersection between two lines.	**(i)** $\overrightarrow{PB}=\overrightarrow{OB}-\overrightarrow{OP}$ $=\begin{pmatrix}-9.5\\6.5\\10.5\end{pmatrix}-\begin{pmatrix}11\\-22\\-10\end{pmatrix}$ $=\begin{pmatrix}-20.5\\28.5\\20.5\end{pmatrix}$ $l_B:\mathbf{r}=\begin{pmatrix}11\\-22\\-10\end{pmatrix}+\lambda\begin{pmatrix}-20.5\\28.5\\20.5\end{pmatrix},\lambda\in\mathbb{R}$ $l_{A'B'}:\mathbf{r}=\begin{pmatrix}-40\\35\\20\end{pmatrix}+\mu\begin{pmatrix}10\\0\\11\end{pmatrix},\mu\in\mathbb{R}$ *B'* is the point of intersection between lines l_B and $l_{A'B'}$ $\overrightarrow{OB'}=\begin{pmatrix}11-20.5\lambda\\-22+28.5\lambda\\-10+20.5\lambda\end{pmatrix}=\begin{pmatrix}-40+10\mu\\35\\20+11\mu\end{pmatrix}$ for some $\lambda,\mu\in\mathbb{R}$ $11-20.5\lambda=-40+10\mu \quad -(1)$ $-22+28.5\lambda=35 \quad -(2)$ $-10+20.5\lambda=20+11\mu \quad -(3)$ From eqn (2), $\lambda=2$ To check with eqn (1) and (2): From (1), $11-20.5(2)=-40+10\mu\Rightarrow\mu=1$ From (2), $-10+20.5(2)=20+11\mu\Rightarrow\mu=1$		Presentation was sloppy for most scripts, only if they attempted this question. Candidates are strongly advised to be mindful of vectors presentation. The following are unacceptable: ✗ $l=\begin{pmatrix}-40\\35\\20\end{pmatrix}+\mu\begin{pmatrix}10\\0\\11\end{pmatrix}$ ✗ $l:\underset{\sim}{\mathbf{r}}=\begin{pmatrix}-40\\35\\20\end{pmatrix}+\mu\begin{pmatrix}10\\0\\11\end{pmatrix}$ Candidates who scored well for this question understood that *B'* is the point of intersection between lines l_B and $l_{A'B'}$ No need for similar triangles of planes as the question did not mention. Some candidates mentioned that $\overrightarrow{A'B'}=\begin{pmatrix}10\\0\\11\end{pmatrix}$ and solve for *B'*. This is no true as it is only right to say that $A'B'$ is parallel to $\begin{pmatrix}10\\0\\11\end{pmatrix}$ and not

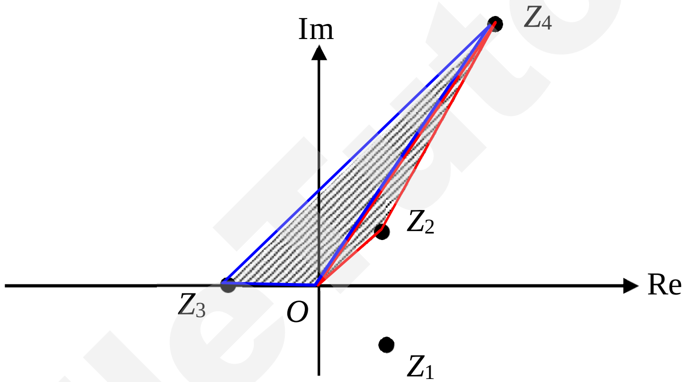
		$\overrightarrow{OB'}=\begin{pmatrix}-30\\35\\31\end{pmatrix}$ Coordinates of *B'* is $(-30,35,31)$		equals to $\begin{pmatrix}10\\0\\11\end{pmatrix}$. Therefore, the correct version should be $\overrightarrow{A'B'}=k\begin{pmatrix}10\\0\\11\end{pmatrix}$ Some candidates claimed that *B* is the midpoint of *B*' and *P* which is not true. When question asked for coordinates, candidates must leave the points as (, ,) as the final answer before they are credited with the answer mark.
	Observing normal of the equation of plane in context, and writing down its equation of plane. Finding foot of perpendicular from a point to a plane.	**(ii)** Since l_O passes through *O* and *P* and is perpendicular to the wall, $\mathbf{n}=\overrightarrow{OP}=\begin{pmatrix}11\\-22\\-10\end{pmatrix}$ Equation of wall: $\mathbf{r}\cdot\begin{pmatrix}11\\-22\\-10\end{pmatrix}=\begin{pmatrix}-40\\35\\20\end{pmatrix}\cdot\begin{pmatrix}11\\-22\\-10\end{pmatrix}=-1410$ $l_O:\mathbf{r}=s\begin{pmatrix}11\\-22\\-10\end{pmatrix},s\in\mathbb{R}$		Some candidates failed to see that since l_O is perpendicular to the wall, it makes $\overrightarrow{OP}=\begin{pmatrix}11\\-22\\-10\end{pmatrix}$ the normal of the wall. Since *A'* and *B'* lies on the wall, one can find the equation of the wall in scalar product form $\underset{\sim}{r}\cdot\underset{\sim}{n}=d$ easily. However, some candidates chose to find the normal by taking $\overrightarrow{OA'}\times\overrightarrow{OB'}$. Remember, the two vectors has to

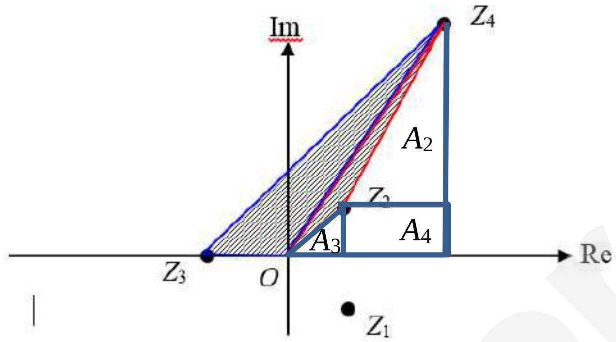
		$\overrightarrow{OO'}=\begin{pmatrix}11s\\-22s\\-10s\end{pmatrix}$ for some $s\in\mathbb{R}$ $\begin{pmatrix}11s\\-22s\\-10s\end{pmatrix}\cdot\begin{pmatrix}11\\-22\\-10\end{pmatrix}=-1410$ $121s+484s+100s=-1410$ $s=-2$ $\overrightarrow{OO'}=\begin{pmatrix}11(-2)\\-22(-2)\\-10(-2)\end{pmatrix}=\begin{pmatrix}-22\\44\\20\end{pmatrix}$ Coordinates of O' is $(-22,\ 44,\ 20)$		be 2 direction vectors parallel to the plane. However, $\overrightarrow{OA'}\times\overrightarrow{OB'}$ is not accept as O does not lie on the plane, making $\overrightarrow{OA'}$ and $\overrightarrow{OB'}$ NOT its direction vectors.
	Finding distance between a point and a plane (with the help of foot of perpendicular).	**(iii)** Distance between point P and screen $\left\lvert\overrightarrow{O'P}\right\rvert=\left\lvert\overrightarrow{OP}-\overrightarrow{OO'}\right\rvert=\left\lvert\begin{pmatrix}11\\-22\\-10\end{pmatrix}-\begin{pmatrix}-22\\44\\20\end{pmatrix}\right\rvert$ $=\left\lvert\begin{pmatrix}33\\-66\\-30\end{pmatrix}\right\rvert=\sqrt{33^2+(-66)^2+(-30)^2}$ $=\sqrt{6345}$		You do not need correct answer for previous parts to continue this question. Some candidates used $\left\lvert\overrightarrow{OP}\right\rvert+\left\lvert\overrightarrow{OO'}\right\rvert$ and this is accepted because *O*, *O'* and *P* are collinear.
	Show two planes are parallel or not	**(iv)** $\mathbf{n}$ of plane $O'A'B'=\begin{pmatrix}11\\-22\\-10\end{pmatrix}$		$\mathbf{n}$ of plane $O'A'B'=\begin{pmatrix}11\\-22\\-10\end{pmatrix}$ is observed from (iii). Candidates just need to use cross product and find the normal for plane *OAB*.

		$\mathbf{n}$ of plane $OAB = \overrightarrow{OA} \times \overrightarrow{OB}$ $= \begin{pmatrix} -23 \\ 16 \\ 10 \end{pmatrix} \times \begin{pmatrix} -9.5 \\ 6.5 \\ 10.5 \end{pmatrix}$ $= \begin{pmatrix} 103 \\ 146.5 \\ 2.5 \end{pmatrix}$ Since $\begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix} \neq k \begin{pmatrix} 103 \\ 146.5 \\ 2.5 \end{pmatrix}$ for some real k, planes *OAB* and $O'A'B'$ are not parallel. Alternative method: By using the fact that if a direction vector of plane *OAB* is not perpendicular to the normal of *O'A'B'*. $\overrightarrow{OA} \cdot \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix}$ $= \begin{pmatrix} -23 \\ 16 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix}$ $= -705 \neq 0$ Since $\overrightarrow{OA}$ (on the plane *OAB*) is not perpendicular to the normal of *O'A'B'*, the two planes are not parallel.		To my horror, some of the candidates did not use their normal to prove. Instead, they used direction vectors on the planes respectively and prove. This is a major concept error as directional vectors on the planes cannot prove whether planes are parallel.  As you can see, $\overrightarrow{AB}$ is a direction vector lying on both planes but that does not mean that both planes are parallel! Candidates must justify their answers using numerical evidence instead of trying to prove using sentences.

11	Complex Numbers			
	Assessment Objectives	**Solution**		**Feedback**
	Able to find all the roots of a polynomial equation degree 3.	**(i)** $$\begin{array}{r} 8z^2-4\sqrt{2}z+2 \\ z+\frac{1}{2}\overline{\big)8z^3+\left(4-4\sqrt{2}\right)z^2+\left(2-2\sqrt{2}\right)z+1} \\ \underline{8z^3+\qquad 4z^2} \qquad\qquad\qquad\qquad \\ -4\sqrt{2}z^2+\left(2-2\sqrt{2}\right)z+1 \\ \underline{-4\sqrt{2}z^2 \qquad -2\sqrt{2}z} \quad \\ 2z \qquad +1 \\ \underline{2z \qquad +1} \\ 0 \end{array}$$ $$8z^3+\left(4-4\sqrt{2}\right)z^2+\left(2-2\sqrt{2}\right)z+1=0$$ $$\left(8z^2-4\sqrt{2}z+2\right)\left(z+\frac{1}{2}\right)=0$$ $$8z^2-4\sqrt{2}z+2=0$$ $$z=\frac{4\sqrt{2}\pm\sqrt{\left(4\sqrt{2}\right)^2-4(8)(2)}}{2(8)}$$ $$=\frac{4\sqrt{2}\pm\sqrt{32-64}}{16}$$ $$=\frac{4\sqrt{2}\pm\sqrt{-32}}{16}$$ $$=\frac{4\sqrt{2}\pm4\sqrt{2}\mathrm{i}}{16}$$ $$=\frac{\sqrt{2}\pm\sqrt{2}\mathrm{i}}{4}$$ $$z_1=\frac{1}{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}},$$		Surprisingly, many candidates claimed that since $z=-\frac{1}{2}$ is a root and therefore so is $z=\frac{1}{2}$ since all coefficients are real. This is only true for non-real root with real coefficients. For those who tried to replace $z=a+b\mathrm{i}$ into the equation found that the expanded expressions were way too complicated to proceed and gave up halfway. These students needed to learn to use alternative plan to solve the problem. Many algebraic slips in solving the quadratic factor and also many of them did not state the z_1, z_2 and z_3 correctly as they did not pay attention to the arguments of these points from the question.

		$z_2 = \frac{1}{2}e^{i\frac{\pi}{4}}, z_3 = \frac{1}{2}e^{i\pi}$		
	Able to find the modulus and argument of a complex number.	$w = \sqrt{2}e^{i\frac{\pi}{24}}$ $z_4 = \frac{w^2}{z_1}$ $= \frac{\left(\sqrt{2}e^{i\frac{\pi}{24}}\right)^2}{\frac{1}{2}e^{-i\frac{\pi}{4}}}$ $= 4e^{i\frac{\pi}{12}+i\frac{\pi}{4}}$ $= 4e^{i\frac{\pi}{3}}$ Modulus = 4, argument = $\frac{\pi}{3}$ $4e^{i\frac{\pi}{3}} = 4\cos\frac{\pi}{3} + 4\sin\frac{\pi}{3}i$ $= 2 + 2\sqrt{3}i$		As the question required the value of z_1, if they did not get it right, they couldn't get this part correctly.

	Able to prove a trigonometry equation using addition formulae.	**(iii)** $$\sin\left(\frac{\pi}{12}\right)=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$ $$=\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)$$ $$=\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$ $$=\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)$$		Those who attempted mostly able to substitute a suitable A and B. Many just omitted this part.
	Able to identify the points on an Argand diagram and use the area of triangle to find the required area.	**(ii), (iv)**  "Hence" Area of quadrilateral $OZ_3Z_4Z_2$ = Area of triangle OZ_2Z_4 + Area of triangle OZ_4Z_3 $$=\frac{1}{2}\left(\frac{1}{2}\right)(4)\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)+\frac{1}{2}\left(\frac{1}{2}\right)(4)\sin\left(\pi-\frac{\pi}{3}\right)$$ $$=\sin\left(\frac{\pi}{12}\right)+\sin\left(\frac{2\pi}{3}\right)$$ $$=\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)+\frac{\sqrt{3}}{2}$$		This part was very poorly answered. Many of them did not label the points clearly with details like the modulus and argument. Due to wrong values obtained in part (i), this part was usually wrong as inherited by part (i). This part proved to be the most challenging part for this question. Many did not reach this question. Only very few students got this part right as they got the correct diagram and applied the hint given (where they considered the triangles carefully).

		"Otherwise"  Area of quadrilateral $OZ_3Z_4Z_2$ = Area of big triangle – A_2 – A_3 – A_4 $=\frac{1}{2}\left(2+\frac{1}{2}\right)2\sqrt{3}-\frac{1}{2}\left(\frac{\sqrt{2}}{4}\right)\left(\frac{\sqrt{2}}{4}\right)-\frac{1}{2}\left(2-\frac{\sqrt{2}}{4}\right)\left(2\sqrt{3}-\frac{\sqrt{2}}{4}\right)-\left(2-\frac{\sqrt{2}}{4}\right)\left(\frac{\sqrt{2}}{4}\right)$ $=\frac{5}{2}\sqrt{3}-\frac{1}{16}-2\sqrt{3}+\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}-\frac{1}{16}-\frac{\sqrt{2}}{2}+\frac{2}{16}$ $=\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)+\frac{\sqrt{3}}{2}$		

Section A: Pure Mathematics [40 marks]

1 (i) The sum of the first n terms of a sequence is given by $S_n = n(2n+7)$. [2]
Show that $u_n = 4n+5$ and prove that the sequence is an Arithmetic Progression.

(ii) Find $\sum_{n=1}^{N} \frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}$ in terms of N, where u_n is the nth term of the arithmetic series in part (i). [4]

(iii) Hence find the exact value of $\frac{1}{\sqrt{53}+\sqrt{49}}+\frac{1}{\sqrt{57}+\sqrt{53}}+\ldots+\frac{1}{\sqrt{361}+\sqrt{357}}$. [2]

2 The origin O, and the points A and B lie in the same plane where $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. It is given that $|\mathbf{a}|=1$, $|\mathbf{b}|=2$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$ radians. The point F is the foot of the perpendicular from A to the line segment passing through O and B.

(i) Find $\overrightarrow{OF}$ in terms of $\mathbf{b}$ only. [4]

(ii) It is given that C is the point of reflection of A about the line segment passing through O and B. Find the position vector of C in terms of $\mathbf{a}$ and $\mathbf{b}$.
Hence, state with a reason, the shape of the quadrilateral $OABC$ and show that its exact area is $\sqrt{3}$. [4]

3 An airport proposes to collect passenger service fee from each traveller to fund an upgrading project for the airport. The upgrading project is expected to cost \$ 1, 000, 000.

The proposal is to collect a fee of \$5 per traveller from 1 January 2019, and increase this fee by \$2.50 on 1 January of every subsequent year. It is given that the airport handles 10, 000 travellers a year and that there is no change in the number of travellers every year.

(i) According to this proposal, find the earliest year in which the airport reaches its target of \$ 1, 000, 000. [4]

However, representatives from the airline industry strongly object to the above proposal. Consequently, the airport decides to allocate funds from its reserve investment revenue every year to finance the upgrading project instead, starting from the year 2019.

The allocated fund in 2019 is \$50, 000. Due to increases in other expenditures, the allocated fund is expected to decrease by 6% of the allocated fund in the previous year, such that the allocated fund is \$47, 000 in 2020, \$44, 180 in 2021 and so on.

(ii) Find the total allocated fund collected in the years from 2024 to 2030, giving your answer correct to nearest dollar. [3]

(iii) Will the airport eventually collect enough to fund the upgrading project? Justify your answer. [3]

4.

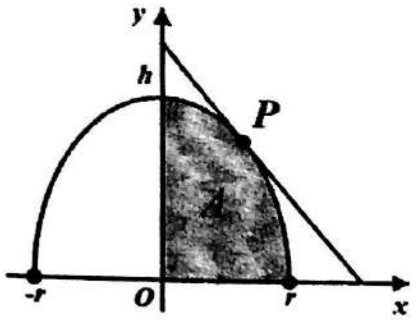


Figure 1

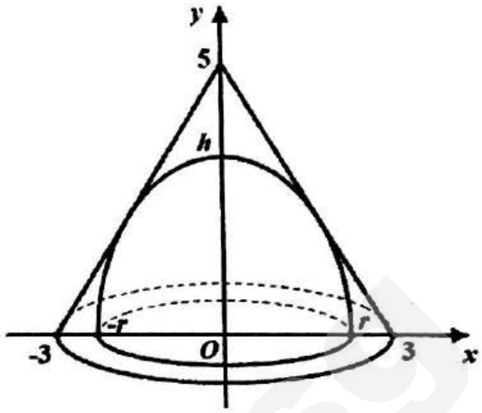


Figure 2

Figure 1 shows the upper-half of an ellipse with equation $\left(\frac{x}{r}\right)^2+\left(\frac{y}{h}\right)^2=1$, and region A bounded by the ellipse and the axes in the first quadrant.

(i) Show that the volume V of the solid of revolution generated by rotating region A through 2π radians about the y-axis, is $V=\frac{2}{3}\pi r^2 h$. [3]

(ii) The ellipse can also be represented by parametric equations $x=r\cos\theta$, $y=h\sin\theta$, where $0\le\theta\le\pi$. Show that the equation of the tangent to the ellipse at point $P\ (r\cos\alpha,\ h\sin\alpha)$, is $y=-\left(\frac{h}{r}\cot\alpha\right)x+h\ \text{cosec}\,\alpha$. [4]

Figure 2 shows a fixed right circular cone of height 5 and base radius 3, which contains an inscribed hemi-ellipsoid generated by rotating the region that is bounded by the ellipse $\left(\frac{x}{r}\right)^2+\left(\frac{y}{h}\right)^2=1$ and the x-axis through π radians about the y axis.

(iii) Given that the line $y=-\frac{5}{3}x+5$, the slanted edge of the cone joining points $(3,\ 0)$ and $(0,\ 5)$, is a tangent to the inscribed ellipse $\left(\frac{x}{r}\right)^2+\left(\frac{y}{h}\right)^2=1$, deduce that the base radius r and height h for the inscribed hemi-ellipsoid is related by the equation $\left(\frac{h}{5}\right)^2+\left(\frac{r}{3}\right)^2=1$. [3]

(iv) Find the exact value of height h for which the volume of the inscribed hemi-ellipsoid is a maximum as h varies. [4]

Section B: Probability and Statistics [60 marks]

5 In a supermarket game, ten randomly selected shoppers are allowed 15 minutes each to search for mystery items hidden amongst the supermarket shelves. Each shopper is allowed to find at most one mystery item. On average the probability that a shopper will find a mystery item is $\frac{1}{p}$ where $1 < p < 2$.

(i) State, in the context of this question, an assumption needed to model the number of shoppers who find a mystery item by a binomial distribution. [1]

Assume now that part (i) holds.

The probability that three of the ten shoppers find a mystery item is $\frac{15}{4}\left(\sqrt{2}-1\right)^7$.

(ii) Find the value of p. [2]

The supermarket conducts this game n times.

(iii) Find the least value of n such that there is a probability of more than 0.01 that in more than two of the games, three of the ten shoppers find a mystery item in a game. [3]

6. Five fair coins are tossed together in one throw. The number of tails and heads obtained in one throw are denoted by T and H respectively. The random variable X denotes $T - H$.

(i) Show that $\mathrm{P}(X=1)=\frac{5}{16}$ and hence find the probability distribution of X. [3]

(ii) State the value of $\mathrm{E}(X)$ and find the value of $\mathrm{Var}(X)$. [2]

A player pays \$1 for one throw. He receives nothing if the difference between the number of tails and heads obtained is less than three, receives \$2 if the difference is equal to three, and receives \$$k$ if the difference is more than three.

(iii) Find the value of \$$k$ if the expectation of the player's profit is \$10. [2]

7 (a) For events A and B, it is given that $\mathrm{P}(B)=\frac{1}{3}$, $\mathrm{P}(A\cap B)=\frac{1}{5}$ and $\mathrm{P}(A'\cap B')=\frac{1}{6}$.

Find $\mathrm{P}(A\mid B')$. [3]

(b) A seven-digit number is formed by writing down the digits $1,2,2,3,4,5,5$ in some order. Find how many of such numbers can be formed if

(i) the two '5's are not next to each other, [2]

(ii) there are exactly three digits between the two '5's, [3]

(iii) the number is an odd number between $1\ 000\ 000$ and $2\ 000\ 000$. [3]

8 In a neuroscience study, researchers investigate the relationship between brain mass, x kilograms and Intelligence Quotient index, y. The table below shows the data of a random sample of 10 people.

x	1.196	1.342	1.399	1.476	1.493	1.504	1.521	1.568	1.582	1.601
y	73	82	98	109	114	119	128	138	142	148

(i) Draw the scatter diagram for these values, labelling the axes clearly. [1]

It is thought that the intelligence quotient, y can be modelled by one of the formulae

$$y = a + bx \quad \text{or} \quad y = c + dx^2$$

where a, b, c and d are constants.

(ii) Find the value of the product moment correlation coefficient between

(a) x and y, [1]

(b) x^2 and y. [1]

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $y = a + bx$ or $y = c + dx^2$ is the better model. Hence, calculate the equation of the least squares regression line of the suitable model you have found. [3]

(iv) Use the regression line found in part **(iii)** to estimate the value of Intelligence Quotient index when the brain mass is 1500 grams.
Comment on the reliability of your answer. [2]

(v) An internet article claims that the neuroscience study shows that "heavier brain mass leads to higher intelligence quotient". Comment briefly on the validity of this statement. [1]

9 In a certain country, heights of males and females follow independent normal distributions. Heights of males have mean 175 cm and standard deviation 10 cm, and heights of females have mean μ cm and standard deviation σ cm.

(i) If the proportion of females shorter than 143 cm and the proportion of females taller than 183 cm are both equal to 0.01114 , write down the value of μ and show that σ is 8.75 , correct to 3 significant figures. [3]

(ii) If two females and one male are randomly chosen, find the probability that the sum of the heights of the two females differ from twice the height of the male by at least 30 cm. [4]

Flight attendants and fighter pilots have strict height requirements due to the nature of their professions. The national carrier of the country has a minimum height requirement of 155 cm and a maximum of 185 cm for female flight attendants. Its air force has a minimum height requirement of 160 cm and a maximum of 192.5 cm for its male fighter pilots.

(iii) Two females and one male are chosen at random. Find the probability that the male meets fighter pilot height requirement whereas only one female meets the female flight attendant height requirement. [3]

During a group interview conducted by the national carrier, all female candidates wear standard shoes with 5 cm heels. A random sample of 15 female candidates is chosen.

(iv) Find the probability that the average height of the sample of female candidates wearing standard shoes is greater than 170 cm. [3]

10 In a large busy restaurant, the mean time taken for a server to clear a table and set it for the next guest is 4.5 minutes. In order to improve the quality of service and maintain a clean environment, the restaurant manager introduced a new routine to clear tables. A random sample of 30 servers is taken and the time taken by each server to clear a table, x minutes, is recorded. The data are summarised as follows:

$$\sum x = 132 \text{ and } \sum x^2 = 583.96 \text{ .}$$

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Test, at the 10% significance level, whether the mean time taken for a server to clear a table has changed. [5]

(iii) Using the results in part (ii), state the range of values of α, where $\alpha\%$ is the level of significance, at which a test would lead the manager to conclude that the mean time has not changed. [1]

(iv) Without carrying out another test, state with reasons, the conclusion if the test at 5% significance level was to determine whether the mean time taken to clear a table is less than 4.5 minutes. [2]

After a few weeks, based on feedback from servers and guests, the manager modifies the routine. Another random sample of 30 servers is taken and their time taken to clear a table has mean $\bar{x}$ minutes and variance 1.01 minutes^2.

(v) Find the set of values of $\bar{x}$ for which a test at 1% significance level concludes that the mean time taken to clear a table is greater than 4.5 minutes, giving your answer to 3 decimal places. [4]

- END OF PAPER -

Section A: Pure Mathematics [40 marks]

1	**Sigma Notation**		
	Assessment Objectives	**Solution**	**Feedback**
	Able to prove an arithmetic sequence.	**(i)** $u_n = S_n - S_{n-1}$ $= n(2n+7)-(n-1)(2n+5)$ $= 2n^2+7n-(2n^2+3n-5)$ $= 4n+5$ $u_1 = 4(1)+5 = 9$ $S_1 = 1[2(1)+7] = 9$ $u_1 = S_1$ $u_n - u_{n-1}$ $= (4n+5)-(4(n-1)+5)$ $= 4n+5-(4n+1)$ $= 4$ Since $u_n - u_{n-1} = 4$ (constant), the sequence is AP.	Majority of the students were able to do this part of the question. Although a large number of students omitted verifying $u_1 = S_1$, they were not penalised. Common Mistakes: 1) A significant number of students assumed an AP and manipulate the form of S_n given to look like $\frac{n}{2}[2a+(n-1)d]$ and solve for *a* and *d*. 2) Concluding AP with $S_n - S_{n-1} = 4n+5$ (**constant**) 3) Algebraic Slips when finding $u_n - u_{n-1}$. 4) Proving AP using $u_3 - u_2 = u_2 - u_1$. 5) Concluding AP with "since $u_1 = S_1$, sequence is AP".

	Able to understand the difference of arithmetic sequences is always a fixed constant. Able to apply method of differences.	**(ii)** $\sum_{n=1}^{N}\frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}$ $=\sum_{n=1}^{N}\frac{\left(\sqrt{u_{n+1}}-\sqrt{u_n}\right)}{\left(\sqrt{u_{n+1}}+\sqrt{u_n}\right)\left(\sqrt{u_{n+1}}-\sqrt{u_n}\right)}$ $=\sum_{n=1}^{N}\frac{\left(\sqrt{u_{n+1}}-\sqrt{u_n}\right)}{u_{n+1}-u_n}$ $=\frac{1}{4}\sum_{n=1}^{N}\left(\sqrt{u_{n+1}}-\sqrt{u_n}\right)$ $=\frac{1}{4}\Big\{\left(\cancel{\sqrt{u_2}}-\sqrt{u_1}\right)+$ $\left(\cancel{\sqrt{u_3}}-\cancel{\sqrt{u_2}}\right)+$ $\vdots$ $\left(\cancel{\sqrt{u_N}}-\cancel{\sqrt{u_{N-1}}}\right)+$ $\left(\sqrt{u_{N+1}}-\cancel{\sqrt{u_N}}\right)\Big\}$ $=\frac{1}{4}\left(\sqrt{u_{N+1}}-\sqrt{u_1}\right)$ $=\frac{1}{4}\left(\sqrt{4N+9}-3\right)$		This part of the question was poorly done. Many students were stuck at manipulating the expression such that method of difference can be applied. A significant number of students attempted to use partial fractions to manipulate the expression.
	Able to rewrite the series using summation.	**(iii)** $\frac{1}{\sqrt{53}+\sqrt{49}}+\frac{1}{\sqrt{57}+\sqrt{53}}+\cdots+\frac{1}{\sqrt{361}+\sqrt{357}}$ $=\frac{1}{\sqrt{4(11)+9}+\sqrt{4(11)+5}}+\frac{1}{\sqrt{4(12)+9}+\sqrt{4(12)+5}}+\cdots+\frac{1}{\sqrt{4(88)+9}+\sqrt{4(88)+5}}$		Many students were able to re-express the series using summation. However, some made the mistakes on the upper and lower limits. Students who were not able to attempt part (ii) often just give up part (iii). A number of students simply used the result in part (ii), ignoring the fact that lower limit is 11 and did not change

		$=\sum_{n=1}^{88}\frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}-\sum_{n=1}^{10}\frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}$ $=\frac{1}{4}\left(\sqrt{4(88)+9}-3\right)-\frac{1}{4}\left(\sqrt{4(10)+9}-3\right)$ $=4-1$ $=3$		to $=\sum_{n=1}^{88}\frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}-\sum_{n=1}^{10}\frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}$.

2	Topic – Vectors (Basic + Algebra)				
	Skill / Concept Assessed	**Solution**			**Feedback**
	Able to find foot of perpendicular from a point to a line.	**(i)**	Equation of line segment passing through O and B: $\mathbf{r}=\lambda\mathbf{b}, \lambda\in\mathbb{R}$ F lies on the line: $\overrightarrow{OF}=\lambda\mathbf{b}$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{AF}=\overrightarrow{OF}-\overrightarrow{OA}=\lambda\mathbf{b}-\mathbf{a}$ $\overrightarrow{AF}\cdot\mathbf{b}=0$ $(\lambda\mathbf{b}-\mathbf{a})\cdot\mathbf{b}=0$ $\lambda\mathbf{b}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b}=0$ $\lambda\lvert\mathbf{b}\rvert^2-\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\frac{\pi}{3}=0$ $\lambda(2)^2-2\left(\frac{1}{2}\right)=0$ $\lambda=\frac{1}{4}$ $\overrightarrow{OF}=\frac{1}{4}\mathbf{b}$		Many students were not able to solve this part of the question properly. Most were unable to find the foot of perpendicular when the vectors **a** and **b** are not explicitly defined in numbers. Many were weak in the properties of vectors. Common mistakes: 1) $\mathbf{b}\cdot\mathbf{b}=\mathbf{b}^2$ 2) Missing "for some $\lambda\in\mathbb{R}$" 3) Fail to recall $\mathbf{a}\cdot\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$ 4) Making a the subject from $\mathbf{a}\cdot\mathbf{b}=1$, i.e. $\mathbf{a}\cdot\mathbf{b}=1\Rightarrow\mathbf{a}=\frac{1}{\mathbf{b}}$.
			Alternative: $\lvert\overrightarrow{OF}\rvert=\cos\frac{\pi}{3}=0.5$ $\therefore\overrightarrow{OF}=\frac{0.5}{2}\mathbf{b}=\frac{1}{4}\mathbf{b}$		
	Able to find point of reflection from a given point to a line. Able to state properties of parallelogram i.e. kite Able to find area of parallelogram using twice area of suitable triangles.	**(ii)**	F is the midpoint of A and C $\overrightarrow{OF}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OC})$ $\overrightarrow{OC}=2\overrightarrow{OF}-\overrightarrow{OA}$ $=\frac{1}{2}\mathbf{b}-\mathbf{a}$ Hence, Since the diagonals AC and OB are perpendicular, $OA=OC=1$ and $AB=CB$, $OACB$ is a kite.		Students who were unable to answer part (i) right could not get $\overrightarrow{OC}$ as well. Many students who answered kite were unable to reason out the properties of a kite or left out the reasoning totally. Majority of the students gave the answer as parallelogram and subsequently used

		Area of $OACB = 2\times$ Area of OAB $= \left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$ $= \left\|\mathbf{a}\times\mathbf{b}\right\|$ $= \left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|\sin\frac{\pi}{3}$ $= 2\left(\frac{\sqrt{3}}{2}\right)$ $= \sqrt{3}$		an incorrect method to find the area. Common Mistakes on area: 1) $\left\|\overrightarrow{OA}\times\overrightarrow{OC}\right\|$ 2) $\frac{1}{2}\times\left\|\overrightarrow{OA}\times\overrightarrow{OF}\right\|\times 4$ 3) $\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\| = \sqrt{1\times 2}$ 4) $\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\| = \sqrt{1+2} = \sqrt{3}$

3	AP and GP				
	Assessment Objectives	**Solution**			**Feedback**
	Able to interpret the contextual problem to find the sum to the first n terms.	**(i)**	Let the amount received in \$ be u_n, where n is the number of years from 2018. $u_1 = 10,000 \times 5.00 = 50,000$ $u_2 = 10,000 \times 7.50 = 75,000$ $u_3 = 10,000 \times 10.00 = 100,000$ ... forms an AP with a = 50,000, d = 25,000 For $S_n \geq 1,000,000$, $\frac{n}{2}\left[2(50,000)+(n-1)(25,000)\right] \geq 1,000,000$ $n\left[(500)+(n-1)(125)\right] \geq 10000$ $n\left[125n+375\right] \geq 10000$ $n\left[n+3\right] \geq 80$ From GC, $n \geq 7.5692$ or $n \leq -10.569$ $n \geq 8$ or $n \leq -11$ rejected as n is positive In the year of 2026. <u>Alternative Method</u> Can just consider fee paid by one traveler i.e. \$100 So use S_n > 100 using a = 5 and d = 2.50		This part was generally well answered. Most candidates recognized this part was an AP, interpreted correctly and proceeded to find the sum of the first n terms greater than or equal to 1,000,000. However, some candidates failed to recall the correct formula for sum to the first n terms and many candidates did not formulate an inequality and simply solved for an equality. In addition, a significant number of candidates simply left the answer as the number of years instead of the required year. These candidates need to learn to read the question more carefully. A pity that some got the context right but used different units in finding the sum. For example, some used the first term as 50, 000 but the common difference was 2.50, instead of 25, 000. A small group of candidates painstakingly listed term by term to generate the sum manually, they gained marks but lost time. They need to consider alternative in this case.
	Able to find the sum of terms using difference of sums.	**(ii)**	Let the amount received in \$ be v_n, where n is the number of years from 2018. $v_1 = 50,000$ $v_2 = 50,000 \times 0.94 = 47,000$		This part caused a lot of problems to candidates due to the nature of the question asking for amount received within certain years. Many couldn't get the difference of sum correctly. Most common errors were

			$v_3 = (50,000 \times 0.94) \times 0.94 = 44,180$ … forms a GP with a = 50, 000, r = 0.94 The total allocated fund received in the years 2024 to 2030, $v_6 + v_7 + \ldots + v_{12}$ $= S_{12} - S_5$ $= \dfrac{50000\left(1-(0.94)^{12}\right)}{1-0.94} - \dfrac{50000\left(1-(0.94)^{5}\right)}{1-0.94}$ $= 1000000\left((0.94)^5 - (0.94)^{12}\right)$ $= 214986.42$ $= 214986$ The total allocated fund received in the years 2024 to 2030 is \$214, 986. Accept answer to 3SF i.e. \$215, 000.		$\dfrac{50000\left(1-(0.94)^{12}\right)}{1-0.94} - \dfrac{50000\left(1-(0.94)^{6}\right)}{1-0.94}$ or $\dfrac{50000\left(1-(0.94)^{11}\right)}{1-0.94} - \dfrac{50000\left(1-(0.94)^{4}\right)}{1-0.94}$. Many failed to leave answer as nearest dollars as required by the question. Similarly, a small group of candidates painstakingly listed term by term to generate the sum manually, they gained marks but lost time. They need to consider alternative in this case. Some were using a short cut to even consider the use of sigma formula from GC, although it was not intended to be solved that way.
	Able to interpret the meaning of sum to infinity in contexts.	**(iii)**	In the long run, as $n \to \infty$, $S_\infty = \dfrac{50000}{1-0.94}$ $= 833,333.33$ The airport would not be able to collect the fund for expansion project as the amount received will never reach \$1, 000, 000.	M1 A1 B1	Most of the candidates knew what to do, although they can improve on their explanation more clearly.

4. Topic: Application of Differentiation, Definite Integrals

Skill / Concept Assessed	Solution	Feedback
Derive the volume of revolution generated by a specified region about an axis : • By constructing a definite integral representing the volume of solid of revolution, • And evaluating the value of a definite integral $\int_a^b \mathrm{f}(y)\,\mathrm{d}y$.	**(i)** Volume of required solid of revolution generated by rotating about the y-axis, $V = \int_0^h \pi x^2 \,\mathrm{d}y$. $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{h}\right)^2 = 1$ $\Rightarrow x^2 = r^2\left(1 - \frac{y^2}{h^2}\right)$ $V = \int_0^h \pi r^2\left(1 - \frac{y^2}{h^2}\right) \mathrm{d}y$ $= \pi r^2\left[y - \frac{y^3}{3h^2}\right]_0^h$ $= \pi r^2\left[\left(h - \frac{h^3}{3h^2}\right) - 0\right]$ $= \frac{2}{3}\pi r^2 h$ (shown)	Overall a majority of candidates (80 – 85 %) attempted at least one part to question 4. About 60 – 70 % of candidates who attempted part (i) obtained partial credit. Notable sources of error include : • Integrating erroneously w.r.t. x using $V = \int_0^r \pi y^2 \,\mathrm{d}x$, producing $\frac{2}{3}\pi r h^2$ ✗. • Writing $x = r\left(1 - \frac{y}{h}\right)$ from $x^2 = r^2\left(1 - \frac{y^2}{h^2}\right)$ ✗, wherein erroneous "formula" $\sqrt{A^2 - B^2} = A - B$ or $\sqrt{P - Q} = \sqrt{P} - \sqrt{Q}$ was applied. In responses where the correct definite integral is formed for the required volume, all intermediate steps leading to the required expression should be included for full credit.

	Derive an equation of the <u>tangent</u> to a parametrically defined curve at a given point : • By finding the <u>tangent's</u> <u>gradient</u> = derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ of a parametrically defined curve at that point, • And constructing a straight line equation for the tangent passing through the point with this gradient.	**(ii)**	Given $x = r\cos\theta$, $y = h\sin\theta$, where $0 \le \theta \le \pi$. $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -r\sin\theta$, $\frac{\mathrm{d}y}{\mathrm{d}\theta} = h\cos\theta$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \Big/ \frac{\mathrm{d}y}{\mathrm{d}\theta}$ $= \frac{h\cos\theta}{-r\sin\theta} = -\frac{h}{r}\cot\theta$ At point $P(r\cos\alpha,\ h\sin\alpha)$, $\theta = \alpha$ gradient of tangent = $-\frac{h}{r}\cot\alpha$. Equation of the tangent to the curve at P : $y - h\sin\alpha = -\frac{h}{r}\cot\alpha\ (x - r\cos\alpha)$, $y = -\left(\frac{h}{r}\cot\alpha\right)x + h\cot\alpha\cos\alpha + h\sin\alpha$, $y = -\left(\frac{h}{r}\cot\alpha\right)x + h\left(\frac{(\cos\alpha)^2}{\sin\alpha} + \sin\alpha\right)$, $y = -\left(\frac{h}{r}\cot\alpha\right)x + h\left(\frac{(\cos\alpha)^2 + (\sin\alpha)^2}{\sin\alpha}\right)$, $y = -\left(\frac{h}{r}\cot\alpha\right)x + h\ \mathrm{cosec}\alpha$. (shown) <u>Alternatively</u>, Equation of the tangent to the curve at P : $y = -\frac{h}{r}\cot\alpha\ x + C$, for some constant C $\because$ $P(r\cos\alpha,\ h\sin\alpha)$ is on this tangent line, $h\sin\alpha = -\frac{h}{r}\cot\alpha\ (r\cos\alpha) + C$ $C = h\cot\alpha\cos\alpha + h\sin\alpha$ $= h\left(\frac{\cos^2\alpha}{\sin\alpha} + \sin\alpha\right) = h\left(\frac{\cos^2\alpha + \sin^2\alpha}{\sin\alpha}\right)$ $= h\left(\frac{1}{\sin\alpha}\right) = h\ \mathrm{cosec}\alpha$ $\therefore$ $y = -\left(\frac{h}{r}\cot\alpha\right)x + h\ \mathrm{cosec}\alpha$. (shown)		Generally well-answered with ≥ 60% of responses to this part obtaining the full credit, and ≥ 90% of responses obtaining at least half of the credit. Notable common errors to be rectified : • Slips in algebraic manipulation [including sign errors etc., e.g. $y = -\left(\frac{h}{r}\cot\alpha\right)x - h\cot\alpha\cos\alpha + h\sin\alpha$] • Omitting intermediate steps in deriving the required equation of the tangent line, which should be included for full credit.

	Recognise that a slanted edge is tangential to an ellipse, and obtain/deduce a relationship between two unknown variables, by comparing different cartesian equations of the tangent line. Use appropriate trigonometric identity to facilitate algebraic manipulation / elimination of a parameter.	**(iii)** [HOT]	When the hemi-ellipsoid is inscribed within the cone, there is a point on the ellipse whose tangent line matches that of the cone's slanted edge. i.e. General equation of tangent line * $y = -\left(\frac{h}{r}\cot\alpha\right)x + h\ \mathrm{cosec}\alpha$ matches $y = -\frac{5}{3}x + 5$, for some α. Comparing coefficients produces $\frac{h}{r}\cot\alpha = \frac{5}{3}$ and $h\ \mathrm{cosec}\alpha = 5$ $\Rightarrow\ \cot\alpha = \frac{5r}{3h}$ and $\mathrm{cosec}\alpha = \frac{5}{h}$ [Obtain relationship between variables *r* and *h*, by eliminating parameter α.] Since $1 + (\cot\alpha)^2 = (\mathrm{cosec}\alpha)^2$, $1 + (\frac{5r}{3h})^2 = (\frac{5}{h})^2$ Multiplying by $(\frac{h}{5})^2$ on both sides produces $\left(\frac{h}{5}\right)^2 + \left(\frac{r}{3}\right)^2 = 1$ (shown) <u>Alternative method</u> Substituting (3,0) and (0,5) into the general eqn. of tangent line produces : $0 = -\left(\frac{h}{r}\cot\alpha\right)(3) + h\ \mathrm{cosec}\alpha \Rightarrow 3\frac{h}{r}\frac{\cos\alpha}{\sin\alpha} = \frac{h}{\sin\alpha}$ $\Rightarrow \cos\alpha = \frac{r}{3}$, and $5 = -\cancel{\left(\frac{h}{r}\cot\alpha\right)}(0) + h\ \mathrm{cosec}\alpha \Rightarrow \sin\alpha = \frac{h}{5}$. Since $(\sin\alpha)^2 + (\cos\alpha)^2 = 1$, $\left(\frac{h}{5}\right)^2 + \left(\frac{r}{3}\right)^2 = 1$ (shown).		$\leq$ 40% of candidates could obtain partial credit for this question part. About half of the candidates who obtained no credit actually left this part unattempted. A significant portion of responses that obtained no credit mistakenly attempted to identify / look for similar Δs which are **not** actually present, erroneously quoting $\frac{h}{5} = \frac{r}{3}$ ✘. Many returned to the equation for the ellipse, which is no longer relevant for this qn. part [e.g. sub. (3,0), (0,5) into the $(\frac{x}{r})^2 + (\frac{y}{h})^2 = 1$, eqn. for the ellipse]. Amongst responses given partial credit, many **correctly used the general equation * for a tangent line** obtained in (ii), extracting equations relating *h*, *r*, α from comparing coefficients / substituting (3,0) and (0,5) into *, but **omitted** further steps involving the elimination of parameter α via a suitable trigo. identity.

Solve a problem involving the optimization (in particular maximization) of a quantity, subject to constraints in context :

- By expressing the quantity to be maximised in terms of just a **single** variable,
- Finding the point(s) at which the required quantity is stationary,
- Selecting the correct point at which the quantity is maximised,
- Verifying the nature of the point at which the designated quantity is maximised, by applying a derivative test ($1^{st}/2^{nd}$ derivative test),

(iv) Hemi-ellipsoid volume $V=\frac{2}{3}\pi r^2 h$.

$$\left(\frac{h}{5}\right)^2+\left(\frac{r}{3}\right)^2=1 \qquad \Rightarrow \quad r^2=3^2\left(1-\frac{h^2}{5^2}\right)=9\left(1-\frac{h^2}{25}\right)$$

$$\therefore\ V=\frac{2}{3}\pi\times 9\left(1-\frac{h^2}{25}\right)h$$

$$=6\pi\left(h-\frac{h^3}{25}\right).$$

When volume V is maximum, it is also stationary w.r.t. h.

$$\therefore\ \frac{\mathrm{d}V}{\mathrm{d}h}=6\pi\left(1-\frac{3h^2}{25}\right)=0.$$

$$\Rightarrow\ h^2=\frac{25}{3},\quad h=\frac{5}{\sqrt{3}}\quad (\because h>0).$$

$$\therefore\ \frac{\mathrm{d}^2V}{\mathrm{d}h^2}=6\pi\left(-\frac{6h}{25}\right)$$

$$=-\frac{36}{25}\pi h$$

At $h=\frac{5}{\sqrt{3}}$, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=-\frac{36}{25}\pi\left(\frac{5}{\sqrt{3}}\right)<0$.

$\therefore$ By the 2^{nd} derivative test, volume V attains a maximum at $h=\frac{5}{\sqrt{3}}$.

Alternatively, use the 1^{st} derivative test :

h	$\left(\frac{5}{\sqrt{3}}\right)^-$	$\frac{5}{\sqrt{3}}$	$\left(\frac{5}{\sqrt{3}}\right)^+$
$\frac{\mathrm{d}V}{\mathrm{d}h}$	>0	0	<0
Slope			

$\therefore$ By the 1^{st} derivative test, volume V attains a maximum at $h=\frac{5}{\sqrt{3}}$.

This part is moderately well-attempted, with $\geq 70\%$ of responses obtaining partial credit.

Responses that failed to obtain any credit have foremost not expressed volume V properly in terms of just a single variable h.

Notable errors from responses include :

- Differentiating $V=\frac{2}{3}\pi r^2 h$, $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{2}{3}\pi r^2$ ✗, erroneously treating r as a constant when it varies with h.
- Writing $\frac{\mathrm{d}r}{\mathrm{d}h}=0$ ✗ for maximising V (instead $\frac{\mathrm{d}V}{\mathrm{d}h}=0$ ✓)
- Quoting / using $\frac{r}{h}=\frac{3}{5}$ ✗.
- Algebraic slips/errors
- Omission of 1^{st} / 2^{nd} derivative test for verifying that V indeed attains a maximum at the value of h at which $\frac{\mathrm{d}V}{\mathrm{d}h}=0$.

Section B: Probability and Statistics [60 marks]

5. Binomial Distribution

Assessment Objectives	Solution			Feedback
Able to state assumptions for binomial distribution to hold.	**(i)**	The probability, $\frac{1}{p}$, of a shopper finding a mystery item is a constant for all shoppers. Or The event where a shopper finds a mystery item is independent of all other shoppers finding a mystery item.		Many candidates gave "Probability of a shopper finding a mystery item is independent of another shopper…" which is incorrect. It the outcome that is independent, not the probability/ chance which is a value/ number. A value/ number cannot be independent. A handful of the candidates did not explain in the context of the question. Some candidates gave two assumptions, actually one will suffice. Strongly recommend that all students use the exact phrasing rather than in their own words ie "Probability is equal, same…"
Able to solve for p using the probability formula for binomial distribution.	**(ii)**	Let X be the random variable denoting the number of shoppers who found a mystery item out of 10 shoppers. Then $X \sim \mathrm{B}(10, \frac{1}{p})$ Given $\mathrm{P}(X=3)=\frac{15}{4}(\sqrt{2}-1)^7$ $^{10}C_3\left(\frac{1}{p}\right)^3\left(1-\frac{1}{p}\right)^7=\frac{15}{4}(\sqrt{2}-1)^7$ $\frac{120}{p^{10}}(p-1)^7=\frac{15}{4}(\sqrt{2}-1)^7$ By observation, $\frac{120}{p^{10}}=\frac{15}{4}\Rightarrow p=\sqrt{2}$ and $(p-1)^7=(\sqrt{2}-1)^7\Rightarrow p=\sqrt{2}$ Hence, $p=\sqrt{2}$		Some candidates did not present and define the random variable. They are strongly recommended to define even though they may jump straight to solve for p. Misconceptions include: • Mix up the understanding of $\mathrm{P}(X=3)$ and $\mathrm{P}(X\le 3)$ (adding up all the terms up to x=3). • Substitute p instead of $\frac{1}{p}$ into the binomial formula. • Comparing terms like that: $120\left(\frac{1}{p}\right)^3=\frac{15}{4}$ & $\left(1-\frac{1}{p}\right)^7=(\sqrt{2}-1)^7$ which does not work for this case.

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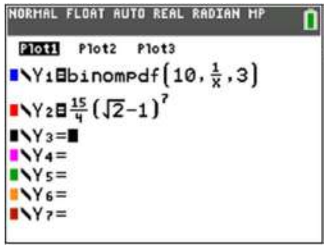
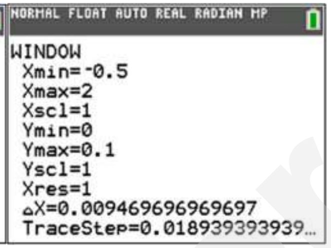
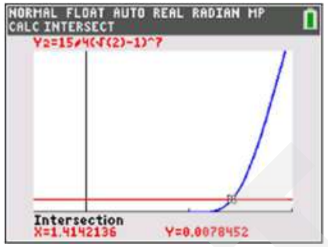
Alternative method

$P(X=3)=\frac{15}{4}(\sqrt{2}-1)^7$







Since $1<p<2$, hence p = 1.41 (to 3 s.f.)

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Some candidates failed to recognize that GC is able to solve for p and did not continue.

A few solve for p using table of values which is not a wise method because p is a value of probability which is a decimal, not an integer.

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<td>Able to solve for unknown number of trials for binomial distribution.</td>
<td>(iii)</td>
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Let Y be the random variable denoting the number of games where three shoppers found a mystery item out of 10 shoppers out of n games.

$Y \sim B(n, \frac{15}{4}(\sqrt{2}-1)^7)$

$P(Y>2)=1-P(Y\le 2)>0.01$

Using GC,

n	$1-P(Y\le 2)$
56	0.0098
57	0.0103
58	0.0108

Hence the least value of n is 57.

OR

$P(Y>2)=1-P(Y\le 2)>0.01$

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Again, some candidates did not present and define the random variable properly. While other candidates, misuse the value of probability to be $\sqrt{2}$ instead of $\frac{15}{4}(\sqrt{2}-1)^7$.

Misconceptions:

- Writing $P(Y>2)=1-P(Y\le 1)$ instead of $P(Y>2)=1-P(Y\le 2)$
- Comparing to other numbers instead of 0.01 or 0.99.

Some candidates did not show the table of values despite using that method, hence did not

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		$P(Y \le 2) < 0.99$ Using GC, <table><tr><td>n</td><td>$P(Y \le 2)$</td></tr><tr><td>56</td><td>0.9902</td></tr><tr><td>57</td><td>0.9897</td></tr><tr><td>58</td><td>0.9892</td></tr></table>Hence the least value of n is 57.		receive the full credit.

6	Discrete Random Variable		
	Assessment Objectives	**Solution**	**Feedback**
	Able to formulate probability distribution table from a given context.	**(i)** X = number of tails – number of heads in one throw. Using probability to obtain the p.d. table $P(X=-5)=P(0\text{ tail}-5\text{ heads})=\left(\frac{1}{2}\right)^5=\frac{1}{32}$ $P(X=-3)=P(1\text{ tail}-4\text{ heads})=\left(\frac{1}{2}\right)^1\times\left(\frac{1}{2}\right)^4\times\frac{5!}{4!}=\frac{5}{32}$ $P(X=-1)=P(2\text{ tails}-3\text{ heads})=\left(\frac{1}{2}\right)^2\times\left(\frac{1}{2}\right)^3\times\frac{5!}{2!3!}=\frac{10}{32}$ $P(X=1)=P(3\text{ tails}-2\text{ heads})=\left(\frac{1}{2}\right)^3\times\left(\frac{1}{2}\right)^2\times\frac{5!}{3!2!}=\frac{10}{32}$ $P(X=3)=P(4\text{ tails}-1\text{ head})=\left(\frac{1}{2}\right)^4\times\left(\frac{1}{2}\right)^1\times\frac{5!}{4!}=\frac{5}{32}$ $P(X=5)=P(5\text{ tails}-0\text{ head})=\left(\frac{1}{2}\right)^5=\frac{1}{32}$	About half the cohort could solve this question. Those who could not did not spend the time to understand the situation. Most students used the alternative method. Once they could figure out P(X=1), the rest of the probabilities were easy to derive. Even those who could not get the probabilities correct were able to find the values for X.

	0T -5H	1T -4H	2T -3H	3T -2H	4T -1H	5T -0H
$X=x$	-5	-3	-1	1	3	5
$P(X=x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

Alternative Method

Taking $X\sim B(5,\frac{1}{2})$

$P(X=1)=\frac{5}{16}$

<table>
<tr><td></td><td></td><td></td><td><table><tr><td></td><td>0T -5H</td><td>1T -4H</td><td>2T -3H</td><td>3T -2H</td><td>4T -1H</td><td>5T -0H</td></tr><tr><td>$X = x$</td><td>-5</td><td>-3</td><td>-1</td><td>1</td><td>3</td><td>5</td></tr><tr><td>$\mathrm{P}(X = x)$</td><td>$\frac{1}{32}$</td><td>$\frac{5}{32}$</td><td>$\frac{10}{32}$</td><td>$\frac{10}{32}$</td><td>$\frac{5}{32}$</td><td>$\frac{1}{32}$</td></tr></table></td><td></td><td></td></tr>
<tr><td></td><td>Able to state $\mathrm{E}(X)$ by using property of symmetry.
Able to find $\mathrm{Var}(X)$ from p.d. table.</td><td>(ii)</td><td>By symmetry, $\mathrm{E}(X) = 0$

Using p.d. table
$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$
$= \frac{(-5)^2 + (-3)^2(5) + (-1)^2(10) + 1^2(10) + 3^2(5) + 5^2}{32} - [0]^2$

$= \frac{160}{32} = 5$</td><td></td><td>Those who got the probability table correct had no problems with this part.</td></tr>
<tr><td></td><td>Able to solve contextualized problems using p.d. table</td><td>(iii)</td><td>Using the table in (i)<table><tr><td>$|X| = |x|$</td><td>1</td><td>3</td><td>5</td></tr><tr><td>Winnings $</td><td>0</td><td>2</td><td>k</td></tr><tr><td>$\mathrm{P}(X = x)$</td><td>$\frac{20}{32}$</td><td>$\frac{10}{32}$</td><td>$\frac{2}{32}$</td></tr></table>
Expected winnings
$= 0 \times \frac{20}{32} + 2 \times \frac{10}{32} + k \times \frac{2}{32}$
$= \frac{20 + 2k}{32} = \frac{10 + k}{16}$
Expected profit $\frac{10 + k}{16} - 1 = 10$
$k = 11 \times 16 - 10 = 166$</td><td></td><td>Many students did not know how to handle the amount received being based on the difference in the number of tails and heads. Many still used the table from (i) and balanced the other probabilities by subtracting from 1.

Those who got the probabilities correct did not account for the $1 payment.</td></tr>
</table>

		Alternative Method (table below) Expected winnings $= (-1)\times\frac{20}{32}+1\times\frac{10}{32}+(k-1)\times\frac{2}{32}$ $=\frac{1}{16}k-\frac{3}{8}$ So, $\frac{1}{16}k-\frac{3}{8}=10$ $k-6=160$ $k=166$		

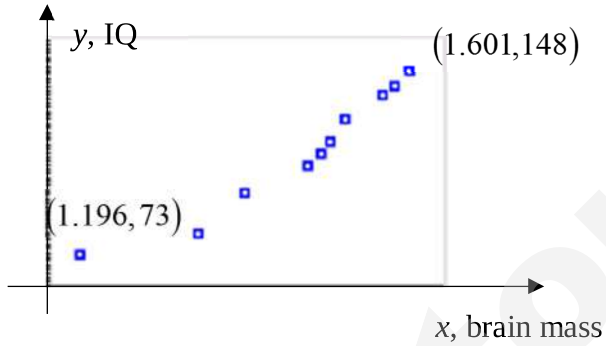
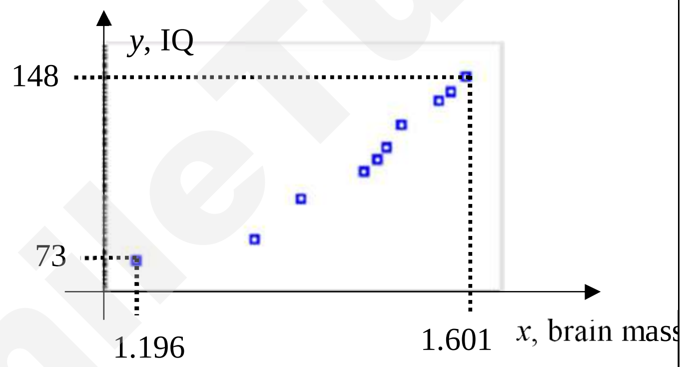
Winnings $	-1	1	*k* - 1
P(*X* = *x*)	$\frac{20}{32}$	$\frac{10}{32}$	$\frac{2}{32}$

7	Permutation and Combination + Probability			
	Assessment Objectives	**Solution**		**Feedback**
	Able to find conditional probability using Venn diagram.	**(a)** Given $P(B)=\frac{1}{3}$. $P(A\cap B)=\frac{1}{5}$ and $P(A'\cap B')=\frac{1}{6}$ $P(A\mid B')$ $=\dfrac{P(A\cap B')}{P(B')}$ $=\dfrac{P(A\cup B)-P(B)}{1-\frac{1}{3}}$ $=\dfrac{1-P(A'\cap B')-\frac{1}{3}}{\frac{2}{3}}$ $=\dfrac{1-\frac{1}{6}-\frac{1}{3}}{\frac{2}{3}}$ $=\frac{3}{4}$		Many students got this part correct. Those who did not were not able to use the given information and a Venn diagram to obtain $P(A\cap B')$ correctly.
		(b) 1, 2, 2, 3, 4, 5, 5 Note that there are three distinct digits and two repeated "2" and two repeated "5"		
	Able to consider complement or use stages to count number of ways.	**(bi)** <u>Method 1: Using complement</u> No. of digits = all possible – two "5"s together $=\dfrac{7!}{2!2!}-\dfrac{6!}{2!}$ $=900$		Surprisingly, quite a number of candidates considered the "5"s as distinguishable "5". As a result, they over-counted the number.
		<u>Method 2: Slotting in method</u> X X X_X X _ _ _ _ _ _ ↑ ↑ ↑ ↑ ↑ ↑		

		No. of ways to arrange three distinct digits and two “2”s $= \frac{5!}{2!} = 60$ No. of ways to choose 2 slots for the two “5”s $= {}^6C_2 = 15$ So, total number $= 60 \times 15 = 900$		
	Able to separate into cases and count number of ways.	**(bii)** $\boxed{\text{“5” \_ \_ \_ “5”}}$ _ _ Case 1: If the two “2”s are between the two “5”s = select one other digit to put in between the two “5” × no. of ways to arrange the two “2”s and one other digit between the two “5”s × no. of ways to arrange one block and 2 distinct digits $= {}^3C_1 \times \frac{3!}{2!} \times 3! = 54$ Case 2: If one “2”s is between the two “5”s = select two digits to put in between the two “5” × no. of ways to arrange one “2”s and two other digits between the two “5”s × no. of ways to arrange one block and 2 distinct digits $= {}^3C_2 \times 3! \times 3! = 108$ Case 3: If no “2”s is between the two “5”s = select three digits to put in between the two “5”s × no. of ways to arrange 3 distinct digits between two “5”s × no. of ways to arrange one block and two “2”s $= {}^3C_3 \times 3! \times \frac{3!}{2!} = 18$ So, total = 54 + 108 + 18 = 180		This part caused more problems as compared to part (b)(i). Many omitted one of the cases, for example, 1 “2” inside the “5”s block and 1 “2” outside the block.
		<u>Alternative solution</u> No. of ways to select and arrange three digits to put between two “5” = ${}^5P_3$ No. of ways to arrange one block of 5 digits and the		

		remaining 2 individual digits = 3! To remove duplication from two "2"s = $\frac{1}{2!}$ Total no. of ways $=\frac{{}^5P_3 \times 3!}{2!}=180$		
Able to count number of ways with restrictions on start and end positions.	**(biii)**	1 _ _ _ _ _ _ _ _ _ _ ↑ {3, 5} Seven-digit number must start with "1" and end with digits 3 or 5 Case 1: Seven-digit number ends with "3" = arrange the remaining digits with repeated "5" and "2" $=\frac{5!}{2!2!}=30$ Case 2: Seven-digit number ends with "5" = arrange the remaining digits with repeated "2" $=\frac{5!}{2!}=60$ Total no. of ways $=30+60=90$		Many candidates did not read the question carefully. These candidates did not realise that the first number has to be "1" only. They considered "2" as the other case. Another common error was they did not consider the indistinguishable "2"s and "5", so many of such candidates tend to over-count the number.
		Alternative solution Number must start with "1" = 1 No. of ways to select the last odd digit out of 3 choices i.e. {3, 5, 5} = ${}^3C_1$ No. of ways to arrange the remaining 5 digits = 5! To remove double counting from two "2" and two "5" = $\frac{1}{2!2!}$ Total no. of ways $=\frac{1\times 5! \times {}^3C_1}{2!2!}$ = 90		

8 Correlation and Regression

	Assessment Objectives	Solution	Feedback
	Able to plot a scatter diagram	**(i)**  OR 	This part was generally well done but some candidates did not label the coordinate of the start and endpoints. Many candidates used the graph paper to draw but it is not required by the question, unless "plot on a graph paper is mentioned." Even if they draw to the scale, they should also label the coordinates of the start and end points. They must always draw dotted lines and label the start and end values for both axes if they are drawing this way. Unsure why some candidates think that $(1.196, 73)$ is an outlier. They should look at the general trend as a big picture. A handful of the candidates swopped the *x* and *y* axes without changing the shape of the graph, that is correct.
	Able to find product moment correlation coefficient for two variables as well as transformed variables using GC	**(ii)(a)** Between *x* and *y*: $r = 0.9645001408 \approx 0.965$ (3 s.f) **(b)** Between x^2 and *y*: $r = 0.9745220219 \approx 0.975$ (3 s.f.)	This part is generally well done. However, some candidates need to check their values on the GC as their values are incorrect. Some did not round up to 3 sig fig, and thus, affecting their accuracy mark.
	Able to decide the suitable model with explanation using product moment correlation coefficient or scatter diagram	**(iii)** From **(i)**, since as *x* increases, *y* **increases** at a increasing rate, the points on the scatter diagram take the shape of the graph of $y = c + dx^2$.	Majority of the candidates did not read the question carefully, hence did not refer to scatter diagram in (i) to explain which is the suitable model.

	Able to find least squares regression line using GC for transformed variable.	And From **(ii)**, the product moment correlation coefficient between x and y^2 is closer to 1, as compared to that between x and y, $\therefore$ the model $y = c + dx^2$ is the better model. $y = -36.5807 + 69.911x^2$ $y = -36.6 + 69.9x^2$		When using the r-value, candidates need to be mindful of which is **"closer to 1"** rather than saying "which is greater". A handful of the candidates give the equation of regression line as: $y = -36.6 + 70.0x^2$ (Round off wrongly) Or $y = -36.6 + 69.9x$ Candidates are also strongly advised to leave the equation in 5 sf rather than rounding to 3 sf straight away, which affects the answer for (iv) slightly.
	Able to estimate value of y given x using least squares regression line. Able to comment reliability of estimated value using interpolation.	**(iv)** $y = -36.5807 + 69.911(1.5)^2$ $= 120.71905$ ≈ 121 Since the value of 1.5 is within the range of values of x and the value of r is close to 1, this estimate is reliable.		Use the equation with 5 sf to calculate. A handful of candidates substituted 1500 instead of 1.5, read the units carefully! For the estimate to be reliable, always refer to which is given. In this case, the value of x is given so use the range of x values to explain reliability, not the calculated y. Majority of the candidates leave out a very important part when explaining reliability, which is "r-value is close to 1". Candidates must always remember that in order to explain reliability, we need BOTH interpolation and r-value to explain. Majority of the candidates wrote interpolation as "intrapolation" and this is incorrect!
	Able to reason that causation does not lead to correlation in the context of the question.	**(v)** This statement may not be valid as strong positive linear correlation between brain mass and IQ as shown by the data does not imply causation. There are other factors affecting intelligence quotient other than brain mass as well.		Candidates need to give the keyword in order to get credit. In this case, they should mention the word "cause", or "causation" and the idea here is to say that correlation does not imply causation.

9	Normal Distribution			
	Assessment Objectives	**Solution**		**Feedback**
	Able to solve for unknown population mean μ and standard deviation σ.	**(i)** Let M and F be the random variable the height of a random female and male respectively. Then $M \sim \text{N}(175, 10^2)$ and $F \sim \text{N}(\mu, \sigma^2)$		
		Given $\text{P}(F < 143) = 0.01114$ and $\text{P}(F < 183) = 0.98886$ Since $\text{P}(F > 183) = 1 - 0.98886 = 0.01114$ Hence by symmetry, $\mu = \frac{183+143}{2} = 163$ Using $\text{P}(F < 143) = 0.01114$ $\frac{143-163}{\sigma} = -2.285560213$ $\sigma = 8.750589849 \approx 8.75$ (shown)		Most were able to get this part correct either by symmetry or formulation of two equations. A small minority could not get invNorm value correctly
	Able to combine normal random variables and calculate probabilities correctly	**(ii)** $F_1 + F_2 - 2M \sim \text{N}(2\times163 - 2\times175, 2\times8.75^2 + 2^2\times10^2)$ $F_1 + F_2 - 2M \sim \text{N}(-24,\ 553.125)$ $\text{P}(\lvert F_1 + F_2 - 2M \rvert > 30)$ $= 1 - \text{P}(-30 < F_1 + F_2 - 2M < 30)$ $= 0.4101527381$ ≈ 0.410		Many students did not take note of the word 'differ' in the question. Those who did consider modulus, many did not know how to break the modulus correctly to obtain the correct probability. Common mistakes: - error in variance $2^2\times8.75^2 + 2\times10^2$ or $2^2\times8.75^2 + 2^2\times10^2$ - $\text{P}(\lvert F_1 + F_2 - 2M \rvert > 30)$ $= \text{P}(-30 < F_1 + F_2 - 2M < 30)$ - $\text{P}(F_1 + F_2 - 2M > 30)$ $= 1 - \text{P}(F_1 + F_2 - 2M \le 29)$ - $\text{P}(\lvert F_1 + F_2 - 2M \rvert > 30)$ $= \text{P}(F_1 + F_2 - 2M < 30) \times \text{P}(F_1 + F_2 - 2M > 30)$ - $\text{P}(\lvert F_1 + F_2 - 2M \rvert > 30)$ $= \text{P}(F_1 + F_2 - 2M < 30) + \text{P}(F_1 + F_2 - 2M > 30)$ - $\text{P}(\lvert F_1 + F_2 - 2M \rvert > 30)$

				$=2\mathrm{P}(F_1+F_2-2M>30)$ (i.e. assumed symmetry even though mean is not at zero)
	Use of PnC and probability in combination with normal distributions.	**(iii)** Prob $=\mathrm{P}(155<F<185)\times[1-\mathrm{P}(155<F<185)]\times 2\times\mathrm{P}(160<M<192.5)$ $= 0.2707250364$ ≈ 0.271		Many students could string the probabilities together but did not consider the permutation of the two women. A minority did not know the difference between addition and multiplication principles and added the probabilities with some even leaving the answer as more than 1.
	Use of adding a constant to a normal variable and sampling distribution.	**(iv)** Let X be the random variable denoting height of a random female candidate wearing standard shoes. So $X=F+5$ Since F is normally distributed, then $\bar{X}\sim \mathrm{N}(163+5,\dfrac{8.75^2}{15})$ $\mathrm{P}(\bar{X}>170)=0.1880099665\approx 0.188$		Many could not solve this part correctly as they did not know how to handle the variance for the average height of female candidates. Many chose to ignore the 'average' part of the question and solved it as though there is only one random female wearing heels.
		Alternative solution 1 $X=F+5\Rightarrow \bar{X}=\bar{F}+5$ $\bar{F}\sim \mathrm{N}(163,\dfrac{8.75^2}{15})$ $\mathrm{P}(\bar{X}>170)=\mathrm{P}(\bar{F}>165)=0.1880099665\approx 0.188$ **Alternative solution 2** $F_1+F_2+...+F_{15}\sim \mathrm{N}(15\times 163, 15\times 8.750589849^2)$ $F_1+F_2+...+F_{15}\sim \mathrm{N}(2445, 1148.592341)$ $P\left(\dfrac{F_1+F_2+...+F_{15}+15\times 5}{15}>170\right)$ $=\mathrm{P}\left(F_1+F_2+...+F_{15}>2475\right)$		

		$= 0.1880260552 \approx 0.188$ **<u>Alternative solution 3</u>** $F_1 + F_2 + ... + F_{15} + 15\times 5$ $\sim \mathrm{N}(15\times 163 + 75,\ 15\times 8.750589849^2)$ $F_1 + F_2 + ... + F_{15} + 75 \sim \mathrm{N}(2520,\ 1148.592341)$ $P\left(F_1 + F_2 + ... + F_{15} + 15\times 5 > 170\times 15\right)$ $= \mathrm{P}\left(F_1 + F_2 + ... + F_{15} + 75 > 2550\right)$ $= 0.1880260552 \approx 0.188$		

10	Hypothesis Testing			
	Assessment Objectives	**Solution**		**Feedback**
	Able to find unbiased estimates of population mean and variance.	**(i)** $\hat{\mu} = \frac{\sum x}{n} = \frac{132.00}{30} = 4.4 \text{ (exact)}$ $s^2 = \frac{1}{n-1}\left[\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right]$ $s^2 = \frac{1}{29}\left[583.96 - \frac{(132.00)^2}{30}\right]$ $= 0.1089655172$ ≈ 0.109		Majority of students scored full marks. A handful of students calculated s^2 erroneously because they substituted 4.4^2 instead of 132.00^2.
	Able to carry out two-tailed test.	**(ii)** $H_0 : \mu = 4.5$ mins, the mean time taken is 4.5 mins $H_1 : \mu \neq 4.5$ mins, the mean time taken is not 4.5 mins Under H_0, since n = 30 is large enough, by Central Limit Theorem $\bar{X} \sim N(4.5, \frac{0.1089655172}{30})$ approximately At 5% level, reject H_0 if p-value < 0.05 Test-statistic, $z = \frac{4.4 - 4.5}{\sqrt{\frac{0.1089655172}{30}}}$ Using GC, z_{test} = - 1.659266271 p-value = 0.0970621091 < 0.10 Hence reject H_0. Alternatively, Since z_{test} = - 1.659266271 < - 1.64485 Hence, reject H_0. Conclude at 10% level that there is sufficient evidence that the mean time taken has changed from 4.5 minutes.		Majority of students scored 3 marks or more. Common errors include: 1. $\bar{X} \sim N(4.4, \frac{0.1089655172}{30})$ 2. p-value = 0.0970621091 > 0.5 (halving significance level unnecessarily) 3. p-value = 0.0485310546 (conducting one tail test instead of two) 4. p-value = 5.0014×10^{-7} (neglecting to take square root of s^2)

	Able to apply the concept of p-value and significance level.	**(iii)**	Since p-value = 0.0970621091, To not reject H_0, $\text{p-value} > \dfrac{\alpha}{100}$ So, $0.0970621091 > \dfrac{\alpha}{100}$ Hence $0 < \alpha < 9.71$		Many did not attempt or formed inequality based on z critical values instead. Of candidates who attempted, mistakes include 1. $\alpha < 9.71$ (neglecting lower bound) 2. $\alpha < 0.0971$ (should be $\alpha\% < 0.0971$) 3. $9.71 < \alpha < 100$ (stating condition for **rejecting H_0** instead)
	Able to change p-value from two-tail to one-tail test.	**(iv)**	In this case, $H_1 : \mu < 4.5$ left-tail test. Using (ii), $\text{p-value} = \dfrac{0.0970621091}{2} = 0.0485310546 < 0.05$ So reject H_0, conclude at 5% level that there is sufficient evidence that the mean taken to clear a table is less than 4.5 minutes.		Many reasoned erroneously that since (ii) is a two-tail test, it is inconclusive as mean time could be less or more than 4.5 Another common error is p-value = 0.0970621091 > 0.5 (neglecting to halve p- value for a one-tail test) Alternative method of comparing z statistic and z critical values is accepted.
	Able to find $\overline{x}$ when given the conclusion of the test.	**(v)**	$s^2 = \dfrac{n}{n-1}(\text{sample variance})$ $= \dfrac{30}{29}(1.01) \approx 1.044827586$ $H_0 : \mu = 4.5$ $H_1 : \mu > 4.5$ Under H_0, since n = 40 is large enough, by Central Limit Theorem $\overline{X} \sim N(4.5, \dfrac{1.01}{29})$ approximately Test-statistic, $z = \dfrac{\overline{x} - 4.5}{\sqrt{\dfrac{1.01}{29}}}$		Majority of students scored 2 marks for this part. Errors include: 1. Substituting $s^2 = 1.01$ (mixing up concepts of sample variance, unbiased estimate of population variance and population variance) 2. Substituting $\mu = \overline{x}$ (confusing sample mean with population mean) 3. Substituting $s^2 = 1.01^2$ (confusing standard deviation with variance) 4. $z = \dfrac{4.5 - \overline{x}}{\sqrt{\dfrac{s^2}{n}}}$ (wrong formula) 5. $z_{test} < 2.326347877$ (left-tail test instead of right)

		To reject H_0 at 1% level for right-tailed test $\Rightarrow z_{test} > 2.326347877$ $\Rightarrow \dfrac{\overline{x}-4.5}{\sqrt{\dfrac{1.01}{29}}} > 2.326347877$ $\Rightarrow \overline{x} > 4.934146542$ $\Rightarrow \overline{x} > 4.934$ Hence, $\{\overline{x} \in R : \overline{x} > 4.934 \text{ minutes}\}$		6. No set notation

Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2)

9758/01

Paper 1

September 2018

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Graph Paper

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	3	7	5	6	8	11	11	9	13	13	14	100

1 If x is a sufficiently small angle, find the first three non-zero terms in the Maclaurin series for $\sin^7(x+\frac{1}{4}\pi)$. [3]

2 **(i)** Using appropriate expansions from the List of Formulae (MF26), show that the series expansion of $\ln\left(1+e^{2x}\right)$ in ascending powers of x, up to and including the term in x^2, is

$\ln 2 + x + \dfrac{x^2}{2}$. [3]

(ii) Find the set of values of x such that the value for the above expansion for $\ln\left(1+e^{2x}\right)$ is within ± 0.3 of the value of $\ln\left(1+e^{2x}\right)$. [2]

(iii) Deduce an expansion for $\dfrac{2}{1+e^{-2x}}$ up to and including the term in x. [2]

3 If $x = 3\cos^2\theta + 6\sin^2\theta$, show that $6-x = 3\cos^2\theta$, and find a similar expression for $x-3$. By using the substitution $x = 3\cos^2\theta + 6\sin^2\theta$, evaluate exactly $\int_3^6 \dfrac{1}{\sqrt{(x-3)(6-x)}}\,dx$. [5]

4 Show that the following inequality

$$\tan x + \cot x > 4 \text{ for } 0 < x < \tfrac{1}{2}\pi$$

can be simplified to

$$0 < \sin 2x < \tfrac{1}{2}.$$ [4]

Hence solve exactly the inequality leaving your answer in terms of π. [2]

5 Functions f and g are defined by

$$\mathrm{f}: x \mapsto 1+\frac{2}{x-1}, \quad x \in \mathbb{R}, \; x<1,$$

$$\mathrm{g}: x \mapsto \ln x, \quad x \in \mathbb{R}, \; 0<x<1.$$

(i) Explain why the composite function gf does not exist. [1]

(ii) Find an expression for $\mathrm{fg}(x)$. Hence or otherwise, find $(\mathrm{fg})^{-1}(0)$. [4]

(iii) Find an expression for $\mathrm{h}(x)$ for each of the following cases:

(a) $\mathrm{gh}(x)=x$, [1]

(b) $\mathrm{hg}(x)=x^2+1$. [2]

6 **(a)** By using the substitution $y=zx^2$, find the general solution of the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}=2xy-y^2, \quad \text{where } x \neq 0.$$

[4]

(i) Sketch the solution curve that passes through $(2,-4)$, indicating any stationary points and asymptotes clearly. [4]

(ii) State the particular solution for which y has no turning point. [1]

(b) A differential equation is of the form $\frac{\mathrm{d}y}{\mathrm{d}x}+y=px+q$, where p and q are constants. Its general solution is $y=4x-1+D\mathrm{e}^{-x}$, where D is an arbitrary constant. Find the values of p and q. [2]

7 (a) Given that $z^* = \dfrac{(2\mathrm{i})^3}{\left(\sqrt{3}+\mathrm{i}\right)^4}$, find the exact value of $|z|$ and $\arg(z)$.

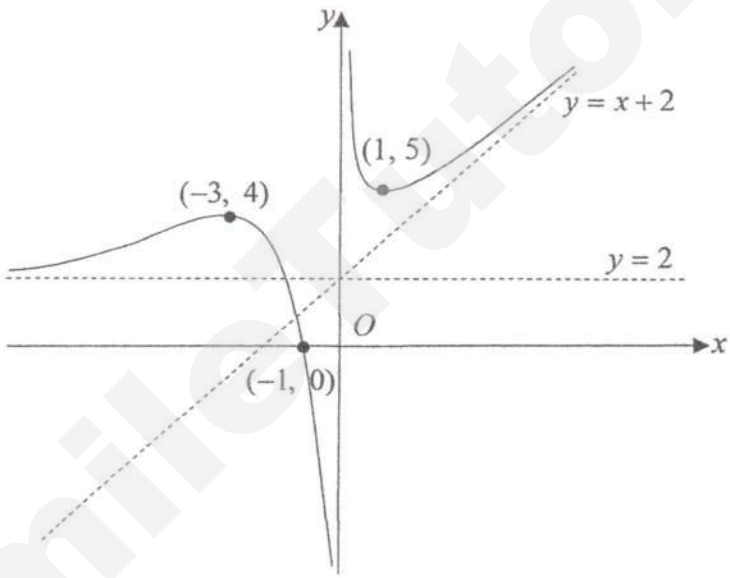
Hence state the smallest positive integer n such that z^n is purely imaginary. [5]

(b) When the polynomial $ax^4+bx^3+cx^2+24x-44$, where a, b and $c \in \mathbb{R}$, is divided by $(x-1)$, $(x+1)$ and $(x-2)$, the remainders are -18, -54 and 0 respectively.

(i) Find the values of a, b and c. [3]

(ii) The equation $ax^4+bx^3+cx^2+24x-44=0$, with the values of a, b and c found in part (i), has a root $3-(\sqrt{2})\mathrm{i}$. Find the other roots of the equation, showing your working clearly. [3]

8



The diagram shows the graph of $y=\mathrm{f}(x)$. The curve has turning points $(-3,\ 4)$ and $(1,\ 5)$ and crosses the x-axis at $(-1,\ 0)$. The curve has asymptotes $x=0, y=2$ and $y=x+2$.

Sketch, on separate diagrams, the graphs of

(i) $y=\mathrm{f}(|x|)$, [2]

(ii) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

including the coordinates of the points where the graphs cross the axes, the turning points and the equations of any asymptotes.

Describe a sequence of three transformations which transforms the graph of $y=\mathrm{f}(x)$ to the graph of $y=\mathrm{f}(ax+b)$, where a and b are constants such that $a<-1$ and $0<b<1$. State the coordinates of the point where the graph of $y=\mathrm{f}(ax+b)$ cuts the x-axis. [4]

9 The line l_1 passes through the point A with the position vector $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and is parallel to $\begin{pmatrix}t\\t^2+1\\3\end{pmatrix}$, while the cartesian equation of the plane p is given by $tx-2y+z=-3$, where t is a real constant. It is known that l_1 and p have no point in common.

(i) Show that $t=-1$. [3]

(ii) Find the distance between l_1 and p. [2]

(iii) The line l_2 has the cartesian equation $2y=z$, $x=3$. Show that l_2 lies on p. [2]

(iv) Given that point B and point C lie on l_1 and l_2 respectively, find $\overrightarrow{BC}$ such that it is perpendicular to both l_1 and l_2. [3]

(v) Find the vector equation of the line of reflection of l_1 in p. [3]

10 Duncan, an aspiring marathon runner, embarks on the following training regime:

1. On Day 1, he runs 5 km.
2. On Day 2, he runs $(1+\alpha)$ times the distance ran in the previous day.
3. On Day 3, he runs α times the distance ran in the previous day.

You may take α to be a positive constant.

For each subsequent day, Duncan repeats parts 2 and 3 of the training regime, in that order. Thus on Day 4, he runs $(1+\alpha)$ times the distance ran on Day 3; and on Day 5, he runs α times the distance ran on Day 4, and so on.

(i) The distances that Duncan runs on odd-numbered days follows a geometric progression. State its common ratio in terms of α. [1]

(ii) By considering just the distances that Duncan runs on even-numbered days, find the range of values of α such that the total distance ran on all even-numbered days is finite. [2]

(iii) By considering the total distances ran on both odd and even numbered days, determine the theoretical maximum total distance, expressing your answer in terms of α. [2]

Duncan decides to fix $\alpha = 0.65$.

(iv) **(a)** Show that the distance he ran on Day 10 is 10.915 km, correct to the nearest 0.001 km. [1]

(b) The distance he ran on Day n first exceeds 42.195 km. Find the value of n. [3]

(c) Duncan aims to complete his training regime on Day n, and instead of following the regime for that day, he plans to only run exactly 42.195 km on that day. Find the total distance that Duncan would have covered at the end of his training regime. [4]

11 In the study of Microeconomics, the price \$$p$ (in thousands) that consumers and producers of a particular product A is willing to pay to consume or produce x quantities (in thousands) of the product is modelled by the following equations:

- Producers (supply curve): $x = t + e^t - 1$, $p = t^2$, where $t > 0$.
- Consumers (demand curve): $\frac{xp}{10} + \sin^{-1}\left(\frac{p}{10}\right) = 1$, where $x > 0$ and $0 < p < 10$.

(i) The price p that consumers are willing to pay to consume x quantities of the product decreases as x increases. Use differentiation to verify this. [2]

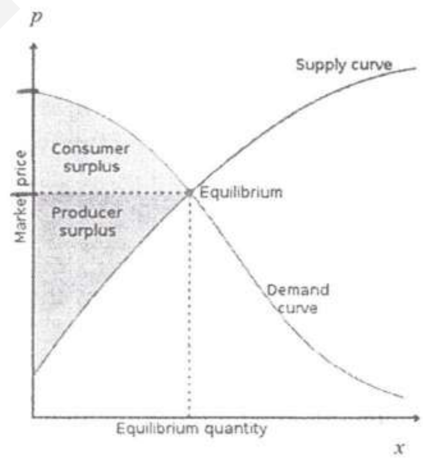
(ii) On the same diagram, sketch the demand and supply curves where x is the horizontal axis and p is the vertical axis. [3]

(iii) Market equilibrium is achieved when the quantity demanded and the quantity supplied are the same at a particular price. Find this price. [2]

Economic surplus, also known as total welfare or Marshallian surplus, refers to two related quantities:

- Consumer surplus is the monetary gain obtained by consumers because they are able to purchase a product for a price that is less than the highest price that they would be willing to pay. Thus the consumer surplus is the area of the region bounded by the demand curve, the p-axis and the horizontal line that passes through the equilibrium point.
- Producer surplus is the amount that producers benefit by selling at a market price that is higher than the least that they would be willing to sell for. Thus the producer surplus is the area of the region bounded by the supply curve, the p-axis and the horizontal line that passes through the equilibrium point.

The diagram below illustrates an example of the surpluses of **another product**.

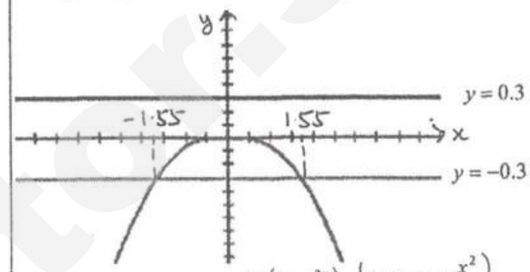


(iv) The total economic surplus is the sum of the consumer and producer surpluses at the market equilibrium. Find the total economic surplus for product A. [4]

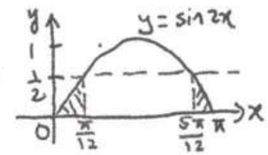
(v) Due to a technological advancement in the manufacturing of product A, the supply curve is now translated a units in the positive x-direction and the new market equilibrium is achieved when $x = 5$. Find the value of a. [3]

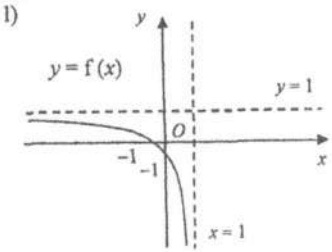
2018 Year 6 H2 Math Prelim Exam (Paper 1)
Solution & Markers' Comments

Suggested Solution	Markers' comments
$\sin^7(x+\frac{1}{4}\pi)$ $=(\sin x\cos\frac{1}{4}\pi+\cos x\sin\frac{1}{4}\pi)^7$ $=\left(\frac{\sin x}{\sqrt{2}}+\frac{\cos x}{\sqrt{2}}\right)^7$ $\approx\frac{1}{8\sqrt{2}}\left[x+\left(1-\frac{x^2}{2}\right)\right]^7$ $=\frac{1}{8\sqrt{2}}\left(1+\left(x-\frac{x^2}{2}\right)\right)^7$ $=\frac{1}{8\sqrt{2}}\left(1+7\left(x-\frac{x^2}{2}\right)+\frac{7\times6}{1\times2}\left(x-\frac{x^2}{2}\right)^2+...\right)$ $\approx\frac{1}{8\sqrt{2}}\left(1+7x+\frac{35}{2}x^2\right)$	• In general, it's easier to use small angle approximation instead of repeated differentiation to find the Maclaurin series. • Common mistake is to assume that $(x+\frac{1}{4}\pi)$ is small. Even though x is small, do note that $\frac{1}{4}\pi$ **is not small.** Hence $\sin^7(x+\frac{1}{4}\pi)\not\approx(x+\frac{1}{4}\pi)^7$. See also Assign8 Q2.
	Total Marks: 3

Suggested Solution	Markers' comments
$\ln(1+e^{2x})=\ln\left(1+1+2x+\frac{(2x)^2}{2!}+...\right)$ $=\ln(2+2x+2x^2+...)$ $=\ln[(2)(1+x+x^2)]$ $=\ln2+\ln[1+(x+x^2+...)]$ $=\ln2+(x+x^2+...)-\frac{(x+x^2+...)^2}{2}+...$ $=\ln2+x+\frac{x^2}{2}+...$ (shown)	• The given instruction is to use appropriate expansions from MF26. Students who differentiated $\ln(1+e^{2x})$ to obtain the Maclaurin series is not given any credit. • Some students wrote $\ln(1+e^{2x})$ $=e^{2x}-\frac{(e^{2x})^2}{2}+...$ $\approx\left[1+2x+\frac{(2x)^2}{2!}\right]-\frac{\left[1+2x+\frac{(2x)^2}{2!}\right]^2}{2}$ Firstly, the RHS will not produce an expression which involves ln 2 (Think of a **different approach** if your method does not lead to the correct answer). Secondly, we will not be able to correctly evaluate the constant term for $\ln(1+e^{2x})$ as the constant 1 will appear in every term of the expansion of e^{2x}.
(ii) $-0.3<\ln(1+e^{2x})-\left(\ln2+x+\frac{x^2}{2}\right)<0.3$ Using GC, $\{x\in\mathbb{R}:-1.55<x<1.55\}$  	• Quite a number of students did not leave their answer in set notation form as required by the question. • The use of GC to solve the inequality $-0.3<\ln(1+e^{2x})-\left(\ln2+x+\frac{x^2}{2}\right)<0.3$ is expected since the question did not forbid the use of GC.
(iii) $\because\ln(1+e^{2x})=\ln2+x+\frac{x^2}{2}+...$ Differentiate both sides wrt x: $\frac{2e^{2x}}{1+e^{2x}}=1+x+...$ $\frac{2e^{2x}/e^{2x}}{(1+e^{2x})/e^{2x}}=1+x+...$ ($\div$ by e^{2x} in num & denom) $\therefore\frac{2}{1+e^{-2x}}=1+x+...$	• Many students either did not realise or were unable to use the result obtained in part (i) to **deduce** part (iii). The word deduce means that we need to use the result in part (i) to solve part (iii). • $[1+(1-2x)+...]^{-1}$ $\neq1+(-1)(1-2x)+...$ • Likely forms of new expression ○ Derivative of the original ○ Integration of the original ○ Replacement of x by kx
	Total Marks: 7

Qn	Suggested Solution	Markers' comments
	$x=3\cos^2\theta+6\sin^2\theta$ $6-x=6-3\cos^2\theta-6\sin^2\theta$ $=6(1-\sin^2\theta)-3\cos^2\theta$ $=6\cos^2\theta-3\cos^2\theta$ $=3\cos^2\theta$ (shown) $x-3=3\cos^2\theta+6\sin^2\theta-3$ $=3(\cos^2\theta-1)+6\sin^2\theta$ $=3(-\sin^2\theta)+6\sin^2\theta$ $=3\sin^2\theta$	• Most students obtain the 2 marks. • Some students tried replacement, which is totally wrong since it is not a change of variable but simple trigonometry.
	$x=3\cos^2\theta+6\sin^2\theta$, $\frac{dx}{d\theta}=[6\cos\theta(-\sin\theta)+12\sin\theta(\cos\theta)]$ $=6\sin\theta\cos\theta$ when $x=3$, $3\cos^2\theta+6\sin^2\theta=3$ $3(\cos^2\theta-1)+6\sin^2\theta=0$ $3(-\sin^2\theta)+6\sin^2\theta=0$ $\sin\theta=0\Rightarrow\theta=0$ when $x=6$, $3\cos^2\theta+6\sin^2\theta=6$ $3\cos^2\theta+6(\sin^2\theta-1)=0$ $3\cos^2\theta+6\cos^2\theta=0$ $\cos\theta=0\Rightarrow\theta=\frac{\pi}{2}$ $\int_3^6\frac{1}{\sqrt{(x-3)(6-x)}}\,dx$ $=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{(3\sin^2\theta)(3\cos^2\theta)}}(6\sin\theta\cos\theta)\,d\theta$ $=\frac{6}{3}\int_0^{\frac{\pi}{2}}1\,d\theta$ $=2[\theta]_0^{\frac{\pi}{2}}=2\left[\frac{\pi}{2}-0\right]$ $=\pi$	• For definite integral involving substitution, remember to change x = 3 and 6 to the corresponding values in θ. • There were a lot of careless mistakes in manipulation. • Students should use GC to verify their final answer
		Total Marks: 5

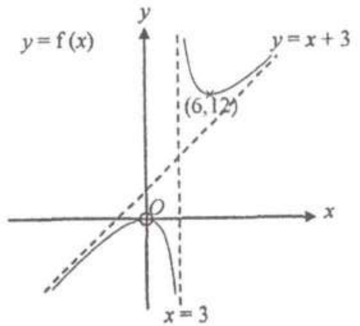
Qn	Suggested Solution	Markers' comments
4	$\tan x+\cot x>4$ $\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}>4$ $\frac{\sin^2x+\cos^2x}{\sin x\cos x}>4$ $\frac{1}{\frac{1}{2}\sin 2x}>4$ $\frac{1}{\sin 2x}>2$ $\therefore 0<\sin 2x<\frac{1}{2}$ (shown) So we have $0<2x<\frac{1}{6}\pi$ or $\frac{5}{6}\pi<2x<\pi$ $0<x<\frac{1}{12}\pi$ or $\frac{5}{12}\pi<x<\frac{1}{2}\pi$ 	• Always refer to the result to be shown for hints on how to start. Since the shown result contains only sin $2x$, you should convert everything in terms of sin and cos (Note the pairing of trigo functions i.e. sin-cos, sec-tan, cosec-cot). • When dealing with inequalities that are not so straight-forward, it is always good to sketch graphs.
		Total Marks : 6

Qn	Suggested Solution	Markers' comments
5(i)	$R_f=(-\infty,1)\not\subset D_g=(0,1)$ Thus gf does not exist. 	• R_f and D_g need to be written explicitly as $R_f=(-\infty,1)$ and $D_g=(0,1)$ in order to convince that R_f is not a subset of D_g. • A handful of students did not get the correct R_f as they did not consider the restriction on the domain of f when it comes to the sketching of the graph of f.
(ii)	$fg(x)=f(\ln x)=1+\frac{2}{\ln x-1}=\frac{\ln x+1}{\ln x-1}$	• This part is generally well done. Students need to know how to use **Method 1** to obtain the answer.

	Method 1 Let $x=(\mathrm{fg})^{-1}(0)$ $\mathrm{fg}(x)=0$ $\dfrac{\ln x+1}{\ln x-1}=0$ $\ln x+1=0$ $\ln x=-1$ $x=\mathrm{e}^{-1}$ $\therefore(\mathrm{fg})^{-1}(0)=\mathrm{e}^{-1}$ **Method 2 (work backwards from final value = 0)** Let $x=(\mathrm{fg})^{-1}(0)$ $\mathrm{fg}(x)=0$ Substitute $\mathrm{f}(x)=0\Rightarrow 0=1+\dfrac{2}{x-1}\Rightarrow x=-1$ Substitute $\mathrm{g}(x)=-1\Rightarrow -1=\ln x\Rightarrow x=\mathrm{e}^{-1}$ $\therefore(\mathrm{fg})^{-1}(0)=\mathrm{e}^{-1}$ $\{\mathrm{e}^{-1}\}\xrightarrow{\mathrm{g}}\{-1\}\xrightarrow{\mathrm{f}}\{0\}$ **Method 3 (find inverse directly)** Let $y=\mathrm{fg}(x)=\dfrac{\ln x+1}{\ln x-1}$ $y\ln x-y=\ln x+1$ $\ln x(y-1)=y+1$ $\ln x=\dfrac{y+1}{y-1}$ $x=\mathrm{e}^{\frac{y+1}{y-1}}$ $(\mathrm{fg})^{-1}(x)=\mathrm{e}^{\frac{x+1}{x-1}}$ $\therefore(\mathrm{fg})^{-1}(0)=\mathrm{e}^{\frac{0+1}{0-1}}=\mathrm{e}^{-1}$	
)	$\mathrm{gh}(x)=x\Rightarrow \mathrm{h}(x)=\mathrm{g}^{-1}(x)=\mathrm{e}^{x}$	• Well-attempted
	Method 1 $\mathrm{hg}(x)=x^2+1$ $\mathrm{hg}\left(\mathrm{g}^{-1}(x)\right)=\left(\mathrm{g}^{-1}(x)\right)^2+1$ $\therefore \mathrm{h}(x)=\left(\mathrm{e}^{x}\right)^2+1=\mathrm{e}^{2x}+1$	• We cannot simply post-multiply both sides by g^{-1} to obtain $\mathrm{h}(x)$. However, we could use **Method 1** to replace x by $\mathrm{g}^{-1}(x)$ since $\mathrm{g}\left(\mathrm{g}^{-1}(x)\right)=x$ and $\mathrm{g}^{-1}(x)=\mathrm{e}^{x}$.

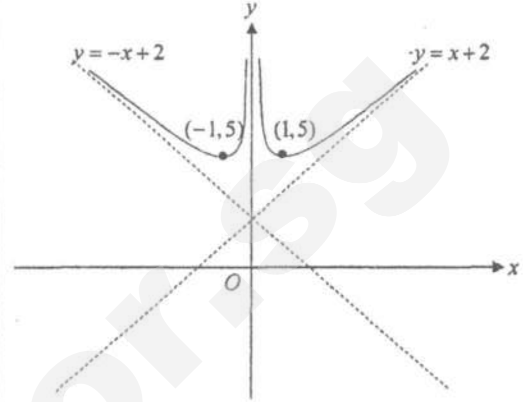
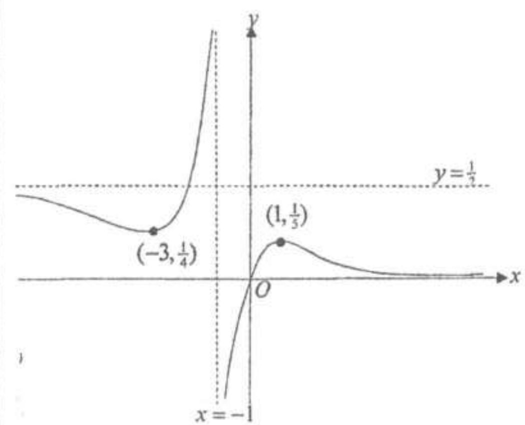
	Method 2 $\mathrm{hg}(x)=x^2+1$ $\mathrm{h}\left(\mathrm{g}(x)\right)=x^2+1$ $\mathrm{h}(\ln x)=\left(\mathrm{e}^{\ln x}\right)^2+1$ $\therefore \mathrm{h}(x)=\left(\mathrm{e}^{x}\right)^2+1=\mathrm{e}^{2x}+1$	
		Total Marks: 8

Qn	Suggested Solution	Markers' comments
6(a)	$x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}=2xy-y^2$ ...(1) Given $y=zx^2$: $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2xz+x^2\dfrac{\mathrm{d}z}{\mathrm{d}x}$ ...(2) Substitute (2) into (1): $x^2\left(2xz+x^2\dfrac{\mathrm{d}z}{\mathrm{d}x}\right)=2x\left(zx^2\right)-\left(zx^2\right)^2$ $x^4\dfrac{\mathrm{d}z}{\mathrm{d}x}=-z^2x^4$ $\dfrac{\mathrm{d}z}{\mathrm{d}x}=-z^2$ $\int\dfrac{1}{z^2}\,\mathrm{d}z=-\int 1\,\mathrm{d}x$ $-\dfrac{1}{z}=-x+C$ $-\dfrac{x^2}{y}=-x+C$ $y=\dfrac{x^2}{x-C}$	• Many students failed to do chain rule to get equation (2), treating z as a constant. Students are reminded that if z is a constant with respect to x, then there is no change of variable involved. • There are some careless mistakes, ranging from sign error (which can be checked by simple differentiation and substituting back) to the following: $\dfrac{1}{z}=x-C$ $z=\dfrac{1}{x}-C$ (wrong, should be $z=\dfrac{1}{x-C}$)
(i)	Given $(2,-4)$, $-4=\dfrac{2^2}{2-C}\Rightarrow C=3$ Hence, $y=\dfrac{x^2}{x-3}=x+3+\dfrac{9}{x-3}$	• Many students who has the correct general equation only drew the bottom half of the graph. Before sketching, it would be good to have an idea of what the graph should look like so that you can spot the missing parts.

Suggested Solution	Markers' comments
	• Students can also use zoom-fit on their GC to see the whole graph. • Close to all students did not omit (0, 0), when it is given that $x \neq 0$. • There are some students who has the correct general equation but evaluated the oblique asymptote incorrectly.
When $C=0$, particular soln is $y=\frac{x^2}{x-0}=x$ which is a straight line and has no turning point.	• Many students didn't answer the question by stating the equation explicitly. • Many did differentiation instead or tried to solve $\frac{dy}{dx} \neq 0$. • Students are reminded to look at the number of mark(s) given for the time/effort needed.
Given $y=4x-1+De^{-x} \Rightarrow \frac{dy}{dx}=4-De^{-x}$ $\frac{dy}{dx}+y=\left(4-De^{-x}\right)+\left(4x-1+De^{-x}\right)=4x+3$ $\therefore p=4,\ q=3$	• Many students got this correct. • There were some students who attempted to integrate y in terms of x to obtain xy, which is completely incorrect.
	Total marks : 11

Qn	Suggested Solution	Markers' comments
7(a)	$z^*=\frac{(2i)^3}{\left(\sqrt{3}+i\right)^4}=\frac{-8i}{\left(\sqrt{3}+i\right)^4}$ $\lvert z\rvert=\lvert z^*\rvert$ $=\left\lvert\frac{-8i}{\left(\sqrt{3}+i\right)^4}\right\rvert=\frac{8}{\left(\sqrt{\left(\sqrt{3}\right)^2+1^2}\right)^4}$ $=\frac{8}{16}=\frac{1}{2}$ $\arg(z)=-\arg\left(z^*\right)$ $=-\arg\left(\frac{-8i}{\left(\sqrt{3}+i\right)^4}\right)$ $=-\left[\arg(-8i)-4\arg\left(\sqrt{3}+i\right)\right]$ $=-\left[-\frac{1}{2}\pi-4\left(\frac{1}{6}\pi\right)\right]$ $=\frac{7}{6}\pi$ $\therefore \arg(z)=\frac{7}{6}\pi-2\pi=-\frac{5}{6}\pi$ **Alternative** $z^*=\frac{(2i)^3}{\left(\sqrt{3}+i\right)^4}=\frac{\left(2e^{i\frac{\pi}{2}}\right)^3}{\left(2e^{i\frac{\pi}{6}}\right)^4}$ $=\frac{8e^{i\frac{3\pi}{2}}}{16e^{i\frac{4\pi}{6}}}=\frac{1}{2}e^{i\left(\frac{3\pi}{2}-\frac{4\pi}{6}\right)}$ $=\frac{1}{2}e^{i\frac{5\pi}{6}}$ $\boxed{\therefore z=\frac{1}{2}e^{-i\frac{5\pi}{6}}}$ $\Rightarrow \lvert z\rvert=\frac{1}{2},\ \arg(z)-\frac{5}{6}\pi$ $z=re^{-i\theta}$ where $r=\lvert z\rvert=\frac{1}{2}$ $\theta=\arg(z)=-\frac{5}{6}\pi$ $\arg(z^n)=n\arg(z)=-\frac{5}{6}n\pi$ Since z^n is purely imaginary, $-\frac{5}{6}n\pi=(2k+1)\left(\frac{1}{2}\pi\right),\ k\in\mathbb{Z}$ $\Rightarrow n=-\frac{3}{5}(2k+1)$ $\therefore$smallest positive integer $n=3$ (when $k=-3$)	• Important to remember the properties of complex numbers: ✓ $\lvert z\rvert=\lvert z^*\rvert$ ✓ $\arg(z)=-\arg\left(z^*\right)$ • Not efficient to expand the denominator $\left(\sqrt{3}+i\right)^4$ • The alternative method is a useful and efficient way of finding the modulus and argument of complicated expressions at one go. • It's wrong to write $\frac{7}{6}\pi=-\frac{5}{6}\pi$. A separate line should be used for this. • On the other hand, for complex numbers which are purely real, arguments would take the form $k\pi, k\in\mathbb{Z}$
(b)(i)	Let $\mathrm{f}(x)=ax^4+bx^3+cx^2+24x-44$ $\mathrm{f}(1)=-18\Rightarrow a+b+c=2$ $\mathrm{f}(-1)=-54\Rightarrow a-b+c=14$ $\mathrm{f}(2)=0\Rightarrow 16a+8b+4c=-4$ From GC : $a=1, b=-6, c=7$	• Instead of long division, it's easier to use remainder and factor theorems to find the 3 equations, before solving simultaneously using GC.
(ii)	$x^4-6x^3+7x^2+24x-44=0$ If $3-(\sqrt{2})i$ is a root, $3+(\sqrt{2})i$ is also a root (since <u>equation has all real coefficients</u> OR by <u>conjugate root theorem</u>)	• It's important to state the <u>reason</u> for existence of the conjugate root $3+(\sqrt{2})i$. ✓Eqn has all real coeff; or ✓Conjugate root theorem

Method 1 Observe $(x-2)$ is a **factor** of the polynomial equation. Compare product of last terms, $[x\boxed{-(3-(\sqrt{2})\mathrm{i})}][x\boxed{-(3+(\sqrt{2})\mathrm{i})}](x\boxed{-2})(x\boxed{+a}) = x^4-6x^3+7x^2+24x\boxed{-44}$ $(3-(\sqrt{2})\mathrm{i})(3+(\sqrt{2})\mathrm{i})(-2)(a)=-44$ $(3^2+(\sqrt{2})^2)(-2)a=-44$ $a=2$ **Method 2** $[x-(3-(\sqrt{2})\mathrm{i})][x-(3+(\sqrt{2})\mathrm{i})]=[(x-3)+(\sqrt{2})\mathrm{i}][(x-3)-(\sqrt{2})\mathrm{i}]$ $=[(x-3)^2+2]$ $=x^2-6x+11$ Since $(x-2)$ is a **factor** of the polynomial equation, $x^4-6x^3+7x^2+24x-44=0$ $\Rightarrow (x^2-6x+11)(x-2)(x+2)=0$ (by inspection) $\therefore$ the other roots are $3+(\sqrt{2})\mathrm{i}$, 2 and -2	• Note the correct description ✓Factor : $(x-2)$ ✓Root : $x=2$ • Note that question states **"…showing your working clearly."**, so use of GC is not allowed to find the rest of the roots. But you can use the GC to verify your answers. • Conclude by listing the **other roots together.**
	Total marks : 11

Qn	Suggested Solution	Markers' comments
8(i)		• A number of students confused the graph of $y=\mathrm{f}(\lvert x\rvert)$ with the graph of $y=\lvert\mathrm{f}(x)\rvert$. • For the former, $y=\mathrm{f}(\lvert x\rvert)$ $=\begin{cases}\mathrm{f}(x) & \text{if } x\ge 0\\ \mathrm{f}(-x) & \text{if } x<0\end{cases}$ Thus the graph to be drawn is to retain the positive x-region and reflect it in the y-axis. • For the latter, $y=\lvert\mathrm{f}(x)\rvert$ $=\begin{cases}\mathrm{f}(x) & \text{if } \mathrm{f}(x)\ge 0\\ -\mathrm{f}(x) & \text{if } \mathrm{f}(x)<0\end{cases}$ Thus the graph to be drawn is to retain the positive y-region and reflect the negative y-region in the x-axis.
(ii)		• Most were able to label the critical features of the graph (Stationary Points, Asymptotes) though quite a number fail to observe that the graph passes through the origin. • For a step to step guide on how to sketch the reciprocal graph, scan the QR code below: (Can also be accessed via team drive)

1. Translate the graph by b units in the negative x-direction. 2. Scale the graph by a factor of $-\frac{1}{a}$ parallel to the x-axis. (Also accept scale factor of $\frac{1}{\|a\|}$. But do not accept scale factor of $\frac{1}{a}$.) 3. Reflect in the y-axis. From original graph, $f(-1)=0$. To find x-intercept of new graph, $f(ax+b)=0$ so $ax+b=0$ $\therefore x=\frac{-1-b}{a}$ Coordinates of the point where the graph of $y=f(ax+b)$ cuts the x-axis is $(\frac{-1-b}{a},0)$ or $(\frac{1+b}{\|a\|},0)$	• Note that for each transformation, only x or y is to be replaced. • Recommended order of transformation of graphs of the form $y=f(x)$ is TSR for x, RST for y while for graphs of the form $g(y)=f(x)$ e.g. $x^2+y^2=1$ is TSR for both x and y. • Since $a<0$, you need to be careful when performing the scaling. The scale factor should be the positive constant $-\frac{1}{a}$. • Note that for this equation, transformations only involved x, the x-intercept in new graph must result from the transformation of the original x-intercept.
	Total Marks: 9

Suggested Solutions	Markers' comments
l_1: $\underset{\sim}{r}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}t\\t^2+1\\3\end{pmatrix},\ \lambda\in\mathbb{R}$ $p: tx-2y+z=-3 \Rightarrow \underset{\sim}{r}\cdot\begin{pmatrix}t\\-2\\1\end{pmatrix}=-3$ The direction vector of l_1 is parallel to p (i.e., perpendicular to the normal of p) and the point A does not lie on p. $\begin{pmatrix}t\\t^2+1\\3\end{pmatrix}\cdot\begin{pmatrix}t\\-2\\1\end{pmatrix}=0 \Rightarrow t^2-2t^2-2+3=0 \Rightarrow t=-1$ or 1	• If the line and the plane have no common point of intersection, the line has to be parallel to the plane and not lying on the plane. Hence have to show that the line is perpendicular to the normal of the plane, and that any point on the line does not lie on the plane. • Since it is a "show" question, sufficient working is expected.

	$\begin{pmatrix}1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}t\\-2\\1\end{pmatrix}\neq-3 \Rightarrow t-6+2\neq-3 \Rightarrow t\neq1$ Therefore, $t=-1$ (shown)	
(ii)	The distance between l_1 and p = shortest distance of A to p $=\frac{\left\|d-\overrightarrow{OA}\cdot\underset{\sim}{n}\right\|}{\|\underset{\sim}{n}\|}$ $=\frac{\left\|-3-\begin{pmatrix}1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\1\end{pmatrix}\right\|}{\sqrt{6}}$ $=\frac{2}{\sqrt{6}}$	• This is finding the length of projection of the vector (joining point A and any point on the plane) onto the normal of the plane. • Instead of using $\overrightarrow{OA}\cdot\underset{\sim}{n}$, some students use $\underset{\sim}{d}_{l_1}\cdot\underset{\sim}{n}$ which is incorrect.
(iii)	l_2: $2y=z, x=3, \Rightarrow \underset{\sim}{r}=\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\2\end{pmatrix}, \mu\in\mathbb{R}$ **Method 1** $\begin{pmatrix}0\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\1\end{pmatrix}=0-2+2=0$ Therefore, l_2 is parallel to p. $\begin{pmatrix}3\\0\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\1\end{pmatrix}=-3$ Therefore, l_2 lies on p (shown). **Method 2** Substitute l_2: $2y=z, x=3$ into equation of p: LHS of $p=-3-2y+2y=-3=$ RHS of p	• Most students get this part correct. • Take note of proper presentation.

Method 1

$\overrightarrow{OB}=\begin{pmatrix}1-\lambda\\3+2\lambda\\2+3\lambda\end{pmatrix}$, for some $\lambda\in\mathbb{R}$

$\overrightarrow{OC}=\begin{pmatrix}3\\\mu\\2\mu\end{pmatrix}$, for some $\mu\in\mathbb{R}$

$\overrightarrow{BC}=\begin{pmatrix}3-1+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}$

Since BC is perpendicular to both l_1 and l_2,

$\overrightarrow{BC}\cdot\underset{\sim}{d}_{l_1}=0$ and $\overrightarrow{BC}\cdot\underset{\sim}{d}_{l_2}=0$

$\begin{pmatrix}3-1+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\3\end{pmatrix}=0$ and $\begin{pmatrix}3-1+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\1\\2\end{pmatrix}=0$

$7\lambda-4\mu=7\quad\cdots(1)\qquad\qquad 8\lambda-5\mu=-7\quad\cdots(2)$

Solve (1) and (2): $\mu=-\dfrac{7}{3}$ and $\lambda=-\dfrac{7}{3}$

$\overrightarrow{OB}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\left(-\dfrac{7}{3}\right)\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}\frac{10}{3}\\-\frac{5}{3}\\-5\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}3\\\mu\\2\mu\end{pmatrix}=\begin{pmatrix}3\\-\frac{7}{3}\\-\frac{14}{3}\end{pmatrix}$

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}3\\-\frac{7}{3}\\-\frac{14}{3}\end{pmatrix}-\begin{pmatrix}\frac{10}{3}\\-\frac{5}{3}\\-5\end{pmatrix}=\begin{pmatrix}-\frac{1}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{pmatrix}$

Method 2

$\underset{\sim}{d}_{l_1}\times\underset{\sim}{d}_{l_2}=\begin{pmatrix}1\\-2\\-3\end{pmatrix}\times\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}-1\\-2\\1\end{pmatrix}$

$\overrightarrow{BC}=\begin{pmatrix}3-1+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}=\begin{pmatrix}2+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}=\beta\begin{pmatrix}-1\\-2\\1\end{pmatrix}$

Solving the equation, $\beta=\dfrac{1}{3},\mu=\lambda=-\dfrac{7}{3}$

Hence $\overrightarrow{BC}=\dfrac{1}{3}\begin{pmatrix}-1\\-2\\1\end{pmatrix}$.

- Since point B and C lie on l_1 and l_2 respectively, we can formulate the vector expression for $\overrightarrow{BC}$.
- Also, since $\overrightarrow{BC}$ is perpendicular to both lines, it would be perpendicular to the director vector of both lines.
- Alternatively, $\overrightarrow{BC}$ is parallel to the normal vector of both lines.
- Be careful when solving for the values of μ and λ using your GC.

(v)

Method 1

Let $l_1{}'$ be the line of reflection of l_1 in p.

Let point B' be the point lying on $l_1{}'$ and is also the reflection of B in p.

$\underset{\sim}{d}_{l_1'}=\underset{\sim}{d}_{l_1}=\begin{pmatrix}-1\\2\\3\end{pmatrix}$ because l_1 is parallel to p.

$\overrightarrow{BB'}=2\overrightarrow{BC}\ \Rightarrow\ \overrightarrow{OB'}=2\overrightarrow{BC}+\overrightarrow{OB}$

$$=2\begin{pmatrix}-\frac{1}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{pmatrix}+\begin{pmatrix}\frac{10}{3}\\-\frac{5}{3}\\-5\end{pmatrix}1$$

$$=\begin{pmatrix}\frac{8}{3}\\-3\\-\frac{13}{3}\end{pmatrix}$$

Equation of line of reflection $l_1{}':\underset{\sim}{r}=\begin{pmatrix}\frac{8}{3}\\-3\\-\frac{13}{3}\end{pmatrix}+\gamma\begin{pmatrix}-1\\2\\3\end{pmatrix},\ \gamma\in\mathbb{R}$

Method 2

Let point F be the foot of perpendicular of point A on plane p.

$\overrightarrow{OF}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$

Since point F lies on plane p,

$\begin{pmatrix}1-\lambda\\3-2\lambda\\2+\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\1\end{pmatrix}=-3$

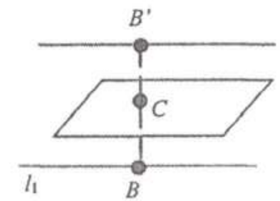
Solving, $\lambda=\dfrac{1}{3}$

$\overrightarrow{OF}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}=\begin{pmatrix}\frac{2}{3}\\\frac{7}{3}\\\frac{7}{3}\end{pmatrix}$

$\overrightarrow{OF}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}\Rightarrow\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}$

$\overrightarrow{OA'}=2\begin{pmatrix}\frac{2}{3}\\\frac{7}{3}\\\frac{7}{3}\end{pmatrix}-\begin{pmatrix}1\\3\\2\end{pmatrix}=\begin{pmatrix}\frac{1}{3}\\\frac{5}{3}\\\frac{8}{3}\end{pmatrix}$

- Recognize that l_1 is parallel to the plane, and so the reflection of the line will also be parallel to the plane.
- Then it suffices to find the point of reflection of any point on l_1 in the plane.
- Since point B lies on l_1, and point C lies on the plane (because point C lies on l_2 which lies on the plane), and that BC is perpendicular to the plane, $\overrightarrow{BB'}=2\overrightarrow{BC}$.



- It is also possible to find the point of reflection of point A using ratio (or mid-point) theorem. See **Method 2**.

Equation of line of reflection $l_1' : \mathbf{r} = \begin{pmatrix} \frac{1}{3} \\ \frac{5}{3} \\ \frac{8}{3} \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $\gamma \in \mathbb{R}$	
	Total marks: 13

Solution	Markers' comments

	For first Odd-no Days	For first Even-no Days
1	5	$5(1+\alpha)$
2	$5(1+\alpha)(\alpha)$	$5(1+\alpha)^2(\alpha)$
3	$5(1+\alpha)^2(\alpha)^2$	$5(1+\alpha)^3(\alpha)^2$
4	$5(1+\alpha)^3(\alpha)^3$	$5(1+\alpha)^4(\alpha)^3$
5	$5(1+\alpha)^4(\alpha)^4$	$5(1+\alpha)^5(\alpha)^4$
	$\vdots$	$\vdots$

The common ratio for both sequences $= \alpha(1+\alpha)$

Majority are able to get this mark.

For GP sum to infinity to exist, need $-1 < \alpha(1+\alpha) < 1$

Since distance for each day is positive, we solve

$$0 < \alpha(1+\alpha) < 1$$

From GC, $0 < \alpha < 0.618$ (3 sf)

Since the question did not ask for exact answers, we can just use GC to solve!

Common Errors:
1. Wrong condition for GP sum to infinity to exist.
2. Forgot the condition $\alpha > 0$.
3. Errors in concluding the correct region in solving inequalities.

Theoretical max total distance $= \dfrac{5}{1-\alpha(1+\alpha)} + \dfrac{5(1+\alpha)}{1-\alpha(1+\alpha)}$

$= \dfrac{5+5(1+\alpha)}{1-\alpha(1+\alpha)}$

$= \dfrac{10+5\alpha}{1-\alpha-\alpha^2}$

Sum to infinity for the total distance run on both even and odd days.

Calculation errors are very common.

Refer to 2011 GCE A Level P1 Q9(ii) for similar phrasing. Students who applied the sum to infinity formula are more successful as compared to those who derived the sum to infinity from S_n.

Distance on Day 10

$= 5(1.65)\left[(1.65)(0.65)\right]^4 = 10.915$ km

Students did well for this part.

(b)

Method 1

For the distance on Day n to **first** exceed 42.195 km, observe that n must be even, where the distance is larger than the previous day.

$5(1.65)\left[(1.65)(0.65)\right]^{\frac{n-2}{2}} > 42.195$, where n is **even**

You should argue for the smallest **even** n in your presentation.

From GC,

n	Distance on Day n
48	$41.26595 < 42.195$
50	$44.25773 > 42.195$

Smallest $n = 50$

Method 2

Distance run on the mth odd-numbered day

$= 5\left[(1.65)(0.65)\right]^{m-1}$

Distance run on the mth even-numbered day

$= 5(1.65)\left[(1.65)(0.65)\right]^{m-1}$

Consider

$5\left[(1.65)(0.65)\right]^{m-1} > 42.195$ vs $5(1.65)\left[(1.65)(0.65)\right]^{m-1} > 42.195$

$m \geq 32$ vs $m \geq 25$

Since m for the even-numbered days is smaller, the distance will first exceed 42.195 on Day $n = 25 \times 2$, i.e. smallest $n = 50$.

Common Errors:
1. Misinterpreted the question and constructed inequality involving the total distance.
2. Students forgot that this distance formula is only applicable for even n and concluded smallest $n = 49$.
3. Students forgot to consider the difference between finding the mth even-numbered day versus finding Day n. A lot concluded $n = 25$.

(c)

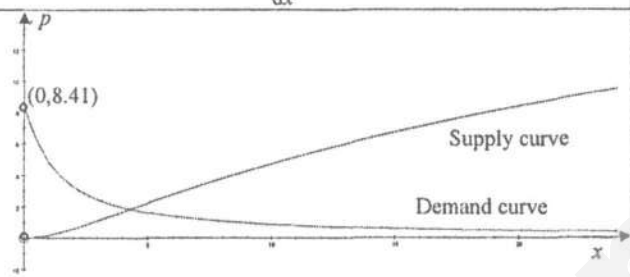
Distance for first 50 days
= Distance for first 25 odd-numbered days
\+ Distance for first 24 even-numbered days
\+ Distance on Day 50

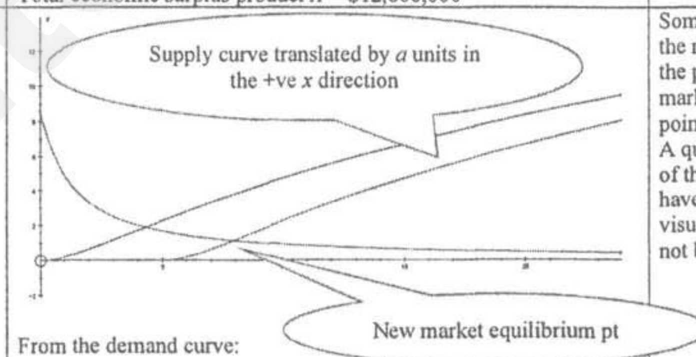
	For first Odd-no Days	For first Even-no Days
1	5	$5(1+\alpha)$
2	$5(1+\alpha)(\alpha)$	$5(1+\alpha)^2(\alpha)$
3	$5(1+\alpha)^2(\alpha)^2$	$5(1+\alpha)^3(\alpha)^2$
4	$5(1+\alpha)^3(\alpha)^3$	$5(1+\alpha)^4(\alpha)^3$
5	$5(1+\alpha)^4(\alpha)^4$	$5(1+\alpha)^5(\alpha)^4$
	$\vdots$	$\vdots$
Total distance	first 25 odd-numbered days $\dfrac{5\left\{\left[(1.65)(0.65)\right]^{25}-1\right\}}{(1.65)(0.65)-1}$	first 24 even-numbered days $\dfrac{5(1.65)\left\{\left[(1.65)(0.65)\right]^{24}-1\right\}}{(1.65)(0.65)-1}$

Common errors:
1. Counting the even- and odd-numbered days wrongly.
2. Didn't take into account the distance run on the last day
3. Error in applying S_n formula.

Note that in $S_n = \dfrac{a(1-r^n)}{1-r}$, n represents the number of terms that you sum.

	Total distance covered $= \frac{5}{(1.65)(0.65)-1}\left\{\left[(1.65)(0.65)\right]^{25}-1\right\}$ $+\frac{5(1.65)}{(1.65)(0.65)-1}\left\{\left[(1.65)(0.65)\right]^{24}-1\right\}+42.195$ $=866.681$ km	
		Total marks: 13

	Suggested Solution	**Markers' comments**
	$\frac{xp}{10}+\sin^{-1}\left(\frac{p}{10}\right)=1$ Note: Remember to include this term Differentiate wrt to x, $\frac{p}{10}+\frac{x}{10}\frac{\mathrm{d}p}{\mathrm{d}x}+\frac{1}{10\sqrt{1-\left(\frac{p}{10}\right)^2}}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)=0$ $\frac{\mathrm{d}p}{\mathrm{d}x}=-\frac{p}{x+\frac{1}{\sqrt{1-\left(\frac{p}{10}\right)^2}}}$ You need to explain why $\frac{\mathrm{d}p}{\mathrm{d}x}<0$. Since $x>0$ and $0<p<10$, $\frac{\mathrm{d}p}{\mathrm{d}x}<0$. (shown)	
		To draw the demand curve, make x the subject in terms of p, draw this as Y1 in the GC and you can use the "Draw" feature in the GC (the prgm button) to draw the inverse curve of Y1 which will give you the p vs x curve. Note: You need to draw the shape of the supply curve with a point of inflexion.
	$\frac{(t+e^t-1)t^2}{10}+\sin^{-1}\left(\frac{t^2}{10}\right)=1$	Take note that if the qn did not mention anything about not

	Using GC, $t=1.3713$, $p=1.3713^2=1.8805=1.88$ (to 3 s.f) At market equilibrium, the price is \$1880.	being able to use the GC means that you can use GC to find the soln.
(iv)	Total economic surplus for product A $=\int_{1.8805}^{8.4147}\frac{10}{p}\left(1-\sin^{-1}\left(\frac{p}{10}\right)\right)\mathrm{d}p+\int_0^{1.3713}2t(t+e^t-1)\,\mathrm{d}t$ $=12.8$ Alternatively, $=\int_{1.8805}^{8.4147}\frac{10}{p}\left(1-\sin^{-1}\left(\frac{p}{10}\right)\right)\mathrm{d}p+\int_0^{1.8805}(\sqrt{p}+e^{\sqrt{p}}-1)\,\mathrm{d}p$ $=12.8$ Total economic surplus product A = \$12,800,000	Again, qn did not exclude the use of the GC.
(v)	 From the demand curve: when $x=5$, from the demand curve, $p=1.6654$ From the translated supply curve: $p=t^2=1.6654 \Rightarrow t=\sqrt{1.6654}=1.2905$ $x=t+e^t-1+a=5 \Rightarrow a=1.07$	Some students have the misconception that the price of the new market equilibrium point will not change. A quick mental sketch of the graph would have helped them to visualise that it will not be true.
		Total marks: 14

Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) 9758/02

Paper 2

September 2018

3 hours

Additional Materials: Answer Paper
Graph Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	5	9	6	8	12	8	9	9	10	12	12	100

Section A: Pure Mathematics [40 marks]

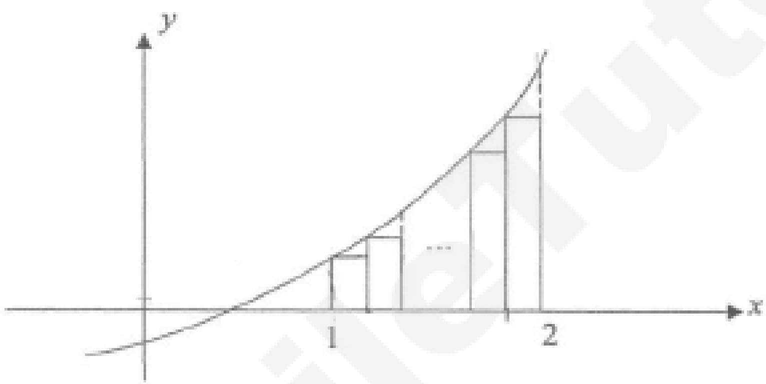
1 With reference to the origin O, the position vectors of point A and B are $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors.

(i) State the shortest distance from B to the line passing through O and A. [1]

(ii) Given that $\mathbf{c} = \mathbf{a} + \mathbf{b}$, state the geometrical meaning of $|\mathbf{b}\times\mathbf{c}|$ and show that $|\mathbf{b}\times\mathbf{c}| = |\mathbf{a}\times\mathbf{b}|$. [2]

(iii) Given that $\mathbf{a}\times 2\mathbf{b} = \mathbf{d}\times 3\mathbf{a}$, find a linear relationship between $\mathbf{a}, \mathbf{b}$ and $\mathbf{d}$. [2]

2 The equation of curve C_1 is given by $y = 3^x - 2$. The graph of C_1 is shown in the diagram below. Rectangles, each of width $\frac{1}{n}$, where n is an integer, are drawn under C_1 for $1 \le x \le 2$.



(i) Show that the total area of all n rectangles, S_n, is given by

$$S_n = -2 + \frac{3}{n}\sum_{r=1}^{n} 3^{\frac{r-1}{n}}.$$

Hence evaluate S_n, leaving your answer in terms of n. [4]

(ii) Find the exact value of $\lim\limits_{n\to\infty} S_n$. [2]

The equation of curve C_2 is given by $y = \frac{x^3}{\sqrt{4+x^2}}$.

(iii) The region R is bounded by C_1, C_2, $x = 1$ and $x = 2$. Find the volume of the solid of revolution formed when R is rotated through 4 right angles about the x-axis, giving your answer correct to 2 decimal places. [3]

3 A sequence u_n is given by

$$u_n = \frac{3}{M-n+1}, \text{ where } n \text{ and } M \text{ are positive integers such that } n < M+1.$$

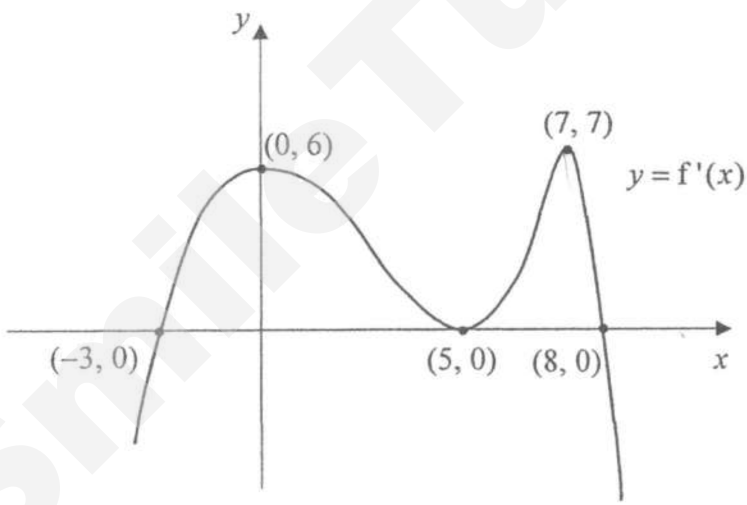
(i) Describe the behaviour of the sequence. [1]

Let S_n denote the sum of the first n terms of u_n.

(ii) Write down S_1, S_2 and S_3. Hence find $\sum_{n=1}^{M} S_n$ in terms of M. [3]

(iii) Show that $S_n > \dfrac{3n}{M}$. [2]

4 The diagram below shows the curve $y = \mathrm{f}'(x)$. It has turning points at (0, 6), (5, 0) and (7, 7) and intersects the x-axis at (–3, 0) and (8, 0).



(i) State the x-coordinates of the stationary points for the curve $y = \mathrm{f}(x)$. Hence determine the nature of these stationary points. [3]

(ii) State the range of values of x such that the curve $y = \mathrm{f}(x)$ is concave downwards. [2]

(iii) A student makes a claim on each of the following statements:

(A) If the curve $y = \mathrm{f}(x)$ passes through the point (–3, 7), then $\mathrm{f}(x) > 7$ for $-3 < x < 5$.

(B) By using integration by parts, $\int_{-3}^{8} x\mathrm{f}(x)\,\mathrm{d}x > 32\mathrm{f}(8) - \frac{9}{2}\mathrm{f}(-3)$.

For each of the above statements, explain briefly whether the student is right to make such a claim. [3]

5 **(a)** A curve has the equation $(x+y)^2 = 4e^{xy}$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [2]

(ii) Given that the curve cuts the positive y-axis at point A, find the equation of the tangent to the curve at A. [2]

(iii) The tangent to the curve at A meets the curve at another point B. Find the coordinates of B. [3]

(b) A closed cylinder is designed to contain a fixed volume of p cm^3 of liquid such that its external surface area is a minimum. Find the radius of the cylinder in terms of p in cm. [5]

Section B: Probability and Statistics [60 marks]

6 A test consists of 15 multiple choice questions, where each question has n possible options, of which only one is correct. A student took the test by randomly choosing an answer to each question. It is known that the probability of answering exactly 3 questions correctly is the same as the probability of answering exactly 4 questions correctly.

(i) By forming an equation in terms of n, find the value of n. [3]

Each correct answer is awarded 3 marks and each incorrect answer carries a penalty of 1 mark. The score is the total marks awarded based on the number of correct and incorrect answers.

(ii) Find the expected score, s, obtained by the student. [3]

(iii) Find the probability that the score obtained by the student is within 4 marks of s. [2]

7 During a symposium, 4 boys and 8 girls are divided into 4 groups of three each for discussion. How many ways are there to divide the 12 participants such that each group consists of exactly 1 boy? [2]

After the discussion, all members of the 4 groups sit at random at a round table.

Find the probability that

(i) the 3 members in each group are next to each other, [2]

(ii) every boy is separated from each other by exactly 2 girls. [3]

For the 12 participants, events A and B are defined by

A: every boy is separated from each other by exactly 2 girls; and
B: none of the boys are seated next to each other.

(iii) Determine if the events A and B are independent. [2]

8 An interactive simulation ride allows a group of 5 riders to take the ride at a time. The ride time, X minutes, follows a normal distribution with mean μ minutes and standard deviation 2 minutes. The ride starts promptly at 10 am daily with no wait time between any groups of 5 riders. There are only 4 scheduled rides every morning. At 10 am on a particular morning, there are already 20 people queuing for the ride. It is assumed that all the people in the queue will take the ride based on the sequence of the queue and the ride times are independent.

(i) Show that $\mu = 14$, correct to the nearest integer, if $\mathrm{P}(\mu < X < 16) = 0.35$. [2]

For the rest of the question, use $\mu = 14$ for your calculations. A ride is considered long if it has a ride time of at least 15 minutes.

(ii) Find the probability that the 12th person in the queue took the ride before 10.30 am on that morning. [3]

(iii) Show that the probability of having at least 2 long rides on that morning is 0.363. [2]

(iv) Given that there are at least 2 long rides on that morning, find the probability that none of these long rides are consecutive. [2]

9 To study the recent relationship between the property price index, p (in %) and the stock index, s (in thousands), of a particular city, Hilton recorded the readings from each of the past 8 quarters in the table below.

Stock Index, s (thousands)	2.12	2.53	2.63	2.70	2.75	2.83	2.87	2.98
Property Price Index, p (%)	115	130	145	140	146	150	155	170

Hilton realised that he recorded one of the values of p incorrectly.

(i) Sketch a scatter diagram for the data and circle the erroneous point X on your diagram. [2]

[For the remaining parts of this question, you should exclude the point X.]

Hilton proposes that s and p can be modelled by one of the formulae:

$$p = a + b s^5 \quad \text{or} \quad \ln p = c + d s,$$

where a, b, c and d are positive constants.

(ii) Determine the better model for the given data, giving a reason for your choice. [3]

(iii) Assuming that the value of s at X is correct, estimate the corresponding value of p. Give two reasons why you would expect this estimate to be reliable. [4]

Hilton concludes that higher stock index will lead to higher property price index. Comment on his conclusion in the context of the question. [1]

10 The discrete random variable X has probability mass function given by

$$\mathrm{P}(X=x)=\begin{cases}\dfrac{ax}{n(n+1)} & \text{for } x=1,2,\ldots,n,\\ 0 & \text{otherwise,}\end{cases}$$

where a is a constant.

(i) Show that $a=2$. [2]

(ii) Find $\mathrm{E}(X)$ in terms of n. You may use the result that $\sum_{r=1}^{n} r^2=\frac{n}{6}(n+1)(2n+1)$. [3]

Two players P_1 and P_2 play a game with $n+1$ tokens labelled 1 to $n+1$. Each player randomly picks one token without replacement and the player who picks the token with the smaller number loses. The amount of money lost by the losing player, in dollars, will be the number on the winning token. For example, if P_1 and P_2 pick the tokens labelled 5 and 3 respectively, P_2 loses \$5 to P_1.

(iii) Explain why the probability of P_2 losing a game is 0.5. [1]

(iv) Given that P_2 loses, find the probability P_2 loses \$$(m+1)$ in terms of m and n, where m is such that $1 \le m \le n$. [3]

(v) Using the result in part **(i)**, when P_2 loses a game, find the amount that he is expected to lose in terms of n. [3]

11 The speeds of cars along a busy stretch of road follow a normal distribution. Studies show that a mean speed of 50 km/h is needed to ensure a smooth flow of traffic. When the mean speed falls below 50 km/h, it may lead to road congestion. If this happens, Wireless Road Pricing (WRP) will be used to charge motorists to discourage them from using the road, hence improving the traffic condition. A random sample of the speeds, x km/h, of 120 cars along the stretch of road is recorded and the data are summarised by

$$\sum x = 5415, \quad \sum x^2 = 351500.$$

(i) Calculate unbiased estimates of the population mean and variance of the speeds of the cars. [2]

(ii) What do you understand by the term 'unbiased estimate'? [1]

(iii) Test at the 3% significance level whether WRP is needed. [4]

The Road Transport Authority decides to implement WRP on the same stretch of road. After the implementation, the speeds of a second sample of 80 cars are recorded with a mean speed of 60 km/h and a variance of 1100 (km/h)^2.

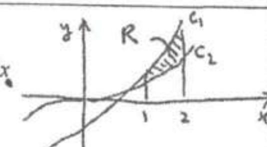
(iv) A test at the 8% significance level of the second sample suggests that the mean speed has increased beyond μ_0. Use an algebraic method to find the maximum value of μ_0. [4]

(v) State one assumption used in obtaining the sample statistics for the second sample. [1]

2018 Year 6 H2 Math Prelim (Paper 2)

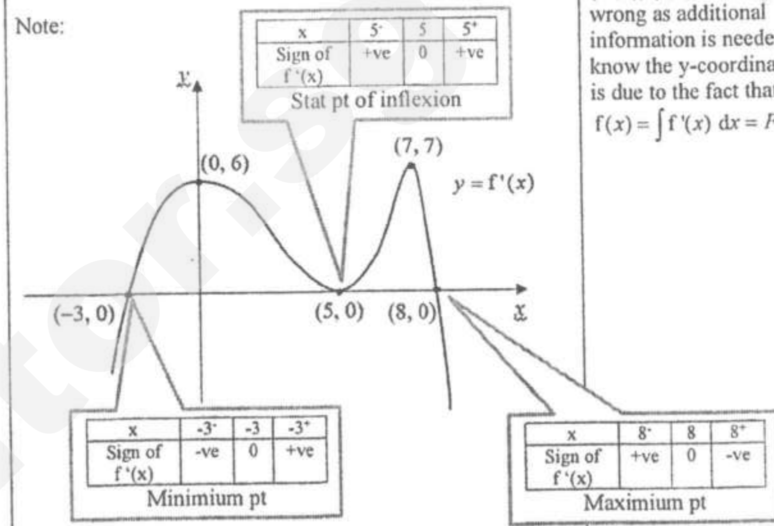
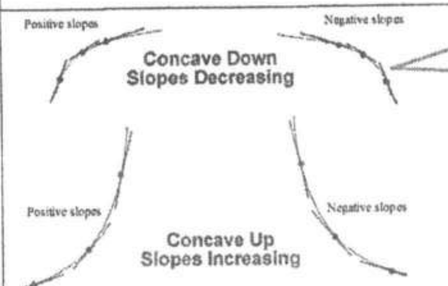
Solution & Markers' Comments

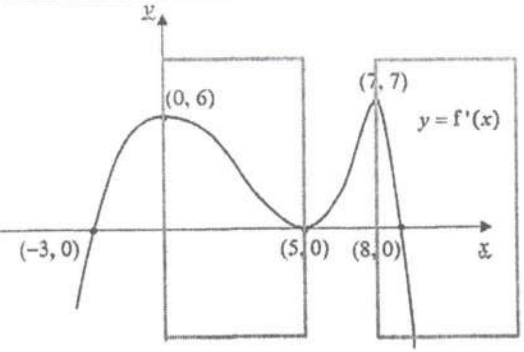
	Suggested Solution	Markers' comments
	Shortest distance $= \lvert \mathbf{b}\times\hat{\mathbf{a}}\rvert$ or $\dfrac{\lvert \mathbf{b}\times\mathbf{a}\rvert}{\lvert\mathbf{a}\rvert}$	• The key word here is 'state' => answer should be quite straightforward. • Quite a handful of students omitted the modulus sign, considered dot product instead or did not consider the unit vector for **a**.
	$\lvert\mathbf{b}\times\mathbf{c}\rvert$ is the area of a parallelogram with sides *OB* and *OC*. $\lvert\mathbf{b}\times\mathbf{c}\rvert = \lvert\mathbf{b}\times(\mathbf{a}+\mathbf{b})\rvert$ $= \lvert(\mathbf{b}\times\mathbf{a})+(\mathbf{b}\times\mathbf{b})\rvert$ $= \lvert\mathbf{b}\times\mathbf{a}\rvert = \lvert-(\mathbf{a}\times\mathbf{b})\rvert = \lvert\mathbf{a}\times\mathbf{b}\rvert$ (shown) Note that the area of the parallelogram with sides *OB* and *OC* is the same as the area of the parallelogram with sides *OA* and *OB* (with same base and vertical height)	• Some students wrote $\lvert\mathbf{b}\times\mathbf{c}\rvert$ as the length of a vector perpendicular to both **b** and **c**. This is not an **interpretation** of the geometrical meaning; it is merely translating in words the meaning of modulus of the cross product of **b** and **c**. • In general, $\lvert\mathbf{b}\times(\mathbf{a}+\mathbf{c})\rvert \neq \lvert\mathbf{b}\times\mathbf{a}\rvert+\lvert\mathbf{b}\times\mathbf{c}\rvert$ • $\mathbf{b}\times\mathbf{b}\neq\lvert\mathbf{b}\rvert^2$ but $\mathbf{b}\bullet\mathbf{b}=\lvert\mathbf{b}\rvert^2$
	$\mathbf{a}\times 2\mathbf{b} = \mathbf{d}\times 3\mathbf{a}$ $(\mathbf{a}\times 2\mathbf{b})-(\mathbf{d}\times 3\mathbf{a})=\mathbf{0}$ $(\mathbf{a}\times 2\mathbf{b})+(3\mathbf{a}\times\mathbf{d})=\mathbf{0}$ $\mathbf{a}\times(2\mathbf{b}+3\mathbf{d})=\mathbf{0}$ Vector **a** is parallel to $2\mathbf{b}+3\mathbf{d}$. $\mathbf{a}=k(2\mathbf{b}+3\mathbf{d}), k\in\mathbb{R}$	• There is no division for vectors. Hence, $\mathbf{a}\times 2\mathbf{b}=-\mathbf{a}\times 3\mathbf{d} \not\Rightarrow 2\mathbf{b}=-3\mathbf{d}$ • 3 vectors parallel to a plane (coplanar vectors) $\Rightarrow \mathbf{a}=\lambda\mathbf{b}+\mu\mathbf{d},\ \lambda,\mu\in\mathbb{R}$ However, the above does not necessarily show a linear relationship as it involves two parameters λ and μ. A linear relationship involves only one parameter e.g. k for this question.
		Total Marks: 5

Qn	Suggested Solution	Markers' comments
2(i)	Total area of n rectangles $=\frac{1}{n}\left[(3^1-2)+\left(3^{1+\frac{1}{n}}-2\right)+\left(3^{1+\frac{2}{n}}-2\right)+\ldots\left(3^{1+\frac{n-1}{n}}-2\right)\right]$ $=\frac{1}{n}\left[(3^1-2)+\left(3\left(3^{\frac{1}{n}}\right)-2\right)+\left(3\left(3^{\frac{2}{n}}\right)-2\right)+\ldots\left(3\left(3^{\frac{n-1}{n}}\right)-2\right)\right]$ $=\frac{1}{n}\sum_{r=1}^{n}\left(3\left(3^{\frac{r-1}{n}}\right)-2\right)$ $=-2+\frac{3}{n}\sum_{r=1}^{n}3^{\frac{r-1}{n}}$ (shown) $-2+\frac{3}{n}\sum_{r=1}^{n}3^{\frac{r-1}{n}}=-2+\frac{3}{n}\left(1+3^{\frac{1}{n}}+\ldots+3^{\frac{n-1}{n}}\right)$ $=-2+\frac{3}{n}\left(\frac{\left(3^{\frac{1}{n}}\right)^n-1}{3^{\frac{1}{n}}-1}\right)$ $=-2+\frac{6}{n\left(3^{\frac{1}{n}}-1\right)}$	• As it is a "show" question, the series that show the sum of the area of the rectangles has to be shown clearly. • The last term of the series has to be shown. Ending the series with "..." means that it is an infinite series, i.e., there are infinite number of rectangles. • The width of the rectangles is a constant at $\frac{1}{n}$, while the breadth increases at $\frac{1}{n}$ for every subsequent rectangle. • This is obviously a geometric progression (GP) with common ratio $3^{\frac{1}{n}}$, and so the sum of n terms of a GP is to be used. • Clearly, method of difference is not applicable.
(ii)	$\lim_{n\to\infty}S_n=\int_1^2 3^x-2\,dx$ $=\left[\frac{3^x}{\ln 3}-2x\right]_1^2$ $=\frac{9}{\ln 3}-4-\left(\frac{3}{\ln 3}-2\right)$ $=\frac{6}{\ln 3}-2$	• Students should be able interpret the meaning of $n\to\infty$ in the context of this question. It means that there is an infinite number of rectangles under the curve, this renders the space between the curve and the rectangles negligible, and so $\lim_{n\to\infty}S_n$ is the area under the curve from $x=1$ to $x=2$.
(iii)	Volume $=\pi\int_1^2(3^x-2)^2-\left(\frac{x^3}{\sqrt{4+x^2}}\right)^2 dx$ $=38.8056$ $=38.81$ unit^3 	• $\pi\int_1^2\left(3^x-2-\frac{x^3}{\sqrt{4+x^2}}\right)^2 dx$ is incorrect because it would be the volume form when the curve

Suggested Solution	Markers' comments
	$y=3^x-2-\dfrac{x^3}{\sqrt{4+x^2}}$ is rotated about the x-axis. • To find the required volume, student has to find the volume generated under curve C_1 first, then subtract away the volume generated under curve C_2. • It is important to learn how to key in the expression correctly into the GC to get the correct answer.
	Total Marks: 9

Suggested Solution	Markers' comments
The sequence is <u>increasing</u> from $\frac{3}{M}$ to 3.	• Since both n and M are positive integers, $n<M+1 \Rightarrow$ last term $n=M$. • As the sequence is finite with M terms, it's incorrect to say that the sequence "converges to 3". We use the term "converge or diverge" for infinite sequence.
$S_1=u_1=\frac{3}{M-1+1}=\frac{3}{M}$ $S_2=u_1+u_2=\frac{3}{M}+\frac{3}{M-2+1}=\frac{3}{M}+\frac{3}{M-1}$ $S_3=u_1+u_2+u_3=\frac{3}{M}+\frac{3}{M-1}+\frac{3}{M-2}$ $\sum_{n=1}^{M}S_n$ $=S_1+S_2+S_3+\ldots+S_M$ $=(u_1)+(u_1+u_2)+(u_1+u_2+u_3)+\ldots+(u_1+\ldots+u_M)$ $=Mu_1+(M-1)u_2+(M-2)u_3+\ldots+2u_{M-1}+u_M$ $=M\left(\frac{3}{M}\right)+(M-1)\left(\frac{3}{M-1}\right)+(M-2)\left(\frac{3}{M-2}\right)+\ldots+3$ $=3M$	• There is no need to combine the terms into a single fraction. • It's important to list down the terms correctly which will enable us to see the pattern of recurring terms
$S_n=u_1+u_2+\ldots+u_n$ $=\frac{3}{M}+\frac{3}{M-1}+\ldots+\frac{3}{M-n+1}$ $>\frac{3}{M}+\frac{3}{M}+\ldots+\frac{3}{M}=\frac{3n}{M}$ (shown)	• Examples : $M-1<M \Rightarrow \frac{3}{M-1}>\frac{3}{M}$, $M-2<M \Rightarrow \frac{3}{M-2}>\frac{3}{M}$
	Total Marks: 6

Qn	Suggested Solution	Markers' comments
4(i)	There are 3 stationary points at $x=-3$, 5 and 8. It has a minimum point at $x=-3$, point of inflexion at $x=5$ and maximum point at $x=8$. Note: 	Many students wrote down the coordinates of the stationary points of $y=f(x)$ as (-3,0), (5,0) and (8,0), this is wrong as additional information is needed to know the y-coordinate. This is due to the fact that $f(x)=\int f'(x)\,dx=F(x)+c$.
(ii)	 The gradient is decreasing.	Students will need to know that terms concave upwards / downwards which can be tested in the A-levels.



ange of x are either $0<x<5$ or $x>7$. **Note that the raph given is a gradient curve which means we are ooking for parts of the curve that decreases as x ncreases.**	
Ie is right because for $-3<x<5$, $f'(x)>0$, is increasing, thus $f(x)>7$	There is a variety of reasons given but the most precise answer is the one given here. Students need to be aware that a positive gradient will imply that as x increases, the change in y will be positive.
$\int_{-3}^{8} x f(x)\,dx = \left[\frac{x^2}{2} f(x)\right]_{-3}^{8} - \int_{-3}^{8} \frac{x^2}{2} f'(x)\,dx$ ince $\frac{x^2}{2} \geq 0$ and $f'(x) \geq 0$ for $-3 \leq x \leq 8$, $\therefore \frac{x^2}{2} f'(x) \geq 0$ and $\int_{-3}^{8} \frac{x^2}{2} f'(x)\,dx > 0$. Many do not know this concept. $\int_{-3}^{8} x f(x)\,dx$ $= 32f(8) - \frac{9}{2} f(-3) - \int_{-3}^{8} \frac{x^2}{2} f'(x)\,dx < 32f(8) - \frac{9}{2} f(-3)$ Ie is wrong to make this statement.	At this point after integration by parts, it should be obvious that you will need to take information from the curve of $y = f'(x)$ in order to answer the qn.
	Total Marks: 8

Qn	Suggested Solution	Markers' comments
5(a)(i)	$(x+y)^2 = 4e^{xy}$ $2(x+y)\left(1+\frac{dy}{dx}\right) = 4e^{xy}\left(x\frac{dy}{dx}+y\right)$ $\frac{dy}{dx} = \frac{2ye^{xy}-y-x}{y+x-2xe^{xy}}$	Note: 1. $e^{xy} \neq e^x e^y = e^{x+y}$ 2. $\ln(4e^{xy}) \neq 4xy$. It should be $\ln(4e^{xy}) = \ln 4 + xy$ 3. Do not attempt to "sq root" the equation as it will result in $(x+y)^2 = 4e^{xy} \Rightarrow x+y = \pm 2e^{\frac{xy}{2}}$ You will then have to choose the correct equation based on the point A in **(ii)**. Do not overcomplicate your working.
(ii)	When $x=0$, $(0+y)^2 = 4e^{(0)y}$ $y = 2 \ (\because y>0)$ When at (0, 2), $\frac{dy}{dx} = \frac{2(2)-2}{2} = 1$ Equation of the tangent to the curve at (0, 2) is $y-2 = 1(x-0)$ $y = x+2$	
(iii)	Substitute $y = x+2$ into $(x+y)^2 = 4e^{xy}$, $(x+x+2)^2 = 4e^{x(x+2)}$ $(2x+2)^2 = 4e^{x^2+2x}$ $(x+1)^2 = e^{x^2+2x}$ Using G.C., $x=-2$ or $x=0$ (reject $\because$ it's point A) $\therefore B(-2,0)$	Quite a number of students used the y-coordinate of the intersection between the two curves $y=(x+1)^2$ and $y=e^{x^2+2x}$ as the y-coordinate of B which is wrong. Note that you are solving for the x-coordinate of B in that equation and you will need the equation of the tangent i.e. $y = x+2$ to get the y-coordinate of B.
(b)	Let r and h be the radius and the height of the cylinder respectively. Fixed vol. $p = \pi r^2 h \Rightarrow h = \frac{p}{\pi r^2}$ Surface area, S $= 2\pi r^2 + 2\pi r h$ $= 2\pi r^2 + 2\pi r\left(\frac{p}{\pi r^2}\right)$ $= 2\pi r^2 + \frac{2p}{r}$ $\frac{dS}{dr} = 4\pi r - \frac{2p}{r^2}$	Students need to have a clear picture of which variables are varying. From the qn, it is clear that p is a constant (i.e. it cannot vary). You may vary r but its variation will affect h since they are connected by the volume equation. So for this scenario, you are to vary r such that you can get the minimum surface area.

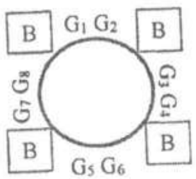
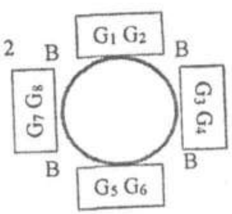
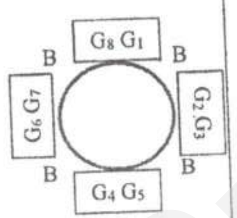
	For min. S, $\frac{dS}{dr}=4\pi r-\frac{2p}{r^2}=0 \Rightarrow r=\left(\frac{p}{2\pi}\right)^{\frac{1}{3}}$ $\frac{d^2S}{dr^2}=4\pi+\frac{4p}{r^3}>0$ since r and p are positive. $\therefore$ S is minimum when $r=\left(\frac{p}{2\pi}\right)^{\frac{1}{3}}$ cm.	Note: $\frac{d}{dr}(2\pi rh)\neq 2\pi h$ as h varies with r. For students who use the first derivative test, they should present the working as $\frac{dS}{dr}=4\pi r-\frac{2p}{r^2}=\frac{2(2\pi r^3-p)}{r^2}$ so that the sign of the derivative can be easily observed by the examiner.
		Total Marks: 12

n	Suggested Solution	Markers' comments
(i)	Let X be the random variable denoting the number of questions answered correctly out of 15. $X \sim \mathrm{B}\left(15,\frac{1}{n}\right)$ [It is always good practice to write down the distribution.] $\mathrm{P}(X=3)=\mathrm{P}(X=4)$ $\binom{15}{3}\left(\frac{1}{n}\right)^3\left(\frac{n-1}{n}\right)^{12}=\binom{15}{4}\left(\frac{1}{n}\right)^4\left(\frac{n-1}{n}\right)^{11}$ [You can refer to MF26 for the pdf for a binomial distribution.] $\left(\frac{n-1}{n}\right)=3\left(\frac{1}{n}\right)$ $n=4$ <u>Using your GC to check the answer</u> From GC, $n=4$	Common errors: 1. Wrong pdf for binomial. In particular, students tend to miss out the $\binom{15}{r}$ part. 2. Confusing n in the question with n in MF26. 3. Errors in solving the equation.

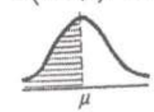
```
NORMAL FLOAT AUTO REAL RADIAN MP
Plot1 Plot2 Plot3
\Y1=binompdf(15,1/X,3)
\Y2=binompdf(15,1/X,4)
\Y3=
\Y4=
\Y5=
\Y6=
```

X	Y1	Y2
1	0	0
2	.01389	.04166
3	.12988	.19482
4	.2252	.2252
5	.25014	.1876
6	.23626	.14175
7	.20862	.10431

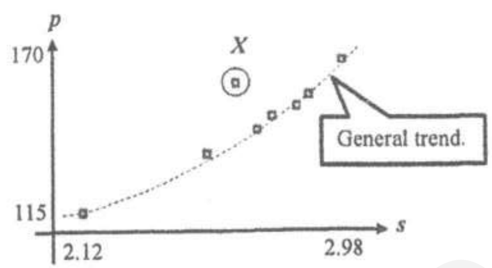
(ii)	$X \sim \mathrm{B}\left(15,\frac{1}{4}\right)$ <u>**Method 1**</u> Let T be the score for 15 questions. Then $T=3X-(15-X)=4X-15$ $s=\mathrm{E}(T)=4\mathrm{E}(X)-15$ $=4\left[15\left(\frac{1}{4}\right)\right]-15=0$ [Refer to MF26 for $\mathrm{E}(X)$ formula.] <u>**Method 2**</u> Let A be the score for 1 question. $\mathrm{E}(A)=3\left(\frac{1}{4}\right)-1\left(\frac{3}{4}\right)=0$ $s=\mathrm{E}(T)=\mathrm{E}(A_1+A_2+\ldots+A_{15})=15\mathrm{E}(A)=0$ <u>**Method 3**</u> Let Y be the random variable denoting the number of questions answered incorrectly out of 15. $Y\sim\mathrm{B}\left(15,\frac{3}{4}\right)$. $s=3\mathrm{E}(X)-\mathrm{E}(Y)$ $=3\left[15\left(\frac{1}{4}\right)\right]-\left[15\left(\frac{3}{4}\right)\right]=0$ [Note that X and Y are not independent. You need to write Y in terms of X if you want to calculate $\mathrm{Var}(3X-Y)$.]	Common Errors: 1. Some students assumed that scores must be non-negative. 2. Students who constructed the pdf table on scores for 15 questions often missed out the case $T=-15$. 3. <u>Limitation of GC</u>: Even with the correct pdf, GC can only give s as -2.3×10^{-13} and not the exact answer 0.
(iii)	$\mathrm{P}(-4<T<4)=\mathrm{P}(-4<4X-15<4)$ $=\mathrm{P}(2.75<X<4.75)$ $=\mathrm{P}(3\le X\le 4)$ [From (i) $\mathrm{P}(X=3)=\mathrm{P}(X=4)$] $=\mathrm{P}(X\le4)-\mathrm{P}(X\le2)$ or $2\mathrm{P}(X=4)$ or $2\mathrm{P}(X=3)$ $=0.450$	**Major Concept Error:** Students assumed T to be a normal random variable when T is a discrete random variable.
		Total Marks: 8

Suggested Solution	Markers' comments
Ways to divide the 12 students with 1 boy in each group $=\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}=2520$ **OR** $\frac{8!}{2!2!2!2!}=2520$ First, put 1 boy in each gp. Next allocate 2 girls to each gp.	There were quite a few students who did this or a variant: $\binom{8}{2}+\binom{6}{2}+\binom{4}{2}+\binom{2}{2}$
P(the 3 members in each group are next to each other) $=\frac{(4-1)!(3!)^4}{(12-1)!}=0.000195$ $(4-1)!=$ ways to sit the 4 gps in a circle $3!=$ ways to arrange the 3 members within each gp	Many students had $(3!)\times 4$ instead of $(3!)^4$
P(every boy is separated from each other by exactly 2 girls) $=\frac{(4-1)!8!}{11!}=\frac{1}{165}$  $(4-1)!=$ ways to sit 4 boys in a circle $8!=$ ways to sit the rest of 8 girls in between the boys OR $\frac{(4-1)!(2520)(2!)^4}{11!}=\frac{1}{165}$ 2520 = ways to sit any 2 girls in each of the 4 slots between the boys. Similar to 1st part's concept to allocate 2 girls to each gp which already has 1 boy. Alternative $\frac{(8-1)!(4!)}{11!}\times 2$ $=\frac{1}{165}$  $(8-1)!=$ ways to sit 8 girls in a circle $4!=$ ways to sit the rest of 4 boys in between the girls	Some students gave $\frac{4!8!}{11!}$ which did not consider the circular nature of the problem. Many who used the Alternative 1 did not multiply by 2. There are some who tried to do probability tree method (see alternative 2) but none got it correct due to its complexity and hence caution is advised.

(iii)	The event "No 2 boys are next to each other" is equivalent to "a least 1 girl between every 2 boys". Thus event A is a subset of event B. Now $P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A)}{P(B)}\neq P(A)$, as $P(B)<1$. Thus A and B are not independent.	Many students did not explain explicitly with mathematical working. Many students did not indicate $P(B)<1$ but marks are only deducted for those who computed $P(B)$ incorrectly. Note that only $P(B)\neq 1$ is required and hence there is no need to compute $P(B)$.
		Total Marks: 9

Qn	Suggested Solution	Markers' comments
8(i)	$P(\mu<X<16)=0.35$ $P(X<16)-P(X<\mu)=0.35$ $P(X<16)=0.85$ $P\left(Z<\frac{16-\mu}{2}\right)=0.85$ Using GC, $\frac{16-\mu}{2}=1.0364$ $\mu=13.927\approx 14$ (shown)	• An analytical method of standardization and inverse norm is expected. • $P(X<\mu)=0.5$ 
(ii)	Let T be the total time taken by the first two groups of riders. $T=X_1+X_2\sim N(14\times 2, 2^2\times 2)\Rightarrow T\sim N(28,8)$ Required Probability $=P(T<30)$ $=0.76025$ $=0.760$ (3 sf)	• The 12th rider belongs to the third group, and so for the 12th rider to ride before 10.30 am, the first two groups must complete their rides within 30 minutes. • $X_1+X_2\neq 2X$ • $2X$ means twice the time taken by one group of riders. • X_1+X_2 means the sum of the time taken by two groups of riders.

i)	Let Y be the number of long rides out of 4. $Y \sim \mathrm{B}(4, \mathrm{P}(X \geq 15)) \Rightarrow Y \sim \mathrm{B}(4, 0.30854)$ $\mathrm{P}(Y \geq 2) = 1 - \mathrm{P}(Y \leq 1)$ $= 1 - 0.63661$ $= 0.36339$ $= 0.363$ (shown)	• Sufficient working is necessary since this is a "show" question.
v)	P(none of these rides are consecutive$\mid Y \geq 2$) $= \dfrac{3[\mathrm{P}(X \geq 15)]^2[\mathrm{P}(X < 15)]^2}{0.363}$ $= \dfrac{0.13655}{0.363}$ $= 0.376$	• With the words "given that", students should recognize that this is a conditional probability question. • The numerator is the probability for at least 2 long rides and none of these rides are consecutive. This means the 3 possible scenarios are LSLS, SLSL, LSSL (L: for long ride; S: for short ride).
		Total Marks: 9

n	Suggested Solution	Markers' comments
i)	 Since there is no controlled variable, both p on s as well as s on p scatter plots are accepted.	Common Errors: 1. Wrong labelling of axes. 2. Careless plotting of points: the points are not always accurately plotted, leading to wrong point X identified. Popular <u>wrong</u> points identified are: the first point, the last point or the 4th point.

(ii)	Omitting X, $r = 0.995$ [between p and s^5] $r = 0.976$ [between $\ln p$ and s] Since the r-value between p and s^5 is closer to 1 than that between $\ln p$ and s (indicating a stronger linear relationship between p and s^5), $p = a + bs^5$ is the better model. Note that $\ln p = c + ds \Rightarrow p = e^c e^{ds}$ curve has the same shape as $p = a + bs^5$. Thus we are not able to decide the better model purely from the scatter plot.	Students tend to forget to delete the point X, or they deleted a neighbouring point. You can press 'trace' button to track the points on the scatter plot. 
(iii)	From GC, $p = 101.449 + 0.279507s^5$ When $s = 2.63$, $p = 136.618 = 137$ (3 sf) Two reasons: 1. The r-value (0.995) is very close to 1, indicating a strong fit of model to the data. 2. $s = 2.63$ lies within data range $2.12 \leq s \leq 2.98$, thus interpolation is used. Interpolation is based on the <u>given</u> value (in this case $s = 2.63$) lying within data range, not the estimated one.	• Instead of finding an estimate for p when s = 2.63, some students proceeded to find the correct data value for p when s = 2.63. Realise that there should not be a follow up question on reasons for the data value to be reliable. • For those who have chosen X to be the first (or last) point, you should realise by now that the estimate is unreliable because it is an extrapolation.
(iv)	High correlation **<u>does not necessarily</u>** imply causation. There could be other factors (e.g. strong economy, increase in population) which can lead to higher property price index. So the higher property price index need not be caused by higher stock index.	Note that both statements need to be stated to give a complete answer in context of the question, which some students were not able to do during the exam.
		Total Marks: 10

n	Solution	Markers' comments
(i)	$\sum \mathrm{P}(X=x)=1$ $\sum_{x=1}^{n}\frac{ax}{n(n+1)}=1$ $\frac{a}{n(n+1)}\sum_{x=1}^{n}x=1$ $\frac{a}{n(n+1)}\left(\frac{n(n+1)}{2}\right)=1$ $a=2$ (shown) <u>Alternative</u> When $n=1, \mathrm{P}(X=1)=\frac{a}{1(2)}=1 \therefore a=2$	• For any discrete random variable X, the sum of probability values for all possible values of x is 1. • Note that in this case, a and n are constants so they can be factored out from the sum in the first method. • The resultant sum is an A.P so apply the A.P formula. (In general, note that it is possible for pure math knowledge to apply in statistics questions)
)	$\mathrm{E}(X)=\sum_{x\in S} x\mathrm{P}(X=x)$ $=\frac{a}{n(n+1)}\sum_{x=1}x^2$ $=\frac{2}{n(n+1)}\left(1^2+2^2+3^2+...+n^2\right)$ $=\frac{2}{n(n+1)}\sum_{r=1}^{n}r^2$ $=\frac{2}{n(n+1)}\left[\frac{n}{6}(n+1)(2n+1)\right]$ $=\frac{1}{3}(2n+1)$	• This part is generally well done.
i)	For this game, the tokens are picked without replacement thus the two tokens picked by the players will be different. For any two possible tokens picked, say a and b, there are two equally possible outcomes: P_1 picks a and P_2 picks b or vice versa, out of which only one where P_2 will lose. Thus the probability of P_2 losing is 0.5. <u>Alternative</u>	• Some elaboration is required here, beyond stating that there will be no draw and rephrasing the statement that both P_1 and P_2 are equally likely to lose.

	$\mathrm{P}(P_2 \text{ loses a game})$ or $\mathrm{P}(P_1 \text{ wins a game})$ $=\left(\frac{1}{n+1}\right)\left(\frac{n}{n}\right)+\left(\frac{1}{n+1}\right)\left(\frac{n-1}{n}\right)+\left(\frac{1}{n+1}\right)\left(\frac{n-2}{n}\right)+...+\left(\frac{1}{n+1}\right)\left(\frac{1}{n}\right)$ Prob P_1 picks the number "$n+1$" Prob P_2 picks a number smaller than "$n+1$" Prob P_1 picks the number "n" Prob P_2 picks a number smaller than "n" Prob P_1 picks the number "2" Prob P_2 picks a number smaller than "2" $=\frac{1}{n(n+1)}(1+2+3+...+n)$ $=\frac{1}{n(n+1)}\left(\frac{n}{2}(1+n)\right)$ $=\frac{1}{2}$	
(iii)	Required probability $=\mathrm{P}(P_2 \text{ loses } \$(m+1) \mid P_2 \text{ loses})$ $=\frac{\mathrm{P}(P_2 \text{ loses } \$(m+1))}{\mathrm{P}(P_2 \text{ loses})}$ $=\frac{\mathrm{P}(P_1 \text{ picks "}m+1\text{"})\times \mathrm{P}(P_2 \text{ picks a no. smaller than "}m+1\text{"})}{\mathrm{P}(P_2 \text{ loses})}$ $=\frac{\left(\frac{1}{n+1}\right)\left(\frac{m}{n}\right)}{\frac{1}{2}}$ $=\frac{2m}{n(n+1)}$	• Note that the required probability is conditional on the event that P_2 loses. • To calculate the denominator i.e. $\mathrm{P}(P_2 \text{ loses } \$m+1)$, note that P_1 must draw the tile $m+1$ and then P_2 must draw a tile smaller than $m+1$. (For convenience's sake, you may assume P_1 draws first but the answer is still the same even if P_2 draws first)

)	Let L be the random variable denoting the amount lost for a game if he loses a game. $L = X + 1$ $\mathrm{E}(L)$ $= \mathrm{E}(X+1)$ $= \left(\frac{1}{3}(2n+1)\right)+1$ $= \frac{1}{3}(2n+4)$	• First define the random variable L, before trying to link it to X. • Note that $0 \le L \le n+1$ and $\mathrm{P}(L = m+1)$ $= \mathrm{P}(X = m)$ • Thus $L - 1 = X$.
		Total Marks : 12

n	Suggested Solution	Markers' comments
(i)	Unbiased estimate of population mean, $\bar{x} = \frac{\sum x}{n} = \frac{5415}{120} = 45.125$ (exact) Unbiased estimate of population variance, $s^2 = \frac{1}{119}\left[351500 - \frac{(5415)^2}{120}\right] = 900.40 = 900$ (3 s.f.)	• Remember to use the exact or 5 s.f. answer for subsequent working.
i)	An estimate is unbiased when the **expected value of the estimator** T used to obtain the estimate is equal to the **value of the population parameter** θ, i.e. $\mathrm{E}(T) = \theta$.	• See Tut19 Q5(ii)
ii)	Let X denote the speed a randomly chosen car (in km/h) with population mean μ. To test $\mathrm{H}_0 : \mu = 50$ vs $\mathrm{H}_1 : \mu < 50$ Conduct 1-tail test at 3% significance level. Under H₀, $\bar{X} \sim \mathrm{N}\left(50, \frac{900.40}{120}\right)$. Use the 5 s.f figure for better accuracy Using a z-test, $p\text{-value} = \mathrm{P}(\bar{X} \le 45.125) = 0.0376$ (3 s.f.) Since p-value > 0.03, we do not reject H_0 and conclude that there is insufficient evidence at 3% significance level that the mean speed is less than 50 km/h. Hence WRP is not needed.	• Important to define the random variable X. • X normal $\Rightarrow \bar{X}$ normal, no need to use CLT. • Important to answer the question at the end i.e. "WRP is not needed".

(iv)	Unbiased estimate of population variance, $s^2 = \frac{80}{79}[1100] = 1113.9 = 1110$ (3 s.f.) To test $\mathrm{H}_0 : \mu = \mu_0$ vs $\mathrm{H}_1 : \mu > \mu_0$ Conduct 1-tail test at 8% significance level. Under H₀, $\bar{X} \sim \mathrm{N}\left(\mu_0, \frac{1113.9}{80}\right)$ If mean speed exceeds $\mu_0 \Rightarrow \mathrm{H}_0$ is rejected, $\Rightarrow p\text{-value} = \mathrm{P}(\bar{X} \ge 60) \le 0.08$ $\mathrm{P}\left(Z \ge \frac{60-\mu_0}{\sqrt{\frac{1113.9}{80}}}\right) \le 0.08$ $\frac{60-\mu_0}{\sqrt{\frac{1113.9}{80}}} > 1.4051$ $\mu_0 < 54.757$ $\therefore$ maximum $\mu_0 = 54.7$	• The sample variance = 1100. We need to find the unbiased estimate by using the formula : $s^2 = \frac{n}{n-1}$(sample var) • Qn requires the use of an algebraic method to find max μ_0 as shown on the left. A GC method is not acceptable.
(v)	Assumption : 1. The speeds of the second sample of cars comes from a random sample, **OR** 2. The speeds of the second sample of cars are independent	
		Total Marks: 12

EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2018

General Certificate of Education Advanced Level

Higher 2

MATHEMATICS **9758/01**

Paper 1 [100 marks]

12 September 2018

3 hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

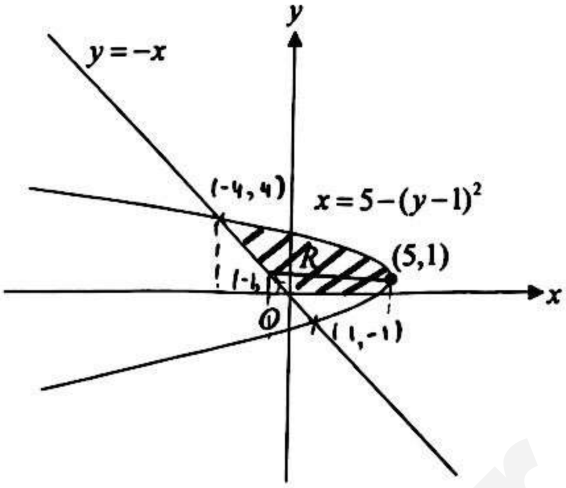
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

1



The shaded region R bounded by the curve $x=5-(y-1)^2$, the line $y=-x$ and the x-axis is rotated about the x-axis through 360°. Find the volume of the solid formed, leaving your answer to 2 decimal places. [4]

2 (i) Solve the inequality $\dfrac{x^2-ax-a}{x-a} \geq a$, where a is a positive real constant, leaving your answer in terms of a. [4]

(ii) Hence, by using a suitable value for a, solve the inequality

$$\frac{4e^{2x}-e^{x}-1}{4e^{x}-1} \geq \frac{1}{4}$$

leaving your answer in exact form. [3]

3 The parametric equations of a curve are $x=at$, $y=at^3$, where a is a positive constant.

(i) The point P on the curve has parameter p and the tangent to the curve at point P cuts the y-axis at S and the x-axis at T. The point M is the midpoint of ST. Find a Cartesian equation of the curve traced by M as p varies. [5]

(ii) Find the exact area bounded by the curve, the line $x=0$, $x=3$ and the x-axis, giving your answer in terms of a. [3]

4 It is given that $y=\sin^{-1}x\cos^{-1}x$, where $-1\le x\le 1$.

(i) Show that $\sqrt{1-x^2}\dfrac{dy}{dx}=\cos^{-1}x-\sin^{-1}x$. [1]

(ii) Show that $\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=-2$ [2]

(iii) Hence find the exact value of A, B and C if y can be expressed as $Ax+Bx^2+Cx^3$, up to (and including) the term in x^3. [4]

(iv) A student used (iii) to estimate that $\sin^{-1}(0.8)\cos^{-1}(0.8)\approx 0.8A+0.8^2B+0.8^3C$. Explain, with working, if his estimate is a good one. [1]

5 (a) Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$. Point C is on the line which contains A and is parallel to $\mathbf{b}$. It is given that the vectors $\mathbf{a}$ and $\mathbf{b}$ are both of magnitude 2 units and are at an angle of $\sin^{-1}(1/6)$ to each other. If the area of triangle OAC is 3 units^2, use vector product to find the possible position vectors of C in terms of $\mathbf{a}$ and $\mathbf{b}$. [5]

(b) Referred to the origin O, the points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ where $\mathbf{p}$ and $\mathbf{q}$ are non-parallel, non-zero vectors. Point R is on PQ produced such that $PQ:QR=1:\lambda$. Point N and M are the mid-points of OP and OR respectively.

(i) Find the position vector of R in terms of λ, $\mathbf{p}$ and $\mathbf{q}$. [1]

F is a point on OQ such that F, P and M are collinear.

(ii) Find the ratio $OF:FQ$, in terms of λ. [4]

6 **Do not use a calculator in answering this question.**

(a) It is given that two complex numbers z and w satisfy the following equations

$$13z = (4-7\mathrm{i})w,$$
$$z - 2w = 5 - 4\mathrm{i}.$$

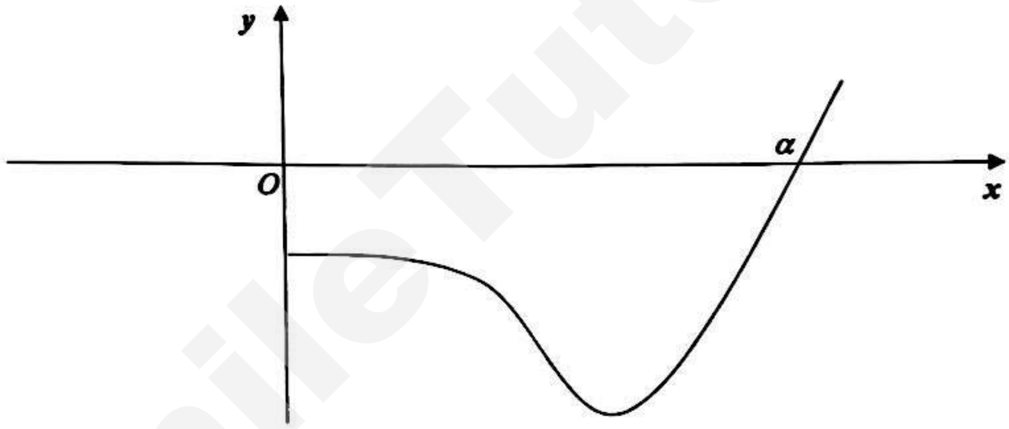
Find z and w. **[4]**

(b) It is given that $q = -\sqrt{3} - \mathrm{i}$.

(i) Find an exact expression for q^6, giving your answer in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. **[3]**

(ii) Find the three smallest positive whole number values of n for which $\dfrac{q^n}{q^*}$ is purely imaginary. **[4]**

7



It is given that $\mathrm{f}(x) = 2x^6 - 4x^4 - 6x^2 - 7$. The diagram shows the curve with equation $y = \mathrm{f}(x)$ for $x \ge 0$. The curve crosses the positive x-axis at $x = \alpha$.

(i) Find the value of α, giving your answer correct to 3 decimal places. **[1]**

(ii) Show that $\mathrm{f}(x) = \mathrm{f}(-x)$ for all real values of x. What can be said about the six roots of the equation $\mathrm{f}(x) = 0$? **[4]**

It is given that $\mathrm{g}'(x) = \mathrm{f}(x)$, for all real values of x.

(iii) Determine the x-coordinates of all the stationary points of graph of $y = \mathrm{g}(x)$ and determine their nature. **[3]**

(iv) For which values of x is the graph of $y = \mathrm{g}(x)$ concave upwards? **[3]**

8 **(a)** **(i)** Show that, for $r \in \mathbb{Z}$, $r \geq 2$,

$$\frac{1}{(r-1)!} - \frac{2}{r!} + \frac{1}{(r+1)!} = \frac{r^2-r-1}{(r+1)!}.$$ [1]

Let $S_n = \sum_{r=2}^{n} \frac{r^2-r-1}{(r+1)!}$.

(ii) Hence find S_n in terms of n. [3]

(iii) Show that S_n converges to a limit L, where L is to be determined. [2]

(iv) Find the least integer value of n such that S_n differs from L by less than 10^{-10}. [2]

(b) **(i)** Suppose that f is a continuous, strictly decreasing function defined on $[1, \infty)$, with $\mathrm{f}(x) > 0$, $x \geq 1$. According to the Maclaurin-Cauchy test, then the infinite series $\sum_{n=1}^{\infty} \mathrm{f}(n)$ is convergent if and only if the integral $\int_1^{\infty} \mathrm{f}(x)\,\mathrm{d}x$ is finite. By applying the Maclaurin-Cauchy test on the function f defined by $\mathrm{f}(x) = \frac{1}{x}$, $x \geq 1$, determine if the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent. [2]

(ii) Let p be a positive number. By considering the Maclaurin-Cauchy test, show that if $p > 1$, the infinite series $1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \frac{1}{n^p} + \ldots$ is convergent. [2]

9 A drilling company plans to install a straight pipeline AB through a mountain. Points (x, y, z) are defined relative to a main control site at the foot of the mountain at $(0,0,0)$, where units are metres. The x-axis points East, the y-axis points North and the z-axis points vertically upwards. Point A has coordinates $(-200,150,10)$ while point B has coordinates $(100,10,a)$, where a is an integer. Point B is at a higher altitude than Point A.

(i) Given that the pipeline AB is of length 337 metres, find the coordinates of B. [3]

A thin flat layer of rock runs through the mountain and is contained in the plane with equation $20x + y + 2z = -837$.

(ii) Find the coordinates of the point where the pipeline meets the layer of rock. [4]

To stabilise the pipeline, the drilling company decides to build 2 cables to join points A and B to the layer of rock. Point A is joined to Point P while point B is joined to Point Q.

(iii) Assuming that the minimum length of cable is to be used, find the length PQ. [2]

(iv) Show that the pipeline is at an angle of 10.8^0 to the horizontal plane. [2]

(v) After the pipeline is completed, a ball bearing is released from point B to roll down the pipeline to check for obstacles. The ball bearing loses altitude at a rate of $0.3t$ metres per second, where t is the time (in seconds) after its release. Find the speed at which the ball bearing is moving along the pipeline 10 seconds after its release. [3]

10 An epidemiologist is studying the spread of a disease, dengue fever, which is spread by mosquitoes, in town A. P is defined as the number of infected people (in thousands) t years after the study begins. The epidemiologist predicts that the rate of increase of P is proportional to the product of the number of infected people and the number of uninfected people. It is known that town A has 10 thousand people of which a thousand were infected initially.

(i) Write down a differential equation that is satisfied by P. [1]

(ii) Given that the epidemiologist projects that it will take 2 years for half the town's population to be infected, solve the differential equation in **(i)** and express P in terms of t. [6]

(iii) Hence, sketch a graph of P against t. [2]

A second epidemiologist proposes an alternative model for the spread of the disease with the following differential equation:

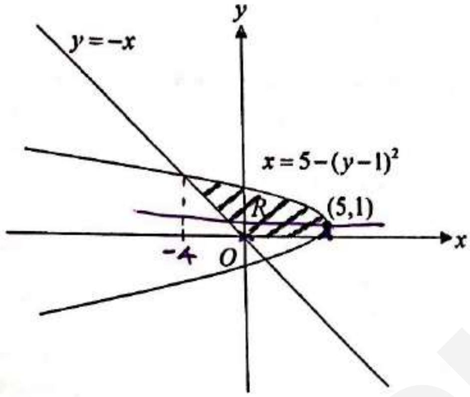
$$\frac{dP}{dt} = \frac{2\cos t}{(2-\sin t)^2} \qquad (*).$$

(iv) Using the same initial condition, solve the differential equation (*) to find an expression of P in terms of t. [3]

(v) Find the greatest and least values of P predicted by the alternative model. [2]

(vi) The government of town A deems the alternative model as a more realistic model for the spread of the disease as it more closely follows the observed pattern of the spread of the disease. What could be a possible factor contributing to this? [1]

1



The shaded region R bounded by the curve $x=5-(y-1)^2$, the line $x=5-(y-1)^2$ and the x-axis is rotated about the x-axis through 360°. Find the volume of the solid formed, leaving your answer to 2 decimal places. [4]

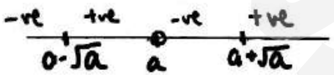
Suggested Solution	Comments
$x=5-(y-1)^2$ $y=1\pm\sqrt{5-x}$ $y=1+\sqrt{5-x}$ intersect $y=-x$ at $(-4,4)$ $y=1-\sqrt{5-x}$ cuts the x-axis at $(4,0)$ Volume required $=\pi\int_{-4}^{5}\left(1+\sqrt{5-x}\right)^2\mathrm{d}x-\pi\int_{4}^{5}\left(1-\sqrt{5-x}\right)^2\mathrm{d}x-\frac{1}{3}\pi(4)^2(4)$ $=201.06$ (to 2 d.p.)	This part is generally well done. Most students were able to recognise that they have to make y the subject. There is a fair number of students who missed out on finding (4, 0). Students who missed out on finding (4,0) also missed out on $\pi\int_{4}^{5}\left(1-\sqrt{5-x}\right)^2$. Most students were able to identify the volume of the cone. It is important for students to be careful in identifying the correct interval and to express the integral accurately, as there were incorrect variations such as $\pi\int_{-4}^{4}\left(1+\sqrt{5-x}\right)^2\mathrm{d}x$ and $\pi\int_{-4}^{5}\left(1+\sqrt{5-x}-(-x)\right)^2\mathrm{d}x$.

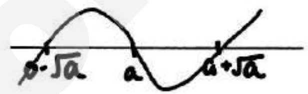
2 (i) Solve the inequality $\dfrac{x^2-ax-a}{x-a} \geq a$, where a is a positive real constant, leaving your answer in terms of a. [4]

(ii) Hence, by using a suitable value for a, solve the inequality

$$\frac{4e^{2x}-e^x-1}{4e^x-1} \geq \frac{1}{4}$$

leaving your answer in exact form. [3]

Suggested Solution	Comments
(i) $\dfrac{x^2-ax-a}{x-a} \geq a \quad (x \neq a)$ $\dfrac{(x^2-ax-a)-a(x-a)}{x-a} \geq 0$ $\dfrac{x^2-2ax+a^2-a}{x-a} \geq 0$ Consider $x^2-2ax+(a^2-a)=0$ $x = \dfrac{-(-2a)\pm\sqrt{(-2a)^2-4(1)(a^2-a)}}{2}$ $= \dfrac{2a\pm\sqrt{4a}}{2} = a\pm\sqrt{a}$	This part was generally well done. Most students were able to recognise the need to express as a single rational function. A glaring error in some students is $\dfrac{(x^2-ax-a)-a(x-a)}{x-a}$ $=\dfrac{x^2-2ax}{x-a}$ Students who attempted the earlier part correctly were mostly able to find the 2 roots $a\pm\sqrt{a}$.
<u>Method 1 (test critical points)</u> $\dfrac{x^2-2ax+a^2-a}{x-a} \geq 0$ can be rewritten as $\dfrac{\left(x-(a+\sqrt{a})\right)\left(x-(a-\sqrt{a})\right)}{x-a} \geq 0$ Using sign test, we can check whether each factor is positive or negative for the different range of values of x  Thus, we have $a-\sqrt{a} \leq x < a$ or $x \geq a+\sqrt{a}$	An error in some students made was to argue that $x^2-2ax+a^2-a$ $=(x-a)^2-a$ is always positive for all real values of x.

<u>Method 2</u> $\dfrac{(x^2-ax-a)-a(x-a)}{x-a} \geq 0$ $(x-a)\left[x^2-2ax+(a^2-a)\right] \geq 0 \quad (x \neq a)$  $a-\sqrt{a} \leq x < a$ or $x \geq a+\sqrt{a}$	Most students were able to demonstrate the correct method to solve the inequality and that $x \neq a$.
(ii) Given $\dfrac{4e^{2x}-e^x-1}{4e^x-1} \geq \dfrac{1}{4}$ $\dfrac{(e^x)^2-\frac{1}{4}(e^x)-\frac{1}{4}}{(e^x)-\frac{1}{4}} \geq \dfrac{1}{4}$ Replace x with e^x and let $a=\dfrac{1}{4}$, $-\dfrac{1}{4} \leq e^x < \dfrac{1}{4}$ or $e^x \geq \dfrac{3}{4}$ $0 < e^x < \dfrac{1}{4}$ or $e^x \geq \dfrac{3}{4}$ $x < \ln\left(\dfrac{1}{4}\right)$ or $x \geq \ln\left(\dfrac{3}{4}\right)$	This part has a few variations and was not as well done. Most students were unable to provide the necessary understanding to demonstrate the correct argument of how the inequality $-\dfrac{1}{4} \leq e^x < \dfrac{1}{4}$ leads to $x < \ln\left(\dfrac{1}{4}\right)$.

3 The parametric equations of a curve are $x = at$, $y = at^3$, where a is a positive constant.

(i) The point P on the curve has parameter p and the tangent to the curve at point P cuts the y-axis at S and the x-axis at T. The point M is the midpoint of ST. Find a Cartesian equation of the curve traced by M as p varies. [5]

(ii) Find the exact area bounded by the curve, the line $x = 0$, $x = 3$ and the x-axis, giving your answer in terms of a. [3]

Suggested Solution	Comments
Solutions:	
(i) $x = at$, $y = at^3$ $\frac{dx}{dt} = a$, $\frac{dy}{dt} = 3at^2$	This question is generally well done. Students are able to write out very good solutions by following the steps to obtain the equation of tangent at point P.
$\frac{dy}{dx} = \frac{3at^2}{a}$ $= 3t^2$ Therefore, gradient of the tangent at $P = 3p^2$	Minority of students have confusion of the symbols, hence unable to find the gradient of tangent at P. For example, they differentiate x with respect to a instead of t, or substitute a by p instead of substitute t by p.
Equation of tangent at P: $y - ap^3 = 3p^2(x - ap)$ $y - ap^3 = 3p^2x - 3ap^3$ $y = 3p^2x - 2ap^3$ For point S, $x = 0, y = -2ap^3$. $\therefore S$ is $(0, -2ap^3)$ For point T, $y = 0, x = \frac{2ap^3}{3p^2} = \frac{2}{3}ap$ $\therefore T$ is $\left(\frac{2}{3}ap, 0\right)$	Majority of the students are able to give tangent at P correctly, again with minority of them having confusion of the symbols a, t, p.
Midpoint $M = \left(\frac{1}{3}ap, -ap^3\right)$. $x = \frac{1}{3}ap$, $y = -ap^3$	Students ought to remember the formula for finding midpoint. Some students made mistakes e.g. subtracting the coordinates of T from S or forget to divide by 2.

$\frac{3x}{a} = p$, $y = -a\left(\frac{3x}{a}\right)^3 = -\frac{27x^3}{a^2}$	Very few students could find the cartesian equation of the curve traced by M as p varies, many students left their answers with y in terms of both p and x.
(ii) Required area $= \int_0^3 y \, dx$ $= \int_0^{\frac{3}{a}} at^3 (a) \, dt$ $= \int_0^{\frac{3}{a}} at^3 (a) \, dt$ $= a^2 \left[\frac{t^4}{4}\right]_0^{\frac{3}{a}} = \frac{81}{4a^2}$ Alternatively, find the cartesian of the given curve and use it to find the required area. $x = at$, $y = at^3$ $\therefore y = a\left(\frac{x}{a}\right)^3 = \frac{x^3}{a^2}$ Required area $= \int_0^3 \frac{x^3}{a^2} dx = \frac{1}{a^2}\left[\frac{x^4}{4}\right]_0^3 = \frac{1}{a^2}\left[\frac{3^4}{4} - 0\right] = \frac{81}{4a^2}$	Not well done. Many students either forget to change the limits or made careless mistakes when substituting limits.

3 The parametric equations of a curve are $x = at,\ y = at^3$, where a is a positive constant.

(i) The point P on the curve has parameter p and the tangent to the curve at point P cuts the y-axis at S and the x-axis at T. The point M is the midpoint of ST. Find a Cartesian equation of the curve traced by M as p varies. [5]

(ii) Find the exact area bounded by the curve, the line $x = 0$, $x = 3$ and the x-axis, giving your answer in terms of a. [3]

Suggested Solution	Comments
Solutions: (i) $x = at,\quad y = at^3$ $\frac{dx}{dt} = a,\quad \frac{dy}{dt} = 3at^2$	This question is generally well done. Students are able to write out very good solutions by following the steps to obtain the equation of tangent at point P.
$\frac{dy}{dx} = \frac{3at^2}{a}$ $= 3t^2$ Therefore, gradient of the tangent at $P = 3p^2$	Minority of students have confusion of the symbols, hence unable to find the gradient of tangent at P. For example, they differentiate x with respect to a instead of t, or substitute a by p instead of substitute t by p.
Equation of tangent at P: $y - ap^3 = 3p^2(x - ap)$ $y - ap^3 = 3p^2x - 3ap^3$ $y = 3p^2x - 2ap^3$ For point S, $x = 0, y = -2ap^3$. $\therefore S$ is $(0, -2ap^3)$ For point T, $y = 0, x = \frac{2ap^3}{3p^2} = \frac{2}{3}ap$ $\therefore T$ is $\left(\frac{2}{3}ap, 0\right)$	Majority of the students are able to give tangent at P correctly, again with minority of them having confusion of the symbols a, t, p.
Midpoint M $= \left(\frac{1}{3}ap, -ap^3\right)$. $x = \frac{1}{3}ap,\quad y = -ap^3$	Students ought to remember the formula for finding midpoint. Some students made mistakes e.g. subtracting the coordinates of T from S or forget to divide by 2.

$\frac{3x}{a} = p,$ $y = -a\left(\frac{3x}{a}\right)^3 = -\frac{27x^3}{a^2}$	Very few students could find the cartesian equation of the curve traced by M as p varies, many students left their answers with y in terms of both p and x.
(ii) Required area $= \int_0^3 y\,dx$ $= \int_0^{\frac{3}{a}} at^3(a)\,dt$ $= \int_0^{\frac{3}{a}} at^3(a)\,dt$ $= a^2\left[\frac{t^4}{4}\right]_0^{\frac{3}{a}} = \frac{81}{4a^2}$ Alternatively, find the cartesian of the given curve and use it to find the required area. $x = at,\quad y = at^3$ $\therefore y = a\left(\frac{x}{a}\right)^3 = \frac{x^3}{a^2}$ Required area $= \int_0^3 \frac{x^3}{a^2}\,dx = \frac{1}{a^2}\left[\frac{x^4}{4}\right]_0^3 = \frac{1}{a^2}\left[\frac{3^4}{4} - 0\right] = \frac{81}{4a^2}$	Not well done. Many students either forget to change the limits or made careless mistakes when substituting limits.

It is given that $y=\sin^{-1}x\cos^{-1}x$, where $-1\le x\le 1$.

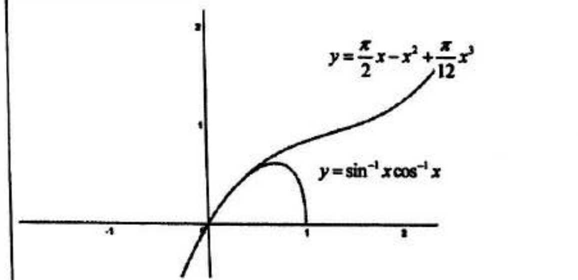
(i) Show that $\sqrt{1-x^2}\dfrac{dy}{dx}=\cos^{-1}x-\sin^{-1}x$. [1]

(ii) Show that $\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=-2$ [2]

(iii) Hence find the exact value of A, B and C if y can be expressed as $Ax+Bx^2+Cx^3$, up to (and including) the term in x^3. [4]

(iv) A student used (iii) to estimate that $\sin^{-1}(0.8)\cos^{-1}(0.8)\approx 0.8A+0.8^2B+0.8^3C$. Explain, with working, if his estimate is a good one. [1]

Suggested Solution	Comments
(i) $y=\sin^{-1}x\cos^{-1}x \Rightarrow \dfrac{dy}{dx}=\dfrac{\cos^{-1}x}{\sqrt{1-x^2}}+\left(\dfrac{-\sin^{-1}x}{\sqrt{1-x^2}}\right)$ $\Rightarrow \sqrt{1-x^2}\dfrac{dy}{dx}=\cos^{-1}x-\sin^{-1}x$---(1) [shown]	This part of the question was generally well done.
(ii) Diff (1) wrt x, $\dfrac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\dfrac{dy}{dx}+\sqrt{1-x^2}\dfrac{d^2y}{dx^2}=\dfrac{-1}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1-x^2}}$ $\Rightarrow(-x)\dfrac{dy}{dx}+\left(1-x^2\right)\dfrac{d^2y}{dx^2}=-1-1=-2\Rightarrow\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=-2$ --(2)[shown]	This part of the question was generally well done. There is still a proportion of student who did not approach the proving via implicit differentiation but adopted a long-winded, usually time consuming method.
(iii) Diff (2) wrt x, $-2x\dfrac{d^2y}{dx^2}+\left(1-x^2\right)\dfrac{d^3y}{dx^3}-\left(\dfrac{dy}{dx}+x\dfrac{d^2y}{dx^2}\right)=0$ When $x=0, y=0, \dfrac{dy}{dx}=\dfrac{\pi}{2}, \dfrac{d^2y}{dx^2}=-2, \dfrac{d^3y}{dx^3}=\dfrac{\pi}{2}$. Therefore, $\sin^{-1}x\cos^{-1}x\approx\left(\dfrac{\pi}{2}\right)x+\dfrac{(-2)}{2!}x^2+\dfrac{\left(\frac{\pi}{2}\right)}{3!}x^3=\dfrac{\pi}{2}x-x^2+\dfrac{\pi}{12}x^3$.	For this part, quite a number of student are not able to handle the implicit differentiation well. Some student evaluated $\cos^{-1}(0)=90$ instead of $\dfrac{\pi}{2}$. Which will lead to a wrong expression. They should just use radian mode.

	One alerting sigh is that some student mistook $\sin^{-1}x$ as $\dfrac{1}{\sin x}$ which is obviously incorrect. $\sin^{-1}x\ne\dfrac{1}{\sin x}=\operatorname{cosec}x$ $\cos^{-1}x\ne\dfrac{1}{\cos x}=\sec x$
(iv) The estimate is not good as $\sin^{-1}(0.8)\cos^{-1}(0.8)=0.597$ (to 3 sf) But $\dfrac{\pi}{2}(0.8)-(0.8)^2+\dfrac{\pi}{12}(0.8)^3=0.751$ (to 3 sf) Alternative Explanation  The graphs illustrated that at $x=0.8$, the two graphs are quite different from each other.	Most of the student are able to provide reasonable evidence or reason. For those who did not change to radian mode will get the wrong answer.

5 **(a)** Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$. Point C is on the line which contains A and is parallel to $\mathbf{b}$. It is given that the vectors $\mathbf{a}$ and $\mathbf{b}$ are both of magnitude 2 units and are at an angle of $\sin^{-1}(1/6)$ to each other. If the area of triangle OAC is 3 units^2, use vector product to find the possible position vectors of C in terms of $\mathbf{a}$ and $\mathbf{b}$. [5] 3

(b) Referred to the origin O, the points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ where $\mathbf{p}$ and $\mathbf{q}$ are non-parallel, non-zero vectors. Point R is on PQ produced such that $PQ:QR = 1:\lambda$. Point M is the mid-point of OP and OR.

(i) Find the position vector of R in terms of λ, $\mathbf{p}$ and $\mathbf{q}$. [1]

F is a point on OQ such that F, P and M are collinear.

(ii) Find the ratio $OF:FQ$, in terms of λ. [4] 1

Suggested Solution	**Comments**
(a) $\overrightarrow{OC} = \mathbf{a} + \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R}$. Area of triangle OAC $= \frac{1}{2}\lvert\overrightarrow{OA} \times \overrightarrow{OC}\rvert$ $= \frac{1}{2}\lvert\mathbf{a} \times (\mathbf{a} + \lambda\mathbf{b})\rvert$ $= \frac{1}{2}\lvert\mathbf{a} \times \mathbf{a} + \lambda(\mathbf{a} \times \mathbf{b})\rvert$ $= \frac{1}{2}\lvert\lambda\rvert\lvert(\mathbf{a} \times \mathbf{b})\rvert$ $= \frac{1}{2}\lvert\lambda\rvert\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$ $= \frac{1}{2}\lvert\lambda\rvert(2)(2)\frac{1}{6} = \frac{1}{3}\lvert\lambda\rvert$ Since area of triangle OAC= 3, $\frac{1}{3}\lvert\lambda\rvert = 3$ $\lambda = 9$ or -9 $\overrightarrow{OC} = \mathbf{a} \pm 9\mathbf{b}$	• A significant number of students wrote statements such as $\overrightarrow{OC} = \mathbf{a} + \lambda\mathbf{b}, \lambda \in \mathbb{R}$, or $l_C = \mathbf{a} + \lambda\mathbf{b}$, for some $\lambda \in \mathbb{R}$, etc. • A large proportion of students equated scalar quantities with vector quantities, numerous examples of such wrong statements (not exhaustive!) include: $\frac{1}{2}(\overrightarrow{OA} \times \overrightarrow{OC}) = 3$ [wrong quote of area formula via vector product], $\lambda(\mathbf{a} \times \mathbf{b}) = 3$, $\frac{1}{2}\lvert OA \times OC\rvert = 3$, etc. • Several students also wrongly stated/implied the area of triangle ***OAB*** to be 3 units^2, e.g. $\frac{1}{2}\lvert\overrightarrow{OA} \times \overrightarrow{OB}\rvert = 3$. • A significant number of candidates wrongly wrote variations of the following statements: $\lvert\mathbf{a} \times \mathbf{a} + \lambda(\mathbf{a} \times \mathbf{b})\rvert = \lvert\mathbf{a} \times \mathbf{a}\rvert + \lvert\lambda(\mathbf{a} \times \mathbf{b})\rvert$, or $\lvert\mathbf{a} \times \mathbf{a} + \lambda(\mathbf{a} \times \mathbf{b})\rvert = \lvert\lvert\mathbf{a}\rvert^2 + \lambda\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta\rvert$, etc. • Majority of candidates neglected to account for the possibility that λ can be a negative number, e.g., through dropping the modulus sign on λ.

(b) By the ratio theorem, $\overrightarrow{OQ} = \frac{\overrightarrow{OR} + \lambda\overrightarrow{OP}}{1+\lambda}$ $\overrightarrow{OR} = (1+\lambda)\mathbf{q} - \lambda\mathbf{p}$	• A significant number of students misinterpreted that R lies on the line segment PR, thus leading to wrong usage of the ratio theorem (not midpoint theorem!).
Since the point F lies on line OQ, $\overrightarrow{OF} = t\mathbf{q}$, for some $t \in \mathbb{R}$. $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP}$ $= \frac{1}{2}\overrightarrow{OR} - \mathbf{p}$ $= \frac{(1+\lambda)}{2}\mathbf{q} - \frac{\lambda}{2}\mathbf{p} - \mathbf{p}$ $= \frac{(1+\lambda)}{2}\mathbf{q} - \left(\frac{\lambda}{2}+1\right)\mathbf{p}$ Since the point F also lies on line PM, $\overrightarrow{OF} = \mathbf{p} + s\overrightarrow{PM}$, for some $s \in \mathbb{R}$. $= \mathbf{p} + s\left[\frac{(1+\lambda)}{2}\mathbf{q} - \left(\frac{\lambda}{2}+1\right)\mathbf{p}\right]$ $= \left(1 - \frac{s\lambda}{2} - s\right)\mathbf{p} + \frac{s(1+\lambda)}{2}\mathbf{q}$ Since $\mathbf{p}$ and $\mathbf{q}$ are non-parallel & non-zero vectors, comparing coefficients of $\mathbf{p}$ and $\mathbf{q}$ against $\overrightarrow{OF} = t\mathbf{q}$, we have $1 - \frac{s\lambda}{2} - s = 0$ $s\left(\frac{\lambda}{2}+1\right) = 1$ $s = \frac{1}{\frac{\lambda}{2}+1} = \frac{2}{\lambda+2}$ $\overrightarrow{OF} = \frac{s(1+\lambda)}{2}\mathbf{q} = \frac{2}{\lambda+2}\left(\frac{1+\lambda}{2}\right)\mathbf{q}$ $= \frac{1+\lambda}{2+\lambda}\mathbf{q} = \frac{1+\lambda}{2+\lambda}\overrightarrow{OQ}$ Thus, $OF:FQ = 1+\lambda:1$	• A significant number of students wrongly presented statements such as vector $\overrightarrow{OF}$ lies on line OQ, etc. • Majority of the candidates are unable to recognise that this part of the question requires them to obtain the intersection point of 2 lines via comparison of the coefficients of $\mathbf{p}, \mathbf{q}$ in 2 separate expressions of $\overrightarrow{OF}$ [via lines OQ and PM] (which are non-parallel and non-zero vectors). Students should note questions based on this similar method can be readily found in the first tutorial of the Vectors topic. • A significant number of students wrongly divided vector quantities, e.g., $\overrightarrow{OF}:\overrightarrow{FQ}$ or $\overrightarrow{OF}/\overrightarrow{FQ}$!

6 Do not use a calculator in answering this question.

-1 **(a)** It is given that two complex numbers z and w satisfy the following equations

$$13z=(4-7\mathrm{i})w,$$
$$z-2w=5-4\mathrm{i}.$$

Find z and w. [4]

(b) It is given that $q=-\sqrt{3}-\mathrm{i}$.

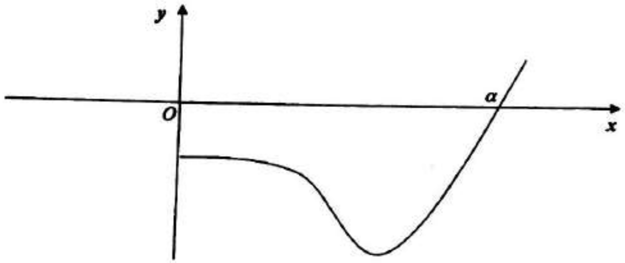
(i) Find an exact expression for q^6, giving your answer in the form $re^{\mathrm{i}\theta}$, where $r>0$ and $0\le\theta<2\pi$. [3]

-1 (ii) Find the three smallest positive whole number values of n for which $\frac{q^n}{q^*}$ is purely imaginary. [4]

Suggested Solution	Comments
(a) From (2): $z=5-4\mathrm{i}+2w$ Sub into equation (1): $13(5-4\mathrm{i}+2w)=(4-7\mathrm{i})w$ $(22+7\mathrm{i})w=-13(5-4\mathrm{i})$ $w=\frac{-13(5-4\mathrm{i})}{22+7\mathrm{i}}\times\frac{22-7\mathrm{i}}{22-7\mathrm{i}}$ $=\frac{-13(110-35\mathrm{i}-88\mathrm{i}-28)}{22^2+7^2}$ $=\frac{-13}{533}(82-123\mathrm{i})=-2+3\mathrm{i}$ Sub into $z=5-4\mathrm{i}+2w$ $z=5-4\mathrm{i}+2(-2+3\mathrm{i})$ $=1+2\mathrm{i}$	Most students used substitution, though some chose the more tedious substitution of $z=\frac{(4-7\mathrm{i})}{13}w$. A common mistake is to use GC to evaluate $w=\frac{-13(5-4\mathrm{i})}{22+7\mathrm{i}}=-2+3\mathrm{i}$. This will lead to loss of marks as calculator is not allowed and so the required rationalising and simplifying steps will need to be shown. A large number of students used $w=a+bi$ after the first substitution step and equated the real and imaginary parts of the equation. While this is not wrong, it is rather tedious as the student will need to solve the resulting simultaneous equations by hand rather than using the GC, and is not recommended in this case.
(b)(i) $q=-\sqrt{3}-\mathrm{i}$	Many students calculated the argument of q wrongly. A common mistake is $\arg(q)=\frac{\pi}{6}$.

$\arg(q)=-\frac{5\pi}{6}$ $\lvert q\rvert=2$ Thus, $q=2e^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}$ $q^6=\left(2e^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}\right)^6=2^6e^{\mathrm{i}(-5\pi)}=64e^{\mathrm{i}(\pi)}$	Many students left the answer as $q^6=64e^{\mathrm{i}(-\pi)}$. This is incomplete as we need $0\le\theta<2\pi$. Several students evaluated $q^6=-64$ directly by hand. This is fine, however, those who used the GC to do this would obtain no marks as calculator is not allowed. From $q^6=-64$, a number of students wrong concluded that $q^6=64e^{\mathrm{i}(0)}$.
(ii) $\arg\left(\frac{q^n}{q^*}\right)=\arg(q^n)-\arg(q^*)$ $=n\arg(q)+\arg(q)$ $=(n+1)\arg(q)$ $=(n+1)\left(-\frac{5\pi}{6}\right)$ $=-(n+1)\frac{5\pi}{6}$ For $\frac{q^n}{q^*}$ to be imaginary, $\arg\left(\frac{q^n}{q^*}\right)=\pm\frac{\pi}{2}$ Thus $-(n+1)\frac{5\pi}{6}=\frac{(2k+1)\pi}{2}, k\in\mathbb{Z}$ $(n+1)=-\frac{3(2k+1)}{5}, k\in\mathbb{Z}$ $n+1=3,9,15...$ $n=2,8,14,..$ The smallest positive whole number values of n are 2, 8, 14.	Students generally were able to show their understanding that $\arg\left(\frac{q^n}{q^*}\right)=(n+1)\arg(q)$ and received credit for that, even though they might have obtained $\arg(q)$ wrongly. Many students were also able to show some understanding that for $\frac{q^n}{q^*}$ to be purely imaginary, $\arg\left(\frac{q^n}{q^*}\right)$ must be of a certain form. However, they were unable to articulate the form required or failed to evaluate the smallest positive whole number values of n required.

7



It is given that $f(x)=2x^6-4x^4-6x^2-7$. The diagram shows the curve with equation $y=f(x)$ for $x\geq 0$. The curve crosses the positive x-axis at $x=\alpha$.

(i) Find the value of α, giving your answer correct to 3 decimal places. [1]

-2 **(ii)** Show that $f(x)=f(-x)$ for all real values of x. What can be said about the six roots of the equation $f(x)=0$? [4]

It is given that $g'(x)=f(x)$, for all real values of x.

(iii) Determine the x-coordinates of all the stationary points of graph of $y=g(x)$ and determine their nature. [3]

-1 **(iv)** For which values of x is the graph of $y=g(x)$ concave upwards? [3]

Suggested Solution	Comments
(i) Using GC, $\alpha=1.804$ (3 d.p)	This is a simple GC question. State that $\alpha>0$ and reject the negative solution. (Note that we do not need to mention the complex roots here)
(ii) $f(x)=2x^6-4x^4-6x^2-7$ $f(-x)=2(-x)^6-4(-x)^4-6(-x)^2-7$ $=2x^6-4x^4-6x^2-7$ $=f(x)$ $\Rightarrow f(x)=f(-x)$ (shown)	This is basic "show" question, start from RHS and obtain LHS. There is no need to waste time explaining about even powers. Students should explicitly conclude that $f(x)=f(-x)$ (shown)

<table>
<tr><td>Since, $f(x)=f(-x)$, two real roots of $f(x)=0$ are 1.804 and -1.804. (From (i))

1
2 The remaining 4 complex roots are in a form of 2 complex conjugate pairs and also are negatives of each other.

The four complex roots can be written as x_1,x_2,x_3,x_4, where
$x_1=-x_3$ while $x_2=-x_4$.
Also $x_1{}^*=x_2$ while $x_3{}^*=x_4$

In other words, the complex roots are of the forms:
$x_1=a+bi$
$x_2=a-bi$
$x_3=-a-bi$
$x_4=-a+bi$</td>
<td>Most students mentioned that there are two real roots.

Those who understood where able to talk about the conjugate pairs. However, students must separate the explanation for real and complex. A majority answered that the all six roots are "symmetrical" (only for real roots) or are "conjugate". This is not accurate. Students must remember that there is no such thing as positive or negative complex numbers.

The last mark to mention the inter-relationship of the complex root is not easy (only achieved by one student). Note that if a complex root that has another complex root that is negative of it ($\because f(x)=f(-x)$), then if the polynomial has real coefficients, this inter-relationship between 4 roots will occur as the conjugate happens to be the negative of one other complex root.</td></tr>
<tr><td>(iii)
$y=g(x)$ has stationary points when $g'(x)=f(x)=0$, i.e. at $x=1.804$ or -1.804.

From the graph, we can see
<table><tr><td></td><td>$x=1.804^-$</td><td>$x=1.804$</td><td>$x=1.804^+$</td></tr><tr><td>$g'(x)=f(x)$</td><td>-ve</td><td>0</td><td>+ve</td></tr></table>$y=g(x)$ has a minimum point at $x=1.804$.

From the graph, we can see
<table><tr><td></td><td>$x=-1.804^-$</td><td>$x=-1.804$</td><td>$x=-1.804^+$</td></tr><tr><td>$g'(x)=f(x)$</td><td>+ve</td><td>0</td><td>-ve</td></tr></table>$y=g(x)$ has a maximum point at $x=-1.804$.</td>
<td>This question revealed "poor reading skills" from a quite a number of students. It was meant to be a straightforward question to "copy" the answer from part (i).

It is a bit shocking to see students presenting that complex numbers are stationary points where this has never ever happened before in tutorials.

There is still very poor presentation of the 1st and 2nd derivative test. More effort is needed on the part of students to score the marks.

The 2nd derivative test is a better approach for this question.</td></tr>
</table>

(iv)	Students are reminded that concavity depends on the rate of change of the gradient and not the gradient itself.
$y = g(x)$ is concave upwards when $g''(x) = f'(x)$ is positive. $f'(x) = 12x^5 - 16x^3 - 12x$ $f'(x) = 0$ Using GC the real roots are $x = 0, -1.37, 1.37$ (these are also the x-coordinates of the stationary points of $f(x)$.) From graph, we can tell that $f'(x)$ is positive for $x \in [-1.37, 0] \cup [1.37, \infty)$	A number of students missed out the $x = 0$ point and were not able to solve the inequality. Quite a number did not realise it is a range of values. Students should be aware that for a polynomial inequality, they should use the graph of the polynomial to solve it (if GC allowed), if GC is not allowed, the polynomial should be factorise-able to allow you to determine the roots.

8 (a) (i) Show that, for $r \in \mathbb{Z}$, $r \geq 2$,

$$\frac{1}{(r-1)!} - \frac{2}{r!} + \frac{1}{(r+1)!} = \frac{r^2-r-1}{(r+1)!}.$$ [1]

Let $S_n = \sum_{r=2}^{n} \frac{r^2-r-1}{(r+1)!}$.

(ii) Hence find S_n in terms of n. [3]

(iii) Show that S_n converges to a limit L, where L is to be determined. [2]

(iv) Find the least integer value of n such that S_n differs from L by less than 10^{-10}. [2]

(b) (i) Suppose that f is a continuous, strictly decreasing function defined on $[1, \infty)$, with $f(x) > 0$, $x \geq 1$. According to the Maclaurin-Cauchy test, then the infinite series $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if the integral $\int_1^{\infty} f(x)\,dx$ is finite. By applying the Maclaurin-Cauchy test on the function f defined by $f(x) = \frac{1}{x}$, $x \geq 1$, determine if the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent. [2]

(ii) Let p be a positive number. By considering the Maclaurin-Cauchy test, show that if $p > 1$, the infinite series $1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \frac{1}{n^p} + \ldots$ is convergent. [2]

Suggested Solution	Comments
(i) For $r \in \mathbb{Z}$, $r \geq 2$, $\frac{1}{(r-1)!} - \frac{2}{r!} + \frac{1}{(r+1)!}$ $= \frac{r(r+1) - 2(r+1) + 1}{(r+1)!}$ $= \frac{r^2+r-2r-2+1}{(r+1)!}$ $= \frac{r^2-r-1}{(r+1)!}$	• This part is generally done well, with most students demonstrating that they are able to work with "factorial" efficiently. • However, some might have skipped too many crucial details, and hence are penalized as it is a "show"-type question.
(ii)	• Again, this part is generally done well, with most students showing that they are able to

$Sn=\sum_{r=2}^{n}\frac{r^2-r-1}{(r+1)!}$ $=\sum_{r=2}^{n}\left[\frac{1}{(r-1)!}-\frac{2}{r!}+\frac{1}{(r+1)!}\right]$ $=\left\{\left[\frac{1}{1!}-\frac{2}{2!}+\frac{1}{3!}\right]\right.$ $+\left[\frac{1}{2!}-\frac{2}{3!}+\frac{1}{4!}\right]$ $+\left[\frac{1}{3!}-\frac{2}{4!}+\frac{1}{5!}\right]$ $+\left[\frac{1}{4!}-\frac{2}{5!}+\frac{1}{6!}\right]$ $\vdots$ $+\left[\frac{1}{(n-3)!}-\frac{2}{(n-2)!}+\frac{1}{(n-1)!}\right]$ $+\left[\frac{1}{(n-2)!}-\frac{2}{(n-1)!}+\frac{1}{n!}\right]$ $\left.+\left[\frac{1}{(n-1)!}-\frac{2}{n!}+\frac{1}{(n+1)!}\right]\right\}$ $=\frac{1}{1!}-\frac{2}{2!}+\frac{1}{2!}+\frac{1}{n!}-\frac{2}{n!}+\frac{1}{(n+1)!}$ $=\frac{1}{2}-\frac{1}{n!}+\frac{1}{(n+1)!}$ $=\frac{1}{2}-\frac{n}{(n+1)!}$	identify the general cancellation pattern involving three terms. • Common mistakes include missing out some, usually $\frac{1}{n!}$, of the six terms that are not "cancelled" out • A handful who tried to further simplify $\frac{1}{2}-\frac{1}{n!}+\frac{1}{(n+1)!}$ had some difficulty with combining $-\frac{1}{n!}+\frac{1}{(n+1)!}$ into a single fraction.
(iii) $Sn=\frac{1}{2}-\frac{n}{(n+1)!}$ As $n\to\infty$, $\frac{n}{(n+1)!}\to 0$. $\therefore S_n\to\frac{1}{2}$, thus S_n converges and $L=\frac{1}{2}$	• Most students are able to obtain the correct value of L based on their answers in the previous part. • However, to gain full credit for this part, students are expected to give a brief explanation as to why the series is convergent.
(iv)	• There are various equivalent formulations of the inequality based on the phrase "S_n

For $\left\|S_n-\frac{1}{2}\right\| < 10^{-10}$ $\frac{n}{(n+1)!}<10^{-10}$ From GC, $n\geq 14$ $\therefore$ the least value of $n=14$.	differs from L by less than 10^{-10}", and they are all accepted. • An example of an inequality that is not accepted in this case would be $S_n-L<10^{-10}$.
(b)(i) $\int_1^\infty \frac{1}{x}\,dx=\left[\ln x\right]_1^\infty$ $\left[\ln x\right]_1^\infty$ is not finite and thus, by the Maclaurin-Cauchy test, $\sum_{n=1}^{\infty}\frac{1}{n}$ is not convergent.	• A significant number of students are able to grasp the gist of this part, and as long as they are able to demonstrate that understanding, full credit is awarded. • Two particular misconceptions about "infinity" seem to be common across the cohort, and for the interested ones, you may test yourself by determining *whether the following statements are true of false:* 1. If a function f is strictly increasing, then as $x\to\infty$, it follows that $f(x)\to\infty$. 2. Since $\frac{1}{n}\to 0$ as $n\to\infty$, it follows that $\sum_{n=1}^{\infty}\frac{1}{n}$ is convergent since we are adding increasingly smaller numbers that are getting closer and closer to 0.
(b)(ii) $\int_1^\infty \frac{1}{x^p}\,dx=\int_1^\infty x^{-p}\,dx$ $=\frac{1}{(-p+1)}\left[x^{-p+1}\right]_1^\infty$ $=0-\frac{1}{(-p+1)}$ $=\frac{1}{p-1}$ is finite, whenever $p>1$. Thus, by the Maclaurin-Cauchy test, whenever $p>1$, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent.	• Quite a number of students had trouble with $\int x^{-p}\,dx$, probably because of the abstractness.

9 A drilling company plans to install a straight pipeline AB through a mountain. Points (x,y,z) are defined relative to a main control site at the foot of the mountain at $(0,0,0)$, where units are metres. The x-axis points East, the y-axis points North and the z-axis points vertically upwards. Point A has coordinates $(-200,150,10)$ while point B has coordinates $(100,10,a)$, where a is an integer. Point B is at a higher altitude than Point A.

(i) Given that the pipeline AB is of length 337 metres, find the coordinates of B. [3]

A thin flat layer of rock runs through the mountain and is contained in the plane with equation $20x+y+2z=-837$.

(ii) Find the coordinates of the point where the pipeline meets the layer of rock. [4]

To stabilise the pipeline, the drilling company decides to build 2 cables to join points A and B to the layer of rock. Point A is joined to Point P while point B is joined to Point Q.

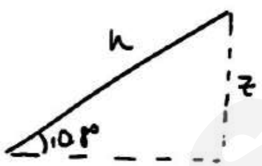
(iii) Assuming that the minimum length of cable is to be used, find the length PQ. [2]

(iv) Show that the pipeline is at an angle of 10.8^0 to the horizontal plane. [2]

(v) After the pipeline is completed, a ball bearing is released from point B to roll down the pipeline to check for obstacles. The ball bearing loses altitude at a rate of $0.3t$ metres per second, where t is the time (in seconds) after its release. Find the speed at which the ball bearing is moving along the pipeline 10 seconds after its release. [3]

Suggested Solution	Comments
(i) Given that length of BA is 337 $\overrightarrow{AB}=\begin{pmatrix}300\\-140\\a-10\end{pmatrix}$ $\sqrt{300^2+(-140)^2+(a-10)^2}=337$ $(a-10)^2=3969$ $a-10=\pm 63$ $a=73$ or -53 Since B is of higher altitude, $a>10$ and thus a= 73. Coordinates of B are $(100,10,73)$.	Majority of students answered well for part (i). Common mistakes: 1) Students have to work out the two values of a, and reject $a=-53$ by understanding the context of the question. 2) Not presenting point B in coordinates.

(ii) $\overrightarrow{AB}=\begin{pmatrix}300\\-140\\63\end{pmatrix}$ Equation of line AB: $\mathbf{r}=\begin{pmatrix}-200\\150\\10\end{pmatrix}+\lambda\begin{pmatrix}300\\-140\\63\end{pmatrix}, \lambda\in\mathbb{R}$ Let C be the point of intersection between the line and plane. Since C is on the plane, $\left[\begin{pmatrix}-200\\150\\10\end{pmatrix}+\lambda\begin{pmatrix}300\\-140\\63\end{pmatrix}\right]\cdot\begin{pmatrix}20\\1\\2\end{pmatrix}=-837$ $-4000+150+20+\lambda(6000-140+126)=-837$ $5986\lambda=2993\Rightarrow\lambda=\frac{1}{2}$ $\overrightarrow{OC}=\begin{pmatrix}-200\\150\\10\end{pmatrix}+\frac{1}{2}\begin{pmatrix}300\\-140\\63\end{pmatrix}=\begin{pmatrix}-50\\80\\41.5\end{pmatrix}$ The coordinates of C are (-50,80,41.5).	This part was also answered well by majority. Common mistakes: 1) Careless algebraic manipulation in the scalar product. 2) Again, a number did not present the intersection point in coordinate form. Do read the question properly and not lose such unnecessary mark.
(iii) Length of projection of AB onto the plane	Students have to understand the need of part (iii), and some clearly did not understand the shortest length of PQ was the length of projection of vector AB onto the plane. A significant number attempted to use $\dfrac{\left\|\overrightarrow{AB}\cdot\begin{pmatrix}20\\1\\2\end{pmatrix}\right\|}{\sqrt{405}}$ which was wrong.

$=\frac{\left\lvert \overrightarrow{AB}\times\begin{pmatrix}20\\1\\2\end{pmatrix}\right\rvert}{\sqrt{405}}$ $=\frac{\left\lvert \begin{pmatrix}300\\-140\\63\end{pmatrix}\times\begin{pmatrix}20\\1\\2\end{pmatrix}\right\rvert}{\sqrt{405}}=\frac{\left\lvert \begin{pmatrix}-343\\660\\3100\end{pmatrix}\right\rvert}{\sqrt{405}}$ $=\frac{\sqrt{10163249}}{\sqrt{405}}=158$ m (to 3 sf)	
(iv) Let acute angle between horizontal plane and AB be α. Normal to horizontal plane is **k**. $\sin\alpha=\frac{\left\lvert \overrightarrow{AB}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\left\lvert\overrightarrow{AB}\right\rvert}$ $\sin\alpha=\frac{\begin{pmatrix}300\\-140\\63\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{337}=\frac{63}{337}$ $\alpha=10.8^0$	The key error was identifying the wrong normal to the horizontal plane. Many students used the normal of the the plane with equation $20x+y+2z=-837$, which showed a lack of contextual understanding.
(v) Let h refer to the sloped distance the ball bearing has moved and z the altitude the ball bearing has moved.  Given that $\frac{dz}{dt}=-0.3t$, we need to find $\frac{dh}{dt}$.	Part (v) was poorly attempted. Many students did not understand the contextual setting. However, there are some good solutions on the use of the vertical component of the vector in the diagram which led to the final answer in a fast way.

$\sin\left(10.8^0\right)=\frac{z}{h}$ $\frac{63h}{337}=z$ Differentiate with respect to t: $\frac{63}{337}\frac{dh}{dt}=\frac{dz}{dt}$ $\frac{dh}{dt}=\frac{337}{63}(-0.3t)$ When t=10 $\frac{dh}{dt}=\frac{337}{63}(-0.3(10))$ $=-16.0ms^{-1}$ Speed of the bearing= 16.0 m/s	

10 An epidemiologist is studying the spread of a disease, dengue fever, which is spread by mosquitoes, in town A. P is defined as the number of infected people (in thousands) t years after the study begins. The epidemiologist predicts that the rate of increase of P is proportional to the product of the number of infected people and the number of uninfected people. It is known that town A has 10 thousand people of which a thousand were infected initially.

(i) Write down a differential equation that is satisfied by P. [1]

(ii) Given that the epidemiologist projects that it will take 2 years for half the town's population to be infected, solve the differential equation in (i) and express P in terms of t. [6]

(iii) Hence, sketch a graph of P against t. [2]

A second epidemiologist proposes an alternative model for the spread of the disease with the following differential equation:

$$\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{2\cos t}{(2-\sin t)^2} \qquad (*).$$

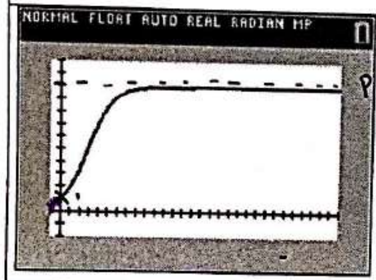
(iv) Using the same initial condition, solve the differential equation (*) to find an expression of P in terms of t. [3]

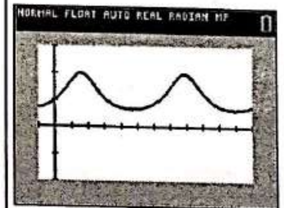
(v) Find the greatest and least values of P predicted by the alternative model. [2]

(vi) The government of town A deems the alternative model as a more realistic model for the spread of the disease as it more closely follows the observed pattern of the spread of the disease. What could be a possible factor contributing to this? [1]

Suggested Solution	Comments
(i) $\dfrac{\mathrm{d}P}{\mathrm{d}t}=kP(10-P)$	Most can formulate the D.E. correctly. Quite a number of students did not read the question carefully and did not realize the units of P was in thousands, wasting marks unnecessarily and also making the working more complex.
(ii) $\dfrac{\mathrm{d}P}{\mathrm{d}t}=kP(10-P)$ **Method 1 to integrate:** $\displaystyle\int\frac{1}{P(10-P)}\mathrm{d}P=k\int\mathrm{d}t$ $\displaystyle\frac{1}{10}\int\frac{1}{P}+\frac{1}{10-P}\mathrm{d}P=k\int\mathrm{d}t$ $\frac{1}{10}\left[\ln\lvert P\rvert-\ln\lvert(10-P)\rvert\right]=kt+C$ $\frac{1}{10}\ln\left\lvert\frac{P}{10-P}\right\rvert=kt+c$	Students need to be very careful in making sure that the integral is in the standard form EXACTLY e.g. $\dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}$ before doing the integration. A missing sign could change the working drastically. Common mistake: $\displaystyle\int\frac{1}{P}+\frac{1}{10-P}\mathrm{d}P$ $=\ln\lvert P\rvert+\ln\lvert 10-P\rvert+c$ Many students did not explain why the modulus sign could be removed and simply removed it without

$\frac{1}{10}\ln\left(\frac{P}{10-P}\right)=kt+C$
 $\ln\left(\frac{P}{10-P}\right)=10kt+C_1$
 $\frac{P}{10-P}=e^{10kt+C_1}=Ae^{10kt}$
 Method 2 to integrate
 $\displaystyle\int\frac{1}{P(10-P)}\mathrm{d}P=k\int\mathrm{d}t$
 $\displaystyle\int\frac{1}{25-(P-5)^2}\mathrm{d}P=k\int\mathrm{d}t$
 $\frac{1}{10}\ln\left\lvert\frac{5+(P-5)}{5-(P-5)}\right\rvert=kt+c$
 $\frac{1}{10}\ln\left\lvert\frac{P}{10-P}\right\rvert=kt+c$
 From either Method 1 or 2,
 since $P>0, 10-P\geq 0$
 $\frac{1}{10}\ln\left(\frac{P}{10-P}\right)=kt+C$
 $\ln\left(\frac{P}{10-P}\right)=10kt+C_1$
 $\frac{P}{10-P}=e^{10kt+C_1}=Ae^{10kt}$
 	
 Substitute in values into solution
 Sub $t=0, P=1$
 $\frac{P}{10-P}=e^{10kt+C_1}=Ae^{10kt}$
 $\frac{1}{9}=Ae^{0}\Rightarrow A=\frac{1}{9}$
 $\frac{P}{10-P}=\frac{1}{9}e^{10kt}$
 	
 Sub $t=2, P=5$
	reason. Such working would not obtain full marks for the solution.

$\frac{5}{10-5}=\frac{1}{9}e^{10(2)k}$ $1=\frac{1}{9}e^{20k}$ $e^{20k}=9\Rightarrow k=\frac{1}{20}\ln(9)\approx 0.10986$ So we have $\frac{P}{10-P}=\frac{1}{9}e^{\frac{t}{2}\ln(9)}$ $9P=(10-P)e^{\frac{t}{2}\ln(9)}$ $P\left(9+e^{\frac{t}{2}\ln(9)}\right)=10e^{\frac{t}{2}\ln(9)}$ $P=\frac{10e^{\frac{t}{2}\ln(9)}}{9+e^{\frac{t}{2}\ln(9)}}$	Note that all final answers must be simplified reasonably. For example, $P=\frac{\frac{10}{9}e^{\frac{t}{2}\ln(9)}}{1+\frac{1}{9}e^{\frac{t}{2}\ln(9)}}$ would not obtain full marks as it contains a fraction within another fraction.
 P=10	Common mistakes: The graph must be drawn in consideration of the context. Thus the graph should NOT show negative values of t. It's also important to show that the graph approaches the asymptote P=10. Inflexion
(iv) $\frac{dP}{dt}=\frac{2\cos t}{(2-\sin t)^2}=(-2)(-\cos t)(2-\sin t)^{-2}$ $P=\frac{-2(2-\sin t)^{-1}}{-1}=\frac{2}{2-\sin t}+c$ Sub $t=0, P=1$	This is not well-done by a number of students who could not recognize the standard form and proceeded to use trigonometric identities which did not help to simplify the problem. Even those who recognised the form $f'(x)(f(x))^n$ were not careful about making sure the integral is exactly in the form before integrating: e.g.

$1=\frac{2}{2-\sin 0}+c$ $c=1-1=0$ Thus $P=\frac{2}{2-\sin t}$	$\int\frac{2\cos t}{(2-\sin t)^2}dt=(2)\int(\cos t)(2-\sin t)^{-2}dt$ $=\frac{2(2-\sin t)^{-1}}{-1}+c=\frac{-2}{(2-\sin t)}+c$ (*INCORRECT*)
(v) Since $-1\le\sin t\le 1$, Largest value of P is when $\sin t=1$ Largest value of P= 2 Smallest value of P is when $\sin t=-1$ Smallest value of P= 2/3	In general, students need to know the behaviour of the trigonometric functions very well. Some incorrectly thought that $0\le\sin t\le 1$ and substituted those values of $\sin t$. Alternative method is to consider $\frac{dP}{dt}=\frac{2\cos t}{(2-\sin t)^2}=0$ but it's a longer as students would need to determine the nature of the turning points.
(vi)  We can use the GC to plot $P=\frac{2}{2-\sin t}$. The second model could be deemed more suitable, as it shows oscillating values of P, which could correspond to the population of the mosquitoes which could vary seasonally. (For example, when the season is hot and rainy, the environment is more conducive for the breeding of mosquitoes.)	A good number of students sketched the graph of P against t to visualise the relationship between P and t even though it was not asked in the question. These students are usually more likely to address the periodic and fluctuating pattern observed. It is a good practice to sketch the graph in order to observe any trends. Many students answered correctly that the second model showed a decrease in P, but did not go on to attempt to explain why P decreased and increased periodically.

EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2018

General Certificate of Education Advanced Level

Higher 2

MATHEMATICS

9758/02

Paper 2 [100 marks]

21 September 2017

3 hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

Section A: Pure Mathematics [40 marks]

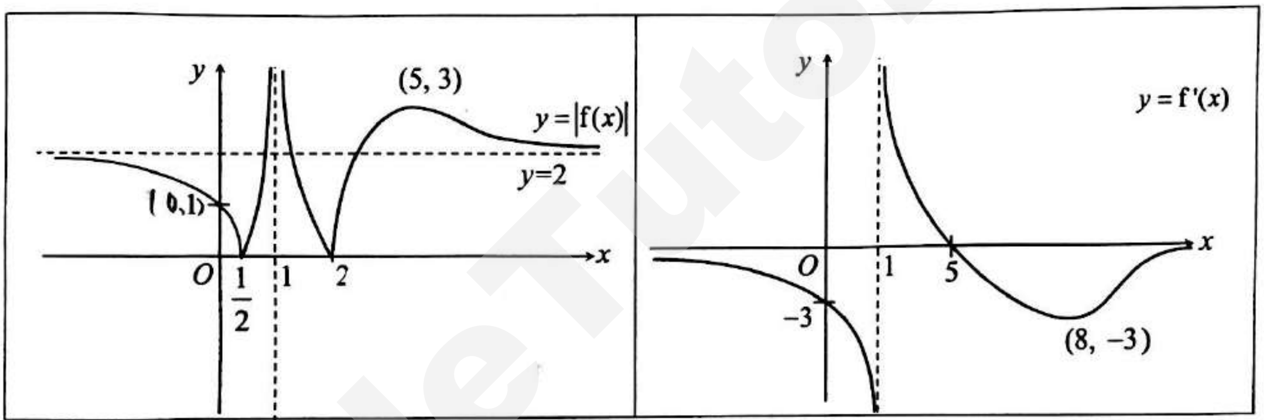
1 (i) Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n\to\infty} \frac{1}{n}\left\{ f\left(\frac{n+1}{n}\right) + f\left(\frac{n+2}{n}\right) + \ldots + f\left(\frac{2n}{n}\right) \right\}$$

is $\int_1^2 f(x)\,dx$ [2]

(ii) Hence evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{n}{n+r}$ exactly. [3]

2 The diagrams below show the graphs of $y = |f(x)|$ and $y = f'(x)$.



On separate diagrams, sketch the graphs of:

(i) $y = |f(2x)| + 1$ [2]

(ii) $y = \frac{1}{f'(x)}$ [3]

(iii) $y = f(x)$ [3]

showing clearly, in each case, the intersection(s) with the axes, the coordinates of the turning point(s) and the equation(s) of the asymptotes.

3 **(a)** A retirement savings account pays a compound interest of 0.2% per month on the amount of money in the account at the end of each month. A one-time principal amount of $\$P$ is deposited to open the account and a monthly pay-out of $\$x$ is withdrawn from the account at the beginning of each month, starting from the month that the account is opened.

(i) Show that the amount in the account at the end of n months after the interest has been added is given by

$$P(1.002^n) - 501x(1.002^n - 1).$$ [4]

(ii) Suppose a fixed monthly pay-out of \$2,000 is to be sustained for at least 25 years, find the minimum principal amount required correct to the nearest dollar. [2]

(iii) If a principal amount of \$600,000 is placed in the account, find the number of years for which a monthly pay-out of \$2,000 per month can be sustained, leaving your answer correct to the nearest whole number. [2]

(b) A different retirement savings account provides an increasing amount of monthly pay-out over a period of 25 years. The pay-out in the first month is $\$a$. The pay-out for each subsequent month is an increment of $\$c$ from the pay-out of the previous month.

The pay-out in the final month is \$4,000, and the total pay-out at the end of 25 complete years is \$751,500. Find the month in which the pay-out is \$2,000. [5]

4 The function f is defined by

$$f(x)=\begin{cases}4-x & \text{for } 1\le x<3,\\ (x-4)^2 & \text{for } 3\le x<4,\end{cases}$$

and it is given that $f(x-3)=f(x)$ for all real values of x.

(i) State a reason why f does not have an inverse. [1]

(ii) Sketch the graph of $y=f(x)$ for $-1<x<6$. [3]

(iii) Evaluate $f(2017)$. [2]

The function g has domain $[1,4)$ and is defined by

$$g(x)=\begin{cases}4-x & \text{for } 1\le x<3,\\ (x-4)^2 & \text{for } 3\le x<4.\end{cases}$$

(iv) By sketching $y=g(x)$ and $y=g^{-1}(x)$ on the same diagram, state the values of x such that $g(x)=g^{-1}(x)$. [3]

The function h, is defined by

$$h(x)=\begin{cases}\sqrt{1-x} & \text{for } 0\le x<1,\\ (x-1)^4 & \text{for } 1\le x\le 3.\end{cases}$$

(v) Explain why hg^{-1} doesn't exist. [1]

(vi) Given that hg exist, define hg in similar form as function h. [2]

(vii) Find the range of hg. [2]

Section B: Probability and Statistics [60 marks]

5 Two fair 4-sided dice each has its faces labelled '1', '2', '3' and '4'. The two dice are thrown and the absolute difference in score on their bottom faces is denoted by X.

(i) Find $P(X = x)$ for all possible values of x. [2]

(ii) Find $E(X)$ and $Var(X)$. [2]

6 **(a)** Two events A and B occur with probabilities a and b respectively.
Let $c = P(A \cap B)$.

(i) Find the minimum value of c, in terms of a and b, if ~~terms of a and b,~~ $a+b>1$. [2]

(ii) Find the minimum value of c if $a = 0.6$ and $b = 0.2$. [1]

(b) For independent events A and B, prove that the events A' and B' are also independent. [3]

7 When Mr. Lee sends a text message to any of his students over the weekend, he gets a reply, on average, about 6 out of 10 times. On a particular weekend, Mr. Lee sends a text message to 25 students.

(i) State, in the context of this question, two assumptions needed to model the number of students that reply by a binomial distribution. [2]

(ii) Explain why one of the assumptions stated in part (i) may not hold in this context. [1]

Assume now that these assumptions do in fact hold. Stating the distribution clearly,

(iii) find the probability that at least half of the students reply. [3]

It is given instead that the probability of a randomly chosen student replying is p. Find the least value of p such that there is at least a 90% chance of more than 20 students replying to the text message. [3]

8 A teacher, Mr. Ku, suspects that the average time a student spends on his or her mobile phone per day is μ_0 minutes. He selected a random sample of 97 students in the school who own mobile phones and recorded the amount of time each student spent on his or her phone in a randomly selected day. The results are displayed in the table below.

Time spent per day (to nearest minute)	60	65	72	90	110	180
Number of people	11	20	32	18	10	6

(i) Calculate unbiased estimates of the population mean and variance of the time a student spends on his or her mobile phone per day. [2]

The null hypothesis that the average time a student spends on his or her mobile phone per day is μ_0 minutes is tested, at 5% level of significance, against the alternative hypothesis that the average time a student spends on his or her mobile phone per day differs from μ_0 minutes.

(ii) Determine the range of values of μ_0 for which the null hypothesis is rejected. [5]

(iii) Explain, in the context of this question, the meaning of 'at 5% level of significance'. [1]

(iv) If the null hypothesis in (ii) is rejected at 5% significance level, can we reject the null hypothesis at 1% level of significance? Explain your answer. [1]

9 The table below gives the world record time, in seconds, for the 100 metre race for the various years.

Year, x	1993	1994	1996	1999	2002	2005	2007	2008	2009
Time, t	9.87	9.85	9.84	9.79	9.78	9.77	9.74	9.69	9.58

(i) Draw the scatter diagram for these values, labelling the axes clearly. [2]

It is thought that the world record time t can be modelled by one of the following formulae

$$t = a + bY \quad \text{or} \quad t = c + dY^3$$

where a, b, c, d are constants and $Y = (x - 2000)$.

(ii) Using the value of the product moment correlation coefficient, explain which of $t = a + bY$ and $t = c + dY^3$ is the better model. [3]

(iii) Using the better model found in part **(ii)**, find the equation of a suitable regression line. Use it to estimate the world record time in the year 2018. Comment on the reliability of this estimate. [3]

(iv) Deduce, with justification, the product moment correlation coefficient between t and x. [1]

10 In this question, you should state clearly the distribution of any random variables that you define.

The volume, S, in ml, of perfume in a randomly chosen small bottle has mean 20 and variance σ^2.

(i) If $\sigma = 15$, explain why S may not be appropriately modelled by a normal distribution. [2]

It is now assumed that S follows a normal distribution

(ii) Given that 6.68% of the small bottles contains more than 23 ml of perfume, find the value of σ. [2]

For the rest of the question, the volume of perfume, S, in ml, in a randomly chosen small bottle follows the distribution $\mathrm{N}(20,4)$ and the volume of perfume, L, in ml, in a randomly chosen large bottle follows the distribution $\mathrm{N}(100,25)$.

(iii) Calculate the probability that 6 randomly chosen small bottles and 9 randomly chosen large bottles contain a total volume of at least 1 litre of perfume. [3]

(iv) Calculate the probability that the volume of perfume in a randomly chosen large bottle differs from 6 times the volume of perfume in a randomly chosen small bottle by more than 25ml. [3]

(v) State, in this context, an assumption needed for your calculations in parts **(iii)** and **(iv)**. [1]

11 A bag contains 5 cards with the letter 'A', 3 cards with the letter 'B' and 2 cards with the letter 'C'.

A A A A A B B B C C

The 10 cards are arranged at random in a row to form a letter sequence. For example, AABBCAAACB is a possible letter sequence.

(i) Find the number of possible letter sequences. [1]

(ii) Find the number of possible letter sequences if no two 'B's are next to each other **and** no two 'C's are next to each other. [4]

(iii) Find the probability that the first two letters are identical given that the second letter is **not** an 'A'. [4]

The 10 cards are now arranged at random in a circle.

(iv) Find the probability that no two 'A's are next to each other. [3]

Section A: Pure Mathematics [40 marks]

1 (i) Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n\to\infty}\frac{1}{n}\left\{f\left(\frac{n+1}{n}\right)+f\left(\frac{n+2}{n}\right)+\ldots+f\left(\frac{2n}{n}\right)\right\}$$

is $\int_1^2 f(x)\,dx$ [2]

(ii) Hence evaluate $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{n}{n+r}$ exactly. [3]

<table>
<tr><th>Suggested Solution</th><th>Comments</th></tr>
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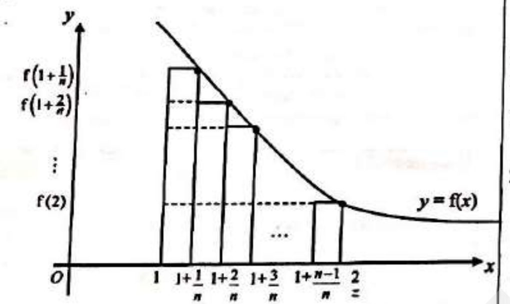
(i)



As shown in the diagram, the area under the curve $y = f(x)$ from $x=1$ to $x=2$ can be approximated by the total areas of the n rectangles with width $\frac{1}{n}$ and heights given by $f\left(1+\frac{1}{n}\right), f\left(1+\frac{2}{n}\right), f\left(1+\frac{3}{n}\right), \ldots, f\left(1+\frac{n}{n}\right)$.

As n increases, the approximations will get better and approach the exact area as a limit,

i.e. $$\lim_{n\to\infty}\frac{1}{n}\left\{f\left(\frac{n+1}{n}\right)+f\left(\frac{n+2}{n}\right)+\ldots+f\left(\frac{2n}{n}\right)\right\}=\int_1^2 f(x)\,dx$$

[Shown]

</td><td>

Part (i) is badly done.

Students ought to take note of important details such as

1) Indicating the correct "right point" height.

2) The area starts from x=1 to x=2.

Good explanation are hardly seen. Many students simple rewrote the given math terms and missed out the explanation in words.

Some students did not use area of n rectangles, but used area of triangles/ trapeziums/squares/blocks/boxes which are unacceptable.

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<tr><td>

(ii) $$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{n}{n+r}=\int_1^2 f(x)\,dx \quad \text{where } f(x)=\frac{1}{x}$$

</td><td>

Students who are able to see that $f(x)=\frac{1}{x}$, often get full marks.

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<tr><td>

$$=\int_1^2 \frac{1}{x}\,dx=\left[\ln|x|\right]_1^2=\ln 2-\ln 1=\ln 2$$

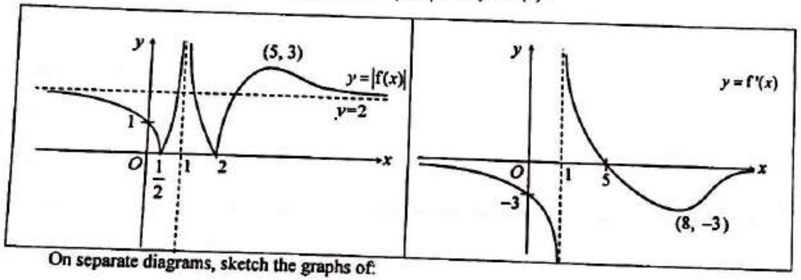
</td><td>

With minority of students doing differentiation instead of integration, which is very shocking.

Many students are confused what to do and perform AP/GP, series formulas or MOD instead of linking it to the question and use integration to find actual area under the curve.

</td></tr>
</table>

2 The diagrams below show the graphs of $y=|f(x)|$ and $y=f'(x)$.



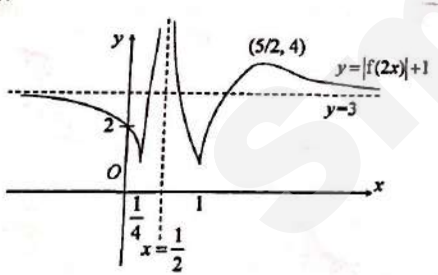
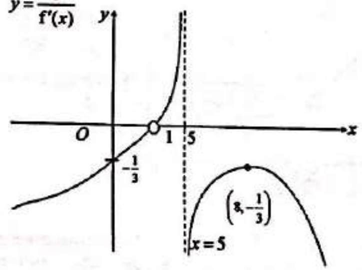
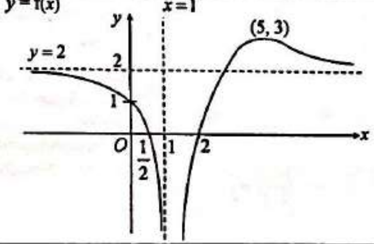
On separate diagrams, sketch the graphs of:

(i) $y=|f(2x)|+1$ [2]

(ii) $y=\dfrac{1}{f'(x)}$ [3]

(iii) $y=f(x)$ [3]

showing clearly, in each case, the intersection(s) with the axes, the coordinates of the turning point(s) and the equation(s) of the asymptotes.

Suggested Solution	Comments
(i) 	Generally well done. Students must read the question carefully on the requirements of indicating **"the intersection(s) with the axes, the coordinates of the turning point(s) and the equation(s) of the asymptotes."** Some students added extra full dotted line $y=1$. They should avoid doing so because marker may be mistaken it as an asymptote.
	Location of the graph can be improved as some students drew the graph lying on the x-axis
(ii) $y=\dfrac{1}{f'(x)}$ 	Generally well done. With minority of the students missing out some intercepts or forget to reciprocate the y-intercept value.
(iii) $y=f(x)$ 	Badly done. Many students either did not make use or misread the graph $y=f'(x)$. Hence unable to draw $y=f(x)$ correctly. Students who gave $y=-2$ as an asymptote lose all marks immediately.

3 **(a)** A retirement savings account pays a compound interest of 0.2% per month on the amount of money in the account at the end of each month. A one-time principal amount of $\$P$ is deposited to open the account and a monthly pay-out of $\$x$ is withdrawn from the account at the beginning of each month, starting from the month that the account is opened.

(i) Show that the amount in the account at the end of n months after the interest has been added is given by

$$P(1.002^n)-501x(1.002^n-1).$$ [4]

→ **(ii)** Suppose a fixed monthly pay-out of \$2,000 is to be sustained for at least 25 years, find the minimum principal amount required correct to the nearest dollar. [2]

→ **(iii)** If a principal amount of \$600,000 is placed in the account, find the number of years for which a monthly pay-out of \$2,000 per month can be sustained, leaving your answer correct to the nearest whole number. [2]

(b) A different retirement savings account provides an increasing amount of monthly pay-out over a period of 25 years. The pay-out in the first month is $\$a$. The pay-out for each subsequent month is an increment of $\$c$ from the pay-out of the previous month.

The pay-out in the final month is \$4,000, and the total pay-out at the end of 25 complete years is \$751,500. Find the month in which the pay-out is \$2,000. [5]

Suggested Solution | **Comments**

(a)(i)

Month	Beginning of month	Balance at the end of the month
1	$P-x$	$(P-x)(1.002)=P(1.002)-x(1.002)$
2	$P(1.002)-x(1.002)-x$	$P(1.002)^2-x(1.002)^2-x(1.002)$
3	$P(1.002)^2-x(1.002)^2-x(1.002)-x$	$P(1.002)^3-x(1.002)^3-x(1.002)^2-x(1.002)$
n	$P(1.002)^{n-1}-x(1.002)^{n-1}$ $-x(1.002)^{n-2}-\cdots-x(1.002)-x$	$P(1.002)^n-x(1.002)^n$ $-x(1.002)^{n-1}-\cdots-x(1.002)$

Balance after n months

$=P(1.002)^n-x(1.002)^n-x(1.002)^{n-1}-\cdots-x(1.002)$

$=P(1.002)^n-\dfrac{x(1.002)(1.002^n-1)}{1.002-1}$

$=P(1.002)^n-501x(1.002^n-1)$ (shown)

Comments:

1) A well-written, neat table is required to show the iterative pattern in order to obtain the "show" marks

2) Students who did not expand the expression for the balance at the end of the first month often wasted time trying to simplify the expression to obtain the "show" part

(a)(ii)

$P(1.002)^{300}-501(2000)(1.002^{300}-1)\geq 0$

$P\geq\dfrac{501(2000)(1.002^{300}-1)}{1.002^{300}}=451761.1356$

Minimum principal amount required is \$451762.

Comments:

3) Students are reminded that since the question wants the minimum amount, the intermediate step should be an equality for $P\geq$

4) Since it is to the nearest dollar, then 451762 is the nearest dollar that is more than 451761.1356

(a)(iii)

Method 1

$(600000)(1.002)^n-501(2000)(1.002^n-1)\geq 0$

n	LHS
457	$214.38>0$
458	$-1789.2<0$

$\therefore$ No. of months $=457$

Method 2

$(600000)(1.002)^n-501(2000)(1.002^n-1)<2000$

n	LHS
456	$2214>2000$
457	$214.38<2000$

$\therefore$ No. of months $=457$

No. of years for which the pay-out can be sustained is $\dfrac{457}{12}=38.083\approx 38$ years.

Comments:

5) Even though the question did not ask for a maximum, the keyword "sustained" indicates that there is a range of years that the pay-out can continue

6) The rationale for solving ≥ 0 is that if there is at least \$0 in the account at the end of that month, it means that there was enough (≥ 2000) at the beginning of the month to pay-out

7) The other interpretation is to solve <2000. This means that at the end of that month, there is not enough to pay-out in the next month

8) GC table method is preferred here and should be clearly shown which value of n satisfies the inequality

9) It is not necessary to consider $n-1$ for ≥ 2000 and then to add the extra month. This is prone to careless mistakes

(b)

Month	Pay-out
1	a
2	a+c
3	a+2c
⋮	⋮
300	4000

Let $\$a$ be the pay-out in the first month.

$a+299c=4000$ --- (1)

$\dfrac{300}{2}(2a+299c)=751500$

$2a+299c=5010$ --- (2)

Solving the simultaneous equations,

$a=1010,\quad c=10$

To find the month n with a pay-out of \$2,000:

$(1010)+(n-1)(10)=2000$

$n=100$

$\therefore$ The pay-out is \$2,000 in the 100th month.

<u>Alternative solution:</u>

Let $\$a$ be the pay-out in the first month.

$a+299c=4000$ --- (1)

Comments:

10) This question was generally well-done with a few minor careless mistakes such as forgetting that the AP formulae has $n-1$ to get 299

11) Overall, students **SHOULD** answer the question when it is a problem sum. There is a large number of students who do not bother to answer the question (E.g. part (ii), leaving it as an inequality). Remember that it is not just about writing the correct number but it is about showing you understanding of the question.

$\frac{300}{2}(a+4000)=751500 \Rightarrow a=1010$. Hence from (1), $c=10$ Consider $U_n=1010+(n-1)(10)=2000 \Rightarrow n=100$ $\therefore$ The pay-out is \$2,000 in the 100th month.	

4 The function f is defined by

$$f(x)=\begin{cases} 4-x & \text{for } 1 \le x < 3, \\ (x-4)^2 & \text{for } 3 \le x < 4, \end{cases}$$

and it is given that $f(x-3)=f(x)$ for all real values of x.

(i) State a reason why f does not have an inverse. [1]

(ii) Sketch the graph of $y=f(x)$ for $-1<x<6$. [3]

(iii) Evaluate $f(2017)$. [2]

The function g has domain $[1,4)$ and is defined by

$$g(x)=\begin{cases} 4-x & \text{for } 1 \le x < 3, \\ (x-4)^2 & \text{for } 3 \le x < 4. \end{cases}$$

(iv) By sketching $y=g(x)$ and $y=g^{-1}(x)$ on the same diagram, state the values of x such that $g(x)=g^{-1}(x)$. [3]

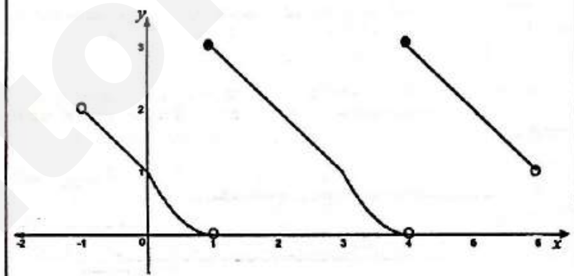
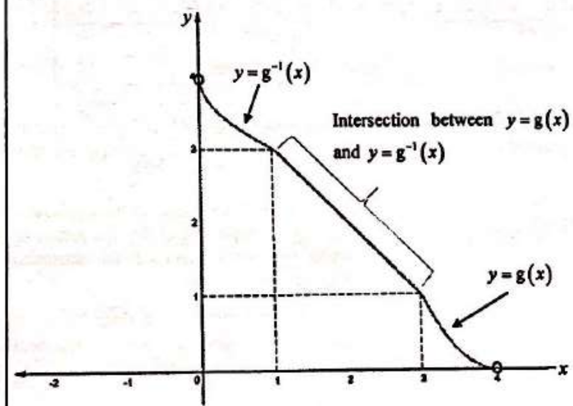
The function h, is defined by

$$h(x)=\begin{cases} \sqrt{1-x} & \text{for } 0 \le x < 1, \\ (x-1)^4 & \text{for } 1 \le x \le 3. \end{cases}$$

(v) Explain why hg^{-1} doesn't exist. [1]

(vi) Given that hg exist, define hg in similar form as function h. [2]

(vii) Find the range of hg. [2]

Suggested Solution	Comments
(a)(i) Since $f(x-3)=f(x)$, we can easily find 2 values of x (for example $f(-3)=f(0)$) to show that it's not 1-1, hence no inverse.	This part was not well done. Many used the reason that "it's not 1-1 hence no inverse" without giving explicit example like $y=1$.
	Some use $y=k, k\in\mathbb{R}$ which is obvious incorrect. Please also not that Piecewise function DOES NOT meant it's not 1-1. Else there will not be (iv).
(ii) 	For this part, students need to pay more attention to the "end points" on whether it's include or not. Those end points not to be included should have a "open circle".
(iii) $f(2017)=f(2014)=f(2011)=\cdots=f(1)=3$	This part is generally well done.
(iv)  	For this part, sutdents are NOT required to find the function of $y=g^{-1}(x)$ explicitly but just a reflection of $y=g(x)$ about the line $y=x$. You can also acquire the graph via GC. For this particular question, $g(x)=g^{-1}(x)$ will give infinitely many solutions hence those who try to use the idea $g(x)=g^{-1}(x) \Rightarrow g(x)=x$ will not work.

From the graph, the values of x such that $g(x)=g^{-1}(x)$ is $1\le x\le 3$.	
(v) $R_{g^{-1}}=[1,4)\not\subset[0,3]=D_h$, hg^{-1} doesn't exist	This part is generally well done.
(vi) $hg(x)=\begin{cases}(3-x)^4 & \text{for } 1\le x<3,\\ \sqrt{1-(x-4)^2} & \text{for } 3\le x<4.\end{cases}$ Note that a more detailed solution would be: $hg(x)=\begin{cases}(3-x)^4 & \text{for } 1\le x<3,\\ \sqrt{1-(x-4)^2} & \text{for } 3<x<4,\\ \left((x-4)^2-1\right)^2 & \text{for } x=3\end{cases}$ or $hg(x)=\begin{cases}(3-x)^4 & \text{for } 1\le x<3,\\ \sqrt{1-(x-4)^2} & \text{for } 3<x<4,\\ 0 & \text{for } x=3\end{cases}$ Alternative solutions can be $hg(x)=\begin{cases}(3-x)^4 & \text{for } 1\le x\le 3,\\ \sqrt{1-(x-4)^2} & \text{for } 3<x<4.\end{cases}$	This part is not well done. Many student didn't pay attention to the range of $g(x)$ for each of the domain $[1,3)$ and $[3,4)$ respectively in order to decide which pieces of $h(x)$ should they substitute in. Only a handful of student realised that when $x=3$, it produces a 3rd piece to the final answer.
(vii) $D_g=[1,4)\xrightarrow{g}R_g=(0,3]\xrightarrow{h}R_{hg}=[0,16]$ Therefore, Range of $hg=[0,16]$	For this part, student using the mapping method is generally well done. A handful of student adopted to substitute the endpoints of the domain into the function which will yield a wrong answer as this method only works if the function is an increasing and continuous function.

Section B: Probability and Statistics [60 marks]

5 Two fair 4-sided dice each has its faces labelled '1', '2', '3' and '4'. The two dice are thrown and the absolute difference in score on their bottom faces is denoted by X.

(i) Find $P(X=x)$ for all possible values of x. [2]

(ii) Find $E(X)$ and $Var(X)$. [2]

Suggested Solution	Comments
(i) Consider the number of outcomes with the respective differences out of 16: $P(X=0)=\frac{4}{16}=\frac{1}{4}$ $P(X=1)=\frac{6}{16}=\frac{3}{8}$ $P(X=2)=\frac{4}{16}=\frac{1}{4}$ $P(X=3)=\frac{2}{16}=\frac{1}{8}$	• This question was generally well-done, although some candidates failed to recognise the question asked for the probability distribution of the absolute difference in scores shown on the bottom daces on the two fair 4-sided dice.
(ii) $E(X)=(0)\left(\frac{1}{4}\right)+(1)\left(\frac{3}{8}\right)+(2)\left(\frac{1}{4}\right)+(3)\left(\frac{1}{8}\right)=1.25$ $Var(X)=E(X^2)-(E(X))^2$ $=(0)^2\left(\frac{1}{4}\right)+(1)^2\left(\frac{3}{8}\right)+(2)^2\left(\frac{1}{4}\right)+(3)^2\left(\frac{1}{8}\right)-(1.25)^2$ $=0.9375$	• A few students wrongly stated and used that $Var(X)=E(X^2)$.

6 (a) Two events A and B occur with probabilities a and b respectively.
Let $c = P(A \cap B)$.

(i) Find the minimum value of c, in terms of a and b, if $a+b>1$. [2]

(ii) Find the minimum value of c if $a=0.6$ and $b=0.2$. [1]

(b) For independent events A and B, prove that the events A' and B' are also independent. [3]

Suggested Solution	Comments
(a) (i) By considering $P(A\cup B)\le 1$ $P(A\cap B')+P(A\cap B)+P(B\cap A')\le 1$ Let $P(A\cap B)=c$ Then we have $a-c+c+b-c\le 1$ $c\ge a+b-1$ Note: Can also use $P(A\cup B)=P(A)+P(B)-P(A\cap B)\le 1 \Rightarrow a+b-c\le 1 \Rightarrow c\ge a+b-1$ Thus, minimum possible value of $P(A\cap B)$ is $a+b-1$. (ii) when $P(A)=0.6$ and $P(B)=0.2$, [note that $P(A)+P(B)<1$]. $P(A\cup B)=P(A)+P(B)-P(A\cap B)\Rightarrow P(A\cap B)=0.8-P(A\cup B)$ minimum possible value of $P(A\cap B)$ is 0, i.e., A and B are disjoint. [i.e. when $P(A\cup B)=P(A)+P(B)$].	• The key trick in (a)(i) is to use the inequality $P(A\cup B)\le 1$ to obtain the minimum value of c, which many students failed to establish inequalities/provide the necessary explanations to eventually deduce that the **minimum value** of $c=a+b-1>0$ in part(a)(i), since $a+b>1$. • A significant number of student wrongly assumed that events A and B are independent in part (a), which did not state so. • A number of students wrongly stated that the minimum value of c in part (a)(ii) is a negative number!
(b) $P(A'\cap B')=1-P(A\cup B)$ $=1-[P(A)+P(B)-P(A\cap B)]$ $=1-P(A)-P(B)+P(A)P(B)$ (since A, B are independent) $=P(A')-P(B)[1-P(A)]$ $=P(A')-P(B)[P(A')]$ $=P(A')[1-P(B)]$ $=P(A')P(B')$ Thus A' and B' are independent as well.	• A significant number of students did not give a general proof, and wrongly used specific figures in parts (a)(i), (ii). • Several students wrongly assumed within their attempted proofs (explicitly/implicitly) that A' and B' are independent, e.g., $P(A'\cap B')=(1-a)(1-b)$ $=P(A')\cdot P(B')$ which was what they exactly were required to prove in the first place!

7 When Mr. Lee sends a text message to any of his students over the weekend, he gets a reply, on average, about 6 out of 10 times. On a particular weekend, Mr. Lee sends a text message to 25 students.

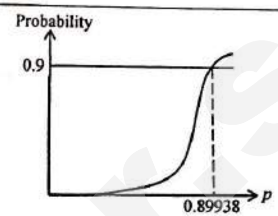
(i) State, in the context of this question, two assumptions needed to model the number of students that reply by a binomial distribution. [2]

(ii) Explain why one of the assumptions stated in part (i) may not hold in this context. [1]

Assume now that these assumptions do in fact hold. Stating the distribution clearly,

(iii) find the probability that at least half of the students reply. [3]

It is given instead that the probability of a randomly chosen student replying is p. Find the least value of p such that there is at least a 90% chance of more than 20 students replying to the text message. [3]

Suggested Solution	Comments
(i) A student replying has a constant probability of 0.6. The event of a student replying is independent of another student. OR Whether a student replies is independent of another student.	Students have to be aware that independence refers to the events instead of the probabilities.
(ii) Some students are more likely to reply a message. Hence the assumption of equal probability might not hold.	Students have to highlight which assumption they are addressing. Most students did well here.
(iii) Let X be the number of students that reply, out of 25. $X \sim \mathrm{B}(25, 0.6)$ Required probability $= \mathrm{P}(X \geq 13)$ $= 1 - \mathrm{P}(X \leq 12)$ $= 0.84623 \quad (5\text{sf})$ $= 0.846 \quad (3\text{sf})$	This is generally well done. Most students were able to identify $X \geq 13$ as the correct range of values.
(iv) Let Y be the number of students that reply, out of 25. Let p be the new probability of a student replying. $Y \sim \mathrm{B}(25, p)$ Given that $\mathrm{P}(Y > 20) \geq 0.9$ (or $\mathrm{P}(Y \leq 20) \leq 0.1$)  Therefore, the least probability of a student replying is 0.899 [accept 0.900].	Students have to be aware that they have to define a new random variable in full. 2 common errors made: (1) $\mathrm{P}(Y > 20)$ is equal to $1 - \mathrm{P}(Y \leq 19)$, which means the value of $x = 20$ is unaccounted for. (2) Interpret "at least a 90% chance" as "= 0.9" or "> 0.9", instead of "≥ 0.9". It is important to remember that when a graphical or tabular approach is used, it is always beneficial to show the working.

8 A teacher, Mr. Ku, suspects that the average time a student spends on his or her mobile phone per day is μ_0 minutes. He selected a random sample of 97 students in the school who own mobile phones and recorded the amount of time each student spent on his or her phone in a randomly selected day. The results are displayed in the table below.

Time spent per day (to nearest minute)	60	65	72	90	110	180
Number of people	11	20	32	18	10	6

→ **(i)** Calculate unbiased estimates of the population mean and variance of the time a student spends on his or her mobile phone per day. [2]

The null hypothesis that the average time a student spends on his or her mobile phone per day is μ_0 minutes is tested, at 5% level of significance, against the alternative hypothesis that the average time a student spends on his or her mobile phone per day differs from μ_0 minutes.

−1 **(ii)** Determine the range of values of μ_0 for which the null hypothesis is rejected. [5]

(iii) Explain, in the context of this question, the meaning of 'at 5% level of significance'. [1]

(iv) If the null hypothesis in (ii) is rejected at 5% significance level, can we reject the null hypothesis at 1% level of significance? Explain your answer. [1]

Suggested Solution	Comments
(i) Using a GC, The unbiased estimate of population mean is $83.134 \approx 83.1$ (3sf) The unbiased estimate of population variance is $29.011^2 \approx 841.64 \approx 842$ (3sf) 	Students have to know the GC skills here to find the unbiased estimates of the population mean and variance. Key L2 as FrequentList. 1-Var Stats $\bar{x}=83.13402062$ $\Sigma x=8064$ $\Sigma x^2=751188$ $Sx=29.0106406$ **Note : use S_x and then square it to get S^2 value.**
(ii) Let X be the time a student spent on their mobile phone per day. Let μ be the population mean time a student spent on their mobile phone per day. Test: $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ Perform a 2-tail test and 5% level of significance	Students need to write down the Null and Alternative hypotheses clearly. This is a 2-tailed test.
Under H_0, since $n = 97$ is large, by Central Limit Theorem, $\bar{X} \sim N\left(\mu_0, \frac{s^2}{n}\right)$ approximately, with unknown μ_0 and $s^2 = 841.64$ At a 5% (2-tail), the Critical Region is $\{\lvert z\rvert > 1.96\}$ Since H_0 is rejected, $\lvert z-value\rvert > 1.96$ z-value < -1.96 or z-value > 1.96 $\frac{83.134-\mu_0}{\sqrt{841.64/97}} < -1.96$ or $\frac{83.134-\mu_0}{\sqrt{841.64/97}} > 1.96$ $\mu_0 > 88.907 \approx 88.9$ or $\mu_0 < 77.361 \approx 77.4$	Many students wrote wrong distribution for $\bar{X}$ Take note: when you do have the populat variance, σ^2 and n normal distribution, ap the Central Limit Theore Identifying the rejecti region was important. z-value < -1.96 or z-value > 1.96 **Remember the correc standardisation for th test statistics: $z = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$**
(iii) 5% level of significance means that there is a 0.05 probability of concluding that the average time a student spent on their mobile phone per day differs from μ_0 when in actual fact it did not.	Key mistake: 1. Students have to write in context of the question and not just writing down the definition of the meaning of level of significance. 2. Not short form writing. What is LOS? It is not recognised.
(iv) Test: $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ Since we reject Ho at 5% level of significance, $p-value = p < 0.05$ But that doesn't imply $p < 0.01$. Hence, it's inconclusive.	This one mark is to check for conceptual understanding for the rejection criteria at different level of significance.

9 The table below gives the world record time, in seconds, for the 100 metre race for the various years.

Year, x	1993	1994	1996	1999	2002	2005	2007	2008	2009
Time, t	9.87	9.85	9.84	9.79	9.78	9.77	9.74	9.69	9.58

(i) Draw the scatter diagram for these values, labelling the axes clearly. [2]

It is thought that the world record time t can be modelled by one of the following formulae

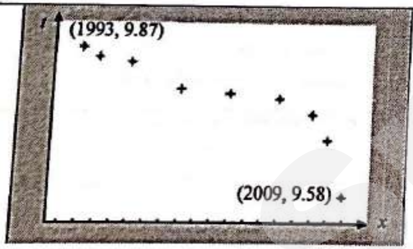
$$t = a + bY \quad \text{or} \quad t = c + dY^3$$

where a, b, c, d are constants and $Y = (x - 2000)$.

(ii) Using the value of the product moment correlation coefficient, explain which of $t = a + bY$ and $t = c + dY^3$ is the better model. [3]

(iii) Using the better model found in part (ii), find the equation of a suitable regression line. Use it to estimate the world record time in the year 2018. Comment on the reliability of this estimate. [3]

(iv) Deduce, with justification, the product moment correlation coefficient between t and x. [1]

Suggested Solution	**Comments**
Overall comments on this question: The topic of linear regression is really focused on the use of the GC. Unfortunately, any wrong answer is penalized without working marks so students need to check their data entry in the GC very carefully. There is also a premium on the accuracy of answers so students are advised to leave their answers to a higher degree of accuracy (at least 5 sf) in intermediate answers and to work with the more accurate answers for subsequent parts. There is a significant minority of students who carelessly round off their answers incorrectly. This will be penalized. For example $-0.89351066 = -0.893510$ (to 6 sf)	
	(i) Many students wrongly labelled the vertical axis as y-axis (not penalised.) The scatter diagram need not be drawn to scale but the general trend needs to be shown. For example, we should not see a straight line. Most students drew the graph quite well, obtaining full marks for this part.
(ii) For $t = a + bY$, $r = -0.89351 \approx -0.894$ (3sf) For $t = c + dY^3$, $r = -0.97163 \approx -0.972$ (3sf) Since -0.972 is closer to -1 than -0.894, $t = c + dY^3$ is the better model. Alt explanation accepted: $\lvert$r-value$\rvert$ for $t = c + dY^3$ is closer to 1 than $\lvert$r-value$\rvert$ for $t = a + bY$.	Not accepted: • r-value for $t = c + dY^3$ is closer to 1 than r-value for $t = a + bY$. (since both r-values are negative) • r-value for $t = c + dY^3$ is higher than r-value for $t = a + bY$. (since both r-values are negative)
(iii) Using GC, $t = 9.7985 - (2.5279 \times 10^{-4})(x - 2000)^3$ When $x = 2018$, $t = 9.7985 - (2.5279 \times 10^{-4})(2018 - 2000)^3$ $= 8.3242 \approx 8.32$ (3sf) The estimate obtained by extrapolation is not reliable since $x = 2018$ lies outside the range of the observed data. $(1993 \le x \le 2009)$	Many students missed out the 10^{-4} part and obtained an outrageous answer for t. That should be a hint for students to double-check their working. Most students could see "extrapolation" as a key reason for the estimate not being reliable, but some answers are not precise about its meaning about extension of the linear relationship for example "It is not reliable because it is extrapolation." is a very vague answer.
(v) Since the PMCC is not affected by linear transformation (mx+C with m > 0), in this case, a translation of 2000 units parallel to the x-axis $r = -0.89351 \approx -0.894$ (3sf)	Many students could recognise that a translation parallel to the x-axis (or y-axis as well) would shift the data points together and not affect the linear relationship.

10 In this question, you should state clearly the distribution of any random variables that you define.

The volume, S, in ml, of perfume in a randomly chosen small bottle has mean 20 and variance σ^2.

(i) If $\sigma = 15$, explain why S may not be appropriately modelled by a normal distribution. [2]

It is now assumed that S follows a normal distribution

(ii) Given that 6.68% of the small bottles contains more than 23 ml of perfume, find the value of σ. [2]

For the rest of the question, the volume of perfume, S, in ml, in a randomly chosen small bottle follows the distribution $N(20,4)$ and the volume of perfume, L, in ml, in a randomly chosen large bottle follows the distribution $N(100,25)$.

(iii) Calculate the probability that 6 randomly chosen small bottles and 9 randomly chosen large bottles contain a total volume of at least 1 litre of perfume. [3]

(iv) Calculate the probability that the volume of perfume in a randomly chosen large bottle differs from 6 times the volume of perfume in a randomly chosen small bottle by more than 25ml. [3]

(v) State, in this context, an assumption needed for your calculations in parts (iii) and (iv). [1]

Suggested Solution	Comments
(i) Let S be the volume of perfume in a randomly chosen small bottle. $S \sim N(20,\sigma^2)$ When $\sigma = 15$, $P(S<0) = 0.0912$ (3sf) This means that there is a significant chance (9.12%) that a small bottle has a negative volume of perfume. Hence, a normal distribution with $\mu = 20, \sigma = 15$ is not appropriate.	• It is important to consider the context when giving the explanation. • Moreover, explanations which are too vague, for instance mentioning that $P(S<0)$ is "significant" without actually calculating the probability, are not awarded full credit as this is a 2-mark question.
(ii) $S \sim N(20,\sigma^2)$ Given $P(S>23) = 0.0668$ $P\left(Z > \frac{23-20}{\sigma}\right) = 0.0668$ Using GC, $\frac{23-20}{\sigma} = 1.5001$ $\sigma = \frac{3}{1.5001}$ $= 1.9999$ (5sf) $= 2.00$ (3sf)	• As σ may not necessarily be an integer, no marks are awarded if "table of values' are used. • When using the graphical method, it is advisable to include a brief sketch of the graphs being used.
(iii) $S \sim N(20,2^2)$ Let L be the volume of perfume in a randomly chosen large bottle. $L \sim N(100,5^2)$ Let $T = S_1 + \cdots + S_6 + L_1 + \cdots + L_9$ $E(T) = E(S_1) + \cdots + E(S_6) + E(L_1) \cdots + E(L_9)$ $= 6E(S) + 9E(L_9)$ $= 6(20) + 9(100)$ $= 1020$ $Var(T) = Var(S_1) + \cdots + Var(S_6)$ $+ Var(L_1) \cdots + Var(L_9)$ $= 6Var(S) + 9Var(L)$ $= 6(4) + 9(25)$ $= 249$ $T \sim N\left(1020, (\sqrt{249})^2\right)$ Required probability $= P(T > 1000)$ $= 0.89750$ (5sf) $= 0.898$ (3sf)	• This part is generally done well.

(iv) $E(6S-L)=6E(S)-E(L)$ $=6(20)-100$ $=20$ $\text{Var}(6S-L)=6^2\text{Var}(S)+\text{Var}(L)$ $=36(4)+25$ $=169$ $6S-L\sim N(20,13^2)$ Required probability $=P(\lvert 6S-L\rvert>25)$ $=1-P(-25<6S-L<25)$ $=0.35053 \quad (5sf)$ $=0.351 \quad (3sf)$	• This part is also generally done well. • However, do note that the random variable $\lvert 6S-L\rvert$ does not necessarily follow a normal distribution.
(v) An assumption needed is that the distribution of volumes of perfume in ALL (between and within S and L) are independent of one another.	• There is a huge variety of answers presented here, some talk about the need for the bottles to have an infinite capacity while others mentioned the need for there to be no spillage/evaporation. However, these answers fail to capture the key point of the question.

11 **11** A bag contains 5 cards with the letter 'A', 3 cards with the letter 'B' and 2 cards with the letter 'C'.

A A A A A B B B C C

The 10 cards are arranged at random in a row to form a letter sequence. For example, AABBCAAACB is a possible letter sequence.

(i) Find the number of possible letter sequences. [1]

(ii) Find the number of possible letter sequences if no two 'B's are next to each other and no two 'C's are next to each other. [4]

(iii) Find the probability that the first two letters are identical given that the second letter is not an 'A'. [4]

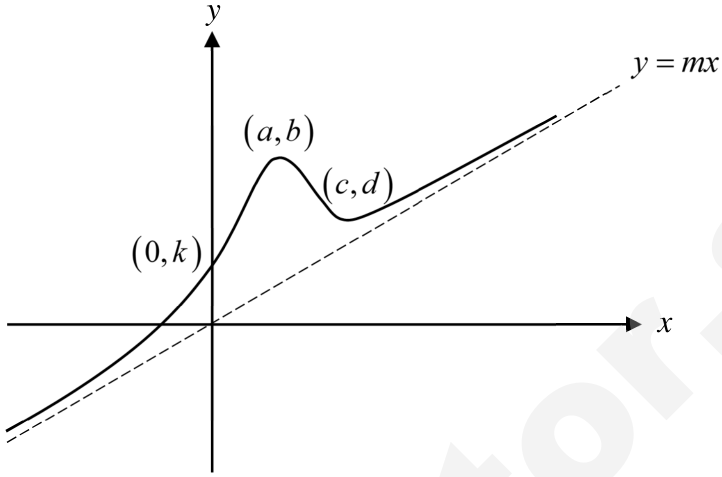
The 10 cards are now arranged at random in a circle.

(iv) Find the probability that no two 'A's are next to each other. [3]

Suggested Solution	Comments
(i) Number of different letter sequences $=\dfrac{10!}{5!3!2!}=2520$	This part was generally well done.
(ii) Number of different letter sequences = (Number with no adjacent 'B's) – (Number with no adjacent 'B's, but with adjacent 'C's) = (No. of ways to permutate 7 units – 5As and 2Cs, and slot in 3Bs in between the 7 units) – (No. of ways to permutate 6 units – 5As and CC, and slot in 3Bs in between the 6 units) $=\left(\dfrac{7!}{5!2!}\cdot{}^8C_3\right)-\left(\dfrac{6!}{5!}\cdot{}^7C_3\right)$ $=966$ <u>Alternative Solution:</u> Case 1: 2 'C's are separated by 'A's Number of such letter sequences $={}^6C_2\,{}^8C_3=840$ Case 2: 2 'C's are not separated by 'A's Then 'C's have to be separated by a 'B' i.e. there is a CBC block. Number of such letter sequences $=\dfrac{6!}{5!}{}^7C_2=126$ Total number of such letter sequences = 840 + 126 = 966	Few students can do this part correctly. Most students were able to find (1) the number of letter sequences with no adjacent 'B's, or/and (2) the number of letter sequences with no adjacent 'C's. However, they were unable to combine the different cases together properly. A number of students misinterpreted that 'B' and 'C' cannot be adjacent, and arrived at the wrong answer.
(iii) P(1st and 2nd letter identical | 2nd letter is not 'A') $=\dfrac{P(BB\text{ or }CC)}{P(AA'\text{ or }A'A')}$ $=\dfrac{P(BB)+P(CC)}{P(AA')+P(A'A')}$ $=\dfrac{\left(\frac{3}{10}\right)\left(\frac{2}{9}\right)+\left(\frac{2}{10}\right)\left(\frac{1}{9}\right)}{\left(\frac{5}{10}\right)\left(\frac{5}{9}\right)+\left(\frac{5}{10}\right)\left(\frac{4}{9}\right)}$ or $=\dfrac{\left(\frac{3}{10}\right)\left(\frac{2}{9}\right)+\left(\frac{2}{10}\right)\left(\frac{1}{9}\right)}{\left(\frac{5}{10}\right)}$ $=\dfrac{8}{45}$ <u>Alternative Solution:</u>	Many students were able to do this part. The common mistakes were: (1) students were not able to find the probability of the first two letters being 'B' or 'C' correctly, (2) students did not consider the conditional probability properly and failed to divide by the probability of the 2nd letter not being 'A'.

P(1st and 2nd letter identical | 2nd letter is not 'A') $=\dfrac{P(BB \text{ or } CC)}{P(\text{2nd letter is } B \text{ or } C)}$ $=\dfrac{\dfrac{\text{No. of permutations starting with } BB \text{ or } CC}{\text{Total no. of permutations}}}{\dfrac{\text{No. of permutations with 2nd letter } B \text{ or } C}{\text{Total no. of permutations}}}$ $=\dfrac{\left(\dfrac{8!}{5!2!}+\dfrac{8!}{5!3!}\right)}{\left(\dfrac{9!}{5!2!2!}+\dfrac{9!}{5!3!}\right)}$ $=\dfrac{8}{45}$	
(iv) Treat all the As as distinct (A_1, A_2, A_3, A_4, A_5), Bs as distinct (B_1, B_2, B_3) and Cs as distinct (C_1, C_2). Probability $=\dfrac{\text{No. of ways to permutate 10 cards in a circle with all As separate}}{\text{No. of ways to permutate 10 cards in a circle}}$ $=\dfrac{\text{No. of ways to permutate } B_1B_2B_3C_1C_2 \text{ in a circle, and slot in the As}}{\text{No. of ways to permutate 10 cards in a circle}}$ $=\dfrac{\left(\dfrac{5!}{5}\right)(5!)}{\left(\dfrac{10!}{10}\right)}=\dfrac{1}{126}$ <u>Alternative Solution:</u> Treat 5 As as identical, 3 Bs as identical and 2 Cs as identical. Since all cards are used, each outcome is equally probable. Probability $=\dfrac{\text{No. of ways of arranging 3Bs and 2Cs in a circle, and slot in the 5As}}{\text{No. of ways of arranging 5As, 3Bs and 2Cs in a circle}}$ $=\dfrac{\left(\dfrac{5!}{5\cdot 3!2!}\right)}{\left(\dfrac{10!}{10\cdot 5!3!2!}\right)}=\dfrac{1}{126}$	Many students were able to do this part. The common mistakes were: (1) students were confused about whether to treat the letters as distinct or identical and treated them as distinct/identical in the numerator and as the other in the denominator, (2) students did not take into account the arrangement around a circle rather than a straight line.

1 The diagram below shows the graph of $y = \mathrm{f}(x)$. The curve has a maximum point at (a, b), a minimum point at (c, d) and cuts the y-axis at $(0, k)$. The equation of the asymptote is $y = mx$, where m is a positive constant.



Sketch the graph of $y = \mathrm{f}'(x)$, giving equation(s) of any asymptotes and coordinate(s) of any intercepts, if it is possible to do so. [2]

2 A drone is programmed to fly from a point O at ground level eastwards to a point C at the top of a building. The point C is 0.1 km vertically above ground level and 0.6 km horizontally from O. The drone passes through two checkpoints A and B before reaching C. The horizontal distances and vertical heights of A and B are shown below.

Point	Horizontal Distance from O (km)	Vertical Height (km)
A	0.15	0.1
B	0.3	0.125

It is given that the flight path of the drone is cubic in nature. Taking O to be the origin and O, A, B and C all lie on the same vertical plane, find the cartesian equation of the flight path. [3]

The drone manufacturer is highly confident of the ability of the drone to keep to its flight path despite external factors such as wind, temperature or humidity. It claims that

> "The drone on this particular programmed flight path will pass through the point at a vertical height of 0.1125 km and a horizontal distance of 0.45 km eastwards from O."

By considering the flight path of the drone, comment on the accuracy of the manufacturer's claim. [1]

3 **(a)** A point Q has position vector $\mathbf{q}$ and a line l has equation $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in$ …. A vector $\mathbf{p}$, where $|\mathbf{p}|=1$, is in the direction of $\mathbf{d}$. Give a geometrical meaning of $|\mathbf{p}\times(\mathbf{a}-\mathbf{q})|$. [1]

(b) The vectors $\mathbf{a}$ and $\mathbf{b}$ form two adjacent sides of a triangle *OAB*, where $\mathbf{a}=\overrightarrow{OA}$ and $\mathbf{b}=\overrightarrow{OB}$. The angle between $\mathbf{a}$ and $\mathbf{b}$ is θ. By considering $(\mathbf{a}-\mathbf{b}).(\mathbf{a}-\mathbf{b})$ and the cosine rule for the triangle *OAB*, show that $\mathbf{a.b}=|\mathbf{a}||\mathbf{b}|\cos\theta$. [4]

4 **(i)** Show that $\frac{3}{2}-\sqrt{a+\frac{1}{4}}<1$, where a is a positive constant. [1]

(ii) Solve the inequality $\frac{x^2-x-a}{x-1}\le 2$, where a is a positive constant. [4]

(iii) Hence solve $\frac{a+x-x^2}{x}\le 2$. [2]

5 Let $\mathrm{f}(r)=\cos\left[\alpha+\left(r+\frac{1}{2}\right)\beta\right]$, $\beta\neq 2k\pi,\ k\in$ ….

(i) Find $\mathrm{f}(r)-\mathrm{f}(r-1)$. [2]

(ii) Show that

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin(\alpha+n\beta)=\frac{\sin(\alpha+p\beta)\sin q\beta}{\sin\frac{1}{2}\beta},$$

where p and q are constants to be determined in terms of n . [4]

(iii) Deduce an expression for

$$\cos\alpha+\cos\left(\alpha+\frac{\pi}{2}\right)+\cos(\alpha+\pi)+\cos\left(\alpha+\frac{3\pi}{2}\right)+\ldots+\cos\left(\alpha+\frac{n\pi}{2}\right).$$ [2]

6 A curve C has parametric equations

$$x=2\sin t,\qquad y=1+\cos t,\qquad 0<t<\pi.$$

(i) Show that the equation of the tangent to C at the point $P(2\sin p, 1+\cos p)$ is $2y+x\tan p=2(1+\sec p)$. [4]

(ii) The tangent at P meets the x-axis at the point A and the y-axis at the point B. The point M is the midpoint of AB. Find the cartesian equation of the curve traced by M as p varies. [5]

7 Mr Wang decides to paint the walls of his room which has a total surface area of 40 m^2. He has two plans:

Plan A: He paints 7 m^2 on the first day, and on each subsequent day, he paints 0.5 m^2 less than the previous day.

Plan B: He paints 7 m^2 on the first day, and on each subsequent day, he paints 20% less than the previous day.

(i) Find an expression for the area painted on the nth day according to Plan A. Give your answer in terms of n. [1]

(ii) Find the number of days required for him to complete painting his room according to Plan A. [2]

(iii) Explain why Plan A cannot be used to paint a wall of arbitrary size. Find the largest area that Plan A can be applied to. [2]

(iv) Find algebraically the number of days needed to paint at least 70% of his room according to Plan B. [3]

(v) State, with a reason, which plan A or B Mr Wang should choose to complete the paint job. [1]

8 The curve G has equation

$$y = \frac{x+2}{x(x+k)} ,$$

where k is a real constant.

(i) Find the range of values of k for which G has no stationary points. [3]

In the rest of the parts of the question, let $k = -2$.

(ii) Sketch G, stating clearly the equations of any asymptotes, the coordinates of the stationary points and the points where G crosses the axes. [3]

(iii) State the values of m for which the line $y = m$ intersects G once. [1]

(iv) By sketching a suitable curve on the diagram in part **(ii)**, show that the equation $x^4 - 4x^3 + 3x^2 + x - 2 = 0$ has exactly two real roots. [2]

9 **(a)** Showing your working clearly, find the complex numbers z and w which satisfy the simultaneous equations

$$4\mathrm{i}z - w = 9\mathrm{i} - 13\ ,$$
$$(4+2\mathrm{i})w^* = z + 3\mathrm{i}\ . \qquad [4]$$

(b) The complex numbers u and v are such that $u = 5\mathrm{e}^{\frac{7}{12}\pi\mathrm{i}}$ and $v = 6\mathrm{i}\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}$ respectively.

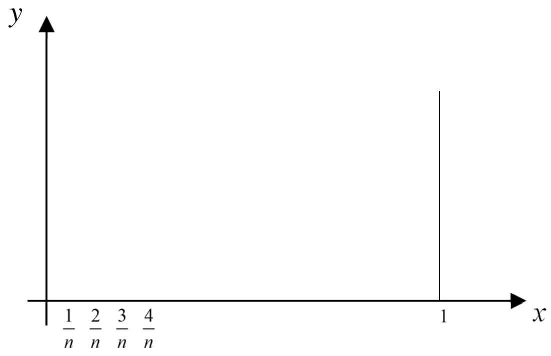
(i) Find an exact expression of $\dfrac{u^2}{v^*}$, giving your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

(ii) Find the three smallest positive integer values of n for which $\dfrac{u^n}{v^*}$ is purely imaginary. [2]

10 **(a)** **(i)** Find $\int x\sqrt{x-1}\ \mathrm{d}x$ using integration by parts. [2]

(ii) The shape of a metal sculpture is formed by rotating the region bounded by the curve $y = \sqrt{a + x\sqrt{x-1}}$, where a is a positive integer, the lines $x = 1$ and $y = \sqrt{a+30}$, through 2π radians about the x-axis. Find the exact volume of the metal sculpture, giving your answer in terms of π. [4]

(b) **(i)** The diagram below shows a sketch of the graph of $y = \ln(1+x)$ for $0 \le x \le 1$. Rectangles each of width $\dfrac{1}{n}$ are drawn under the curve for $0 \le x \le 1$.

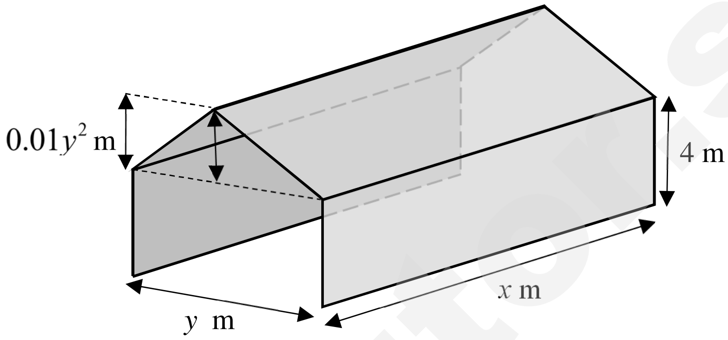


Show that A, the total area of all the rectangles, is given by

$$A = \frac{1}{n}\ln\left[\frac{(n+1)(n+2)(n+3)\ldots(2n-1)}{n^{n-1}}\right]. \qquad [2]$$

(ii) Find the exact value of $\lim\limits_{n\to\infty} A$. [3]

11 To celebrate Singapore's Bicentennial in 2019, the organising committee plans to hold a banquet at the Padang. A tent for the banquet consisting of two rectangular vertical sides and two pieces of the roof is to be constructed as shown in the diagram below. The tent has a length of x m and width y m, and a total floor area of 4000 m^2. The vertical sides of the tent are 4 m tall, and the roof adds another $0.01y^2$ m to the overall height of the tent.



(i) Show that A, the total external surface area of the tent is given by

$$A = 8x + 4000\sqrt{\frac{6400}{x^2} + 1}.$$ [3]

(ii) Show that if A has a stationary value for some x, then x satisfies the equation $\frac{a}{x^6} = \frac{6400}{x^2} + 1$, where a is a constant to be determined. [3]

(iii) If the material for the tent costs \$2.50 per m^2, estimate the minimum total cost of the material for the whole tent. [3]

(iv) A scale model of the tent is to be 3D-printed in a process where the total external surface area of this model always satisfies the equation $A = 8x + 4000\sqrt{\frac{6400}{x^2} + 1}$.

If x increases at a rate of 2 units per minute, use differentiation to find the rate of change in the total external surface area at the instant when $x = 200$ units. [3]

12 An experiment is conducted at room temperature where two substances, *A* and *B*, react in a chemical reaction to form *X* as shown below:

$$A + B \to X .$$

The initial concentrations in mol/dm^3 of substances *A* and *B* are *a* and *b* respectively. At time *t* seconds, the concentration of *A* and *B* are each reduced by x, where x denotes the concentration of *X* at time *t*.

(i) State the concentrations of *A* and *B* at time *t*. [1]

(ii) It is known that the rate of change of concentration of *X* at time *t* is proportional to the product of concentration of *A* and *B* at time *t* with a constant of proportionality k. Write down a differential equation involving x, a, b, t and k. [1]

(iii) State the maximum value of *x* if $a \le b$. Justify your answer. [2]

In the rest of the parts of the question, assume $a = b$.

(iv) The initial concentration of *X* is zero. Solve the differential equation in part **(ii)**, leave your answer in terms of x, a, t and k.

Express the solution in the form $x = \mathrm{f}(t)$ and sketch $x = \mathrm{f}(t)$ relevant in this context. Label the graph as S_1. [5]

It is known that the rate of change of concentration of *X* is doubled with every $10^\circ\mathrm{C}$ rise in the temperature. The experiment above is repeated but at a temperature $20^\circ\mathrm{C}$ above the room temperature. The concentration of *X* for this 2nd experiment at time *t* is now denoted by x_2. Let S_2 be the solution curve for the 2nd experiment.

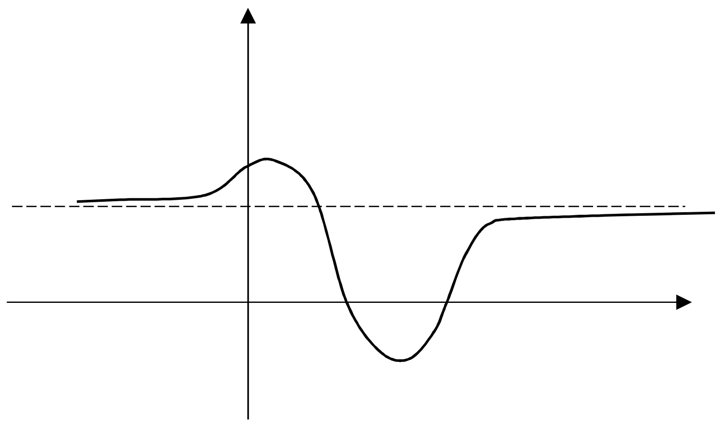
(v) State an equation relating the rate of change of concentration of x and the rate of change of concentration of x_2 at time *t*. [1]

There is no need to solve for S_2 for the rest of the parts of the question.

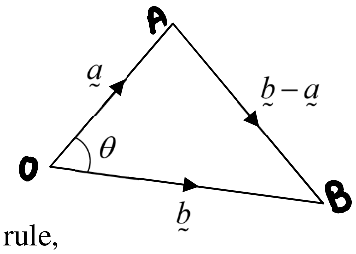
(vi) On the same diagram as in part **(iv)**, sketch the solution curve for S_2. Show clearly the relative positions of S_1 and S_2 and their behaviour when $t \to \infty$. [2]

(vii) It is given that S_1 passes through the point $(1, \alpha)$ and S_2 passes through the point $(1, \beta)$. Using the rate of change of concentration of *X* for the two experiments, state the inequality relating α and β in this context, justifying your answer. [2]

Solutions with Comments for HCI 2018 Prelim Paper 1

Qn	Suggested Solution
1 **[2]**	
2 **[4]**	Let required equation be $y = ax^3 + bx^2 + cx + d$, $a, b, c, d \in \mathbb{R}$. Substitute $(0, 0)$ into equation, $\therefore\ d = 0$ $0.15^3 a + 0.15^2 b + 0.15c = 0.1$ $0.3^3 a \ + 0.3^2 b \ + 0.3c \ = 0.125$ $0.6^3 a \ + 0.6^2 b \ + 0.6c \ = 0.1$ Using GC, $a = \frac{50}{27},\ b = -\frac{5}{2}\ ,\ c = 1$ Required equation is $y = \frac{50}{27}x^3 - \frac{5}{2}x^2 + x \quad \ldots(*)$ Substitute $x = 0.45$ into (*), $y = 0.1125$ $\therefore$ the manufacturer's claim is accurate.
3(a) **[1]**	$\left\lvert \underset{\sim}{p} \times (\underset{\sim}{a} - \underset{\sim}{q}) \right\rvert$ is the shortest distance (or perpendicular distance) from Q to l.

Solutions with Comments for HCI 2018 Prelim Paper 1

Qn	Suggested Solution
3(b) **[4]**	 By cosine rule, $AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos\theta$ $\lvert\underset{\sim}{b}-\underset{\sim}{a}\rvert^2 = \lvert\underset{\sim}{a}\rvert^2 + \lvert\underset{\sim}{b}\rvert^2 - 2\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta$ Since $\mathbf{a}\cdot\mathbf{a} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $= (x_1)^2 + (x_2)^2 + (x_3)^2$ $= \left[\sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}\right]^2$ $= \lvert\mathbf{a}\rvert^2$ We have $\mathbf{b}\cdot\mathbf{b} = \lvert\mathbf{b}\rvert^2$ and $(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \lvert(\mathbf{b}-\mathbf{a})\rvert^2$ $(\underset{\sim}{b}-\underset{\sim}{a}).(\underset{\sim}{b}-\underset{\sim}{a}) = \underset{\sim}{a}.\underset{\sim}{a} + \underset{\sim}{b}.\underset{\sim}{b} - 2\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta$ $\underset{\sim}{b}.\underset{\sim}{b} - \underset{\sim}{b}.\underset{\sim}{a} - \underset{\sim}{a}.\underset{\sim}{b} + \underset{\sim}{a}.\underset{\sim}{a} = \underset{\sim}{a}.\underset{\sim}{a} + \underset{\sim}{b}.\underset{\sim}{b} - 2\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta$ $-2\underset{\sim}{a}.\underset{\sim}{b} = -2\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta$ $\underset{\sim}{a}.\underset{\sim}{b} = \lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta$ (shown)
4(i) **[7]**	$\sqrt{\frac{1}{4}} = \frac{1}{2}$ Since $a > 0$, $a + \frac{1}{4} > \frac{1}{4} > 0$ $\sqrt{a+\frac{1}{4}} > \frac{1}{2}$ $-\sqrt{a+\frac{1}{4}} < -\frac{1}{2}$ $\therefore\ \frac{3}{2} - \sqrt{a+\frac{1}{4}} < 1$ (shown)

Solutions with Comments for HCI 2018 Prelim Paper 1

Qn	Suggested Solution
(ii)	

$$\frac{x^2-x-a}{x-1}\le 2$$

$$\frac{x^2-x-a}{x-1}-2\le 0$$

$$\frac{x^2-x-a-2x+2}{x-1}\le 0$$

$$\frac{x^2-3x+2-a}{x-1}\le 0$$

Method 1: (Algebraic method)

$$\frac{(x-\frac{3}{2})^2-(\frac{3}{2})^2+2-a}{x-1}\le 0$$

$$\frac{(x-\frac{3}{2})^2-(\frac{1}{4}+a)}{x-1}\le 0$$

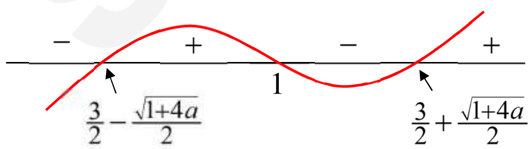
$$\frac{(x-\frac{3}{2})^2-(\sqrt{\frac{1+4a}{4}})^2}{x-1}\le 0$$

$$\frac{(x-\frac{3}{2}+\frac{\sqrt{1+4a}}{2})(x-\frac{3}{2}-\frac{\sqrt{1+4a}}{2})}{x-1}\le 0$$

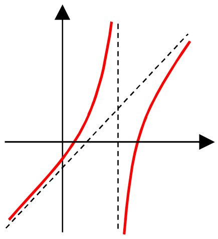
$$\frac{\left[x-\left(\frac{3}{2}-\sqrt{\frac{1}{4}+a}\right)\right]\left[x-\left(\frac{3}{2}+\sqrt{\frac{1}{4}+a}\right)\right]}{x-1}\le 0$$

Since a is a positive constant, $\left(\frac{3}{2}-\sqrt{\frac{1}{4}+a}\right)<1$ (from (i))

And $\left(\frac{3}{2}+\sqrt{\frac{1}{4}+a}\right)>\frac{3}{2}>1$



$\therefore\ x\le\frac{3}{2}-\frac{\sqrt{1+4a}}{2}$ or $1<x\le\frac{3}{2}+\frac{\sqrt{1+4a}}{2}$

Qn	Suggested Solution
	Method 2: (Graphical method) $x-2-\frac{a}{x-1}\le 0$ $$\begin{array}{r} x-2 \\ x-1\overline{)x^2-3x+2-a} \\ \underline{x^2-x} \\ -2x+2-a \\ \underline{-2x+2} \\ -a \end{array}$$  Let $\frac{x^2-3x+2-a}{x-1}=0$ $\therefore\ x^2-3x+2-a=0$ $\therefore\ x=\frac{3\pm\sqrt{9-4(2-a)}}{2}$ $=\frac{3}{2}\pm\frac{\sqrt{1+4a}}{2}$ Since $\frac{x^2-3x+2-a}{x-1}\le 0$, $\therefore\ x\le\frac{3}{2}-\frac{\sqrt{1+4a}}{2}$ or $1<x\le\frac{3}{2}+\frac{\sqrt{1+4a}}{2}$
(iii)	Replace x in $\frac{x^2-x-a}{x-1}\le 2$ with $1-x$, we obtain $\frac{(1-x)^2-(1-x)-a}{(1-x)-1}\le 2$ $\frac{1-2x+x^2-1+x-a}{-x}\le 2$ $\frac{x^2-x-a}{-x}\le 2$ $\frac{a+x-x^2}{x}\le 2$ Hence $1-x\le\frac{3}{2}-\frac{\sqrt{1+4a}}{2}$ or $1<1-x\le\frac{3}{2}+\frac{\sqrt{1+4a}}{2}$ $\therefore\ x\ge-\frac{1}{2}+\frac{\sqrt{1+4a}}{2}$ or $-\frac{1}{2}-\frac{\sqrt{1+4a}}{2}\le x<0$

Solutions with Comments for HCI 2018 Prelim Paper 1

Qn	Suggested Solution
5(i) **[8]**	$\mathrm{f}(r)-\mathrm{f}(r-1)=\cos[\alpha+(r+\frac{1}{2})\beta]-\cos[\alpha+(r-\frac{1}{2})\beta)]$ $=-2\sin(\alpha+r\beta)\sin\frac{1}{2}\beta$ (By factor formula)
5(ii)	$\sum_{r=0}^{n}[\mathrm{f}(r)-\mathrm{f}(r-1)]=\sum_{r=0}^{n}[-2\sin(\alpha+r\beta)\sin\frac{1}{2}\beta]$ $=-2\sin\frac{1}{2}\beta\sum_{r=0}^{n}\sin(\alpha+r\beta)$ $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+...+\sin(\alpha+n\beta)$ $=\sum_{r=0}^{n}\sin(\alpha+r\beta)$ $\sum_{r=0}^{n}\sin(\alpha+r\beta)$ $=-\frac{1}{2\sin\frac{1}{2}\beta}\sum_{r=0}^{n}[\mathrm{f}(r)-\mathrm{f}(r-1)]$ $=-\frac{1}{2\sin\frac{1}{2}\beta}\begin{bmatrix}\cancel{\mathrm{f}(0)}-\mathrm{f}(-1)\\+\mathrm{f}(1)-\cancel{\mathrm{f}(0)}\\+...\\+\cancel{\mathrm{f}(n-1)}-\mathrm{f}(n-2)\\+\mathrm{f}(n)-\cancel{\mathrm{f}(n-1)}]\end{bmatrix}$ $=-\frac{1}{2\sin\frac{1}{2}\beta}[\mathrm{f}(n)-\mathrm{f}(-1)]$ $=-\frac{1}{2\sin\frac{1}{2}\beta}\{\cos[\alpha+(n+\frac{1}{2})\beta]-\cos(\alpha-\frac{1}{2}\beta)\}$ $=-\frac{1}{2\sin\frac{1}{2}\beta}[-2\sin(\alpha+\frac{n}{2}\beta)\sin(\frac{n+1}{2})\beta]$ $=\frac{\sin(\alpha+\frac{n}{2}\beta)\sin(\frac{n+1}{2})\beta}{\sin\frac{1}{2}\beta}---(1)$ Where $p=\frac{n}{2}$ and $q=\frac{n+1}{2}$.

Solutions with Comments for HCI 2018 Prelim Paper 1

Qn	Suggested Solution
5(iii)	Sub $\beta=\frac{\pi}{4}$ Differentiate (1) wrt α : $\cos\alpha+\cos\left(\alpha+\frac{\pi}{2}\right)+\cos(\alpha+\pi)+\cos\left(\alpha+\frac{3\pi}{2}\right)+...+\cos\left(\alpha+\frac{n\pi}{2}\right)$ $=\frac{\cos\left(\alpha+\frac{n\pi}{4}\right)\sin\left[\left(\frac{n+1}{4}\right)\pi\right]}{\sin\frac{\pi}{4}}$ $=\sqrt{2}\cos\left(\alpha+\frac{n\pi}{4}\right)\sin\left[\left(\frac{n+1}{4}\right)\pi\right]$
6(i) **[4]**	$\frac{\mathrm{d}x}{\mathrm{d}t}=2\cos t$ $\frac{\mathrm{d}y}{\mathrm{d}t}=-\sin t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sin t}{2\cos t}=-\frac{\tan t}{2}$ At P, $t=p$ Equation of tangent at P: $y-1-\cos p=-\frac{\tan p}{2}(x-2\sin p)$ $y=1+\cos p-\frac{\tan p}{2}x+\frac{\sin^2 p}{\cos p}$ $=1+\frac{\cos^2 p+\sin^2 p}{\cos p}-\frac{\tan p}{2}x$ $=1+\sec p-\frac{\tan p}{2}x$ $2y+x\tan p=2(1+\sec p)$ (shown)
6(ii) **[5]**	When $y=0$, $\frac{\tan p}{2}x=1+\sec p$ $x=\frac{2+2\sec p}{\tan p}$ When $x=0$, $y=1+\sec p$
	Method 1: Coordinates of $M=\left(\frac{1+\sec p}{\tan p},\frac{1+\sec p}{2}\right)$.

Solutions with Comments for HCI 2018 Prelim Paper 1

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td></td><td>

$y=\dfrac{1+\sec p}{2}\Rightarrow \sec p=2y-1$

$x=\dfrac{1+\sec p}{\tan p}=\dfrac{2y}{\tan p}$

Using $1+\tan^2 p=\sec^2 p$,

$1+\left(\dfrac{2y}{x}\right)^2=(2y-1)^2$

$1+\dfrac{4y^2}{x^2}=4y^2-4y+1$

$y=yx^2-x^2$

$x^2=y(x^2-1)$

$y=\dfrac{x^2}{x^2-1}$

</td></tr>
<tr><td></td><td>

Method 2:

Coordinates of $M=\left(\cot p+\operatorname{cosec} p,\dfrac{1+\sec p}{2}\right)$.

$y=\dfrac{1+\sec p}{2}\Rightarrow \sec p=2y-1$

$x=\cot p+\operatorname{cosec} p=\dfrac{\cos p+1}{\sin p}$

Using $\sin^2 p+\cos^2 p=1$,

$\left(\dfrac{\cos p+1}{x}\right)^2+\left(\dfrac{1}{2y-1}\right)^2=1$

$\dfrac{\left(\dfrac{1}{2y-1}+1\right)^2}{x^2}+\dfrac{1}{(2y-1)^2}=1$

$\dfrac{(2y)^2}{(2y-1)^2x^2}+\dfrac{1}{(2y-1)^2}=1$

$4y^2+x^2=(4y^2-4y+1)x^2$

$y^2=y^2x^2-yx^2$

$x^2=y(x^2-1)$

$y=\dfrac{x^2}{x^2-1}$

</td></tr>
<tr><td>**7 [11] (i)**</td><td>

By Plan A, area painted on the nth day
$= 7-(n-1)0.5$

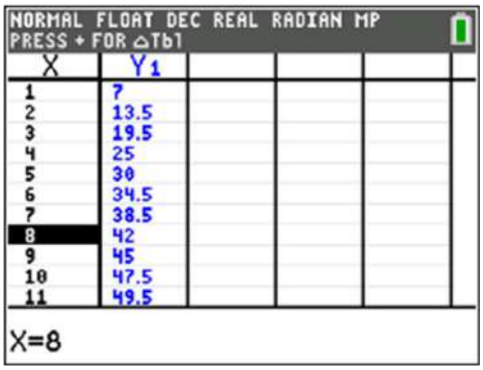
</td></tr>
</table>

Qn	Suggested Solution
	$= 7.5 - 0.5n$

(ii)

Total area painted on n days $= \frac{n}{2}[2(7)-(n-1)0.5] \geq 40$

Method 1:



n	Area
7	$38.5 < 40$
8	$42 > 40$

He will finish painting his room on the 8th day.

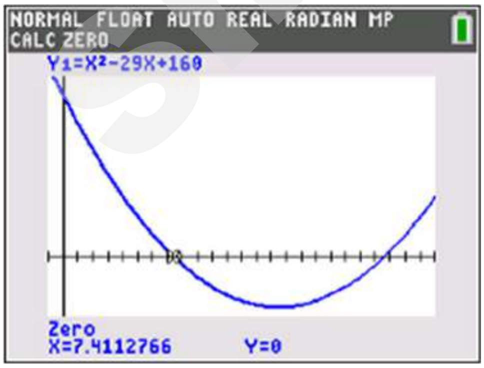
Method 2:

Total area painted on n days =

$$\frac{n}{2}[2(7)-(n-1)0.5] \geq 40$$

$$28n - n(n-1) \geq 160$$

$$n^2 - 29n + 160 \leq 0$$



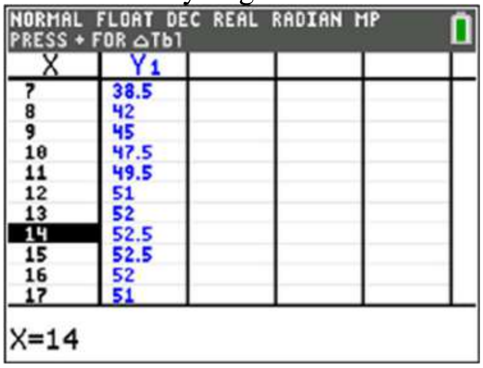
$7.41 \leq n \leq 21.6$

But $7.5 - 0.5n > 0 \Rightarrow n < 15$

So $8 \leq n \leq 14$. Least $n = 8$.

He will finish painting his room on the 8th day.

Solutions with Comments for HCI 2018 Prelim Paper 1

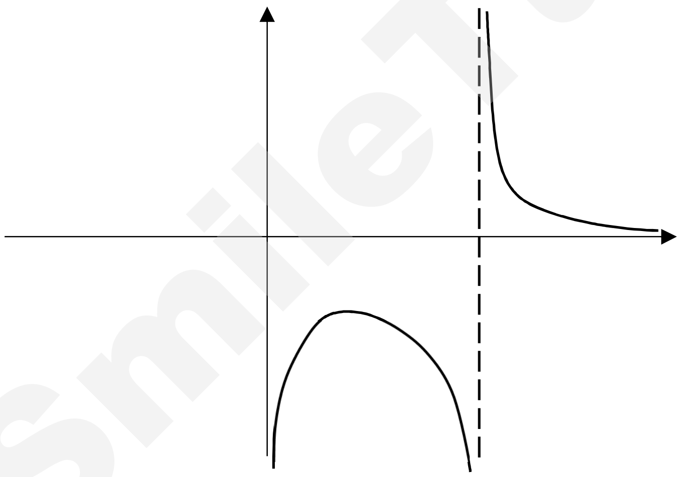
<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td></td><td></td></tr>
<tr><td>(iii)</td><td>For $n \geq 15$, area painted on the nth day = 0 and painting stops. Therefore Plan A cannot be applied to an arbitrarily large wall.

<table>
<tr><th>X</th><th>Y1</th></tr>
<tr><td>7</td><td>38.5</td></tr>
<tr><td>8</td><td>42</td></tr>
<tr><td>9</td><td>45</td></tr>
<tr><td>10</td><td>47.5</td></tr>
<tr><td>11</td><td>49.5</td></tr>
<tr><td>12</td><td>51</td></tr>
<tr><td>13</td><td>52</td></tr>
<tr><td>14</td><td>52.5</td></tr>
<tr><td>15</td><td>52.5</td></tr>
<tr><td>16</td><td>52</td></tr>
<tr><td>17</td><td>51</td></tr>
</table>

By GC or using $\frac{14}{2}\left[2(7)-(14-1)0.5\right]=52.5$, largest area that Plan A can be applied to is 52.5 m^2.</td></tr>
<tr><td>(iv)</td><td>$7\frac{1-0.8^n}{1-0.8} \geq 0.7 \times 40$
$\frac{1-0.8^n}{0.2} \geq 4$
$1-0.8^n \geq 0.8$
$0.8^n \leq 0.2$
$n \ln 0.8 \leq \ln 0.2$
$n \geq 7.21$
He needs 8 days to finish painting at least 70% of his room.</td></tr>
<tr><td>(v)</td><td>If he chooses Plan B, total area painted after an infinite number of days = $\frac{7}{1-0.8}=35<40$. He cannot finish painting his room. Therefore he should choose Plan A.</td></tr>
<tr><td>8[10]
(i)</td><td>$y=\frac{x+2}{x(x+k)}$
$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x(x+k)-(x+2)(2x+k)}{\left[x(x+k)\right]^2}$
$=\frac{x^2+kx-2x^2-kx-4x-2k}{\left[x(x+k)\right]^2}$
$=-\frac{x^2+4x+2k}{\left[x(x+k)\right]^2}$</td></tr>
</table>

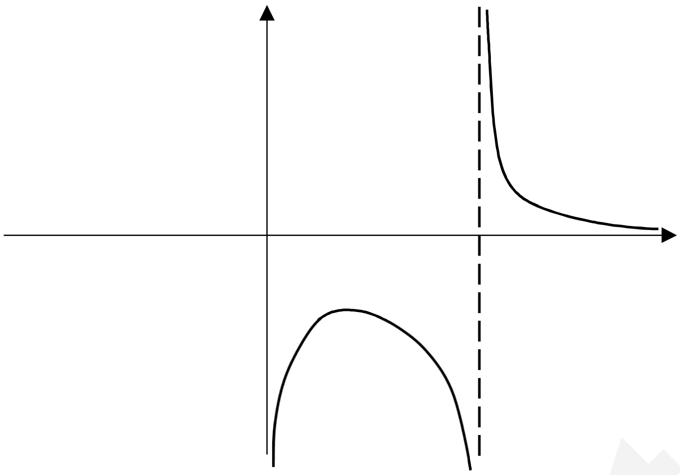
Solutions with Comments for HCI 2018 Prelim Paper 1

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td></td><td>Let $\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow x^2+4x+2k=0$
For no stationary points,
Discriminant <0
$16-4(2k)(1)<0$
$k>2$

When $k=2$, $y=\frac{x+2}{x(x+2)}=\frac{1}{x}, x\neq -2$
$\Rightarrow$ No stationary points when $k=2$

$\therefore k\geq 2$</td></tr>
<tr><td>8(ii)</td><td></td></tr>
<tr><td>8(iii)</td><td>$m=0, -0.0858$ or -2.91 (to 3s.f)</td></tr>
<tr><td>8(iv)</td><td>$x^4-4x^3+3x^2+x-2=0$
$x^4-4x^3+3x^2+2x=x+2$
$x(x-2)(x^2-2x-1)=x+2$
$x^2-2x-1=\frac{x+2}{x(x-2)}$</td></tr>
</table>

Solutions with Comments for HCI 2018 Prelim Paper 1

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td></td><td>Sketch the graph of $y=x^2-2x-1$ in (ii)

Since the graph $y=x^2-2x-1$ intersect the graph $y=\dfrac{x+2}{x(x-2)}$ at two distinct points, there are 2 real roots to the equation $x^4-4x^3+3x^2+x-2=0$.</td></tr>
<tr><td>9[12]
(a)</td><td>$4\mathrm{i}z-w=9\mathrm{i}-13$ ------- (1)
$(4+2\mathrm{i})w^*=z+3\mathrm{i}$
$z=(4+2\mathrm{i})w^*-3\mathrm{i}$ -----(2)

Sub (2) into (1), we have
$4\mathrm{i}\left[(4+2\mathrm{i})w^*-3\mathrm{i}\right]-w=9\mathrm{i}-13$

Let $w=x+\mathrm{i}y$
$4\mathrm{i}[(4+2\mathrm{i})(x-\mathrm{i}y)-3\mathrm{i}]-(x+\mathrm{i}y)=9\mathrm{i}-13$
$4\mathrm{i}[(4x+2x\mathrm{i}-4\mathrm{i}y+2y-3\mathrm{i}]-x-\mathrm{i}y)=9\mathrm{i}-13$
$16\mathrm{i}x-8x+16y+8y\mathrm{i}+12-x-\mathrm{i}y=9\mathrm{i}-13$
$(-9x+16y)+(16x+7y)\mathrm{i}=9\mathrm{i}-25$

Compare real and imaginary parts
$16y-9x=-25$ ------- (3)
$16y+8y-y=9$
$16x+7y=9$ ---------- (4)

Using GC,
$x=1,\ \ y=-1$

$\therefore w=1-\mathrm{i}$</td></tr>
</table>

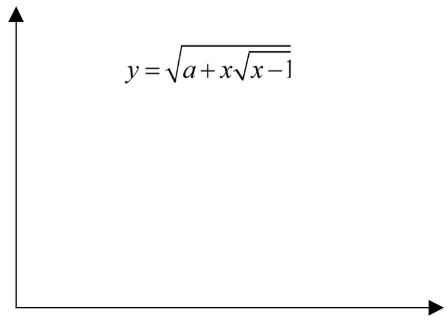
Solutions with Comments for HCI 2018 Prelim Paper 1

Qn	Suggested Solution
	$z=(4+2\mathrm{i})(1+\mathrm{i})-3\mathrm{i}=2+3\mathrm{i}$
9(bi)	$v=6\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}=6\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}=6\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$, since $\mathrm{i}=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ (modulus 1 and argument $\frac{\pi}{2}$ for i) $\therefore \lvert v\rvert=6,\ \arg(v)=\frac{\pi}{6}$ $\Rightarrow\ \lvert v^*\rvert=6,\ \arg v^*=-\frac{\pi}{6}$ $\left\lvert\frac{u^2}{v^*}\right\rvert=\frac{\lvert u\rvert^2}{\lvert v^*\rvert}=\frac{25}{6}$ $\arg\left(\frac{u^2}{v^*}\right)=2\arg u-\arg v^*$ $=2\left(\frac{7\pi}{12}\right)-\left(-\frac{\pi}{6}\right)$ $=\frac{4\pi}{3}$ $\arg\left(\frac{u^2}{v^*}\right)=-2\pi+\frac{4\pi}{3}=-\frac{2\pi}{3}$ Hence $\frac{u^2}{v^*}=\frac{25}{6}\mathrm{e}^{-\mathrm{i}\left(\frac{2\pi}{3}\right)}$
9 (bii)	$\frac{u^n}{v^*}=\frac{\left(5\mathrm{e}^{\frac{7}{12}\pi\mathrm{i}}\right)^n}{6\mathrm{e}^{-\frac{\pi}{6}\mathrm{i}}}$ $=\frac{5^n\mathrm{e}^{\frac{7}{12}n\pi\mathrm{i}}}{6\mathrm{e}^{-\frac{\pi}{6}\mathrm{i}}}$ $=\frac{5^n}{6}\mathrm{e}^{\left(\frac{7}{12}n\pi+\frac{\pi}{6}\right)\mathrm{i}}$

Solutions with Comments for HCI 2018 Prelim Paper 1

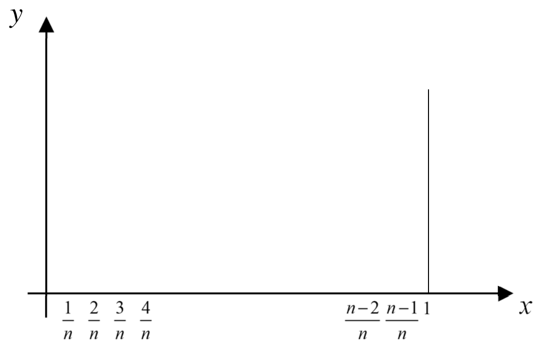
Qn	Suggested Solution
	$\therefore \arg\left(\frac{u^n}{v^*}\right)=\frac{7n\pi}{12}+\frac{\pi}{6}=\frac{(7n+2)\pi}{12}$ Since for $\frac{u^n}{v^*}$ to be purely imaginary, $\arg\left(\frac{u^n}{v^*}\right)=\frac{(2k+1)}{2}\pi$, where k is an integer (whole number), therefore we have $\frac{7n\pi+2\pi}{12}=\frac{(2k+1)\pi}{2},\ k\in\mathbb{Z}$ $12k+4=7n$ Using GC, $k=2,\ n=4$ $k=9,\ n=16$ $k=16,\ n=28$ Therefore the 3 smallest positive integers n are 4, 16 and 28.
10[14] (a)(i)	$\int x\sqrt{x-1}\,\mathrm{d}x$ $=\frac{2x}{3}(x-1)^{\frac{3}{2}}-\int\frac{2}{3}(x-1)^{\frac{3}{2}}\,\mathrm{d}x$ $=\frac{2x}{3}(x-1)^{\frac{3}{2}}-\frac{4}{15}(x-1)^{\frac{5}{2}}+c$ $u=x \Rightarrow u'=1$ $v'=\sqrt{x-1} \Rightarrow v=\frac{2}{3}(x-1)^{\frac{3}{2}}$

Solutions with Comments for HCI 2018 Prelim Paper 1

Qn	Suggested Solution
10 (a)(ii)	 $\sqrt{a+30}=\sqrt{a+x\sqrt{x-1}}$ $a+30=a+x\sqrt{x-1}$ $900=x^2(x-1)$ $x^3-x^2-900=0$ $x=10$ $V_c=\int_1^{10} a+x\sqrt{x-1}\,\mathrm{d}x$ $=\pi\left[ax\right]_0^{10}+\int_1^{10} x\sqrt{x-1}\,\mathrm{d}x$ $=9a\pi+\int_1^{10} x\sqrt{x-1}\,\mathrm{d}x$ $=9a\pi+\left[\frac{2}{3}x(x-1)^{\frac{3}{2}}\right]_1^{10}-\frac{2}{3}\int_1^{10}(x-1)^{\frac{3}{2}}\,\mathrm{d}x$ $=9a\pi+\pi\left[\frac{2}{3}x(x-1)^{\frac{3}{2}}-\frac{4}{15}(x-1)^{\frac{5}{2}}\right]_1^{10}$ $=9a\pi+\pi\left[\frac{2700}{15}-\frac{972}{15}\right]$ $=\left(9a+\frac{1728}{15}\right)\pi$ units^3 $V=9\pi\left(\sqrt{a+30}\right)^2-V_c$ $=9\pi(a+30)-(9a+\frac{1728}{15})\pi$ $=\frac{774\pi}{5}$ units^3

Solutions with Comments for HCI 2018 Prelim Paper 1

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>10 (c)(i)</td><td>



Area of 1st rectangle = $\frac{1}{n}\ln\left(1+\frac{1}{n}\right)$

Area of 2nd rectangle = $\frac{1}{n}\ln\left(1+\frac{2}{n}\right)$

Area of 3rd rectangle = $\frac{1}{n}\ln\left(1+\frac{3}{n}\right)$

$\vdots$

Area of $(n-1)$ th rectangle = $\frac{1}{n}\ln\left(1+\frac{n-1}{n}\right)$

$$A=\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\ln\left(1+\frac{3}{n}\right)+\cdots+\ln\left(1+\frac{n-1}{n}\right)\right]$$
$$=\frac{1}{n}\ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\cdots\left(1+\frac{n-1}{n}\right)\right]$$
$$=\frac{1}{n}\ln\left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\left(\frac{n+3}{n}\right)\cdots\left(\frac{2n-1}{n}\right)\right]$$
$$=\frac{1}{n}\ln\left[\frac{(n+1)(n+2)(n+3)\ldots(2n-1)}{n^{n-1}}\right]$$

</td></tr>
<tr><td>10 (c)(ii)</td><td>

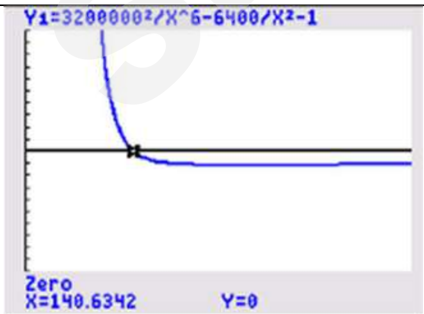
$u=\ln(1+x) \Rightarrow u'=\frac{1}{1+x}$

$v'=1 \Rightarrow v=x$

$$\lim_{n\to\infty} A=\int_0^1 \ln(1+x)\,\mathrm{d}x$$
$$=\left[x\ln(1+x)\right]_0^1-\int_0^1 1-\frac{1}{1+x}\,\mathrm{d}x$$

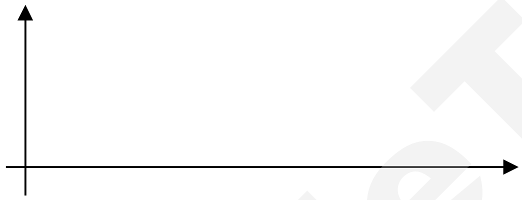
</td></tr>
</table>

Solutions with Comments for HCI 2018 Prelim Paper 1

Qn	Suggested Solution
	$=\left[x\ln(1+x)\right]_0^1-[x-\ln(1+x)]_0^1$ $=\ln 2-[1-\ln 2]$ $=2\ln 2-1\text{ units}^2$
11(i)	Floor area $xy=4000\Rightarrow y=\dfrac{4000}{x}$ Area $A=2(4x)+2x\sqrt{\left(0.01y^2\right)^2+\left(\dfrac{y}{2}\right)^2}$ $=8x+2x\sqrt{\left(0.01\dfrac{4000^2}{x^2}\right)^2+\left(\dfrac{4000}{2x}\right)^2}$ $=8x+2x\sqrt{\dfrac{160000^2}{x^4}+\dfrac{2000^2}{x^2}}$ $=8x+4000\sqrt{\dfrac{6400}{x^2}+1}$
11(ii)	$\dfrac{\mathrm{d}A}{\mathrm{d}x}=8+2000\left(\dfrac{6400}{x^2}+1\right)^{-1/2}\left(-\dfrac{12800}{x^3}\right)=0$ $\left(\dfrac{6400}{x^2}+1\right)^{-1/2}\dfrac{25600000}{x^3}=8$ $\dfrac{3200000}{x^3}=\sqrt{\dfrac{6400}{x^2}+1}$ $\dfrac{1.024\times10^{13}}{x^6}=\dfrac{6400}{x^2}+1$ $a=1.024\times10^{13}$
11(iii)	 By GC, x = 140.6342 = 141 to 3 s.f. **Method 1:**

Qn	Suggested Solution
	<table><tr><td>x</td><td>140.63^-</td><td>140.63</td><td>140.63^+</td></tr><tr><td>$\frac{dA}{dx}$</td><td>–ve</td><td>0</td><td>+ve</td></tr><tr><td></td><td>\</td><td>–</td><td>/</td></tr></table> **Method 2:** $\frac{d^2A}{dx^2}=\left(\frac{6400}{x^2}+1\right)^{-1/2}\left(\frac{76800000}{x^4}\right)$ $+\left(\frac{6400}{x^2}+1\right)^{-3/2}\left(-\frac{12800000}{x^3}\right)\left(-\frac{12800}{x^3}\right)$ >0 when $x = 140.6342$ Hence A is minimum when $x = 141$ to 3 s.f. Minimum Cost = $\left[8(140.6342)+4000\sqrt{\frac{6400}{140.6342^2}+1}\right]2.50$ = \$14317.43 (nearest cents) = \$14300 (to 3 s.f.)
11(iv)	From **(ii)**, $\frac{dA}{dx}=8-\left(\frac{6400}{x^2}+1\right)^{-1/2}\frac{25600000}{x^3}$ When $x = 200$, $\frac{dA}{dt}=\frac{dA}{dx}\frac{dx}{dt}$ $=\left[8-\left(\frac{6400}{200^2}+1\right)^{-1/2}\frac{25600000}{200^3}\right]2$ $=10.0577$ $=10.1\ \text{unit}^2/\text{min}\ \ (3\text{ s.f.})$
12(i)	the concentration of A and B at time t are $(a-x)$ and $(b-x)$ mol/dm^3 respectively
12(ii)	$\frac{dx}{dt}=k(a-x)(b-x),\ \ k\in \mathbb{R}^+$
12(iii)	Max value for x is a, $\because \frac{dx}{dt}=0$ and after $x=a$ there is no more concentration of substance A for reaction to continue.
12(iv)	$\frac{dx}{dt}=k(a-x)^2$ $\int (a-x)^{-2}dx = kt$ $(a-x)^{-1}=kt+C$ $x=a-\frac{1}{kt+C}$ ---(1) When $t=0, x=0 \Rightarrow c=\frac{1}{a}$

Solutions with Comments for HCI 2018 Prelim Paper 1

Qn	Suggested Solution
	$x = a - \dfrac{1}{kt + \dfrac{1}{a}}$ $x = a - \dfrac{a}{akt+1}$
12(v)	$\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = 4\dfrac{\mathrm{d}x}{\mathrm{d}t}$
12(vi)	
12(vii)	From the graph, $\alpha < \beta$. $\because \dfrac{\mathrm{d}x_2}{\mathrm{d}t}$ for $S_2 > \dfrac{\mathrm{d}x}{\mathrm{d}t}$ for S_1 and both curves start from the origin.

Section A: Pure Mathematics [40 marks]

1 **(a)** Find $\int \frac{x+1}{x^2+3x+9}\,\mathrm{d}x$. [4]

(b) Use the substitution $x=\cos\theta$ to find the exact value of $\int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x$. [4]

2 **(i)** Using the standard series from the list of Formulae (MF26), show that the series expansion for $\frac{1}{1+\sin x}$ can be approximated to $1-x+x^2-\frac{5}{6}x^3$ when x is sufficiently small . [3]

(ii) Let $\mathrm{f}(x)=\frac{1}{1+\sin x^3}$. Using the expansion in part **(i)**, find $\mathrm{f}^{(9)}(0)$. [2]

(iii) " The series expansion of $\left(1+\frac{x}{a}\right)^b$ is equal to the series expansion of $\frac{1}{1-\sin x}$ as far as the term in x^3 where a and b are constants."

Write down the series expansion of $\left(1+\frac{x}{a}\right)^b$ and $\frac{1}{1-\sin x}$ as far as the term in x^3.

Hence justify if the above statement is valid. [4]

3 The function f is defined as

$$\mathrm{f}: x \mapsto \begin{cases} a\mathrm{e}^{a-x} & \text{for } 0 \le x < a, \\ \left|\frac{1}{a}(x-a)^2 - a\right| & \text{for } x \ge a. \end{cases}$$

(i) Sketch the graph of $y=\mathrm{f}(x)$, indicating clearly the axial intercepts. Show that f^{-1} does not exist. [4]

(ii) If the domain of f is restricted to $[0, k]$, determine the largest value of k in terms of a such that f^{-1} exists. [1]

Use the domain found in part **(ii)** for the rest of the question.

(iii) Define f^{-1} in similar form. [4]

(iv) Show that $\mathrm{ff}\left(\frac{a}{2}\right)=a\mathrm{e}^{\frac{a}{2}}\left(2-e^{\frac{a}{2}}\right)$ where $0<a\le\ln 2$. [2]

4 A line l has equation $x-1=\dfrac{y}{2}=z-3$, and a plane p has equation

$$\mathbf{r}=\begin{pmatrix}5\\-2\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-3\end{pmatrix}+\mu\begin{pmatrix}k\\1\\4\end{pmatrix},$$

where k is a real constant, $\lambda, \mu \in \mathbb{R}$.

(i) Given that l and p are parallel, show that $k=-1$. [3]

Use $k=-1$ for the rest of the parts of the question.

(ii) Hence show that l and p do not intersect. [2]

(iii) Find the exact distance between l and p. [3]

(iv) A point A on l has coordinates $(2,2,4)$ and N is the foot of the perpendicular from A to p. Find the coordinates of N. Hence find the coordinates of the reflection of N in l. [4]

Section B: Statistics [60 marks]

5 A discrete random variable Y takes non-negative integer values with probabilities given as follows:

y	0	1	2	…	n	…
$\mathrm{P}(Y=y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	…	$\frac{1}{2^{n+1}}$	…

(i) Find the probability that Y is odd. [2]

(ii) Y_1 and Y_2 are two independent observations of Y. Find the probability that the sum of Y_1 and Y_2 is less than 4, given that their sum exceeds 2. [4]

6 Six married couples are to be seated in a row at a concert. Find the number of ways they can sit if

(i) each couple is to sit together, [2]

(ii) all women are next to one another and all men are next to one another, such that no man can sit next to his wife. [3]

After the concert, one particular married couple leaves. The rest go to a restaurant where they sit at a round table. Find the probability that each man sits next to his wife, and men and women alternate. [2]

7 An engineering team from a car manufacturer wants to test their cars' braking system. The car travels along a stretch of road with speed v km/h. When the brakes are applied, the car comes to rest after travelling a further distance of d metres. A random sample of 6 pairs of values of v and d collected by a trainee mechanic from the engineering team is shown below.

v	30	40	50	60	70	80
d	5.00	5.30	6.45	8.50	17.00	25.90

(i) Draw a scatter diagram for these values, labelling the axes clearly. [2]

It is thought that the distance travelled d can be modelled by one of the following models.

Model I : $d = av + b$ or

Model II : $d = \mathrm{e}^{pv+q}$

where a, b, p and q are constants.

(ii) Find the value of the product moment correlation coefficient between

(a) v and d,

(b) v and $\ln d$. [2]

(iii) The trainee mechanic proposed that Model II is a better model than Model I. Use your answers to parts **(i)** and **(ii)** to explain why the trainee mechanic is right. [2]

(iv) Find the equation of the regression line of $\ln d$ on v. [1]

(v) Using the regression line in part **(iv)**, find the value of v if the driver applies his brakes immediately upon seeing an obstacle that is 10 metres away and stops just in time before crashing into it. [1]

(vi) The original data set contains 7 pairs of data with regression line $d = 0.4256v - 11.74$. The trainee mechanic found that he does not have the value of d when $v = 75$ from his record. Find the missing value of d correct to 2 decimal places. [3]

[Turn over

8 Mrs Lee claimed that the mean time taken by students to finish a meal during recess is not more than 20 minutes. Two students, Jack and Jill, decided to work together to test if Mrs Lee's claim is true. A total of 50 students were selected. The time, x minutes, by each of the 50 students to finish a meal during recess was recorded. The results are summarised by

$$\sum x = 1380, \quad \sum x^2 = 83000.$$

(i) Find unbiased estimates of the population mean and variance of the time taken by a student to finish a meal during recess. [2]

(ii) Stating a necessary assumption, carry out a test of Mrs Lee's claim at the 5% level of significance. [5]

(iii) Explain, in the context of the question, the meaning of "at the 5% level of significance". [1]

(iv) Jack and Jill have just learnt hypothesis testing. Jack carried out the test as in part **(ii)** while Jill performed a 2-tail test at 5% level of significance. Without performing any further test, explain whether Jill has the same conclusion as Jack. [2]

9 A bag contains 2 red balls, 3 yellow balls and 1 blue ball. Sue and Ben play a game where each takes turns to draw a ball from the bag, with replacement. The number of red balls obtained in n fixed draws from the bag is denoted by R.

(i) State, in context, an assumption satisfied by R for it to be well modelled by a binomial distribution. [1]

(ii) Sue and Ben each draws n times from the bag. Find the least n such that the probability of both getting a total of at most 10 red balls is not more than 0.5. [3]

(iii) Sue and Ben each draws 5 times from the bag. The player with more red balls drawn wins. Otherwise, the game ends in a draw. Find the probability that Sue wins the game if she draws more than 3 red balls. [3]

In a variation of the game, Sue draws balls at random from the bag, one at a time without replacement, and stops when she obtains 2 yellow balls. The total number of balls Sue has to draw from the bag before she stops is denoted by T. Find $\mathrm{E}(T)$ and $\mathrm{Var}(T)$. [5]

[Turn over

10 Each morning, Tony drives from his home to his office and has to pass 4 traffic lights on his way. He has to reach his office by 8.50 am. The driving time to his office and the time held up at a traffic light junction, in minutes, may be assumed to follow normal distributions with means and standard deviations as summarized below:

	Mean	Standard deviation
Driving time	14	2.1
Time held up at a traffic light junction	μ	0.2

Tony leaves home at 8.30 am. If the probability that Tony is not late is 0.713, show that $\mu = 1.2$, correct to 1 decimal place. State an assumption needed in your calculations. [4]

(i) For 10 mornings, Tony leaves home at 8.30 am. Find the probability that he arrives late at his office for the third time on the 10th day. [2]

(ii) Find the probability that Tony's driving time to his office is less than 10 times the time he is held up at a traffic light junction. [2]

(iii) Find the probability that Tony's driving time to his office and the total time he is held up at the 4 traffic light junctions differs by more than 8 minutes. [3]

Assume μ is unknown.

(iv) The time held up at a traffic light junction is recorded on n randomly chosen occasions. Find the smallest n so that it is at least 98% certain that the sample mean time Tony is held up at a traffic light junction is within 5 seconds of μ. [3]

Solutions with Comments for HCI 2018 Prelim Paper 2

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>1 [8]
(a)</td><td>

$$\int \frac{x+1}{x^2+3x+9}\,\mathrm{d}x$$

$$=\frac{1}{2}\int \frac{2x+2+1-1}{x^2+3x+9}\,\mathrm{d}x \qquad =\frac{1}{2}\int \frac{2x+3}{x^2+3x+9}\,\mathrm{d}x-\frac{1}{2}\int \frac{1}{x^2+3x+9}\,\mathrm{d}x$$

$$=\frac{1}{2}\ln\left|x^2+3x+9\right|-\frac{1}{2}\int \frac{1}{(x+\frac{3}{2})^2+9-\frac{9}{4}}\,\mathrm{d}x$$

$$=\frac{1}{2}\ln\left|x^2+3x+9\right|-\frac{1}{2}\int \frac{1}{(x+\frac{3}{2})^2+\left(\frac{\sqrt{27}}{2}\right)^2}\,\mathrm{d}x$$

$$=\frac{1}{2}\ln\left(x^2+3x+9\right)-\frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{2x+3}{3\sqrt{3}}\right)+C$$, since x^2+3x+9 is always positive.

</td></tr>
<tr><td>1(b)</td><td>

$\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\sin\theta$, $\frac{1}{\sqrt{2}}=\cos\theta \Rightarrow \theta=\frac{\pi}{4}$

$0=\cos\theta \Rightarrow \theta=\frac{\pi}{2}$

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x=\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos^2\theta}{\sqrt{1-\cos^2\theta}}\cdot(-\sin\theta)\,\mathrm{d}\theta$$

$$=\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\left(-\cos^2\theta\right)\mathrm{d}\theta$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta$$

$$=\left[-\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\frac{\pi}{4}-\left(\frac{\pi}{8}+\frac{1}{4}\right)$$

$$=\frac{\pi}{8}-\frac{1}{4}$$

</td></tr>
</table>

Solutions with Comments for HCI 2018 Prelim Paper 2

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>2[9]
(i)</td><td>

$$\frac{1}{1+\sin x}=(1+\sin x)^{-1}=1-(\sin x)+(\sin x)^2-(\sin x)^3+\ldots$$

$$=1-(x-\frac{x^3}{3!})+(x-\frac{x^3}{3!})^2-(x-\frac{x^3}{3!})^3+\ldots$$

$$=1-x+x^2-\frac{5}{6}x^3+\ldots$$

</td></tr>
<tr><td>2(ii)</td><td>

Replace x by x^3,

$$\frac{1}{1+\sin x^3}=1-x^3+x^6-\frac{5}{6}x^9+\ldots$$

$$\Rightarrow \frac{f^9(0)}{9!}=-\frac{5}{6}$$

$$\Rightarrow f^9(0)=-\frac{5}{6}\times 9!=-302400$$

</td></tr>
<tr><td>2(iii)</td><td>

$$(1+\frac{x}{a})^b=1+b(\frac{x}{a})+\frac{b(b-1)}{2}(\frac{x}{a})^2+\frac{b(b-1)(b-2)}{6}(\frac{x}{a})^3+\ldots$$

$$\frac{1}{1-\sin x}=1+x+x^2+\frac{5}{6}x^3+\ldots$$

Coefficient of x: $\frac{b}{a}=1\Rightarrow a=b---(1)$

Coefficient of x^2 : $\frac{b(b-1)}{2a^2}=1---(2)$

Coefficient of x^3 : $\frac{b(b-1)(b-2)}{6a^3}=\frac{5}{6}---(3)$

Sub (1) into (2):

$$\frac{b(b-1)}{2b^2}=1$$

$$\Rightarrow \frac{(b-1)}{2b}=1$$

$$\Rightarrow b-1=2b$$

$$\Rightarrow b=-1---(3)$$

$$\therefore a=b=-1$$

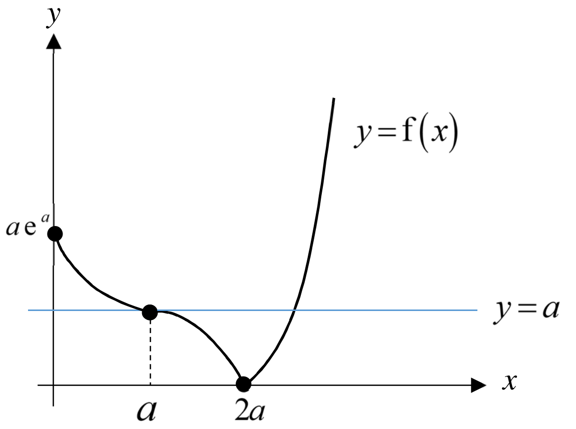
Sub into (3):

$$\text{LHS}=\frac{(-1)(-1-1)(-1-2)}{6(-1)^3}=1\neq\frac{5}{6}$$

There is no value of a and b that satisfies the three equations. Hence the statement is invalid.

</td></tr>
</table>

Solutions with Comments for HCI 2018 Prelim Paper 2

Qn	Suggested Solution
3[11] (i)	 Since the line $y=a$ cuts the graph $y=\mathrm{f}\left(x\right)$ more than once, f is not a one-one function and therefore f^{-1} does not exist.
3(ii)	Largest value of $k = 2a$
3(iii)	Let $y=\mathrm{f}\left(x\right)\Rightarrow \mathrm{f}^{-1}\left(y\right)=x$ 1. $y=a-\frac{1}{a}\left(x-a\right)^{2}$ $\left(x-a\right)^{2}=a^{2}-ay$ $x=a\pm\sqrt{a^{2}-ay}$ $\left(\text{reject } x=a-\sqrt{a^{2}-ay}\because x\geq a\right)$ $\therefore x=a+\sqrt{a^{2}-ay}$ 2. $y=a\mathrm{e}^{a-x}$ $a-x=\ln\frac{y}{a}$ $x=a-\ln\frac{y}{a}$

Solutions with Comments for HCI 2018 Prelim Paper 2

Qn	Suggested Solution
	$\therefore \mathrm{f}^{-1} : x \mapsto \begin{cases} a+\sqrt{a^2-ax} & \text{for } 0 \le x \le a \\ a-\ln\dfrac{x}{a} & \text{for } a < x \le a\mathrm{e}^a \end{cases}$
3(iv)	$\mathrm{f}\left(\dfrac{a}{2}\right) = a\mathrm{e}^{a-\frac{a}{2}} = a\mathrm{e}^{\frac{a}{2}}$ $\mathrm{ff}\left(\dfrac{a}{2}\right) = \mathrm{f}\left[\mathrm{f}\left(\dfrac{a}{2}\right)\right]$ $= \mathrm{f}\left(a\mathrm{e}^{\frac{a}{2}}\right)$ $= a-\dfrac{1}{a}\left(a\mathrm{e}^{\frac{a}{2}}-a\right)^2$ $= a-a\left(\mathrm{e}^{\frac{a}{2}}-1\right)^2$ $= a\mathrm{e}^{\frac{a}{2}}\left(2-\mathrm{e}^{\frac{a}{2}}\right)$ Since $0 < a \le \ln 2$ $0 < \dfrac{a}{2} \le \ln\sqrt{2}$ $1 < \mathrm{e}^{\frac{a}{2}} \le \sqrt{2}$ $a < a\mathrm{e}^{\frac{a}{2}} \le a\sqrt{2} < 2a$ Alternatively, $\mathrm{f}(x) = \begin{cases} a\mathrm{e}^{a-x} & \text{for } 0 \le x < a \\ a-\dfrac{1}{a}(x-a)^2 & \text{for } a \le x \le 2a \end{cases} \qquad \mathrm{R_f} = \begin{cases} \left(a, a\mathrm{e}^a\right] \\ [0,a] \end{cases}$ $\therefore \mathrm{ff}(x) = \begin{cases} a-\dfrac{1}{a}\left(a\mathrm{e}^{a-x}-a\right)^2 = a-a\left(\mathrm{e}^{a-x}-1\right)^2 & \text{for } 0 \le x < a \\ a\mathrm{e}^{a-a+\frac{1}{a}(x-a)^2} = a\mathrm{e}^{\frac{1}{a}(x-a)^2} & \text{for } a \le x \le 2a \end{cases}$ $\mathrm{ff}\left(\dfrac{a}{2}\right) = a-a\left(\mathrm{e}^{\frac{a}{2}}-1\right)^2 = a\mathrm{e}^{\frac{a}{2}}\left(2-\mathrm{e}^{\frac{a}{2}}\right)$

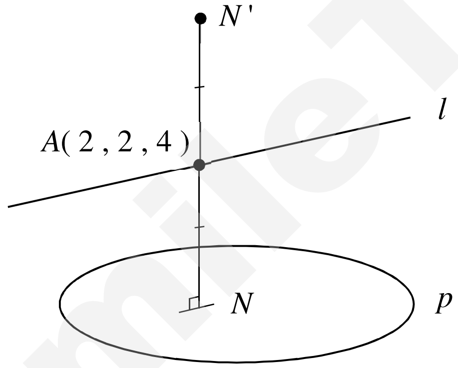
Solutions with Comments for HCI 2018 Prelim Paper 2

Qn	Suggested Solution
	Note for students: for ff to exist, $\mathrm{R_f} \subseteq \mathrm{D_f}$ $\mathrm{R_f} = \left[0, a\mathrm{e}^a\right]$ $\mathrm{D_f} = [0, 2a]$ $\Rightarrow \quad a\mathrm{e}^a \le 2a$ $a\left(e^a - 2\right) \le 0$ Since $a > 0$, $\mathrm{e}^a - 2 \le 0$ $a \le \ln 2$ $\therefore 0 < a \le \ln 2$
4[12] (i)	Let $t = x - 1 = \frac{y}{2} = z - 3$, $t \in \mathbb{R}$. $\therefore \; x = 1 + t$ $y = 0 + 2t$ $z = 3 + t$ Hence $l : \underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$. Normal of p is $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} k \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ -8-3k \\ 2-k \end{pmatrix}$ Since l and p are parallel, $\begin{pmatrix} 7 \\ -8-3k \\ 2-k \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ are perpendicular. Hence $\begin{pmatrix} 7 \\ -8-3k \\ 2-k \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0$ $7 - 16 - 6k + 2 - k = 0$ $\therefore \quad k = -1 \qquad$ (shown)

Solutions with Comments for HCI 2018 Prelim Paper 2

Qn	Suggested Solution
4(ii)	<u>Method 1</u>: (using equation of p in dot product form) $p: \underset{\sim}{r} \cdot \begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix} = 48$ Substitute a point $(1,0,3)$ on l into p, $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix} = 16 \neq 48$ Hence l does not lie on p. $\therefore$ l and p do not intersect. (shown) <u>Method 2</u>: (using equation of p in parametric form) If l and p intersect, $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ $\begin{pmatrix} 1+t \\ 2t \\ 3+t \end{pmatrix} = \begin{pmatrix} 5+2\lambda-\mu \\ -2+\lambda+\mu \\ 1-3\lambda+4\mu \end{pmatrix}$ $\therefore \quad 2\lambda-\mu-t=-4$ $\lambda+\mu-2t=2$ $-3\lambda+4\mu-t=2$ Solving using GC, no solution. $\therefore$ l and p do not intersect. (shown)
4(iii)	<u>Method 1</u>: (using $\left\lvert \underset{\sim}{a} \cdot \underset{\sim}{\hat{b}} \right\rvert$) Using points $(1,0,3)$ on l and $(5,-2,1)$ on p, $\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$ Hence required distance $= \left\lvert \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} \cdot \frac{1}{\sqrt{83}} \begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix} \right\rvert = \frac{32}{\sqrt{83}} = \frac{32\sqrt{83}}{83}$ units

Solutions with Comments for HCI 2018 Prelim Paper 2

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td></td><td>
<u>Method 2</u>: (using intersection of $\perp$ line with p)
Using the point $(1,0,3)$ on l,
Equation of line through $(1,0,3)$ and perpendicular to p is
$\underset{\sim}{r}=\begin{pmatrix}1\\0\\3\end{pmatrix}+s\begin{pmatrix}7\\-5\\3\end{pmatrix},\quad s\in\mathbb{R}$.
When line intersects p , $\begin{pmatrix}1+7s\\-5s\\3+3s\end{pmatrix}\cdot\begin{pmatrix}7\\-5\\3\end{pmatrix}=48$
$7+49s+25s+9+9s=48$
$\therefore\ s=\frac{32}{83}$
Hence required distance $=\left|\frac{32}{83}\begin{pmatrix}7\\-5\\3\end{pmatrix}\right|=\frac{32\sqrt{83}}{83}$ units
</td></tr>
<tr><td>4(iv)</td><td>


<u>Method 1</u>: (using part (iii))
From (iii), $\overrightarrow{AN}=\frac{32\sqrt{83}}{83}\left[\frac{1}{\sqrt{83}}\begin{pmatrix}7\\-5\\3\end{pmatrix}\right]=\frac{32}{83}\begin{pmatrix}7\\-5\\3\end{pmatrix}$
$\therefore\ \overrightarrow{ON}=\overrightarrow{OA}+\overrightarrow{AN}=\begin{pmatrix}2\\2\\4\end{pmatrix}+\frac{32}{83}\begin{pmatrix}7\\-5\\3\end{pmatrix}=\frac{1}{83}\begin{pmatrix}390\\6\\428\end{pmatrix}$
Hence $N\left(\frac{390}{83},\frac{6}{83},\frac{428}{83}\right)$ [or $N(4.70,0.0723,5.16)$]

<u>Method 2</u>: (using intersection of $\perp$ line through A with p)
</td></tr>
</table>

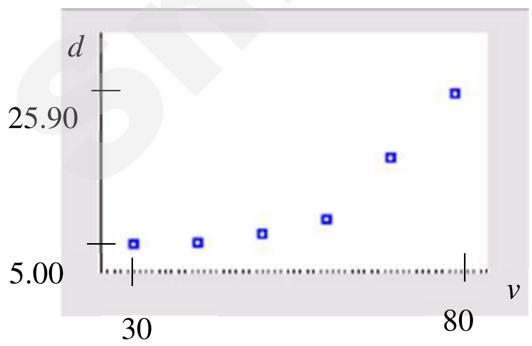
Solutions with Comments for HCI 2018 Prelim Paper 2

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td></td><td>Equation of line through $(2,2,4)$ and perpendicular to p is
$\underset{\sim}{r}=\begin{pmatrix}2\\2\\4\end{pmatrix}+\gamma\begin{pmatrix}7\\-5\\3\end{pmatrix},\quad \gamma\in\mathbb{R}$.

When line intersects p , $\begin{pmatrix}2+7\gamma\\2-5\gamma\\4+3\gamma\end{pmatrix}\cdot\begin{pmatrix}7\\-5\\3\end{pmatrix}=48$
$14+49\gamma-10+25\gamma+12+9\gamma=48$
$\therefore\ \gamma=\frac{32}{83}$
$\therefore\ \overrightarrow{ON}=\begin{pmatrix}2\\2\\4\end{pmatrix}+\frac{32}{83}\begin{pmatrix}7\\-5\\3\end{pmatrix}=\frac{1}{83}\begin{pmatrix}390\\6\\428\end{pmatrix}$
Hence $N\left(\frac{390}{83},\frac{6}{83},\frac{428}{83}\right)$ [or $N(4.70,0.0723,5.16)$ 3 s.f]

Let $N'(x,y,z)$ be the reflection of N in l .
$\therefore\ \begin{pmatrix}2\\2\\4\end{pmatrix}=\frac{1}{2}\begin{pmatrix}x+\frac{390}{83}\\y+\frac{6}{83}\\z+\frac{428}{83}\end{pmatrix}$
$x=-\frac{58}{83}$, $y=\frac{326}{83}$, $z=\frac{236}{83}$
$\therefore\ N'(-\frac{58}{83},\frac{326}{83},\frac{236}{83})$ [or $N'(-0.699,3.93,2.84)$]</td></tr>
<tr><td>**5(i)**</td><td>$\mathrm{P}(Y\text{ is odd})=\dfrac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{1}{3}$</td></tr>
</table>

Solutions with Comments for HCI 2018 Prelim Paper 2

Qn	Suggested Solution
5(ii)	$\mathrm{P}\left(Y_1+Y_2<4 \mid Y_1+Y_2>2\right)$ $=\dfrac{\mathrm{P}\left(2<Y_1+Y_2<4\right)}{\mathrm{P}\left(Y_1+Y_2>2\right)}=\dfrac{\mathrm{P}\left(Y_1+Y_2=3\right)}{1-\mathrm{P}\left(Y_1+Y_2\le 2\right)}$ $=\dfrac{\mathrm{P}\left(\left(Y_1,Y_2\right)=(0,3),(1,2),(2,1),(3,0)\right)}{1-\mathrm{P}\left(\left(Y_1,Y_2\right)=(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)\right)}$ $=\dfrac{2\left[\frac{1}{2}\frac{1}{16}+\frac{1}{4}\frac{1}{8}\right]}{1-\left[\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)+\frac{1}{4}\left(\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{8}\frac{1}{2}\right]}=\dfrac{\frac{1}{8}}{\frac{5}{16}}=\dfrac{2}{5}$
6(i)	No of ways = $2^6 6!=46080$
6(ii)	<u>Method 1</u> No of ways = $6!\left({}^5C_1\right)5!(2)=864000$ <u>Method 2</u> No of ways with all women next to one another and all men next to one another = 6! 6! 2 = 1036800 No of ways with couple seated at 6th and 7th seat together, and all women next to one another and all men next to one another = 5! 5!(6)(2) = 172800 Required no of ways = $1036800-172800=864000$
	No of arrangement = $(5-1)!2=48$ Probability required = $\dfrac{48}{9!}=\dfrac{1}{7560}$ (or 0.000132 (3 s.f.))
7(i)	
7(ii) (a)	0.902 (3 s.f.)
7(ii) (b)	0.955 (3 s.f.)

Qn	Suggested Solution
7(iii)	From the scatter plot of the data, it seems to fulfil the traits of exponential curve rather than a linear curve. For Model II, the r value is closer to 1 as compared to Model I. Thus Model II is better.
7(iv)	$d = e^{cv+d}$ $\ln d = cv + d$ Regression line of $\ln d$ on v: $\ln d = 0.0342758025v + 0.34294554177$ $\ln d = 0.0343v + 0.343$ (3 s.f.)
7(v)	When $d = 10$, $\ln 10 = 0.0342758025v + 0.34294554177$ $v = 57.2$ km/h
7(vi)	$d = 0.4256v - 11.74$ $(\bar{d}, \bar{v})$ satisfies the regression line. $\Rightarrow \bar{d} = 0.4256\bar{v} - 11.74$ Let d be the distance travelled after brakes are applied. $\Rightarrow \frac{(68.15+d)}{7} = 0.4256\frac{(330+75)}{7} - 11.74$ $\Rightarrow 68.15 + d = (0.4256)(405) - (11.74)(7)$ $\Rightarrow d = 22.04$ metres
8(i)	Let X denote the time taken to finish a meal. $\bar{x} = \frac{1}{n}\sum x = \frac{1380}{50} = 27.6$ $s^2 = \frac{1}{n-1}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)$ $= \frac{1}{49}\left(83000 - \frac{1380^2}{50}\right)$ $= \frac{44912}{49} = \frac{6416}{7} = 916.57$ $= 917$ (3 s.f.)
8(ii)	Let μ be the population mean time taken for a student to finish a meal during recess. $H_0 : \mu = 20$ $H_1 : \mu > 20$

Solutions with Comments for HCI 2018 Prelim Paper 2

Qn	Suggested Solution
	Under H_0, test statistic, $Z = \dfrac{\overline{X} - 20}{\dfrac{\sqrt{44912/49}}{\sqrt{50}}} \sim \mathrm{N}(0,\ 1)$ approximately by Central Limit Theorem, since $n = 50$ is sufficiently large. $p\text{-value} = 0.0379$ (3 s.f.) Since p-value = 0.0379 < 0.05, we reject H_0 at 5% level of significance. There is sufficient evidence to conclude that the time taken by students is more than 20 minutes. Assumption: Time taken by each student is independent of each other and that the students are chosen randomly.
8(iii)	5% level of significance means there is a probability of 0.05 of concluding that the population mean time taken by a student to finish a meal during recess is more than 20 minutes when in fact, the population mean time is not more than 20 minutes.
8(iv)	Jack did the Z-test with 1-tail test and rejected H_0. Jill did the Z-test with 2-tail test. p-value for 1-tail test = 0.0379, which is < 0.05. With the same test statistic, under Z-test 2-tail test, the p-value will be doubled, i.e. 0.037943 × 2 = 0.0759 > 0.05. Jill will not be rejecting H_0 and have a different conclusion from Jack.
9(i)	Binomial distribution. The colour of a ball chosen is independent of another.
9(ii)	Let Y denote no of red balls obtained in $2n$ draws from the bag. $Y \sim \mathrm{B}\left(2n, \frac{1}{3}\right)$ $\mathrm{P}(Y \le 10) \le 0.5$ When $n = 15$, $\mathrm{P}(Y \le 10) = 0.5848 > 0.5$ When $n = 16$, $\mathrm{P}(Y \le 10) = 0.4836 < 0.5$ Thus least $n = 16$.
9(iii)	Let S denote no of red balls obtained by Sue. $S \sim \mathrm{B}\left(5, \frac{1}{3}\right)$ Let B denote no of red balls obtained by Ben. $B \sim \mathrm{B}\left(5, \frac{1}{3}\right)$ Required probability

Solutions with Comments for HCI 2018 Prelim Paper 2

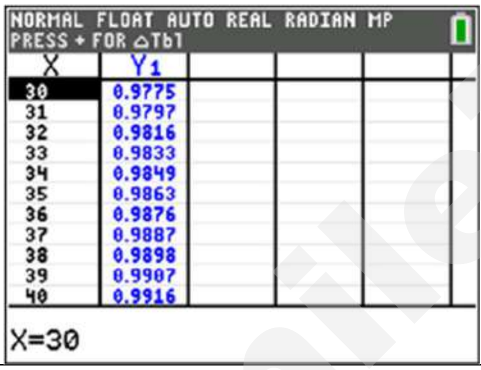
Qn	Suggested Solution
	$=\mathrm{P}(\text{Sue wins} \mid \text{gets} > 3 \text{ red balls})$ $=\dfrac{\mathrm{P}(\text{Sue wins and gets} > 3 \text{ red balls})}{\mathrm{P}(\text{gets} > 3 \text{ red balls})}$ $=\dfrac{\mathrm{P}(S=4)\mathrm{P}(B\le 3)+\mathrm{P}(S=5)\mathrm{P}(B\le 4)}{1-\mathrm{P}(S\le 3)}$ $=\dfrac{0.0433877}{0.045267}$ $=0.958$
9(iv)	Let T denote number of balls drawn before Sue obtains 2 yellow balls. $\mathrm{P}(T=2)=\mathrm{P}(YY)=\frac{3}{6}.\frac{2}{5}=\frac{1}{5}$ $\mathrm{P}(T=3)=\mathrm{P}(YY'Y,Y'YY)=\frac{3}{6}.\frac{3}{5}.\frac{2}{4}.2=\frac{3}{10}$ $\mathrm{P}(T=4)=\mathrm{P}(YY'Y'Y,Y'YY'Y,Y'Y'YY)=\frac{3}{6}.\frac{3}{5}.\frac{2}{4}.\frac{2}{3}.3=\frac{3}{10}$ $\mathrm{P}(T=5)$ $=\mathrm{P}(YY'Y'Y'Y,Y'YY'Y'Y,Y'Y'YY'Y,Y'Y'Y'YY)$ $=\frac{3}{6}.\frac{3}{5}.\frac{2}{4}.\frac{1}{3}.\frac{2}{2}.4=\frac{1}{5}$ $\mathrm{E}(T)$ $=(2)\left(\frac{1}{5}\right)+(3)\left(\frac{3}{10}\right)+(4)\left(\frac{3}{10}\right)+(5)\left(\frac{1}{5}\right)=3.5$ $\mathrm{Var}(T)=\mathrm{E}(T^2)-[\mathrm{E}(T)]^2$ $=(2)^2\left(\frac{1}{5}\right)+(3)^2\left(\frac{3}{10}\right)+(4)^2\left(\frac{3}{10}\right)+(5)^2\left(\frac{1}{5}\right)-(3.5)^2$ $=13.3-(3.5)^2$ $=1.05$ or $\frac{21}{20}$
10	Let D be the time taken to drive to office. $D\sim \mathrm{N}(14, 2.1^2)$ Let T be the time held up at a traffic light junction. $T\sim \mathrm{N}(\mu, 0.2^2)$ $D+T_1+T_2+T_3+T_4 \sim \mathrm{N}(14+4\mu, 4.57)$

Solutions with Comments for HCI 2018 Prelim Paper 2

Qn	Suggested Solution
	$\mathrm{P}(D+T_1+T_2+T_3+T_4 \le 20)=0.713$ $\mathrm{P}(Z \le \frac{20-(14+4\mu)}{\sqrt{4.57}}) = 0.713$ $\frac{6-4\mu}{\sqrt{4.57}} = 0.5621702875$ $\mu = 1.19955$ $= 1.2$ (1 d.p.) The driving time and the time held up at each of the traffic light junction are independent of one another.
10(i)	Probability Tony is late on 2 days in the first 9 days $= {}^9C_2\left(0.713\right)^7\left(1-0.713\right)^2 = 0.2777749021$ P(Tony is late at his office for the third time on the 10th day) $= {}^9C_2\left(0.713\right)^7\left(1-0.713\right)^2\left(1-0.713\right)$ =0.0797 (3 s.f.)
10(ii)	$D-10T \sim \mathrm{N}(2,\ 8.41)$ $\mathrm{P}(D<10T) = \mathrm{P}(D-10T<0) = 0.245$ (3 s.f.)
10(iii)	$D-(T_1+T_2+T_3+T_4) \sim \mathrm{N}(9.2,\ 4.57)$ $\mathrm{P}\left(\lvert D-(T_1+T_2+T_3+T_4)\rvert > 8\right)$ $=1-\mathrm{P}(-8<D-(T_1+T_2+T_3+T_4)<8)$ $=1-0.28728$ $=0.713$ (3 s.f.)
10(iv)	Let μ be the population mean time held up at a traffic light junction, $\overline{X}$ be the mean time held up at a traffic light in a sample of size *n*. $\overline{X} \sim \mathrm{N}\left(\mu, \frac{0.2^2}{n}\right)$ Given $\mathrm{P}\left(\lvert \overline{X}-\mu \rvert \le \frac{1}{12}\right) \ge 0.98$ $\mathrm{P}\left(\lvert Z\rvert \le \frac{\frac{1}{12}}{\frac{0.2}{\sqrt{n}}}\right) \ge 0.98$ $\mathrm{P}\left(-\frac{5\sqrt{n}}{12} \le Z \le \frac{5\sqrt{n}}{12}\right) \ge 0.98$

Solutions with Comments for HCI 2018 Prelim Paper 2

Qn	Suggested Solution
	$\frac{5\sqrt{n}}{12} \geq 2.326347877$ $n \geq 31.17$ Smallest $n = 32$ OR Using GC Tables, $n=31, \mathrm{P}\left(-\frac{5\sqrt{n}}{12} \leq Z \leq \frac{5\sqrt{n}}{12}\right)=0.9797$ $n=32, \mathrm{P}\left(-\frac{5\sqrt{n}}{12} \leq Z \leq \frac{5\sqrt{n}}{12}\right)=0.9816$



X	Y1
30	0.9775
31	0.9797
32	0.9816
33	0.9833
34	0.9849
35	0.9863
36	0.9876
37	0.9887
38	0.9898
39	0.9907
40	0.9916

INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CIVICS GROUP | INDEX NUMBER

Mathematics — 9758/01

Paper 1

24 August 2018

3 hours

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

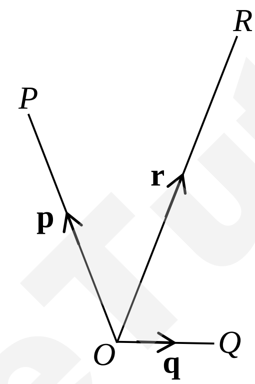
1 The variables x and y are related by

$$\frac{dy}{dx} = \sqrt{y^2 + e^{2x}}.$$

It is given that the curve of y passes through the point $(0,1)$. Find the Maclaurin series for y in ascending powers of x, up to and including the term in x^2. Give the coefficients in exact form. [4]

2 Find $\int_0^{\pi} x^2 \sin(nx)\, dx$, where n is an even integer. Leave your answer in the form $\frac{k\pi^2}{n}$, where the value of k is to be determined. [4]

3



The origin O and the points P, Q and R lie in the same plane, where $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OQ} = \mathbf{q}$ and $\overrightarrow{OR} = \mathbf{r}$ (see diagram).

(i) Explain why $\mathbf{r}$ can be expressed as $\mathbf{r} = \alpha\mathbf{p} + \beta\mathbf{q}$, for constants α and β. [1]

(ii) The point X is on PR such that $PX : XR = 2:1$. It is given that the area of triangle OPX is equal to the area of triangle OQR, find the ratio $\alpha : \beta$ in the case where α and β are positive. [4]

4 **(a)** Solve the inequality $x^2(x-5) \geq (x-5)(2kx - k^2)$, given that k is a constant and $k < 5$. [3]

(b) It is given that $f(x) = a + bx - x^2$ and $g(x) = |x - c|$ where a, b and c are constants and $2 < c < 4$. Given further that $f(2) = g(2)$, $f(5) = g(5)$ and $f(4) - g(4) = \frac{8}{3}$, find the values of a, b and c. [3]

Hence, find the exact value of the area bounded by the graphs of $y = f(x)$ and $y = g(x)$ for $2 \leq x \leq 3$. [3]

5 **(a)** A sequence u_1, u_2, u_3, … is given by

$$u_n = \frac{1}{n!} \text{ and } u_{n+1} = u_n - \frac{1}{(n-1)!+n!} \text{ for } n \geq 1.$$

(i) Find a simplified expression for $\sum_{r=1}^{N} \frac{1}{(r-1)!+r!}$. [2]

(ii) Hence show that $\sum_{r=4}^{N} \frac{1}{(r-1)!+r!} < \frac{1}{24}$. [2]

(b) D'Alembert's ratio test states that a series of the form $\sum_{r=0}^{\infty} a_r$ converges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, and diverges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$. When $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=0}^{\infty} \frac{(-1)^r \pi^{2r}}{(2r)!}$ converges and state the sum to infinity of this series. [4]

6 It is given that $f(x) = \frac{x}{\sqrt{2+x^2}}$.

(i) Find $\int_0^k f(x)\,dx$ in terms of k, where k is a positive constant. [2]

(ii) It is now given that $g(x)$ is the first three terms, in ascending powers of x, of the series expansion of $f(x)$. Find $g(x)$ and the set of values of x for the expansion to be valid. [5]

(iii) Given that $\left|\int_0^k f(x)\,dx - \int_0^k g(x)\,dx\right| < 0.005$, where $k > 0$, find the range of values of k correct to 4 significant figures. [2]

7 Two swimmers are training for a long distance swimming competition. They are to swim a distance of 4 km by swimming 80 laps at a swimming pool, where 1 lap covers 50 m from one end of the pool to the other end. Both swimmers aim to complete the distance in between $2\frac{1}{3}$ hours and $2\frac{5}{6}$ hours inclusive.

(i) Swimmer *A* swims the first lap in *T* seconds and each subsequent lap takes 1.5 seconds longer than the previous lap. Find the set of values of *T* which will enable *A* to complete the distance within the required time interval. [3]

(ii) Swimmer *B* swims the first lap in *t* seconds and the time for each subsequent lap is 1.5% more than the time for the previous lap. Find the set of values of *t* which will enable *B* to complete the distance within the required time interval. Leave your answers correct to 2 decimal places. [3]

(iii) Assuming each swimmer completes a distance of 1.8 km in exactly 50 minutes, determine which swimmer is faster in their 80th lap. Justify your answer. [4]

8 **(a)** Without using a calculator, find the complex numbers *z* and *w* which satisfy the simultaneous equations

$$(3+\mathrm{i})z+3w=-5\mathrm{i} \quad \text{and} \quad (\mathrm{i}-2)z-6\mathrm{i}w=1-3\mathrm{i}\,. \qquad [4]$$

(b) **(i)** Given that $-\frac{1}{2}(1+\mathrm{i})$ is a root of the equation

$$k\omega^4-2\omega^3+5\omega^2+6\omega+4=0\,,$$

find the value of the real number *k* and the other roots in exact form. [5]

(ii) The roots of the equation in part **(i)** are denoted by ω_1, ω_2, ω_3 and ω_4, where $\arg\omega_1<\arg\omega_2<\arg\omega_3<\arg\omega_4$.

Find $\dfrac{\omega_3}{\omega_4}$ in polar form $r(\cos\theta+\mathrm{i}\sin\theta)$, where $r>0$ and $-\pi<\theta\le\pi$.

Give *r* and θ in exact form. [3]

9 The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{2x^2 + kx + 8}{x - 4} \text{ and } k \text{ is a constant.}$$

(i) Find the equation of the tangent to C at the point where $x = 1$. [4]

(ii) Find the range of values of k for which C has more than one stationary point. [2]

Let $k = -7$ for the remaining parts of the question.

(iii) Sketch C, stating the coordinates of the turning point(s) and the points of intersection with the axes and the equations of any asymptotes. [3]

(iv) On separate diagrams, sketch the graphs of

(a) $y = \mathrm{f}\left(|x|\right)$, [2]

(b) $y = \mathrm{f}'(x)$, [2]

stating the equations of any asymptotes and the coordinates of any points of intersection with the axes where appropriate.

10 A curve C_k has parametric equations

$$x = 1 + k\cos\theta, \qquad y = -2 + \frac{1}{2}k\sin\theta,$$

where k is a positive constant.

(i) Find the cartesian equation of C_k and show that its gradient function is $\dfrac{1-x}{4(y+2)}$. [4]

(ii) On the same diagram, sketch the graphs of C_1 and C_4. Label the two graphs clearly. [3]

On a map, the curves C_1, C_2, C_3 and C_4 represent the contours of a mountain. A stream flows down the mountain. Its path on the map is always at right angles to the contour it is crossing.

(iii) Explain why the path of the stream is modelled by the differential equation

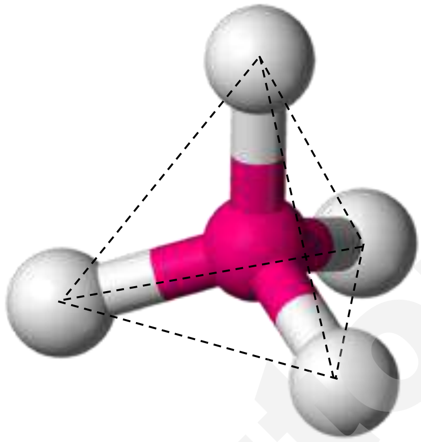
$$\left(\frac{1}{y+2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{x-1}.$$

By considering $\int \frac{1}{y+2}\,\mathrm{d}y = \int \frac{4}{x-1}\,\mathrm{d}x$, show that the path of the stream on the map is represented by the general solution $y = A(x-1)^4 - 2$, where A is an arbitrary constant. [5]

(iv) The path of the stream on the map passes through the point $(-1, -1)$. find the equation of the path. [1]

11 In Chemistry, the molecular structure of chemical compounds is often of interest to chemists as this will aid them in predicting the chemical properties of the compound.

In studying the molecular structure of *silicon tetrachloride*, it is found that this compound takes the form of a regular tetrahedron, that is, it consists of a *silicon* atom at the centre with four *chlorine* atoms symmetrically positioned at the corners of the regular tetrahedron (see diagram).



Suppose the centers of the *chlorine* atoms are at the points *A*, *B*, *C* and *D* with coordinates $(5,-2,5)$, $(5,4,-1)$, $(-1,-2,-1)$ and $(7,-4,-3)$ respectively, where *ABCD* forms a regular tetrahedron.

(i) Verify that triangle *ABC* is an equilateral triangle. [2]

(ii) Find a vector that is perpendicular to the plane containing triangle *ABC*. [1]

(iii) π_1 is a plane that is perpendicular to $\overrightarrow{AB}$ and passes through the mid-point of the line segment *AB*. Find the cartesian equation of π_1. [2]

(iv) π_2 is a plane that is perpendicular to $\overrightarrow{BC}$ and passes through the mid-point of the line segment *BC*. Given that π_1 and π_2 meet in the line *l*, find a vector equation for *l*. [3]

(v) The position of the *silicon* atom is at the point *G*, where *G* is equidistant from *A*, *B*, *C* and *D*. Find the coordinates of *G*. [3]

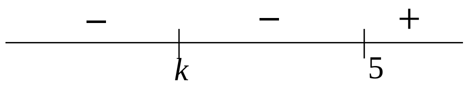
(vi) The angle *AGD* is also known as the bonding angle of the compound. Find the bonding angle. Show your workings clearly. [2]

Innova Junior College
H2 Mathematics
2018 Prelim Exam Paper 1 Solution

Q1	Suggested Solution
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{\left(y^2+\mathrm{e}^{2x}\right)} \Rightarrow \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=y^2+\mathrm{e}^{2x}$ --- (1) Differentiating (1) with respect to x. $2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2y\frac{\mathrm{d}y}{\mathrm{d}x}+2\mathrm{e}^{2x}$ --- (2) When $x=0$, $y=1$ (given) $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{1+\mathrm{e}^0}=\sqrt{2}$ From (2): $2\sqrt{2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2(1)(\sqrt{2})+2\mathrm{e}^0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\sqrt{2}+1}{\sqrt{2}}$ Therefore the Maclaurin series for y is $y=1+x\sqrt{2}+\left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)x^2+\ldots$

Q2	Suggested Solution
	$\int_0^{\pi} x^2\sin(nx)\,\mathrm{d}x$ $=\left[-x^2\frac{\cos(nx)}{n}\right]_0^{\pi}+\frac{1}{n}\int_0^{\pi}2x\cos(nx)\ \mathrm{d}x$ --- (1) $=-\frac{\pi^2}{n}\cos(n\pi)+\frac{1}{n}\left\{\left[(2x)\frac{\sin(nx)}{n}\right]_0^{\pi}-\int_0^{\pi}(2)\frac{\sin(nx)}{n}\mathrm{d}x\right\}$ $=-\frac{\pi^2}{n}-\frac{2}{n^2}\int_0^{\pi}\sin(nx)\ \mathrm{d}x$ --- (2) $=-\frac{\pi^2}{n}-\frac{2}{n^2}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi}$ $=-\frac{\pi^2}{n}+\frac{2}{n^3}\left[\cos(n\pi)-\cos 0\right]$ $=-\frac{\pi^2}{n}$ $\therefore\ k=-1$

Q3	Suggested Solution
(i)	Since **p** and **q** are non-parallel vectors, for some α and β, the sum of $\alpha\mathbf{p}$ and $\beta\mathbf{q}$ is **r** by law of parallelogram for vector addition.
(ii)	Using ratio theorem, $\overrightarrow{OX} = \mathbf{x} = \frac{1}{3}\mathbf{p} + \frac{2}{3}\mathbf{r} = \frac{1}{3}\mathbf{p} + \frac{2}{3}\alpha\mathbf{p} + \frac{2}{3}\beta\mathbf{q}$ Area of triangle *OPX* $= \frac{1}{2}\lvert\mathbf{p}\times\mathbf{x}\rvert$ $= \frac{1}{2}\left\lvert\mathbf{p}\times\left(\frac{1}{3}\mathbf{p} + \frac{2}{3}\alpha\mathbf{p} + \frac{2}{3}\beta\mathbf{q}\right)\right\rvert$ $= \frac{1}{3}\beta\lvert\mathbf{p}\times\mathbf{q}\rvert$ Area of triangle *OQR* $= \frac{1}{2}\lvert\mathbf{q}\times\mathbf{r}\rvert$ $= \frac{1}{2}\lvert\mathbf{q}\times(\alpha\mathbf{p} + \beta\mathbf{q})\rvert$ $= \frac{1}{2}\alpha\lvert\mathbf{p}\times\mathbf{q}\rvert$ Since the area of the triangles are the same, $\frac{1}{3}\beta\lvert\mathbf{p}\times\mathbf{q}\rvert = \frac{1}{2}\alpha\lvert\mathbf{p}\times\mathbf{q}\rvert$ $\frac{\alpha}{\beta} = \frac{2}{3}$ The ratio required is 2:3.

Q4	Suggested Solution
(a)	$x^2(x-5) \geq (x-5)\left(2kx - k^2\right)$ $(x-5)\left[x^2 - \left(2kx - k^2\right)\right] \geq 0$ $(x-5)\left(x^2 - 2kx + k^2\right) \geq 0$ $(x-5)(x-k)^2 \geq 0$  $x = k$ or $x \geq 5$

(b)

$\mathrm{f}(x) = a + bx - x^2$; $\mathrm{g}(x) = |x - c|$ where $2 < c < 4$

$2 < c \Rightarrow \mathrm{g}(2) = |2 - c| = c - 2$

$c < 4 \Rightarrow \mathrm{g}(5) = |5 - c| = 5 - c$

$$\mathrm{f}(2) = \mathrm{g}(2) \Rightarrow a + 2b - 4 = c - 2$$
$$a + 2b - c = 2 \quad \cdots\cdots(1)$$

$$\mathrm{f}(5) = \mathrm{g}(5) \Rightarrow a + 5b - 25 = 5 - c$$
$$a + 5b + c = 30 \quad \cdots\cdots(2)$$

$$\mathrm{f}(4) - \mathrm{g}(4) = \frac{8}{3} \Rightarrow a + 4b - 16 - (4 - c) = \frac{8}{3}$$
$$a + 4b + c = \frac{68}{3} \cdots\cdots(3)$$

$$a = -\frac{29}{3}; \quad b = \frac{22}{3}; \quad c = 3$$

Area bounded

$$\int_2^3 -\frac{29}{3} + \frac{22}{3}x - x^2 + (x - 3)\,\mathrm{d}x$$

$$= \left[-\frac{29}{3}x + \frac{11}{3}x^2 - \frac{x^3}{3} + \frac{1}{2}x^2 - 3x\right]_2^3$$

$$= \left(-29 + 33 - 9 + \frac{9}{2} - 9\right) - \left(-\frac{58}{3} + \frac{44}{3} - \frac{8}{3} + 2 - 6\right)$$

$$= \frac{11}{6}$$

Q5	**Suggested Solution**

(a)

$$\sum_{r=1}^{N} \frac{1}{(r-1)! + r!} = \sum_{r=1}^{N} (u_r - u_{r+1})$$

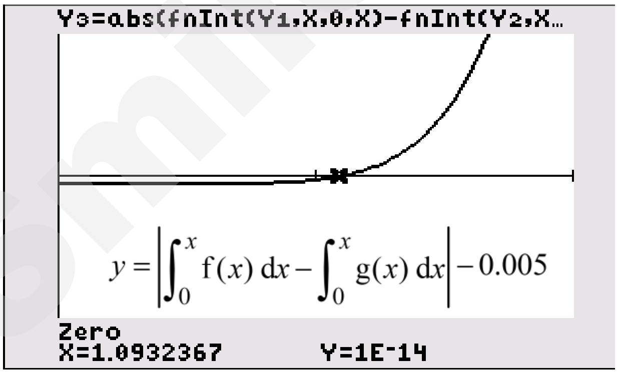
$$\begin{aligned} &= u_1 - \cancel{u_2} \\ &+ \cancel{u_2} - \cancel{u_3} \\ &\quad \vdots \\ &+ \cancel{u_{N-1}} - \cancel{u_N} \\ &+ \cancel{u_N} - u_{N+1} \end{aligned}$$

$$= u_1 - u_{N+1}$$

$$= 1 - \frac{1}{(N+1)!}$$

	$\sum_{r=4}^{N}\frac{1}{(r-1)!+r!}=\sum_{r=1}^{N}\frac{1}{(r-1)!+r!}-\sum_{r=1}^{3}\frac{1}{(r-1)!+r!}$ $=\left(1-\frac{1}{(N+1)!}\right)-\left(1-\frac{1}{4!}\right)$ $=\frac{1}{24}-\frac{1}{(N+1)!}$ $<\frac{1}{24}\quad\left(\text{since }\frac{1}{(N+1)!}>0\right)$
(b)	Let $a_n=\frac{(-1)^n\pi^{2n}}{(2n)!}$ $\frac{a_{n+1}}{a_n}=\frac{(-1)^{n+1}\pi^{2n+2}}{(2n+2)!}\times\frac{(2n)!}{(-1)^n\pi^{2n}}$ $=-\frac{\pi^2}{(2n+1)(2n+2)}$ $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert=\lim_{n\to\infty}\frac{\pi^2}{(2n+1)(2n+2)}=0<1$ Hence by ratio test, $\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r)!}$ converges. $\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r)!}=\cos\pi=-1$

Q6	Suggested Solution
(i)	$\int_0^k \mathrm{f}(x)\,\mathrm{d}x=\int_0^k x\left(2+x^2\right)^{-1/2}\,\mathrm{d}x$ $=\frac{1}{2}\int_0^k 2x\left(2+x^2\right)^{-1/2}\,\mathrm{d}x$ $=\frac{1}{2}\left[\frac{\left(2+x^2\right)^{1/2}}{1/2}\right]_0^k$ $=\sqrt{2+k^2}-\sqrt{2}$

(ii)	$\mathrm{f}(x)=x\left(2+x^2\right)^{-1/2}$ $=x\left[2\left(1+\dfrac{x^2}{2}\right)\right]^{-1/2}$ $=\dfrac{1}{\sqrt{2}}x\left(1-\dfrac{1}{2}\cdot\dfrac{x^2}{2}+\dfrac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2!}\left(\dfrac{x^2}{2}\right)^2+\ldots\right)$ $=\dfrac{1}{\sqrt{2}}x\left(1-\dfrac{1}{4}x^2+\dfrac{3}{32}x^4+\ldots\right)$ $\mathrm{g}(x)=\dfrac{1}{\sqrt{2}}x-\dfrac{1}{4\sqrt{2}}x^3+\dfrac{3}{32\sqrt{2}}x^5$ The expansion is valid when $\left\lvert\dfrac{x^2}{2}\right\rvert<1$, $\Rightarrow \quad x^2<2$ $\lvert x\rvert<\sqrt{2}$ $\left\{x: x\in\mathbb{R}, -\sqrt{2}<x<\sqrt{2}\right\}$
(iii)	$\left\lvert\int_0^k \mathrm{f}(x)\,\mathrm{d}x-\int_0^k \mathrm{g}(x)\,\mathrm{d}x\right\rvert<0.005$ $\left\lvert\int_0^k \mathrm{f}(x)\,\mathrm{d}x-\int_0^k \mathrm{g}(x)\,\mathrm{d}x\right\rvert-0.005<0$  From GC, $0<k<1.093$ (4 s.f.)

Q7	Suggested Solution
(i)	1st term $=T$ & Common difference $=1.5$ Total time taken to complete the distance of 4 km (i.e. 80 laps) $=\dfrac{80}{2}\left[2T+(80-1)1.5\right]=80T+4740$ To complete within the required time interval, $\left(2\tfrac{1}{3}\right)(60)(60)\le 80T+4740\le\left(2\tfrac{5}{6}\right)(60)(60)$

	$8400 \le 80T + 4740 \le 10200$ $45.75 \le T \le 68.25$ Set of values of T is $\{T \in \mathbb{R} : 45.75 \le T \le 68.25\}$
(ii)	1st term $= t$ & Common ratio $=1.015$ Total time taken to complete the distance of 4 km (i.e. 80 laps) $= \dfrac{t\left(1.015^{80} - 1\right)}{1.015 - 1} = \dfrac{200t}{3}\left(1.015^{80} - 1\right)$ To complete within the required time interval, $8400 \le \dfrac{200t}{3}\left(1.015^{80} - 1\right) \le 10200$ $55.0059 \le t \le 66.7928$ $55.01 \le t \le 66.79$ (2 dec pl) Set of values of t is $\{t \in \mathbb{R} : 55.01 \le t \le 66.79\}$ (to 2d.p.)
(iii)	Completing a distance of 1.8 km is equivalent to swimming 36 laps For swimmer A: $\dfrac{36}{2}\left[2T + (36-1)1.5\right] = (50)(60) \Rightarrow T = 57.083333$ $\therefore$ time taken to swim the 80th lap $= 57.083333 + (80-1)(1.5) = 175.58333$ For swimmer B: $\dfrac{t\left(1.015^{36} - 1\right)}{1.015 - 1} = (50)(60) \Rightarrow t = 63.457186$ $\therefore$ time taken to swim the 80th lap $= 63.457186(1.015)^{80-1} = 205.73024$ Swimmer A is faster in his 80th lap.

Q8	Suggested Solution
(a)	$(3+\mathrm{i})z + 3w = -5\mathrm{i}$ --- (1) $(\mathrm{i}-2)z - 6\mathrm{i}w = 1 - 3\mathrm{i}$ --- (2) $(1)\times 2\mathrm{i}$: $2\mathrm{i}(3+\mathrm{i})z + 6\mathrm{i}w = -5\mathrm{i}(2\mathrm{i})$ --- (3) $(2)+(3)$:

$$(\mathrm{i}-2)z+2\mathrm{i}(3+\mathrm{i})z=1-3\mathrm{i}-5\mathrm{i}(2\mathrm{i})$$

$$z\left(\mathrm{i}-2+6\mathrm{i}-2\right)=1-3\mathrm{i}+10$$

$$\begin{aligned}
z&=\frac{11-3\mathrm{i}}{-4+7\mathrm{i}}\times\frac{-4-7\mathrm{i}}{-4-7\mathrm{i}}\\
&=\frac{-44-77\mathrm{i}+12\mathrm{i}-21}{16+49}\\
&=\frac{-65-65\mathrm{i}}{65}\\
&=-1-\mathrm{i}
\end{aligned}$$

Substitute $z=-1-\mathrm{i}$ into (1):

$$\begin{aligned}
(3+\mathrm{i})(-1-\mathrm{i})+3w&=-5\mathrm{i}\\
3w&=-5\mathrm{i}-\left(-3-3\mathrm{i}-\mathrm{i}+1\right)\\
w&=\frac{2-\mathrm{i}}{3}
\end{aligned}$$

(b)(i)

Since the coefficients of the equation are real, $-\frac{1}{2}\left(1-\mathrm{i}\right)$ is another root of the equation.

$$\begin{aligned}
\text{Quadratic factor }&=\left(\omega+\tfrac{1}{2}+\tfrac{1}{2}\mathrm{i}\right)\left(\omega+\tfrac{1}{2}-\tfrac{1}{2}\mathrm{i}\right)\\
&=\left(\omega+\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\mathrm{i}\right)^2\\
&=\omega^2+\omega+\tfrac{1}{4}+\tfrac{1}{4}\\
&=\omega^2+\omega+\tfrac{1}{2}
\end{aligned}$$

By inspection,

$$k\omega^4-2\omega^3+5\omega^2+6\omega+4=\left(\omega^2+\omega+\tfrac{1}{2}\right)\left(k\omega^2+p\omega+8\right)$$

Comparing ω: $6=8+\frac{1}{2}p \Rightarrow p=-4$

Comparing ω^3: $-2=p+k \Rightarrow k=-2+4=2$

Solving $2\omega^2-4\omega+8=0$:

$$\omega=\frac{4\pm\sqrt{16-4(2)(8)}}{2(2)}=\frac{4\pm4\mathrm{i}\sqrt{3}}{4}=1\pm\mathrm{i}\sqrt{3}$$

(b)(ii)

Note that $\omega_3=1+\mathrm{i}\sqrt{3}$ and $\omega_4=-\frac{1}{2}\left(1-\mathrm{i}\right)$.

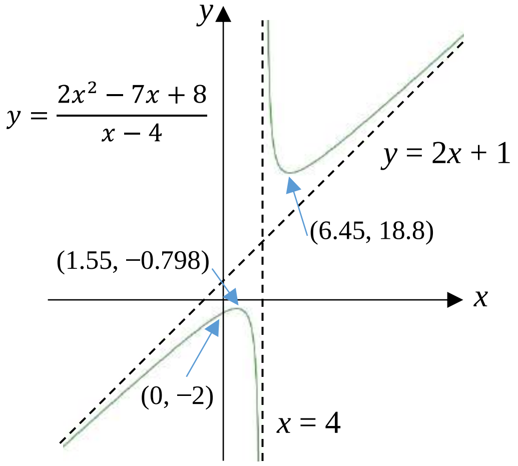
$$\left|\frac{\omega_3}{\omega_4}\right|=\frac{|\omega_3|}{|\omega_4|}=\frac{2}{\sqrt{2}/2}=\frac{4}{\sqrt{2}}$$

$$\arg\left(\frac{\omega_3}{\omega_4}\right)=\arg\omega_3-\arg\omega_4=\frac{\pi}{3}-\frac{3\pi}{4}=-\frac{5\pi}{12}$$

	$\frac{\omega_3}{\omega_4}=\frac{4}{\sqrt{2}}\left[\cos\left(-\frac{5\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{12}\right)\right]$

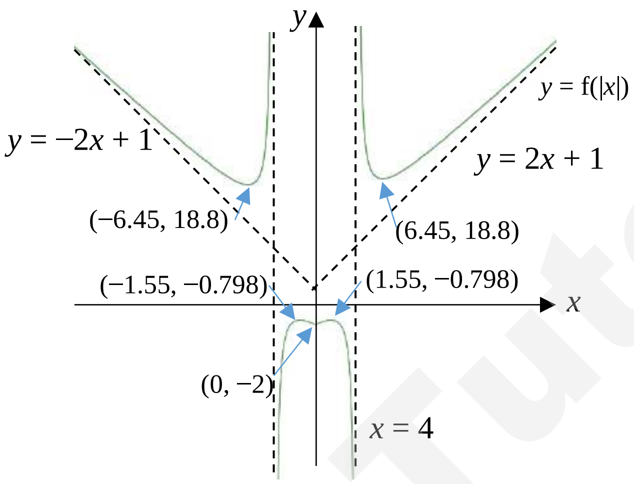
9	**Suggested Solution**
(i)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x-4)(4x+k)-(2x^2+kx+8)}{(x-4)^2}$ $=\frac{4x^2+(k-16)x-4k-2x^2-kx-8}{(x-4)^2}$ $=\frac{2x^2-16x-4k-8}{(x-4)^2}$ When x = 1, $y=\frac{10+k}{-3}$, and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2-16-4k-8}{(-3)^2}=\frac{-22-4k}{9}$ Equation of tangent to C at x = 1: $y-\frac{10+k}{-3}=\frac{-22-4k}{9}(x-1)$ $9y+30+3k=(-22-4k)x+22+4k$ $9y+2(11+2k)x=k-8$ Or $y=-\frac{2(11+2k)}{9}x+\frac{k-8}{9}$
(ii)	For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\frac{2x^2-16x-4k-8}{(x-4)^2}=0$ $x^2-8x-2k-4=0$ For more than 1 stationary point, this equation must have real and distinct roots, $(-8)^2-4(-2k-4)>0$ $64+8k+16>0$ $k>-10$

(iii)



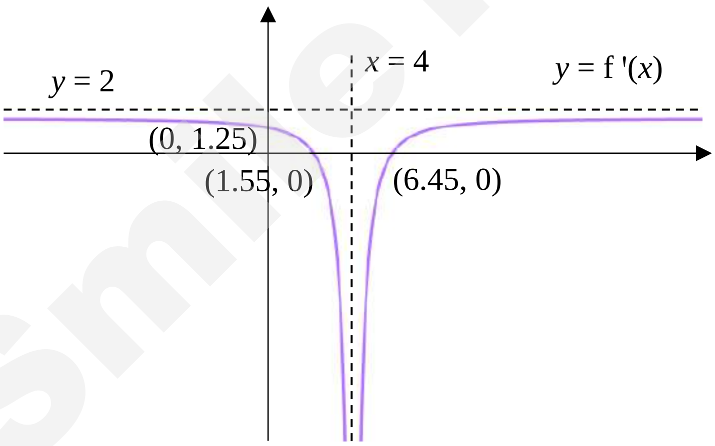
y
$y = \dfrac{2x^2 - 7x + 8}{x - 4}$
y = 2x + 1
(6.45, 18.8)
(1.55, −0.798)
x
(0, −2)
x = 4

(iv)
(a)

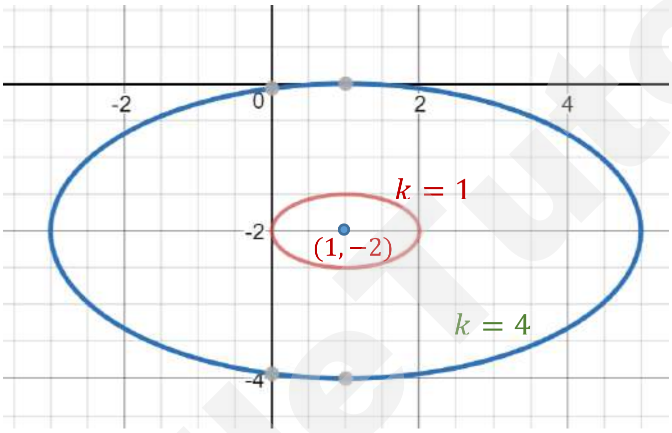


y
y = f(|x|)
y = −2x + 1
y = 2x + 1
(−6.45, 18.8)
(6.45, 18.8)
(−1.55, −0.798)
(1.55, −0.798)
x
(0, −2)
x = 4

(b)



x = 4
y = f '(x)
y = 2
(0, 1.25)
(1.55, 0)
(6.45, 0)

Q10	Suggested Solution
(i)	$x = 1 + k\cos\theta \Rightarrow \cos\theta = \dfrac{(x-1)}{k}$ $y = -2 + \dfrac{1}{2}k\sin\theta \Rightarrow \sin\theta = \dfrac{2(y+2)}{k}$ $\cos^2\theta + \sin^2\theta = 1 \Rightarrow \dfrac{(x-1)^2}{k^2} + \dfrac{4(y+2)^2}{k^2} = 1$ Cartesian eqn of C_k is $(x-1)^2 + 4(y+2)^2 = k^2$ Applying implicit differentiation, $2(x-1) + 8(y+2)\dfrac{dy}{dx} = 0$ $\dfrac{dy}{dx} = -\dfrac{(x-1)}{4(y+2)} = \dfrac{1-x}{4(y+2)}$
(ii)	
(iii)	The path of the stream on the map is perpendicular to the ellipses implies that (gradient of the path)×(grad of ellipse) = −1. Since gradient of ellipse $= -\dfrac{(x-1)}{4(y+2)}$, We have $\dfrac{dy}{dx}\left[-\dfrac{(x-1)}{4(y+2)}\right] = -1$ Thus $\dfrac{1}{y+2} \times \dfrac{dy}{dx} = \dfrac{4}{x-1}$. Using the given result, $\displaystyle\int\left(\frac{1}{y+2}\right)dy = \int \frac{4}{x-1}\,dx$ $\ln\|y+2\| = 4\ln\|x-1\| + C$ $\|y+2\| = e^{4\ln\|x-1\|+C}$ $= \left(e^{\ln\|x-1\|^4}\right)\left(e^C\right)$

	$y+2=\pm\left(e^{C}\right)(x-1)^4$ $y=A(x-1)^4-2$ where $A=\pm e^{C}$
(iv)	The path passes through the point $(-1,-1)$, $-1=A(-1-1)^4-2$ $A=\dfrac{1}{16}$ Eqn of the path is $y=\dfrac{1}{16}(x-1)^4-2$

Q11	Suggested Solution
(i)	$\overrightarrow{AB}=\begin{pmatrix}5\\4\\-1\end{pmatrix}-\begin{pmatrix}5\\-2\\5\end{pmatrix}=6\begin{pmatrix}0\\1\\-1\end{pmatrix}\Rightarrow\left\lvert\overrightarrow{AB}\right\rvert=6\sqrt{2}$ $\overrightarrow{AC}=\begin{pmatrix}-1\\-2\\-1\end{pmatrix}-\begin{pmatrix}5\\-2\\5\end{pmatrix}=-6\begin{pmatrix}1\\0\\1\end{pmatrix}\Rightarrow\left\lvert\overrightarrow{AC}\right\rvert=6\sqrt{2}$ $\overrightarrow{BC}=\begin{pmatrix}-1\\-2\\-1\end{pmatrix}-\begin{pmatrix}5\\4\\-1\end{pmatrix}=-6\begin{pmatrix}1\\1\\0\end{pmatrix}\Rightarrow\left\lvert\overrightarrow{AC}\right\rvert=6\sqrt{2}$ Since *AB* = *BC* = *CA*, triangle *ABC* is an equilateral triangle.
(ii)	$\begin{pmatrix}0\\1\\-1\end{pmatrix}\times\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$
(iii)	Since π_1 is perpendicular to $\overrightarrow{AB}$, the normal vector of π_1 is $\begin{pmatrix}0\\1\\-1\end{pmatrix}$. By symmetry, π_1 will pass through *C*. $\pi_1 : \underset{\sim}{r}\bullet\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\-2\\-1\end{pmatrix}\bullet\begin{pmatrix}0\\1\\-1\end{pmatrix}=-1$ Cartesian equation of π_1 is: $y-z=-1$
(iv)	Since π_2 is perpendicular to $\overrightarrow{BC}$, the normal vector of π_2 is $\begin{pmatrix}1\\1\\0\end{pmatrix}$. By symmetry, π_2 will pass through *A*.

	$\pi_2 : \underset{\sim}{r} \bullet \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}5\\-2\\5\end{pmatrix} \bullet \begin{pmatrix}1\\1\\0\end{pmatrix} = 3$ Cartesian equation of π_2 is: $x + y = 3$ Using GC, the equation of the line of intersection of the two planes is $l: \underset{\sim}{r} = \begin{pmatrix}4\\-1\\0\end{pmatrix} + \lambda \begin{pmatrix}-1\\1\\1\end{pmatrix}$, where $\lambda \in \mathbb{R}$.
(v)	Note that point *G* lies on the line *l* found in part **(iv)**. Since *G* lies on *l*, $\overrightarrow{OG} = \begin{pmatrix}4\\-1\\0\end{pmatrix} + \lambda \begin{pmatrix}-1\\1\\1\end{pmatrix}$ for some $\lambda \in \mathbb{R}$ $\overrightarrow{AG} = \begin{pmatrix}-1-\lambda\\1+\lambda\\-5+\lambda\end{pmatrix}$ & $\overrightarrow{DG} = \begin{pmatrix}-3-\lambda\\3+\lambda\\\lambda+3\end{pmatrix}$ Given that $\left\|\overrightarrow{DG}\right\| = \left\|\overrightarrow{AG}\right\|$, $2(1+\lambda)^2 + (\lambda-5)^2 = 3(\lambda+3)^2$ $2(\lambda^2 + 2\lambda + 1) + (\lambda^2 - 10\lambda + 25) = 3(\lambda^2 + 6\lambda + 9)$ $4\lambda + 2 - 10\lambda + 25 = 18\lambda + 27$ $\lambda = 0$ $\overrightarrow{OG} = \begin{pmatrix}4\\-1\\0\end{pmatrix}$ Thus, coordinates of *G* are (4, -1, 0).
(vi)	$\cos\angle AGD = \dfrac{\overrightarrow{DG} \bullet \overrightarrow{AG}}{\left\|\overrightarrow{DG}\right\|\left\|\overrightarrow{AG}\right\|} = \dfrac{3\begin{pmatrix}-1\\1\\1\end{pmatrix} \bullet \begin{pmatrix}-1\\1\\-5\end{pmatrix}}{(3\sqrt{3})(\sqrt{27})} = -\dfrac{1}{3}$ $\angle AGD = 109.5^\circ$
	Alternative: Let angle *AGX* be α . $\cos\alpha = \dfrac{1}{3} \Rightarrow \alpha = 70.52^\circ \Rightarrow \theta = 180^\circ - \alpha = 109.5^\circ$

INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME			
CLASS		INDEX NUMBER	

MATHEMATICS 9758/02

Paper 2 **11 September 2018**

3 hours

Additional Materials: Answer Paper
Cover Page
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages and **1** blank page.

Section A: Pure Mathematics [40 marks]

1 It is given that

$$f(x)=\begin{cases}2x-8 & \text{for } 0\le x<2,\\ -x-2 & \text{for } 2\le x<6,\end{cases}$$

and that $f(x)=f(x+6)$ for all real values of x.

(i) Evaluate $f(-21)+f(49)$. [2]

(ii) Sketch the graph of $y=f(x)$ for $-6\le x\le 10$. [3]

(iii) Find $\displaystyle\int_{-2}^{7} f(x)\,dx$. [3]

2 The function f is defined by

$$f(x)=\cos x-\sqrt{3}\sin x\text{, where } -\frac{\pi}{3}\le x\le\frac{2\pi}{3}.$$

(i) Express $f(x)$ in the form $R\cos(x+\alpha)$ where R and α are exact positive constants to be found. State the range of f. [3]

(ii) Show that $f(x)$ decreases as x increases. [2]

(iii) Define f^{-1}. [2]

(iv) The composite function fg is defined by

$$fg(x)=x-3\text{, where } 1\le x\le 3.$$

Find $g(x)$. [2]

3 Liquid is being poured into a cylindrical tank at a constant rate of 1200 $\text{cm}^3\ \text{s}^{-1}$ and is leaking out of a hole in the base at a rate proportional to the square root of the height of liquid in the tank. The tank is initially empty, and at time t seconds after pouring, the liquid in the tank has volume V cm^3 and height h cm. The circular surface area of the liquid in the tank is 3000 cm^2.

(i) Write down a differential equation expressing $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of h. Hence show that $\frac{\mathrm{d}h}{\mathrm{d}t} = 0.4 - k\sqrt{h}$, where k is a positive constant. [3]

When $h = 36$, liquid is leaking out of the hole at 360 $\text{cm}^3\ \text{s}^{-1}$.

(ii) Show that $k = 0.02$. [1]

(iii) By using the substitution $\sqrt{h} = 20 - x$, find the particular solution of t in terms of h. [5]

(iv) Hence find the time taken for the liquid to reach a height of 100 cm, giving your answer in minutes and seconds, correct to the nearest second. [2]

4 Given that $\mathrm{f}(x) = \frac{1}{\sqrt{1+x^2}}$ and $\mathrm{g}(x) = \frac{a}{\sqrt{1+(0.5x-1)^2}}$ where a is a constant greater than 2, describe fully a sequence of transformations which would transform the graph of $y = \mathrm{f}(x)$ onto the graph of $y = \mathrm{g}(x)$. [3]

The region R is bounded by the curves $y = \mathrm{f}(x)$, $y = \mathrm{g}(x)$, the line $x = 2$ and the y-axis. A sculpture is made in the shape of the solid of revolution formed by rotating R through 2π radians about the x-axis.

(i) Find the exact volume of the sculpture, giving your answer in terms of a and π. [4]

(ii) Another region S is bounded by the curve $y = \mathrm{f}(x)$, the line $x = 2$ and the x- and y- axes. A second sculpture takes the shape of the solid of revolution formed by rotating S through 2π radians about the y-axis. Find the exact volume of the second sculpture. [3]

(iii) Given that the volume of the first sculpture found in part **(i)** is at least 50 times the volume of the second sculpture found in part **(ii)**, find the smallest integer value of a. [2]

Section B: Probability and Statistics [60 marks]

5 The random variable X has the distribution $\mathrm{B}(25, p)$, where $0 < p < 1$. Given that $\mathrm{P}(X \le 1) = 0.15$, write down an equation for the value of p and find this value numerically. Hence find $\mathrm{Var}(X)$. [4]

6 Find the number of ways in which the letters of the word APPRECIATE can be arranged if

(i) vowels (A, E, I) and consonants (P, R, C, T) must alternate, [2]

(ii) between the two Es, there are exactly two other letters and at least one of which must be an A. [3]

7 The discrete random variable X takes values 0, 1, 2 and 3 only. The probability distribution of X is shown in the table, where p is a constant and $0 < p < \frac{1}{10}$.

x	0	1	2	3
$\mathrm{P}(X = x)$	$1-6p$	$3p$	$2p$	p

(i) Given that $\mathrm{Var}(X) = 0.75$, find the value of $\mathrm{E}(X)$. [3]

(ii) The random variable S is the sum of n independent observations of X, where n is large. Given that the probability that S exceeds 150 is at least 0.75, find the set of possible values of n. [3]

8 For events A, B and C, it is given that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.45$ and $\mathrm{P}(C) = 0.35$.

It is further given that $\mathrm{P}(B \mid C) = 0.5$, $\mathrm{P}(A \cap C) = 0.15$ and $\mathrm{P}(A \cap B \cap C) = 0.1$.

(i) Find $\mathrm{P}(A' \cap B \cap C)$. [2]

(ii) Given also that events A and B are independent, find $\mathrm{P}(A \cup B)$. [2]

(iii) Given instead that events A and B are **not** independent, find the greatest and least possible values of $\mathrm{P}(A' \cap B' \cap C')$. [4]

9 Bottles of tomato juice produced by a company are said to contain 250 ml, with a standard deviation of 10 ml. After receiving feedback from some consumers regarding the volume of tomato juice per bottle, the manager takes a random sample of 50 bottles to test whether the mean volume has been overstated. He measures the volume, x ml of tomato juice in each bottle and the sample mean volume is found to be 247.5 ml.

(i) State appropriate hypotheses for the test, defining any symbols you use. [2]

(ii) Find the p-value of the test and state the meaning of this p-value in context. [2]

(iii) State, giving a reason, whether it is necessary to assume a normal distribution for this test to be valid. [1]

The company installs a new machine to produce smaller bottles of tomato juice with mean volume μ_0 ml. A random sample of these smaller bottles of tomato juice is taken. The sample size is 60 and the volumes, y ml, are summarised as follows.

$$\sum y = 10\,757 \qquad \sum y^2 = 1\,931\,597$$

(iv) Calculate unbiased estimates of the population mean and variance of the volume of smaller bottles of tomato juice. [2]

(v) A two-tail test is to be carried out at the 5% significance level by the manager. Find the range of values of μ_0, correct to 1 decimal place, such that the null hypothesis will not be rejected. [3]

10 A company wants to investigate the effect of using strong acid solution in reducing the weight of metal plates. Eight metal plates are randomly selected and immersed in a strong acid solution for different lengths of time, t hours. The percentages of weight loss, w %, are calculated and the results are shown in the table below.

t	100	150	200	250	300	350	400	450
w	0.80	1.40	2.00	2.31	2.53	2.65	2.71	2.77

(i) Calculate the product moment correlation coefficient between t and w, and explain whether your answer suggests that a linear model is appropriate. [3]

(ii) Draw the scatter diagram for these values, labelling the axes clearly. Explain which of the following equations, where a and b are constants and $b > 0$, provides the most accurate model of the relationship between t and w.

(A) $w = a + b \ln t$

(B) $w = a + bt^2$

(C) $w = a + \dfrac{b}{t^2}$

[2]

(iii) Using the model you chose in part **(ii)**, write down the equation for the relationship between t and w, giving the numerical values of the coefficients. State the product moment correlation coefficient for this model and comment on its value. [3]

(iv) Given that a metal plate being immersed in the strong acid solution for t hours has a weight loss of 2.4%, estimate the value of t. Give two reasons why this estimate is reliable. [3]

(v) Given that 1 day = 24 hours, re-write your equation from part **(iii)** so that it can be used to estimate the percentage weight loss of metal plates when the length of time of immersing the metal plates in the strong acid solution is measured in days. [1]

11 **(a)** Flash Electrics is a company which specializes in installing electricity meters in new houses in Central City. The time taken, T minutes, by its employees to install an electricity meter may be assumed to be normally distributed with mean μ and standard deviation σ.

(i) Given that $\mathrm{P}(T < 40) = \mathrm{P}(T > 50) = 0.36$, find the values of μ and σ. [3]

(ii) A random sample of 18 new houses in Central City with electricity meters installed by Flash Electrics is taken. Find the probability that at least 5 but fewer than 10 houses in this sample have an installation time of at most 50 minutes. [3]

(b) The electricity consumption, measured in kilowatt hour (kWh), of the households in Central City has a normal distribution with mean 520 and standard deviation 35.

(i) Find the probability that the electricity consumption of a randomly chosen household is more than 500 kWh. [1]

(ii) Two households in Central City are randomly chosen. Find the probability that both households each have electricity consumption of less than 500 kWh. [2]

(iii) The probability that the total electricity consumption of two randomly chosen households is less than 1000 kWh is denoted by p. Without calculating its value, explain why p will be greater than your answer to part **(ii)**. [1]

The electricity consumption of the households in Star City has a normal distribution with mean 475 kWh and standard deviation 25 kWh. It is known that electricity costs \$0.18 per kWh and \$0.15 per kWh in Central City and Star City respectively.

Let X represent the electricity bill of a randomly chosen household in Central City.

Let Y represent the electricity bill of a randomly chosen household in Star City.

(iv) Find $\mathrm{P}(4Y - 3X > 7)$ and explain, in the context of this question, what your answer represents. [5]

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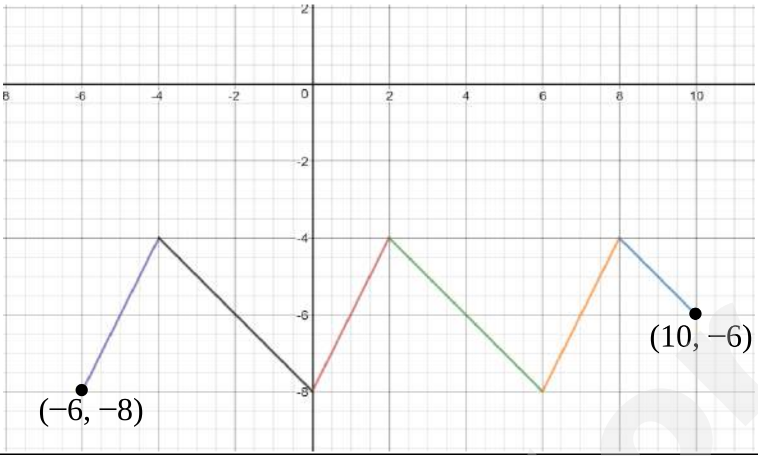
2018 IJC H2 Math Prelim Paper 2 Solution

1	Solution

(i)

$$\begin{aligned} \mathrm{f}(-21)+\mathrm{f}(49) &= \mathrm{f}(-21+24)+\mathrm{f}(49-48) \\ &= \mathrm{f}(3)+\mathrm{f}(1) \\ &= (-3-2)+(2-8) \\ &= -11 \end{aligned}$$

(ii)



(iii)

Method 1

$\int_{-2}^{7} \mathrm{f}(x)\ \mathrm{d}x$

$= -\{\text{sum of areas of 4 trapezia}\}$

$= -\left[\frac{1}{2}(6+8)2\right] - \left[\frac{1}{2}(8+4)2\right] - \left[\frac{1}{2}(4+8)4\right] - \left[\frac{1}{2}(8+6)\right]$

$= -14-12-24-7$

$= -57$

Method 2

$\int_{-2}^{7} \mathrm{f}(x)\ \mathrm{d}x$

$= \int_{4}^{6} (-x-2)\ \mathrm{d}x + \int_{0}^{2} (2x-8)\ \mathrm{d}x + \int_{2}^{6} (-x-2)\ \mathrm{d}x + \int_{0}^{1} (2x-8)\ \mathrm{d}x$

$= -57$

Method 3

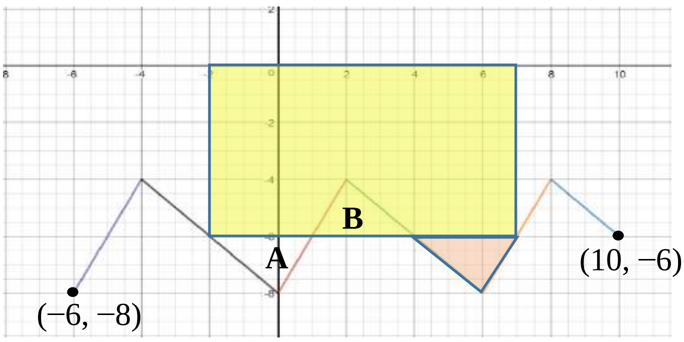
Since areas of triangles A and B are identical,

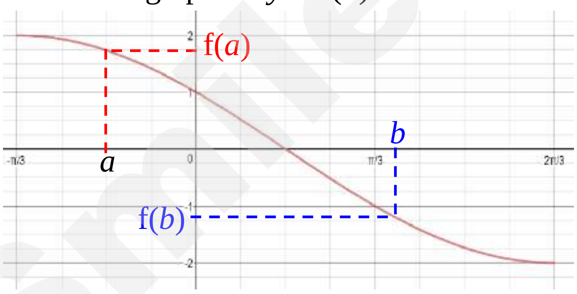
$\int_{-2}^{7} \mathrm{f}(x)\ \mathrm{d}x = -[\text{Area of rectangle + area of triangle}]$

$= -\left[9\times 6 + \frac{1}{2}\times 2\times 3\right]$

$= -57$

Note: Areas of triangles A and B are identical.



2	Solutions
(i)	$R=\sqrt{1+3}=2$ $\alpha=\tan^{-1}\sqrt{3}=\dfrac{\pi}{3}$ $\cos x-\sqrt{3}\sin x=2\cos\left(x+\dfrac{\pi}{3}\right)$ $R_{\mathrm{f}}=[-2,2]$
(ii)	$\mathrm{f}:x\mapsto\cos x-\sqrt{3}\sin x,\quad -\dfrac{\pi}{3}\le x\le\dfrac{2\pi}{3}$ **<u>Method 1:</u>** $\mathrm{f}(x)=2\cos\left(x+\dfrac{\pi}{3}\right)\quad\Rightarrow\quad \mathrm{f}'(x)=-2\sin\left(x+\dfrac{\pi}{3}\right)$ $-\dfrac{\pi}{3}\le x\le\dfrac{2\pi}{3}\quad\Rightarrow\quad 0\le x+\dfrac{\pi}{3}\le\pi$ For $0<x+\dfrac{\pi}{3}<\pi\quad\Rightarrow\quad \sin\left(x+\dfrac{\pi}{3}\right)>0$ $\mathrm{f}'(x)<0$ When $x+\dfrac{\pi}{3}=0$ *or* $\pi,\quad \sin\left(x+\dfrac{\pi}{3}\right)=0$ $\Rightarrow\quad \mathrm{f}'(x)=0$ The end-points are stationary points. Thus f(x) decreases as x increases. **<u>Method 2:</u>** Sketch the graph of $y=\mathrm{f}(x)$.  From the graph, for $-\dfrac{\pi}{3}\le a<b\le\dfrac{2\pi}{3}$, $\mathrm{f}(a)>\mathrm{f}(b)$. Thus f($x$) decreases as x increases **<u>Method 3:</u>** For $-\dfrac{\pi}{3}\le a<b\le\dfrac{2\pi}{3}$ $\mathrm{f}(a)-\mathrm{f}(b)=2\cos\left(a+\dfrac{\pi}{3}\right)-2\cos\left(b+\dfrac{\pi}{3}\right)$ $=-4\sin\left[\dfrac{1}{2}\left(a+b+\dfrac{2\pi}{3}\right)\right]\sin\left[\dfrac{1}{2}(a-b)\right]$ $=-4\sin\left(\dfrac{a+b}{2}+\dfrac{\pi}{3}\right)\sin\left[\dfrac{1}{2}(a-b)\right]$ Since $-\dfrac{\pi}{3}<\dfrac{a+b}{2}<\dfrac{2\pi}{3},\quad\Rightarrow\quad 0<\dfrac{a+b}{2}+\dfrac{\pi}{3}<\pi$

	$\sin\left(\frac{a+b}{2}+\frac{\pi}{3}\right)>0$ Since $-\frac{\pi}{3}\le a<b\le\frac{2\pi}{3}$, $\Rightarrow -\frac{\pi}{3}-\frac{2\pi}{3}\le a-b<0$ $-\frac{\pi}{2}\le\frac{a-b}{2}<0$ $-1\le\sin\left(\frac{a-b}{2}\right)<0$ Thus $\mathrm{f}(a)-\mathrm{f}(b)>0$ i.e. For $a<b$, $\mathrm{f}(a)>\mathrm{f}(b)$ Therefore, f(x) decreases as x increases.
(iii)	Let $y=2\cos\left(x+\frac{\pi}{3}\right)$ $\cos\left(x+\frac{\pi}{3}\right)=\frac{y}{2}$ $x=\cos^{-1}\left(\frac{y}{2}\right)-\frac{\pi}{3}$ $\mathrm{f}^{-1}(x)=\cos^{-1}\left(\frac{x}{2}\right)-\frac{\pi}{3},\ -2\le x\le 2$
(iv)	$\mathrm{fg}(x)=x-3,\ 1\le x\le 3$ $\mathrm{f}^{-1}\mathrm{fg}(x)=\mathrm{f}^{-1}(x-3)$ $\mathrm{g}(x)=\cos^{-1}\left(\frac{x-3}{2}\right)-\frac{\pi}{3}$ **<u>Alternatively</u>** $\mathrm{f}(\mathrm{g}(x))=x-3,\ 1\le x\le 3$ $2\cos\left(\mathrm{g}(x)+\frac{\pi}{3}\right)=x-3$ $\mathrm{g}(x)+\frac{\pi}{3}=\cos^{-1}\left(\frac{x-3}{2}\right)$ $\mathrm{g}(x)=\cos^{-1}\left(\frac{x-3}{2}\right)-\frac{\pi}{3}$

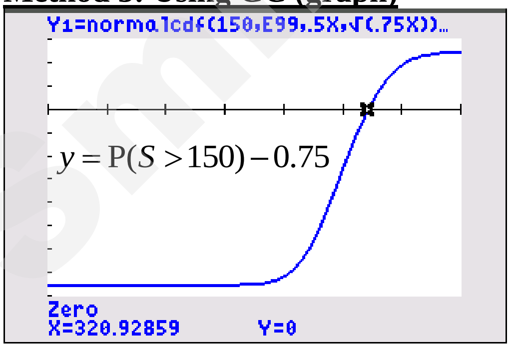
3	**Solutions**
(i)	$\dfrac{dV}{dt}=1200-A\sqrt{h}$, where A is a positive constant. $V=\pi r^2 h=3000h \quad \Rightarrow \quad \dfrac{dV}{dt}=3000\dfrac{dh}{dt}$ $\dfrac{dh}{dt}=\dfrac{1}{3000}\left(1200-A\sqrt{h}\right)$ $\quad =\dfrac{1200}{3000}-\dfrac{A}{3000}\sqrt{h}$ $\quad =0.4-k\sqrt{h} \qquad \text{where } k=\dfrac{A}{3000}>0$
(ii)	When $h=36$, $A\sqrt{36}=360 \quad \Rightarrow \quad A=60$ $k=\dfrac{60}{3000}=0.02$
(iii)	$h=(20-x)^2 \quad \cdots\cdots(1)$ $\Rightarrow \quad \dfrac{dh}{dt}=-2(20-x)\dfrac{dx}{dt} \quad \cdots\cdots(2)$ Substituting (1) & (2) into $\dfrac{dh}{dt}=0.4-0.02\sqrt{h}$, $-2(20-x)\dfrac{dx}{dt}=0.4-0.02(20-x)$ $(x-20)\dfrac{dx}{dt}=0.01x \quad \cdots\cdots(*)$ $\int \dfrac{x-20}{x}\,dx=\int 0.01\,dt$ $\int 1-\dfrac{20}{x}\,dx=\int 0.01\,dt$ $x-20\ln\lvert x\rvert=0.01t+C$ $t=100\left[x-20\ln\lvert x\rvert-C\right]$ $\quad =100\left[\left(20-\sqrt{h}\right)-20\ln\left\lvert 20-\sqrt{h}\right\rvert-C\right]$ $\quad =100\left[-\sqrt{h}-20\ln\left\lvert 20-\sqrt{h}\right\rvert+C'\right] \qquad \text{where } C'=20-C$ When $t=0, h=0 \quad \Rightarrow \quad C'=20\ln 20$ $\dfrac{dh}{dt}>0 \Rightarrow 0.4-0.02\sqrt{h}>0$ $0.02\left(20-\sqrt{h}\right)>0$ $20-\sqrt{h}>0$ Thus $t=100\left[-\sqrt{h}+20\ln\dfrac{20}{20-\sqrt{h}}\right]$
(iv)	When $h=100$, $t=100\left[-10+20\ln 2\right]=386.294$ It takes 6 mins and 26 seconds for the height to reach 100 cm.

4	Solutions
	Method 1: $\mathrm{g}(x) = a\mathrm{f}(0.5x-1)$ Sequence of transformations involved: A: A scaling of the graph of $y=\mathrm{f}(x)$ parallel to the y-axis with scale factor a B: A translation of 1 unit in the positive x-direction C: A scaling parallel to the x-axis with scale factor 2. **Method 2:** $\mathrm{g}(x) = a\mathrm{f}\left(0.5(x-2)\right)$ Sequence of transformations involved: A: A scaling of the graph of $y=\mathrm{f}(x)$ parallel to the y-axis with scale factor a B: A scaling parallel to the x-axis with scale factor 2. C: A translation of 2 units in the positive x-direction Acceptable alternative order of sequence: BCA.
(i)	Volume $= \pi \int_0^2 \frac{a^2}{1+(0.5x-1)^2} - \frac{1}{1+x^2} \; \mathrm{d}x$ $= \pi \left[\frac{a^2}{0.5} \tan^{-1}(0.5x-1) - \tan^{-1} x \right]_0^2$ $= \pi \left[\left(2a^2 \tan^{-1} 0 - \tan^{-1} 2\right) - \left(2a^2 \tan^{-1}(-1) - \tan^{-1} 0\right) \right]$ $= \pi \left(\frac{\pi}{2} a^2 - \tan^{-1} 2 \right)$
(ii)	$y = \frac{1}{\sqrt{1+x^2}} \Rightarrow x^2 = \frac{1}{y^2} - 1$ Volume of 2nd sculpture $= Vol\ of\ cylinder + \pi \int_{\frac{1}{\sqrt{5}}}^{1} \frac{1}{y^2} - 1 \; \mathrm{d}y$ $= \pi\left(2^2\right)\frac{1}{\sqrt{5}} + \pi\left[-\frac{1}{y} - y \right]_{\frac{1}{\sqrt{5}}}^{1}$ $= \pi\left(\frac{4}{\sqrt{5}}\right) + \pi\left[(-2) - \left(-\sqrt{5} - \frac{1}{\sqrt{5}} \right) \right]$ $= 2\pi\left(\sqrt{5}-1\right)$
(iii)	$\pi\left(\frac{\pi}{2}a^2 - \tan^{-1} 2\right) \geq 50\left[2\pi\left(\sqrt{5}-1\right)\right]$ $a^2 \geq \frac{2}{\pi}\left[100\left(\sqrt{5}-1\right) + \tan^{-1} 2\right]$ $a \geq 8.91$ since $a > 2$ Smallest integer value of a is 9

5	**Solutions**
5	$\mathrm{P}(X \le 1) = 0.15$ $\binom{25}{0}(1-p)^{25} + \binom{25}{1}p(1-p)^{24} = 0.15$ $(1-p)^{25} + 25p(1-p)^{24} = 0.15$ $\left[\text{or } \ (1-p)^{24}(1+24p) = 0.15\right]$ Using GC, $p = 0.12865$ $= 0.129$ (to 3 s.f.) $\mathrm{Var}(X) = np(1-p)$ $= 25(0.12865)(1-0.12865)$ $= 2.80$ (to 3 s.f.)

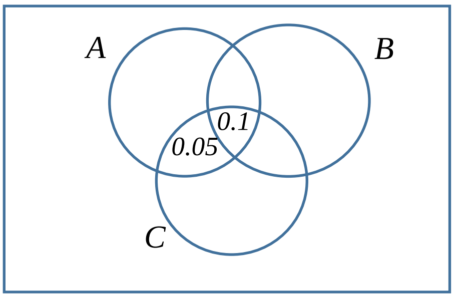
6	**Solutions**
(i)	AA EE I PP R C T c_c_c_c_c_ _c_c_c_c_c Number of ways = $\left(\frac{5!}{2!2!}\right)\left(\frac{5!}{2!}\right) \times 2! = 3600$
(ii)	E_ _ E _ _ _ _ _ _ Case 1 :2 As No of ways $= \frac{7!}{2!} = 2520$ Case 2 : 1A with 1P No of ways= $2!7! = 10080$ Case 3 : 1A without P No of ways= $\left({}^4C_1 \times 2!\right) \times \frac{7!}{2!} = 20160$ Total no of ways = 32760

7	**Solutions**
(i)	Given $\mathrm{Var}(X) = 0.75$, $\mathrm{E}\left(X^2\right) = (1)^2(3p) + (2)^2(2p) + (3)^2(p) = 20p$ $\mathrm{E}(X) = 1(3p) + (2)(2p) + (3)(p) = 10p$ $20p - (10p)^2 = 0.75$ $100p^2 - 20p + 0.75 = 0$ $p = \frac{1}{20}$ *or* $p = \frac{3}{20}$ (Reject since $0 < p < \frac{1}{10}$)

	$\therefore p = \dfrac{1}{20}$ $(or\ \ 0.05)$, $\mathrm{E}(X) = \dfrac{1}{2}$ or 0.5
(ii)	$S = X_1 + X_2 + X_3 + ... + X_n$ Since n is large, **by Central Limit Theorem,** $S \sim N(\,0.5n\,, 0.75n\,)$ approximately. **<u>Method 1: Algebraic method</u>** $\mathrm{P}(S > 150) \geq 0.75$ $\mathrm{P}\left(Z < \dfrac{150 - 0.5n}{\sqrt{0.75n}}\right) \leq 0.25$ $\dfrac{150 - 0.5n}{\sqrt{0.75n}} \leq -0.6744897$ $150 - 0.5n \leq -0.6744897\sqrt{0.75n}$ $0.5n - 0.6744897\sqrt{0.75n} - 150 \geq 0$ $\sqrt{n} \leq -16.746$ (reject since $\sqrt{n} > 0$) or $\sqrt{n} \geq 17.914$ $n \geq 320.93$ Thus $\{n:\ n \in \mathbb{Z}^+,\ \ n \geq 321\}$ **<u>Method 2: Using GC (table)</u>** $\mathrm{P}(S > 150) \geq 0.75$ When $n = 320$, $\mathrm{P}(S > 150) = 0.7407 < 0.75$ When $n = 321$, $\mathrm{P}(S > 150) = 0.7507 > 0.75$ When $n = 322$, $\mathrm{P}(S > 150) = 0.7605 > 0.75$ Thus $\{n:\ n \in \mathbb{Z}^+,\ \ n \geq 321\}$ **<u>Method 3: Using GC (graph)</u>**  From the graph, $n \geq 320.92859$ Thus $\{n:\ n \in \mathbb{Z}^+,\ \ n \geq 321\}$

8 Solutions

(i)



Given $\mathrm{P}(B \mid C) = 0.5$,

$$\mathrm{P}(B \mid C) = \frac{\mathrm{P}(B \cap C)}{\mathrm{P}(C)}$$

$$0.5 = \frac{\mathrm{P}(B \cap C)}{0.35}$$

$$\mathrm{P}(B \cap C) = 0.175$$

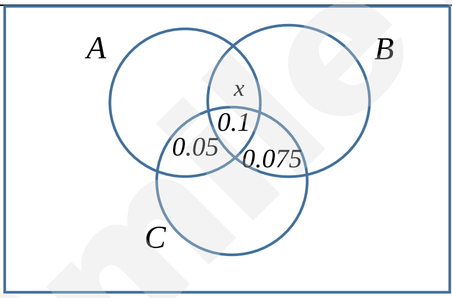
$$\begin{aligned}\mathrm{P}(A' \cap B \cap C) &= 0.175 - 0.1 \\ &= 0.075\end{aligned}$$

(ii)

Since A and B are independent events,

$$\begin{aligned}\mathrm{P}(A \cup B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) \\ &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A) \times \mathrm{P}(B) \\ &= 0.5 + 0.45 - 0.5(0.45) \\ &= 0.725\end{aligned}$$

(iii)



$$\begin{aligned}&\mathrm{P}(A' \cap B' \cap C') \\ &= 1 - \mathrm{P}(A \cup B \cup C) \\ &= 1 - 0.35 - \big(0.5 - 0.15 - x) - (0.45 - 0.175 - x\big) - x \\ &= 0.025 + x\end{aligned}$$

Max x occurs when $x + 0.1 + 0.075 = 0.45$

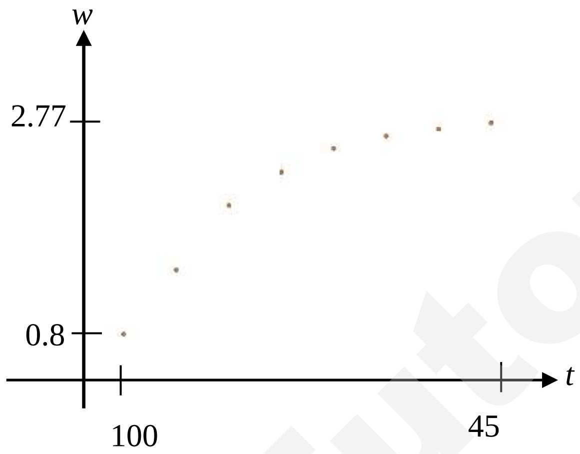
$x = 0.275$,

greatest possible value of $\mathrm{P}(A' \cap B' \cap C') = 0.3$

when $x = 0$,

least possible value of $\mathrm{P}(A' \cap B' \cap C') = 0.025$

9	**Solutions**
(i)	Let X be the volume of tomato juice in a randomly chosen bottle (in ml) Let μ represents the population mean volume of tomato juice per bottle $H_0: \mu = 250$ $H_1: \mu < 250$
9(ii)	Under H_0, Since $n = 50$ is large, by Central Limit Theorem, $\bar{X} \sim N\left(250, \frac{10^2}{50}\right)$ approximately. Using GC, p-value = 0.0385498886 ≈ 0.0385 (3 s.f.) There is 0.0385 probability of drawing a random sample of 50 bottles of tomato juice and obtaining a sample mean volume of 247.5 ml or less, assuming that the population mean volume is 250 ml.
(iii)	No, it is not necessary to assume a normal distribution for the test to be valid, since $n = 50$ is large, Central Limit Theorem can be applied for $\bar{X}$ to be normally distributed.
(iv)	Unbiased estimate of population mean of smaller bottles of tomato juice $= \frac{10757}{60} = 179.283 \approx 179$ (3 s.f.) Unbiased estimate of population variance $= \frac{1}{59}\left[1931597 - \frac{10757^2}{60}\right]$ $= 51.63022599$ ≈ 51.6 (3 s.f.)
(v)	$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$ Test at 5% significance level. Under H_0, since $n = 60$ is large, by CLT, $\bar{Y} \sim N\left(\mu_0, \frac{51.6302}{60}\right)$ approximately Test statistic $Z = \frac{\bar{Y} - \mu_0}{s/\sqrt{60}} \sim N(0,1)$ H_0 is not rejected $\Rightarrow$ The test statistic lies outside the critical region. $-1.959963986 < \frac{179.283 - \mu_0}{\sqrt{\frac{51.630226}{60}}} < 1.959963986 \quad \cdots\cdots(*)$ $-1.959963986\left(\sqrt{\frac{51.630226}{60}}\right) < 179.283 - \mu_0 < 1.959963986\left(\sqrt{\frac{51.630226}{60}}\right)$ $-1.8181281 < 179.283 - \mu_0 < 1.8181281$ $-1.8181281 - 179.283 < -\mu_0 < 1.8181281 - 179.283$ $177.4652052 < \mu_0 < 181.1014614$ $177.5 < \mu_0 < 181.1$ (1 d.p.)

10	**Solutions**
(i)	r = 0.925 (3 s.f.) Acceptable answers: • Since the product moment correlation coefficient is close to 1, indicating a **strong positive <u>linear</u>** correlation, it suggests that a linear model is appropriate. • A linear model with positive linear correlation would suggest that the weight loss may exceed 100%, which is impossible. Thus a linear model is not appropriate.
(ii)	 (A) $w = a + b\ln t$, from the scatter diagram, as length of time increases, percentage of weight loss also **increases at a decreasing rate**.
(iii)	$w = -5.31 + 1.35\ln t$ The product moment correlation coefficient between ln$t$ and w is $r = 0.9828402622$ $= 0.983 \quad (3 s.f.)$ Since r = 0.983 is closer to 1 than the previous product moment correlation coefficient of 0.925, indicating stronger positive linear correlation between w and ln$t$ compared to the linear model.
(iv)	$2.4 = -5.3074 + 1.3522\ln t$ $\ln t = \dfrac{2.4 + 5.3074}{1.3522}$ $t = e^{5.6999}$ $t = 298.84 \approx 299$ (3 s.f.) This estimate is reliable since w = 2.4 is within the data range of w and the product moment correlation coefficient between ln t and w is $r = 0.983$ which is very close to 1, showing a very strong positive linear correlation between ln t and w.
(v)	1 day = 24 hours $w = -5.31 + 1.35\ln(24t)$

11	Solutions
(a) (i)	$\mu = \frac{40+50}{2} = 45$ $P(T < 40) = 0.36$ $P\left(Z < \frac{40-45}{\sigma}\right) = 0.36$ $\frac{-5}{\sigma} = -0.3584588$ $\sigma = 13.9486 = 13.9$ (to 3 s.f.)
(a) (ii)	$P(T \le 50) = 1 - 0.36 = 0.64$ Let C be the number of houses, out of 18, for which it takes at most 50 min to install an electricity meter. $C \sim B(18, 0.64)$ $P(5 \le C < 10) = P(C \le 9) - P(C \le 4)$ $= 0.160$ (3 s.f.)
(b) (i)	Let E be the electricity consumption of the households in Central City $E \sim N(520,\ 35^2)$ $P(E > 500) = 0.7161454588 = 0.716$ (to 3 s.f.)
(b) (ii)	Required probability $= [P(E < 500)]^2$ $= [1 - 0.7161454588]^2$ $= 0.0806$ (to 3 s.f.)
(b) (iii)	Part **(ii)** is a subset of the event where the total electricity consumption for two randomly chosen households is less than 1000 kWh.
(b) (iv)	Let S be the electricity consumption of the households in Star City. $S \sim N(475,\ 25^2)$ $X = 0.18E \Rightarrow E(X) = 0.18 \times 520 = 93.6$ $\text{Var}(X) = 0.18^2 \times 35^2 = 39.69$ $Y = 0.15S \Rightarrow E(Y) = 0.15 \times 475 = 71.25$ $\text{Var}(Y) = 0.15^2 \times 25^2 = 14.0625$ $E(4Y - 3X) = 4 \times 71.25 - 3 \times 93.6 = 4.2$ $\text{Var}(4Y - 3X) = 4^2 \times 14.0625 + 3^2 \times 39.69$ $= 582.21$ $4Y - 3X \sim N(4.2,\ 582.21)$ $P(4Y - 3X > 7) = 0.454$ (3 s.f.) It means that there is 0.454 probability that 4 times the electricity bill of a randomly chosen household in Star City exceeds 3 times the electricity bill of a randomly chosen household in Central City by more than \$7.

Name:__________________________ Class:_____________

JURONG JUNIOR COLLEGE

JC2 Preliminary Examinations 2018

MATHEMATICS
Higher 2

Paper 1

9758/01

28 August 2018

3 hours

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 To watch the daily matches of a badminton tournament in a stadium, each spectator is to buy either a daily normal ticket for an adult spectator or a daily concession ticket for a student spectator. Tickets are purchased from either ticket booth A, B or C. The number of tickets sold and the total amount of money collected by each ticket booth are shown in the following table.

Ticket booth	Number of daily normal tickets sold	Number of daily concession tickets sold	Total amount collected
A	$5n$	n	\$5976
B	$7n$	$2n$	\$8712
C	357	51	\$5763

Find the price of each daily normal ticket and each daily concession ticket and determine the value of n. [4]

2 The curve C has equation $y = \dfrac{ax^2 + bx + c}{x + d}$, where $x \in \mathbb{R}$, $x \neq -d$ and a, b, c and d are constants.

It is given that C has stationary points at $x = 0$ and $x = -2$. The lines $x = -1$ and $y = x$ are asymptotes to C.

(i) Write down the value of d, and determine the values of a, b and c. [6]

With the values of a, b, c and d found in **(i)**,

(ii) find the range of values that y can take using an algebraic method, [4]

(iii) sketch the graph of $y = \dfrac{x + d}{ax^2 + bx + c}$, indicating clearly the coordinates of the points where the graph crosses the axes, the turning points and the equations of any asymptotes. [3]

3 **(i)** Find $\displaystyle\int \frac{x}{\left(4 + 3x^2\right)^2}\, dx$. [2]

(ii) Hence find the exact value of $\displaystyle\int_0^{\frac{2}{\sqrt{3}}} \frac{2x^2}{\left(4 + 3x^2\right)^2}\, dx$. [4]

4 A curve C has parametric equations

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad \text{for } 0 < \theta < 2\pi.$$

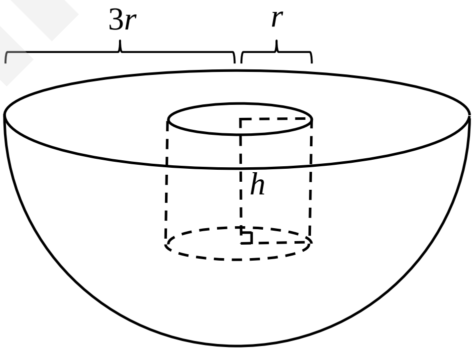
(i) Find the equation of the tangent that is parallel to the x-axis. [3]

(ii) The normal to the curve at the point with parameter $\frac{2\pi}{3}$ meets the x- and y-axes at P and Q respectively. Show that the equation of the normal is $y = -\sqrt{3}x + \frac{2\sqrt{3}\pi}{3}$.
Hence find the exact area of the triangle OPQ. [5]

(iii) Given that θ is increasing at a rate of 2 radians per second, find the rate of change of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $\theta = \frac{\pi}{3}$. [3]

5 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

A toy is constructed from a hemisphere with radius $3r$ cm by removing a circular cylinder of radius r cm and height h cm where $h < 3r$ as shown in the diagram below. As r and h vary, the total cost of coating the surface with a protective film on the entire toy is a constant \$$C$. The cost of coating on the flat surfaces is \$$k$ per cm^2 and that on the curved surfaces is \$$2k$ per cm^2, where k is a positive constant.



Show that the volume, V cm^3, of the toy is

$$V = \frac{117}{4}\pi r^3 - \frac{Cr}{4k}.$$ [3]

(i) Find the value of r in terms of C and k which gives a stationary value of V. [2]

(ii) Find also the ratio of the height to the radius, $\frac{h}{r}$, in this case, simplifying your answer. [2]

(iii) Explain why it is not possible for this toy to have a stationary value of V. [1]

6 The position vectors of points A, B and C of a triangle are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, relative to an origin O.

(i) By considering the area of triangle ABC, show that the shortest distance from B to AC is
$$\frac{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|}{|\mathbf{c}-\mathbf{a}|}.$$ [4]

R is a point on AB such that $\overrightarrow{AR}=\frac{1}{3}\overrightarrow{AB}$. S is a point on AC such that $\overrightarrow{AS}=\frac{2}{3}\overrightarrow{AC}$. $OACB$ is a kite with $OA=OB$, $CA=CB$ and OC is perpendicular to AB.

(ii) Show that $\angle SRA=90°$. [6]

7 **(a)** A function f is said to be self-inverse if $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ for all x in the domain of f.

The function f is defined by

$$\mathrm{f}:x\mapsto\frac{3x+k}{x-b},\ x\in\mathbb{R},\ x\neq b\text{, where } k \text{ and } b \text{ are constants.}$$

(i) Find the value of b and the set of values of k such that f is self-inverse. [3]

Using the value of b found in (i), another function g is defined such that

$$\mathrm{fg}:x\mapsto 2x-1,\ x\neq 2.$$

(ii) Find in terms of k, an expression for $\mathrm{g}(x)$. [2]

(b) The function h is defined as follows:

$$\mathrm{h}(x)=\begin{cases}-4x+8, & \text{for } 1\le x<2,\\ -x^2+8x-12, & \text{for } 2\le x<4,\end{cases}$$

and that $\mathrm{h}(x+3)=\mathrm{h}(x)$ for all real values of x.

(i) Sketch the graph of $y=\mathrm{h}(x)$ for $-4\le x\le 6$, indicating the axial intercepts and endpoints clearly. [3]

(ii) Find $\int_{-\frac{3}{2}}^{3}\mathrm{h}(x)\,\mathrm{d}x$. [3]

8 When a plague of locusts attacks a wheat crop, the proportion of the crop destroyed after t hours is denoted by x. In a model, it is assumed that the rate at which the crop is destroyed is proportional to $x(1 - x)$. A plague of locusts is discovered in a wheat crop when one-third of the crop has been destroyed and the rate of destruction at this instant is $\frac{1}{6}$.

(i) Show that $\frac{\mathrm{d}x}{\mathrm{d}t} = kx(1-x)$, where k is a constant to be determined. [3]

(ii) Find the percentage of the crop destroyed two hours after the plague of locusts is first discovered. [9]

9 **Do not use a calculator in answering this question.**

(a) Find the roots of the equation $z^2 + (\mathrm{i}-4)z + (6-2\mathrm{i}) = 0$, giving your answers in cartesian form $a + \mathrm{i}b$. [2]

(b) The complex number w has modulus r and argument θ, where $0 < \theta < \frac{\pi}{2}$, and w^* denotes the conjugate of w. State the modulus and argument of p, where $p = \frac{w}{w^*}$. [2]

Given that p^6 is real and positive, find the possible values of θ. [3]

(c) The polynomial $\mathrm{P}(z)$ of degree 4 has real coefficients. Two of the roots of the equation $\mathrm{P}(z) = 0$ are $z = 1 + \mathrm{i}$ and $z = 2$.

(i) State the number of complex roots of $\mathrm{P}(z) = 0$, justifying your answer. [1]

(ii) By expressing $\mathrm{P}(z)$ as a product of linear factors, find the remaining roots of the equation $\mathrm{P}(z) = 0$ given that $\mathrm{P}(\mathrm{i}) = 10 + 10\mathrm{i}$. [5]

10 Building contractors are constructing a rock climbing wall at the corner wall of a gymnasium. Points (x, y, z) are defined relative to a ground anchor point at $(0,0,0)$, where units are metres. Support beams are laid in straight lines and the thickness of the support beams and rock climbing wall can be neglected.

The three support beams of the rock climbing wall, S_1, S_2 and S_3 start at the ground anchor point and go in the direction $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$, and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ respectively. The support beams S_1 and S_2 are on the ground level. The vertices A, B and C of the rock climbing wall lie on the support beams S_1, S_2 and S_3 respectively. The rock climbing wall lies on the plane π with vector equation $\mathbf{r} = \begin{pmatrix}1\\5\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-12\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-7\end{pmatrix}$, where λ, $\mu \in \mathbb{R}$.

(i) Find the cartesian equation of the plane π and hence show that the coordinates of A are $(4,0,0)$. [4]

One of the building safety standards stipulates that the rock climbing wall should be inclined to the horizontal ground at an acute angle not exceeding $80°$.

(ii) Determine if this building safety standard is met. [3]

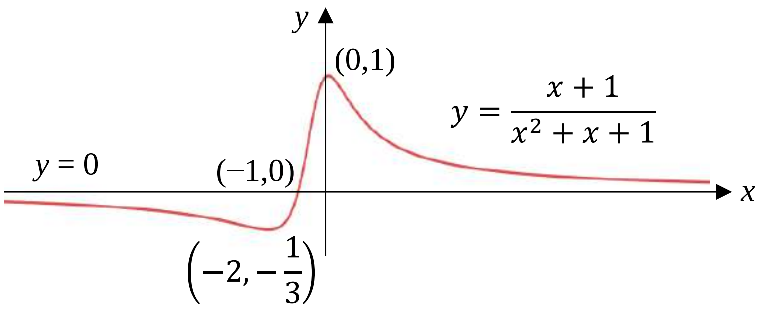
For additional stability, a fourth support beam from the ground anchor point to a point N on the rock climbing wall is laid. This support beam is the shortest in length.

(iii) Find the coordinates of N and the exact length of this support beam. [5]

Jurong Junior College
2018 JC2 H2 Mathematics Prelim Paper 1 Solution

Qn	Solution
1	Let \$$x$ and \$$y$ be the price of each daily normal ticket and each daily concession ticket respectively. Booth A : $5nx + ny = 5976$ -----(1) Booth B : $7nx + 2ny = 8712$ -----(2) Booth C : $357x + 51y = 5763$ -----(3)
	$5x + y - 5976\left(\frac{1}{n}\right) = 0 \quad \cdots\cdots \quad (1)$ $7x + 2y - 8712\left(\frac{1}{n}\right) = 0 \quad \cdots\cdots \quad (2)$ $357x + 51y + 0\left(\frac{1}{n}\right) = 5763 \quad \cdots\cdots \quad (3)$
	Or $\frac{5x+y}{7x+2y} = \frac{5976}{8712} \Rightarrow 1728x - 3240y = 0$
	From GC : $x = 15$, $y = 8$, $\frac{1}{n} = \frac{1}{72}$ Each daily normal ticket costs \$15, and each daily concession ticket costs \$8 and $n = 72$.

Qn	Solution	Marks	Remarks
2(i)	Since $x = -1$ is an asymptote, $\Rightarrow \quad d = 1$	[B1]	
	Since $y = x$ is an asymptote, $y = x + \frac{A}{x+1} = \frac{x^2 + x + A}{x+1} = \frac{ax^2 + bx + c}{x+1}$	[M1]	Use long division $y = \frac{ax^2 + bx + c}{x+1}$ $= ax + b - a + \frac{a + c - b}{x+1}$
	$\Rightarrow \quad a = 1 \quad$ and $\quad b = 1$	[A2]	A1 for $a = 1$ A1 for $b = 1$
	$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{A}{(x+1)^2}$ when $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\Rightarrow \quad A = 1$	[M1]	Or : when $x = -2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
	$\Rightarrow \quad c = 1$	[A1]	
(ii)	$y = \frac{x^2 + x + 1}{x+1}$ $y(x+1) = x^2 + x + 1$ $x^2 + (1-y)x + (1-y) = 0$	[M1]	Form a quadratic equation in *x*.
	For real *x*, $(1-y)^2 - 4(1)(1-y) \geq 0$	[M1]	Use Discriminant

	$(1-y)^2-4(1)(1-y)\geq 0$ $1-2y+y^2-4+4y\geq 0$ $y^2+2y-3\geq 0$ $(y+3)(y-1)\geq 0$	[M1]	Solve y
	$\Rightarrow \quad \underline{\underline{y\leq -3 \quad \text{or} \quad y\geq 1}}\,\text{(ans)}$	[A1]	
(iii)		[G3]	G1 for shape G1 for $y=0$ & $(-1,0)$ G1 for $(0,1)$ & $\left(-2,-\dfrac{1}{3}\right)$

Qn	Solution	Marks	Remarks
3(i)	$\displaystyle\int \frac{x}{\left(4+3x^2\right)^2}\,\mathrm{d}x = \frac{1}{6}\int (6x)\left(4+3x^2\right)^{-2}\,\mathrm{d}x$	[M1]	
	$\displaystyle = -\frac{1}{6\left(4+3x^2\right)}+c$	[A1]	
(ii)	$u=2x \qquad \dfrac{\mathrm{d}v}{\mathrm{d}x}=\dfrac{x}{\left(4+3x^2\right)^2}$ $\dfrac{\mathrm{d}u}{\mathrm{d}x}=2 \qquad v=-\dfrac{1}{6\left(4+3x^2\right)}$ $\displaystyle\int_0^{\frac{2}{\sqrt{3}}} \frac{2x^2}{\left(4+3x^2\right)^2}\,\mathrm{d}x = \left[-\frac{2x}{6\left(4+3x^2\right)}\right]_0^{\frac{2}{\sqrt{3}}}+\frac{1}{3}\int_0^{\frac{2}{\sqrt{3}}}\frac{1}{4+3x^2}\,\mathrm{d}x$	[M1]	Integration by parts with the correct u and $\dfrac{\mathrm{d}v}{\mathrm{d}x}$.
	$\displaystyle = -\frac{1}{12\sqrt{3}}+\frac{1}{3\sqrt{3}}\int_0^{\frac{2}{\sqrt{3}}}\frac{\sqrt{3}}{2^2+\left(\sqrt{3}x\right)^2}\,\mathrm{d}x$ $\displaystyle = -\frac{1}{12\sqrt{3}}+\left[\frac{1}{6\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right)\right]_0^{\frac{2}{\sqrt{3}}}$	[M1]	Award mark for $\tan^{-1}\left(\dfrac{\sqrt{3}x}{2}\right)$
	$\displaystyle = \frac{\pi}{24\sqrt{3}}-\frac{1}{12\sqrt{3}}$	[M1]	Substitution of limits to the two anti-derivatives
	$\displaystyle = \frac{\pi-2}{24\sqrt{3}}$	[A1]	

Qn	Solution	Marks	Remarks
4(i)	$\dfrac{dy}{dx}=\dfrac{\sin\theta}{1-\cos\theta}$	[M1]	
	$\dfrac{dy}{dx}=0\Rightarrow \sin\theta=0\Rightarrow\theta=\pi$	[M1]	Award mark for $\dfrac{dy}{dx}=0$ and attempt to solve for θ
	$y=1-\cos\pi=2$	[A1]	
(ii)	At $\theta=\dfrac{2\pi}{3}$, $x=\dfrac{2\pi}{3}-\dfrac{\sqrt{3}}{2}$, $y=\dfrac{3}{2}$, $\dfrac{dy}{dx}=\dfrac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}}=\dfrac{\sqrt{3}}{3}$.	[M1]	
	Gradient of normal $=-\sqrt{3}$	[A1]	
	Eqn of normal is $y-\dfrac{3}{2}=-\sqrt{3}\left[x-\left(\dfrac{2\pi}{3}-\dfrac{\sqrt{3}}{2}\right)\right]$ $y=-\sqrt{3}x+\dfrac{2\sqrt{3}\pi}{3}$ (Shown)	[M1]	Show working. AG
	$x=0,\ y=\dfrac{2\sqrt{3}\pi}{3}\Rightarrow Q\left(0,\ \dfrac{2\sqrt{3}\pi}{3}\right)$ $y=0,\ x=\dfrac{2\pi}{3}\quad\Rightarrow P\left(\dfrac{2\pi}{3},\ 0\right)$	[M1]	
	Area of triangle = $\dfrac{1}{2}\times\dfrac{2\pi}{3}\times\dfrac{2\sqrt{3}\pi}{3}=\dfrac{2\sqrt{3}\pi^2}{9}$ units^2	[A1]	
(iii)	$\dfrac{dy}{dx}=\dfrac{\sin\theta}{1-\cos\theta}$ $\dfrac{d\left(\frac{dy}{dx}\right)}{d\theta}=\dfrac{\cos\theta(1-\cos\theta)-\sin^2\theta}{(1-\cos\theta)^2}=\dfrac{1}{\cos\theta-1}$	[M1]	Differentiate wrt θ Accept $\dfrac{d^2y}{d\theta^2}=\dfrac{1}{\cos\theta-1}$
	$\dfrac{d\left(\frac{dy}{dx}\right)}{dt}=\dfrac{d\left(\frac{dy}{dx}\right)}{d\theta}\cdot\dfrac{d\theta}{dt}=\dfrac{1}{\cos\frac{\pi}{3}-1}(2)$	[M1]	Use chain rule $\dfrac{d\left(\frac{dy}{dx}\right)}{d\theta}\cdot\dfrac{d\theta}{dt}$
	$=-4$ units/s	[A1]	

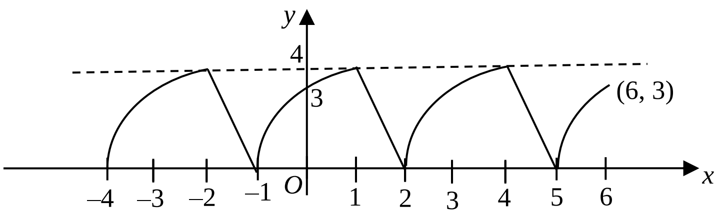
Qn	Solution	Marks	Remarks
5	$C=\pi(3r)^2k+2\pi(3r)^2(2k)+2\pi rh(2k)$ $=45\pi r^2k+4\pi rhk$	[M1]	
	$\Rightarrow h=\dfrac{C-45\pi r^2k}{4\pi rk}$	[M1]	

	$V=\frac{2}{3}\pi(3r)^3-\pi r^2h$ $=18\pi r^3-\pi r^2\left(\frac{C}{4\pi rk}-\frac{45}{4}r\right)$ $=\frac{117}{4}\pi r^3-\frac{Cr}{4k}$	[M1]	Show working. AG
(i)	$\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{117}{4}\pi\left(3r^2\right)-\frac{C}{4k}$	[M1]	
	At stationary value of V, $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ $\frac{117}{4}\pi\left(3r^2\right)-\frac{C}{4k}=0$ $\Rightarrow r^2=\frac{C}{351\pi k}\Rightarrow r=\sqrt{\frac{C}{351\pi k}}$	[A1]	
(ii)	$\frac{h}{r}=\frac{C-45\pi r^2k}{4\pi r^2k}=\frac{C}{4\pi r^2k}-\frac{45}{4}$ $=\frac{C}{4\pi k\left(\frac{C}{351\pi k}\right)}-\frac{45}{4}$	[M1]	
	$=\frac{153}{2}$	[A1]	
(iii)	Since $h=76.5r$ does not satisfy $h<3r$, $\therefore$ it is not possible for this toy to have a stationary value of V.	[B1]	

Qn	Solution	Marks	Remarks
6(i)	Area of triangle $ABC=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$	[M1]	
	$=\frac{1}{2}\left\|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\right\|$ $=\frac{1}{2}\left\|\mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}-\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{a}\right\|$	[M1]	
	$=\frac{1}{2}\left\|\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a}\right\|$	[A1]	
	$\Rightarrow\frac{1}{2}\left\|\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a}\right\|=\frac{1}{2}\left\|\mathbf{c}-\mathbf{a}\right\|\times\text{height}$ shortest distance from B to $AC=\frac{\left\|\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a}\right\|}{\left\|\mathbf{c}-\mathbf{a}\right\|}$ (Shown)	[M1]	Show working AG
6(ii)	$\overrightarrow{RA}=\frac{1}{3}\overrightarrow{BA}=\frac{1}{3}(\mathbf{a}-\mathbf{b})$	[B1]	

	$\overrightarrow{RS}=\overrightarrow{OS}-\overrightarrow{OR}$ $=\left(\dfrac{2\mathbf{c}+\mathbf{a}}{3}\right)-\left(\dfrac{2\mathbf{a}+\boldsymbol{b}}{3}\right)$ $=-\dfrac{1}{3}\mathbf{a}-\dfrac{1}{3}\mathbf{b}+\dfrac{2}{3}\mathbf{c}$ or $\overrightarrow{RS}=\overrightarrow{RA}+\overrightarrow{AS}$ $=\dfrac{1}{3}(\mathbf{a}-\mathbf{b})+\dfrac{2}{3}(\mathbf{c}-\mathbf{a})$ $=-\dfrac{1}{3}\mathbf{a}-\dfrac{1}{3}\mathbf{b}+\dfrac{2}{3}\mathbf{c}$	[M1] [A1]	
	$\overrightarrow{RA}\bullet\overrightarrow{RS}=\dfrac{1}{3}(\mathbf{a}-\mathbf{b})\bullet\left(-\dfrac{1}{3}\mathbf{a}-\dfrac{1}{3}\mathbf{b}+\dfrac{2}{3}\mathbf{c}\right)$ $=\dfrac{1}{9}\left[(\mathbf{a}-\mathbf{b})\bullet(-\mathbf{a}-\mathbf{b}+2\mathbf{c})\right]$ $=\dfrac{1}{9}\left[-\mathbf{a}\bullet\mathbf{a}-\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}+2(\mathbf{a}-\mathbf{b})\bullet\mathbf{c}\right]$	[M1]	
	$=\dfrac{1}{9}\left[-\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2+2(\mathbf{a}-\mathbf{b})\bullet\mathbf{c}\right]$	[M1]	
	$OACB$ is a kite with $OA=OB$, $CA=CB$ and $\overrightarrow{BA}\perp\overrightarrow{OC}$ $\Rightarrow \lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert$ and $(\mathbf{a}-\mathbf{b})\bullet\mathbf{c}=0$ $\therefore \overrightarrow{RA}\bullet\overrightarrow{RS}=0$ (Shown)	[M1]	

Qn	Solution	Marks	Remarks
7(a) (i)	$y=\dfrac{3x+k}{x-b}$ $xy-by=3x+k$ $x(y-3)=by+k$ $x=\dfrac{by+k}{y-3}$ $\mathrm{f}^{-1}(x)=\dfrac{bx+k}{x-3}$	[M1]	
	For $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, $\dfrac{3x+k}{x-b}=\dfrac{bx+k}{x-3}$ $\therefore b=3$	[A1]	
	Also, $3x+k\neq m(x-3)$ since f is a one to one function. $\therefore\ k\neq -9$	[A1]	
(ii)	$\mathrm{fg}(x)=2x-1$ $\mathrm{g}(x)=\mathrm{f}^{-1}(2x-1)$ $=\mathrm{f}(2x-1)$ $=\dfrac{3(2x-1)+k}{(2x-1)-3}$	[M1]	
	$=\dfrac{6x-3+k}{2x-4},\ x\neq 2$	[A1]	

7(b) (i)		[G3]	M1: Correct shape, maximum and minimum points on the interval $1 \le x \le 4$. A1: y-intercept at (0, 3) and interpreting $\text{h}(x+3)=\text{h}(x)$ A1: Endpoints at (–4,0) and (6, 3)
(ii)	$\int_{-1}^{2} \text{h}(x)\,\text{d}x = \int_{2}^{4}(-x^2+8x-12)\,\text{d}x + \frac{1}{2}(1)(4)$ $= \frac{22}{3}$	[M1]	Finding $\int_{-1}^{2} \text{h}(x)\,\text{d}x$ or equivalent
	$\int_{-\frac{3}{2}}^{3} \text{h}(x)\,\text{d}x = \frac{1}{2}\left(\frac{1}{2}\right)(2)+\left(\frac{22}{3}\right)+\int_{2}^{3}(-x^2+8x-12)\,\text{d}x$	[M1]	
	$= \frac{19}{2} = 9.5$	[A1]	

Qn	Solution	Marks	Remarks
8(i)	$\frac{\text{d}x}{\text{d}t} = kx(1-x)$	[B1]	
	When $x=\frac{1}{3}$, $\frac{\text{d}x}{\text{d}t}=\frac{1}{6}=k\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$	[M1]	
	$\therefore\ k=\frac{3}{4}$ i.e. $\frac{\text{d}x}{\text{d}t}=\frac{3}{4}x(1-x)$	[A1]	
(ii)	$\int \frac{1}{x(1-x)}\,\text{d}x = \int \frac{3}{4}\,\text{d}t$	[M1]	
	$\int\left(\frac{1}{x}+\frac{1}{1-x}\right)\text{d}x = \int \frac{3}{4}\,\text{d}t$	[M1]	Use of partial fractions (or equivalent)
	$\ln\lvert x\rvert - \ln\lvert 1-x\rvert = \frac{3}{4}t+c$	[M1]	
	$\ln\left\lvert\frac{x}{1-x}\right\rvert = \frac{3}{4}t+c$ $\frac{x}{1-x}=\text{e}^{\frac{3}{4}t+c}$ $\frac{x}{1-x}=A\text{e}^{\frac{3}{4}t}$, where $A=\text{e}^c$	[M1]	
	when $t=0$, $x=\frac{1}{3}$, $\frac{1}{2}=A\text{e}^0$	[M1]	
	$A=\frac{1}{2}$ i.e. $\frac{x}{1-x}=\frac{1}{2}\text{e}^{\frac{3}{4}t}$	[A1]	

	when $t=2$, $\dfrac{x}{1-x}=\dfrac{1}{2}e^{\frac{3}{2}}$	[M1]	
	$2x=e^{\frac{3}{2}}-xe^{\frac{3}{2}}$ $x\left(2+e^{\frac{3}{2}}\right)=e^{\frac{3}{2}}$ $x=\dfrac{e^{\frac{3}{2}}}{2+e^{\frac{3}{2}}}$	[M1]	
	% of crop destroyed = $\dfrac{e^{\frac{3}{2}}}{2+e^{\frac{3}{2}}}\times 100=69.1\%$	[A1]	

Qn	Solution	Marks	Remarks
9(a)	$z^2+(\mathrm{i}-4)z+(6-2\mathrm{i})=0$ $z=\dfrac{-(\mathrm{i}-4)\pm\sqrt{(\mathrm{i}-4)^2-4(6-2\mathrm{i})}}{2}$	[M1]	
	$z=\dfrac{-\mathrm{i}+4\pm\sqrt{\mathrm{i}^2-8\mathrm{i}+16-24+8\mathrm{i}}}{2}$ $z=2+\mathrm{i},\ 2-2\mathrm{i}$	[A1]	
9(b)	$w=re^{\mathrm{i}\theta}$ and $w^*=re^{-\mathrm{i}\theta}$ $p=\dfrac{re^{\mathrm{i}\theta}}{re^{-\mathrm{i}\theta}}=e^{\mathrm{i}(2\theta)}$	[M1]	
	$\lvert p\rvert=1$ and $\arg(p)=2\theta$	[A1]	
	$p^6=e^{\mathrm{i}(12\theta)}=\cos(12\theta)+\mathrm{i}\sin(12\theta)$	[M1]	
	$0<\theta<\dfrac{\pi}{2}\Rightarrow\ 0<12\theta<6\pi$ For p^6 to be real, $\sin(12\theta)=0$, i.e. $12\theta=\pi,\ 2\pi,\ 3\pi,\ 4\pi,\ 5\pi$	[M1]	
	For p^6 to be positive $\Rightarrow\cos(12\theta)>0$ $\Rightarrow 12\theta=2\pi,\ 4\pi$ $\theta=\dfrac{\pi}{6},\ \dfrac{\pi}{3}$	[A1]	
9(c) (i)	2 complex roots. Since $\mathrm{P}(z)$ has real coefficients and $z=1+\mathrm{i}$ is a complex root, its conjugate is another root. There cannot be a third complex root since $z=2$ is a real root.	[B1]	
(ii)	$\mathrm{P}(z)=\big(z-(1+\mathrm{i})\big)\big(z-(1-\mathrm{i})\big)(z-2)(az+c)$	[M1]	Accept $(z+c)$ Or $\mathrm{P}(z)=k\big(z-(1+\mathrm{i})\big)$ $\big(z-(1-\mathrm{i})\big)(z-2)(z+c)$
	$\mathrm{P}(\mathrm{i})=\big(\mathrm{i}-(1+\mathrm{i})\big)\big(\mathrm{i}-(1-\mathrm{i})\big)(\mathrm{i}-2)(a\mathrm{i}+c)=10+10\mathrm{i}$	[M1]	
	$(-1)(-1+2\mathrm{i})(\mathrm{i}-2)(a\mathrm{i}+c)=10+10i$ $(5\mathrm{i})(a\mathrm{i}+c)=10+10\mathrm{i}$ $-5a+5c\mathrm{i}=10+10\mathrm{i}$	[M1]	

	Comparing real and imaginary parts, $a=-2$ and $c=2$	[A1]	
	$\therefore \quad -2z+2=0 \;\Rightarrow z=1$ Hence the other 2 roots are $z=1-\mathrm{i}$ and $z=1$.	[A1]	

Qn	Solution	Marks	Remarks
10 (i)	$\begin{pmatrix}2\\3\\-12\end{pmatrix}\times\begin{pmatrix}1\\2\\-7\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix}$	[M1]	
	$\begin{pmatrix}1\\5\\-1\end{pmatrix}\bullet\begin{pmatrix}3\\2\\1\end{pmatrix}=12$	[M1]	
	Cartesian equation of plane $\pi: 3x+2y+z=12$	[A1]	
	At support beam S_1: $y=z=0 \Rightarrow$ vertex $A=(4,0,0)$	[M1]	AG
10 (ii)	Acute angle of inclination of wall $=\cos^{-1}\dfrac{\left\lvert\begin{pmatrix}0\\0\\1\end{pmatrix}\bullet\begin{pmatrix}3\\2\\1\end{pmatrix}\right\rvert}{\sqrt{9+4+1}}$	[M2]	M1 for using formula to find angle between 2 vectors M1 for $\begin{pmatrix}0\\0\\1\end{pmatrix}$
	$=74.498^\circ \simeq 74.5^\circ$ Since $74.5^\circ < 80^\circ$, the safety standard is met.	[A1]	
10 (iii)	$\overrightarrow{ON}=\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}$ for some λ	[B1]	
	$\Rightarrow \begin{pmatrix}3\lambda\\2\lambda\\\lambda\end{pmatrix}\bullet\begin{pmatrix}3\\2\\1\end{pmatrix}=12$ $\lambda=\dfrac{6}{7}$	[M1]	
	Point $N: \left(\dfrac{18}{7},\dfrac{12}{7},\dfrac{6}{7}\right)$	[A1]	
	Length of 4th support beam $=\dfrac{6}{7}\sqrt{9+4+1}$	[M1]	
	$=\dfrac{6\sqrt{14}}{7}$	[A1]	

Name:__________________________ Class:_____________

JURONG JUNIOR COLLEGE

JC2 Preliminary Examinations 2018

MATHEMATICS
Higher 2

Paper 2

9758/02

12 September 2018

3 hours

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

Section A: Pure Mathematics [40 marks]

1 Given that $\mathrm{f}(x)=\mathrm{e}^{\sin x}$, use the standard series to find the series expansion for $\mathrm{f}(x)$ in the form $a+bx+cx^2+dx^3$, where *a*, *b*, *c* and *d* are constants to be determined.

Hence show that the first three non-zero terms for the expansion of $\dfrac{1}{\left(\mathrm{e}^{\sin x}\right)^2}$ in ascending powers of *x* is $1-2x+2x^2$. [4]

The function $y=\mathrm{g}(x)$ satisfies $4\dfrac{\mathrm{d}y}{\mathrm{d}x}=\left(y+1\right)^2$ and $y=1$ at $x=0$.

(i) By further differentiation, find the series expansion for $\mathrm{g}(x)$, up to and including the term in x^3.

Hence show that when *x* is small,

$$\mathrm{g}\left(x\right)-\mathrm{f}\left(x\right)\approx\frac{1}{4}x^3. \qquad [5]$$

(ii) By using the result in **(i)**, justify whether $\mathrm{f}(x)$ is a good approximation to $\mathrm{g}(x)$ for values of *x* close to zero. [1]

2 In 2004, 10 000 cases of obesity among Singaporeans aged 18-30 years old were reported. Each year after that, the number of cases reported increased by 7%. If this pattern were to continue, how many obesity cases would be reported in 2018? Leave your answer to the nearest whole number. [3]

John, who was obese, started on a weight-loss programme. The number of calories he burned in the first week of his exercise regime was a. As the intensity of the exercise regime increased, the number of calories John burned each week was increased by *d*. On the other hand, the number of calories John consumed each week is a geometric sequence such that the numbers of calories he consumed in the first, second and third week equal the numbers of calories he burned through exercising in the seventh, third and first week respectively.

(i) Show that $d=\dfrac{a}{2}$. [2]

(ii) John burned 3000 calories in the seventh week through exercising. Find the least number of weeks required for the total number of calories John burned to exceed the total number of calories he consumed by at least 200 000. [5]

3 **(i)** Using the method of differences, find $\sum_{r=1}^{n} \frac{1}{r(r+1)}$. [3]

Hence find $\sum_{r=1}^{n} \left[3^{-r} - \frac{1}{r(r+1)} \right]$. [3]

(ii) Use your result in part **(i)** to show

$$\sum_{r=3}^{2N} \left[3^{1-r} - \frac{1}{r(r-1)} \right] = -\frac{1}{3} + \frac{1}{2}\left(\frac{1}{N} - \frac{1}{3^{2N-1}} \right).$$ [3]

Hence find $\sum_{r=3}^{\infty} \left[3^{1-r} - \frac{1}{r(r-1)} \right]$. [1]

4 **(i)** By using the substitution $x = \frac{1}{3}\sin^2\theta$, where $0 \le \theta < \frac{\pi}{2}$, find the exact value of

$$\int_0^{\frac{1}{4}} \sqrt{\frac{x}{1-3x}}\,\mathrm{d}x.$$ [5]

The region R is bounded by the curve $y = \sqrt{\frac{x}{1-3x}}$, the line $y = 1$ and the y-axis.

(ii) Using your answer in **(i)**, find the exact value of the area of R. [2]

(iii) Find the volume of revolution when R is rotated completely about the y-axis. Give your answer correct to 4 decimal places. [3]

Section B: Probability and Statistics [60 marks]

5 There are ten boys and twelve girls in a school table tennis club. A team of seven boys and seven girls will be selected randomly to represent the school in a table tennis friendly match.

(i) In how different ways can the team be formed? [2]

(ii) Jason is the youngest boy and Joyce is the youngest girl in the club. What is the probability that the team includes both Jason and Joyce? [2]

(iii) Joel is the oldest boy in the club. Given that Joel is selected for the team, what is the probability that the team includes Jason or Joyce, but not both? [4]

[Turn over

6 Alice and Betty each throw a fair cubical die simultaneously.

The random variable X is the larger number shown on the two dice or the common number of the dice if the numbers are equal.

(i) Show that $\mathrm{P}(X \le x) = \left(\dfrac{x}{6}\right)^2$, for $x = 1, 2, \ldots, 6$. [2]

(ii) Deduce that $\mathrm{P}(X = x) = \dfrac{2x-1}{36}$, for $x = 1, 2, \ldots, 6$. [1]

(iii) Show that $\mathrm{E}(X) = \dfrac{161}{36}$ and $\mathrm{Var}(X) = \dfrac{2555}{1296}$. [3]

(iv) Forty independent observations of X are taken. Using a suitable approximation, estimate the probability that the mean of these observations is at least 4.5. [3]

7 The number of employees, y, who stay back and continue to work in the office t minutes after 5 pm on a particular day in a company is recorded. The results are shown in the table.

t	15	30	45	60	75	90	105
y	30	19	15	13	12	11	10

(i) Draw a scatter diagram for these values, labeling the axes clearly. [1]

(ii) Find, correct to 4 decimal places, the product moment correlation coefficient between

(a) t and y,

(b) $\sqrt{t}$ and y,

(c) $\dfrac{1}{t}$ and y.

Hence, state with a valid reason, which of the above models is the most appropriate model of the relationship between t and y. [4]

(iii) Using the model you chose in part **(ii)**, find the equation for the relationship between t and y. [2]

(iv) Predict, to the nearest whole number, the number of employees who stay back and continue to work in the office at 7 pm on that particular day. Comment on the reliability of your prediction. [2]

8 A jar contains 10 blue and 8 red marbles. Five marbles are randomly drawn from the box, one by one and without replacement.

(i) Explain why it is inappropriate to model the number of blue marbles by a binomial distribution. [1]

(ii) Find the probability that exactly three marbles are blue. [3]

Another jar contains 20 blue and 12 red marbles. n marbles are randomly drawn from the box, one by one and with replacement. The number of red marbles drawn is denoted by R.

(iii) Given that the mean of R is 4.5, find n and $\mathrm{P}(R > 4)$. [3]

(iv) Given instead that $\mathrm{P}(R = 0 \text{ or } 1) < 0.01$, write down an inequality for n and find the least value of n. [3]

9 Durians and melons are sold by weight. The masses, in kg, of durians and melons are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean Mass	Standard Deviation
Durians	2.1	0.25
Melons	0.6	0.16

Durians are sold at \$15 per kg and melons at \$6 per kg.

(i) Find the probability that the mass of a randomly chosen durian is less than four times the mass of a randomly chosen melon. [3]

(ii) Two durians and eight melons are randomly selected. Find the probability that the average mass of these ten fruits exceeds 1 kg. [4]

(iii) Find the probability that the total selling price of a randomly chosen durian and a randomly chosen melon is less than \$40. [4]

(iv) Without any further calculation, explain why the probability of the event that both a randomly chosen durian has a selling price less than \$35 and a randomly chosen melon has a selling price less than \$5 is less than the answer to part **(iii)**. [1]

[Turn over

10 There was a complaint that the average waiting time for a patient to see a doctor in a local polyclinic is longer than 60 minutes. A public relation officer in the polyclinic investigated the waiting times, x minutes, for 70 randomly chosen patients. The data are summarised by

$$\sum(x-50)=1071, \quad \sum(x-50)^2=73158.$$

(i) Explain whether the public relation officer should use a 1-tail or a 2-tail test. [1]

(ii) Explain why the public relation officer is able to carry out a hypothesis test without knowing anything about the population distribution of the waiting times for the patients to see a doctor. [1]

(iii) Find unbiased estimates of the population mean and variance. [2]

(iv) Test, at the 5% significance level, whether the complaint is valid. [4]

In another test, using the same set of data and also at the 5% significance level, the hypotheses are as follows:

H_0 : the population mean waiting time is equal to k minutes.
H_1 : the population mean waiting time is not equal to k minutes.

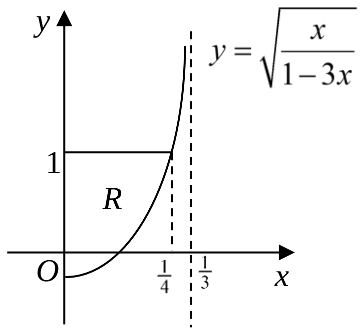
(v) Given that the null hypothesis is rejected in favour of the alternative hypothesis, find the set of possible values of k. [4]

Qn	Solution
1	$\mathrm{f}(x) = \mathrm{e}^{\sin x}$ $= \mathrm{e}^{x-\frac{x^3}{3!}+\ldots}$ $=1+\left(x-\frac{x^3}{3!}\right)+\frac{\left(x-\frac{x^3}{3!}\right)^2}{2!}+\frac{\left(x-\frac{x^3}{3!}\right)^3}{3!}+\ldots..$ $=1+x-\frac{x^3}{6}+\frac{x^2}{2}+\frac{x^3}{6}+\ldots$ $=1+x+\frac{x^2}{2}+\ldots.$ $\therefore\ a=1,\ b=1,\ c=\frac{1}{2}$ and $d=0$.
	$\frac{1}{\left(\mathrm{e}^{\sin x}\right)^2}\approx\left(1+x+\frac{x^2}{2}\right)^{-2}$
	$=1+(-2)\left(x+\frac{x^2}{2}\right)+\frac{(-2)(-3)}{2!}\left(x+\frac{x^2}{2}\right)^2+\ldots..$ $\approx 1-2x+2x^2$
(i)	$4\frac{\mathrm{d}y}{\mathrm{d}x}=(y+1)^2$ Differentiating with respect to x, $4\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2(y+1)\frac{\mathrm{d}y}{\mathrm{d}x}$ $4\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=2(y+1)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$
	Sub $x=0$, $y=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{3}{2}$
	Using Maclaurin's formula, $\mathrm{g}(x)=1+x+\frac{x^2}{2!}+\left(\frac{3}{2}\right)\frac{x^3}{3!}+\ldots..$
	$\mathrm{g}(x)=1+x+\frac{x^2}{2}+\frac{x^3}{4}+\ldots..$
	$\mathrm{g}(x)-\mathrm{f}(x)=\left(1+x+\frac{x^2}{2}+\frac{x^3}{4}+\ldots..\right)-\left(1+x+\frac{1}{2}x^2+\ldots..\right)$ $\approx\frac{x^3}{4}$
(ii)	As $x\to 0$, $\mathrm{g}(x)-\mathrm{f}(x)\approx\frac{1}{4}x^3\to 0$. Therefore, $\mathrm{f}(x)$ is a good approximation to $\mathrm{g}(x)$ for values of x close to zero.

2	GP: $a=10000,\ r=1.07$
	$U_{15}=10000(1.07)^{14}$
	$U_{15}=25785.34\approx 25785$
(i)	First three terms of G.P.: $a+6d,\ a+2d,\ a$ $\Rightarrow \dfrac{a+2d}{a+6d}=\dfrac{a}{a+2d}$
	$a^2+4ad+4d^2=a^2+6ad$ $4d^2=2ad$ $d\neq 0\Rightarrow d=\dfrac{a}{2}$
(ii)	Given $a+6d=3000$, where $d=\dfrac{a}{2}$ from (i) $a+3a=3000\Rightarrow a=\dfrac{3000}{4}=750$
	Total calories loss $S_n=\dfrac{n}{2}\left[2(750)+(n-1)(375)\right]$ $=\dfrac{375n}{2}(3+n)$
	G.P. $U_1=3000,\ r=\dfrac{a}{a+2\left(\dfrac{a}{2}\right)}=\dfrac{1}{2}$ Total calories gain $S_n=\dfrac{3000\left(1-\left(\dfrac{1}{2}\right)^n\right)}{1-\dfrac{1}{2}}=6000\left(1-\left(\dfrac{1}{2}\right)^n\right)$
	$\dfrac{375n}{2}(3+n)-6000\left(1-\left(\dfrac{1}{2}\right)^n\right)\geq 200000$ $\dfrac{3n}{16}(3+n)-6\left(1-\left(\dfrac{1}{2}\right)^n\right)\geq 200$
	From GC, $n\geq 31.68$ (or 32) Least number of weeks $=32$.

3(i)	$$\sum_{r=1}^{n}\frac{1}{r(r+1)}=\sum_{r=1}^{n}\left[\frac{1}{r}-\frac{1}{(r+1)}\right]$$ $$=\left\{\begin{array}{l}\frac{1}{1}-\cancel{\frac{1}{2}}\\+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\\+\cancel{\dots}\\+\cancel{\frac{1}{n-1}}-\cancel{\frac{1}{n}}\\+\cancel{\frac{1}{n}}-\frac{1}{n+1}\end{array}\right\}$$ $$=1-\frac{1}{n+1}$$
	$$\sum_{r=1}^{n}\left[3^{-r}-\frac{1}{r(r+1)}\right]=\sum_{r=1}^{n}3^{-r}-\sum_{r=1}^{n}\frac{1}{r(r+1)}$$ $$=\frac{\frac{1}{3}\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}}-\left(1-\frac{1}{n+1}\right)$$ $$=\frac{1}{2}\left(1-\frac{1}{3^n}\right)-1+\frac{1}{n+1}$$ $$=-\frac{1}{2}\left(\frac{1}{3^n}\right)+\frac{1}{n+1}-\frac{1}{2}$$
(ii)	$$\sum_{r=3}^{2N}\left[3^{1-r}-\frac{1}{r(r-1)}\right]=\sum_{r=2}^{2N-1}\left[3^{-r}-\frac{1}{r(r+1)}\right]$$ $$=\sum_{r=1}^{2N-1}\left[3^{-r}-\frac{1}{r(r+1)}\right]-\left(\frac{1}{3}-\frac{1}{2}\right)$$ $$=-\frac{1}{2}\left(\frac{1}{3^{2N-1}}\right)+\frac{1}{2N}-\frac{1}{2}+\frac{1}{6}$$ $$=-\frac{1}{3}+\frac{1}{2}\left(\frac{1}{N}-\frac{1}{3^{2N-1}}\right)\quad\text{[Shown]}$$
	$$\sum_{r=3}^{\infty}\left[3^{1-r}-\frac{1}{r(r-1)}\right]=\lim_{N\to\infty}\left[-\frac{1}{3}+\frac{1}{2}\left(\frac{1}{N}-\frac{1}{3^{2N-1}}\right)\right]=-\frac{1}{3}$$

4(i)	$x=\frac{1}{3}\sin^2\theta \quad\Rightarrow\quad \frac{\mathrm{d}x}{\mathrm{d}\theta}=\frac{2}{3}\sin\theta\cos\theta$ When x = 0, θ = 0; when $x=\frac{1}{4}$, $\theta=\frac{\pi}{3}$. $$\int_0^{\frac{1}{4}}\sqrt{\frac{x}{1-3x}}\,\mathrm{d}x=\int_0^{\frac{\pi}{3}}\sqrt{\frac{\frac{1}{3}\sin^2\theta}{\cos^2\theta}}\left(\frac{2}{3}\sin\theta\cos\theta\,\mathrm{d}\theta\right)$$

	$= \dfrac{2}{3\sqrt{3}}\int_0^{\frac{\pi}{3}} \sin^2\theta \, \mathrm{d}\theta$ $= \dfrac{2}{3\sqrt{3}}\int_0^{\frac{\pi}{3}} \dfrac{1-\cos 2\theta}{2} \, \mathrm{d}\theta$
	$= \dfrac{1}{3\sqrt{3}}\left[\theta - \dfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}}$
	$= \dfrac{1}{3\sqrt{3}}\left[\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}\right]$
(ii)	
	Area of $R = \dfrac{1}{4} - \dfrac{1}{3\sqrt{3}}\left[\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}\right]$ $= \dfrac{1}{3} - \dfrac{\pi}{9\sqrt{3}}$
(iii)	$y = \sqrt{\dfrac{x}{1-3x}} \quad \Rightarrow y^2 = \dfrac{x}{1-3x}$ $y^2 - 3xy^2 = x$ $x(1+3y^2) = y^2$ $x = \dfrac{y^2}{1+3y^2}$
	Volume $= \pi\displaystyle\int_0^1 \left(\dfrac{y^2}{1+3y^2}\right)^2 \mathrm{d}y$ $= 0.0761$ (4 dp)

5(i)	No. of different ways $= {}^{10}\mathrm{C}_7 \times {}^{12}\mathrm{C}_7$
	$= 95040$
(ii)	No. of teams including Jason and Joyce $= {}^{9}\mathrm{C}_6 \times {}^{11}\mathrm{C}_6$

	Required probability $= \dfrac{{}^{9}C_6 \times {}^{11}C_6}{95040}$ $= \dfrac{38808}{95040}$ $= \dfrac{49}{120}$ or 0.408
(iii)	No. of teams including Joel $= {}^{9}C_6 \times {}^{12}C_7$ $= 66528$
	No. of teams including Joel and Jason but not Joyce $= {}^{8}C_5 \times {}^{11}C_7$ $= 18480$ No. of teams including Joel and Joyce but not Jason $= {}^{8}C_6 \times {}^{11}C_6$ $= 12936$
	Required probability $= \dfrac{18480 + 12936}{66528}$
	$= \dfrac{17}{36}$ or 0.472

6(i)	Table of outcomes: (see table below) P(one number $\le x$) = $\dfrac{x}{6}$
	$P(X \le x) = $ P(both numbers $\le x$), $x = 1, 2, \dots, 6$. $= P(X_1 = 1, 2, \dots x \;\&\; X_2 = 1, 2, \dots x)$ $= \left(\dfrac{x}{6}\right)\left(\dfrac{x}{6}\right)$ $= \left(\dfrac{x}{6}\right)^2$
(ii)	$P(X = x) = P(X \le x) - P(X \le x-1)$ $= \left(\dfrac{x}{6}\right)^2 - \left(\dfrac{x-1}{6}\right)^2$ $= \dfrac{2x-1}{36}$

X_1 \ X_2	**1**	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

(iii)

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$\mathrm{E}(X) = \sum_{\text{all } r} x\mathrm{P}(X = x)$$

$$= 1\times\frac{1}{36}+2\times\frac{3}{36}+3\times\frac{5}{36}+4\times\frac{7}{36}+5\times\frac{9}{36}+6\times\frac{11}{36} = \frac{161}{36} \quad \text{[Shown]}$$

$$\mathrm{E}(X^2) = \sum_{\text{all } r} x^2\mathrm{P}(X = x)$$

$$=1^2\times\frac{1}{36}+2^2\times\frac{3}{36}+3^2\times\frac{5}{36}+4^2\times\frac{7}{36}+5^2\times\frac{9}{36}+6^2\times\frac{11}{36}$$

$$=\frac{791}{36}$$

$$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$$

$$=\frac{791}{36}-\left(\frac{161}{36}\right)^2$$

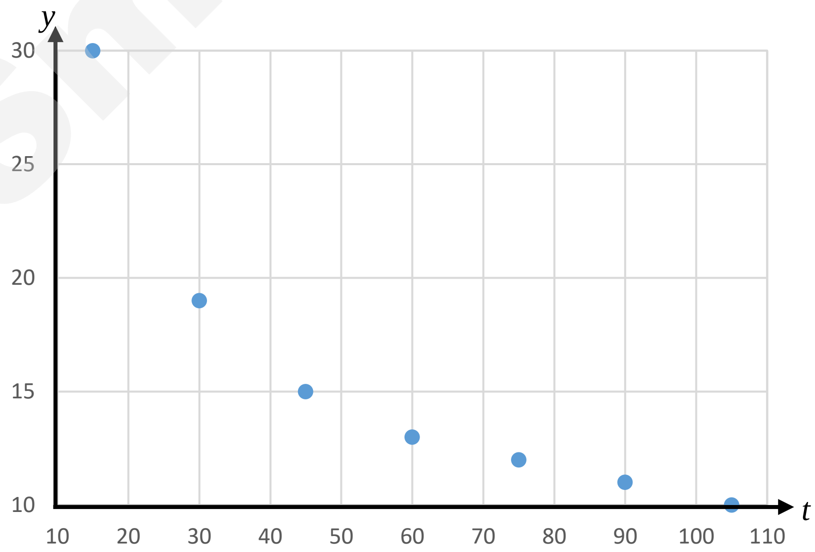
$$=\frac{2555}{1296} \quad \text{[Shown]}$$

(iv) Since $n = 40$ is large, by Central Limit Theorem,

$\overline{X} \sim \mathrm{N}\left(\frac{161}{36}, \frac{2555}{1296\times 40}\right)$ approximately.

$\mathrm{P}(\overline{X} \geq 4.5) = 0.45021 \approx 0.450$ (3 sig figs)

7(i)



(ii)	From G.C., (a) $\underline{\underline{r = -0.8745 \quad \text{(ans)}}}$ (b) $\underline{\underline{r = -0.9288 \quad \text{(ans)}}}$ (c) $\underline{\underline{r = 0.9993 \quad \text{(ans)}}}$
	Model (c) is the most appropriate model for the relationship between *t* and *y* since its value of $\|r\|$ is closest to1.
(iii)	From G.C., $y = 7.2048 + 344.60\left(\frac{1}{t}\right)$ $\underline{\underline{y = 7.20 + 345\left(\frac{1}{t}\right) \quad \text{(3s.f.)(ans)}}}$
(iv)	When $t = 120$, $y = 7.2048 + 344.60\left(\frac{1}{120}\right) = 10.076$ $\underline{\underline{y \simeq 10 \quad \text{(ans)}}}$
	Since $t = 120$ is outside the given range of the values of *t*, this is an extrapolation and thus the prediction may not be reliable.
8(i)	It is inappropriate to model the number of blue marbles by a binomial distribution because the marbles are drawn without replacement, the colour of the marbles depends on that of the previous draw.
(ii)	Required probability $= \frac{10}{18} \times \frac{9}{17} \times \frac{8}{16} \times \frac{8}{15} \times \frac{7}{14} \times \frac{5!}{3! \times 2!}$ or $\frac{{}^{10}C_3 \times {}^{8}C_2}{{}^{18}C_5}$
	$= \frac{20}{51}$ or 0.392
(iii)	$R \sim \mathrm{B}\left(n, \frac{12}{32}\right) = \mathrm{B}\left(n, \frac{3}{8}\right)$ E(R) = 4.5 $\Rightarrow \frac{3}{8}n = 4.5$ $n = 12$
	P($R > 4$) = 1 – P($R \le 4$) = 0.48972 $\approx$ 0.490 (3 sig figs)
(iv)	$R \sim \mathrm{B}\left(n, \frac{3}{8}\right)$ P(R = 0 or 1) < 0.01 $\Rightarrow \mathrm{P}(R = 0) + \mathrm{P}(R = 1) < 0.01$
	$\Rightarrow \left(\frac{5}{8}\right)^n + n\left(\frac{3}{8}\right)\left(\frac{5}{8}\right)^{n-1} < 0.01$
	From GC, least n = 15

9(i)	Let D = Mass of a durian and R = Mass of a melon. Then $D \sim \mathrm{N}(2.1, 0.25^2)$ and $R \sim \mathrm{N}(0.6, 0.16^2)$. $D-4R \sim \mathrm{N}\left(2.1-4(0.6),\ 0.25^2+4^2\times 0.16^2\right)$
	$D-4R \sim \mathrm{N}\left(-0.3, 0.4721\right)$
	$\mathrm{P}\left(D-4R<0\right) \approx 0.66881 = 0.669$ (to 3 sig figs)
(ii)	Let $M = \dfrac{D_1+D_2+R_1+R_2+\cdots+R_8}{10}$ $\mathrm{E}(M) = \dfrac{1}{10}[2(2.1)+8(0.6)] = 0.9$ $\mathrm{Var}(M) = \dfrac{1}{10^2}[2(0.25)^2+8(0.16)^2] = 0.003298$
	$M \sim \mathrm{N}\left(0.9, 0.003298\right)$
	$\mathrm{P}(M>1) = 0.040815 = 0.0408$ (to 3 sig figs)
(iii)	Let $S = 15D + 6R$ $15D \sim \mathrm{N}\left(15(2.1),\ 15^2\times 0.25^2\right)$ and $6R \sim \mathrm{N}\left(6(0.6),\ 6^2\times 0.16^2\right)$
	$S \sim \mathrm{N}\left(31.5+3.6,\ 14.0625+0.9216\right) = \mathrm{N}\left(35.1,\ 14.9841\right)$
	$\mathrm{P}(S<40) = 0.89722 = 0.897$
(iv)	Event in **part (iii)** includes the event in **part (iv)** plus some other cases. [For example, the case where $15D < 33$ and $6R < 7$ is included in **(iii)** but not in **(iv)**.]

10(i)	A 1-tail test should be used because he is investigating for an average time longer than 60 minutes
(ii)	Since the sample size is large, the public relation officer can apply the Central Limit Theorem to approximate the distribution of the sample mean ($\overline{X}$) by a normal distribution to conduct a hypothesis test.
(iii)	unbiased estimate of population mean $= \bar{x}$ $= 50+\dfrac{1071}{70}$ $= \dfrac{653}{10}$
	Unbiased estimate of population variance $= s^2$ $= \dfrac{1}{69}\left(73158-\dfrac{1071^2}{70}\right)$ $= \dfrac{189239}{230}$

(iv)

Let X be the waiting time for a randomly chosen patient to see a doctor
Let μ be the population mean of waiting time of patients to see a doctor

$H_0 : \mu = 60$

$H_1 : \mu > 60$

Under H_0, since n = 70 is large, by the Central Limit Theorem,

$\bar{X} \sim N\left(60, \dfrac{189239}{230(70)}\right)$ approximately .

Test Statistic : $Z = \dfrac{\bar{X} - 60}{\sqrt{\dfrac{189239}{230(70)}}} \sim N(0,1)$ approximately

Level of significance: 5%

By using G.C.,
$p-\text{value} = 0.0611$ (3 s.f)

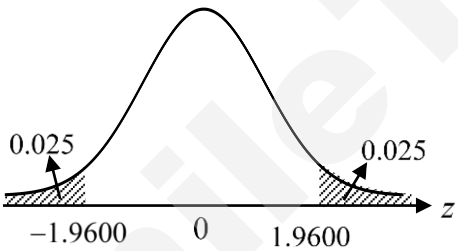
Since p-value = 0.0611 > 0.05, we do not reject H_0 at the 5% level of significance and conclude that there is insufficient evidence that the population mean waiting time is longer than 60 minutes. i.e. The complaint is not valid.

(v)

$H_0 : \mu = k$

$H_1 : \mu \neq k$

Level of significance: 5%



For H_0 to be rejected,
$z \leq -1.9600$ or $z \geq 1.9600$

$\dfrac{\bar{x} - k}{s/\sqrt{70}} \leq -1.9600$ or $\dfrac{\bar{x} - k}{s/\sqrt{70}} \geq 1.9600$

$\dfrac{65.3 - k}{\sqrt{822.778/70}} \leq -1.9600$ or $\dfrac{65.3 - k}{\sqrt{822.778/70}} \geq 1.9600$

$65.3 - k \leq -6.7197$ or $65.3 - k \geq 6.7197$

$\Rightarrow$ $k \leq 58.6$ or $k \geq 72.0$ (ans)

MERIDIAN JUNIOR COLLEGE
JC2 Preliminary Examination
Higher 2

H2 Mathematics 9758/01

Paper 1 **14 September 2018**

3 Hours

Additional Materials: Writing paper
Graph Paper
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 Express $\dfrac{-x^2+11x-11}{x^2-4x+4}+3$ as a single simplified fraction.

Hence, without using a calculator, solve the inequality

$$\frac{x^2-11x+11}{x^2-4x+4}<3.$$ [5]

2 **(a)** Interpret geometrically what $\mathbf{a}\times\mathbf{b}$ means, given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. [1]

(b) Show that a formula for the area of triangle OAB can be given as $k\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|$, where k is a constant to be determined. [2]

Hence give the geometrical meaning of $\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|$ in relation to an appropriate quadrilateral. [1]

3 **(i)** Show that $1-\mathrm{e}^{\mathrm{i}n\theta}=\mathrm{e}^{\frac{1}{2}\mathrm{i}n\theta}\left(-2\mathrm{i}\sin\dfrac{n\theta}{2}\right)$, and $1+\mathrm{e}^{\mathrm{i}n\theta}=\mathrm{e}^{\frac{1}{2}\mathrm{i}n\theta}\left(2\cos\dfrac{n\theta}{2}\right)$ where $n\in\mathbb{R}$.

[2]

(ii) It is given that $z=\mathrm{e}^{\mathrm{i}\theta}$, where $0\le\theta\le\dfrac{\pi}{2}$, using **(i)**, show that

$$\left|1-z+z^2-z^3\right|=4\sin\frac{\theta}{2}\cos\theta.$$ [4]

4 Find

(a) $\int x\sqrt{5-x^2}\ \mathrm{d}x$, [2]

(b) $\int \sin(\ln x)\ \mathrm{d}x$ where $x>0$, [3]

(c) the exact value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{12}}\cos x\left|\sin x\right|\ \mathrm{d}x$. [4]

5 It is given that

$$\mathrm{f}: x \mapsto \left|\frac{1}{x-3}\right|, \text{ where } x \in \mathbb{R}, x \neq 3,$$
$$\mathrm{g}: x \mapsto \ln x, \text{ where } x \in \mathbb{R}, x > 0.$$

(i) Explain why the composite function gf exists and find gf in a similar form. [3]

(ii) Explain why f does not have an inverse. [1]

(iii) If the domain of f is further restricted to $x < k$, state the maximum value of k such that f^{-1} will exist. Hence find f^{-1} in a similar form. [4]

(iv) State the geometrical relationship between f and f^{-1}. [1]

6 **(a)** **(i)** Using standard series from the List of Formulae (MF26), find the first three non-zero terms of the Maclaurin's series for $y = \dfrac{1}{\sqrt{1+\ln(1+3x)}}$. [3]

(ii) Deduce the approximate value of $\int_0^1 y \, \mathrm{d}x$. Explain why the approximation is not good. [2]

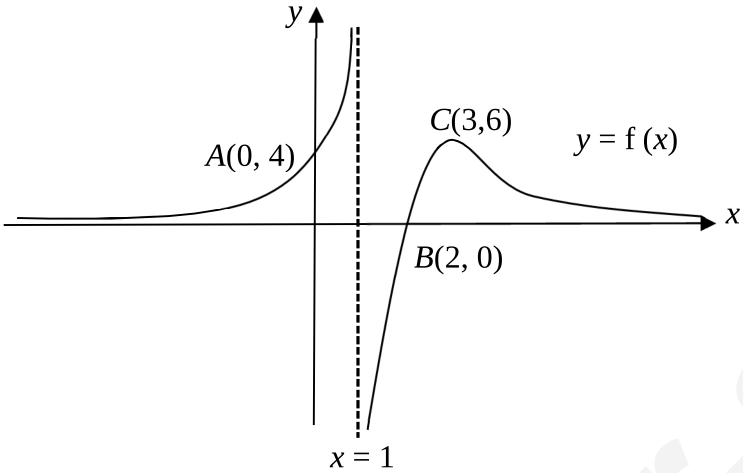
(iii) State the equation of the tangent to the curve $y = \dfrac{1}{\sqrt{1+\ln(1+3x)}}$ at $x = 0$. [1]

(b) Show that, when x is sufficiently small for x^3 and higher powers of x to be neglected,

$$\frac{\cos 2x}{1-\sin x} \approx a + bx + cx^2,$$

where a, b and c are constants to be determined. [3]

7 **(a)**



The diagram above shows the curve $y = \mathrm{f}(x)$ with a maximum point at $C(3, 6)$. The curve crosses the axes at the points $A(0, 4)$ and $B(2, 0)$. The lines $x = 1$ and $y = 0$ are the asymptotes of the curve.

Sketch on separate diagrams, the graphs of

(i) $y = \mathrm{f}'(x)$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

stating clearly, where applicable, the equations of the asymptotes, the axial intercepts and the coordinates of the points corresponding to A, B and C.

(b) Show that the equation $y = \dfrac{3x^2 - 6x}{x - 1}$ can be written as $y = A(x-1) + \dfrac{B}{(x-1)}$, where A and B are constants to be found. Hence state a sequence of transformations that will transform the graph of $y = \dfrac{1}{x} - x$ to the graph of $y = \dfrac{3x^2 - 6x}{x - 1}$. [4]

8 The points A and B have position vectors $-2\mathbf{i}-7\mathbf{j}+3\mathbf{k}$ and $\mathbf{j}+\mathbf{k}$ respectively. The plane p has equation $x-y=5$.

(i) Find a vector equation of line l passing through the points A and B. [2]

(ii) Find the acute angle between l and p. [2]

(iii) Verify that the point A lies on the plane p. Given that the point C is the reflection of the point B in the plane p, describe the shape formed by the points A, B and C. [2]

(iv) Find a vector equation of the line which is a reflection of the line l in the plane p. [4]

9 An analyst is studying how the population of bluegill fish in a lake changes over time. By considering their natural birth and death rates, he found that the rate of change of the bluegill fish population is proportional to $6x-x^2$, where x is the population, in thousands, of bluegill fish in the lake at time t years. It is given that the initial population of the bluegill fish in the lake is 12000 and there were 8600 bluegill fish after 1 month.

(i) Show that $x=\dfrac{12\mathrm{e}^{12t\ln\frac{43}{26}}}{2\mathrm{e}^{12t\ln\frac{43}{26}}-1}$. [6]

(ii) Find the time taken for the bluegill fish population to decrease to 75% of its initial population. Leave your answer in years to 3 significant figures. [2]

(iii) Deduce the long term implication on the population of bluegill fish in a lake following this model and state an assumption for this model to hold in the long term. [3]

10 **(a)** The sum of the first n terms of a sequence $\{u_n\}$ is given by $S_n = kn^2 - 3n$, where k is a non-zero real constant.

(i) Prove that the sequence $\{u_n\}$ is an arithmetic sequence. [3]

(ii) Given that u_2, u_3 and u_6 are consecutive terms in a geometric sequence, find the value of k. [3]

(b) A zoology student observes jaguars preying on white-tailed deer in the wild. He observes that when a jaguar spots its prey from a distance of d m away, it starts its chase. At the same time, the white-tailed deer senses danger and starts escaping.

He models the predator-prey movements as follows:

The jaguar starts its chase with a leap distance of 6 m. Subsequently, each leap covers a distance of 0.1 m less than its preceding leap.

The white-tailed deer starts its escape with a leap distance of 9 m. Subsequently, each leap covers a distance of 5% less than its preceding leap.

(i) Find the total distance travelled by a white-tailed deer after n leaps. Deduce the maximum distance travelled by a white-tailed deer. [3]

(ii) Assume that both predator and prey complete the same number of leaps in the same duration of time. Given that d = 11 m, find the least value of n for a jaguar to catch a white-tailed deer within n leaps. [3]

11 A curve C has parametric equations

$$x = 1 - \cos^3\theta, \qquad y = 1 - 3\sin\theta\cos^2\theta, \qquad \text{for } 0 \le \theta \le \frac{1}{2}\pi.$$

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\tan\theta - \cot\theta$. [3]

(ii) Show that the exact coordinates of the turning point is at $\left(1-\left(\sqrt{\dfrac{2}{3}}\right)^3,\ 1-\dfrac{2}{\sqrt{3}}\right)$ and explain why it is a minimum. [6]

(iii) Show that the equation of the normal to the curve at $\theta = \dfrac{\pi}{6}$ is $y = 1.73205x - 0.73205$ (rounded off to 5 decimal places) and hence evaluate the area of the region bounded by C, the y-axis and the normal to the curve at $\theta = \dfrac{\pi}{6}$. [6]

End of Paper

MJC
2018 H2 MATH (9758/01) JC 2 PRELIM
SUGGESTED SOLUTIONS

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>1</td><td>Equations and Inequalities</td></tr>
<tr><td></td><td>

$$\frac{-x^2+11x-11}{x^2-4x+4}+3$$
$$=\frac{-x^2+11x-11+3\left(x^2-4x+4\right)}{x^2-4x+4}$$
$$=\frac{-x^2+11x-11+3x^2-12x+12}{x^2-4x+4}$$
$$=\frac{2x^2-x+1}{x^2-4x+4}$$

</td></tr>
<tr><td></td><td>

$$\frac{x^2-11x+11}{x^2-4x+4}<3$$
$$\frac{-x^2+11x-11}{x^2-4x+4}+3>0$$
$$\frac{2x^2-x+1}{\left(x-2\right)^2}>0$$

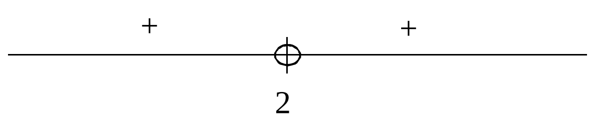
<u>Method 1</u>

Since the discriminant of $2x^2-x+1$ is $\left(-1\right)^2-4\left(2\right)\left(1\right)=-7<0$, and the coefficient of $x^2=2$ is positive, $2x^2-x+1>0$ for all $x\in\mathbb{R}$.

<u>Method 2</u>

$2x^2-x+1=2\left[\left(x^2-\frac{1}{2}x+\frac{1}{2}\right)\right]=2\left(x-\frac{1}{4}\right)^2+\frac{7}{8}>0$ for all $x\in\mathbb{R}$.

Therefore, solving $\frac{1}{\left(x-2\right)^2}>0$, we have $x\in\mathbb{R},\ x\neq 2.$



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</table>

Qn	Solution
2	**Vectors**
(a)	$\mathbf{a} \times \mathbf{b}$ is a vector that is perpendicular to <u>both</u> **a** and **b.** **Other possible answers:** Normal vector of a plane that is <u>parallel</u> to both **a** and **b (**no marks for normal vector of a plane containing vectors **a** and **b)**
(b)	Area of triangle *OAB* $= 0.5\left(\left\lvert\overrightarrow{OA}\right\rvert\right)\left(\left\lvert\overrightarrow{OB}\right\rvert\right)\sin\angle AOB$ $= 0.5\left\lvert\overrightarrow{OA}\times\overrightarrow{OB}\right\rvert$
	$\left\lvert\overrightarrow{OA}\times\overrightarrow{OB}\right\rvert$ is the area of a parallelogram with <u>*OA* and *OB* as its adjacent sides</u>. **Alternative:** $\left\lvert\overrightarrow{OA}\times\overrightarrow{OB}\right\rvert$ is area of parallelogram *OACB* (and include a diagram with *O*,*A*, a fourth point *C,* and *B,* with the 2 parallel sides also indicated clearly in diagram). Cannot accept $\left\lvert\overrightarrow{OA}\times\overrightarrow{OB}\right\rvert$ is area of sum of two triangles, because question asked for geometrical meaning in relation to an appropriate quadrilateral.

Qn	Solution
3	**Complex Numbers**
(i)	$1-e^{in\theta}$ $=e^{\frac{1}{2}in\theta}\left(e^{-\frac{1}{2}in\theta}-e^{\frac{1}{2}in\theta}\right)$ $=e^{\frac{1}{2}in\theta}\left(\cos\frac{n\theta}{2}-i\sin\frac{n\theta}{2}-\left(\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}\right)\right)$ $=e^{\frac{1}{2}in\theta}\left(-2i\sin\frac{n\theta}{2}\right)$ $1+e^{in\theta}$ $=e^{\frac{1}{2}in\theta}\left(e^{-\frac{1}{2}in\theta}+e^{\frac{1}{2}in\theta}\right)$ $=e^{\frac{1}{2}in\theta}\left(\cos\frac{n\theta}{2}-i\sin\frac{n\theta}{2}+\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}\right)$ $=e^{\frac{1}{2}in\theta}\left(2\cos\frac{n\theta}{2}\right)$
(ii)	**Method 1** $\left\|1-z+z^2-z^3\right\|$ $=\left\|(1-z)+(1-z)z^2\right\|$ $=\left\|(1-z)(1+z^2)\right\|$ $=\left\|1-z\right\|\left\|1+z^2\right\|$ $=\left\|e^{\frac{1}{2}i\theta}\left(-2i\sin\frac{\theta}{2}\right)\right\|\left\|e^{i\theta}(2\cos\theta)\right\|$ $=2\sin\frac{\theta}{2}(2\cos\theta)$ $\quad\left(\because \left\|e^{\frac{1}{2}i\theta}\right\|=1,\ \left\|e^{i\theta}\right\|=1, \text{ and } 0\le\theta\le\frac{\pi}{2}\Rightarrow\sin\frac{\theta}{2}>0 \text{ and } \cos\theta>0\right)$ $=4\sin\frac{\theta}{2}\cos\theta$

Method 2

$$1-z+z^2-z^3=\frac{1-(-z)^4}{1-(-z)}=\frac{1-z^4}{1+z}=\frac{1-e^{i4\theta}}{1+e^{i\theta}}$$

Using **(i)**,

$1-z+z^2-z^3$

$$=\frac{1-e^{i4\theta}}{1+e^{i\theta}}$$

$$=\frac{e^{i2\theta}\left(-2i\sin 2\theta\right)}{e^{i\frac{\theta}{2}}\left(2\cos\frac{\theta}{2}\right)}$$

$$=\frac{e^{i\frac{3\theta}{2}}\left(-2i\sin\theta\cos\theta\right)}{\left(\cos\frac{\theta}{2}\right)}$$

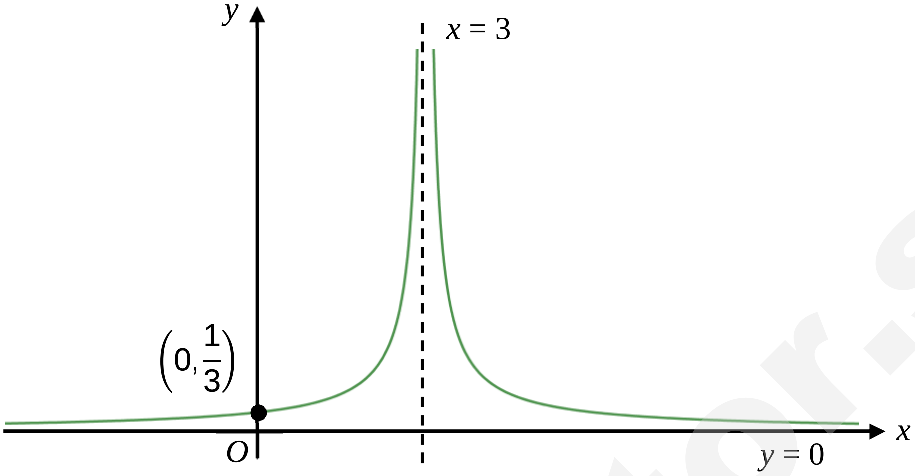
$$=\frac{e^{i\frac{3\theta}{2}}\left(-4i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta\right)}{\left(\cos\frac{\theta}{2}\right)}$$

$$=e^{i\frac{3\theta}{2}}\left(-4i\sin\frac{\theta}{2}\cos\theta\right)$$

Since $\left|e^{i\frac{3\theta}{2}}\right|=1$, and $0\le\theta\le\frac{\pi}{2}\Rightarrow\sin\frac{\theta}{2}>0$ and $\cos\theta>0$

$\left|1-z+z^2-z^3\right|=4\sin\frac{\theta}{2}\cos\theta$ (shown).

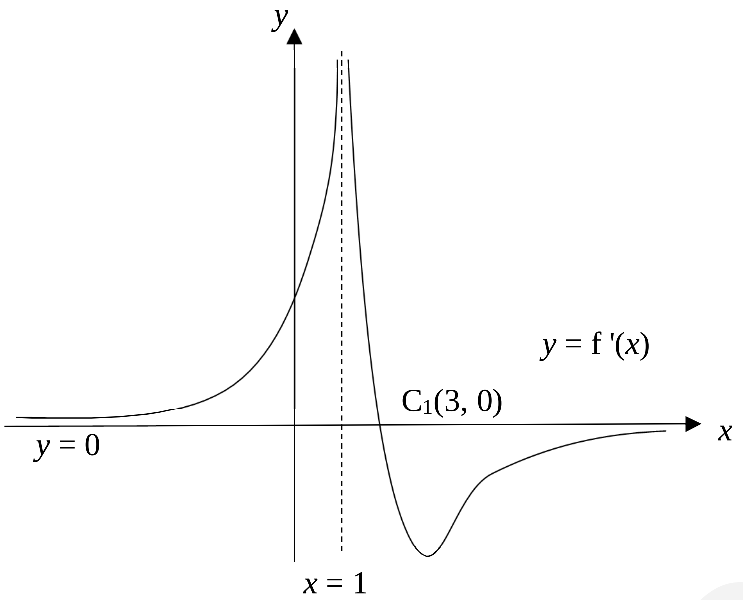
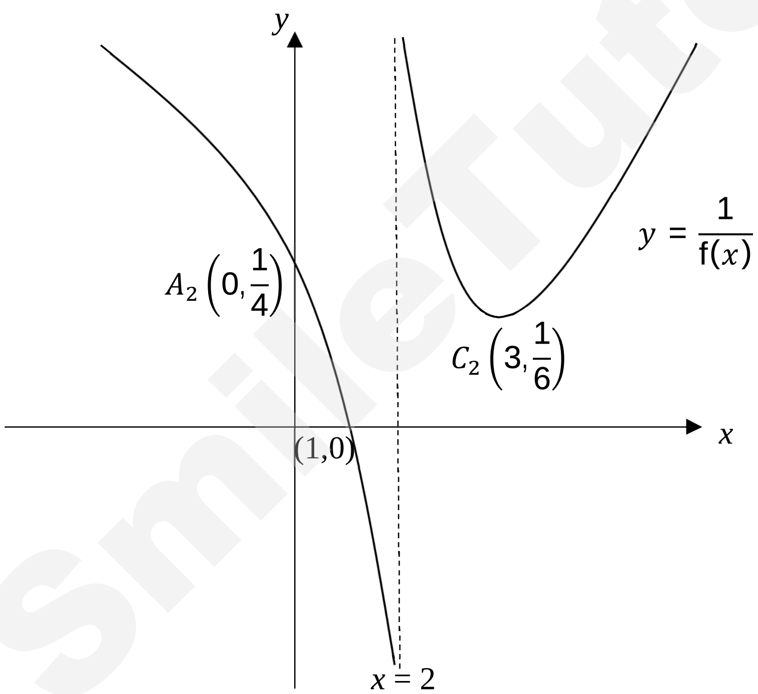
Qn	Solution
4	**Techniques of integration**
(a)	$\int x\sqrt{5-x^2}\ \mathrm{d}x = -\frac{1}{2}\int -2x\sqrt{5-x^2}\ \mathrm{d}x$ $= -\frac{1}{2}\frac{\left(5-x^2\right)^{\frac{3}{2}}}{\frac{3}{2}} + c$ $= -\frac{1}{3}\left(5-x^2\right)^{\frac{3}{2}} + c$
(b)	$\int \sin(\ln x)\ \mathrm{d}x$ $= \int (1)\left(\sin(\ln x)\right)\ \mathrm{d}x$ $= x\sin(\ln x) - \int x\cos(\ln x)\frac{1}{x}\ \mathrm{d}x$ $= x\sin(\ln x) - \int \cos(\ln x)\ \mathrm{d}x$ $= x\sin(\ln x) - \left[x\cos(\ln x) + \int x\sin(\ln x)\frac{1}{x}\ \mathrm{d}x\right]$ $= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\ \mathrm{d}x$ $\Rightarrow 2\int \sin(\ln x)\ \mathrm{d}x = x\sin(\ln x) - x\cos(\ln x)$ $\Rightarrow \int \sin(\ln x)\ \mathrm{d}x = \frac{1}{2}\left[x\sin(\ln x) - x\cos(\ln x)\right] + c$
(c)	$\int_{-\frac{\pi}{4}}^{\frac{\pi}{12}} \cos x\lvert\sin x\rvert \mathrm{d}x$ $= \int_{-\frac{\pi}{4}}^{0} \cos x(-\sin x)\mathrm{d}x + \int_{0}^{\frac{\pi}{12}} \cos x(\sin x)\mathrm{d}x$ $= -\frac{1}{2}\int_{-\frac{\pi}{4}}^{0} 2\sin x\cos x\mathrm{d}x + \frac{1}{2}\int_{0}^{\frac{\pi}{12}} 2\sin x\cos x\mathrm{d}x$ $= -\frac{1}{2}\int_{-\frac{\pi}{4}}^{0} \sin 2x\mathrm{d}x + \frac{1}{2}\int_{0}^{\frac{\pi}{12}} \sin 2x\mathrm{d}x$ $= -\frac{1}{2}\left[\frac{-\cos 2x}{2}\right]_{-\frac{\pi}{4}}^{0} + \frac{1}{2}\left[\frac{-\cos 2x}{2}\right]_{0}^{\frac{\pi}{12}}$ $= -\frac{1}{4}(-1-0) + \frac{1}{4}\left(-\frac{\sqrt{3}}{2}+1\right)$ $= \frac{1}{2} - \frac{\sqrt{3}}{8}$ Recall that $\lvert \mathrm{f}(x)\rvert = \begin{cases} -\mathrm{f}(x) & \text{when } \mathrm{f}(x) < 0 \\ \mathrm{f}(x) & \text{when } \mathrm{f}(x) \geq 0 \end{cases}$ Hence $\lvert \sin x\rvert = \begin{cases} -\sin x \text{ when } -\frac{\pi}{4} \leq x < 0 \\ \sin x \text{ when } 0 \leq x \leq \frac{\pi}{12} \end{cases}$

Qn	Solution
5	**Function**
(i)	Since $R_f = (0,\infty) \subseteq D_g = (0,\infty)$, gf exists. $\text{gf} : x \mapsto \ln\left\lvert\frac{1}{x-3}\right\rvert$, where $x \in \mathbb{R}, x \neq 3$
(ii)	 Since $y = 3$ cuts the graph of f twice, f is not one-one. Hence f^{-1} does not exist.

(iii)	Maximum $k = 3$ For $\frac{1}{x-3} < 0$, Let $y = -\frac{1}{x-3}$ $x - 3 = -\frac{1}{y}$ $x = 3 - \frac{1}{y}$ $\therefore f^{-1}(x) = 3 - \frac{1}{x}$ $D_{f^{-1}} = R_f = (0,\infty)$ $f^{-1} : x \mapsto 3 - \frac{1}{x},\ x \in \mathbb{R}, x > 0$
(iv)	The graph of $y = f(x)$ is a reflection of the graph of $y = f^{-1}(x)$ in the line $y = x$.

Qn	Solution
6	**Binomial Theorem and Maclaurin's Series**

(a) (i)	$y=\left[1+\ln(1+3x)\right]^{-\frac{1}{2}}$ $=\left[1+(3x)-\frac{(3x)^2}{2}+\ldots\right]^{-\frac{1}{2}}$ $\approx\left(1+(3x-\frac{9}{2}x^2)\right)^{-\frac{1}{2}}$ $=1+\left(-\frac{1}{2}\right)\left(3x-\frac{9}{2}x^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(3x-\frac{9}{2}x^2\right)^2+\ldots$ $=1-\frac{3}{2}x+\frac{9}{4}x^2+\frac{3}{8}\left(9x^2\right)+\ldots$ $=1-\frac{3}{2}x+\frac{45}{8}x^2+\ldots$
(a) (ii)	$\int_0^1 y\ \mathrm{d}x\approx\int_0^1 1-\frac{3}{2}x+\frac{45}{8}x^2\ \mathrm{d}x$ $=\frac{17}{8}$ Approximation is not good as $x=1$ is not close to 0 **OR** the two graphs deviate significantly in the interval [0,1].
(a) (iii)	$y=1-\frac{3}{2}x$
(b)	$\frac{\cos(2x)}{1-\sin x}$ $\approx\frac{1-\frac{1}{2}(2x)^2}{1-x}$ $=\frac{1-2x^2}{1-x}$ $=(1-2x^2)(1-x)^{-1}$ $=(1-2x^2)(1+x+x^2)+\ldots$ $=1+x-x^2+\ldots$ $a=1,\ b=1,\ c=-1$

Qn	Solution
7	**Graphing Techniques - Transformations**
(a) (i)	
(a) (ii)	

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(b)	$y=\dfrac{3x^2-6x}{x-1}=3(x-1)-\dfrac{3}{x-1}$ $\therefore A=3,\ B=-3$ A sequence of transformation from the graph of $y=\dfrac{1}{x}-x$ to the graph of $y=\dfrac{3x^2-6x}{x-1}$ is: 1. a. **Reflect** graph **abou**t the x-axis (Replace y with $(-y)$, get $y=x-\dfrac{1}{x}$) OR b. **reflect** graph **about** the y-axis (Replace x with $(-x)$, get $y=x-\dfrac{1}{x}$) 2. **Translate** graph 1 unit in the positive x-direction (Replace x with $(x-1)$, get $y=(x-1)-\dfrac{1}{x-1}$) 3. **Scaling** parallel to y-axis by **factor** of 3 (Replace y with $\dfrac{y}{3}$, get $y=3(x-1)-\dfrac{3}{x-1}$) Other possible transformations: 2, 1a, 3 or 3,1,2 or 2, 3, 1a

Qn	Solution
8	**Vectors 3**
(i)	$\overrightarrow{OA}=\begin{pmatrix}-2\\-7\\3\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\overrightarrow{AB}=\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}-2\\-7\\3\end{pmatrix}=\begin{pmatrix}2\\8\\-2\end{pmatrix}=2\begin{pmatrix}1\\4\\-1\end{pmatrix}$ $l:\quad \mathbf{r}=\begin{pmatrix}-2\\-7\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\4\\-1\end{pmatrix}, \lambda\in\mathbb{R}$
(ii)	Let θ be the acute angle between *l* and *p*. $\sin\theta=\dfrac{\left\|\begin{pmatrix}1\\4\\-1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\|}{\sqrt{1^2+4^2+(-1)^2}\sqrt{1^2+(-1)^2}}$ $=\dfrac{\|1-4\|}{\sqrt{18}\sqrt{2}}=\dfrac{3}{6}$ $\Rightarrow\theta=30°$
(iii)	$p:\quad \mathbf{r}\bullet\begin{pmatrix}1\\-1\\0\end{pmatrix}=5$ Subst. $\overrightarrow{OA}$ into *p*, $\text{LHS}=\begin{pmatrix}-2\\-7\\3\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\0\end{pmatrix}=-2+7=5=\text{RHS}$ Hence *A* lies on *p*. Note: $\angle BAC=2(30°)=60°$ and *BA* = *CA* (since reflection) *ABC* is an <u>equilateral</u> triangle.
(iv)	Let *F* be the foot of perpendicular from *B* to *p* $\overrightarrow{OF}=\begin{pmatrix}0\\1\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}\mu\\1-\mu\\1\end{pmatrix}$, for some $\mu\in\mathbb{R}$ $\begin{pmatrix}\mu\\1-\mu\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\0\end{pmatrix}=5$ $\Rightarrow\mu-1+\mu=5$ $\Rightarrow\mu=3$ $\therefore\overrightarrow{OF}=\begin{pmatrix}3\\1-3\\1\end{pmatrix}=\begin{pmatrix}3\\-2\\1\end{pmatrix}$

	$\overrightarrow{AF}=\begin{pmatrix}3\\-2\\1\end{pmatrix}-\begin{pmatrix}-2\\-7\\3\end{pmatrix}=\begin{pmatrix}5\\5\\-2\end{pmatrix}$ By ratio theorem, $\overrightarrow{AC}=2\overrightarrow{AF}-\overrightarrow{AB}$ $=2\begin{pmatrix}5\\5\\-2\end{pmatrix}-\begin{pmatrix}2\\8\\-2\end{pmatrix}=\begin{pmatrix}8\\2\\-2\end{pmatrix}=2\begin{pmatrix}4\\1\\-1\end{pmatrix}$ $\therefore$ Equation of reflection of line l in p is $\mathbf{r}=\begin{pmatrix}-2\\-7\\3\end{pmatrix}+k\begin{pmatrix}4\\1\\-1\end{pmatrix}, k\in\mathbb{R}$

Qn	Solution
9	**Differential Equations**
(i)	$\frac{dx}{dt}=k\left(6x-x^2\right)$ $=k\left[-(x^2-6x)\right]$ $=k\left[-(x-3)^2+9\right]$ $=k\left[9-(x-3)^2\right]$ $\int \frac{1}{3^2-(x-3)^2}\ dx=\int k\ dt$ $\frac{1}{2(3)}\ln\left\|\frac{3+(x-3)}{3-(x-3)}\right\|=kt+c$ $\ln\left\|\frac{x}{6-x}\right\|=6kt+6c$ $\frac{x}{6-x}=Ae^{6kt}$ where $A=\pm e^{6c}$ When $t=0$, $x=12$ $\frac{12}{6-12}=Ae^{6(0)}$ $A=-2$ $\frac{x}{6-x}=-2e^{6kt}$ When $t=\frac{1}{12}$, $x=8.6$ $\frac{8.6}{6-8.6}=-2e^{6k\left(\frac{1}{12}\right)}$ $-\frac{43}{13}=-2e^{0.5k}$ $\ln\frac{43}{26}=0.5k$ $k=2\ln\frac{43}{26}$ $\frac{x}{6-x}=-2e^{12t\ln\frac{43}{26}}$ $x=-2e^{12t\ln\frac{43}{26}}\left(6-x\right)$ $x-2e^{12t\ln\frac{43}{26}}x=-12e^{12t\ln\frac{43}{26}}$ $x=\frac{-12e^{12t\ln\frac{43}{26}}}{1-2e^{12t\ln\frac{43}{26}}}$ $=\frac{12e^{12t\ln\frac{43}{26}}}{2e^{12t\ln\frac{43}{26}}-1}$

(ii)	When $x = 0.75(12) = 9$, $\frac{9}{6-9} = -2e^{12t\ln\frac{43}{26}}$ $\frac{3}{2} = e^{12t\ln\frac{43}{26}}$ $\ln\frac{3}{2} = 12t\ln\frac{43}{26}$ $t = 0.0672$ years (to 3s.f.)
(iii)	$x = \frac{-12e^{6.0372t}}{1-2e^{6.0372t}}$ $= \frac{-12}{\frac{1}{e^{6.0372t}} - 2}$ As $t \to \infty, \frac{1}{e^{6.0372t}} \to 0, x \to 6$ In the long run, the population of the bluegill fish in the lake will **<u>decrease and stabilise</u>** at 6000. The assumption is that there are no other external factors such as pollution or diseases that may affect the population of bluegill fish in the lake.

Qn	Solution
10	**APGP**
(a)(i)	$u_n = S_n - S_{n-1}$ $= kn^2 - 3n - \left[k(n-1)^2 - 3(n-1)\right]$ $= 2kn - k - 3$ $u_n - u_{n-1} = 2kn - k - 3 - \left[2k(n-1) - k - 3\right]$ $= 2k$, a constant $\Rightarrow \{u_n\}$ is in AP.
(ii)	$\dfrac{u_3}{u_2} = \dfrac{u_6}{u_3}$ $(u_3)^2 = (u_2)(u_6)$ $\left[2k(3) - k - 3\right]^2 = \left[2k(2) - k - 3\right]\left[2k(6) - k - 3\right]$ $(5k-3)^2 = (3k-3)(11k-3)$ $8k^2 - 12k = 0$ $4k(2k-3) = 0$ $k = 0$ (rej since $k \neq 0$) or $k = \dfrac{3}{2}$
(b)(i)	Distance travelled by white-tailed deer, $S_n = \dfrac{9\left(1-0.95^n\right)}{1-0.95}$ $= 180\left(1-0.95^n\right)$ As $n \to \infty$, $S_n \to 180$. $\therefore$ max distance is 180 m.
(ii)	Distance travelled by jaguar after n leaps, $S_n = \dfrac{n}{2}\left[2(6) + (n-1)(-0.1)\right]$ For jaguar to catch white-tailed deer within n leaps, Let $D = \dfrac{n}{2}\left[12 + (n-1)(-0.1)\right] - 180\left(1-0.95^n\right) - 11 \geq 0$ When $n = 49, D = -0.21 < 0$ When $n = 50, D = 0.3501 > 0$ $\therefore$ min $n = 50$

Qn	Solutions
11	**Differentiation + Integration**

(i)

$x = 1-\cos^3\theta, \qquad y = 1-3\sin\theta\cos^2\theta$

$\dfrac{dx}{d\theta} = -3\cos^2\theta(-\sin\theta) = 3\cos^2\theta\sin\theta$

$\dfrac{dy}{d\theta} = -\left[\sin\theta 6\cos\theta(-\sin\theta)+3\cos^2\theta(\cos\theta)\right]$

$= 6\sin^2\theta\cos\theta - 3\cos^3\theta$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{6\sin^2\theta\cos\theta-3\cos^3\theta}{3\cos^2\theta\sin\theta}$

$= 2\tan\theta - \cot\theta$ (shown)

(ii)

$2\tan\theta - \cot\theta = 0$

$2\tan\theta = \dfrac{1}{\tan\theta}$

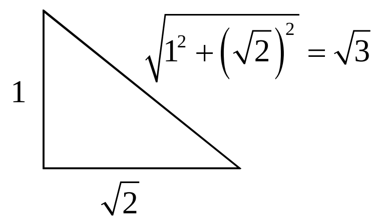
$2\tan^2\theta = 1$

$\tan\theta = \pm\dfrac{1}{\sqrt{2}}$

$\tan\theta = \dfrac{1}{\sqrt{2}} \left(\because \tan\theta \text{ is positive as } 0\le\theta\le\dfrac{\pi}{2}\right)$

$\theta = \tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

Since $\tan\theta = \dfrac{1}{\sqrt{2}}$ and $0\le\theta\le\dfrac{\pi}{2}$, ←



$\tan\theta = \dfrac{1}{\sqrt{2}}, \ \therefore \cos\theta = \dfrac{\sqrt{2}}{\sqrt{3}}$ and $\sin\theta = \dfrac{1}{\sqrt{3}}$

$\cos\theta = \dfrac{\sqrt{2}}{\sqrt{3}}$ and $\sin\theta = \dfrac{1}{\sqrt{3}}$

$\therefore x = 1-\cos^3\theta = 1-\left(\sqrt{\dfrac{2}{3}}\right)^3,$

$y = 1-3\sin\theta\cos^2\theta = 1-3\left(\dfrac{1}{\sqrt{3}}\right)\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)^2 = 1-\dfrac{2}{\sqrt{3}}$

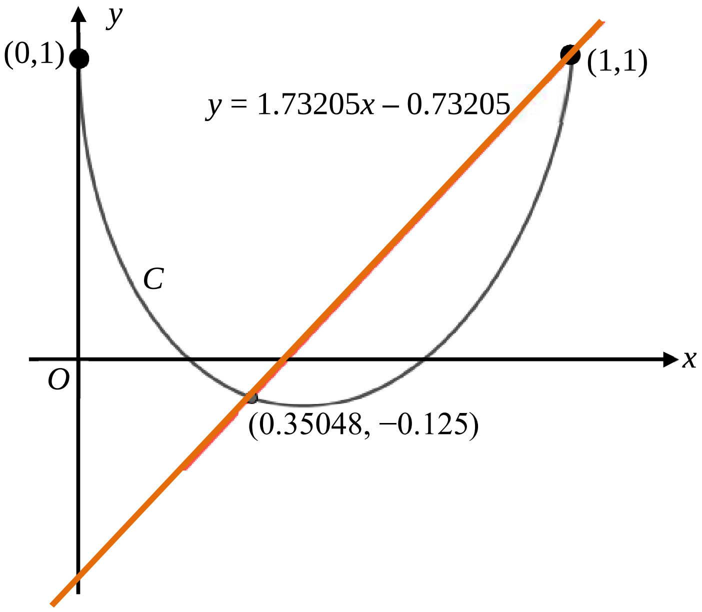
To show minimum point

x	$\left(1-\left(\sqrt{\frac{2}{3}}\right)^3\right)^-$	$1-\left(\sqrt{\frac{2}{3}}\right)^3$	$\left(1-\left(\sqrt{\frac{2}{3}}\right)^3\right)^+$
$\dfrac{dy}{dx}$	$-$	0	$+$

$\therefore$ At $\left(1-\left(\sqrt{\dfrac{2}{3}}\right)^3, \ 1-\dfrac{2}{\sqrt{3}}\right)$, it is a minimum point.

Alternative:

	$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$ $=\left[\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\right]\left(\dfrac{\mathrm{d}\theta}{\mathrm{d}x}\right)$ $=\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(2\tan\theta-\cot\theta\right)\dfrac{1}{3\cos^2\theta\sin\theta}$ $=\left(2\sec^2\theta+\operatorname{cosec}^2\theta\right)\dfrac{1}{3\cos^2\theta\sin\theta}$ Using GC, $\left[\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(2\tan\theta-\cot\theta\right)\Big\|_{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}\right]\left[\dfrac{1}{3\cos^2\theta\sin\theta}\Big\|_{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}\right]$ $=5.19616>0$ $\therefore$ At $\left(1-\left(\sqrt{\dfrac{2}{3}}\right)^3,\ 1-\dfrac{2}{\sqrt{3}}\right)$, it is a minimum point.
(iii)	The gradient of normal is $-\dfrac{1}{2\tan\theta-\cot\theta}\Big\|_{\frac{\pi}{6}}=-\dfrac{1}{0.577351}=1.73205$ Equation of normal is $y-(-0.125)=1.73205\left(x-(0.35048)\right)$ $y=1.73205x-0.73205$



Coordinates of point on C at $\theta = \frac{\pi}{6}$ is $(0.35048, -0.125)$

Area of area of the region bounded by C, the y-axis and the equation of the normal to the curve at $\theta = \frac{\pi}{6}$

$= \int_{-0.125}^{1} x \, dy \; + \text{area of triangle}$

$= \int_{\frac{\pi}{6}}^{0} \left(1-\cos^3\theta\right)\left(6\sin^2\theta\cos\theta - 3\cos^3\theta\right) d\theta$

$\quad + \frac{1}{2}(0.73205-0.125)(0.35048)$

$= 0.0997667 + 0.106379$

$= 0.206$

Alternative:

$= \int_{0}^{\frac{\pi}{6}} \left(1-3\sin\theta\cos^2\theta\right)\left(3\cos^2\theta\sin\theta\right) d\theta$

$\quad - \int_{0}^{0.35048} (1.73205x - 0.73205)\, dx$

$= 0.206$

MERIDIAN JUNIOR COLLEGE
JC2 Preliminary Examination
Higher 2

H2 Mathematics 9758/02

Paper 2 **19 September 2018**

3 Hours

Additional Materials: Writing paper
Graph Paper
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 **(a)** Consider the polynomial $\mathrm{P}(z) = a_0 + a_1 z + a_2 z^2$, where $a_0, a_1, a_2 \in \mathbb{R}, a_2 \neq 0$. Show that if the equation $\mathrm{P}(z) = 0$ has a complex root *w*, then its complex conjugate w^* must also be a root to the equation. [3]

(b) The polynomial $\mathrm{P}(z)$ has real coefficients. The equation $\mathrm{P}(z) = 0$ has a root $z_1 = -a + a\mathrm{i}$, where *a* is a positive real number.

(i) Write down a second root, z_2, in terms of *a*. [1]

(ii) Hence sketch the points Z_1 and Z_2, representing z_1 and z_2 respectively, on an Argand diagram. [2]

(iii) Deduce the area of OZ_1Z_2, in terms of *a*. [1]

2 **(i)** Using the formulae for $\cos(A \pm B)$, prove that

$$\cos\left(r + \frac{1}{2}\right)\theta - \cos\left(r - \frac{1}{2}\right)\theta \equiv -2\sin r\theta \sin\frac{1}{2}\theta. \quad [2]$$

(ii) Hence show that $\sum_{r=1}^{n} \sin r\theta = -\frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta\left[\cos\left(n + \frac{1}{2}\right)\theta - \cos\frac{1}{2}\theta\right]$ [3]

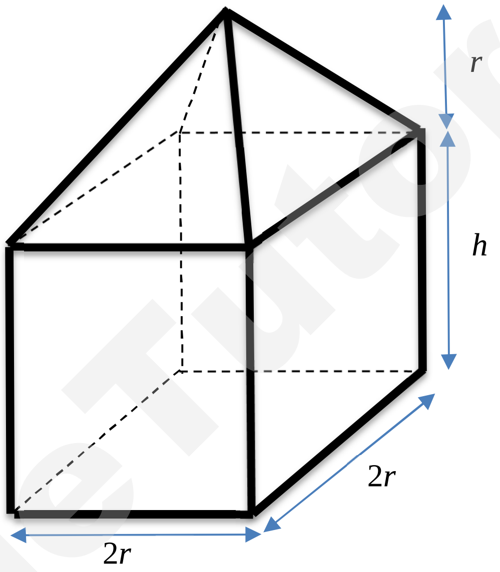
(iii) Using the result in **(ii)**, find the exact value for $\sum_{r=5}^{62} \sin(r-2)\theta$ when $\theta = \frac{\pi}{3}$. [4]

3 **(a)** A semicircle has radius r cm, perimeter P cm and area A cm^2. Show that

$$\frac{\mathrm{d}P}{\mathrm{d}A} = \frac{2+\pi}{\pi r}.$$

Determine the exact value of the radius when the area of the semicircle is increasing at a constant rate of 3 cm^2/s and the perimeter is increasing at a constant rate of $\frac{3}{5}$ cm/s. [4]

(b) An architectural firm wants to make a model of a greenhouse as shown.



The model is to be made up of three parts.

- The roof is modelled by a pyramid with a square base $2r$ cm by $2r$ cm and whose apex is r cm directly above the center of its base.
- The walls are modelled by four rectangles measuring $2r$ cm by h cm.
- The floor is modelled by a square measuring $2r$ cm by $2r$ cm.

The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.

It is given that the external surface area of the model has a fixed value of 160 cm^2. Show that $V = 80r - \left(\frac{2}{3} + 2\sqrt{2}\right)r^3$. Hence, using differentiation, find the value of r which gives the maximum value of V. (You do not need to verify that the volume is a maximum for this value of r.) [Volume of Pyramid = $\frac{1}{3} \times \text{base area} \times \text{height}$] [7]

4 **(a)** **(i)** Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{f}\left(\frac{1}{n}\right)+\mathrm{f}\left(\frac{2}{n}\right)+\ldots+\mathrm{f}\left(\frac{n}{n}\right)\right\}$$

is $\int_0^1 \mathrm{f}(x)\mathrm{d}x$. [2]

(ii) Hence evaluate $\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{e}^{\frac{3}{n}}+\mathrm{e}^{\frac{6}{n}}+\ldots+\mathrm{e}^{3}\right\}$, leaving your answer in exact form. [3]

(b) The function f is defined by

$$\mathrm{f}(x)=\begin{cases}(x+1)^2+a & \text{for } -1<x\le 1,\\ (4+a)(2-x) & \text{for } 1<x\le 2,\end{cases}$$

where a is a positive real constant and that $\mathrm{f}(x)=\mathrm{f}(x+3)$ for all real values of x.

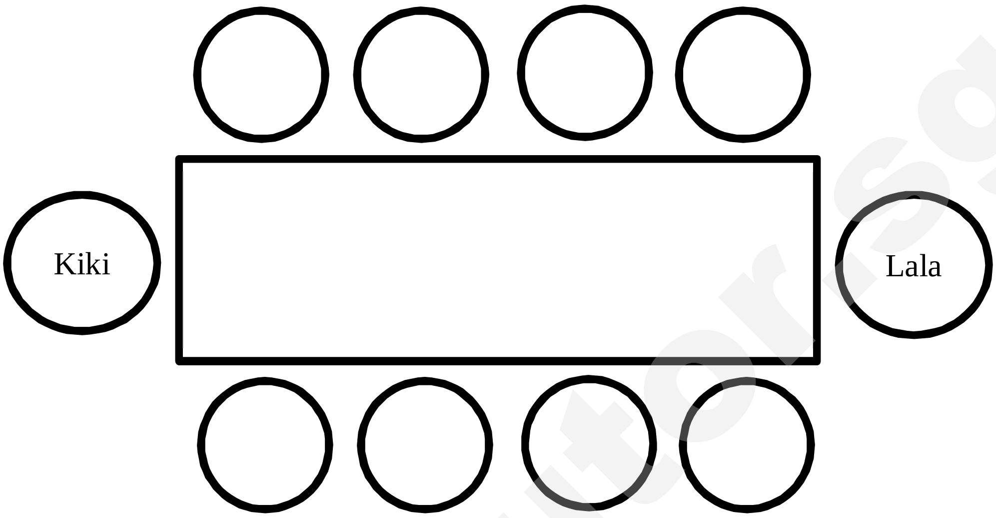
(i) Evaluate $\mathrm{f}(-41)$ and $\mathrm{f}(2018)$. [2]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $-4<x\le 6$. [3]

(iii) Hence find the value of $\int_{-4}^{4}\mathrm{f}(x)\,\mathrm{d}x$ in terms of a. [3]

Section B: Statistics [60 marks]

5 Kiki and Lala host a dinner for four other married couples. They sit at a rectangular table with Kiki and Lala at the left and right ends of the table respectively as shown in the diagram below.



Find the number of ways to seat the four couples such that

(i) there are no restrictions. [1]

(ii) each married couple is seated directly facing each other on opposite sides. [2]

(iii) each married couple is seated directly facing each other on opposite sides and two particular ladies cannot be seated next to each other on the same side. [3]

6 A bag contains five balls numbered 1 to 5. A game consists of a player taking two balls from the bag consecutively, with replacement. The number on each ball is noted down. His score X is found by taking the absolute value of the difference between the two numbers.

(i) Obtain the probability distribution of X. [2]

(ii) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

A player pays \$ a to play this game. He wins an amount (in dollars) corresponding to twice his score. Determine the range of values of a if the player is expected to make a loss. [2]

7 During the National Day Rally in 2017, PM Lee cited that an average of one in nine Singaporeans has diabetes. Suppose we examine 24 patients who are randomly selected from a polyclinic. Let X be the number of these patients who have diabetes.

(i) State, in the context of this question, two assumptions needed to model X by a binomial distribution. [2]

(ii) Explain why one of the assumptions stated in part **(i)** may not hold in this context. [1]

Assume now that these assumptions do in fact hold.

(iii) Find the probability that in a sample of 25 randomly selected patients, the 25$^{\text{th}}$ patient is the third patient who has diabetes. [2]

(iv) Find the least value of n such that the probability of having at most n diabetic patients in the random sample of 24 patients is greater than 0.9. [3]

8 A circular disc is divided into ten equal sectors where five sectors are coloured blue, three sectors are coloured green and two sectors are coloured red.

In a game, a player is given a maximum of three times to spin a pointer pivoted at the centre of the disc.

- If the pointer lands on a blue sector, the player wins \$2 and the game continues.
- If the pointer lands on a green sector, the game ends and the player loses all of his winnings.
- If the pointer lands on a red sector in the first spin, the game ends and the player wins the grand prize of \$20.
- If the pointer lands on a red sector in the second or third spin, the player wins double the amount of his total winnings from all his previous spins, and the game continues. For example, if the pointer lands on a blue sector in the first spin and on a red sector in the second spin, the total winnings for the first two spins will be \$6.

(i) Construct a probability tree showing this information. [3]

(ii) Find the exact probability that the player wins \$12 or more when the game ends. [2]

(iii) Find the exact probability that the pointer in the second spin lands on a blue sector given that the player wins \$12 or more when the game ends. [2]

(iv) Suppose there is no limit to the number of times a player can spin, find the exact probability that the player has no spins in which the pointer lands on a red sector and wins nothing when the game ends. [2]

9 Based on past records, shoppers spend an average of 1.5 hours at MJ mall per visit. After a revamp and expansion to include more stores, the mall management wishes to conduct a hypothesis test to check if the duration that shoppers spend in the mall per visit has increased. A random sample of 50 shoppers were surveyed and the duration of time, x, in hours per visit is recorded. The sample sum is 82.5 hours and the sample variance is 0.425 hours^2.

(i) State appropriate hypotheses for the test and find the unbiased estimates of the population mean and variance. [3]

(ii) Explain why the mall management is able to carry out a hypothesis test without making any assumptions about the distribution of the duration shoppers spend at MJ mall per visit. [1]

(iii) Calculate the p-value and state its meaning in context of the question. Hence state the set of values of the level of significance α % for which the management's hypothesis is valid. [4]

(iv) The mall management believes that the mean spending per shopper per visit is \$150. A test at 5% level of significance found that there is significant evidence that the population mean spending is less than \$150. Using only this information, explain if the following statements are necessarily true, necessarily false, or neither necessarily true nor necessarily false.

(a) There is significant evidence at the 10% level of significance that the population mean spending is less than \$150.

(b) There is significant evidence at the 5% level of significance that the population mean spending differs from \$150. [2]

10 Biologists monitor the population of wild rabbits in a dry grassland region of the Australian outback. The population of wild rabbits, y, in hundreds, in month x are as follows.

Month x	1	3	6	9	11	13	15
Population (in hundreds) y	40.3	35.2	36.3	40.8	39.7	41.5	42.1

(i) Draw a scatter diagram showing these data. [1]

(ii) Suggest a possible reason, in context, why one of the data points does not seem to follow the trend. [1]

(iii) It is desired to predict the population of rabbits in the future months. Explain why, in this context, a linear model is not appropriate. [1]

After removing the outlier, the biologists decided to fit a model of the form $\ln(M-y)=a+bx$, where M is a suitable constant for the remaining data points. The product moment correlation coefficient between x and $\ln(M-y)$ is denoted by r. The following table gives values of r for some possible values of M.

M	44	45	46
r	− 0.945 943		− 0.945 994

(iv) Calculate the value of r for $M = 45$, giving your answer correct to 6 decimal places. [1]

(v) Use the table and your answer in part **(iv)** to suggest with a reason which of 44, 45 or 46 is the most appropriate value for M. [1]

(vi) Using the value for M, calculate the values of a and b, and use them to predict the population of the wild rabbits after 2 years, to the nearest whole number. [4]

(vii) Give an interpretation, in context, of the value of M. [1]

11 **(a)** The time T (in minutes) taken by Kathy to drive from her house to the nearest supermarket has a mean of 5 and a variance of 8.

Explain why T is unlikely to be normally distributed. [2]

(b) In this question you should state clearly the values of the parameters of any normal distribution you use.

In a supermarket, the masses in kilograms of chickens have the distribution $\mathrm{N}\left(2.4, 0.5^2\right)$ and the masses in kilograms of ducks have the distribution $\mathrm{N}\left(4.3, 1.8^2\right)$.

(i) Find the probability that the total mass of 2 randomly chosen chickens is more than 5kg. [2]

(ii) Find the probability that the mean mass of a randomly chosen duck and 2 randomly chosen chickens is more than 3.20kg. [3]

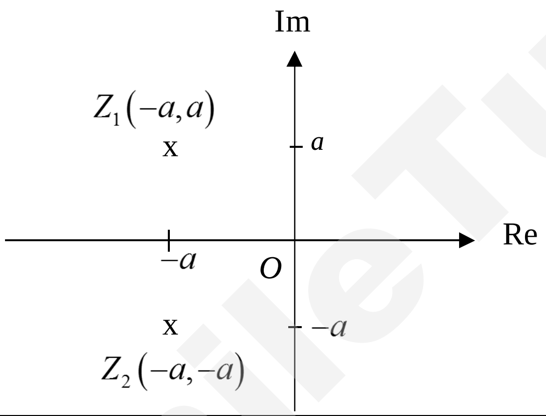
Chickens are sold at $7 per kilogram and ducks are sold at $13 per kilogram. The supermarket is conducting their annual sale and has a discount of 10% and 25% off the prices of chickens and ducks respectively.

(iii) Find the probability that the total cost of 2 randomly chosen chickens and a randomly chosen duck is less than $80. [4]

End of Paper

Section A : Pure Mathematics [40 marks]

Qn	Solution
1	**Complex No.**
(a)	If w is a complex root to the equation $\mathrm{P}(z)=0$, then $\mathrm{P}(w)=0$. $a_0+a_1w+a_2w^2=0$ Taking conjugate on both sides of the equation (*): $\left(a_0+a_1w+a_2w^2\right)^*=0$ $\left(a_0\right)^*+\left(a_1w\right)^*+\left(a_2w^2\right)^*=0$ Since $a_0, a_1, a_2 \in \mathbb{R}$, $a_0+a_1w^*+a_2\left(w^*\right)^2=0$ $\mathrm{P}\left(w^*\right)=0$ Therefore, its complex conjugate w^* is also a root of $\mathrm{P}(z)=0$. **<u>Alternative method</u>** Let $w=x+y\mathrm{i}$ where $x, y \in \mathbb{R}$ then $w^*=x-y\mathrm{i}$. $a_0+a_1w+a_2w^2=0$ $a_0+a_1\left(x+y\mathrm{i}\right)+a_2\left(x+y\mathrm{i}\right)^2=0$ $a_0+a_1\left(x+y\mathrm{i}\right)+a_2\left(x^2+2xy\mathrm{i}-y^2\right)=0$ Comparing real and imaginary parts: $a_0+a_1x+a_2\left(x^2-y^2\right)=0$ $a_1y+a_2\left(2xy\right)=0$

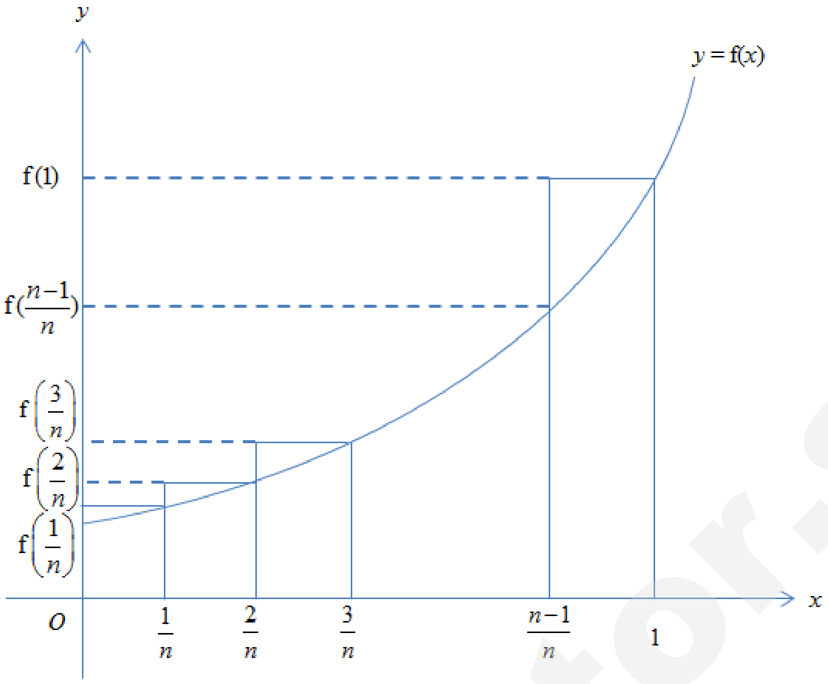
	Subst $w^* = x - y\text{i}$ into $\text{P}(z)$: $a_0 + a_1 w^* + a_2 \left(w^*\right)^2$ $= a_0 + a_1 (x - y\text{i}) + a_2 (x - y\text{i})^2$ $= a_0 + a_1 (x - y\text{i}) + a_2 \left(x^2 - 2xy\text{i} - y^2\right)$ $= a_0 + a_1 x + a_2 \left(x^2 - y^2\right) - \left[a_1 y + a_2 (2xy)\right]\text{i}$ Since $a_0 + a_1 x + a_2 \left(x^2 - y^2\right) = 0$ and $a_1 y + a_2 (2xy) = 0$ $a_0 + a_1 w^* + a_2 \left(w^*\right)^2$ $= 0 - 0\text{i}$ $=0$ $\therefore \text{P}\left(w^*\right) = 0$
(b)(i)	$z_2 = -a - a\text{i}$
(b)(ii)	
(b)(iii)	Area of $OZ_1Z_2 = \frac{1}{2} \times (2a) \times a$ $= a^2$

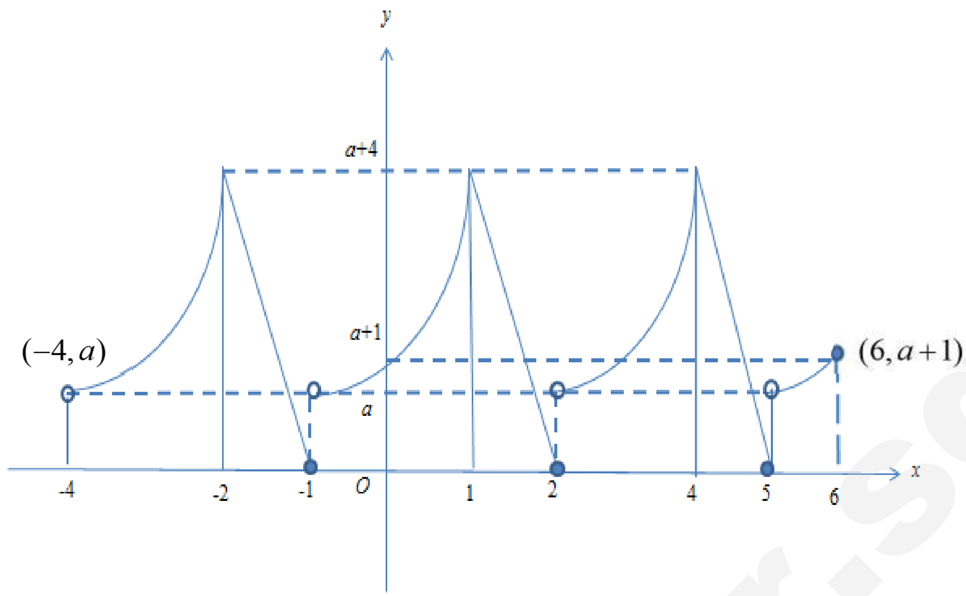
Qn	Solution
2	**Sequences & Series**
(i)	$\text{LHS} = \cos\left(r+\frac{1}{2}\right)\theta - \cos\left(r-\frac{1}{2}\right)\theta$ $= \left[\cos r\theta\cos\frac{1}{2}\theta - \sin r\theta\sin\frac{1}{2}\theta\right] - \left[\cos r\theta\cos\frac{1}{2}\theta + \sin r\theta\sin\frac{1}{2}\theta\right]$ $= -2\sin r\theta\sin\frac{1}{2}\theta = \text{RHS} \quad (\text{shown})$
(ii)	$\sum_{r=1}^{n}\sin r\theta$ $= \sum_{r=1}^{n}\frac{\cos\left(r+\frac{1}{2}\right)\theta - \cos\left(r-\frac{1}{2}\right)\theta}{-2\sin\frac{1}{2}\theta}$ $= \frac{1}{-2\sin\frac{1}{2}\theta}\sum_{r=1}^{n}\left[\cos\left(r+\frac{1}{2}\right)\theta - \cos\left(r-\frac{1}{2}\right)\theta\right]$ $= \frac{1}{-2\sin\frac{1}{2}\theta}\left[\cancel{\cos\frac{3}{2}\theta} - \cos\frac{1}{2}\theta\right.$ $+\cancel{\cos\frac{5}{2}\theta} - \cancel{\cos\frac{3}{2}\theta}$ $+\cancel{\cos\frac{7}{2}\theta} - \cancel{\cos\frac{5}{2}\theta}$ $+\ldots$ $+\cancel{\cos\left(n-\frac{1}{2}\right)\theta} - \cancel{\cos\left(n-\frac{3}{2}\right)\theta}$ $\left.+\cos\left(n+\frac{1}{2}\right)\theta - \cancel{\cos\left(n-\frac{1}{2}\right)\theta}\right]$ $= \frac{1}{-2\sin\frac{1}{2}\theta}\left[\cos\left(n+\frac{1}{2}\right)\theta - \cos\frac{1}{2}\theta\right]$ $= -\frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta\left[\cos\left(n+\frac{1}{2}\right)\theta - \cos\frac{1}{2}\theta\right]$
(iii)	$\sum_{r=5}^{62}\sin(r-2)\theta$ $= \sum_{r=3}^{60}\sin r\theta \qquad (\text{replace } r \text{ by } r+2)$ $= \sum_{r=1}^{60}\sin r\theta - \sin\theta - \sin 2\theta$ $= \frac{1}{-2\sin\frac{1}{2}\theta}\left[\cos\left(60+\frac{1}{2}\right)\theta - \cos\frac{1}{2}\theta\right] - \sin\theta - \sin 2\theta$ When $\theta = \frac{\pi}{3}$,

	$$\sum_{r=5}^{62}\sin\left[(r-2)\left(\frac{\pi}{3}\right)\right]=\frac{1}{-2\sin\frac{\pi}{6}}\left[\cos\left(20\pi+\frac{\pi}{6}\right)-\cos\frac{\pi}{6}\right]-\sin\frac{\pi}{3}-\sin\frac{2\pi}{3}$$ $$=-1\left[\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right]-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}=-\sqrt{3}$$

Qn	Solutions
3	**Applications of Differentiation: Rate of Change, Maxima/Minima**
(a)	$P = 2r + \pi r \qquad \Rightarrow \dfrac{\mathrm{d}P}{\mathrm{d}r} = 2 + \pi$ $A = \dfrac{\pi r^2}{2} \qquad \Rightarrow \dfrac{\mathrm{d}A}{\mathrm{d}r} = \pi r$ $\therefore \dfrac{\mathrm{d}P}{\mathrm{d}A} = \dfrac{\frac{\mathrm{d}P}{\mathrm{d}r}}{\frac{\mathrm{d}A}{\mathrm{d}r}} = \dfrac{2+\pi}{\pi r} \quad \text{(Shown)}$
	Given that $\dfrac{\mathrm{d}P}{\mathrm{d}t} = \dfrac{3}{5}$ cm/s and $\dfrac{\mathrm{d}A}{\mathrm{d}t} = 3\,\text{cm}^2/\text{s}$ $\dfrac{\mathrm{d}P}{\mathrm{d}A} = \left(\dfrac{\mathrm{d}P}{\mathrm{d}t}\right)\left(\dfrac{\mathrm{d}t}{\mathrm{d}A}\right)$ $\dfrac{2+\pi}{\pi r} = \left(\dfrac{3}{5}\right)\left(\dfrac{1}{3}\right)$ $5(2+\pi) = \pi r$ $r = \dfrac{5(2+\pi)}{\pi}$ cm
(b)	Let S be the external surface area of the model. $S = 4\left(\dfrac{1}{2}\right)(2r)\left(\sqrt{2}r\right) + 4(2r)h + (2r)(2r)$ $= 4\left(\sqrt{2}\right)r^2 + 8rh + 4r^2$ $= 4\left(\sqrt{2}+1\right)r^2 + 8rh = 160$ $h = \dfrac{160 - 4\left(\sqrt{2}+1\right)r^2}{8r}$ $= \dfrac{40 - \left(\sqrt{2}+1\right)r^2}{2r}$ Volume of model $V = \dfrac{1}{3}(2r)(2r)r + (2r)(2r)h$ $= \dfrac{4}{3}r^3 + 4r^2\left(\dfrac{40-\left(\sqrt{2}+1\right)r^2}{2r}\right)$ $= \dfrac{4}{3}r^3 + 2r\left(40 - \left(\sqrt{2}+1\right)r^2\right)$ $= \dfrac{4}{3}r^3 + 80r - 2\left(\sqrt{2}+1\right)r^3$ $= 80r - \left(\dfrac{2}{3} + 2\sqrt{2}\right)r^3$ For V to be maximum,

	$\frac{\mathrm{d}V}{\mathrm{d}r} = 80 - 3\left(\frac{2}{3} + 2\sqrt{2}\right)r^2 = 0$ $\left(4 - 6\left(\sqrt{2} + 1\right)\right)r^2 + 80 = 0$ Since $r > 0$, $r = \sqrt{\frac{-80}{4 - 6\left(\sqrt{2} + 1\right)}}$ $= 2.7622$ ≈ 2.76 cm

Qn	Solution
4	**Applications of Integration**
(a)(i)	 $\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{f}\left(\frac{1}{n}\right)+\mathrm{f}\left(\frac{2}{n}\right)+\ldots+\mathrm{f}\left(\frac{n}{n}\right)\right\}$ $=\lim_{n\to\infty}$ (Total area of the n rectangles) = Area under the curve y = f(x) from x = 0 to x =1 $=\int_0^1 \mathrm{f}(x)\mathrm{d}x$
(a)(ii)	Let $\mathrm{f}(x)=\mathrm{e}^{3x}$ $\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{e}^{\frac{3}{n}}+\mathrm{e}^{\frac{6}{n}}+\ldots+\mathrm{e}^{3}\right\}$ $=\int_0^1 \mathrm{e}^{3x}\mathrm{d}x$ $=\frac{1}{3}\left[\mathrm{e}^{3x}\right]_0^1$ $=\frac{1}{3}(\mathrm{e}^3-1)$
(b)(i)	$\mathrm{f}(-41)=\mathrm{f}(1)=4+a$ $\mathrm{f}\left(2018\right)=\mathrm{f}(2)=0$

(ii)	
(iii)	$\int_{-4}^{4} \mathrm{f}(x)\mathrm{d}x$ $= 3\int_{-1}^{1}\left((x+1)^2 + a\right)\mathrm{d}x + 2\left(\frac{1}{2}(1)(4+a)\right)$ $= \left[(x+1)^3 + 3ax\right]_{-1}^{1} + 4 + a$ $= (8+3a) - (0-3a) + 4 + a$ $= 12 + 7a$

Section B: Probability and Statistics [60 marks]

Qn	Solution
5	**Permutations and Combinations**
(i)	Number of ways = 8! = 40320
(ii)	Number of ways = ${}^{8}C_1 \times 1 \times {}^{6}C_1 \times 1 \times {}^{4}C_1 \times 1 \times {}^{2}C_1 \times 1 = 384$ Alternative: $4! \times (2!)^4 = 384$
(iii)	Number of ways = $384 - {}^{6}C_1 \times 2! \times 4 \times 1 \times 2 \times 1 = 288$

Qn	Suggested Solutions
6	**D.R.V.**
(i)	(tables below)

X	**1**	**2**	**3**	**4**	**5**
1	0	1	2	3	4
2	1	0	1	2	3
3	2	1	0	1	2
4	3	2	1	0	1
5	4	3	2	1	0

Probability Distribution of X

x	0	1	2	3	4
$\mathrm{P}(X=x)$	$\frac{5}{25}=\frac{1}{5}$	$\frac{8}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{2}{25}$

(ii)

Using GC, $\mathrm{E}(X)=1.6, \mathrm{Var}(X)=1.44$

OR

$$\mathrm{E}(X)=0\left(\frac{1}{5}\right)+1\left(\frac{8}{25}\right)+2\left(\frac{6}{25}\right)+3\left(\frac{4}{25}\right)+4\left(\frac{2}{25}\right)=1.6$$

$$\mathrm{E}(X^2)=0^2\left(\frac{1}{5}\right)+1^2\left(\frac{8}{25}\right)+2^2\left(\frac{6}{25}\right)+3^2\left(\frac{4}{25}\right)+4^2\left(\frac{2}{25}\right)=4$$

$$\mathrm{Var}(X)=\mathrm{E}(X^2)-\left[\mathrm{E}(X)\right]^2=4-1.6^2=1.44$$

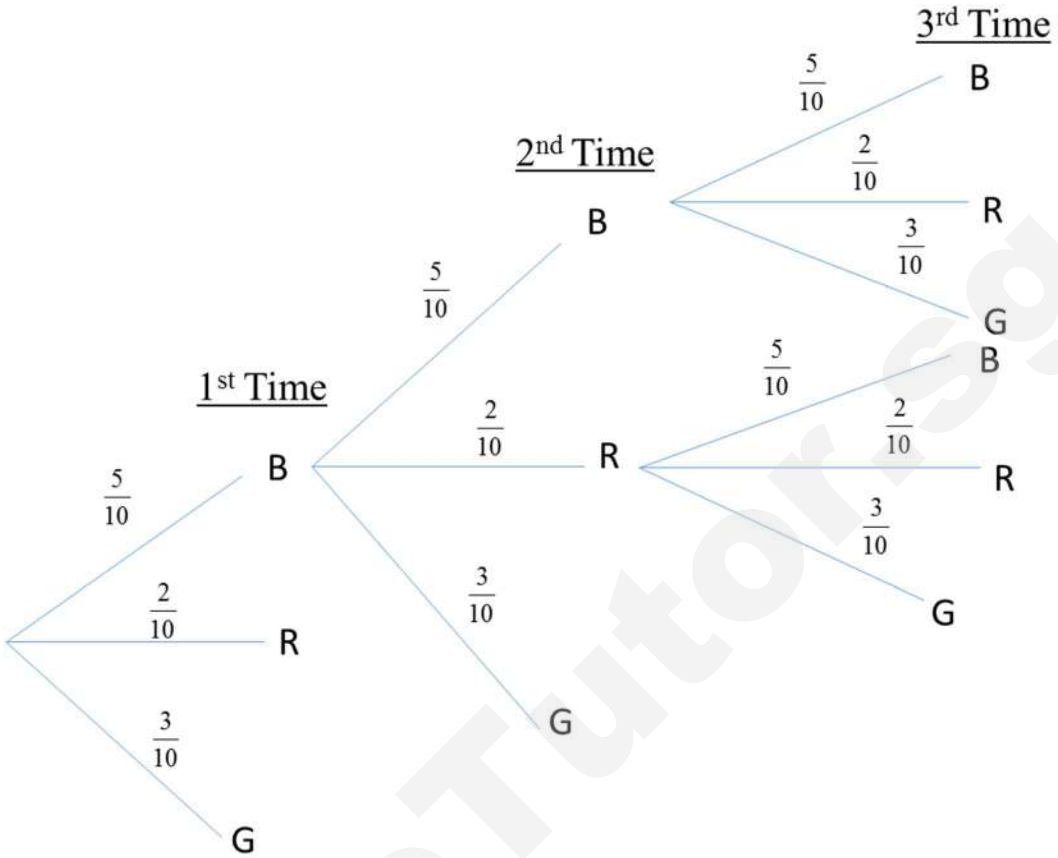
(iii)

$\mathrm{E}(2X-a)<0$

$2(1.6)-a<0$

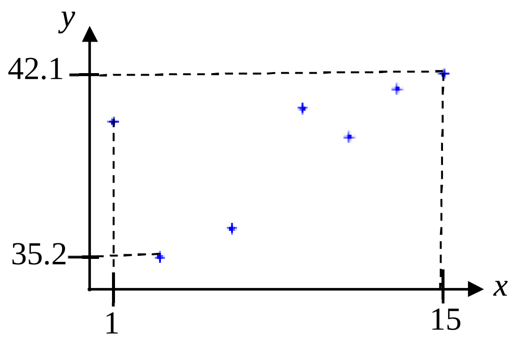
$\therefore a>3.2$

Qn	Suggested Solutions
7	**Binomial Distribution**
(i)	(I) Whether a patient has diabetes is independent of any other patient who has diabetes. (S) The probability that a patient has diabetes is constant for all patients.
(ii)	(I) Patients who are selected may be family members hence each of them having diabetes are not independent of each other. **<u>OR</u>** (S) The probability of a patient having diabetes is not a constant as the elderly have a higher probability of having diabetes. **<u>OR</u>** any reasonable factors e.g. lifestyle, diet, race, age
(iii)	Let X be the number of patients, out of 24, who have diabetes. $X \sim \mathrm{B}\left(24, \frac{1}{9}\right)$ Probability that in a sample of 25, the 25$^{\text{th}}$ patient is the third patient who has diabetes $= \mathrm{P}(X = 2)\mathrm{P}(\text{the 25th patient is the third patient who has diabetes})$ $= 0.25531 \times \frac{1}{9} = 0.0284$
(iv)	$\mathrm{P}(X \le n) > 0.9$ Using GC, When $n = 4$, $\mathrm{P}(X \le 4) = 0.87974 < 0.9$ When $n = 5$, $\mathrm{P}(X \le 5) = 0.95653 > 0.9$ Therefore least $n = 5$

Qn	Solution
8	**Probability (Tree Diagram)**
(i)	
(ii)	P(player wins \$12 or more when the game ends) = P(BRR) + P(BBR) +P(R) $=\frac{5}{10}\left(\frac{2}{10}\right)\left(\frac{2}{10}\right)+\frac{5}{10}\left(\frac{5}{10}\right)\left(\frac{2}{10}\right)+\left(\frac{2}{10}\right)$ $=\frac{27}{100}$
(iii)	P (second spin lands on a blue sector | the player wins \$12 or more when the game ends) $=\frac{\left(\frac{5}{10}\right)\left(\frac{5}{10}\right)\left(\frac{2}{10}\right)}{0.27}=\frac{5}{27}$
(iv)	P (no spins result on a red sector and wins nothing when game ends) $=\frac{3}{10}+\left(\frac{5}{10}\right)\left(\frac{3}{10}\right)+\left(\frac{5}{10}\right)^2\left(\frac{3}{10}\right)+\left(\frac{5}{10}\right)^3\left(\frac{3}{10}\right)+\ldots$ $=\frac{\frac{3}{10}}{1-\frac{5}{10}}$ $=\frac{3}{5}$

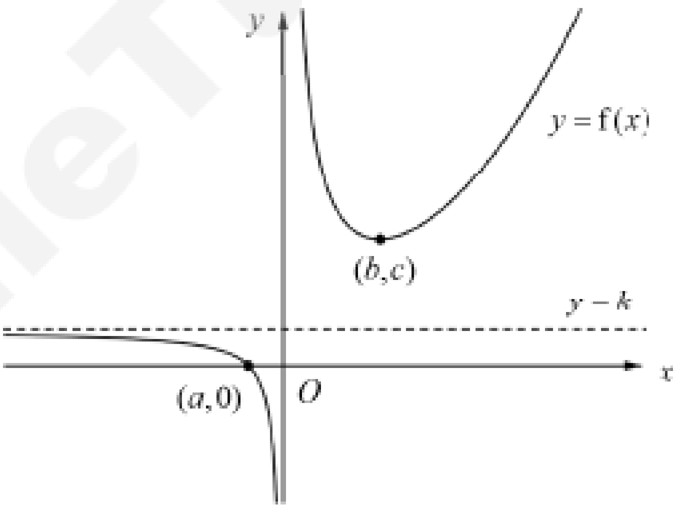
Qn	Solution
9	**Hypothesis Testing**
(i)	Let X be the duration a randomly chosen shopper spends at MJ mall per visit (in hours). Let μ denote the population mean duration spent by shoppers at MJ mall per visit (in hours). H_0: $\mu = 1.5$ H_1: $\mu > 1.5$ $\bar{x} = \frac{\sum x}{n} = \frac{82.5}{50} = 1.65$ $s^2 = \frac{50}{49}(0.425) = 0.43367 \text{ (to 5 s.f.)} = 0.434 \text{ (to 3 s.f.)}$
(ii)	Since sample size is large, by Central Limit Theorem, the sample mean duration spent by 50 shoppers is approximately normal. $\bar{X} \sim \mathrm{N}\left(1.5, \frac{0.43367}{50}\right)$ approximately. Hence there is no need to assume that the distribution of the duration shoppers spend at MJ mall per visit is normal.
(iii)	Using GC, Under H_0, $p\text{-value} = 0.053630 \text{ (to 5 s.f.)}$ $= 0.0536 \text{ (to 3 s.f.)}$ 0.0536 is the probability of obtaining a sample mean more than or equal to 1.65, assuming that the mean duration spent by shoppers at MJ mall is 1.5 hours. For management's hypothesis to be valid, H_0 is to be rejected. $p\text{-value} = 0.053630 < \frac{\alpha}{100}$ $\Rightarrow \alpha > 5.3630$ $\therefore$ Required set of values is $\{\alpha \in \mathbb{R} : 5.36 < \alpha \leq 100\}$ (to 3 s.f.)
(iv)	H_0: $\mu = 150$ H_1: $\mu < 150$ Given: p-value<0.05 **(a)** This is necessarily true because p-value< 0.05 < 0.1 **(b)** This is not necessarily true nor necessarily false. Now, H_0: $\mu = 150$ H_1: $\mu \neq 150$ The new p-value is double of the original p-value. Case 1: If the original p-value was more than 0.025, for example 0.03, then the new p-value would be 0.06 >0.05, hence in such a case there would not be significant evidence at the 5% level of significance that the population mean spending differs from \$150.

	Case 2: If the original p-value was less than 0.025, for example 0.02, then the new p-value would be $0.04 < 0.05$, hence in such a case there would be significant evidence at the 5% level of significance that the population mean spending differs from \$150.

Qn	Solution
10	**Correlation & Relation**
(i)	
(ii)	The outlier is (1, 40.3). Any appropriate reason, some possible reasons: • A fire in the dry grassland killed the wild rabbits • Drought/Famine in the region causes wild rabbits to die • Disease/Epidermic in the region killed the wild rabbits • Introduction of a new predator like coyote or dogs that hunts the wild rabbits, thus leading to a sudden decrease in the population. • Hunting season lead to a sudden decrease in the wild rabbit population.
(iii)	The population of wild rabbits <u>cannot increase or decrease indefinitely</u> in a region, hence a linear model is not appropriate or the population of wild rabbits should plateau due to limited resources.
(iv)	Using G.C., When $M = 45$, $r = -0.946\ 203$ (to 6 decimal places)
(v)	Since $M = 45$ gives a value of r that is closer to -1, it would be the most appropriate value for M.
(vi)	Using G.C., $\ln(45-y) = 2.6356 - 0.10435x$ $\Rightarrow \ln(45-y) = 2.64 - 0.104x \quad (3\text{ s.f.})$ $\Rightarrow a = 2.64, \quad b = -0.104$ When $x = 24$, $y = 45 - e^{2.6356-0.10435(24)} = 43.8598$ The population of the wild rabbits after 2 years is 4386.
(vii)	$M = 45$ i.e. 4500 rabbits is the <u>maximum population</u> of wild rabbits which this region can support in the long run, assuming all factors stay the same.

Qn	Solution
11	**Normal Distribution & Sampling**
(a)	If $T \sim \mathrm{N}(5,8)$, then P (T < 0) = 0.0385, which means that there is a non-zero probability that the time taken could be negative. However, this is impossible. Hence it is not suitable. OR If $T \sim \mathrm{N}(5,8)$, then 99.7% of the values would lie within $5 \pm 3\sqrt{8}$, which contains a significant range of negative values.
(b) (i)	Let X and Y be the masses of a randomly chosen chicken and a randomly chosen duck in kilograms respectively. $X \sim \mathrm{N}(2.4, 0.5^2)$ $Y \sim \mathrm{N}(4.3, 1.8^2)$ $X_1 + X_2 \sim \mathrm{N}(4.8, 0.5)$ $\mathrm{P}(X_1 + X_2 > 5) = 0.38865 \approx 0.389$ (3s.f.)
(b) (ii)	$\dfrac{X_1 + X_2 + Y}{3} \sim \mathrm{N}\left(\dfrac{91}{30}, \dfrac{187}{450}\right)$ $\mathrm{P}\left(\dfrac{X_1 + X_2 + Y}{3} > 3.2\right) = \mathrm{P}\left(\dfrac{X_1 + X_2 + Y}{3} > 3.2\right)$ $= 0.39799 \approx 0.398$ (3s.f.) Alternatively (Consider sum) $X_1 + X_2 + Y \sim \mathrm{N}\left(9.1, \dfrac{187}{50}\right)$ $\mathrm{P}\left(\dfrac{X_1 + X_2 + Y}{3} > 3.2\right) = \mathrm{P}(X_1 + X_2 + Y > 9.6) = 0.39799 \approx 0.398$ (3s.f.)
(b) (iii)	$T = 0.9(7)(X_1 + X_2) + 0.75(13)Y \sim \mathrm{N}(72.165, 327.8475)$ $\mathrm{P}(T < 80) \approx 0.667$ (3s.f.)

2018 MI H2 Math Prelim Paper 1

1	By using the series expansion of e^x from the List of Formulae (MF26), find $\sum_{r=2}^{\infty} \frac{3^r}{r!}$. [3]
2	Without using a calculator, solve the inequality $$\frac{x-7}{x^2-x-6} \leq 1.$$ [4] Hence solve the inequality $\frac{\ln x - 7}{(\ln x)^2 - \ln x - 6} - 1 \leq 0$. [2]
3	The diagram shows a sketch of the graph of $y = \mathrm{f}(x)$. The curve intersects the x-axis at the point $(a, 0)$ and has a turning point at the point (b, c), where a, b and c are constants. The y-axis and the line $y = k$, where k is a constant, are the asymptotes of the curve.  Sketch, on separate diagrams, the graphs of (i) $y = \frac{1}{\mathrm{f}(x)}$, [3] (ii) $y = \mathrm{f}'(x)$, [3] stating clearly, where possible, the equations of any asymptotes and coordinates of any turning points and axial intercepts.
4	The curve C has equation $y = \frac{2x^2 + 13x + 23}{x + \lambda}$, where λ is a positive constant.

	(i) Sketch the graph of C, stating clearly the coordinates of any points of intersection with the axes and equations of any asymptotes. [3] **(ii)** For $\lambda = 3$, find algebraically the set of values that y can take. [3]
5	**(i)** Given that $\ln y = \sin 2x$, show that $\frac{dy}{dx} = 2y\cos 2x$. Hence find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 0$. [3] **(ii)** Write down the first three non-zero terms in the Maclaurin series for y. [1] **(iii)** It is given that the second and third non-zero terms in part **(ii)** are equal to the first and second non-zero terms in the series expansion of $e^{px}\ln(1+qx)$ respectively. Using appropriate expansions from the List of Formulae (MF26), find the values of the constants p and q. [3] Hence state the range of values of x for which the series expansion of $e^{px}\ln(1+qx)$ is valid. [1]
6	A curve C has parametric equations $$x = 1 - e^{-t}, \qquad y = 1 + t^2.$$ **(i)** Find the exact gradient of C when $t = 1$. [2] **(ii)** By first finding an expression for $\frac{d^2y}{dx^2}$ in terms of the parameter t, determine if C is concave upward for all real values of t. [2] **(iii)** Find the cartesian equation of C. [2] **(iv)** Hence find the volume of revolution when the region bounded by C, the x-axis and the lines $x = -\frac{1}{2}$ and $x = \frac{3}{5}$ is rotated completely about the x-axis, giving your answer correct to 3 decimal places. [2]
7	**(a)** **(i)** Find $\int \cos 2x \sin^5 2x \, dx$. [3] **(ii)** Use the substitution $u = e^x$ to find $\int \frac{e^x}{\sqrt{1-2e^{2x}}} \, dx$. [3]

(b) **(i)** Show that $\int x^2 \sin nx \, dx = \frac{2}{n^3}\cos nx + \frac{2x}{n^2}\sin nx - \frac{x^2}{n}\cos nx + c$, where n is a positive integer and c is an arbitrary constant. [3]

(ii) Hence find $\int_{\pi}^{2\pi} x^2 \sin nx \, dx$, giving your answer in the form $\frac{p}{n^3} - \frac{q\pi^2}{n}$, where the possible values of p and q are to be determined. [3]

8 The function f is defined by

$$\mathrm{f} : x \mapsto \sqrt{3}\cos x - \sin x, \quad x \in \mathbb{R}, \ -\pi \le x \le \pi.$$

(i) Express $\mathrm{f}(x)$ as $R\cos(x+\alpha)$, where R and α are constants to be found. [1]

(ii) Sketch the graph of $y = \mathrm{f}(x)$, stating clearly the exact coordinates of the end-points and turning points. Hence explain why f^{-1} does not exist. [2]

(iii) If the domain of f is further restricted to $-\frac{\pi}{6} \le x \le k$, state the maximum value of k for which f^{-1} exists. [1]

(iv) Using the value of k found in part **(iii)**, find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

Hence sketch, on the same diagram, the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$. [3]

The function g is defined by

$$\mathrm{g}(x) = \frac{1}{1-x}, \quad x \in \mathbb{R}, \ x \ne 0, \ x \ne 1.$$

Given that $\mathrm{g}^2(x) = \frac{x-1}{x}$ and $\mathrm{g}^3(x) = x$ for $x \in \mathbb{R}, \ x \ne 0, \ x \ne 1,$ determine the value of $\mathrm{g}^{2018}(2)$. [2]

9 A toy all-terrain vehicle, Terra is designed to handle multiple terrains. In a test run, the vehicle moves on a path that consists of different sections of increasing difficulty. Terra completes the first section, second section, third section and so on until all sections have been completed. The total time (in seconds) taken by Terra to complete n sections is given by $\frac{3}{2}n^2 + \frac{13}{2}n$.

(i) Find the time taken by Terra to complete the nth section. [2]

Terra was made to travel up a man-made hill of height 35 m, from the foot of the hill to the peak. Terra is designed to cover a vertical distance of 5 m in the first ten minutes. For each subsequent ten-minute period, Terra covers 80% of the vertical distance covered in the previous ten-minute period.

(ii) The vertical distances covered by Terra in the ten-minute periods form a geometric sequence. Explain if this sequence is convergent. [1]

(iii) Explain if Terra is able to reach the peak of the hill. [2]

After the completion of the fourth ten-minute period, Terra's vertical distance covered is changed such that in each subsequent ten-minute period, Terra covers 90% of the vertical distance covered in the previous ten-minute period.

(iv) Find the minimum number of ten-minute periods required for Terra to reach the peak of the hill. [3]

After installing the brake system on Terra, the design team wants to test its effectiveness. In this test, Terra moves with increasing speed from a stationary position for 4 seconds before the brake system is engaged. Terra then has to stop moving in 2 seconds. The test is done many times and is repeated immediately after the previous one.

Terra's motion is modelled by the function given by

$$\mathrm{f}(t)=\begin{cases} \dfrac{3}{2}t & , \quad 0 \le t \le 4, \\ \dfrac{3}{2}t^2-18t+54 & , \quad 4 < t \le 6, \end{cases}$$

where t denotes the time (in seconds) after the start of the first test and $\mathrm{f}(t)$ denotes Terra's speed (in metres per second) at time t. The repetition of the test can be represented by the relationship given by $\mathrm{f}(t)=\mathrm{f}(t+6)$ for $t \ge 0$.

(v) Sketch the graph of $y=\mathrm{f}(t)$ for $0 \le t \le 14$. [2]

(vi) The distance covered by Terra is given by the area under the graph of $y=\mathrm{f}(t)$. Find the distance travelled by Terra from $t=4$ to $t=11$. [2]

10	Relative to the origin O, the position vectors of the points A and B are $\begin{pmatrix} 0 \\ 5 \\ 15 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 7 \\ 12 \end{pmatrix}$ respectively. The line l_1 has equation $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$. The line l_2 is parallel to the vector $\begin{pmatrix} -1 \\ 6 \\ 12 \end{pmatrix}$ and passes through A. **(i)** Given that l_1 and l_2 intersect at the point C, find $\overrightarrow{OC}$. [2] The point B lies on l_1 and is the foot of the perpendicular from A to l_1. **(ii)** Find the cartesian equation of the line of reflection of l_2 in the line l_1. [2] The equations of two planes π_1 and π_2 are as follows: $\pi_1: \; x + y - 2z = -1,$ $\pi_2: \; ax + y + 3z = 5,$ where a is a constant. **(iii)** Find the foot of the perpendicular from A to π_1 and hence, find the perpendicular distance from A to π_1. [4] **(iv)** Given that the angle between l_1 and π_2 is 30°, find the value(s) of a. [2] **(v)** Given instead that the line of intersection of π_1 and π_2 has the equation $\mathbf{r} = \mathbf{v} + \beta \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$, $\beta \in \mathbb{R}$, where $\mathbf{v}$ is the position vector of the point V on the line of intersection, determine the value of a and a possible position vector of V. [3]
11	A game designer created a computer game that involves an alien race inhabiting and developing a planet. **(i)** The population of the aliens (in thousands) on the planet at time n years is denoted by x_n. The population of the aliens changes according to the relationship that the game designer devised as follows:

$$x_n = x_{n-1} + \frac{13}{2} - n \text{ for } n \geq 1.$$

The initial population of the aliens on the planet is 8000. By considering $\sum_{r=1}^{n}(x_r - x_{r-1})$, find an expression for x_n in terms of n. [4]

(ii) The game designer believes that the relationship devised in part **(i)** is overly simplistic and decides to use another model. The population of the aliens (in thousands) at time t years is denoted by x. It is set that the growth rate of x is proportional to $50 - 2t - x$.

It is given that the initial population of the alien on the planet is 8000 and the population grows at a rate of 14000 per year initially.

Show that the growth rate of x at time t years can be modelled by the differential equation

$$\frac{dx}{dt} = \frac{1}{3}(50 - 2t - x).$$ [2]

Using the substitution $u = 2t + x$, find x in terms of t. [5]

(iii) To add more variation into the game play, the game designer decides to use an alternative model that is given by

$$\frac{d^2x}{dt^2} = -\frac{100}{(t+1)^3}, \ t \geq 0,$$

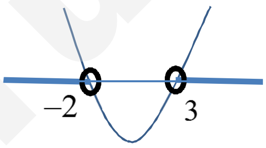
where x is the population of the aliens (in thousands) at time t years.

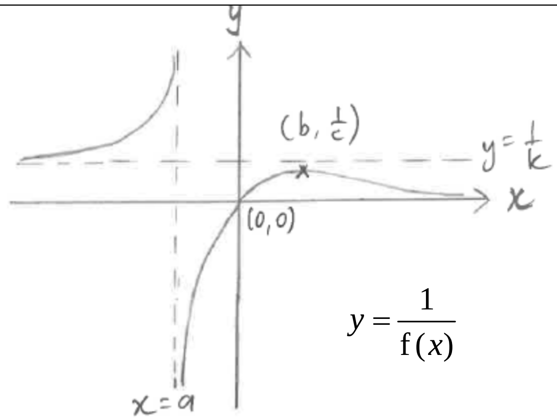
It is given that the initial population of the aliens on the planet is 8000 and the population grows at a rate of $41\frac{2}{3}$ thousand per year initially.

Find x in terms of t. [3]

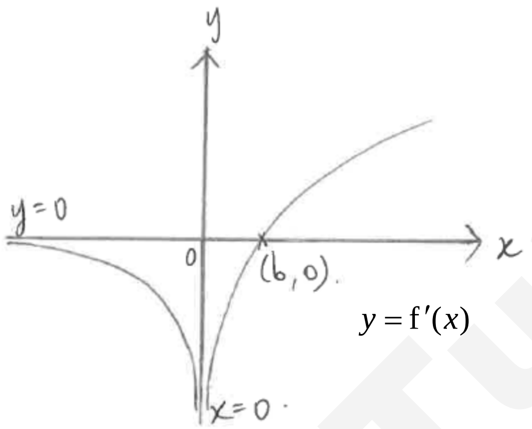
End

2018 MI H2 Math Prelim Paper 1 Solution

1	$\displaystyle\sum_{r=2}^{\infty}\frac{3^r}{r!}=\sum_{r=0}^{\infty}\frac{3^r}{r!}-\sum_{r=0}^{1}\frac{3^r}{r!}$ $\displaystyle=\sum_{r=0}^{\infty}\frac{3^r}{r!}-\frac{3^0}{0!}-\frac{3^1}{1!}$ $=\mathrm{e}^3-4$
2	$\dfrac{x-7}{x^2-x-6}\le 1$ $\dfrac{x-7-(x^2-x-6)}{x^2-x-6}\le 0$ $\dfrac{-x^2+2x-1}{x^2-x-6}\le 0$ $\dfrac{x^2-2x+1}{x^2-x-6}\ge 0$ $\dfrac{(x-1)^2}{(x+2)(x-3)}\ge 0$ For $(x-1)^2>0$, $\dfrac{1}{(x+2)(x-3)}>0$ $(x+2)(x-3)>0$ $x<-2$ or $x>3$  For $(x-1)^2=0$, $x=1$ $\therefore x<-2$ or $x>3$ or $x=1$ $\dfrac{\ln x-7}{(\ln x)^2-\ln x-6}\le 1$ Replacing x with $\ln x$: $\ln x<-2$ or $\ln x>3$ or $\ln x=1$ $0<x<\mathrm{e}^{-2}$ or $x>\mathrm{e}^3$ or $x=\mathrm{e}$
3	**(i)**



(ii)



4 **(i)**

$$y=\frac{2x^2+13x+23}{x+\lambda}$$

$$=2x+13-2\lambda+\frac{23-13\lambda+2\lambda^2}{x+\lambda}$$

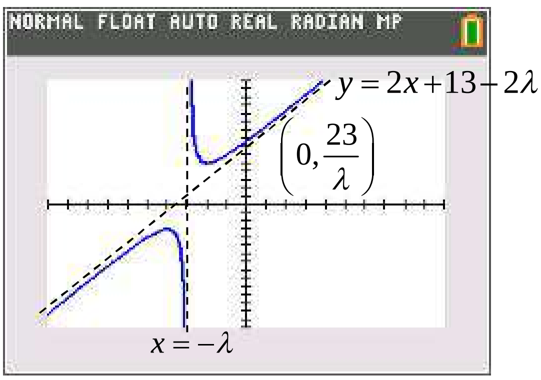
The asymptotes are $y=2x+13-2\lambda$ and $x=-\lambda$

When $x=0,\ y=\frac{23}{\lambda}$

When $y=0,\ \frac{2x^2+13x+23}{x+\lambda}=0$

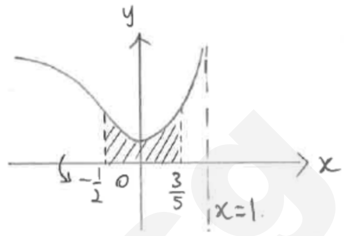
$$2x^2+13x+23=0$$

Using GC, there are no real roots $\Rightarrow$ no $x-$intercepts



	(ii) $y=\dfrac{2x^2+13x+23}{x+3}$ $y(x+3)=2x^2+13x+23$ $2x^2+(13-y)x+23-3y=0$ To determine the range of values of that y can take : For real roots, $b^2-4ac\geq 0$ $(13-y)^2-4(2)(23-3y)\geq 0$ $y^2-2y-15\geq 0$ $(y+3)(y-5)\geq 0$ $y\leq -3 \quad \text{or} \quad y\geq 5 \Rightarrow \{y\in\mathbb{R}: y\leq -3 \quad \text{or} \quad y\geq 5\}$
5	**(i)** <u>Method 1</u> $\ln y=\sin 2x$ Differentiating both sides with respect to x: $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\cos 2x$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2y\cos 2x$ (shown) <u>Method 2</u> $\ln y=\sin 2x$ $y=\mathrm{e}^{\sin 2x}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\cos 2x\ \mathrm{e}^{\sin 2x}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2y\cos 2x$ (shown) Differentiating both sides with respect to x : $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\cos 2x\dfrac{\mathrm{d}y}{\mathrm{d}x}-4y\sin 2x$ When $x=0,\ y=1,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=2,\ \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=4$ **(ii)** $y=1+2x+4\left(\dfrac{x^2}{2!}\right)+\ldots=1+2x+2x^2+\ldots$ **(iii)**

	$e^{px}\ln(1+qx)$ $=\left(1+px+\frac{(px)^2}{2!}+\dots\right)\left(qx-\frac{(qx)^2}{2}+\dots\right)$ $=\left(1+px+\frac{p^2x^2}{2}+\dots\right)\left(qx-\frac{q^2x^2}{2}+\dots\right)$ $=qx-\frac{q^2x^2}{2}+pqx^2+\dots$ $=qx+\left(pq-\frac{q^2}{2}\right)x^2+\dots$ By comparison, $q=2,\ pq-\frac{q^2}{2}=2\Rightarrow p=2$ The expansion is valid for $-1<2x\le 1\Rightarrow -\frac{1}{2}<x\le\frac{1}{2}$
6	**(i)** $x=1-e^{-t}$, $\quad y=1+t^2$ $\frac{dx}{dt}=e^{-t}$, $\quad \frac{dy}{dt}=2t$ $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=2t\cdot\frac{1}{e^{-t}}=2te^{t}$ When $t=1$, $\frac{dy}{dx}=2e$ The required gradient is $2e$ **(ii)** $\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}=\left(2te^{t}+2e^{t}\right)\cdot\frac{1}{e^{-t}}=2e^{2t}(t+1)$ Since $2e^{2t}>0$ for $t\in\mathbb{R}$ but $t+1>0$ only for $t>-1$, $\frac{d^2y}{dx^2}=2e^{2t}(t+1)>0$ only for $t>-1$. $\therefore$ The curve is not concave upwards for all real values of t. **(iii)**

	$x=1-\mathrm{e}^{-t} \Rightarrow \mathrm{e}^{-t}=1-x \Rightarrow t=-\ln(1-x)$ Substituting $t=-\ln(1-x)$ into $y=1+t^2$, $y=1+\left[\ln(1-x)\right]^2$ **(iv)** The required volume $=\pi\displaystyle\int_{-\frac{1}{2}}^{\frac{3}{5}}\left(1+\left[\ln(1-x)\right]^2\right)^2\,\mathrm{d}x$ $= 4.6699 = 4.670 \text{ units}^3$ (3 d.p) 
7	**(a)(i)** $\displaystyle\int \cos 2x \sin^5 2x\,\mathrm{d}x = \frac{1}{2}\int 2\cos 2x\sin^5 2x\,\mathrm{d}x$ $\displaystyle= \frac{1}{2}\cdot\frac{\sin^{5+1}2x}{5+1}+c$ $\displaystyle= \frac{1}{12}\sin^6 2x + c$ **(a)(ii)** $\displaystyle\int \frac{\mathrm{e}^x}{\sqrt{1-2\mathrm{e}^{2x}}}\,\mathrm{d}x$ $\displaystyle u=\mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=u$ <u>Method 1</u> $\displaystyle\int \frac{\mathrm{e}^x}{\sqrt{1-2\mathrm{e}^{2x}}}\,\mathrm{d}x = \int \frac{u}{\sqrt{1-2u^2}}\cdot\frac{1}{u}\,\mathrm{d}u$ $\displaystyle= \int \frac{1}{\sqrt{1-2u^2}}\,\mathrm{d}u$ $\displaystyle= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\frac{1}{2}-u^2}}\,\mathrm{d}u$ $\displaystyle= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2-u^2}}\,\mathrm{d}u$ $\displaystyle= \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{2}u\right)+c$ $\displaystyle= \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{2}\mathrm{e}^x\right)+c$

Method 2

$$\int \frac{e^x}{\sqrt{1-2e^{2x}}}\,dx = \int \frac{u}{\sqrt{1-2u^2}}\cdot\frac{1}{u}du$$

$$= \int \frac{1}{\sqrt{1-2u^2}}\,du$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}}{\sqrt{1-\left(\sqrt{2}u\right)^2}}\,du$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{2}u\right)+c$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{2}e^x\right)+c$$

(b)(i)

$$\int x^2 \sin nx\,dx$$

$$= x^2\left(-\frac{\cos nx}{n}\right)+\frac{2}{n}\int x\cos nx\,dx$$

$$= -\frac{x^2}{n}\cos nx+\frac{2}{n}\left[x\left(\frac{\sin nx}{n}\right)-\frac{1}{n}\int \sin nx\,dx\right]$$

$$= -\frac{x^2}{n}\cos nx+\frac{2}{n}\left[\frac{x}{n}\sin nx+\frac{1}{n^2}\cos nx\right]+c$$

$$= \frac{2}{n^3}\cos nx+\frac{2x}{n^2}\sin nx-\frac{x^2}{n}\cos nx+c$$

(b)(ii)

$$\int_{\pi}^{2\pi} x^2\sin nx\,dx$$

$$= \left[\frac{2}{n^3}\cos nx-\frac{x^2}{n}\cos nx+\frac{2x}{n^3}\sin nx\right]_{\pi}^{2\pi}$$

$$= \frac{2}{n^3}\cos 2\pi n-\frac{(2\pi)^2}{n}\cos 2\pi n+\frac{2}{n^3}(2\pi)\sin 2\pi n$$
$$-\left[\frac{2}{n^3}\cos\pi n-\frac{\pi^2}{n}\cos\pi n+\frac{2}{n^3}(\pi)\sin\pi n\right]$$

$$= \frac{2}{n^3}(1)-\frac{4\pi^2}{n}(1)+0-\left[\frac{2}{n^3}\cos\pi n-\frac{\pi^2}{n}\cos\pi n+0\right]$$

$$= \frac{2}{n^3}-\frac{4\pi^2}{n}-\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right)\cos\pi n$$

$$\cos\pi n = \begin{cases} -1 & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases}$$

For n odd,

$$\frac{2}{n^3}-\frac{4\pi^2}{n}-\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right)\cos\pi n$$

$$=\frac{2}{n^3}-\frac{4\pi^2}{n}-\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right)(-1)$$

$$=\frac{4}{n^3}-\frac{5\pi^2}{n} \;\Rightarrow\; p=4, q=5$$

For n even,

$$\frac{2}{n^3}-\frac{4\pi^2}{n}-\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right)\cos\pi n$$

$$=\frac{2}{n^3}-\frac{4\pi^2}{n}-\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right)(1)$$

$$=-\frac{3\pi^2}{n} \;\Rightarrow\; p=0, q=3$$

$$\therefore \int_{\pi}^{2\pi} x^2 \sin nx \, \mathrm{d}x = \begin{cases} \dfrac{4}{n^3}-\dfrac{5\pi^2}{n} & \text{for } n \text{ odd} \\ -\dfrac{3\pi^2}{n} & \text{for } n \text{ even} \end{cases}$$

8 **(i)**

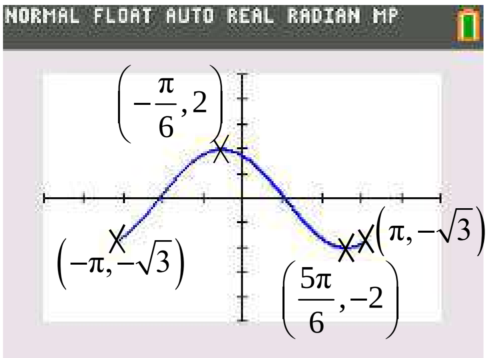
$R\cos\alpha=\sqrt{3}$; $R\sin\alpha=1$

$R^2=\sqrt{3}^2+1^2 \Rightarrow R=2$ $(R>0)$

$\tan\alpha=\dfrac{1}{\sqrt{3}} \Rightarrow \alpha=\dfrac{\pi}{6}$ (α is acute)

$\mathrm{f}(x)=\sqrt{3}\cos x-\sin x=2\cos\left(x+\dfrac{\pi}{6}\right)$

(ii)



A horizontal line $y=k,\ k\in(-2,2)$ intersects the graph of $y=\mathrm{f}(x)$ at more than one point. f is not a one-one function. f^{-1} does not exist.

(iii)

From the graph, the maximum value of k is $\dfrac{5\pi}{6}$

(iv)

Let $y = \mathrm{f}(x)$

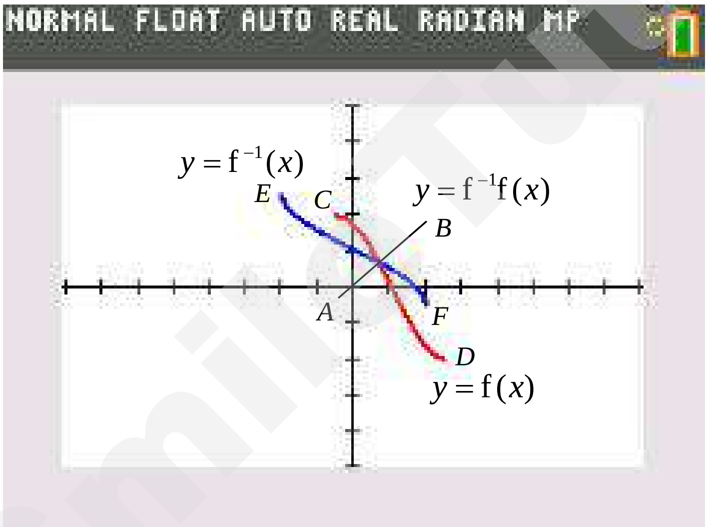
$$y = 2\cos\left(x + \frac{\pi}{6}\right)$$

$$x = \cos^{-1}\left(\frac{y}{2}\right) - \frac{\pi}{6}$$

$$\mathrm{f}^{-1}(y) = \cos^{-1}\left(\frac{y}{2}\right) - \frac{\pi}{6}$$

$$\mathrm{f}^{-1}(x) = \cos^{-1}\left(\frac{x}{2}\right) - \frac{\pi}{6}$$

Domain of f^{-1} = Range of f $= [-2, 2]$



Endpoints for $y = \mathrm{f}^{-1}\mathrm{f}(x)$: $A\left(-\frac{\pi}{6}, -\frac{\pi}{6}\right), B\left(\frac{5\pi}{6}, \frac{5\pi}{6}\right)$

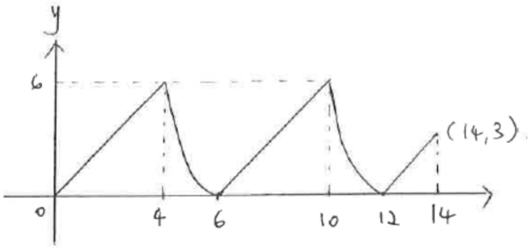
Endpoints for $y = \mathrm{f}(x)$: $C\left(-\frac{\pi}{6}, 2\right), D\left(\frac{5\pi}{6}, -2\right)$

Endpoints for $y = \mathrm{f}^{-1}(x)$: $E\left(-2, \frac{5\pi}{6}\right), F\left(2, -\frac{\pi}{6}\right)$

(v)

	$\mathrm{g}^{2018}(2) = \mathrm{g}^3\mathrm{g}^{2015}(2)$ $= \mathrm{g}^{2015}(2)$ $= \mathrm{g}^3\mathrm{g}^{2012}(2)$ $= \mathrm{g}^{2012}(2)$ $= \ldots$ $= \mathrm{g}^3\mathrm{g}^2(2)$ $= \mathrm{g}^2(2)$ $= \dfrac{2-1}{2} = \dfrac{1}{2}$
9	**(i)** $T_n = S_n - S_{n-1}$ $= \frac{3}{2}n^2 + \frac{13}{2}n - \left[\frac{3}{2}(n-1)^2 + \frac{13}{2}(n-1)\right]$ $= \frac{3}{2}n^2 + \frac{13}{2}n - \left[\frac{3}{2}n^2 - 3n + \frac{3}{2} + \frac{13}{2}n - \frac{13}{2}\right]$ $= 3n + 5$ **(ii)** The common ratio, $r = 0.8$ Since $-1 < r < 1,$ the sequence is convergent **(iii)** $S_\infty = \dfrac{5}{1-0.8} = 25 < 35$ $\therefore$ Terra will not reach the peak of the hill **(iv)** <u>Method 1</u>

	$T_1,\ T_2,\quad T_3,\quad T_4,\qquad T_5$ $5, 5(0.8), 5(0.8^2), 5(0.8^3), 5(0.8^3)(0.9)$ $U_1 = 5(0.8^3)(0.9), U_2, ..., U_n$ $T_5 = U_1$ i.e. T_5 is the first term of the GP after r is changed from 0.8 to 0.9. $\left(T_1+T_2+T_3+T_4\right)+\left(U_1+U_2+U_3+....+U_n\right)=35$ $\frac{5(1-0.8^4)}{1-0.8}+\frac{5(0.8^3)(0.9)(1-0.9^n)}{1-0.9}=35$ $\frac{5(0.8^3)(0.9)(1-0.9^n)}{1-0.9}=20.24$ $1-0.9^n=0.87847$ $0.9^n=0.12153$ $n=\frac{\ln 0.12153}{\ln 0.9}=20.004$ The minmimum no. of 10-minute periods required $=4+21=25$ Method 2 $T_1,\ T_2,\quad T_3,\quad T_4$ $5, 5(0.8), 5(0.8^2), 5(0.8^3)$ $U_1 = 5(0.8^3), U_2 = 5(0.8^3)(0.9), ..., U_n$ $T_4 = U_1$ i.e. T_4 is the first term of the GP. $\left(T_1+T_2+T_3\right)+\left(U_1+U_2+U_3+....+U_n\right)=35$ $\frac{5(1-0.8^3)}{1-0.8}+\frac{5(0.8^3)(1-0.9^n)}{1-0.9}=35$ $1-0.9^n=0.890625$ $0.9^n=0.109375$ $n=\frac{\ln 0.109375}{\ln 0.9}=21.004$ The minimum number of 10-minute periods required =3+22 =25 **(v)**

	 **(vi)** Method 1 The distance covered $=\int_4^6 \frac{3}{2}t^2-18t+54\ \mathrm{d}t+\frac{1}{2}(4)(6)+\int_4^5 \frac{3}{2}t^2-18t+54\ \mathrm{d}t$ $=19.5$ m Method 2 The distance covered $=\int_4^6 \frac{3}{2}t^2-18t+54\ \mathrm{d}t+\frac{1}{2}(4)(6)+\int_4^5 \frac{3}{2}t^2-18t+54\ \mathrm{d}t$ $=\left(\frac{1}{2}t^3-9t^2+54t\right)_4^6+12+\left(\frac{1}{2}t^3-9t^2+54t\right)_4^5$ $=(108-104)+12+\left(\frac{215}{2}-104\right)=19\frac{1}{2}$ m
10	**(i)** $l_1:\mathbf{r}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix},\lambda\in\mathbb{R}$ $l_2:\mathbf{r}=\begin{pmatrix}0\\5\\15\end{pmatrix}+\alpha\begin{pmatrix}-1\\6\\12\end{pmatrix},\alpha\in\mathbb{R}$

Equating the equations of both lines,

$$\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}0\\5\\15\end{pmatrix}+\alpha\begin{pmatrix}-1\\6\\12\end{pmatrix}$$

$$\begin{pmatrix}-2-\lambda\\1+2\lambda\\3+3\lambda\end{pmatrix}=\begin{pmatrix}-\alpha\\5+6\alpha\\15+12\alpha\end{pmatrix}$$

$2+\lambda=\alpha \Rightarrow \alpha-\lambda=2$

$1+2\lambda=5+6\alpha \Rightarrow 3\alpha-\lambda=-2$

Solving both equations simultaneously,

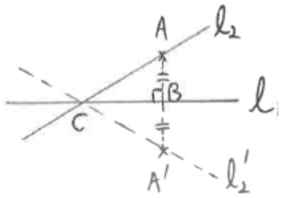
$\alpha=-2, \lambda=-4$

$$\overrightarrow{OC}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+(-4)\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}2\\-7\\-9\end{pmatrix}$$

(ii)

By Ratio Theroem, $\overrightarrow{OB}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$

$$\begin{pmatrix}-5\\7\\12\end{pmatrix}=\frac{1}{2}\left[\begin{pmatrix}0\\5\\15\end{pmatrix}+\overrightarrow{OA'}\right]$$



$$\overrightarrow{OA'}=\begin{pmatrix}-10\\9\\9\end{pmatrix}$$

$$\overrightarrow{CA'}=\begin{pmatrix}-10\\9\\9\end{pmatrix}-\begin{pmatrix}2\\-7\\-9\end{pmatrix}=\begin{pmatrix}-12\\16\\18\end{pmatrix}=2\begin{pmatrix}-6\\8\\9\end{pmatrix}$$

<u>Method 1</u>

A vector equation of the line of intersection is

$$\mathbf{r}=\begin{pmatrix}-10\\9\\9\end{pmatrix}+s\begin{pmatrix}-6\\8\\9\end{pmatrix},\ s\in\mathbb{R}$$

Cartesian equation of the line of intersection is

$$\frac{-x-10}{6}=\frac{y-9}{8}=\frac{z-9}{9}$$

Method 2

A vector equation of the line of intersection is

$$\mathbf{r} = \begin{pmatrix} 2 \\ -7 \\ -9 \end{pmatrix} + t\begin{pmatrix} -6 \\ 8 \\ 9 \end{pmatrix},\ t \in \mathbb{R}$$

Cartesian equation of the line of intersection is

$$\frac{2-x}{6} = \frac{y+7}{8} = \frac{z+9}{9}$$

(iii)

$$x + y - 2z = -1 \Leftrightarrow \mathbf{r}\bullet\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -1$$

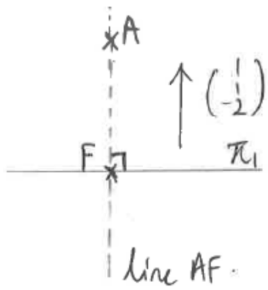
Let the point F be the foot of the perpendicular from A to π_1

$$\text{Line } AF:\ \mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ 15 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\pi_1 : x + y - 2z = -1 \Leftrightarrow \mathbf{r}\bullet\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -1$$

By substitution,

$$\left[\begin{pmatrix} 0 \\ 5 \\ 15 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\right]\bullet\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -1$$



$$\begin{pmatrix} \mu \\ 5+\mu \\ 15-2\mu \end{pmatrix}\bullet\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -1$$

$$6\mu - 25 = -1 \Rightarrow \mu = 4$$

$$\overrightarrow{OF} = \begin{pmatrix} 0 \\ 5 \\ 15 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 7 \end{pmatrix} \Rightarrow F \text{ is } (4,9,7)$$

$$\overrightarrow{AF} = \begin{pmatrix} 4 \\ 9 \\ 7 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 15 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -8 \end{pmatrix}$$

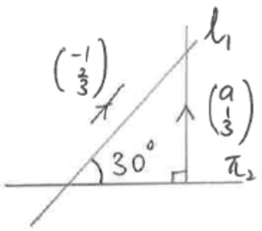
The required perpendicular distance

$$= \left|\overrightarrow{AF}\right| = \sqrt{4^2 + 4^2 + (-8)^2} = \sqrt{96} = 4\sqrt{6} \text{ units}$$

(iv)

$$\sin 30^{\circ} = \frac{\begin{pmatrix}-1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}a\\1\\3\end{pmatrix}}{\sqrt{14}\sqrt{a^2+10}}$$

$$\left(\frac{1}{2}\right)\sqrt{14}\sqrt{a^2+10} = -a+11$$



$$\sqrt{14(a^2+10)} = 2(11-a)$$

$$14(a^2+10) = 4(11-a)^2$$

$$10a^2+88a-344=0$$

$a = 2.9321$ or -11.732

$a = 2.93$ or -11.7 (3 s.f)

(v)

$$\pi_2 : ax+y+3z=5 \Leftrightarrow \mathbf{r}\bullet\begin{pmatrix}a\\1\\3\end{pmatrix}=5$$

A direction vector parallel to the line of intersection

$$=\begin{pmatrix}1\\1\\-2\end{pmatrix}\times\begin{pmatrix}a\\1\\3\end{pmatrix}=\begin{pmatrix}5\\-2a-3\\1-a\end{pmatrix}$$

By comparison to $\begin{pmatrix}5\\-1\\2\end{pmatrix}$, $-2a-3=-1 \Rightarrow a=-1$

(verified by $1-a=2 \Rightarrow a=-1$)

$\pi_1 :\ x+y-2z=-1$

$\pi_2 :\ -x+y+3z=5$

Let $z=0$, $x+y=-1$,

$-x+y=5$

Solving both equations simultaneously, $y=2,\ x=-3$.

$(-3,2,0)$ lies on the line of intersection.

A possible position vector of V is $\begin{pmatrix}-3\\2\\0\end{pmatrix}$.

11 **(i)**

$$x_{n+1}-x_n=\frac{13}{2}-n$$

$$\sum_{r=1}^{n}\left(x_r-x_{r-1}\right)=\sum_{r=1}^{n}\left(\frac{13}{2}-r\right)$$

$$\begin{pmatrix} x_1-x_0 \\ +x_2-x_1 \\ +x_3-x_2 \\ \ldots \\ +x_{n-1}-x_{n-2} \\ +x_n-x_{n-1} \end{pmatrix}=\sum_{r=1}^{n}\frac{13}{2}-\sum_{r=1}^{n}r$$

$$x_n-x_0=\frac{13}{2}n-\frac{n}{2}(n+1)$$

$$x_n-8=6n-\frac{1}{2}n^2$$

$$x_n=-\frac{1}{2}n^2+6n+8$$

(ii)

$$\frac{dx}{dt}=k\left(50-2t-x\right)$$

Given that $\frac{dx}{dt}=14$, $x=8$ when $t=0$,

$$14=k\left(50-0-8\right)$$

$$k=\frac{1}{3}$$

$$\frac{dx}{dt}=\frac{1}{3}\left(50-2t-x\right) \text{ (shown)}$$

$$\frac{dx}{dt}=\frac{1}{3}\left(50-2t-x\right)$$

$$u=2t+x \Rightarrow x=u-2t$$

$$\frac{dx}{dt}=\frac{du}{dt}-2$$

$$\frac{du}{dt}-2=\frac{1}{3}\left[50-2t-(u-2t)\right]$$

$$\frac{du}{dt}=\frac{1}{3}\left(50-u\right)+2$$

$$\frac{du}{dt}=\frac{1}{3}\left(56-u\right)$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{1}{3}(56-u)$$

$$\frac{1}{56-u}\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{1}{3}$$

$$\int\frac{1}{56-u}\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t=\int\frac{1}{3}\mathrm{d}t$$

$$-\ln|56-u|=\frac{1}{3}t+C_1$$

$$\ln|56-u|=-\frac{1}{3}t+C_2 \text{ where } C_2=-C_1$$

$$|56-u|=\mathrm{e}^{-\frac{1}{3}t+C_2}$$

$$56-u=\pm\mathrm{e}^{C_2}\mathrm{e}^{-\frac{1}{3}t}$$

$$56-u=A\mathrm{e}^{-\frac{1}{3}t} \text{ where } A=\pm\mathrm{e}^{C_2}$$

$$u=56-A\mathrm{e}^{-\frac{1}{3}t}$$

$$2t+x=56-A\mathrm{e}^{-\frac{1}{3}t}$$

$$x=56-A\mathrm{e}^{-\frac{1}{3}t}-2t$$

Given that $x=8$ when $t=0$,

$$8=56-A\mathrm{e}^{-\frac{1}{3}(0)}-2(0)\Rightarrow A=48$$

$$x=56-48\mathrm{e}^{-\frac{1}{3}t}-2t$$

(iii)

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{100}{(t+1)^3}$$

$$\int\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathrm{d}t=-100\int(t+1)^{-3}\mathrm{d}t$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=50(t+1)^{-2}+C$$

$$\int\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t=50\int(t+1)^{-2}\mathrm{d}t$$

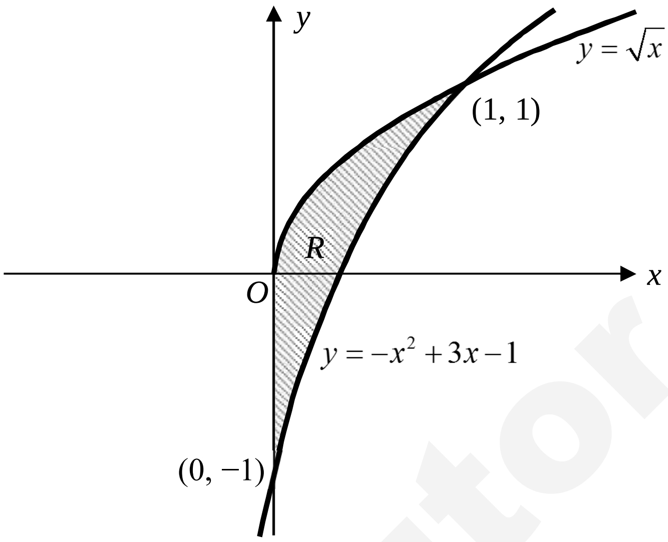
$$x=-50(t+1)^{-1}+Ct+D$$

$$x=-\frac{50}{t+1}+Ct+D$$

	Given that $\frac{\mathrm{d}x}{\mathrm{d}t}=41\frac{2}{3}=\frac{125}{3}$, $x=8$ when $t=0$, $\frac{125}{3}=50\left(0+1\right)^{-2}+C \Rightarrow C=-\frac{25}{3}$ $8=-\frac{50}{0+1}+0+D \Rightarrow D=58$ $x=-\frac{50}{t+1}-\frac{25}{3}t+58$

2018 MI H2 Math Prelim Paper 2
Section A: Pure Mathematics [40 marks]

1 In the diagram, the region R is bounded by the curves $y = -x^2 + 3x - 1$, $y = \sqrt{x}$ and the y-axis.



Without using a graphing calculator, find the volume of the solid generated when R is rotated through 2π radians about the y-axis. [4]

2 **(a)** A curve C has parametric equations

$$x = k\cos t, \quad y = k\sin 2t, \quad \text{for } -\frac{\pi}{4} \le t \le \frac{\pi}{4},$$

where k is a positive constant.

(i) Find the equation of the normal to C at the point $(k\cos p, k\sin 2p)$, where p is a constant such that $-\frac{\pi}{4} < p < \frac{\pi}{4}$. [3]

(ii) Deduce the equation of the tangent to C at the point where $t = 0$. [2]

(b) A rectangle of length $2|x|$ and breadth $2|y|$ is inscribed in the ellipse $\frac{x^2}{36} + \frac{y^2}{9} = 1$ where (x, y) is any point on the ellipse.

(i) Show that the area of the rectangle, A, is given by $A = 8\sqrt{9y^2 - y^4}$. [2]

(ii) Find the maximum area of the rectangle. [4]

3	**(a)**	Given that $a-\mathrm{i}$ is a root of the equation $z^3+4(1+\mathrm{i})z^2+(-2+9\mathrm{i})z-5+\mathrm{i}=0$, where a is a real constant, show that $(a^3+4a^2+3a)+(a^2+a)\mathrm{i}=0$ and find all the roots of the equation. [4]
		Deduce the roots of the equation $z^3+8(1+\mathrm{i})z^2+4(-2+9\mathrm{i})z+8(-5+\mathrm{i})=0$. [2]
	(b)	It is given that $w=\left(1-\mathrm{i}\sqrt{3}\right)^4$.
		(i) Without using a graphing calculator, find the modulus and argument of w. [2]
		(ii) Hence find the three smallest positive whole number values of n for which $\dfrac{w^n}{w^*}$ is a real number. [4]

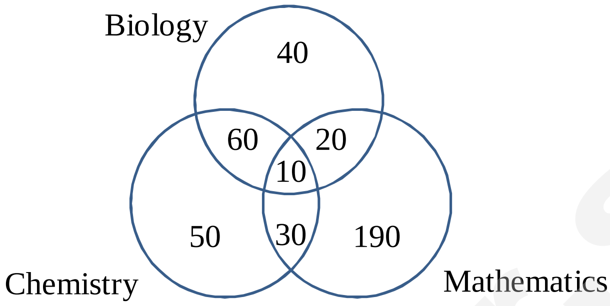
4		Relative to the origin O, the position vectors of points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. It is given that $\mathbf{a}$ and $\mathbf{b}-\mathbf{a}$ are perpendicular and C lies on AB produced such that $AC:AB=4:3$.
	(i)	If $\mathbf{a}$ is a unit vector, show that $\lvert\mathbf{b}\rvert > 1$. [3]
		Given further that $\lvert\mathbf{b}\rvert = 2$, find the angle between $\mathbf{a}$ and $\mathbf{b}$. [1]
	(ii)	The direction cosines of $\mathbf{b}$ are 0.6, λ, μ and $\mathbf{b}$ is perpendicular to the y-axis. Find
		(a) the angle $\mathbf{b}$ makes with the x-axis, [1]
		(b) λ and μ. [2]
	(iii)	By expressing $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, show that $\lvert\mathbf{c}\cdot\mathbf{a}\rvert = \mathbf{a}\cdot\mathbf{a}$. [3]
		Hence state the length of projection of $\mathbf{c}$ onto $\mathbf{a}$ in terms of $\mathbf{a}$. [1]
	(iv)	Give a geometrical interpretation of $\lvert\mathbf{c}\times\mathbf{a}\rvert$ and hence evaluate $\dfrac{\lvert\mathbf{c}\times\mathbf{a}\rvert}{\lvert\mathbf{b}\times\mathbf{a}\rvert}$. [2]

Section B: Probability and Statistics [60 marks]

5	A biased die is such that the probability of getting a score of 1, 2, 3, 4, 5 and 6 is $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$, p and q respectively. **(i)** If the mean score is 3.5, find p and q. [2] A game is played by throwing the biased die and a fair coin together. If the coin shows a head, the player wins \$1 more than the score on the die. If the coin shows a tail, the player wins \$1 less than the score on the die. The winnings from one game is denoted by W. **(ii)** Find $\text{P}(W = w)$ for all possible values of w. [2] **(iii)** Without any further calculations, state whether you would play this game if you need to pay \$4 at the start of the game, justifying your answer. [1]

6	In the 2018 FIFA World Cup, the Morocco football team started the opening match against Iran with 1 goalkeeper, 3 defenders, 4 midfielders and 3 forwards. **(i)** Before the match, the 11 selected players, together with their coach stand in a line for a photo shoot. In how many ways can this be done if the coach and the goalkeeper must stand at either ends and one particular defender and one particular forward must not stand together? [2] **(ii)** During half-time, the coach gathered the 11 players at a round table with 12 numbered seats to discuss strategies for the second half of the match. In how many ways can this be done if the 3 defenders must be seated together, the 4 midfielders must be seated together and the 3 forwards must be seated together? [3] **(iii)** After the match, the coach decides to construct four-lettered code-words from the 7 letters of the word MOROCCO. How many such code-words are there? [3]

7	The Venn diagram below shows the number of students studying Biology, Chemistry and Mathematics in a junior college.



One of the students is chosen at random.

B is the event that the student studies Biology
C is the event that the student studies Chemistry
M is the event that the student studies Mathematics

(i) Determine if B and C are independent, justifying your answer. [1]

(ii) Find

(a) $\mathrm{P}(B \cup C')$ and [1]

(b) $\mathrm{P}(C \mid M \cap B)$. [1]

The junior college also offers Further Mathematics as a subject. A student studying Further Mathematics must also study Mathematics. Given that the event that a randomly chosen student studying Further Mathematics is independent of C, find the largest possible number of students studying Further Mathematics. [3]

8 The random variable X has a normal distribution with mean μ and standard deviation 2.

(i) Given that $\mathrm{P}(X \le 1) = 0.1587$, find the value of μ, giving your answer to the nearest integer. [2]

(ii) Given further that $\mathrm{P}(X > 3k) = \mathrm{P}(X < 9k + 4)$ and using the answer obtained in part **(i)**, find the value of k. [1]

The random variable Y is related to X by the formula $Y = 10 - X$.

(iii) Find $\mathrm{P}(\overline{Y} > 6)$, where $\overline{Y}$ is the mean of two independent observations of Y. [3]

9	A sample of 10 students are selected from a mixed school to participate in a survey on the school sports facilities. The number of male students in a sample of 10 is denoted by the random variable X and the proportion of male students in the mixed school is p. **(i)** State, in the context of this question, two assumptions required for X to be well modelled by a binomial distribution. [2] Assume now that these assumptions do in fact hold. **(ii)** Given that $\mathrm{P}(X \le 1) = 0.05,$ write down an equation for the value of p, and find this value numerically. [2] **(iii)** 8 such samples of 10 students are selected to participate in the household income survey. Find the probability that exactly 7 of the samples have at least 2 male students. [2] For the rest of the question, take $p = 0.4$. **(iv)** Find the most probable number of **female** students selected in a sample of 10 students. [2] **(v)** 60 samples of 10 students each are selected to participate in the school climate survey. By using a suitable approximation, estimate the probability that the total number of male students selected exceeds 230. [3]

10	The distributor claims that the mean mass of cereal in a randomly chosen packet is 600 grams. A retailer suspects that the mean mass of the cereal is being overstated. He takes a random sample of 50 packets of cereal and weighs the content, x grams, in each packet. The results are summarised as follows: $$\sum(x-600)=-8 \text{ and } \sum(x-600)^2=11.3$$ **(i)** Find unbiased estimates of the population mean and variance of the mass of cereal in a packet. [2] **(ii)** Test, at the 1% level of significance, whether the distributor's claim has been overstated. [4] **(iii)** State whether there is a need to assume a normal population in conducting the test in part **(ii)**, justifying your answer. [1] **(iv)** Explain, in this context, the meaning of the p-value of the test obtained in part **(ii)**. [1] The packaging process has been changed so that the mass of cereal in a randomly chosen packet follows a normal distribution with a standard deviation of 0.5 grams. The distributor now claims that the mean mass of cereal in a randomly chosen packet is μ_0 grams. The retailer selects a new random sample of 25 packets of cereal and the sample mean mass is found to be 600.6 grams. Find the range of possible values of μ_0 so that the retailer's suspicion that the mean mass differs from μ_0 grams is valid at the 5% level of significance. Give your answer correct to 2 decimal places. [4]

11	During the National Step Challenge period, participants are able to redeem rewards if they obtained 750 healthpoints in the first tier. In any day during the period, participants will earn 10 healthpoints when they clocked 5000 steps, and 25 healthpoints when they clocked 7500 steps and a maximum of 40 healthpoints when they clocked 10000 steps. Kenny recorded his steps count using his step tracker in the first two weeks as follows:

Day x	1	3	5	7	9	11	13
Steps y (in thousands)	3.8	5.9	6.7	7.5	6.9	8.2	8.5

(i) Draw a scatter diagram showing the above data. [1]

(ii) Suggest a possible reason why one of the step counts does not seem to follow the trend. [1]

Assume that the reason given in part **(ii)** is valid, the outlier is removed.

(iii) Using the remaining six points, find both equations of the least squares regression line of y on x and that of x on y. [2]

(iv) Interpret, in the context of this question, the gradient of the least squares regression line of y on x obtained in part **(iii)**. [1]

(v) Use a suitable regression line found in part **(iii)** to estimate the number of days taken by Kenny to clock 10000 steps. Justify the choice of the regression line used. [2]

It is decided to fit a model of the form $\ln(L-y)=a+bx$, where L is a suitable constant. The product moment correlation coefficient between x and $\ln(L-y)$ is denoted by r. The following table gives values of r for some possible values of L.

L	10.1	10.2	10.3
r		-0.98424	-0.98318

(vi) Calculate the value of r for $L=10.1$, giving your answer correct to 5 decimal places. Hence, with the help of the table, suggest with a reason which of 10.1, 10.2 or 10.3 is the most appropriate value for L. [2]

(vii) Using this value of L, calculate the values of a and b, and use them to predict the number of steps clocked by Kenny on Day 10. Comment on the reliability of this estimate. [3]

2018 MI H2 Math Prelim Paper 2

Section A: Pure Mathematics [40 marks]

Qn	Solution
1	$y=-x^2+3x-1 \Rightarrow x^2-3x+1+y=0$ $\Rightarrow x=\dfrac{-(-3)\pm\sqrt{(-3)^2-4(1)(1+y)}}{2(1)}$ $\Rightarrow x=\dfrac{3\pm\sqrt{5-4y}}{2}$ Since $x<\dfrac{3}{2}$, $x=\dfrac{3-\sqrt{5-4y}}{2}$ Required volume = $\pi\left[\int_{-1}^{1}\left(\dfrac{3-\sqrt{5-4y}}{2}\right)^2 \mathrm{d}y-\int_0^1\left(y^2\right)^2 \mathrm{d}y\right]$ $=\pi\left[\dfrac{1}{4}\int_{-1}^{1} 9-6\sqrt{5-4y}+(5-4y)\,\mathrm{d}y-\int_0^1 y^4\,\mathrm{d}y\right]$ $=\pi\left[\dfrac{1}{2}\int_{-1}^{1} 7-2y-3(5-4y)^{\frac{1}{2}}\,\mathrm{d}y-\int_0^1 y^4\,\mathrm{d}y\right]$ $=\dfrac{\pi}{2}\left[7y-y^2+\dfrac{1}{2}(5-4y)^{\frac{3}{2}}\right]_{-1}^{1}-\left[\dfrac{\pi}{5}y^5\right]_0^1$ $=\dfrac{\pi}{2}\left[\left(7-1+\dfrac{1}{2}\right)-\left(-7-1+\dfrac{1}{2}9^{\frac{3}{2}}\right)\right]-\dfrac{\pi}{5}$ $=\dfrac{\pi}{2}\left(\dfrac{13}{2}-\dfrac{11}{2}\right)-\dfrac{\pi}{5}=\dfrac{\pi}{2}-\dfrac{\pi}{5}=0.3\pi$ units^3.

2(a) (i)	$x=k\cos t \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t}=-k\sin t$ $y=k\sin 2t \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}t}=2k\cos 2t$ $\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}t}\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{2k\cos 2t}{-k\sin t}=-\dfrac{2\cos 2t}{\sin t}$. At $(k\cos p, k\sin 2p)$, $t=p \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{2\cos 2p}{\sin p}$. Gradient of normal = $\dfrac{\sin p}{2\cos 2p}$. Equation of normal: $y-k\sin 2p=\dfrac{\sin p}{2\cos 2p}(x-k\cos p)$
2(a) (ii)	From **(a)(i)**, when $t=p=0$, equation of normal at $(k\cos 0, k\sin 2(0))$, i.e. $(k,0)$

is $y - k\sin 2(0) = \dfrac{\sin 0}{2\cos 2(0)}(x - k\cos 0)$, i.e. $y = 0$.

Hence, equation of tangent at $(k, 0)$ is $x = k$.

2(b) (i)

$$A = \left(2|x|\right)\left(2|y|\right) = 4\sqrt{x^2}\sqrt{y^2} = 4\sqrt{x^2y^2}$$

$$= 4\sqrt{36\left(1 - \frac{y^2}{9}\right)y^2}$$

$$= 4\sqrt{\left(36 - 4y^2\right)y^2}$$

$$= 8\sqrt{9y^2 - y^4} \text{ (shown)}$$

2(b) (ii)

At stationary points,

$$\frac{\mathrm{d}A}{\mathrm{d}y} = \frac{8}{2}\left(9y^2 - y^4\right)^{-\frac{1}{2}}\left(18y - 4y^3\right) = \frac{8y\left(9 - 2y^2\right)}{\sqrt{9y^2 - y^4}} = 0$$

$$\Rightarrow 8y\left(9 - 2y^2\right) = 0$$

$$\Rightarrow y = 0 \text{ (rejected as } y \neq 0) \text{ or } y = \pm\sqrt{\frac{9}{2}} = \pm\frac{3}{\sqrt{2}} = \pm\frac{3}{2}\sqrt{2}.$$

Method 1: First derivative test

y	$\left(\pm\frac{3}{2}\sqrt{2}\right)^-$	$\pm\frac{3}{2}\sqrt{2}$	$\left(\pm\frac{3}{2}\sqrt{2}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}y}$	+ve	0	-ve
Slope			

Method 2: Second derivative test

$$\sqrt{9y^2 - y^4}\,\frac{\mathrm{d}A}{\mathrm{d}y} = 8(9y - 2y^3)$$

$$\sqrt{9y^2 - y^4}\,\frac{\mathrm{d}^2A}{\mathrm{d}y^2} + \frac{\mathrm{d}A}{\mathrm{d}y}\left[\frac{\mathrm{d}}{\mathrm{d}y}\left(\sqrt{9y^2 - y^4}\right)\right] = 8(9 - 6y^2)$$

When $y = \pm\frac{3}{2}\sqrt{2}$, $4.5\frac{\mathrm{d}^2A}{\mathrm{d}y^2} + 0 = 8(-18) \Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}y^2} = -32 < 0.$

Hence, A is a maximum when $y = \pm\frac{3}{2}\sqrt{2}$.

Required area $= 8\sqrt{9\left(\pm\frac{3}{2}\sqrt{2}\right)^2 - \left(\pm\frac{3}{2}\sqrt{2}\right)^4}$

$= 8\sqrt{9\left(\frac{9}{2}\right) - \frac{81}{4}} = 8\sqrt{\frac{81}{4}}$

$= 36$ units^2.

3(a) $a - \mathrm{i}$ is a root of $z^3 + 4(1+\mathrm{i})z^2 + (-2+9\mathrm{i})z - 5 + \mathrm{i} = 0$

	$\Rightarrow (a-\mathrm{i})^3+4(1+\mathrm{i})(a-\mathrm{i})^2+(-2+9\mathrm{i})(a-\mathrm{i})-5+\mathrm{i}=0$ $\Rightarrow (a^3-3a^2\mathrm{i}-3a+\mathrm{i})+4(1+\mathrm{i})(a^2-2a\mathrm{i}-1)\ -2a+4+(3+9a)\mathrm{i}=0$ $\Rightarrow a^3-5a+4-3a^2\mathrm{i}+4\mathrm{i}+9a\mathrm{i}+4(a^2-2a\mathrm{i}-1+a^2\mathrm{i}+2a-\mathrm{i})=0$ $\Rightarrow (a^3+4a^2+3a)+(a^2+a)\mathrm{i}=0$. (shown) Comparing real and imaginary parts, $a(a+1)(a+3)=0 \Rightarrow a=0$ or $a=-1$ or $a=-3$ (N.A) $a(a+1)=0 \Rightarrow a=0$ or $a=-1$. Let the third root be w. Then $z^3+4(1+\mathrm{i})z^2+(-2+9\mathrm{i})z-5+\mathrm{i}$ $=[z-(-\mathrm{i})][z-(-1-\mathrm{i})][z-(w)]=(z+\mathrm{i})(z+1+\mathrm{i})(z-w)$ Comparing constant term, $\mathrm{i}(1+\mathrm{i})(-w)=-5+\mathrm{i}$ $\Rightarrow w=\dfrac{-5+\mathrm{i}}{-\mathrm{i}(1+\mathrm{i})}=\dfrac{-5+\mathrm{i}}{1-\mathrm{i}}\times\dfrac{1+\mathrm{i}}{1+\mathrm{i}}=\dfrac{-5+\mathrm{i}-5\mathrm{i}-1}{2}=-3-2\mathrm{i}.$ Hence, all the roots are $-\mathrm{i},\ -1-\mathrm{i}$ and $-3-2\mathrm{i}$. $z^3+8(1+\mathrm{i})z^2+4(-2+9\mathrm{i})z+8(-5+\mathrm{i})=0$ $\dfrac{1}{8}z^3+(1+\mathrm{i})z^2+\dfrac{1}{2}(-2+9\mathrm{i})z+(-5+\mathrm{i})=0$ $\left(\dfrac{1}{2}z\right)^3+4(1+\mathrm{i})\left(\dfrac{1}{2}z\right)^2+(-2+9\mathrm{i})\left(\dfrac{1}{2}z\right)+(-5+\mathrm{i})=0$ $\Rightarrow \dfrac{1}{2}z=-\mathrm{i}$ or $-1-\mathrm{i}$ or $-3-2\mathrm{i}$ (from first part) $\Rightarrow z=-2\mathrm{i}$ or $-2-2\mathrm{i}$ or $-6-4\mathrm{i}$
3(b) (i)	$\left\lvert\left(1-\sqrt{3}\mathrm{i}\right)^4\right\rvert=\left\lvert 1-\sqrt{3}\mathrm{i}\right\rvert^4=\left(\sqrt{1^2+(-\sqrt{3})^2}\right)^4=2^4=16.$ $\arg\left(1-\sqrt{3}\mathrm{i}\right)^4=4\arg\left(1-\sqrt{3}\mathrm{i}\right)=4\left(-\dfrac{\pi}{3}\right)=-\dfrac{4}{3}\pi.$ $\therefore \arg\left(1-\sqrt{3}\mathrm{i}\right)^4=-\dfrac{4\pi}{3}+2\pi=\dfrac{2}{3}\pi.$ (for principal argument)
3(b) (ii)	$\dfrac{w^n}{w^*}$ is real $\Rightarrow \arg\left(\dfrac{w^n}{w^*}\right)=k\pi$, where $k\in\mathbb{Z}$ $\Rightarrow \arg\left(w^n\right)-\arg\left(w^*\right)=k\pi$ $\Rightarrow n\arg\left(w\right)+\arg\left(w\right)=k\pi$ $\Rightarrow \dfrac{2\pi}{3}(n+1)=k\pi$ (from first part) OR $n=\dfrac{3}{2}k-1$ Hence, the required values of n are 2, 5 and 8. **Alternatively**, $\dfrac{w^n}{w^*}$ is real $\Rightarrow \mathrm{Im}\left(\dfrac{w^n}{w^*}\right)=0 \Rightarrow \sin\left[\arg\left(\dfrac{w^n}{w^*}\right)\right]=0$

	$\Rightarrow \sin\left[\frac{2\pi}{3}(n+1)\right]=0 \Rightarrow \frac{2\pi}{3}(n+1)=k\pi$, where $k\in\mathbb{Z}$. Hence, the required values of n are 2, 5 and 8.

4(i)	$\mathbf{a} \perp \mathbf{b}-\mathbf{a} \Rightarrow \mathbf{a}\cdot(\mathbf{b}-\mathbf{a})=0$ $\Rightarrow \mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2=1$ (since $\lvert\mathbf{a}\rvert=1$) $\Rightarrow \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta=1$ where θ is the angle between **a** and **b** $\Rightarrow \cos\theta=\frac{1}{\lvert\mathbf{b}\rvert}<1 \Rightarrow \lvert\mathbf{b}\rvert>1$. (shown) $\lvert\mathbf{b}\rvert=2 \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=60^{\circ}$.
4(ii) (a)	Let α be the required angle. $\cos\alpha=0.6 \Rightarrow \alpha=\cos^{-1}0.6=53.1^{\circ}$. (1 d.p.)
4(ii) (b)	$\lambda=\cos 90^{\circ}=0$. $0.6^2+0^2+\mu^2=1 \Rightarrow \mu^2=1-0.36=0.64 \Rightarrow \mu=\pm 0.8$.
4(iii)	By Ratio Theorem, $\mathbf{b}=\frac{\mathbf{a}+3\mathbf{c}}{4} \Rightarrow \mathbf{c}=\frac{4\mathbf{b}-\mathbf{a}}{3}$. $\lvert\mathbf{c}\cdot\mathbf{a}\rvert=\left\lvert\left(\frac{4\mathbf{b}-\mathbf{a}}{3}\right)\cdot\mathbf{a}\right\rvert=\left\lvert\frac{4}{3}(\mathbf{b}\cdot\mathbf{a})-\frac{1}{3}(\mathbf{a}\cdot\mathbf{a})\right\rvert$ $=\left\lvert\frac{4}{3}(\mathbf{a}\cdot\mathbf{a})-\frac{1}{3}(\mathbf{a}\cdot\mathbf{a})\right\rvert$ (from (**i**)) $=\lvert\mathbf{a}\cdot\mathbf{a}\rvert=\left\lvert\lvert\mathbf{a}\rvert^2\right\rvert=\lvert\mathbf{a}\rvert^2=\mathbf{a}\cdot\mathbf{a}$. (shown) Required length = $\frac{\lvert\mathbf{c}\cdot\mathbf{a}\rvert}{\lvert\mathbf{a}\rvert}=\frac{\mathbf{a}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert}=\frac{\lvert\mathbf{a}\rvert^2}{\lvert\mathbf{a}\rvert}=\lvert\mathbf{a}\rvert$.
4(iv)	$\lvert\mathbf{c}\times\mathbf{a}\rvert$ = Twice the area of triangle OAC $\lvert\mathbf{b}\times\mathbf{a}\rvert$ = Twice the area of triangle OAB = $(AB)h$, where h is the height of triangle OAB with AB as the base. Since A, B and C are collinear, $\lvert\mathbf{c}\times\mathbf{a}\rvert=(AC)h$. Hence, $\frac{\lvert\mathbf{c}\times\mathbf{a}\rvert}{\lvert\mathbf{b}\times\mathbf{a}\rvert}=\frac{(AC)h}{(AB)h}=\frac{AC}{AB}=\frac{4}{3}$.

Section B: Probability and Statistics [60 marks]

5(i)	Mean score = $\frac{1}{5}+\frac{2}{6}+\frac{3}{7}+\frac{4}{8}+5p+6q=3.5 \Rightarrow 5p+6q=\frac{214}{105}$ … (1) $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+p+q=1 \Rightarrow p+q=\frac{307}{840}$ … (2) From graphing calculator, $p=\frac{13}{84}$ and $q=\frac{59}{280}$.
5(ii)	Let X be the score of the die and Y be the outcome of coin.

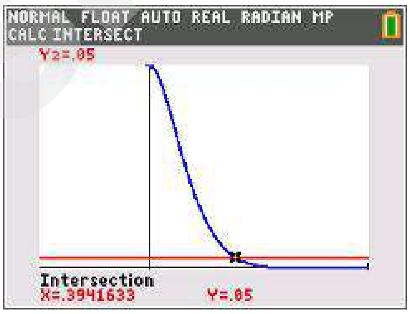
	For $w = 0, 1$, $P(W = w) = P(X = w+1, Y = \text{tail}) = \frac{1}{2}P(X = w+1)$. For $w = 2, 3, 4, 5$, $P(W = w) = P(X = w+1, Y = \text{tail}) + P(X = w-1, Y = \text{head})$ $= \frac{1}{2}\left[P(X = w-1) + P(X = w+1)\right]$ For $w = 6, 7$, $P(W = w) = P(X = w-1, Y = \text{head}) = \frac{1}{2}P(X = w-1)$. The probability distribution of W is given by: <table><tr><td>w</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td></tr><tr><td>$P(W = w)$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{12}$</td><td>$\frac{6}{35}$</td><td>$\frac{7}{48}$</td><td>$\frac{25}{168}$</td><td>$\frac{47}{280}$</td><td>$\frac{13}{168}$</td><td>$\frac{59}{560}$</td></tr></table>
5(iii)	Since the mean score of the die is 3.5 and the coin is fair, the expected winnings is \$3.5, hence I would not play this game since amount pay = \$4 > \$3.5 = expected winnings.

6(i)	Number of ways = $8! \times {}^9C_2 \times 2! \times 2! = 5806080$. OR Number of ways = $10! \times 2! - 9! \times 2! \times 2 = 5806080$.
6(ii)	Number of ways = $(5-1)! \times 3! \times 4! \times 3! \times 12 = 248832$.
6(iii)	Case 1: All 4 letters are distinct Number of code-words = ${}^4C_4 \times 4! = 24$. Case 2: Exactly 3 distinct letters Number of code-words = $\left({}^2C_1 \times {}^3C_2\right) \times \left(\frac{4!}{2!}\right) = 72$. Case 3: Exactly 2 distinct letters Number of code-words = $\left({}^4C_1 \times 3\right) + \left({}^4C_2\right) = 12 + 6 = 18$. Total number of code-words = 24 + 72 + 18 = 114.

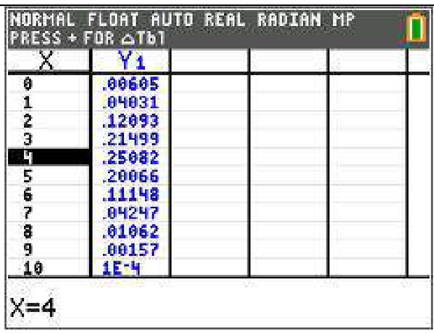
7(i)	$P(B) = \frac{130}{400} = \frac{13}{40}$ and $P(C) = \frac{150}{400} = \frac{3}{8}$. Since $P(B \cap C) = \frac{70}{400} = \frac{7}{40} \neq \frac{39}{320} = \frac{13}{40} \times \frac{3}{8} = P(B)P(C)$, B and C are not independent.
7(ii)(a)	$P\left(B \cup C'\right) = \frac{320}{400} = \frac{4}{5}$.
7(ii)(b)	$P\left(C \mid M \cap B\right) = \frac{10}{30} = \frac{1}{3}$.
7	Let x be the number of students studying Further Mathematics out of the 400 students and F be the event that the chosen student is studying Further Mathematics. $P(F \cap C) = P(F)\ P(C)$ (since F and C are independent)

	$= \frac{3}{8} \times \frac{x}{400} = \frac{3x}{3200}$ Since $F \subseteq M, F \cap C \subseteq M \cap C$. Hence, $\mathrm{P}(F) \le \mathrm{P}(M)$ and $\mathrm{P}(F \cap C) \le \mathrm{P}(M \cap C)$ $\Rightarrow \mathrm{P}(F) \le \frac{250}{400} = \frac{5}{8}$ and $\mathrm{P}(F \cap C) \le \frac{40}{400} = \frac{1}{10}$ $\Rightarrow \frac{x}{400} \le \frac{5}{8}$ and $\frac{3x}{3200} \le \frac{1}{10}$ $\Rightarrow x \le 250$ and $x \le \frac{320}{3} \Rightarrow x \le 106\frac{2}{3}$. Hence, required number of students is 106.

8(i)	Given: $X \sim \mathrm{N}\left(\mu, 2^2\right)$. $\mathrm{P}\left(X \le 1\right) = \mathrm{P}\left(Z \le \frac{1-\mu}{2}\right) = 0.1587$, where $Z \sim \mathrm{N}(0,1)$ $\Rightarrow \frac{1-\mu}{2} \approx -0.9998 \Rightarrow \mu = 3$. (nearest integer)
8(ii)	By symmetry, $\frac{(3k)+(9k+4)}{2} = \mu = 3 \Rightarrow k = \frac{1}{6}$.
8(iii)	E($\bar{Y}$) = E(Y) = 10 − E(X) = 10 − 3 = 7. Var($\bar{Y}$) = $\frac{\mathrm{Var}(Y)}{2} = \frac{\mathrm{Var}(X)}{2} = \frac{4}{2} = 2$. $\mathrm{P}\left(\bar{Y} > 6\right) \approx 0.76025 = 0.760$. (3 s.f.)

9(i)	Two assumptions are: 1. The probability of <u>a student being male</u> is the same for **each** <u>student</u>. 2. Whether <u>a student is male</u> is independent of other <u>students</u>.
9(ii)	$\mathrm{P}(X \le 1) = 0.05 \Rightarrow (1-p)^{10} + 10p(1-p)^9 = 0.05$ $\Rightarrow (1-p)^9(1+9p) = 0.05$  From graphing calculator, p = 0.394. (3 s.f.)
9(iii)	Let Y be the number of samples of 10 students, out of 8, with at least 2 male students. Then $Y \sim \mathrm{B}(8, \mathrm{P}(X \ge 2))$, i.e. $Y \sim \mathrm{B}(8, 0.95)$. Required probability = P(Y = 7) ≈ 0.27933 = 0.279. (3 s.f.) OR ${}^{8}\mathrm{C}_{7}(0.95)^7(0.05) = 0.279$. (3 s.f.)

9(iv)



x	$P(X = x)$
3	0.21499
4	0.25082
5	0.20066

From graphing calculator, the highest probability occurs when $X = 4$, so most probable number of male students in a sample of 10 is 4, i.e. most probable number of female students = 10 − 4 = 6.

9(v)

Let S be the sum of 60 independent observations of X.
$E(S) = 60E(X) = 60(10)(0.4) = 240$.
$\text{Var}(S) = 60\text{Var}(X) = 60(10)(0.4)(0.6) = 144$.
Since $n = 60$ is large, by Central Limit Theorem,
$S \sim N(240, 144)$ approximately.
$P(S > 230) \approx 0.79767 = 0.798$. (3 s.f.)

10(i)

Unbiased estimate of population mean mass,

$$\bar{x} = \frac{\sum(x-600)}{50} + 600 = \frac{-8}{50} + 600 = 599.84.$$

Unbiased estimate of population variance,

$$s^2 = \frac{1}{50-1}\left[\sum(x-600)^2 - \frac{1}{50}\left(\sum(x-600)\right)^2\right]$$

$$= \frac{1}{49}\left[11.3 - \frac{1}{50}(-8)^2\right] = \frac{10.02}{49} = \frac{501}{2450} = 0.204. \text{ (3 s.f.)}$$

10(ii)

Let X be the mass, in grams, of a randomly chosen packet of cereal.
$H_0 : \mu = 600$ (distributor's claim)
$H_1 : \mu < 600$ (claim is overstated)
Under H_0, since $n = 50$ is large, by Central Limit Theorem,

$$\bar{X} \sim N\left(600, \frac{1}{50} \times \frac{501}{2450}\right), \text{i.e. } N\left(600, \frac{5.01}{1225}\right) \text{ approximately.}$$

Use a z-test at $\alpha = 0.01$.
From graphing calculator, p-value = 0.00618 (3 s.f.)
Since p-value = $0.00618 < 0.01 = \alpha$, we reject H_0.
There is sufficient evidence at 1% level of significance to conclude that the distributor has overstated the claim.

10(iii)

No. Since sample size <u>$n = 50$ is large</u>, <u>by Central Limit Theorem</u>, the sample mean mass of a packet of cereal will be <u>approximately normal</u>.

10(iv)

A p-value of <u>0.00618</u> means that there is a probability of <u>0.00618</u> of obtaining <u>a sample mean mass of 599.84 grams</u> <u>**or less**</u>, when the <u>population mean mass of cereals per packet is assumed to be 600 grams</u>.

Let Y be the mass, in grams, of a randomly chosen packet of cereal, after the change in packaging process.

$H_0 : \mu = \mu_0$ (distributor's claim)

$H_1 : \mu \neq \mu_0$ (retailer's suspicion)

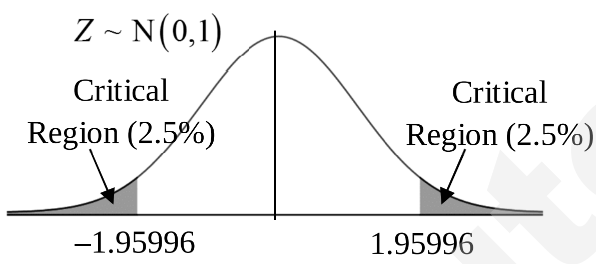
Under H_0, $\bar{Y} \sim N(\mu_0, \frac{0.5^2}{25})$, i.e. $N(\mu_0, 0.01)$.

Standardising the test statistic, $Z = \frac{\bar{Y}-\mu_0}{\sqrt{\frac{0.5^2}{25}}} \sim N(0,1)$.

Corresponding test statistic value: $z = \frac{600.6-\mu_0}{\sqrt{\frac{0.5^2}{25}}} = \frac{600.6-\mu_0}{\sqrt{0.01}}$

Critical values: –1.9599 and 1.9599

Critical region: $z \leq -1.95996$ or $z \geq 1.95996$

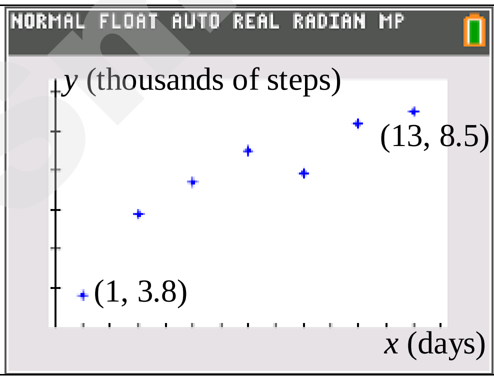


Since the retailer's suspicion is valid at 5% significance level, we **reject** H_0, $z = \frac{600.6-\mu_0}{\sqrt{\frac{0.5^2}{25}}}$ **falls in** the critical region.

$$\Rightarrow \frac{600.6-\mu_0}{\sqrt{0.01}} \leq -1.95996 \text{ or } \frac{600.6-\mu_0}{\sqrt{0.01}} \geq 1.95996$$

$$\Rightarrow 600.7960 \leq \mu_0 \text{ or } 600.4040 \geq \mu_0$$

$$\Rightarrow \mu_0 \leq 600.40 \text{ or } \mu_0 \geq 600.80 \text{ (2 d.p.)}$$

11(i)	
11(ii)	A possible reason is that the step tracker is not working properly on Day 9.
11(iii)	y on x line: $y = 0.34969x + 4.4354$ (5 s.f.), i.e. $y = 0.350x + 4.44$ (3 s.f.). x on y line: $x = 2.4769y - 10.094$ (5 s.f.), i.e. $x = 2.48y - 10.1$ (3 s.f.).
11(iv)	350 is the **expected** increase in the number of steps when the time increases by 1 day. OR Kenny is expected to increase 350 steps every day.

11(v)	Since x is the independent variable and y is the dependent variable, we should use the y on x regression line. When y = 10, $10 = 0.34969x + 4.4354 \Rightarrow x = \dfrac{10-4.4354}{0.34969} \approx 15.913 = 16$. (nearest whole number) The estimated number of days = 16.
11(vi)	For L = 10.1, r = –0.98533 (5 d.p.) The most appropriate value for L is 10.1 since its corresponding value of $\lvert r \rvert$ is closest to 1 as compared to that of 10.2 and 10.3.
11 (vii)	The required equation is $\ln(10.1 - y) = 1.8184 - 0.10859x$ (5 s.f.) Hence, a = 1.82. (3 s.f.) and b = – 0.109. (3 s.f.). When x = 10, $\ln(10.1 - y) = 1.8184 - 0.10859(10)$ $\Rightarrow y = 10.1 - e^{1.8184-1.0859} \approx 8.01973 = 8020$. (3 s.f.) Kenny is expected to clock 8020 steps on Day 10. This estimate is reliable since $\lvert r \rvert$ = 0.98533 is close to 1 and x = 10 is within the range of the data set.

NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

MATHEMATICS	**9758/01**
Paper 1	**10th September 2018**
	3 Hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

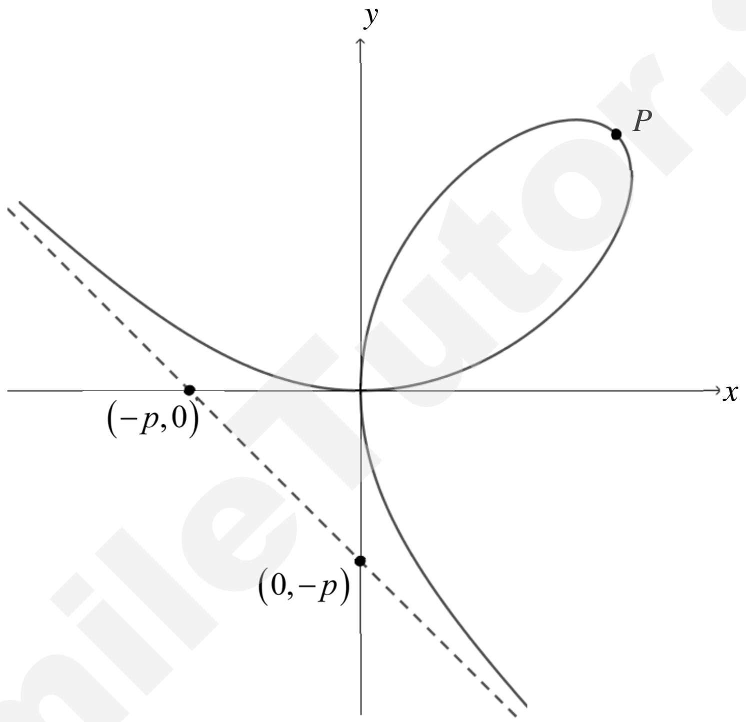
At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

NANYANG JUNIOR COLLEGE
Internal Examinations

1 A departmental store sells a pair of jeans at \$46.90, blouses at \$29.00 each and a pair of shoes at \$19.90. Items that are priced at more than \$25 are sold at a further discount of 15%. Betty bought twice as many blouses as jeans and she charged \$257.93 to her credit card for a total of 10 items. How many pairs of jeans, shoes and blouses did she buy? [4]

2 The diagram below shows the folium of Descartes curve with equation $x^3 + y^3 = 3pxy$, where $p > 0$, and the asymptote of the curve passes through the points $(-p, 0)$ and $(0, -p)$.



Point *P* lies on the curve such that the tangent at *P* is parallel to the asymptote of the curve. Find the coordinates of point *P* in terms of *p*. [6]

3 A curve *C* has equation $y^2 + 14y + 4x^2 + 16x + 16xy + 13 = 0$.

(i) If a real value of *x* is substituted into the equation, it becomes a quadratic equation in *y*. Given that there are two distinct values of *y* for this equation, show that $5x^2 + 8x + 3 > 0$, and hence find the set of possible values of *x*. [5]

(ii) Find the coordinates of the points where *C* cuts the *y*-axis. State with a reason whether *C* is a graph of a function. [2]

4 A curve D has equation $y = x|x-a|, a > 0$.

(i) Describe a pair of transformations which transforms the graph of D on to the graph of $y = 2 - x|x+a|$. [2]

(ii) Sketch D, giving the coordinates of the axial intercepts and turning point in terms of a. [2]

(iii) On a separate diagram, sketch the curve $y = \frac{1}{x|x-a|}$, giving the coordinates of the turning point and the equations of the asymptotes in terms of a. [3]

(iv) State the range of values of k if $\frac{1}{x|x-a|} = k$has exactly one solution. [1]

5 The line l passes through the points A and B with coordinates (5, 2, 4) and (4, −1, 3) respectively. The plane p has equation $4x + 7y + 5z = 24$.

(i) The point C lies on l such that the foot of perpendicular of C onto p has coordinates (3, 1, 1). Find the coordinates of C. [4]

Plane p_1 has equation $3x - 2y + \lambda z = \mu$.

(ii) What can be said about the values of λ and μ if l does not intersectp_1 ? [2]

(iii) Hence find the exact values of μ if the distance between p_1 and l is 2 units. [3]

6 The functions f and g are defined by

$$\mathrm{f} : x \mapsto \sin x \cos x,\ x \in \mathbb{R},\ -\frac{\pi}{2} < x \leq \frac{\pi}{2},$$

$$\mathrm{g} : x \mapsto \frac{1}{x}, \qquad x \in \mathbb{R}, 0 < x \leq 1.$$

(i) A function h is said to be odd if $h(-x) = -h(x)$ for all x in the domain of h. Show that g is not odd and determine if f is odd. [2]

(ii) Explain why f does not have an inverse. If the domain of f is further restricted to $0 < x \leq a$, where $a \in \mathbb{R}$, the function f^{-1}will exist. State the largest possible exact value of a. [2]

Use the value of a in **(ii)** for the rest of the question.

(iii) Sketch the graphs of f, $f^{-1}$and ff^{-1}on the same diagram. [3]

(iv) State with reasons, whether the composite functions fg and gf exist. If the composite function exists, find the rule, domain and range. [3]

7 **(a)** Referred to the origin O, points A and B have position vectors **a** and **b** respectively, where **a** and **b** are unit vectors.

(i) By using scalar product, show that the vector $\boldsymbol{a} + \boldsymbol{b}$ is the bisector of angle AOB. [3]

(ii) If the area of the triangle AOB is $\frac{1}{\sqrt{10}}$ units^2, state the exact value of the sine of angle AOB. [1]

(b) Referred to the origin O, points C and D have position vectors $7\boldsymbol{i}-8\boldsymbol{j}+7\boldsymbol{k}$ and $4\boldsymbol{i}+7\boldsymbol{j}+4\boldsymbol{k}$ respectively.

(i) Using vector product, find the exact shortest distance of the line, passing through points C and D, from the origin. [4]

(ii) Find angle OCD. [2]

8 A curve C is represented by the parametric equations

$$x=t^2(t+6),\ y=t^2+t-6,\ \text{for}\ \ t<0.$$

(i) Find the equation(s) of the tangent to the curve C which is parallel to the y-axis. [3]

(ii) Sketch C, showing clearly the axial intercepts. [2]

(iii) Let R be the finite region bounded by C and the line $x=16$. Find the area of R. [5]

9 **(a)** The sum, S_n, of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by

$$S_n=\frac{6}{13}\left(1-\frac{1}{3^{3n}}\right).$$

(i) Given that the series $\sum u_r$ converges, find the smallest integer n for which S_n is within 10^{-8} of the sum to infinity. [3]

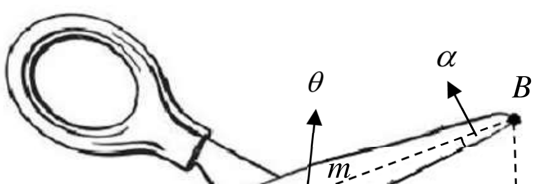
(ii) Find a formula for u_n in simplified form. [2]

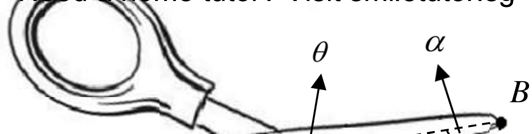
(b) Using the formulae for $sin(A \pm B)$, prove that

(i) $sin(2r+1)\theta - sin(2r-1)\theta = 2\cos 2r\theta \sin\theta$. [1]

(ii) Hence find an expression for $\sum_{r=1}^{n} sin^2 r\,\theta$, giving your answer in terms of $cos(n+1)\theta$, $sin\, n\,\theta$, $sin\,\theta$ and n, where $\theta \neq k\pi, k \in \mathbb{Z}$. [5]

10 The two blades of a pair of scissors are fastened at the point A. The distance from A to the tip of the blade at point B is m cm. Let the angle formed by the line AB and the bottom edge of the blade BC be α radians and the angle between AB and AC be θ radians (Figure 1). A piece of paper resting at point C is cut. As the paper is being cut, the blades come closer to each other and the length of AC increases as shown in Figure 2.





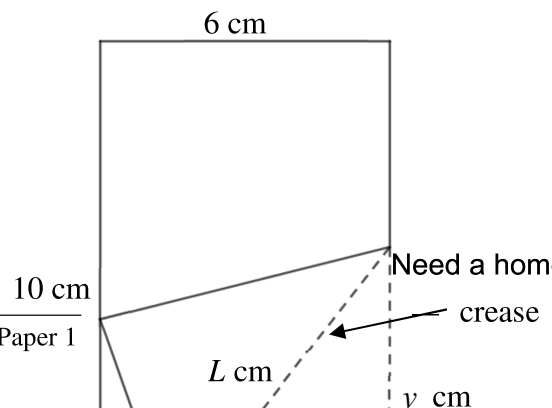
(i) By letting $AC = l$ cm, show that $l = \frac{m \sin \alpha}{\sin(\theta + \alpha)}$. [1]

(ii) Find $\frac{dl}{d\theta}$ in terms of m, θ and α. [1]

(iii) It is given that $m = 15$, $\alpha = \frac{\pi}{60}$ and θ is decreasing at a rate of $\frac{5\pi}{9}$ radians per second. Find the rate at which the paper is being cut at the instant when $\theta = \frac{\pi}{9}$. [3]

Question 10 continues on the next page

A triangle is cut out from a rectangular piece of paper measuring 10 cm by 6 cm using the scissors. To form this triangle, the right-hand corner is folded over so as to reach the left-most edge of the paper, forming a crease for the scissors to cut along, as shown in the diagram below. Let the length of the crease be L cm, the base of the triangle to be folded be x cm and the height of the triangle be y cm.



(iv) Show that $L = \sqrt{x^2 + \frac{3x^2}{x-3}}$. [3]

(v) Using differentiation, find the minimum length of the resulting crease. [4]

11 A patient in the hospital is being administrated a certain drug through an intravenous (IV) drip at a constant rate of 30 mg per hour. The rate of loss of the drug from the patient's body is proportional to *x*, where *x* (in mg) is the amount of drug in the patient's body at time *t* (in hours). The amount of drug in the patient's body needs to reach 120 mg for the treatment to be effective.

(i) Explain why the rate of change of *x* needs to be positive. [1]

(ii) Initially, there are no traces of the drug in the patient's body, and after 4 hours, the amount of drug in the patient's body is 82.6 mg. Show that

$x = A\left(1 - e^{-\frac{t}{5}}\right)$, where *A* is a constant to be determined. [5]

(iii) Find the time needed for the amount of drug in the patient's body to reach 120 mg. [1]

A medical worker, who studied mathematical biology, proposed that the rate of change of the amount of drug in the patient's body actually satisfies the differential equation

$$\frac{d^2x}{dt^2} = \frac{1}{\sqrt{2500-9t^2}}.$$

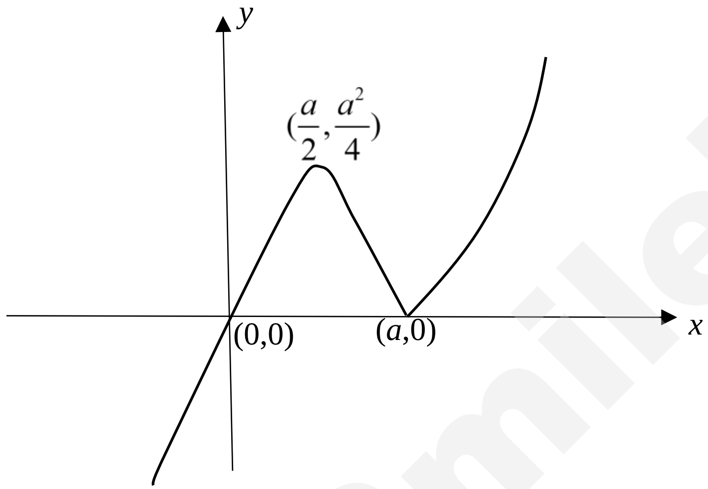
(iv) Find the general solution for the proposed differential equation, given that $x = 0$ when $t = 0$. [6]

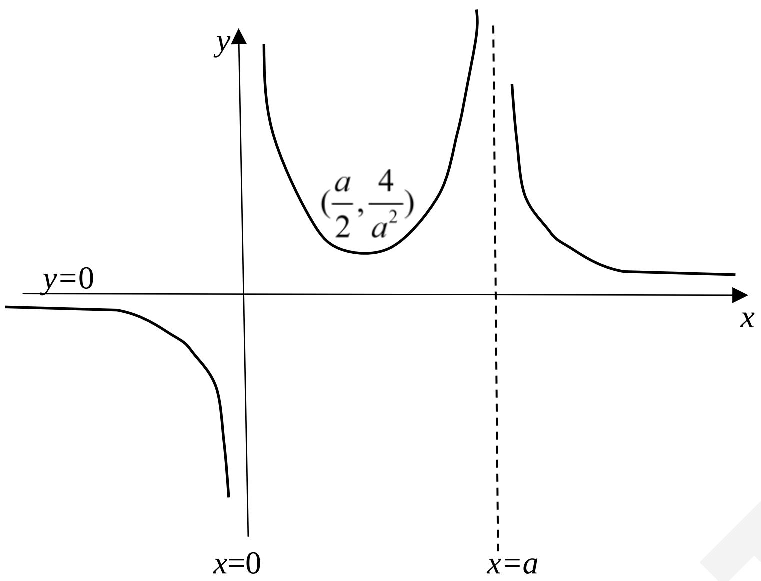
2018 NYJC J2 H2 Mathematics Preliminary exam 9758/1 Marking Guide

Qn	Suggested Answers			Guidance
1	Let x, y and z be the no. of jeans, blouses and pairs of shoes Betty bought. $46.90 \times 0.85x + 29 \times 0.85y + 19.9z = 257.93$ $x + y + z = 10$ $2x - y = 0$ Using GC, $x = 2$, $y = 4$, $z = 4$ Hence, Betty bought 2 pairs of jeans, 4 blouses and 4 pairs of shoes.			This question is generally very well attempted. Just a few pointers for accuracy (which you weren't penalised): For the equation on the total amount paid, it is not very accurate writing it as: $46.90 \times (1 - 85\%)x + 29 \times (1 - 85\%)y + 19.9z = 257.93$ It is always good to conclude your answers writing 2 pairs of jeans, 4 blouses and 4 pairs of shoes.

Qn	Suggested Answers			Guidance
2	$x^3+y^3=3pxy$ Implicit differentiation w.r.t. x: $3x^2+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=3px\frac{\mathrm{d}y}{\mathrm{d}x}+3py$ Gradient of asymptote = -1 Let $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$. $3x^2-3y^2=-3px+3py$ $x^2-y^2=-px+py$ $(x+y)(x-y)=-p(x-y)$ $(x-y)(x+y+p)=0$ $x=y$ or $x+y=-p$ (rejected since $p>0$ and $x>0$, $y>0$) Since $x=y$, $x^3+x^3=3px^2$ $x^2(2x-3p)=0$ $x=0$ (rejected) or $x=\frac{3p}{2}$ Hence, P has coordinates $\left(\frac{3p}{2},\frac{3p}{2}\right)$.			Most students are able to differentiate the equation implicitly. However, many stumbled in simplifying the algebraic expression after the step $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$. It is helpful to remember the identity difference of squares: $a^2-b^2=(a-b)(a+b)$ Another point to note is that the diagram is NOT drawn to scale. As such no marks are awarded to students who claimed/observed (or guessed) that normal at the point P passing through origin (even though it is true). A rigorous method (algebraic) is essential. Similarly for students who claimed that the curve is symmetric about $y=x$.

Qn	Suggested Answers			Guidance
3(i)	Let $x=k$ $y^2+14y+4k^2+16k+16ky+13=0$ $y^2+y(14+16k)+(4k^2+16k+13)=0$ For 2 distinct values of y, $(14+16k)^2-4(4k^2+16k+13)>0$ $196+448k+256k^2-16k^2-64k-52>0$ $240k^2+384k+144>0$ Since $x=k$, $5x^2+8x+3>0$ $(5x+3)(x+1)>0$ $x<-1$ or $x>-\frac{3}{5}$ Therefore the set of x is $\{x : x\in\mathbb{R}, x<-1 \text{ or } x>-\frac{3}{5}\}$			Most students are not able to present the solution in set notation.
3(ii)	When $x=0$ $y^2+14y+13=0$ $(y+13)(y+1)=0 \Rightarrow y=-1$ or -13 The coordinates are $(0, -1)$ and $(0, -13)$ Curve C is not graph of a function as the vertical line $x=0$ cuts the curve at 2 points.			Checking of whether curve is not a function is poorly attempted.

<table>
<tr><th>Qn</th><th>Suggested Answers</th><th></th><th></th><th>Guidance</th></tr>
<tr><td>4(i)</td><td>Reflection in the y-axis (Replace x with –x)
$y=-x\left|-x-a\right|$
$\quad=-x\left|x+a\right|$
Translation of 2 units in the direction of y-axis
$y=2-x\left|x+a\right|$.</td><td></td><td></td><td>Inappropriate terms like ‘Flipping’ and ‘transforming’ are used.
Scaling by -1 unit is wrong. It should scaling by factor -1.
Cambridge is very strict on this.</td></tr>
<tr><td>4(ii)</td><td>
</td><td></td><td></td><td>The sharp point at (a,0) is not apparent.</td></tr>
</table>

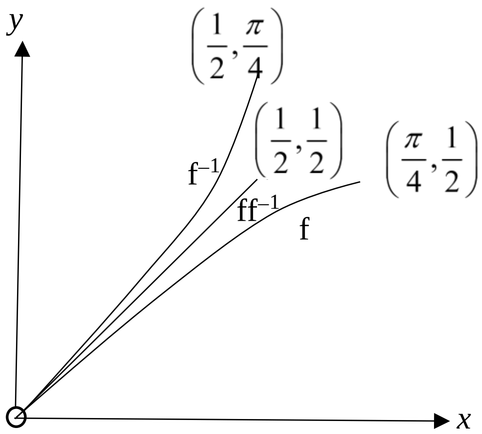
4(iii)				
4(iv)	$0 < k < \frac{4}{a^2}$ or $k < 0$			

Qn	Suggested Answers			Guidance
5(i)	$l:\mathbf{r}=\begin{pmatrix}5\\2\\4\end{pmatrix}+\beta\begin{pmatrix}1\\3\\1\end{pmatrix},\beta\in\mathbb{R}$			Many students wrongly stated : $\overrightarrow{FC}\perp$ line l. In fact $\overrightarrow{FC}//\begin{pmatrix}4\\7\\5\end{pmatrix}$.
	(Suggest not to use the symbols λ or μ, as the following parts of the question use these 2 symbols.)			Another method to find C is to let $\overrightarrow{OC}=\begin{pmatrix}5\\2\\4\end{pmatrix}+\beta\begin{pmatrix}1\\3\\1\end{pmatrix}$ for some $\beta\in\mathbb{R}$.
	Vector equation of the line perpendicular to plane passing through (3, 1, 1) is $\mathbf{r}=\begin{pmatrix}3\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}4\\7\\5\end{pmatrix}$			$\overrightarrow{FC}=\begin{pmatrix}5\\2\\4\end{pmatrix}+\beta\begin{pmatrix}1\\3\\1\end{pmatrix}-\begin{pmatrix}3\\1\\1\end{pmatrix}=\begin{pmatrix}2+\beta\\1+3\beta\\3+\beta\end{pmatrix}$
	The 2 lines will intersect at *C*. $\begin{pmatrix}5+\beta\\2+3\beta\\4+\beta\end{pmatrix}=\begin{pmatrix}3+4\alpha\\1+7\alpha\\1+5\alpha\end{pmatrix}$ $\beta-4\alpha=-2$ $3\beta-7\alpha=-1$ $\beta=2,\alpha=1$ $\overrightarrow{OC}=\begin{pmatrix}5\\2\\4\end{pmatrix}+2\begin{pmatrix}1\\3\\1\end{pmatrix}=\begin{pmatrix}7\\8\\6\end{pmatrix}$ or $\overrightarrow{OC}=\begin{pmatrix}3\\1\\1\end{pmatrix}+1\begin{pmatrix}4\\7\\5\end{pmatrix}=\begin{pmatrix}7\\8\\6\end{pmatrix}$ Hence, point *C* has coordinates (7, 8, 6)			and let $\overrightarrow{FC}=k\begin{pmatrix}4\\7\\5\end{pmatrix}=\begin{pmatrix}2+\beta\\1+3\beta\\3+\beta\end{pmatrix}$, to get $\beta=2$. Or: $\begin{pmatrix}2+\beta\\1+3\beta\\3+\beta\end{pmatrix}\times\begin{pmatrix}4\\7\\5\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ to get $\beta=2$. Answer is (7, 8, 6), not $\begin{pmatrix}7\\8\\6\end{pmatrix}$.

5(ii)	Since l does not intersect p_1, so line l // plane p_1. The direction vector of l is perpendicular to the normal vector of the plane p_1, so $\begin{pmatrix}1\\3\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\\lambda\end{pmatrix}=0 \Rightarrow 3-6+\lambda=0 \Rightarrow \lambda=3$ Also, points A and B are not on the plane p_1, so $\begin{pmatrix}5\\2\\4\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\3\end{pmatrix}\neq\mu$ or $\begin{pmatrix}4\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\3\end{pmatrix}\neq\mu$. This gives: $\mu\neq 23$			Many students did not give the answer for $\mu\neq 23$
5(iii)	Using $\lambda=3$ $\dfrac{\left\lvert\begin{pmatrix}5\\2\\4\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\3\end{pmatrix}-\mu\right\rvert}{\left\lvert\begin{pmatrix}3\\-2\\3\end{pmatrix}\right\rvert}=2$ or $\dfrac{\left\lvert\begin{pmatrix}4\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\3\end{pmatrix}-\mu\right\rvert}{\left\lvert\begin{pmatrix}3\\-2\\3\end{pmatrix}\right\rvert}=2$ $\lvert 23-\mu\rvert=2\sqrt{22}$ $23-\mu=2\sqrt{22}$ or $23-\mu=-2\sqrt{22}$ $\mu=23+2\sqrt{22}$ or $23-2\sqrt{22}$			Can use $\overrightarrow{OC}=\begin{pmatrix}7\\8\\6\end{pmatrix}$ instead of $\begin{pmatrix}5\\2\\4\end{pmatrix}$ or $\begin{pmatrix}4\\-1\\3\end{pmatrix}$ to get μ as shown: $\dfrac{\left\lvert\begin{pmatrix}7\\8\\6\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\3\end{pmatrix}-\mu\right\rvert}{\left\lvert\begin{pmatrix}3\\-2\\3\end{pmatrix}\right\rvert}=2$

				If you obtained the wrong values for $\overrightarrow{OC}$, then you would not get the correct value for μ .

Qn	Suggested Answers			Guidance
6(i)	Although $g(-x)=\frac{1}{(-x)}=-\frac{1}{x}=-g(x)$, the domain of g does not include negative values. Therefore g is not odd. Since $f\left(-\frac{\pi}{2}\right)=-f\left(\frac{\pi}{2}\right)=0$, but $f\left(-\frac{\pi}{2}\right)$ is undefined. Therefore, f is not odd.			Many could not identify the values that g and f are not defined.
6(ii)	Since $f(0)=f\left(\frac{\pi}{2}\right)=0$, there are two values of x that give the same value of f(x), which means f is not one-one. OR: The line y = 0 cuts the graph y = f(x) at 2 points. Therefore f has no inverse. $a=\frac{\pi}{4}$			A counter-example is sufficient to infer that f is not one-one and hence the inverse does not exist. Although some students could routinely state that the line $y = k$, for $-\frac{1}{2}<k<\frac{1}{2}$, cuts the curve at more than one point, they could not identify the value of a correctly. Many gave $a=\frac{\pi}{2}$ which produces a function that is not one-one and yet proceed to draw the inverse of f.

6(iii)				The domain of ff^{-1} was often wrongly stated.
6(iv)	Domain of f = $\left(0,\frac{\pi}{4}\right]$ and Domain of g = $(0,1]$ Range of f = $\left(0,\frac{1}{2}\right]$ and range of g = $[1,\infty)$ Since range of g is not a subset of domain of f, fg does not exist. Range of f is a subset of domain of g, therefore gf exists. $\mathrm{gf}(x)=\mathrm{g}(\sin x\cos x)=\dfrac{1}{\sin x\cos x}=2\operatorname{cosec}2x$ Domain of gf is $\left(0,\frac{\pi}{4}\right]$ and range of gf is $[2,\infty)$.			Checking for the existence of the composite functions were usually done correctly although a handful used the original domain of f instead of the one stated in part (ii).

Qn	Suggested Answers			Guidance
7(a)(i)	Let the angle between **a** + **b** and **a** be α $(\mathbf{a}+\mathbf{b})\cdot\mathbf{a} = \lvert\mathbf{a}+\mathbf{b}\rvert\lvert\mathbf{a}\rvert\cos\alpha$ $\cos\alpha = \dfrac{(\lvert\mathbf{a}\rvert^2+\mathbf{b}\cdot\mathbf{a})}{\lvert\mathbf{a}+\mathbf{b}\rvert} = \dfrac{(1+\mathbf{b}\cdot\mathbf{a})}{\lvert\mathbf{a}+\mathbf{b}\rvert}$ Let the angle between $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}$ be β $(\mathbf{a}+\mathbf{b})\cdot\mathbf{b} = \lvert\mathbf{a}+\mathbf{b}\rvert\lvert\mathbf{b}\rvert\cos\beta$ $\cos\beta = \dfrac{(\mathbf{a}\cdot\mathbf{b}+\lvert\mathbf{b}\rvert^2)}{\lvert\mathbf{a}+\mathbf{b}\rvert} = \dfrac{(\mathbf{a}\cdot\mathbf{b}+1)}{\lvert\mathbf{a}+\mathbf{b}\rvert}$ Since $\cos\alpha = \cos\beta$, $\mathbf{a}+\mathbf{b}$ is the angle bisector of AOB			Common Mistakes: 1. $\lvert\mathbf{a}+\mathbf{b}\rvert = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert = 2$ 2. $\lvert\mathbf{a}+\mathbf{b}\rvert = \sqrt{\lvert\mathbf{a}\rvert+\lvert\mathbf{b}\rvert} = \sqrt{2}$
7(a)(ii)	$\lvert\mathbf{a}\times\mathbf{b}\rvert = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin AOB = \sin AOB$ Since area of triangle $= \dfrac{1}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert = \dfrac{1}{\sqrt{10}}$ $\therefore \sin AOB = \dfrac{2}{\sqrt{10}}$			
7(b)(i)	$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$ $= \begin{pmatrix}4\\7\\4\end{pmatrix} - \begin{pmatrix}7\\-8\\7\end{pmatrix} = \begin{pmatrix}-3\\15\\-3\end{pmatrix}$			

	Perpendicular distance of O from line $= \dfrac{\left\|\begin{pmatrix}-3\\15\\-3\end{pmatrix}\times\begin{pmatrix}4\\7\\4\end{pmatrix}\right\|}{\left\|\begin{pmatrix}-3\\15\\-3\end{pmatrix}\right\|}$ $= \dfrac{\left\|\begin{pmatrix}81\\0\\-81\end{pmatrix}\right\|}{\sqrt{9+225+9}}$ $= \dfrac{81\sqrt{2}}{9\sqrt{3}} = 3\sqrt{6}$			
7(b)(ii)	$\overrightarrow{CO}\cdot\overrightarrow{CD} = \left\|\overrightarrow{CO}\right\|\left\|\overrightarrow{CD}\right\|\cos\theta$ $\begin{pmatrix}-7\\8\\-7\end{pmatrix}\cdot\begin{pmatrix}-3\\15\\-3\end{pmatrix} = \sqrt{7^2+8^2+7^2}\sqrt{3^2+15^2+3^2}\cos\theta$ $21+120+21 = \sqrt{162}\sqrt{243}\cos\theta$ $\theta = 35.3°$ or 0.615 radians Alternatively, $\left\|\overrightarrow{OC}\right\| = \sqrt{49+64+49} = \sqrt{162}$ Let F be foot of perpendicular from O to line CD. Then OF = $3\sqrt{6}$ $\theta = \sin^{-1}\left(\dfrac{3\sqrt{6}}{\sqrt{162}}\right) = 35.3^{\circ}$			Many still make the mistake of not considering the direction of the 2 vectors carefully. Common mistakes are either $\overrightarrow{OC}\cdot\overrightarrow{CD}$ or $\overrightarrow{CO}\cdot\overrightarrow{DC}$

<table>
<tr><th>Qn</th><th>Suggested Answers</th><th></th><th></th><th>Guidance</th></tr>
<tr><td>8(i)</td><td>

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} / \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2t+1}{3t^2+12t}$

For tangent parallel to the *y*-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined.

Hence $3t(t+4)=0$

$t=-4, 0$ (reject $t=0$)

Equation of tangent is $x = 32$

</td><td></td><td></td><td>

Many students **wrongly** stated that $\frac{\mathrm{d}y}{\mathrm{d}x}=0$.

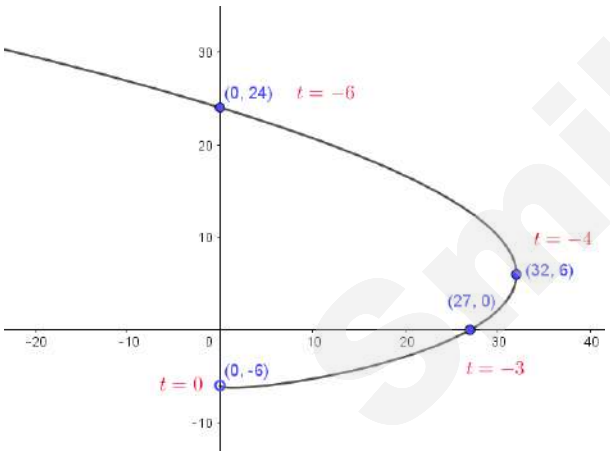
Many students did not reject $t=0$.

Note that the tangent parallel to the *y*-axis is a vertical line.

</td></tr>
<tr><td>8(ii)</td><td>

When $x=0$, $t=0$ or -6 corresponding to (0,–6) or (0,24) respectively

When $y=0$, $t=2$ or -3 (reject $t=2$) corresponding to (27,0) .



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Remember to adjust window setting to reflect the correct range of *t* values.

Point corresponding to (0, 6) must be excluded from the graph.
Many students **incorrectly** drew the graph such that the point (32, 6) lies on the *x*-axis.

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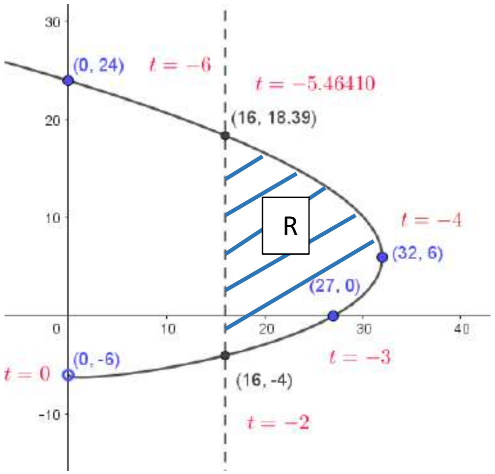
</table>

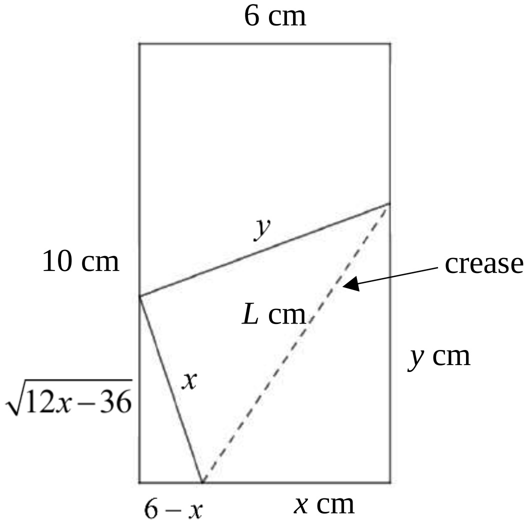
8(iii)	When $x=16$, $t^2(t+6)=16$ $t=-5.4641$ or -2 or 1.4641 (reject $t=1.4641$ since $t<0$) The line $x=16$ cuts C at $(16,-4)$ and $(16,18.392)$  Method 1 (Recommended): Required area $= \int_{y=-4}^{y=18.392} x\,\mathrm{d}y - \underbrace{16(18.39-(-4))}_{\text{Area of rectangle}}$ $= \int_{t=-2}^{t=-5.4641} (t^3+6t^2)(2t+1)\,\mathrm{d}t - 16(18.39-(-4))$			Leave intermediate answers to 5 s.f. Many students interpreted the region R wrongly. Read the question carefully. Do not assume the symmetry of area for this question. Brackets are important. Many wrote $\int_{-5.46410}^{-4} t^2+t-6(3t^2+12t)\,\mathrm{d}t$ instead of $\int_{-5.46410}^{-4} (t^2+t-6)(3t^2+12t)\,\mathrm{d}t$ which resulted in a wrong answer from GC. Limits for integration must be substituted correspondingly. Marks are usually lost due to that.

	$= 241.058$ $\simeq 241$ (3 s.f.) Method 2 (Using idea of "upper – lower curve" and $\int_{x=16}^{x=32} y \, \mathrm{d}x$): Required area $= \int_{-5.4641}^{-4} \left(t^2+t-6\right)\left(3t^2+12t\right) \mathrm{d}t - \int_{-2}^{-4} \left(t^2+t-6\right)\left(3t^2+12t\right) \mathrm{d}t$ $= 241.058$ $\simeq 241$ (3 s.f.)			

Qn	Suggested Answers			Guidance
9(a)(i)	$S_\infty = \frac{6}{13}$ $\lvert S_n - S_\infty \rvert < 10^{-8}$ $\left\lvert \frac{6}{13} \times \frac{1}{3^{3n}} \right\rvert < 10^{-8}$ Using GC, smallest $n = 6$			
9(a)(ii)	$u_n = S_n - S_{n-1}$ $= \frac{6}{13}\left(1 - \frac{1}{3^{3n}}\right) - \frac{6}{13}\left(1 - \frac{1}{3^{3n-3}}\right)$ $= \frac{6}{13}\left(\frac{1}{3^{3n-3}} - \frac{1}{3^{3n}}\right)$ $= \frac{6}{13} \times \frac{1}{3^{3n}}\left(\frac{1}{3^{-3}} - 1\right)$ $= \frac{6}{13} \times \frac{1}{3^{3n}}(26)$ $= \frac{12}{3^{3n}} = \frac{4}{3^{3n-1}}$			Note that $u_n = S_n - S_{n-1}$ holds for any series. You are required to leave your answer in simplified form as well.
9(b)(i)	$\sin(2r+1)\theta - \sin(2r-1)\theta$ $= \sin 2r\theta \cos\theta + \cos 2r\theta \sin\theta - \sin 2r\theta \cos\theta + \cos 2r\theta \sin\theta$ $= 2\cos 2r\theta \sin\theta$ (shown)			Rewriting $\sin(2r+1)\theta - \sin(2r-1)\theta$ as $\sin(2r\theta+\theta) - \sin(2r\theta-\theta)$ will help you do the expansion needed by the question.
9(b)(ii)	$\sum_{r=1}^{n} \sin^2 r\theta$			Note the use of double angle formula $\cos 2r\theta = 1 - 2\sin^2 r\theta$ in order to use result from part (i)

	$=\frac{1}{2}\sum_{r=1}^{n}\left(1-\cos 2r\theta\right)$			
	$=\frac{1}{2}\sum_{r=1}^{n}1-\frac{1}{2}\sum_{r=1}^{n}\cos 2r\theta$ $=\frac{1}{2}n-\frac{1}{4\sin\theta}\sum_{r=1}^{n}\left[\sin(2r+1)\theta-\sin(2r-1)\theta\right], \theta\neq k\pi$			Any term without ‘r’ is considered a constant and hence can be factored out.
	$=\frac{1}{2}n-\frac{1}{4\sin\theta}[\sin 3\theta-\sin\theta$ $+\sin 5\theta-\sin 3\theta$ $+\sin 7\theta-\sin 5\theta$ . . $+\sin\left(2n+1\right)\theta-\sin\left(2n-1\right)\theta]$ $=\frac{1}{2}n-\frac{1}{4\sin\theta}\left[\sin\left(2n+1\right)\theta-\sin\theta\right]$ $=\frac{1}{2}n-\frac{1}{4\sin\theta}\left[2\cos\left(n+1\right)\theta\sin n\theta\right]$ $=\frac{1}{2}n-\frac{1}{2\sin\theta}\left[\cos\left(n+1\right)\theta\sin n\theta\right]$			Use factor formula $\sin\left(2n+1\right)\theta-\sin\theta$ $=2\cos\frac{1}{2}\left[\left(2n+1\right)\theta+\theta\right]\times$ $\sin\frac{1}{2}\left[\left(2n+1\right)\theta-\theta\right]$ to get the final answer

Qn	Suggested Answers			Guidance
10(i)	$\dfrac{m}{\sin\left(\pi-(\theta+\alpha)\right)}=\dfrac{l}{\sin\alpha}$ $l=\dfrac{m\sin\alpha}{\sin\left(\pi-(\theta+\alpha)\right)}$ $=\dfrac{m\sin\alpha}{\sin(\theta+\alpha)}\text{(shown)}$			Since the angle given in the question is in radian, students should use π instead of 180 degrees.
10(ii)	$l=\dfrac{m\sin\alpha}{\sin(\theta+\alpha)}$ $\dfrac{\mathrm{d}l}{\mathrm{d}\theta}=\dfrac{-(m\sin\alpha)\cos(\theta+\alpha)}{\sin^2(\theta+\alpha)}$ Alternatively, $\dfrac{\mathrm{d}l}{\mathrm{d}\theta}=-(m\sin\alpha)\operatorname{cosec}(\theta+\alpha)\cot(\theta+\alpha)$			Many students did quotient rule wrongly. Note that: $\sin^{-1}x\neq\dfrac{1}{\sin x}$ $(\sin x)^{-1}=\dfrac{1}{\sin x}$ Many failed to realise that α is a constant.
10(iii)	Using chain rule, $\dfrac{\mathrm{d}l}{\mathrm{d}t}=\dfrac{\mathrm{d}l}{\mathrm{d}\theta}\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ $=\dfrac{-15\left(\sin\dfrac{\pi}{60}\right)\cos\left(\dfrac{\pi}{9}+\dfrac{\pi}{60}\right)}{\sin^2\left(\dfrac{\pi}{9}+\dfrac{\pi}{60}\right)}\left(-\dfrac{5\pi}{9}\right)$ $=8.2611\text{ cm / sec}$ $\approx 8.26\text{ cm / sec}$			Common mistake: $\dfrac{\mathrm{d}l}{\mathrm{d}t}=\dfrac{\mathrm{d}l}{\mathrm{d}\theta}\div\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ This is **WRONG**.
10(iv)	$L=\sqrt{x^2+y^2}$			Many students used similar triangles wrongly. In order to use

	$\sqrt{x^2-(6-x)^2}=\sqrt{x^2-36+12x-x^2}=\sqrt{12x-36}$  Area of rectangle = $6\times10=60\ \text{cm}^2$ $60=\frac{1}{2}(6-x)\left(\sqrt{12x-36}\right)+2\left(\frac{1}{2}\right)(xy)$ $+\frac{1}{2}(6)\left(10-\sqrt{12x-36}+10-y\right)$ $=\frac{1}{2}(6-x)\left(\sqrt{12x-36}\right)+xy+60-3\sqrt{12x-36}-3y$ $0=\frac{1}{2}(6-x)\left(\sqrt{12x-36}\right)+xy-3\sqrt{12x-36}-3y$ $3y-xy=\left(3-\frac{x}{2}\right)\left(\sqrt{12x-36}\right)-3\sqrt{12x-36}$			similar triangle, you need to ensure that the corresponding three angles of the two triangles are the same.

	$y=\dfrac{-\frac{x}{2}\left(\sqrt{12x-36}\right)}{3-x}$ $=\dfrac{x\left(\sqrt{12x-36}\right)}{2x-6}$ $L=\sqrt{x^2+y^2}$ $=\sqrt{x^2+\left(\dfrac{x\left(\sqrt{12x-36}\right)}{2x-6}\right)^2}$ $=\sqrt{x^2+\left(\dfrac{x^2\left(12x-36\right)}{\left(2x-6\right)^2}\right)}$ $=\sqrt{x^2+\left(\dfrac{12x^2\left(x-3\right)}{4\left(x-3\right)^2}\right)}=\sqrt{x^2+\dfrac{3x^2}{x-3}}$ **Alternative method** $y^2=6^2+\left(y-\sqrt{12x-36}\right)^2$ $y^2=36+y^2-2y\sqrt{12x-36}+12x-36$ $2y\sqrt{12x-36}=12x$ $y=\dfrac{12x}{2\sqrt{12x-36}}$ $y^2=\dfrac{3x^2}{x-3}$			

	$L=\sqrt{x^2+y^2}$ $=\sqrt{x^2+\frac{3x^2}{x-3}}$			
10(v)	$\frac{\mathrm{d}L}{\mathrm{d}x}=\frac{1}{2}\left(x^2+\frac{3x^2}{x-3}\right)^{-\frac{1}{2}}\left(2x+\frac{(x-3)(6x)-(3x^2)}{(x-3)^2}\right)$ $=\frac{2x+\frac{3x^2-18x}{(x-3)^2}}{2\sqrt{x^2+\frac{3x^2}{x-3}}}=0$ $2x+\frac{3x^2-18x}{(x-3)^2}=0$ $x\left(2+\frac{3x-18}{(x-3)^2}\right)=0$ $2+\frac{3x-18}{(x-3)^2}=0$ or $x=0$ (rejected) $2+\frac{3x-18}{(x-3)^2}=0$ $\frac{2(x-3)^2+3x-18}{(x-3)^2}=0$ $2(x^2-6x+9)+3x-18=0$ $2x^2-9x=0$ $x=\frac{9}{2}$ or $x=0$ (rejected)			Many students applied the quotient rule wrongly.

By first derivative test,

x	4.45	$\frac{9}{2}$	4.55
$\frac{\mathrm{d}L}{\mathrm{d}x}$	-0.0604	0	0.0553
Slope	\	–	/

$x=\frac{9}{2}$ gives a minimum L.

Hence,

$$\text{Min.}\,L=\sqrt{\left(\frac{9}{2}\right)^2+\frac{3\left(\frac{9}{2}\right)^2}{\left(\frac{9}{2}\right)-3}}=7.79\text{cm}$$

Many did not use the first or the second derivative test to check if the point gives a maximum or a minimum.

Need to read the question carefully. Question is asking for the minimum length of the resulting crease which is denoted by L, not x.

Qn	Suggested Answers			Guidance
11(i)	If the rate of change of x is not positive, it is not possible for the amount of drug to increase to reach 120 mg in the body for the treatment to be effective.			
11(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t}=30-kx$, where $k>0$ $\Rightarrow \int \frac{1}{30-kx}\,dx=\int dt$ $\Rightarrow -\frac{1}{k}\ln(30-kx)=t+C$, since $\frac{\mathrm{d}x}{\mathrm{d}t}=30-kx>0$ $\Rightarrow \ln(30-kx)=-kt-kC$ $\Rightarrow (30-kx)=e^{-kt-kC}=e^{-kC}e^{-kt}=Be^{-kt}$, $B>0$ When $x=0, t=0$, thus $B=30$ When $t=4, x=82.6$, $30-k(82.6)=30e^{-4k}$, Solving using GC. $k=0.200$ $x=150\left(1-\mathrm{e}^{-\frac{t}{5}}\right)$, where $A=150$			
11(iii)	When $x=120, k=\frac{1}{5}$, $120=150\left(1-e^{-\frac{t}{5}}\right)$ $\Rightarrow t=8.05$ hrs			
11(iv)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{1}{\sqrt{2500-9t^2}}$			Common mistake seen: $d^2x=\int\frac{1}{\sqrt{2500-9x^2}}\,dt^2$

	$\frac{\mathrm{d}x}{\mathrm{d}t}=\int \frac{1}{\sqrt{2500-9t^2}}\,\mathrm{d}t=\frac{1}{3}\sin^{-1}\left(\frac{3t}{50}\right)+\mathrm{C}$			The above expression is wrong. We ***cannot*** do variable separable for 2nd order DE.
	$x=\int \left[\frac{1}{3}\sin^{-1}\left(\frac{3t}{50}\right)+\mathrm{C}\right]\mathrm{d}t$			There are students who could not recognise it is integration by parts. The common weakness was also inability to identify the correct $\mathrm{f}'(x)$ in the use of $\int \mathrm{f}'(x)\mathrm{f}^n(x)\mathrm{d}x$ formula.
	$=\frac{1}{3}\left[t\sin^{-1}\left(\frac{3t}{50}\right)-\int t\cdot\frac{1}{\sqrt{1-\left(\frac{3t}{50}\right)^2}}\left(\frac{3}{50}\right)\mathrm{d}t\right]+\mathrm{C}t+\mathrm{D}$			
	$=\frac{1}{3}\left[t\sin^{-1}\left(\frac{3t}{50}\right)-\int t\cdot\frac{3}{\sqrt{50^2-(3t)^2}}\,\mathrm{d}t\right]+\mathrm{C}t+\mathrm{D}$ $=\frac{1}{3}\left[t\sin^{-1}\left(\frac{3t}{50}\right)+\frac{1}{6}\int \frac{-18t}{\sqrt{2500-9t^2}}\,\mathrm{d}t\right]+\mathrm{C}t+\mathrm{D}$			
	$=\frac{1}{3}\left[t\sin^{-1}\left(\frac{3t}{50}\right)+\frac{1}{3}\sqrt{2500-9t^2}\right]+\mathrm{C}t+\mathrm{D}$ $=\frac{1}{3}t\sin^{-1}\left(\frac{3t}{50}\right)+\frac{1}{9}\sqrt{2500-9t^2}+\mathrm{C}t+\mathrm{D}$ When $t=0$, $x=0$, $\mathrm{D}=-\frac{50}{9}$ $x=\frac{1}{3}t\sin^{-1}\left(\frac{3t}{50}\right)+\frac{1}{9}\sqrt{2500-9t^2}+\mathrm{C}t-\frac{50}{9}$			

NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

MATHEMATICS **9758/02**

Paper 2 **17th September 2018**

3 Hours

Additional Materials: Answer Paper

Graph paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **8** printed pages.

Section A: Pure Mathematics [40 marks]

1 Sarah carried out a series of experiments which involved using decreasing amounts of a chemical. In the first experiment, she used 4 grams of the chemical and the amount of chemical used formed a geometric progression. In the 25th experiment, she used 1 gram of the chemical.

(i) Find the total amount of chemical she used in the first 25 experiments. [4]

(ii) Show that the theoretical maximum total amount of chemical she would use will not exceed 71.3 grams. [1]

Robert carried out the same series of experiments. He also used decreasing amounts of the same chemical but the amount of chemical used formed an arithmetic progression with common difference *d*. If the total amount of chemical that both Sarah and Robert used for the first 25 experiments were the same, and the amount of chemical Robert used for the 25th experiment was still 1 gram, find the value of *d* and the amount of chemical he used for the first experiment. [4]

2 **(a)** **(i)** Evaluate $\int \frac{2x-4}{x^2-2x+4}\,\mathrm{d}x$. [3]

(ii) Without the use of a graphic calculator, evaluate $\int_1^4 \frac{|2x-4|}{x^2-2x+4}\,\mathrm{d}x$, leaving your answer in logarithmic form. [4]

(b) Given that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cos^2 x}\right) = \frac{2\sin x}{\cos^3 x}$, evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^3 x}\,\mathrm{d}x$, leaving your answer in exact form. [3]

3 **Do not use a calculator in answering this question.**

(i) The equation $z^3+4z^2+8z+8=0$ has a root $z=-2$. Find the other two roots z_1 and z_2 where $-\pi < \arg z_2 < \arg z_1 \le \pi$. [2]

(ii) Find the modulus and argument of *w*, where $w = \frac{z_1}{z_2}$. [3]

(iii) Find the set of positive integers *n* for which w^n is real, and show that, for these values of *n*, w^n is 1. [3]

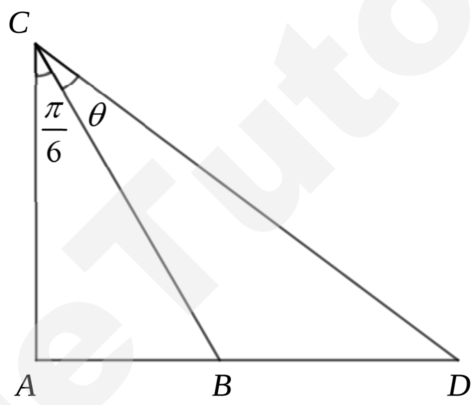
(iv) Express $w^{100} - (w^*)^{100}$ in the form *k*i, giving the exact value of *k* in non-trigonometrical form. [2]

4 **(a)** Hyperbolic functions are commonly used to study fluid dynamics and electromagnetic theory, where integrals with a $\sqrt{x^2+1}$ term occurs. The inverse function of the hyperbolic function $\sinh x$ is $\sinh^{-1} x$. It is given that $y = \sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$.

(i) Show that $\mathrm{e}^y \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \mathrm{e}^y \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \left(x^2+1\right)^{-\frac{1}{2}} - x^2\left(x^2+1\right)^{-\frac{3}{2}}$. [2]

(ii) Hence, by further differentiation, find the first two non-zero terms of the Maclaurin's series for $\sinh^{-1} x$ in ascending powers of x. [4]

(b) The diagram shows a right angled triangle ABC with angle $ACB = \dfrac{\pi}{6}$ radians. D lies on AB produced such angle $BCD = \theta$ radians.



(i) Show that $\dfrac{AB}{AD} = \dfrac{1}{\sqrt{3}\tan\left(\dfrac{\pi}{6}+\theta\right)}$. [1]

(ii) Given that θ is sufficiently small for θ^3 and higher powers of θ to be neglected, show that

$$\frac{AB}{AD} \approx 1 + a\theta + b\theta^2,$$

where a and b are exact constants to be determined. [4]

Section B: Statistics [60 marks]

5 This question is about arrangements of all ten letters in the word EXCELLENCE.

(i) Find the number of arrangements in which the letters are **not** in alphabetical order. [2]

The letters are now arranged in a circle.

(ii) Find the number of arrangements that can be made with all E's together and no other adjacent letters the same. [4]

6 A game is played using a fair six-sided die, a pawn and a simple board as shown below.

S	1	2	3	4	5	**E**

Initially, the pawn is placed on square **S**. The game is played by throwing the die and moving the pawn in the following manner:

S 1 2 3 4 5 **E** 5 4 3 2 1 2 3 4 5 **E**………

Thus, for example, if the first and second throw of the die gives a "5" and "4" respectively, the final position of the pawn will be on square "3".

The game will stop when the pawn stops at square **E**.

Let X be the random variable denoting the number of throws of the die required to move the pawn such that it stops at square **E**.

(i) Show $\mathrm{P}(X = 2) = \frac{5}{36}$. [1]

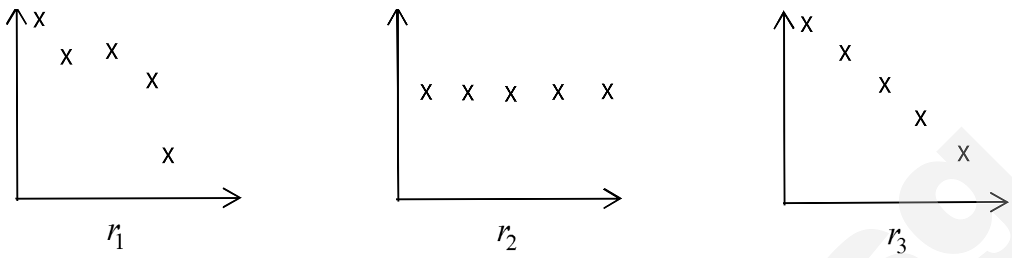
(ii) Find the probability that more than two throws of the die are needed for the pawn to stop at square **E** given that the first throw of the die gives an even number. [3]

It is now given that for each game, a player has a maximum of 3 throws of the die and a special prize is given to any player who uses not more than two throws for the pawn to stop at square **E**.

(iii) Find the probability of a player winning a special prize in at least three but not more than eight games out of ten games. [3]

(iv) Find the least number of games needed so that the probability of winning at least a special prize is at least 0.998. [3]

7 **(a)** The following three scatter plots have product moment correlation coefficients as r_1, r_2 and r_3 respectively.



State, with justifications, an inequality that relates r_1, r_2 and r_3 that best describes the correlations associated with the scatter plots (A), (B) and (C). [2]

(b) A motoring magazine published the following data on the engine capacity measured in cubic centimetres (x) and the prices in thousand dollars (y) of ten new car models.

Car Model	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*J*	*K*
x	1000	1270	1750	2230	1990	600	650	1500	1450	1650
y	139	142	151.6	169.8	169.3	121.9	121.9	141.6	130.5	161.5

(i) Plot a scatter diagram on graph paper for these values using appropriate scales for the x- and y-axes. On your diagram, indicate the car model for each point. [2]

(ii) Find the equation of the regression line of y on x. [1]

(iii) Calculate the product moment correlation coefficient and comment on the relationship between x and y. [2]

One particular consumer regards low price and large engine capacity as the two most important factors in choosing a car. By drawing the regression line on the scatter diagram, suggest one model which will give particularly good value for money for this consumer. Which three models would you advise the consumer not to buy? Justify your answers. [3]

8 The masses of manufactured links of a chain are normally distributed with mean 800 grams and standard deviation 20 grams.

(i) Find the probability that the mass of a randomly chosen link is more than 805 grams. [1]

To close the gate opening in a link, a locking sleeve is attached to it and it increases the mass of a link by 10%.

(ii) By writing down the distribution of the masses of links with locking sleeves, find the probability that the mass of a randomly chosen link with a locking sleeve is between 865.35 and 895.5 grams. [3]

Hooks with mean masses 750 grams are manufactured to attach to the links. The masses of hooks are normally distributed such that 15% of them have mass less than 735.6 grams.

(iii) Find the standard deviation of the masses of hooks. [2]

Five independent links with locking sleeves and a hook are packed in a wooden box with a fixed mass of 1 kilogram. The probability that the mean mass of n such wooden boxes with its contents more than 6190 grams exceeds 0.013.

(iv) Find the greatest value of n, stating the parameters of any distribution that you use. [4]

Question 9 is printed on the next page

9 During a Mathematics lesson, Ms Kim wanted her pupils to build a model. She carried two indistinguishable boxes of bricks into class. The bricks are identical and indistinguishable except for colours. The number of coloured bricks found in each box is as follows.

Colour of Bricks	Box 1	Box 2
Blue	2	4
Red	3	3
Yellow	5	3

A pupil, Donald, has to draw two bricks from these boxes randomly to build a model. He draws a brick randomly from one of the boxes. The brick is not replaced. He then draws a second brick randomly from one of the boxes.

(i) Show the probability that Donald does not draw any yellow brick is $\frac{25}{72}$. [2]

(ii) Find the probability that Donald draws two yellow bricks. [3]

(iii) Tabulate the probability distribution table for the number of yellow bricks drawn and find the expected number of yellow bricks drawn by Donald. [2]

Ms Kim decided to combine the two boxes of bricks into a single box.

(iv) Donald draws 5 bricks **with replacement** from the single box. Find the probability that he draws less than two yellow bricks. [2]

(v) Donald wanted to build the model with a yellow brick. He randomly draws a brick from the single box one at a time **with replacement** until he gets a yellow brick. Find the probability that it will not take him more than nine draws to get a yellow brick. [3]

Question 10 is printed on the next page

10 The distances thrown, x metres, by a discus thrower have been recorded over a long period. The results for a random sample of 60 throws are summarised by $\sum(x-65)=120$ and $\sum(x-65)^2=3810$.

(i) State what it means for a sample to be random in this context. [1]

(ii) Calculate the unbiased estimates of the population mean and variance of the distances thrown by the discus thrower. [2]

An analysis of all the data collected over the long period gives a mean of 68 metres and a standard deviation of 7.5 metres.

The Sports Council wishes to study if a new technique will improve the mean distance thrown by this thrower. Using this new technique, the results of a random sample of n throws give a mean of $\bar{x}$ metres. A test is carried out, at the 2% level of significance, to determine whether the mean distance thrown by the thrower has improved. You may assume that the distances thrown by the thrower follows a normal distribution.

(iii) State appropriate hypotheses for the test, defining any symbols you use. [1]

(iv) Given that n = 30, find the range of values of $\bar{x}$ for which the result of the test would be to reject the null hypothesis. [3]

(v) It is given instead that $\bar{x}=70.1$ and the result of the test is that the null hypothesis is not rejected. Obtain an inequality involving n, and hence find the set of values that n can take. [3]

A sports magazine published a claim about the mean distance thrown using this new technique. It is assumed that the distances of the discuses thrown follow a normal distribution with known population variance.

Two discus thrower coaches, Ang and Tan, decide to each conduct a hypothesis test, at the 5% level of significance, to determine whether the magazine has **overstated** the mean distance thrown using this new technique. Ang obtained a sample with mean $\bar{x}$ and concluded that there is sufficient evidence, at the 5% level of significance, that the magazine has overstated the mean distance thrown. Tan took a different sample with sample size **four times** that of Ang's and obtained the same sample mean.

Will Tan's test yield the same conclusion? Justify your answer. [2]

Qn	Suggested Answers			Guidance
1(i)	$u_1 = a = 4$, $u_{25} = 1 \Rightarrow ar^{24} = 1$ $\Rightarrow r = \sqrt[24]{0.25}$ or 0.94387 $S_{25} = \dfrac{a\left(1-r^{25}\right)}{1-r} = \dfrac{4\left[1-\left(\sqrt[24]{0.25}\right)^{25}\right]}{1-\sqrt[24]{0.25}} \approx 54.451 \approx 54.5$ or $S_{25} = \dfrac{a\left(1-r^{25}\right)}{1-r} = \dfrac{4\left[1-(0.94387)^{25}\right]}{1-0.94387} \approx 54.449 \approx 54.4$			Common error: Mistaken that $S_{25} = \dfrac{a\left(1-r^{24}\right)}{1-r}$. The correct S_{25} is $\dfrac{a\left(1-r^{25}\right)}{1-r}$
1(ii)	$S_\infty = \dfrac{a}{1-r} = \dfrac{4}{1-\sqrt[24]{0.25}} \approx 71.269 < 71.3$ or $S_\infty = \dfrac{a}{1-r} = \dfrac{4}{1-0.94387} \approx 71.263 < 71.3$ Hence theoretical maximum total amount of chemical she uses will not exceed 71.3 grams.			Students **should compute** S_∞ and **show** that it is less than 71.3.
	$u_{25} = 1 \Rightarrow a + 24d = 1$ -----(1) $S_{25} = \dfrac{25}{2}\left(2a+24d\right) = 54.451$ $\Rightarrow 25a + 300d = 54.451 -----(2)$ Solving the 2 equations, $d \approx -0.0982$ and $a \approx 3.36$			Students should ensure that they give $\boldsymbol{a}$ and $\boldsymbol{d}$, as required by the question.

	or $S_{25} = \frac{25}{2}(2a+24d) = 54.449 - - - - - (2)$ $\Rightarrow 25a + 300d = 54.449$ Solving the 2 equations, $d \approx -0.0982$ and $a \approx 3.36$ or $S_{25} = \frac{25}{2}(2a+24d) = 54.45 - - - - - (2)$ $\Rightarrow 25a + 300d = 54.45$ Solving the 2 equations, $d \approx -0.0982$ and $a \approx 3.36$			

<table>
<tr><th>Qn</th><th>Suggested Answers</th><th></th><th></th><th>Guidance</th></tr>
<tr><td>2(a)(i)</td><td>$$\int \frac{2x-4}{x^2-2x+4}\,dx = \int \frac{2x-2}{x^2-2x+4}\,dx - \int \frac{2}{x^2-2x+4}\,dx$$ $$= \int \frac{2x-2}{x^2-2x+4}\,dx - \int \frac{2}{(x-1)^2+3}\,dx$$ $$= \ln(x^2-2x+4) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}} + C$$</td><td></td><td></td><td>Well done. Only some students were unable to recall the formula $\int \frac{f'(x)}{(f(x))^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a}+C$</td></tr>
<tr><td>2(a)(ii)</td><td>$x^2-2x+4=(x-1)^2+3>0$ since $(x-1)^2 \geq 0$.
Therefore, $\int_1^4 \frac{|2x-4|}{x^2-2x+4}\,dx = -\int_1^2 \frac{2x-4}{x^2-2x+4}\,dx + \int_2^4 \frac{2x-4}{x^2-2x+4}\,dx$ $$= -\left[\ln(x^2-2x+4) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}}\right]_1^2 + \left[\ln(x^2-2x+4) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}}\right]_2^4$$ $$= -\left[\left(\ln 4 - \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}\right) - \left(\ln 3 - \frac{2}{\sqrt{3}}\tan^{-1}0\right)\right] + \left[\left(\ln 12 - \frac{2}{\sqrt{3}}\tan^{-1}\frac{3}{\sqrt{3}}\right) - \left(\ln 4 - \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}\right)\right]$$ $$= -\left[\left(\ln 4 - \frac{2}{\sqrt{3}}\left(\frac{\pi}{6}\right)\right) - (\ln 3 - 0)\right] + \left(\ln 12 - \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}\right)\right) - \left(\ln 4 - \frac{2}{\sqrt{3}}\left(\frac{\pi}{6}\right)\right)$$ $$= -2\ln 4 + \ln 3 + \ln 12 + \frac{4}{\sqrt{3}}\left(\frac{\pi}{6}\right) - \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}\right)$$ $$= \ln\left(\frac{9}{4}\right) = 2\ln\left(\frac{3}{2}\right)$$</td><td></td><td></td><td>Some students used $\int_a^b \frac{|f(x)|}{g(x)}\,dx = \left|\int_a^b \frac{f(x)}{g(x)}\,dx\right|$
This is correct only when $g(x) \geq 0$ for $a \leq x \leq b$.

In general, the above is not true.

Many students were unable to get the exact answers.</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td></tr>
</table>

2(b)	$\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^3 x}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{2\sin x}{\cos^3 x}(\sin x)\,dx$ $= \frac{1}{2}\left[\left[\frac{\sin x}{\cos^2 x}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{\cos x}\,dx\right]$ $= \frac{1}{2}\left[\left[\frac{\sin x}{\cos^2 x}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec x\,dx\right]$ $= \frac{1}{2}\left[\frac{\sin x}{\cos^2 x} - \ln(\sec x + \tan x)\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{\sqrt{2}} - \frac{1}{2}\ln\left(\sqrt{2}+1\right)$ $u = \sin x$, $\frac{dv}{dx} = \frac{2\sin x}{\cos^3 x}$ $\frac{du}{dx} = \cos x$, $v = \frac{1}{\cos^2 x}$			Some students wasted their time to prove $\frac{d}{dx}\left(\frac{1}{\cos^2 x}\right) = \frac{2\sin x}{\cos^3 x}$ which is not required. Some students were unable to use the correct ***u*** and ***v*** for integration by parts. Some students stated wrongly: $\frac{1}{\cos x} = \cos^{-1} x$. In fact, $y = \cos^{-1} x$ is the inverse function of $y = \cos x$. $\frac{1}{\cos x} = (\cos x)^{-1} \neq \cos^{-1} x$.

Qn	Suggested Answers			Guidance
3(i)	$z^3+4z^2+8z+8=(z+2)(z^2+az+4)$ Comparing coefficient of z^2: $4 = a + 2 \Rightarrow a = 2$ $\therefore z^2+2z+4$ is a factor of z^3+4z^2+8z+8 $(z+2)(z^2+2z+4)=0$ $z+2=0$ or $z^2+2z+4=0$ $z=-2$ or $z=\dfrac{-2\pm\sqrt{4-16}}{2}=-1\pm \mathrm{i}\sqrt{3}$ $\therefore z_1=-1+\mathrm{i}\sqrt{3}$ and $z_2=-1-\mathrm{i}\sqrt{3}$			This question does not allow the use of calculator (and GC); answers unsupported by workings will not gain any marks. Use an Argand diagram to identify z_1 and z_2 correctly.
3(ii)	$\lvert w\rvert=\dfrac{\lvert z_1\rvert}{\lvert z_2\rvert}=1$ $\arg w=\arg\left(\dfrac{z_1}{z_2}\right)=\arg z_1-\arg z_2$ $=\dfrac{2\pi}{3}-\left(-\dfrac{2\pi}{3}\right)=\dfrac{4\pi}{3}$ $\therefore \arg w=\dfrac{4\pi}{3}-2\pi=-\dfrac{2\pi}{3}$			Do not work out z_1/z_2. Use properties of modulus and argument to get modulus and argument of w. Note that you have to give the arg(w) in the principal range.
3(iii)	$w^n=\mathrm{e}^{-\mathrm{i}\frac{2n\pi}{3}}=\cos\left(-\dfrac{2n\pi}{3}\right)+\mathrm{i}\sin\left(-\dfrac{2n\pi}{3}\right)$ Since w^n is real, $\sin\left(-\dfrac{2n\pi}{3}\right)=0$ $\sin\left(\dfrac{2n\pi}{3}\right)=0$ $\dfrac{2n\pi}{3}=0,\pm\pi,\pm2\pi,\pm3\pi,\pm4\pi,\pm5\pi,.....$ Since $n\in\mathbb{Z}^+$, $n=3,6,9,....$			

	Set of positive integers n is $\{n : n = 3p,\ p \in \mathbb{Z}^+\}$ Or Since w^n is real, $\arg w^n = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \pm 5\pi, ...$ $-\dfrac{2n\pi}{3} = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \pm 5\pi, ...$ Since $n \in \mathbb{Z}^+$, $n = 3, 6, 9, ...$ Set of positive integers n is $\{n : n = 3p,\ p \in \mathbb{Z}^+\}$ $w^n = \mathrm{e}^{-\mathrm{i}\frac{2n\pi}{3}} = \mathrm{e}^{-\mathrm{i}2p\pi} = \cos(-2p\pi) + \mathrm{i}\sin(-2p\pi) = 1$ (shown)			$w^n = 1$ is true for n which are multiples of 3. You have to give answer in set notation (which not many students manage to give) and not just for a few values of n. Note: $w^n \neq \lvert w^n \rvert$. To show $w^n = 1$, both modulus and argument of w^n must be shown.
3(iv)	$w^n = \mathrm{e}^{-\mathrm{i}\frac{2n\pi}{3}} = \cos\left(-\dfrac{2n\pi}{3}\right) + \mathrm{i}\sin\left(-\dfrac{2n\pi}{3}\right)$ $w^{100} - (w^*)^{100} = w^{100} - \left(w^{100}\right)^*$ $= 2\mathrm{i}\sin\left(-\dfrac{200\pi}{3}\right)$ $= 2\mathrm{i}\left(-\dfrac{\sqrt{3}}{2}\right) = -\sqrt{3}\mathrm{i}$ Alternatively, $w^{100} - (w^*)^{100} = w^{100} - \left(w^{100}\right)^*$ $= w - w^* = \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}} - \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ $= -2\mathrm{i}\sin\left(\dfrac{2\pi}{3}\right) = -2\mathrm{i}\left(\dfrac{\sqrt{3}}{2}\right) = -\sqrt{3}\mathrm{i}$			Note: $w^{100} - (w^*)^{100} \neq (w - w^*)^{100}$ $w^{100} = w^{99} \times w = w$ since $w^n = 1$ if n is a multiple of 3.

Qn	Suggested Answers			Guidance
4(a)(i)	$y=\ln\left(x+\sqrt{x^2+1}\right)$ $e^y = x+\sqrt{x^2+1}$ Differentiating both sides w.r.t. x $e^y\frac{dy}{dx}=1+\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}(2x)$ $\Rightarrow e^y\frac{dy}{dx}=1+x\left(x^2+1\right)^{-\frac{1}{2}}$ Differentiating both sides w.r.t. x $e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^2=\left(x^2+1\right)^{-\frac{1}{2}}+x\left(-\frac{1}{2}\right)\left(x^2+1\right)^{-\frac{3}{2}}(2x)$ $\Rightarrow e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^2=\left(x^2+1\right)^{-\frac{1}{2}}-x^2\left(x^2+1\right)^{-\frac{3}{2}}$ (shown)			It would be easier to do an implicit differentiation.
4(a)(ii)	$e^y\frac{d^3y}{dx^3}+e^y\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)+e^y\left(\frac{dy}{dx}\right)^3+e^y2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$ $=-\frac{1}{2}\left(x^2+1\right)^{-\frac{3}{2}}(2x)-2x\left(x^2+1\right)^{-\frac{3}{2}}-x^2\left(-\frac{3}{2}\right)\left(x^2+1\right)^{-\frac{5}{2}}(2x)$ $\Rightarrow e^y\frac{d^3y}{dx^3}+3e^y\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)+e^y\left(\frac{dy}{dx}\right)^3$ $=-x\left(x^2+1\right)^{-\frac{3}{2}}-2x\left(x^2+1\right)^{-\frac{3}{2}}+3x^3\left(x^2+1\right)^{-\frac{5}{2}}$ When $x=0$, $y=0,\frac{dy}{dx}=1,\frac{d^2y}{dx^2}=0,\frac{d^3y}{dx^3}=-1$ $\therefore \sinh^{-1}x=x-\frac{x^3}{6}+\dots$			

4(b)(i)	$\frac{AB}{AD}=\frac{AC\tan\frac{\pi}{6}}{AC\tan\left(\frac{\pi}{6}+\theta\right)}=\frac{1}{\sqrt{3}\tan\left(\frac{\pi}{6}+\theta\right)}$ (shown)			
4(b)(ii)	$\frac{AB}{AD}=\frac{1}{\sqrt{3}\tan\left(\frac{\pi}{6}+\theta\right)}=\frac{1-\tan\frac{\pi}{6}\tan\theta}{\sqrt{3}\left(\tan\frac{\pi}{6}+\tan\theta\right)}$ $=\frac{1}{\sqrt{3}}\left(\frac{1-\frac{\tan\theta}{\sqrt{3}}}{\frac{1}{\sqrt{3}}+\tan\theta}\right)=\frac{1}{\sqrt{3}}\left(1-\frac{\tan\theta}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}+\tan\theta\right)^{-1}$ $=\left(1-\frac{\tan\theta}{\sqrt{3}}\right)\left(1+\sqrt{3}\tan\theta\right)^{-1}\approx\left(1-\frac{\theta}{\sqrt{3}}\right)\left(1+\sqrt{3}\theta\right)^{-1}$ $=\left(1-\frac{\theta}{\sqrt{3}}\right)\left(1+(-1)\sqrt{3}\theta+\frac{(-1)(-2)}{2}\left(\sqrt{3}\theta\right)^2+\ldots\right)$ $=\left(1-\frac{\theta}{\sqrt{3}}\right)\left(1-\sqrt{3}\theta+3\theta^2+\ldots\right)$ $=1-\sqrt{3}\theta+3\theta^2-\frac{\theta}{\sqrt{3}}+\theta^2+\ldots\approx1+\left(-\frac{4\sqrt{3}}{3}\right)\theta+4\theta^2$ $a=-\frac{4\sqrt{3}}{3},b=4$			$\left(\frac{\pi}{6}+\theta\right)$ is not a small angle. There is a need to use the addition formulae to split up the angles. It would be easier to use $\tan\theta\approx\theta$, rather than changing $\tan\theta=\frac{\sin\theta}{\cos\theta}\approx\frac{\theta}{1-\frac{\theta^2}{2}}$. Note that the question wants up to term in θ^2. Many students did not do binomial expansion to sufficient no. of terms. Poor algebraic manipulation. Example: $a\left(\frac{2+3}{4+5}\right)=\frac{2a+3a}{4+5}$ $a\left(\frac{2+3}{4+5}\right)\neq\frac{2a+3a}{4a+5a}$

Qn	Suggested Answers			Guidance
5(i)	No. of arrangements $= \dfrac{10!}{2!2!4!} - 1$ $= 37799$			To approach the question directly is difficult as there are many arrangements. As such, the principle of complementation is used. The hint in question was the word "not" in **bold**. Note that there is only one way of arranging it in alphabetical order, CCEEEELLNX. This is similar to A level question 2015/2/11ii. Please attempt it.
5(ii)	No. of arrangements =Total no. of ways with all E's together – No. of ways all E's together and C's and L's together – No. of ways all E's together and C's but L's separated – No. of ways all E's together and L's but C's separated $= \dfrac{(7-1)!}{2!2!} - (5-1)! - (4-1)!\,{}^4C_2 - (4-1)!\,{}^4C_2$ $= 180 - 24 - 36 - 36 = 84$			Note that the letters are now arranged in a **circle**. Many students missed this out. A point to note is that no other adjacent letters being the same **does not mean** all the Ls and Cs must be separated. "LCLC" or "LC", "CL" strings are perfectly fine.

Qn	Suggested Answers			Guidance
6(i)	Let *Di* be the random variable denoting the score of the *i*th throw of the die for $i = 1, 2$ $\mathrm{P}(X=2)$ $= \mathrm{P}(D_1=1, D_2=5)+\mathrm{P}(D_1=2, D_2=4)+\mathrm{P}(D_1=3, D_2=3)$ $+\mathrm{P}(D_1=4, D_2=2)+\mathrm{P}(D_1=5, D_2=1)$ $=\frac{1}{6}\left(\frac{1}{6}\right)5=\frac{5}{36}$			This is a Answer Given (AG) question. Clear presentation is necessary. Just writing $\frac{1}{6}\left(\frac{1}{6}\right)5=\frac{5}{36}$ or similar would not secure any marks.
6(ii)	$\mathrm{P}(X>2 \mid D_1 \text{ is even})$ $=\frac{\mathrm{P}(X>2 \text{ and } D_1 \text{ is even})}{\mathrm{P}(D_1 \text{ is even})}$ $=\frac{\mathrm{P}(D_1=2, D_2\neq 4)+\mathrm{P}(D_1=4, D_2\neq 2)}{\mathrm{P}(D_1 \text{ is even})}$ $=\frac{\frac{1}{6}\left(\frac{5}{6}\right)+\frac{1}{6}\left(\frac{5}{6}\right)}{\frac{3}{6}}=\frac{5}{9}$			Note that $\mathrm{P}(X>2 \mid D_1 \text{ is even})$ $\neq \mathrm{P}(X>2)\mathrm{P}(D_1 \text{ is even})$ The events are not independent.
6(iii)	Let W be the random variable denoting number of games, out of 10, that a special prize is won. P(winning a special prize) $=\mathrm{P}(X=1)+\mathrm{P}(X=2)=\frac{1}{6}+\frac{5}{36}=\frac{11}{36}$ $W\sim \mathrm{B}\left(10,\frac{11}{36}\right)$ Probability required $=\mathrm{P}(3\le W\le 8)=\mathrm{P}(W\le 8)-\mathrm{P}(W\le 2)$ $=0.63173=0.632$			Note that "not more than eight games" include eight games also. This is a common mistake made. Another group of students struggled in managing $\mathrm{P}(3\le W\le 8)$. Drawing a simple number line would help.

Also,

$\mathrm{P}(3 \le W \le 8)$

$\ne \mathrm{P}(3 \le W)\mathrm{P}(W \le 8)$

6(iv)

Let n be the number of games needed.
Let Y be the random variable denoting number of games, out of n, that a special prize is won.

$$Y \sim \mathrm{B}\left(n, \frac{11}{36}\right)$$

$$\mathrm{P}(Y \ge 1) \ge 0.998 \Rightarrow 1 - \mathrm{P}(Y = 0) \ge 0.998$$

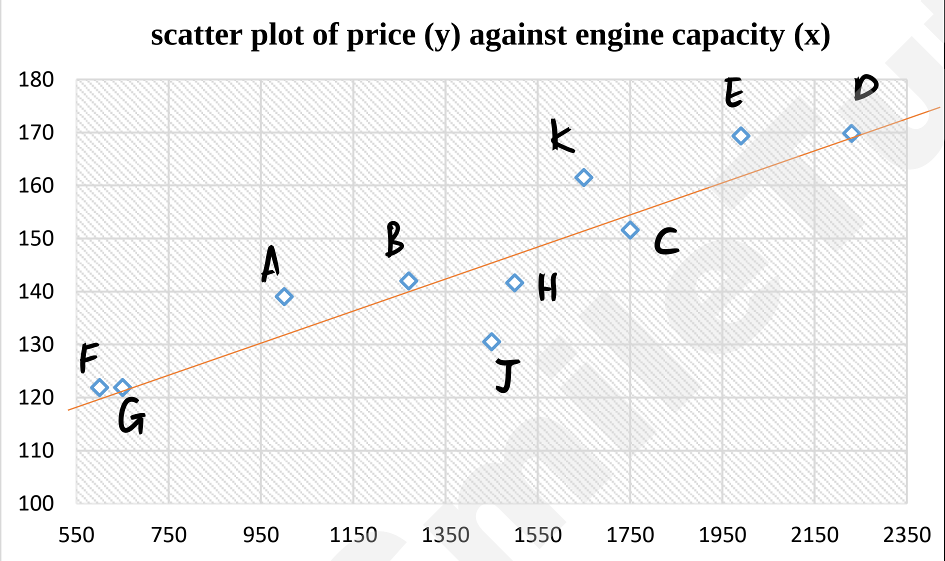
Using GC,

n	$1-\mathrm{P}(Y=0)$	$1-\mathrm{P}(Y=0)-0.998$
17	0.99797	-3×10^{-5}
18	0.99859	5.9×10^{-4}

Least n = 18

The main issue in this question is in the reading of the GC. It is essential to scroll the navigating button to the column that shows the probability. This is especially if students are using the key strokes that is shown in the column that is not highlighted. GC will round off to 0.998 when n = 17 and mislead the user.

The method shown in the highlighted column is better as it shows sign changes.

Qn	Suggested Answers			
7(a)	Scatter plot (C) shows perfect negative correlation, so $r_3 = -1$. Scatter plot (A) shows strong, but not perfect, negative correlation, so $-1 < r_1 < 0$. Scatter plot (B) shows no correlation, so $r_2 = 0$. Therefore, $r_3 < r_1 < r_2$.			Most students were able to get 2 marks. Quite a handful said the value of r is undefined for plot (B).
7(b)(i)				Quite a number of students forgotten to label the points and to draw the regression line. Not drawing on graph paper resulted in loss of 3 marks. So please read the question.
7(b)(ii)	Equation of regression line of *y* on *x* is $y = 0.03019188x + 102.30964 \approx 0.0302x + 102$			
7(b)(iii)	$r = 0.91336 \approx 0.913$			

	The value of r implies(suggests) a strong positive linear relationship between x and y.			
	The regression line to be drawn on the scatter diagram passing through D and G. Point corresponding to model J lies well below the line, which implies that its **price is lower** than it would be expected given its engine capacity. This would be good value for the consumer. Points corresponding to models A, E and K are well above the line, which implies that their **prices are higher** than it would be expected given their engine capacity and would not be recommended in this case.			Quite a number of students just state the models without justification.

Qn	Suggested Answers			Guidance
8(i)	Let L denotes the mass of a randomly chosen link. L~N(800,20^2) P(L > 805) ≈ 0.40129 ≈ 0.401			
8(ii)	Let S denotes the mass of a randomly chosen link with locking sleeve S = 1.1L , S~N(880,22^2) P(865.35 < S < 895.5) ≈ 0.50672 ≈ 0.507			Common mistake Var(S) = 1.1 x 20^2 Var(S) = 20^2
8(iii)	Let H denotes the mass of a randomly chosen hook $H \sim \text{N}(750,\sigma^2)$ P(H < 735.6) = 0.15 $\frac{735.6-750}{\sigma}=-1.0364$ $\sigma \approx 13.894 \approx 13.9$			
8(iv)	Let C denotes the mass of a wooden box and its contents $C = S_1 + S_2 + S_3 + S_4 + S_5 + H + 1000$ Let $\overline{C}=\frac{C_1+C_2+...C_n}{n}$ $\overline{C} \sim N\left(6150,\frac{2613.0}{n}\right)$ $P\left(\overline{C}>6190\right)>0.013$ $P\left(Z>\frac{6190-6150}{\sqrt{\frac{2613.0}{n}}}\right)>0.013$ $40\sqrt{\frac{n}{2613.0}}<2.2262$ $n<8.09$ Greatest n = 8			Common mistakes E(C) = E(S_1 + … + S_5 + H_1 +…+H_5 +1000) Var(C) = Var(5S+5H+1000) = 25Var(S)+25Var(H) +1000

Qn	Suggested Answers			
9(i)	Let X be the random variable denoting the number of yellow bricks drawn by Donald. P(no yellow bricks) $=\mathrm{P}(X=0)$ $=\underbrace{\frac{1}{2}\left(\frac{5}{10}\right)\frac{1}{2}\left(\frac{4}{9}\right)}_{\substack{\text{R or B from Box 1 then}\\ \text{R or B from Box 1}}}+\underbrace{\frac{1}{2}\left(\frac{5}{10}\right)\frac{1}{2}\left(\frac{7}{10}\right)}_{\substack{\text{R or B from Box 1 then}\\ \text{R or B from Box 2}}}+\underbrace{\frac{1}{2}\left(\frac{7}{10}\right)\frac{1}{2}\left(\frac{5}{10}\right)}_{\substack{\text{R or B from Box 2 then}\\ \text{R or B from Box 1}}}+\underbrace{\frac{1}{2}\left(\frac{7}{10}\right)\frac{1}{2}\left(\frac{6}{9}\right)}_{\substack{\text{R or B from Box 2 then}\\ \text{R or B from Box 2}}}$ $=\frac{25}{72}$ (shown)			Need to multiply probability of choosing box 1 or 2 which is $\frac{1}{2}$ due to the statement "He draws a brick **randomly** from one of the boxes". There is no need to consider the red and blue bricks separately. They can be viewed as non-yellow bricks which makes the working neater
9(ii)	$\mathrm{P}(X=2)$ $=\frac{1}{2}\left(\frac{5}{10}\right)\frac{1}{2}\left(\frac{4}{9}\right)+\frac{1}{2}\left(\frac{5}{10}\right)\frac{1}{2}\left(\frac{3}{10}\right)+\frac{1}{2}\left(\frac{3}{10}\right)\frac{1}{2}\left(\frac{5}{10}\right)+\frac{1}{2}\left(\frac{3}{10}\right)\frac{1}{2}\left(\frac{2}{9}\right)$ $=\frac{53}{360}$			
9(iii)	$\begin{array}{\|c\|c\|c\|c\|}\hline x & 0 & 1 & 2 \\ \hline \mathrm{P}(X=x) & \frac{25}{72} & 1-\frac{25}{72}-\frac{53}{360}=\frac{91}{180} & \frac{53}{360} \\ \hline\end{array}$ *Direct computation P(X = 1):*			

	$\mathrm{P}(X=1)=\mathrm{P}(Y,Y')+\mathrm{P}(Y',Y)$ $=\frac{1}{2}\left(\frac{5}{10}\right)\left[\frac{1}{2}\left(\frac{5}{9}\right)+\frac{1}{2}\left(\frac{7}{10}\right)\right]+\frac{1}{2}\left(\frac{3}{10}\right)\left[\frac{1}{2}\left(\frac{5}{10}\right)+\frac{1}{2}\left(\frac{7}{9}\right)\right]$ $+\frac{1}{2}\left(\frac{5}{10}\right)\left(\frac{1}{2}\left(\frac{5}{9}\right)+\frac{1}{2}\left(\frac{3}{10}\right)\right)+\frac{1}{2}\left(\frac{7}{10}\right)\left(\frac{1}{2}\left(\frac{5}{10}\right)+\frac{1}{2}\left(\frac{3}{9}\right)\right)$ $=\frac{113}{720}+\frac{23}{240}+\frac{77}{720}+\frac{7}{48}=\frac{91}{180}$ $\mathrm{E}(X)=0+\frac{91}{180}+2\left(\frac{53}{180}\right)=\frac{4}{5}$			Students **should not** round the expected number to an integer.
9(iv)	Let W be the random variable denoting the number of yellow bricks drawn out of 5. $W \sim \mathrm{B}\left(5, \frac{8}{20}\right)$ P(draws less than 2 yellow bricks) $=\mathrm{P}(W<2)=\mathrm{P}(W\le 1)=0.33639=0.337$ (3 s.f.) Alternatively, P(draws less than 2 yellow bricks) $=\left(\frac{12}{20}\right)^5+{}^5C_4\left(\frac{12}{20}\right)^4\left(\frac{8}{20}\right)=\frac{1053}{3125}$ (or 0.33696)			Students should define another random variable as it has a different distribution as (iii). Read the question carefully, there is replacement. Many students who used the alternative method did not factor in the ordering (i.e. the ${}^5C_4$).
9(v)	P(not more than 9 draws to get the first yellow brick) =P(first yellow brick is drawn on the 1st , 2nd, …or 9th draw)			There are 9 terms in the GP not 8.

	$=\left(\frac{8}{20}\right)+\left(\frac{12}{20}\right)^{1}\left(\frac{8}{20}\right)+\left(\frac{12}{20}\right)^{2}\left(\frac{8}{20}\right)+\ldots+\left(\frac{12}{20}\right)^{8}\left(\frac{8}{20}\right)$ $=\frac{0.4\left(1-0.6^{9}\right)}{1-0.6}$ =0.98992 = 0.990 (to 3 sig. fig.) Alternatively: P(not more than 9 draws to get the first yellow brick) = 1 – P(10 or more draws to get the first yellow brick) $= 1-\left(\frac{12}{20}\right)^{9}=0.98992=0.900$ (to 3 sig.fig.)			This is not a binomial distribution as the number of trials is not fixed (in fact you can keep drawing from the box forever and not get a yellow brick).

Qn	Suggested Answers			
10(i)	The sample is random would mean that the result of each throw has the same chance of being selected. The outcome of each throw is also independent of one another.			Many wrote about equal probability of attaining the various distances rather than the outcome of each throw having an equal probability of being selected.
10(ii)	Unbiased estimate of the population mean, $\bar{x}=\frac{120}{60}+65=67$ Unbiased estimate of the population variance, $s^2=\frac{1}{59}\left[\sum(x-65)^2-\frac{\left(\sum(x-65)\right)^2}{60}\right]=60.50847=60.5$ (to 3 s.f.)			Quite a big number of students quoted the wrong formula from MF26 in calculating of s^2. They used the formula $s^2=\frac{n}{n-1}\left[\frac{\sum\left(x-\bar{x}\right)^2}{n}\right]$, whereas the summary of data was given with assumed mean of 65 instead.
10(iii)	Let μ be the population mean distance of the throws. Null hypothesis , H_0 : $\mu=68$ Alt hypothesis, H_1 : $\mu>68$			Many did not define μ.
10(iv)	Let X be the random variable denoting the distances thrown by the thrower. Under H_0, $\bar{X}\sim \mathrm{N}\left(68,\frac{7.5^2}{30}\right)$ and test statistic $Z=\frac{\bar{X}-68}{7.5/\sqrt{30}}\sim N(0,1)$ Critical Region : Reject H_0 if $Z_{calc}\geq 2.05375$			Many made the mistake of using Central Limit Theorem for this part when the question specified that the "distances thrown by the thrower follows a

	Calculations : $Z_{\text{calc}} = \dfrac{\bar{x}-68}{7.5/\sqrt{30}}$ Since H_0 is rejected, $\dfrac{\bar{x}-68}{7.5/\sqrt{30}} \geq 2.05375$ $\bar{x} \geq 70.8122$ $\bar{x} \geq 70.9$			**normal distribution**". This is a conceptual error to use CLT when the distribution is normal. For the test statistic, many neglected the equal sign that comes with the inequality.
10(v)	If H_0 is not rejected, $Z_{\text{calc}} < 2.05375$ $\dfrac{70.1-68}{7.5/\sqrt{n}} < 2.05375$ $\sqrt{n} < 7.33482$ $n < 53.79954$ Set of values of n is $\{n \in \mathbb{Z}^+ : n \leq 53\}$ or $\{n \in \mathbb{Z} : 1 \leq n \leq 53\}$			
	Null hypothesis , H_0 : $\mu = \mu_0$ Alt hypothesis, H_1 : $\mu < \mu_0$ Reject H_0 if $Z_{\text{calc}} \leq -1.64485$ For Ang's conclusion (H_0 is rejected) : $Z_{\text{calc}} = \dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \leq -1.64485$ For Tan's case : $Z'_{\text{calc}} = \dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{4n}} = 2\left(\dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)$			Only a handful wrote the correct hypotheses or wrote the hypotheses for this part of the question. The usual argument for testing would involve the correct set of hypotheses before proceeding to justify the conclusion. Most students continue with the same set of hypotheses for this part which is incorrect.

	Since $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \le -1.64485$, $2\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right) \le 2(-1.64485) \le -1.64485$, ie $Z'_{\text{calc}} \le -1.64485$, which leads to H_0 being rejected. Therefore, Tan's test would yield the same conclusion.			This part was not done well.

NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2 PRELIMINARY EXAMINATION
Higher 2

MATHEMATICS 9758/01

Paper 1

24 August 2018

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in the brackets [ ] at the end of each question or part question.

This document consists of **8** printed pages.

National Junior College

1 The n^{th} term of a sequence is given by $u_n = n!(n-2)$, for all positive integers n where $n \geq 2$.

Show that

$$u_n - u_{n-1} = (n-1)!\left(an^2 + bn + c\right)$$

for some real constants a, b and c to be determined. [2]

Hence find

$$\sum_{n=3}^{N+1}\left[(n-1)!\left(2n^2 - 6n + 6\right)\right].$$ [3]

2 Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point C is on AB produced such that $AB : BC = 1 : k$, where k is a real constant.

Given that $OA = 2OC$, $\angle AOB = \frac{\pi}{2}$, and $\angle AOC = \frac{3\pi}{4}$, find the exact value of k, showing your working clearly. [5]

3 By means of the substitution $u = \sin\theta$, find

$$\int \frac{2\cos\theta - 3\sin 2\theta}{1 + \sin^2\theta}\,\mathrm{d}\theta.$$ [5]

4 The curve C has equation $9x^2 - y^2 + 3xy - 5 = 0$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. Hence, explain why C has no turning points. [4]

(ii) Find the equations of the tangent and the normal to C at the point $P(1, -1)$. [2]

(iii) The tangent and normal to C at the point $P(1, -1)$ meets the y-axis at the points Q and R respectively. State the area of triangle PQR. [1]

5 Given that a is a positive constant. A curve C_1 has parametric equations

$$x = \frac{a}{t}, \quad y = 1 + t.$$

Sketch C_1, labelling the coordinates of the point(s) where the curve crosses the x- and y-axes, and the equations of the asymptote(s) in terms of a, if any. [2]

Another curve C_2 has equation $y = \sqrt{1 + \frac{x^2}{a^2}}$.

(i) Show algebraically that the y-coordinates of the point(s) of intersection of C_1 and C_2 satisfies the equation $(y-1)^2(y^2-1) - 1 = 0$. [2]

(ii) Sketch C_2 on the same diagram as C_1, labelling the coordinates of the point(s) where the curve crosses the x- and y-axes, and the equations of the asymptotes in terms of a, if any.

Find the coordinates of point(s) of intersections of C_1 and C_2 and label the coordinates in this diagram, leaving the answers correct to 3 significant figures, in terms of a. [4]

6 **(a)** The function f is defined by

$$\mathrm{f} : x \mapsto 2x^2 - \lambda x - 3, \text{ where } x \in \mathbb{R}, \ \frac{7}{4} < x < 5,$$

where λ is a real constant. Find the set of possible values of λ such that f^{-1} exists. [2]

(b) The function g is defined by

$$\mathrm{g}(x) = \begin{cases} (x-1)^2, & \text{for } 0 \le x \le 1, \\ 2\log_2 x, & \text{for } 1 < x \le 2. \end{cases}$$

(i) Sketch the graph of $y = \mathrm{g}(x)$, labelling clearly the coordinates of the end-points and the points where the curve crosses the x-axis, if any. [2]

(ii) Hence solve the inequality $1 < \mathrm{g}(x) \le 2$ exactly. [2]

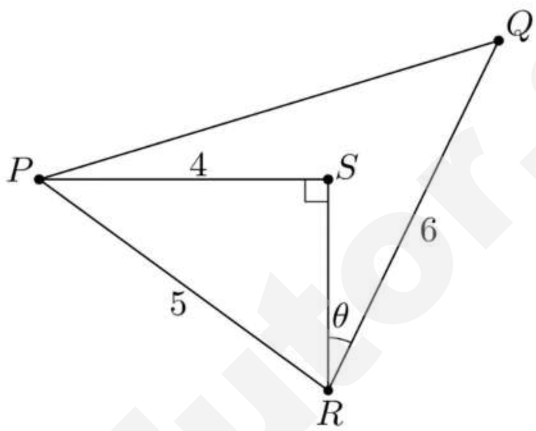
(iii) Given that g^2 exists, define g^2 in a similar form as g. [3]

7 **(a)** Given that $y+2=(x+1)^{\ln(x+1)}$, where $x>-1$, show that

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x}=2(y+2)\ln(x+1). \qquad [2]$$

By repeated differentiation of the above result, find the Maclaurin series of y up to and including the term in x^2. [3]

(b)



In the diagram above, $QR=6$, $PS=4$, $PR=5$, $\angle PSR=\frac{\pi}{2}$ and $\angle QRS=\theta$ radians.

(i) Show that $PQ=(61-36\cos\theta+48\sin\theta)^{\frac{1}{2}}$. [2]

(ii) Given that θ is a sufficiently small angle, show that

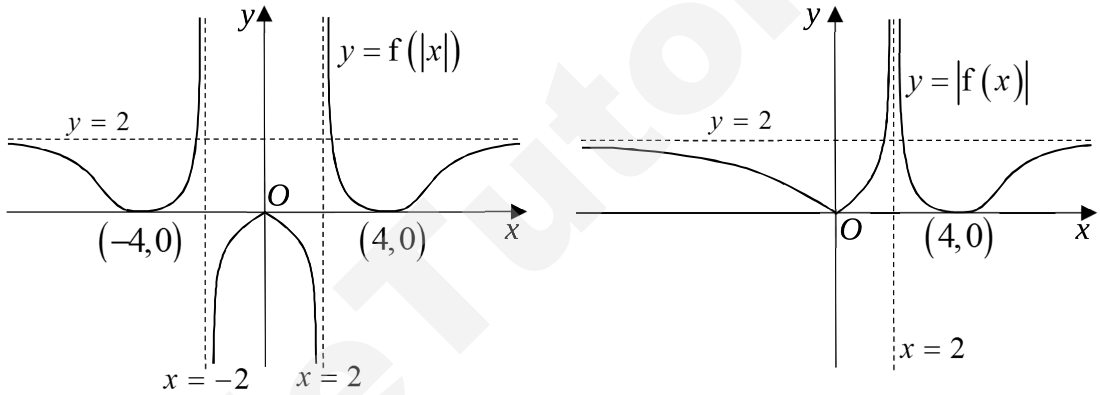
$$PQ\approx 5+p\theta+q\theta^2$$

for some rational constants p and q to be determined exactly. [4]

8 **(a)** Describe a sequence of exactly three transformations that will transform the curve with equation $y=\dfrac{1}{x-3}$ onto the curve with equation $y=\dfrac{2x+a}{3-x}$, where a is a positive constant. [3]

Given instead that $a<-6$, use a non-graphical method to determine the range of values of x where the graph of $y=\dfrac{2x+a}{3-x}$ is concave upwards. [2]

(b) The diagrams below show the graphs of $y=\mathrm{f}\left(|x|\right)$ and $y=\left|\mathrm{f}\left(x\right)\right|$, where the equations of the asymptotes and the coordinates of the turning points are given. The gradient of the graph with equation $y=\mathrm{f}\left(x\right)$ at the origin is $-\dfrac{3}{4}$.



On separate diagrams, sketch the graphs of

(i) $y=\mathrm{f}(x)$, [2]

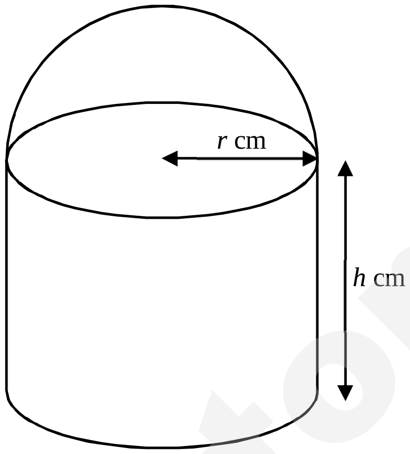
(ii) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

(iii) $y=\mathrm{f}'(x)$, [3]

including the coordinates of the points where the graphs cross the x- and y- axes and the equations of the asymptotes, if any.

9 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

A manufacturer produces closed hollow cans of fixed volume k cm^3 as shown in the diagram below. The top part is a hemisphere made of tin. The bottom part is a cylinder made of aluminium of cross-sectional radius r cm and height h cm. There is no material between the cylinder and the hemisphere so that any fluid can move freely within the container.



(i) If tin costs 4 cents per cm^2 and aluminium costs 6 cents per cm^2, use differentiation to find the values of r and h such that the total cost of producing the cans is minimised, giving your answers in terms of k. Simplify your answers. [7]

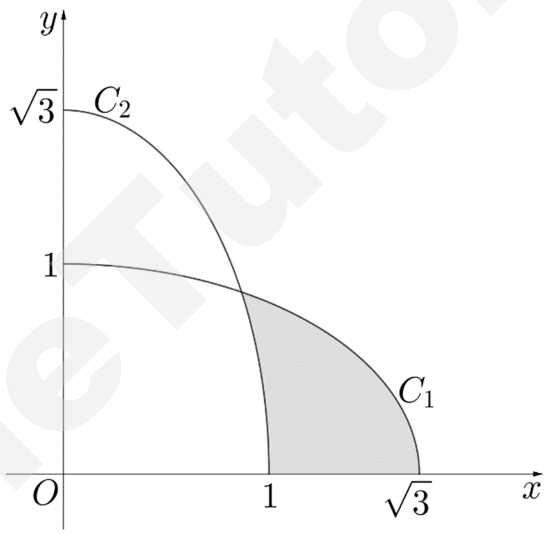
(ii) At the beginning of an experiment, a similar-shaped can of dimensions $r = 4$ and $h = 10$, is filled to its capacity with water. Due to a hole at its base, water is leaking at a constant rate of 2 cm^3 s^{-1} when the can is standing upright. Find the exact rate at which the height of the water is decreasing 80 seconds after the start of the experiment. [5]

10 **(i)** Show that for any real constant p, $\dfrac{p^2x^2}{\sqrt{1-p^2x^2}} = \dfrac{1}{\sqrt{1-p^2x^2}} - \sqrt{1-p^2x^2}$.

Hence, or otherwise, prove that for any constant n such that $0 < n < \dfrac{1}{p}$,

$$\int_n^{\frac{1}{p}} \sqrt{1-p^2x^2}\,\mathrm{d}x = \frac{\pi}{4p} - \frac{1}{2p}\sin^{-1}(pn) - \frac{1}{2}n\sqrt{1-p^2n^2}.$$ [6]

(ii) The curves C_1 and C_2 are ellipses with the origin O as their common centre. It is given that the points $(0, 1)$ and $\left(\sqrt{3}, 0\right)$ are vertices of C_1 and the points $\left(0, \sqrt{3}\right)$ and $(1, 0)$ are vertices of C_2. The diagram below shows the parts of the two curves in the first quadrant.



Use the result in part **(i)** to find the area of the shaded region exactly, giving your answer in the form $\dfrac{\pi\sqrt{3}}{m}$, where m is an integer constant to be determined. [6]

[**Question 11 is printed on the next page.**]

11 A swimming pool contains 375000 litres of pure water. Water containing s milligrams of free chlorine per litre flows into the pool at a rate of 10 litres per minute. The pool is also draining at a rate of 10 litres per minute, such that the volume of the water in the pool remains constant.

(i) Show that the rate at which the mass of the free chlorine, x grams, is changing in the pool over time, t minutes, can be modelled by the differential equation

$$\frac{dx}{dt} = \frac{375s - x}{37500},$$

State a necessary assumption for the above model to be valid. [3]

(ii) Given that no free chlorine is present in the pool initially, find x in terms of t and s. [4]

(iii) Find, in terms of s, the mass of free chlorine present in the pool after one hour. [1]

(iv) Sketch the graph of x against t which is relevant to the context, labelling the point(s) where the curve crosses the axes and the equation(s) of the asymptote(s). Hence determine the mass of free chlorine present in the pool after a long period in terms of s. [3]

To prevent health complications, the recommended safe level of free chlorine to be used is between 375 grams and 1125 grams in a pool which contains 375 000 litres of water. Find the range of values of s that should be used. [2]

– END OF PAPER –

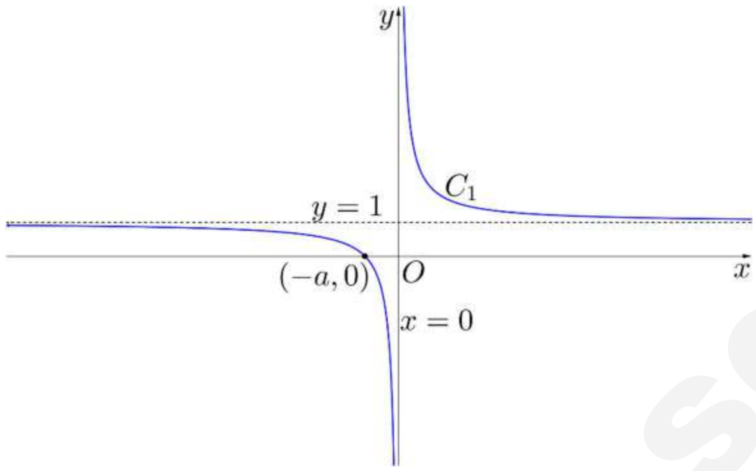
2018 NJC H2 Math Prelim Paper 1 Solution

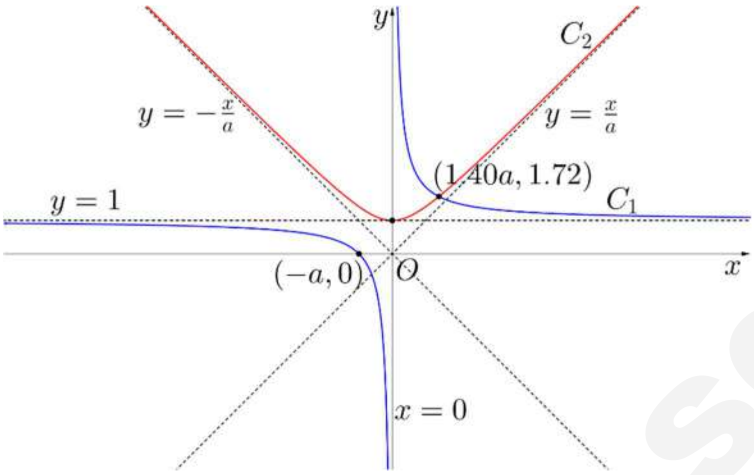
1	$u_n - u_{n-1}$ $= n!(n-2)-(n-1)!(n-3)$ $=(n-1)!n(n-2)-(n-1)!(n-3)$ $=(n-1)!(n^2-2n-n+3)$ $=(n-1)!(n^2-3n+3)$ $\sum_{n=3}^{N+1}\left[(n-1)!\left(2n^2-6n+6\right)\right].$ $2\sum_{n=3}^{N+1}\left((n-1)!\left(n^2-3n+3\right)\right)$ $=2\sum_{n=3}^{N+1}\left(u_n-u_{n-1}\right)$ $=2\ [\cancel{u_3}-u_2$ $+\cancel{u_4}-\cancel{u_3}$ $+\vdots$ $+\cancel{u_N}-\cancel{u_{N-1}}$ $+u_{N+1}-\cancel{u_N}]$ $=2\left(u_{N+1}-u_2\right)$ $=2\left[(N+1)!(N-1)-2!(0)\right]$ $=2(N+1)!(N-1)$

2	Let $OC = \mathbf{c}$. $\mathbf{b} = \dfrac{k\mathbf{a}+\mathbf{c}}{1+k}$ $(1+k)\mathbf{b} = k\mathbf{a}+\mathbf{c}$ $\mathbf{a}\cdot\mathbf{b}(1+k) = \mathbf{a}\cdot(k\mathbf{a}+\mathbf{c})$ $0 = k\lvert\mathbf{a}\rvert^2 + \mathbf{a}\cdot\mathbf{c}$ $k\lvert\mathbf{a}\rvert^2 = -\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert\cos\dfrac{3\pi}{4}$ $k\lvert\mathbf{c}\rvert^2 = -\lvert\mathbf{a}\rvert\cdot\dfrac{1}{2}\lvert\mathbf{a}\rvert\cos\dfrac{3\pi}{4}$ since $\lvert\mathbf{a}\rvert = 2\lvert\mathbf{c}\rvert$ $k = -\dfrac{1}{2}\cos\dfrac{3\pi}{4}$ $k = \dfrac{\sqrt{2}}{4}$

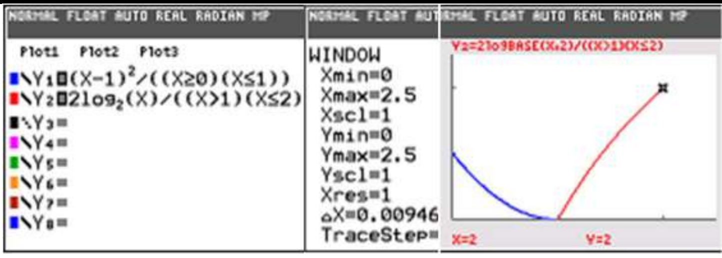
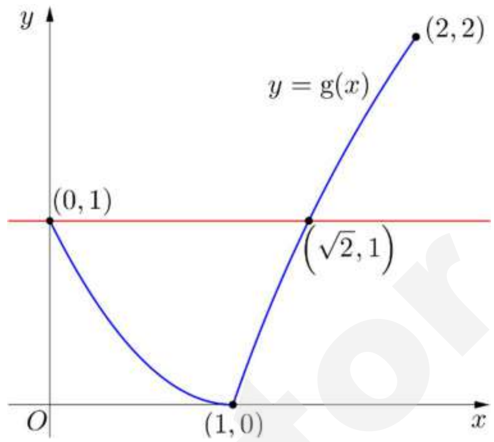
3	$u=\sin\theta \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}\theta}=\cos\theta$ $\displaystyle\int \frac{2\cos\theta-3\sin 2\theta}{1+\sin^2\theta}\,\mathrm{d}\theta$ $\displaystyle=\int \frac{2\cos\theta-6\sin\theta\cos\theta}{1+\sin^2\theta}\,\mathrm{d}\theta$ $\displaystyle=\int\left(\frac{2-6\sin\theta}{1+\sin^2\theta}\right)\cos\theta\,\mathrm{d}\theta$ $\displaystyle=\int \frac{2-6u}{1+u^2}\,\mathrm{d}u$ $\displaystyle=\int \frac{2}{1+u^2}\,\mathrm{d}u-3\int \frac{2u}{1+u^2}\,\mathrm{d}u$ $=2\tan^{-1}u-3\ln\left(1+u^2\right)+c$ $=2\tan^{-1}\left(\sin\theta\right)-3\ln\left(1+\sin^2\theta\right)+c$ $=2\tan^{-1}\left(\sin\theta\right)-3\ln\left(1+\sin^2\theta\right)+c$

4 (i)	Differentiating $9x^2 - y^2 + 3xy - 5 = 0$ w.r.t. x, $18x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 0$ $2y\frac{\mathrm{d}y}{\mathrm{d}x} - 3x\frac{\mathrm{d}y}{\mathrm{d}x} = 18x + 3y$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{18x+3y}{2y-3x}$ Suppose (x, y) is a turning point of C. Then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{18x+3y}{2y-3x} = 0 \Rightarrow 18x + 3y = 0 \Rightarrow y = -6x$ Substituting $y = -6x$ into $9x^2 - y^2 + 3xy - 5 = 0$, we get $9x^2 - (-6x)^2 + 3x(-6x) - 5 = 0$ $9x^2 - 36x^2 - 18x^2 - 5 = 0$ $x^2 = \frac{5}{-45} = -\frac{1}{9}$ which is impossible since $x^2 \geq 0$. Therefore C has no turning points.
4 (ii)	At $P(1,-1)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{18-3}{-2-3} = -3$ Then equation of tangent is $\frac{y+1}{x-1} = -3 \Rightarrow y = -3x + 2$ Then equation of normal is $\frac{y+1}{x-1} = \frac{1}{3} \Rightarrow 3y + 3 = x - 1 \Rightarrow y = \frac{1}{3}x - \frac{4}{3}$
4 (iii)	Area of triangle PQR $= \frac{1}{2}\left(2 + \frac{4}{3}\right)(1)$ $= \frac{5}{3}$

5 (1st part)	
5 (i)	$x=\dfrac{a}{t}$, $y=1+t \Rightarrow x=\dfrac{a}{y-1}$ Substitute $x=\dfrac{a}{y-1}$ into $y=\sqrt{1+\dfrac{x^2}{a^2}}$: $y=\sqrt{1+\dfrac{\left(\dfrac{a}{y-1}\right)^2}{a^2}}$ $y^2=1+\dfrac{1}{(y-1)^2}$ $y^2(y-1)^2=(y-1)^2+1$ $y^2(y-1)^2-(y-1)^2-1=0$ $(y-1)^2(y^2-1)-1=0$ **Alternatively**, $y=\sqrt{1+\dfrac{\left(\frac{a}{t}\right)^2}{a^2}}=\sqrt{1+\dfrac{1}{t^2}}$ Squaring both sides, $y^2=1+\dfrac{1}{t^2}$ $y^2=1+\dfrac{1}{(y-1)^2}$ $y^2-1=\dfrac{1}{(y-1)^2}$ $(y^2-1)(y-1)^2=1$ $(y^2-1)(y-1)^2-1=0$ (shown)

5 (ii)	 Using GC to solve, $(y-1)^2(y^2-1)-1=0$, $y=-1.106919$ (rejected) or 1.716673 When $y = 1.716673$, $x = 1.39534a$ $\therefore$ The point of intersection is $(1.40a, 1.72)$.

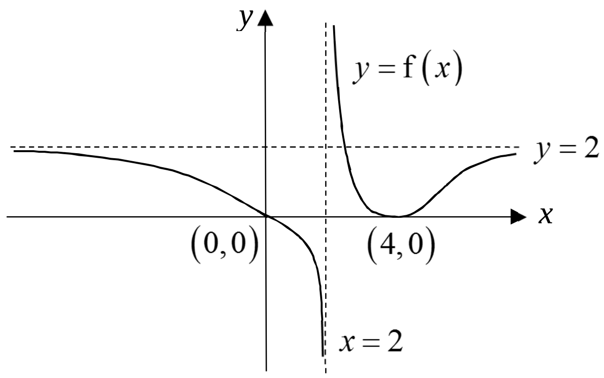
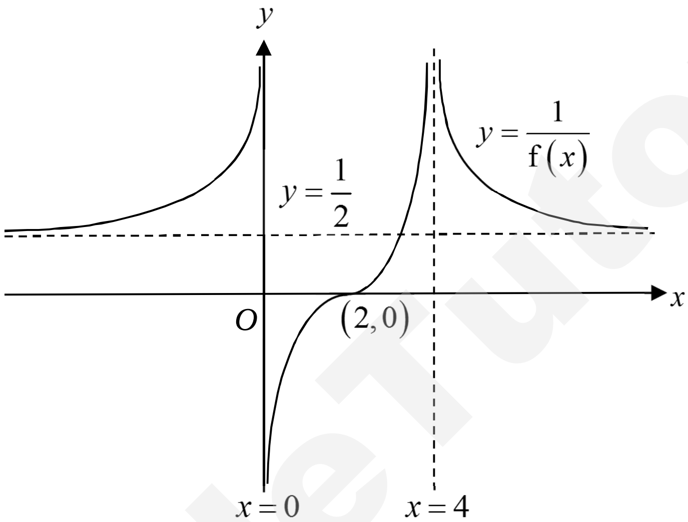
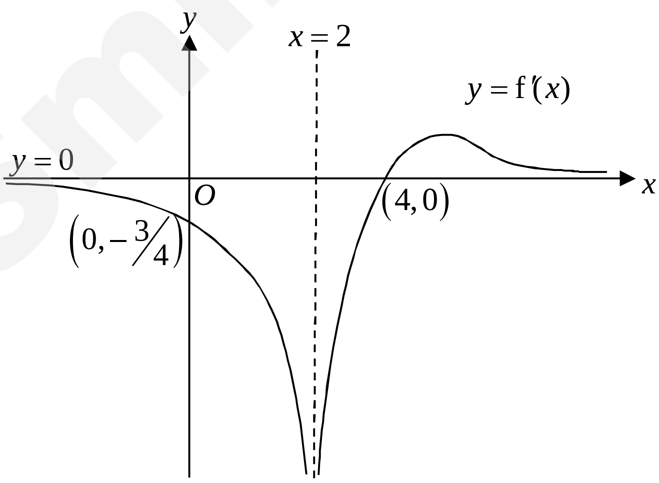
6 (a)	$2x^2-\lambda x-3=2\left(x^2-\frac{\lambda}{2}x\right)-3$ $=2\left(x^2+2\left(-\frac{\lambda}{4}\right)x+\left(-\frac{\lambda}{4}\right)^2\right)-3-2\left(-\frac{\lambda}{4}\right)^2$ $=2\left(x-\frac{\lambda}{4}\right)^2-\frac{\lambda^2+24}{8}$ OR $\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^2-\lambda x-3\right)=0$ $4x-\lambda=0$ $x=\frac{\lambda}{4}$ For f^{-1} to exist, the turning point of $y=2x^2-\lambda x-3$ **cannot** lie in the interval $\frac{7}{4}<x<5$. Therefore, $\frac{\lambda}{4}\le\frac{7}{4}$ or $\frac{\lambda}{4}\ge 5\Rightarrow\lambda\le 7$ or $\lambda\ge 20$

6 (b)(i)	 
6 (b)(ii)	From the sketch in **(b)(i)**, Point of intersection between $y = \mathrm{g}(x)$ and $y = 1$ occurs at the points (0, 1) and where $2\log_2 x = 1 \Rightarrow \log_2 x = \frac{1}{2}$ $\Rightarrow x = 2^{\frac{1}{2}} = \sqrt{2}$ Therefore, $1 < \mathrm{g}(x) \le 2 \Rightarrow \sqrt{2} < x \le 2$
6 (b) (iii)	For $0 \le x \le 1$, $\mathrm{g}^2(x) = \mathrm{g}\left((x-1)^2\right) = \left((x-1)^2 - 1\right)^2 = \left(x^2 - 2x\right)^2$ Considering the effect of part **(b)(ii),** For $1 < x \le \sqrt{2}$, $\mathrm{g}^2(x) = \mathrm{g}\left(2\log_2 x\right) = \left(2\log_2 x - 1\right)^2$ For $\sqrt{2} < x \le 2$, $\mathrm{g}^2(x) = \mathrm{g}\left(2\log_2 x\right) = 2\log_2\left(2\log_2 x\right)$ Therefore, $\mathrm{g}^2(x) = \begin{cases} \left(x^2 - 2x\right)^2 & \text{for } 0 \le x \le 1, \\ \left(2\log_2 x - 1\right)^2 & \text{for } 1 < x \le \sqrt{2}, \\ 2\log_2\left(2\log_2 x\right) & \text{for } \sqrt{2} < x \le 2 \end{cases}$

7 (a)	$y+2=(x+1)^{\ln(x+1)}$ $\ln(y+2)=\ln(x+1)^{\ln(x+1)}$ $\ln(y+2)=(\ln(x+1))^2$ $\frac{1}{y+2}\frac{\mathrm{d}y}{\mathrm{d}x}=2(\ln(x+1))\left(\frac{1}{x+1}\right)$ $(x+1)\frac{\mathrm{d}y}{\mathrm{d}x}=2(y+2)\ln(x+1)$ (shown) $(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2(y+2)}{x+1}+2\frac{\mathrm{d}y}{\mathrm{d}x}\ln(x+1)$ $(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\left[1-2\ln(x+1)\right]=\frac{2(y+2)}{x+1}$ When $x=0$, $y+2=1$ $\therefore y=-1$ $\frac{1}{-1+2}\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{2(-1+2)}{0+1}$ $\therefore \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2$ $y\approx -1+\frac{x^2}{2!}(2)$ $\therefore y=-1+x^2$

7 (b)(i)	$PQ^2 = PR^2 + QR^2 - 2PR \cdot QR\cos\angle PRQ$ $= 5^2 + 6^2 - 2(5)(6)\cos\left(\angle PRS + \theta\right)$ $= 25 + 36 - 60\left[\cos\left(\angle PRS\right)\cos\theta - \sin\left(\angle PRS\right)\sin\theta\right]$ $= 61 - 60\left(\frac{3}{5}\cos\theta - \frac{4}{5}\sin\theta\right)$ $= 61 - 36\cos\theta + 48\sin\theta$ $\therefore PQ = \left(61 - 36\cos\theta + 48\sin\theta\right)^{\frac{1}{2}}$ (shown)
7 (b) (ii)	$PQ = \left(61 - 36\cos\theta + 48\sin\theta\right)^{\frac{1}{2}}$ $\approx \left(61 - 36\left(1 - \frac{\theta^2}{2}\right) + 48\theta\right)^{\frac{1}{2}}$ $= \left(25 + 48\theta + 18\theta^2\right)^{\frac{1}{2}}$ $= 5\left(1 + \frac{48}{25}\theta + \frac{18}{25}\theta^2\right)^{\frac{1}{2}}$ $\approx 5\left(1 + \frac{1}{2}\left(\frac{48}{25}\theta + \frac{18}{25}\theta^2\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{48}{25}\theta\right)^2\right)$ $= 5\left(1 + \frac{24}{25}\theta + \frac{9}{25}\theta^2 - \frac{288}{625}\theta^2\right)$ $= 5\left(1 + \frac{24}{25}\theta - \frac{63}{625}\theta^2\right)$ $= 5 + \frac{24}{5}\theta - \frac{63}{125}\theta^2,\ p = \frac{24}{5},\ q = -\frac{63}{125}$

8 (a) (i)	1. Reflect graph about the *x*-axis 2. Scale graph parallel to the *y*-axis by factor of (6 + *a*) 3. Translate graph along/in the negative *y*-direction by 2 units
8 (a) (ii)	$y = \frac{2x+a}{3-x}$ $= -\frac{2x+a}{x-3}$ $= -2 - \frac{6+a}{x-3}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{6+a}{(x-3)^2}(-1)$ $= \frac{6+a}{(x-3)^2}$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{6+a}{(x-3)^3}(-2)$ $= -\frac{2(6+a)}{(x-3)^3}$ Where the graph is concave upwards, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$. Since $a < -6$, $6 + a < 0$. Therefore, $(x-3)^3 > 0 \Rightarrow x > 3$

8 (b) (i)	
8 (b) (ii)	
8 (b) (iii)	

9 (i)	Consider the volume of a can $\frac{2}{3}\pi r^3+\pi r^2 h=k$ $\pi r h=\frac{k}{r}-\frac{2}{3}\pi r^2$ $h=\frac{k}{\pi r^2}-\frac{2}{3}r$ --- (1) Cost of a can, $C=4\left(2\pi r^2\right)+6\left(2\pi r h\right)+6\left(\pi r^2\right)$ $=14\left(\pi r^2\right)+12\left(\pi r h\right)$ $=14\left(\pi r^2\right)+\frac{12k}{r}-8\pi r^2$ (from (1)) $=6\pi r^2+\frac{12k}{r}$ $\frac{\mathrm{d}C}{\mathrm{d}r}=12\pi r-\frac{12k}{r^2}=0$ $r^3=\frac{k}{\pi}$ $\therefore r=\sqrt[3]{\frac{k}{\pi}}$ $\frac{\mathrm{d}^2C}{\mathrm{d}r^2}=12\pi+\frac{24k}{r^3}=36\pi>0$ or $\frac{\mathrm{d}^2C}{\mathrm{d}r^2}=12\pi+\frac{24k}{r^3}>0$ since $r,k>0$ When $r=\sqrt[3]{\frac{k}{\pi}}$, $h=\frac{k}{\pi\left(\sqrt[3]{\frac{k}{\pi}}\right)^2}-\frac{2}{3}\left(\sqrt[3]{\frac{k}{\pi}}\right)$ $=\sqrt[3]{\frac{k}{\pi}}-\frac{2}{3}\left(\sqrt[3]{\frac{k}{\pi}}\right)=\frac{1}{3}\left(\sqrt[3]{\frac{k}{\pi}}\right)$ The cheapest can is manufactured when $r=\sqrt[3]{\frac{k}{\pi}}$ and $h=\frac{1}{3}\left(\sqrt[3]{\frac{k}{\pi}}\right)$.

9 (ii)	Volume of hemisphere $=\frac{2}{3}\pi r^3=\frac{128}{3}\pi$ After 80 seconds, volume of water leaked $=160>\frac{128}{3}\pi$ Hence we only consider the rate of change of the height of water in the **cylinder** after 80 seconds Let H be the height in cm of the water in the can. $V=\pi r^2 H$ $\quad=16\pi H$ $\frac{\mathrm{d}V}{\mathrm{d}t}=16\pi\frac{\mathrm{d}H}{\mathrm{d}t}$ $\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{1}{16\pi}\frac{\mathrm{d}V}{\mathrm{d}t}$ $\quad=\frac{-2}{16\pi}$ $\quad=-\frac{1}{8\pi}$ The height of the liquid is decreasing at $\frac{1}{8\pi}$ cm s^{-1} , 80 seconds after the start of the experiment.

10 (i)	$\text{RHS}=\dfrac{1}{\sqrt{1-p^2x^2}}-\sqrt{1-p^2x^2}$ $=\dfrac{1-\left(1-p^2x^2\right)}{\sqrt{1-p^2x^2}}=\dfrac{p^2x^2}{\sqrt{1-p^2x^2}}=\text{LHS (shown)}$
	Method 1 $\int_n^{\frac{1}{p}}\sqrt{1-p^2x^2}\ \mathrm{d}x=\int_n^{\frac{1}{p}}\dfrac{1}{\sqrt{1-p^2x^2}}-\dfrac{p^2x^2}{\sqrt{1-p^2x^2}}\ \mathrm{d}x$ $=\int_n^{\frac{1}{p}}\dfrac{1}{\sqrt{1-p^2x^2}}\ \mathrm{d}x+\dfrac{1}{2}\int_n^{\frac{1}{p}}x\cdot\dfrac{-2p^2x}{\sqrt{1-p^2x^2}}\ \mathrm{d}x$ $=\dfrac{1}{p}\int_n^{\frac{1}{p}}\dfrac{p}{\sqrt{1-(px)^2}}\ \mathrm{d}x$ $+\left[\dfrac{1}{2}x\cdot\dfrac{\sqrt{1-p^2x^2}}{\frac{1}{2}}\right]_n^{\frac{1}{p}}-\dfrac{1}{2}\int_n^{\frac{1}{p}}\dfrac{\sqrt{1-p^2x^2}}{\frac{1}{2}}\ \mathrm{d}x$ $=\dfrac{1}{p}\left[\sin^{-1}(px)\right]_n^{\frac{1}{p}}+\left[0-n\sqrt{1-p^2n^2}\right]-\int_n^{\frac{1}{p}}\sqrt{1-p^2x^2}\ \mathrm{d}x$ $=\dfrac{1}{p}\left[\sin^{-1}(1)-\sin^{-1}(pn)\right]-n\sqrt{1-p^2n^2}-\int_n^{\frac{1}{p}}\sqrt{1-p^2x^2}\ \mathrm{d}x$ $=\dfrac{1}{p}\left[\dfrac{\pi}{2}-\sin^{-1}(pn)\right]-n\sqrt{1-p^2n^2}-\int_n^{\frac{1}{p}}\sqrt{1-p^2x^2}\ \mathrm{d}x$ $=\dfrac{\pi}{2p}-\dfrac{1}{p}\sin^{-1}\left(pn\right)-n\sqrt{1-p^2n^2}-\int_n^{\frac{1}{p}}\sqrt{1-p^2x^2}\ \mathrm{d}x$ Hence, $2\int_n^{\frac{1}{p}}\sqrt{1-p^2x^2}\ \mathrm{d}x=\dfrac{\pi}{2p}-\dfrac{1}{p}\sin^{-1}\left(pn\right)-n\sqrt{1-p^2n^2}$ $\int_n^{\frac{1}{p}}\sqrt{1-p^2x^2}\ \mathrm{d}x=\dfrac{\pi}{4p}-\dfrac{1}{2p}\sin^{-1}\left(pn\right)-\dfrac{1}{2}n\sqrt{1-p^2n^2}$

10 (i)	**Method 2**

$$\int_n^{\frac{1}{p}} \sqrt{1-p^2x^2}\ \mathrm{d}x = \int_n^{\frac{1}{p}} 1\cdot\sqrt{1-p^2x^2}\ \mathrm{d}x$$

$$= \left[x\sqrt{1-p^2x^2}\right]_n^{\frac{1}{p}} - \int_n^{\frac{1}{p}} x\cdot\frac{1}{2\sqrt{1-p^2x^2}}\cdot\left(-2p^2x\right)\mathrm{d}x$$

$$= \left(0-n\sqrt{1-p^2n^2}\right) + \int_n^{\frac{1}{p}} \frac{p^2x^2}{\sqrt{1-p^2x^2}}\ \mathrm{d}x$$

$$= -n\sqrt{1-p^2n^2} + \int_n^{\frac{1}{p}} \frac{1}{\sqrt{1-p^2x^2}} - \sqrt{1-p^2x^2}\ \mathrm{d}x$$

$$= -n\sqrt{1-p^2n^2} + \int_n^{\frac{1}{p}} \frac{1}{\sqrt{1-p^2x^2}}\ \mathrm{d}x - \int_n^{\frac{1}{p}} \sqrt{1-p^2x^2}\ \mathrm{d}x$$

Hence,

$$2\int_n^{\frac{1}{p}} \sqrt{1-p^2x^2}\ \mathrm{d}x = -n\sqrt{1-p^2n^2} + \int_n^{\frac{1}{p}} \frac{1}{\sqrt{1-p^2x^2}}\ \mathrm{d}x$$

$$\Rightarrow \int_n^{\frac{1}{p}} \sqrt{1-p^2x^2}\ \mathrm{d}x$$

$$= \frac{1}{2}\left[-n\sqrt{1-p^2n^2} + \frac{1}{p}\int_n^{\frac{1}{p}} \frac{p}{\sqrt{1-(px)^2}}\ \mathrm{d}x\right]$$

$$= -\frac{1}{2}n\sqrt{1-p^2n^2} + \frac{1}{2p}\left[\sin^{-1}(px)\right]_n^{\frac{1}{p}}$$

$$= \frac{1}{2p}\left[\sin^{-1}1 - \sin^{-1}(pn)\right] - \frac{1}{2}n\sqrt{1-p^2n^2}$$

$$= \frac{1}{2p}\left[\frac{\pi}{2} - \sin^{-1}(pn)\right] - \frac{1}{2}n\sqrt{1-p^2n^2}$$

$$= \frac{\pi}{4p} - \frac{1}{2p}\sin^{-1}(pn) - \frac{1}{2}n\sqrt{1-p^2n^2}\ \text{(shown)}$$

10 (ii)

$$C_1 : \frac{x^2}{3} + y^2 = 1 \Rightarrow y^2 = 1 - \frac{1}{3}x^2$$

$$C_2 : x^2 + \frac{y^2}{3} = 1 \Rightarrow y^2 = 3 - 3x^2$$

At the point of intersection,

$$1 - \frac{1}{3}x^2 = 3 - 3x^2 \Rightarrow \left(3 - \frac{1}{3}\right)x^2 = 3 - 1$$

$$\Rightarrow \frac{8}{3}x^2 = 2$$

$$\Rightarrow x^2 = \frac{3}{4} \Rightarrow x = \frac{\sqrt{3}}{2} \quad (\because x > 0)$$

Therefore, area of shaded region is given by

$$\int_{\frac{\sqrt{3}}{2}}^{\sqrt{3}} \sqrt{1 - \frac{1}{3}x^2}\ \mathrm{d}x - \int_{\frac{\sqrt{3}}{2}}^{1} \sqrt{3 - 3x^2}\ \mathrm{d}x$$

$$= \int_{\frac{\sqrt{3}}{2}}^{\sqrt{3}} \sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2 x^2}\ \mathrm{d}x - \sqrt{3}\int_{\frac{\sqrt{3}}{2}}^{1} \sqrt{1 - x^2}\ \mathrm{d}x$$

$$= \frac{\pi}{4\left(\frac{1}{\sqrt{3}}\right)} - \frac{1}{2\left(\frac{1}{\sqrt{3}}\right)} \sin^{-1}\left(\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}\right) - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2 \left(\frac{\sqrt{3}}{2}\right)^2}$$

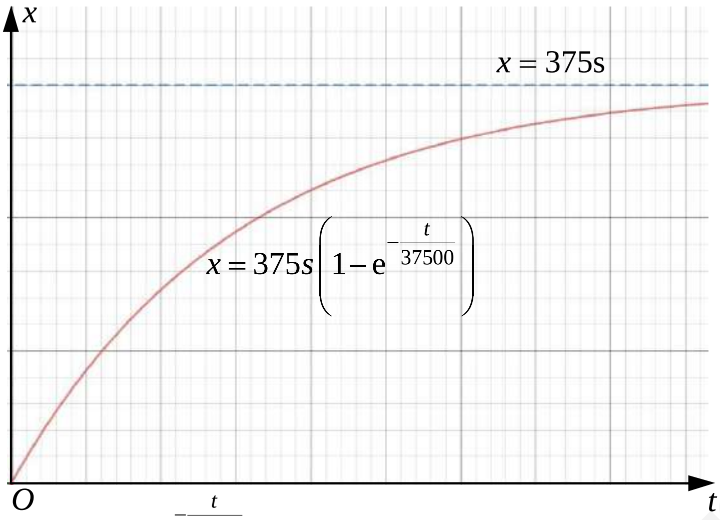
$$-\sqrt{3}\left[\frac{\pi}{4 \cdot 1} - \frac{1}{2 \cdot 1} \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= \frac{\pi\sqrt{3}}{4} - \frac{\sqrt{3}}{2} \sin^{-1}\left(\frac{1}{2}\right) - \frac{\sqrt{3}}{4}\sqrt{\frac{3}{4}}$$

$$-\sqrt{3}\left[\frac{\pi}{4} - \frac{1}{2} \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{4}\sqrt{\frac{1}{4}}\right]$$

$$= -\frac{\sqrt{3}}{2} \cdot \frac{\pi}{6} - \frac{3}{8} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{3} + \frac{3}{8} = \frac{\pi\sqrt{3}}{12}$$

11 (i)	Mass of free chlorine entering per minute $=\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{\text{in}}=10s\text{ mg}=\frac{10s}{1000}=\frac{s}{100}\text{ g}$ Mass of free chlorine leaving per minute $=\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{\text{out}}=\frac{x}{375000}(10)\text{ g}=\frac{x}{37500}\text{ g}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{\text{in}}-\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{\text{out}}$ $=\frac{10s}{1000}-\frac{10x}{375000}$ $\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=\frac{375s-x}{37500}$ (shown)
	Assume that the mixture/concentration of free chlorine and water is uniform/homogeneous/well-mixed in the pool.
11 (ii)	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{375s-x}{37500}$ $\int \frac{1}{375s-x}\,\mathrm{d}x=\int \frac{1}{37500}\,\mathrm{d}t$ $\Rightarrow -\ln\lvert 375s-x\rvert=\frac{t}{37500}+C$ $\Rightarrow -\ln(375s-x)=\frac{t}{37500}+C$ (since $375s-x>0$) $\Rightarrow \ln(375s-x)=-\frac{t}{37500}-C$ $\Rightarrow 375s-x=\mathrm{e}^{-\frac{t}{37500}}\cdot \mathrm{e}^{-C}$ $\Rightarrow x=375s-A\mathrm{e}^{-\frac{t}{37500}}$, where $A=\mathrm{e}^{-C}$ When $t=0$, $x=0$. $0=375s-A\Rightarrow A=375s$ $\therefore x=375s\left(1-\mathrm{e}^{-\frac{t}{37500}}\right)$
11 (iii)	In 60 minutes, $x=375s\left(1-\mathrm{e}^{-\frac{60}{37500}}\right)=0.599520s$ Amount of free chlorine in the pool after one hour is 0.600s or $375s\left(1-\mathrm{e}^{-\frac{1}{625}}\right)$ grams (or 600s milligrams).

11 (iv)	 As $t \to \infty$, $e^{-\frac{t}{37500}} \to 0$, $x \to 375s$.
11 (last part)	$375 < 375s < 1125$ $\Rightarrow 1 < s < 3$

H

NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2 PRELIMINARY EXAMINATION
Higher 2

MATHEMATICS 9758/02

Paper 2 **11 September 2018**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in the brackets [ ] at the end of each question or part question.

National Junior College

Section A: Pure Mathematics [40 marks]

1 **(a)** In a city, a traveller may use a mobile phone application to choose a variety of modes of transport to travel from the international airport *A*, to a specific hotel *Z*: taking a High Speed Rail, taking a Subway train or walking.

Along the way, he will need to make transfer via various interchange stations or subway stations *B*, *C*, *D*, *E*, *F* or *G* and walk within stations to travel in another mode of transport.

When the traveller uses the travelling application, he is given 3 choices to travel from the international airport to the hotel:

<table>
<tr><td></td><td colspan="4">International Airport ⟶ Hotel</td></tr>
<tr><td>Choice I</td><td rowspan="2">Take the High Speed Rail at A for 60km to Interchange Station B</td><td>Walk 0.3 km to Subway Station within B</td><td>Take the Subway train within B to C that travels 10 km.</td><td>Walks 0.5 km to Z.</td></tr>
<tr><td>Choice II</td><td>Within B, cross over to another platform at a negligible distance away to take the Subway train and travel 5 km to D.</td><td>Walks 1.2 km to Z.</td><td></td></tr>
<tr><td>Choice III</td><td>Takes the Subway train at A for 20 km to Interchange Station E.</td><td>Within E, walks 0.2 km to F.</td><td>Take the Subway train at F to G for 40 km</td><td>Walks 0.2 km to Z.</td></tr>
</table>

Given that the travelling time taken for Choices I, II and III are 51.7 minutes, 56.3 minutes and 69.6 minutes and assuming that the waiting time for the high speed rail or subway is negligible, find the average speed of the high speed rail, subway train and the walking pace of the traveller. Leave your answers in km/h. [4]

(b) **(i)** Solve the inequality $\dfrac{2x-1}{2x^2-1} \le 1$ exactly. [3]

(ii) Hence, solve the inequality $\dfrac{2(\cos x)-1}{\cos 2x} \le 1$ exactly for $0 \le x \le \pi$. [3]

2 The planes p_1 and p_2, have equations $2x+3y+6z=0$ and $\mathbf{r}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=6$ respectively.

(i) Find a vector equation of the line of intersection, *l*, between p_1 and p_2. [2]

The line *m* passes through the points $A(2,1,1)$ and $B(5,4,2)$.

(ii) Verify that *A* lies on p_2. [1]

(iii) Find the coordinates of the points on *m* that are equidistant from planes p_1 and p_2. [5]

3 **(a)** The sum, S_n, of the first *n* terms of a sequence is given by

$$S_n=\pi\left(n-\frac{1}{2}\right)^2-\frac{\pi}{4}+n\pi^2.$$

Show that the sequence follows an arithmetic progression and state the value of the common difference. [5]

(b) **(i)** A kangaroo spotted her runaway joey when she was 20 metres away from her and jumped straight towards her joey. The kangaroo's 1st jump was 2 metres and each successive jump was 1.1 times that of her previous jump. At the same time, the joey jumped away from her mother along the same imaginary straight line, such that its first jump was 1 metre and each successive jump was 0.1 metres more than its previous jump.

Assuming that the time taken for the kangaroo and her joey for each jump is the same, set up an inequality for the least number of jumps, *k*, the kangaroo had to take in order to catch up with her joey. Hence, find the value of *k*. [4]

(ii) Find the distance between the kangaroo and her joey just after the *k*th jump. [1]

4 **(a)** It is given that $z^* = \dfrac{(-1-\mathrm{i})^3}{1-\mathrm{i}\sqrt{3}}$, where z^* is the conjugate of a complex number z.

(i) Find the exact values of the modulus and argument of $\dfrac{1}{z}$. [5]

(ii) Hence determine the exact values of a and b (where $-\pi < b \leq \pi$) in the equation

$$e^{2a+ib} = \frac{1}{z^4}.$$ [3]

(b) The complex variables u and v satisfy the equations

$$\mathrm{i}u - v = 3 \quad \text{and} \quad u^* + (1-\mathrm{i})v = 7 + 4\mathrm{i}.$$

Find the values of u and v, giving your answers in the form $x + \mathrm{i}y$. [4]

Section B: Probability and Statistics [60 marks]

5 In this question you should state clearly the values of the parameters of any normal distribution you use.

The masses in grams of dragon fruits and mangoes are independent random variables with the distributions $\mathrm{N}(350, 196)$ and $\mathrm{N}(250, 98)$ respectively. Dragon fruits and mangoes are priced at \$2.40 per kilogram and \$12 per kilogram respectively.

Find the probability that the total cost of 5 randomly chosen dragon fruits and 3 randomly chosen mangoes is less than \$13.50. [4]

6 A school's concert band comprises 24 woodwind players, n brass players and 10 percussion players. $\frac{1}{3}$ of all woodwind players, $\frac{2}{5}$ of all brass players and $\frac{4}{5}$ of all percussion players are Senior High students, while the rest are Junior High students. No student in the band plays more than one type of instrument.

One student from the concert band is selected at random. Find, in terms of n, the probability that the student is neither a percussion player nor a Senior High student. [3]

Suppose instead that two students from the concert band are randomly selected. It is given that the probability that one of them is a Senior High student and the other is a Junior High student is $\frac{1}{2}$. Show that

$$n^2 + pn + q = 0$$

for some integer constants p and q to be determined, and hence find the value of n. [3]

7 Ten circular stickers for temperature taking purposes, each of them indistinguishable apart from their colours, are placed in an opaque box.

(a) It is given that four of the stickers are purple, two are blue and the remaining stickers are pink, orange, yellow and green. Suppose four stickers are given to 4 people, such that each person receives exactly one sticker. Find the number of ways this can be done if

(i) all four stickers are of different colours, [1]

(ii) there are no restrictions on the colours of the stickers. [3]

(b) The 10 stickers labelled with distinct alphabets “A” to “J” are to be packed into zip-lock bags, which may come in different sizes. Bags of the same size are considered to be indistinguishable.

(i) Suppose five zip-lock bags of different sizes are used to contain 2 stickers each. How many ways can this be done? [1]

(ii) Suppose instead that a large-sized bag is used to contain 4 stickers, two medium-sized bags are used to contain 2 stickers each and a small-sized bag is used to contain the remaining 2 stickers. How many ways can this be done? [2]

8 Mary and Jerry play a game using two six-sided dice. One of the dice is fair with ‘1’ to ‘6’ on each face. The other die is weighted such that the score, denoted by Y, has a probability distribution given as follows.

$$\mathrm{P}(Y=y)=\begin{cases} \dfrac{1}{6} & \text{for } y=1, \\ \dfrac{1}{6}(y-1) & \text{for } y=2, 3, \\ \dfrac{1}{36}(y-1) & \text{for } y=4, 5, 6. \end{cases}$$

Mary throws the dice. Jerry pays Mary \$3 if the total score from the two dice is at least “9” and Mary pays Jerry \$4 if the total score from the two dice is at most “4”. Otherwise, there is no monetary transaction between both parties. Let \$$X$ be the amount of money Mary gains after one game.

Show that $\mathrm{P}(X=3)=\dfrac{25}{108}$. Tabulate the probability distribution of X. [4]

The game is played n times, where n is large. Find the least number of games that Mary needs to play so that there is a probability of more than 0.99 for her average winnings in the n games to differ from her expected winnings by at most \$1. State clearly the parameters used in your calculations. [5]

9 Factory Nayma produces balls used by the World Ball Association with diameters that follow a normal distribution. A random sample of 30 balls is chosen and their diameters in centimetres, x, are summarised by

$$\sum(x-22)=-27 \text{ and } \sum(x-22)^2=321.26.$$

(i) Calculate unbiased estimates of the population mean and population variance. [2]

(ii) Test, at the 10% significance level, whether the population mean diameter of a ball produced by Factory Nayma is 22 cm. [5]

(iii) Another random sample of 30 balls is selected, with mean diameter $\bar{x}$. Use an algebraic method to calculate the set of values of $\bar{x}$ for which there is insufficient evidence to conclude that the population mean diameter of all balls is greater than 22 cm at the 10% level of significance. (Answers obtained by trial and improvement from a calculator will obtain no marks.) [3]

10 5.2% of all insurance agents from a large insurance company, Prodential, have an advanced diploma in insurance (ADI). A random sample of 30 agents from Prodential is obtained.

(i) State, in context, two assumptions for the number of insurance agents with ADI in this sample to be well-modelled by a binomial distribution. [2]

Assume now that these assumptions do in fact hold.

(ii) Find the probability that at least three of the insurance agents in this sample have an ADI each. [2]

100p% of all insurance agents from another large insurance company, Avila, have an advanced diploma in insurance (ADI), where $p < 0.5$. A sample of 10 agents from Avila is obtained. It is given that the number of insurance agents with ADI in this sample can be modelled by a binomial distribution.

(iii) It is given that the probability that 5 of the agents in this sample have an ADI each is 0.12294, correct to 5 decimal places. Show that p satisfies an equation of the form $p(1-p)=k$ for some real constant k to be determined, and hence find the value of p correct to 2 decimal places. [3]

(iv) Suppose instead that $p = 0.24$ and forty samples of 10 Avila insurance agents each are obtained. Find the probability that the average number of insurance agents with ADI of the forty samples is between 2.3 and 2.5. [3]

(v) Explain, stating a reason, how increasing the number of samples of 10 Avila insurance agents each will affect your answer in part **(iv)**. [2]

11 **(i)** Sketch a scatter diagram, that might be expected when x and y are related approximately, for each of the cases (A) and (B) below. In each case your diagram should include 6 points, approximately equally spaced with respect to x, and with all x- and y- values positive.

(A) $y = a + bx^2$, where $a, b \in \mathbb{R}^+$

(B) $y = c + \frac{d}{x}$, where $c \in \mathbb{R}^+$ and $d \in \mathbb{R}^-$. [2]

Obesity is becoming increasingly prevalent across the globe. To investigate the effects of obesity on one's health, a study was conducted to determine if the blood pressure of adults aged between 40 and 50 years old is dependent on their Body Mass Index (BMI). Data from six patients in this age-group from a hospital are collected. Their BMI, m kg/m^2, and systolic blood pressure, s, in mmHg, are as follows.

m	22	27	31	36	40	44
s	120	150	168	172	179	183

(ii) Sketch the scatter diagram for these values, labelling the axes clearly. [2]

(iii) With reference to your scatter diagram in part **(ii)**, explain why model (B) in part **(i)** is more appropriate than model (A) for modelling these values and calculate the product moment correlation coefficient for model (B). [2]

(iv) Find the equation of the least-squares regression line of s on $\frac{1}{m}$ and use it to estimate the BMI of another patient (of a similar age profile) whose systolic blood pressure is 110 mmHg. Comment on the reliability of your estimate. [3]

(v) Explain why the regression line of $\frac{1}{m}$ on s should not be used for your calculations in part **(iv)**. [1]

(vi) State, in context, a limitation of using the regression equation in part **(iv)** to estimate the systolic blood pressure of *other* people with BMI within the range $22 < m < 44$. [1]

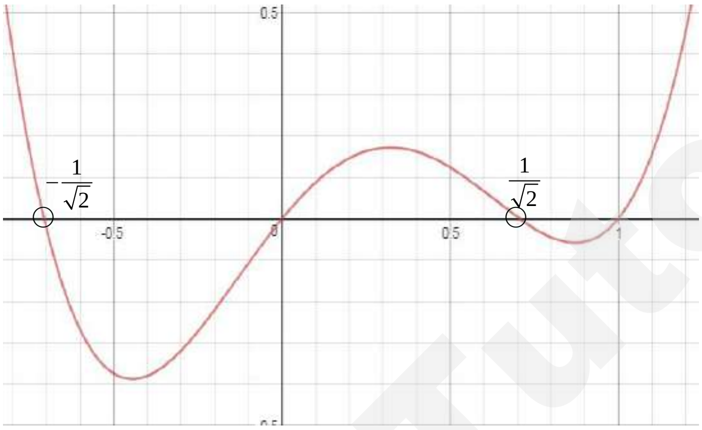
(vii) Suppose a new data pair $(\bar{m}, \bar{s})$ is added to the table above, where $\bar{m}$ and $\bar{s}$ are the mean BMI (in kg/m^2) and the mean systolic blood pressure (in mmHg) of the adults in the study respectively, based on the data above. Without any calculations, explain whether the equation of the regression line you have obtained in part **(iv)** would change. [1]

– END OF PAPER –

2018 NJC H2 Math Prelim Paper 2 Solution

Section A : Pure Mathematics [40 marks]

1 (a)	Let a, b and c be the average speed of the high speed rail, subway train and walking pace of a person. For Choice 1, $\frac{60}{a}+\frac{0.3}{c}+\frac{10}{b}+\frac{0.5}{c}=\frac{51.7}{60} \Rightarrow \frac{60}{a}+\frac{10}{b}+\frac{0.8}{c}=\frac{51.7}{60}$ …(1) For Choice 2, $\frac{60}{a}+\frac{5}{b}+\frac{1.2}{c}=\frac{56.3}{60}$ …(2) For Choice 3, $\frac{20}{b}+\frac{0.2}{c}+\frac{40}{b}+\frac{0.2}{c}=\frac{69.6}{60} \Rightarrow \frac{60}{b}+\frac{0.4}{c}=\frac{69.6}{60}$ …(3) Solving (1), (2) and (3) simultaneously, $\frac{1}{a}=\frac{1}{160}$, $\frac{1}{b}=\frac{1}{60}$ and $\frac{1}{c}=\frac{2}{5}$. Thus, the average speed, in km/h, of the high speed rail, subway train and person are 160, 60 and 2.5 respectively.

1 (b) (i)	$\dfrac{2x-1}{2x^2-1} \le 1 \quad , \quad x \ne \pm\dfrac{1}{\sqrt{2}}$ $\dfrac{2x-1-2x^2+1}{2x^2-1} \le 0$ $2\left(x-x^2\right)\left(2x^2-1\right) \le 0$ $x\left(1-x\right)\left(2x^2-1\right) \le 0$ $x\left(1-x\right)\left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right) \le 0$ $\left(\sqrt{2}x+1\right)\left(x\right)\left(x-1\right)\left(\sqrt{2}x-1\right) \ge 0$  $x < -\dfrac{1}{\sqrt{2}}$ or $0 \le x < \dfrac{1}{\sqrt{2}}$ or $x \ge 1$. (since $x \ne \pm\dfrac{1}{\sqrt{2}}$)
1 (b) (ii)	$\dfrac{2\left(\cos x\right)-1}{\cos 2x} \le 1$ for $0 \le x \le \pi$ $\dfrac{2\left(\cos x\right)-1}{2\left(\cos x\right)^2-1} \le 1$ $\cos x < -\dfrac{1}{\sqrt{2}}$ or $0 \le \cos x < \dfrac{1}{\sqrt{2}}$ or $\cos x \ge 1$ For $\cos x \ge 1$, this is only possible when $x = 0$. For $\cos x < -\dfrac{1}{\sqrt{2}}$, we have $-1 \le \cos x < -\dfrac{1}{\sqrt{2}}$, thus $\dfrac{3\pi}{4} < x \le \pi$. For $0 \le \cos x < \dfrac{1}{\sqrt{2}}$, we have $\dfrac{\pi}{4} < x \le \dfrac{\pi}{2}$. Hence, the solution is $x = 0$ or $\dfrac{\pi}{4} < x \le \dfrac{\pi}{2}$ or $\dfrac{3\pi}{4} < x \le \pi$.

2 (i)	$p_2:\ x+2y+2z=6$ Using GC, $$\mathbf{r}=\begin{pmatrix}-18\\12\\0\end{pmatrix}+\lambda\begin{pmatrix}-6\\2\\1\end{pmatrix},\ \lambda\in\mathbb{R}$$
2 (ii)	Since $2+2(1)+2(1)=6$, or $\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=2+2+2=6,$ the point A lies on p_2.
2 (iii)	Let the point that is equidistant from both planes be C. $\begin{pmatrix}5\\4\\2\end{pmatrix}-\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\1\end{pmatrix}$ $\overrightarrow{OC}=\begin{pmatrix}2\\1\\1\end{pmatrix}+t\begin{pmatrix}3\\3\\1\end{pmatrix}$ for some $t\in\mathbb{R}$ Distance of C from p_1 = Distance of C from p_2 $$\frac{\left\|\left[\begin{pmatrix}2+3t\\1+3t\\1+t\end{pmatrix}-\begin{pmatrix}2\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}\right\|}{\sqrt{1^2+2^2+2^2}}=\frac{\left\|\left[\begin{pmatrix}2+3t\\1+3t\\1+t\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\cdot\begin{pmatrix}2\\3\\6\end{pmatrix}\right\|}{\sqrt{2^2+3^2+6^2}}$$ $$\frac{\lvert 3t+6t+2t\rvert}{3}=\frac{\lvert 4+6t+3+9t+6+6t\rvert}{7}$$ $$\frac{11\lvert t\rvert}{3}=\frac{\lvert 13+21t\rvert}{7}$$ $$77\lvert t\rvert=\lvert 39+63t\rvert$$ $$77t=-39-63t\quad\text{or}\quad 77t=39+63t$$ $$140t=-39\quad\text{or}\quad 14t=39$$ $$t=-\frac{39}{140}\quad\text{or}\quad t=\frac{39}{14}$$ $\overrightarrow{OC}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\left(-\frac{39}{140}\right)\begin{pmatrix}3\\3\\1\end{pmatrix}=\frac{1}{140}\begin{pmatrix}163\\23\\101\end{pmatrix}$ or $\overrightarrow{OC}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\left(\frac{39}{14}\right)\begin{pmatrix}3\\3\\1\end{pmatrix}=\frac{1}{14}\begin{pmatrix}145\\131\\53\end{pmatrix}$ The two points are $\left(\frac{163}{140},\frac{23}{140},\frac{101}{140}\right)$ and $\left(\frac{145}{14},\frac{131}{14},\frac{53}{14}\right)$

3 (a)	$u_n = S_n - S_{n-1}, \quad n \geq 2$ $= \pi\left(n-\frac{1}{2}\right)^2 - \frac{\pi}{4} + n\pi^2 - \pi\left((n-1)-\frac{1}{2}\right)^2 + \frac{\pi}{4} - (n-1)\pi^2$ $= \pi\left(n^2 - n + \frac{1}{4}\right) - \frac{\pi}{4} + n\pi^2 - \pi\left(n^2 - 3n + \frac{9}{4}\right) + \frac{\pi}{4} - n\pi^2 + \pi^2$ $= -n\pi + \frac{1}{4}\pi + 3n\pi - \frac{9}{4}\pi + \pi^2$ $= 2n\pi - 2\pi + \pi^2, \quad n \geq 2$ $\because u_1 = S_1 = \pi\left(1-\frac{1}{2}\right)^2 - \frac{\pi}{4} + \pi^2 = \pi^2 = 2(1)\pi - 2\pi + \pi^2$ $\therefore u_n = 2n\pi - 2\pi + \pi^2, \quad n \geq 1$ Method 1: Comparing *n*th term of AP $\because u_n = 2n\pi - 2\pi + \pi^2, \quad n \geq 1$ $= \pi^2 + 2\pi(n-1), \quad n \geq 1$ Method 2: Calculating common difference $u_n - u_{n-1} = 2n\pi - 2\pi + \pi^2 - \left(2(n-1)\pi - 2\pi + \pi^2\right)$ $= 2\pi$ Sequence is follows a arithmetic progression with common difference 2π .
3 (b) (i)	$\frac{2\left((1.1)^k - 1\right)}{1.1-1} \geq \frac{k}{2}\left(2(1) + (k-1)(0.1)\right) + 20$ $20\left((1.1)^k - 1\right) \geq \frac{k}{2}(1.9 + 0.1k) + 20$ $20(1.1)^k - 20 \geq 0.05k^2 + 0.95k + 20$ $20(1.1)^k - 0.05k^2 - 0.95k \geq 40$ By G.C., $k = 11$.
3 (b) (ii)	Required distance $= \frac{2\left((1.1)^{11} - 1\right)}{1.1-1} - \frac{11}{2}\left(2(1) + (11-1)(0.1)\right) - 20$ $= 0.56233$ $= 0.562$m (3 s.f.)

4 (a) (i)	$\lvert z^*\rvert=\left\lvert\dfrac{(-1-\mathrm{i})^3}{1-\mathrm{i}\sqrt{3}}\right\rvert=\dfrac{\lvert-1-\mathrm{i}\rvert^3}{\lvert 1-\mathrm{i}\sqrt{3}\rvert}=\dfrac{2\sqrt{2}}{2}=\sqrt{2}$ $\arg\left(z^*\right)=\arg\left(\dfrac{(-1-\mathrm{i})^3}{1-\sqrt{3}\mathrm{i}}\right)$ $=3\arg(-1-\mathrm{i})-\arg\left(1-\sqrt{3}\mathrm{i}\right)$ $=3\left(-\dfrac{3}{4}\pi\right)-\left(-\dfrac{1}{3}\pi\right)$ $=-\dfrac{23}{12}\pi\equiv\dfrac{1}{12}\pi$ $\left\lvert\dfrac{1}{z}\right\rvert=\dfrac{1}{\lvert z\rvert}=\dfrac{1}{\lvert z^*\rvert}=\dfrac{1}{\sqrt{2}}$ $\arg\left(\dfrac{1}{z}\right)=-\arg(z)=\arg\left(z^*\right)=-\dfrac{23}{12}\pi$ or $\dfrac{1}{12}\pi$
	$-1-\mathrm{i}=\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}},\ 1-\mathrm{i}\sqrt{3}=2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$ $z^*=\dfrac{\left(\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}\right)^3}{2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}}=\dfrac{2\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{9\pi}{4}}}{2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}}=\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{9\pi}{4}-\mathrm{i}\left(-\frac{\pi}{3}\right)}=\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{23\pi}{12}}$ $\dfrac{1}{z}=\dfrac{1}{\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{23\pi}{12}}}=\dfrac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\frac{23\pi}{12}}$. Therefore, $\left\lvert\dfrac{1}{z}\right\rvert=\dfrac{1}{\sqrt{2}},\ \arg\left(\dfrac{1}{z}\right)=-\dfrac{23}{12}\pi$
4 (a) (ii)	$\dfrac{1}{z^4}=\left(\dfrac{1}{z}\right)^4=\left(\dfrac{1}{\sqrt{2}}\mathrm{e}^{-\frac{23}{12}\pi\mathrm{i}}\right)^4=\dfrac{1}{4}\mathrm{e}^{-\frac{23}{3}\pi\mathrm{i}}=\dfrac{1}{4}\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$ $\mathrm{e}^{2a+\mathrm{i}b}=\dfrac{1}{z^4}\Rightarrow\mathrm{e}^{2a}\cdot\mathrm{e}^{\mathrm{i}b}=\dfrac{1}{4}\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$ Therefore we have $\mathrm{e}^{2a}=\dfrac{1}{4}\Rightarrow 2a=\ln\dfrac{1}{4}\Rightarrow a=\ln\dfrac{1}{2}$ or $-\ln 2$ $\mathrm{e}^{\mathrm{i}b}=\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\Rightarrow b=\dfrac{1}{3}\pi$

4 (b)	$\mathrm{i}u - v = 3 \Rightarrow v = \mathrm{i}u - 3$ Then substituting $w = \mathrm{i}z - 3$ into the other equation, $u^* + (1-\mathrm{i})(\mathrm{i}u - 3) = 7 + 4\mathrm{i}$ $u^* + \mathrm{i}u - 3 - \mathrm{i}^2 u + 3\mathrm{i} = 7 + 4\mathrm{i}$ $u^* + \mathrm{i}u + u = 10 + \mathrm{i}$ $2a + \mathrm{i}(a + \mathrm{i}b) = 10 + \mathrm{i}$ $2a - b + \mathrm{i}a = 10 + \mathrm{i}$ Comparing the real and imaginary parts, we get $a = 1$ and $2a - b = 10 \Rightarrow 2 - b = 10 \Rightarrow b = -8$. Therefore $u = 1 - 8\mathrm{i}$ and $v = \mathrm{i}(1 - 8\mathrm{i}) - 3 = 5 + \mathrm{i}$
	Alternatively, $\mathrm{i}u - v = 3 \quad (1)$ $u^* + (1-\mathrm{i})v = 7 + 4\mathrm{i}$ Let $u = a + b\mathrm{i}$ and $v = c + d\mathrm{i}$ where $a, b, c, d \in \mathbb{R}$. Equation (1) becomes $\mathrm{i}(a + b\mathrm{i}) - (c + d\mathrm{i}) = 3$ $(-b - c) + \mathrm{i}(a - d) = 3 + 0\mathrm{i}$ Comparing the real and imaginary parts, $-b - c = 3 \qquad (3)$ $a - d = 0 \qquad (4)$ Equation (2) becomes $(a - b\mathrm{i}) + (1 - \mathrm{i})(c + d\mathrm{i}) = 7 + 4\mathrm{i}$ $(a + c + d) + \mathrm{i}(-b + d - c) = 7 + 4\mathrm{i}$ Comparing the real and imaginary parts, $a + c + d = 7 \qquad (5)$ $-b - c + d = 4 \qquad (6)$ Using GC to solve (3), (4), (5) and (6) simultaneously, $a = 1,\ b = -8,\ c = 5$ and $d = 1$ Therefore, $u = 1 - 8\mathrm{i}$ and $v = 5 + \mathrm{i}$.

Section B: Probability and Statistics [60 marks]

5	Let X and Y denote the mass in grams of a randomly chosen dragonfruit and mango respectively, and C be the total cost of 5 randomly chosen dragon fruits and 3 randomly chosen mangoes. Then $$C = \frac{2.4}{1000}\sum_{i=1}^{5} X_i + \frac{12}{1000}\sum_{j=1}^{3} Y_j$$ Therefore, $\mathrm{E}(C) = \frac{2.4}{1000}\times 5\times 350 + \frac{12}{1000}\times 3\times 250 = 13.2$ and $\mathrm{Var}(C) = \left(\frac{2.4}{1000}\right)^2 \times 5\times 196 + \left(\frac{12}{1000}\right)^2 \times 3\times 98 = 0.0479808$ $C \sim \mathrm{N}(13.2,\ 0.04798)$ $\mathrm{P}\left(C < 13.5\right) = 0.915$ (to 3 s.f.)

<table>
<tr><td>6</td><td>

	Senior High	Junior High	**Total**
Woodwind	8	16	**24**
Brass	$\frac{2}{5}n$	$\frac{3}{5}n$	$\boldsymbol{n}$
Percussion	8	2	**10**
Total	$\frac{2}{5}n+16$	$\frac{3}{5}n+18$	$\boldsymbol{n+34}$

</td></tr>
<tr><td>6 (1st part)</td><td>

P(neither SH student nor percussion player)

= P(either JH woodwind player or JH brass player)

$=\dfrac{\text{No. of JH woodwind players}+\text{No. of JH brass players}}{\text{Total no. of concert band students}}$

$=\dfrac{16+\frac{3}{5}n}{24+n+10}$

$=\dfrac{\frac{3}{5}n+16}{n+34}$

$=\dfrac{3n+80}{5n+170}$ or $\dfrac{3n+80}{5(n+34)}$ or $1-\dfrac{2n+90}{5n+170}$

</td></tr>
<tr><td></td><td>

Alternatively,

P(neither SH student nor percussion player)

= P(either JH woodwind player or JH brass player)

$=\dfrac{24}{24+n+10}\times\dfrac{2}{3}+\dfrac{n}{24+n+10}\times\dfrac{3}{5}$

$=\dfrac{16}{n+34}+\dfrac{3n}{5(n+34)}$

$=\dfrac{3n+80}{5n+170}$

</td></tr>
</table>

6 (2nd part)	P(one is a SH playe and the other is a JH player) $= \frac{1}{2}$ $\frac{\frac{2}{5}n+16}{n+34}\times\frac{\frac{3}{5}n+18}{n+33}\times 2! = \frac{1}{2}$ $\frac{2n+80}{5n+170}\times\frac{3n+90}{5n+165} = \frac{1}{4}$ $4\cdot 2(n+40)\cdot 3(n+30) = (5n+170)(5n+165)$ $24(n^2+70n+1200) = 25n^2+1675n+28050$ $n^2-5n-750=0$ $n=-25$ (rej) or 30
	Alternatively, $\frac{\binom{0.4n+16}{1}\binom{0.6n+18}{1}}{\binom{n+34}{2}} = \frac{1}{2}$ $\frac{(0.4n+16)(0.6n+18)}{\frac{(n+34)(n+33)}{2!}} = \frac{1}{2}$

7 (a) (i)	Number of ways if all 4 stickers are of different colours $=\binom{6}{4}4!=360$
7 (a) (ii)	Case 1 : All 4 different colours are used No. of ways $=360$ Case 2 : 2 are of the same colour, the remaining 2 are different colours No. of ways $=\binom{2}{1}\binom{5}{2}\frac{4!}{2!}=240$ Case 3 : 3 of them have the same colour and the last one has a different colour No. of ways $=\binom{1}{1}\binom{5}{1}\frac{4!}{3!}=20$ Case 4 : 2 are of the same colour, the other 2 have the same colour (only two purple and 2 blue) No. of ways $=\binom{2}{2}\frac{4!}{2!2!}=6$ Case 5 : All 4 stickers are of the same colour. No. of ways $=1$ Total number of ways $=360+240+20+6+1$ $=627$
7 (b) (i)	Number of ways $=\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}=113400$
7 (b) (ii)	No. of ways $=\dfrac{\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{4}}{2!}=9450$

8 (1st part)

Presenting all the possible outcomes on the total score of the dice :

Fair Die $\left(\frac{1}{6}\right)$

		1	**2**	**3**	**4**	**5**	**6**
Biased Die	(1/6) **1**	2	3	4	5	6	7
	(1/6) **2**	3	4	5	6	7	8
	(2/6) **3**	4	5	6	7	8	9
	(3/36) **4**	5	6	7	8	9	10
	(4/36) **5**	6	7	8	9	10	11
	(5/36) **6**	7	8	9	10	11	12

$$\mathrm{P}(X=3)=\left(\frac{2}{6}\times\frac{1}{6}\times 1\right)+\left(\frac{3}{36}\times\frac{1}{6}\times 2\right)+\left(\frac{4}{36}\times\frac{1}{6}\times 3\right)+\left(\frac{5}{36}\times\frac{1}{6}\times 4\right)$$

$$=\frac{25}{108}$$

$$\mathrm{P}(X=-4)=\left(\frac{1}{6}\times\frac{1}{6}\times 5\right)+\left(\frac{2}{6}\times\frac{1}{6}\times 1\right)$$

$$=\frac{7}{36}$$

$$\mathrm{P}(X=0)=1-\frac{25}{108}-\frac{7}{36}=\frac{31}{54}$$

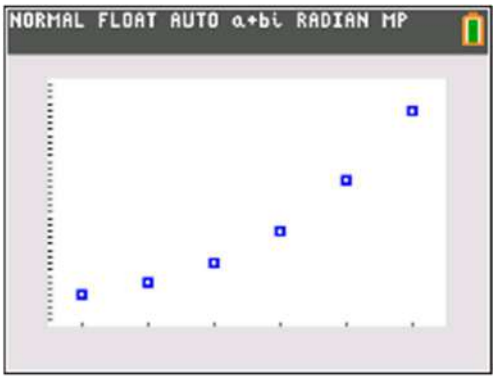
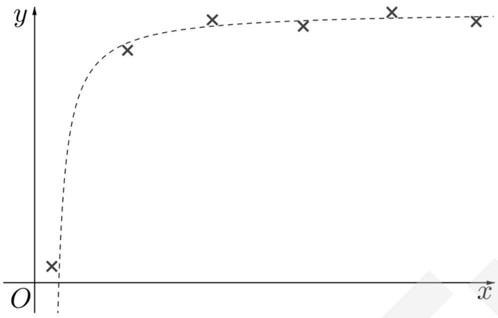
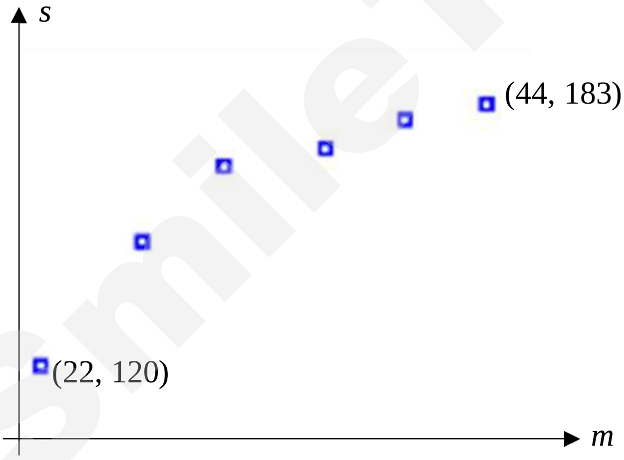
Distribution of X are as follows :

x	3	0	–4
$\mathrm{P}(X=x)$	$\frac{25}{108}$	$\frac{31}{54}$	$\frac{7}{36}$

8 (2nd part)	$\mathrm{E}(X)=3\left(\frac{25}{108}\right)+0\left(\frac{31}{54}\right)-4\left(\frac{7}{36}\right)=-\frac{1}{12}$ $\mathrm{E}(X^2)=3^2\left(\frac{25}{108}\right)+0^2\left(\frac{31}{54}\right)+(-4)^2\left(\frac{7}{36}\right)=\frac{187}{36}$ $\mathrm{Var}(X)=\frac{187}{36}-\left(\frac{1}{12}\right)^2=\frac{83}{16}$ Since ***n* is large, by Central Limit Theorem**, $\bar{X}\sim \mathrm{N}\left(-\frac{1}{12},\frac{83}{16n}\right)$ **approximately**. $\mathrm{P}\left(\left\vert\bar{X}-\left(-\frac{1}{12}\right)\right\vert\le 1\right)>0.99$ Standardising, $\mathrm{P}\left(\vert Z\vert\le\frac{1}{\sqrt{83/16n}}\right)>0.99$ $\mathrm{P}\left(\vert Z\vert\le\frac{4\sqrt{n}}{\sqrt{83}}\right)>0.99$ $\mathrm{P}\left(Z\le-\frac{4\sqrt{n}}{\sqrt{83}}\right)<0.005$ $-\frac{4\sqrt{n}}{\sqrt{83}}<-2.575829$ $n>34.41$ Smallest possible number of games required is 35.

9 (i)	$\bar{x}=\dfrac{-27}{30}+22=21.1$ $s^2=\dfrac{1}{29}\left(321.26-\dfrac{(-27)^2}{30}\right)$ $=10.24$
9 (ii)	Test $\mathrm{H}_0:\mu=22$ against $\mathrm{H}_1:\mu\neq 22$ Level of significance = 10% (two-tailed) Under H_0, $\bar{X}\sim \mathrm{N}\left(22,\dfrac{10.24}{30}\right)$ approximately. Hence $Z=\dfrac{\bar{X}-22}{\sqrt{\dfrac{10.24}{30}}}\sim \mathrm{N}(0,1)$ approximately. Critical region: $\lvert z\rvert>1.64485$ (6 s.f.) Observed test statistic, $z=\dfrac{21.1-22}{\sqrt{\dfrac{10.24}{30}}}=-1.54046\text{ (6 s.f.)}>-1.64485$ (hence do not reject H_0) OR p-value = 0.123446 (6 s.f.) >0.1 (hence do not reject H_0) ***Step 5:*** We conclude that there is <u>insufficient evidence</u> at the <u>10% significance level</u> to claim that the <u>mean diameter of the balls is not 22 cm</u>.
9 (iii)	To not reject H_0, the observed test statistic, $z\leq 1.28155$ (6 s.f.) $\therefore \dfrac{\bar{x}-22}{\sqrt{\dfrac{10.24}{30}}}\leq 1.28155$ $0<\bar{x}\leq 22.7$ (3 s.f.)

10 (i)	The event that each agent has an ADI occurs independently for all agents in the sample. The probability that each agent has an ADI is constant.
10 (ii)	Let Y denote the number of insurance agents with ADI out of 30 randomly chosen Prodential insurance agents. Then $Y \sim \mathrm{B}(30, 0.052)$. $\mathrm{P}(Y \geq 3) = 1 - \mathrm{P}(Y \leq 2)$ $= 0.2032366271$ $= 0.203$ (3sf)
10 (iii)	Let W denote the number of insurance agents with ADI out of 10 randomly chosen Avila insurance agents. Then $W \sim \mathrm{B}(10, p)$. $\mathrm{P}(W = 5) = 0.12294$ $\binom{10}{5} p^5 (1-p)^5 = 0.12294$ $p(1-p) = \left(\frac{0.12294}{252}\right)^{\frac{1}{5}} = 0.21760$, i.e., $k = 0.21760$ $p^2 - p + 0.21760 = 0$ $p = 0.68$ or 0.32 Since $p < 0.5$, $p = 0.32$
10 (iv)	$\mathrm{E}(W) = 10 \times 0.24 = 2.4$ $\mathrm{Var}(W) = 10 \times 0.24 \times (1 - 0.24) = 1.824$ Since sample size, 40, is large, by Central Limit Theorem, $\overline{W} \sim \mathrm{N}\left(2.4, \frac{1.824}{40}\right)$ approximately. Therefore, $\mathrm{P}\left(2.3 < \overline{W} < 2.5\right) = 0.36042 = 0.360$ (to 3 s.f.).
10 (v)	With a larger sample size, the variance of $\overline{W}$ will *decrease*. This means that the distribution of $\overline{W}$ will have a higher concentration about its mean, 2.4, and therefore the value of $\mathrm{P}\left(2.3 < \overline{W} < 2.5\right)$ will increase.

11 (i)	$y=a+bx^2$  $y=c+\dfrac{d}{x}$ 
11 (ii)	
11 (iii)	The equation $y=c+\dfrac{d}{x}$, is more appropriate because from the scatter diagram, as m increases, <u>s increases at a decreasing rate.</u> $r=-0.982$

11 (iv)	Regression line of s on $\frac{1}{m}$: $s = 248.61 - 2731.4\left(\frac{1}{m}\right)$ $\approx 249 - \frac{2730}{m}$ Since systolic blood pressure depends on weight, we should use the regression line of s on $\frac{1}{m}$: $110 = 248.61 - 2731.4\left(\frac{1}{m}\right)$ $m \approx 19.706$ Since $s = 110$ lies outside the range of values of s, **extrapolation** is required which gives an **unreliable** estimate.
11 (v)	Since s depends on m, m is the controlled variable. Therefore, the regression line of $\frac{1}{m}$ on s should **not** be used.
11 (vi)	• Too few patients are selected for the equation of the regression line to be reliable in estimation. • The data is only valid for estimating the blood pressure for people of a similar age profile. • The data is only valid for estimating the blood pressure of patients with similar medical conditions. • A person's blood pressure is not fixed and is influenced by other factors at time of measurement, such as physical activity and/or varying emotional states like anxiety.
11 (vii)	Since the data point $(\bar{m}, \bar{s})$ does not lie on the regression line, the answer in part **(iii)** will change as the equation of the regression line is affected.

Pioneer Junior College 2018 Preliminary Examinations: H2 Mathematics 9758 Paper 1

1 It is given that $\mathrm{f}(x) = 5x^3 + 7x^2 - kx + 2$, where k is a constant.

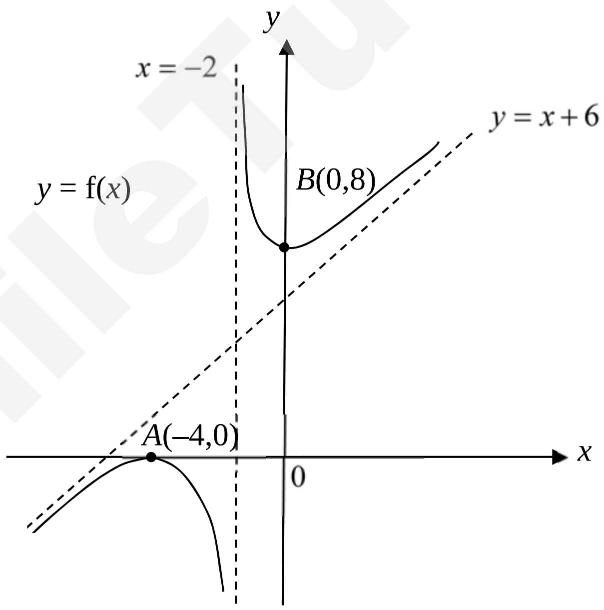
(i) If the graph of $y = \mathrm{f}(x)$ is strictly increasing, find the range of values of k. [3]

(ii) If $k = 0$, state the range of values of x where the graph of $y = \mathrm{f}(x)$ is concave downward. [1]

2 **(a)** Find $\int \frac{1}{1-4x^2}\,\mathrm{d}x$. [2]

(b) By writing $\sin^3 x$ as $\sin x(1-\cos^2 x)$, find $\int \sin^3 x\,\mathrm{d}x$. [2]

3. The diagram below shows the graph of $y = \mathrm{f}(x)$. The curve has a maximum point at A and a minimum point at B. The lines $x = -2$ and $y = x + 6$ are asymptotes of the graph.



Sketch, on separate diagrams, the graphs of

(i) $y = \frac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \mathrm{f}'(x)$, [3]

showing clearly in each case, where appropriate, the asymptotes and co-ordinates of the points corresponding to A and B.

4 **(i)** Obtain the expansion of $\sqrt{1-x+x^2}$ up to and including the term in x^2. [3]

(ii) Given that $f(x) = \ln(a + bx)$, where a is a positive constant and b is a non-zero constant, find the first three terms in the Maclaurin series for $f(x)$. [3]

(iii) The first two terms of the series in (i) are equal to the first two terms in the series expansion of $f(x)$, find a and b, leaving your answers in the exact form. [2]

5 **(i)** Sketch the graphs of $ay = x - b$ and $y = \frac{x-b}{x}$ on the same diagram, where a and b are constants such that $0 < a < 1 < b$. State clearly the exact coordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

(ii) Hence, solve the inequality $\frac{x-b}{x} > \frac{x-b}{a}$. [3]

(iii) Deduce the solution to the inequality $\frac{x}{x+b} > \frac{x}{a}$. [2]

6 A curve C has equation $3y^2 - 2xy + 3x^2 - 96 = 0$.

(i) Find the exact x- coordinates of the stationary points of C. [4]

(ii) For the stationary point with $x < 0$, determine whether it is a maximum or minimum. [3]

(iii) Find the equation(s) of the tangent(s) to C which are parallel to the y-axis. [3]

7 A particle moves on the Cartesian plane in such a way that at time t, its position is given by the parametric equations $x = t\cos t$, $y = t\sin t$. The particle starts moving from the origin.

(i) Find the equation of the normal at which the particle first crosses the y-axis, leaving your answer in the exact form. [5]

(ii) It is given that the instantaneous speed of the particle at time t is defined as $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Find the exact instantaneous speed of the particle when it crosses the x-axis for the second time. [4]

(iii) Explain if the particle will ever come to rest. [1]

8 **(a)** The shape of the mould used to make Ah Gong lava cake is formed by rotating the region bounded by the line $y = 2$, the x-axis and the curves $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, where $a > 2$ and $1 < b < 2$, through π radians about the y-axis. Find the volume of the mould, giving your answer in the form $\pi\left(pa^2 + q + rb^3\right)$, where p, q, r are constants to be determined. [3]

(b) A curve has parametric equations

$$x = 2\sin\theta, \quad y = 3\sin\left(2\theta + \frac{\pi}{2}\right), \quad \text{for } 0 \le \theta \le \frac{\pi}{2}.$$

(i) Sketch the curve. [2]

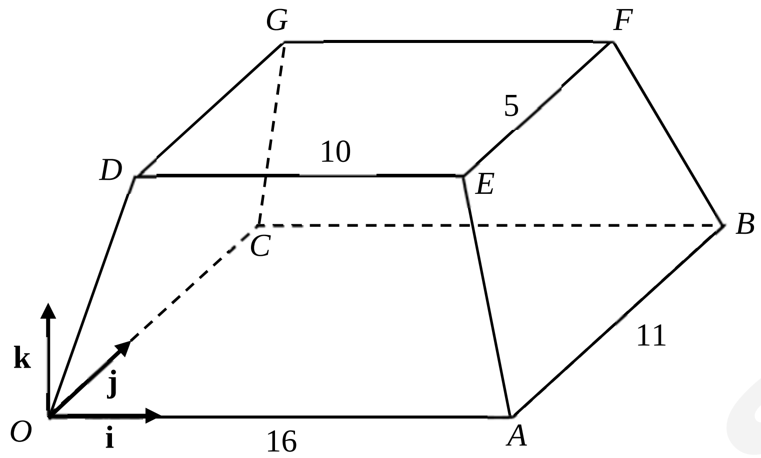
(ii) Find the exact area of the region bounded by the curve and the axes. [6]

9 **(a)** The product of the first three terms of a convergent geometric progression is 1000. If 6 is added to the second term and 7 is added to the third term, the three terms are now consecutive terms of an arithmetic progression. Find the first term and common ratio of the geometric progression. [4]

(b) **(i)** A man takes a loan of $\$P$ for a house from a bank at the beginning of a month. The interest rate is 0.5 % per month so that at the start of every month, the amount of outstanding loan is increased by 0.5%. Equal instalment is paid to the bank at the end of every month. Find his monthly instalment if he would like to repay the loan in 20 years, leaving your answer in the form $0.005P\dfrac{k^n}{k^n - 1}$, where k and n are constants to be determined. [5]

(ii) If the man takes a loan of \$1 000 000 for the house from another bank which charges interest at the start of every month, he will have to pay a monthly instalment of \$10 000 at the end of every month over 20 years to repay the loan. Find the monthly interest rate charged by this bank. [3]

10



The diagram shows a container with a horizontal rectangular base *OABC*, where $OA = 16$ cm and $AB = 11$ cm. The top of the container *DEFG* is also a horizontal rectangle, where $DE = 10$ cm and $EF = 5$ cm. The 4 sloping faces (e.g. *OAED*) of the container, each a trapezium, are inclined at the same angle to the horizontal such that the distance between the base and the top is 6 cm. The point *O* is taken as origin and perpendicular unit vectors **i**, **j**, **k** are such that **i** and **j** are parallel to *OA* and *OC* respectively.

(i) Show that a cartesian equation of plane *BCD* is $3y + 4z = 33$. [3]

(ii) Find the cartesian equations of the planes such that the perpendicular distance from each plane to plane *BCD* is 10 cm. [4]

(iii) Find the coordinates of the point on plane *BCD* which is closest to point *G*. [3]

(iv) Hence or otherwise, determine the acute angle between *CG* and plane *BCD*. [3]

11 Environmental conditions such as acidity, temperature, oxygen levels and toxins influence the rate of growth of microorganisms. A biologist investigates the change of population of a particular type of microorganism of size *n* thousand at time *t* days under different conditions. In both models I and II, the initial population of the microorganism is 3000 and the population reaches 2000 after 1 day.

(i) Under model I, the biologist observes that the rate of growth of microorganism is a constant whereas the death rate is proportional to its population. He also observes that when the population of the microorganism is 1000, it remains at this constant value. By setting up and solving a differential equation, show that $n = 1 + 2^{1-t}$. [8]

(ii) Under model II, the biologist observes that *n* and *t* are related by the differential equation $\dfrac{\mathrm{d}^2 n}{\mathrm{d}t^2} = 4 - 6t$. Find the particular solution of this differential equation.

[3]

(iii) By sketching the graphs of *n* against *t* for both model I and II, state and explain which of the two sets of conditions is more harmful for the growth of this type of microorganism. [2]

Pioneer Junior College 2018 Preliminary Examinations: H2 Mathematics 9758 Paper 1 <u>Solutions</u>

1(i) $f(x) = 5x^3 + 7x^2 - kx + 2$

$f'(x) = 15x^2 + 14x - k$

For f(x) to be strictly increasing, $f'(x) > 0$ i.e. $15x^2 + 14x - k > 0$ for all x

Since coefficient of x^2 is positive, we need discriminant < 0:

$$(14)^2 - 4(15)(-k) < 0$$

$$k < -\frac{49}{15} \quad \text{or } (-3.27)$$

1(ii) $k = 0$: $f'(x) = 15x^2 + 14x$

For the graph is concave downward, $f''(x) < 0$

Hence $30x + 14 < 0$

$\therefore$ Graph is concave downward when $x < -\dfrac{7}{15}$ or $x < -0.467$

2(a)
$$\int \frac{1}{1-4x^2}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{1}{2}\right)^2 - x^2}\,dx$$

$$= \frac{1}{4}\left(\frac{1}{2\left(\frac{1}{2}\right)}\right)\ln\left|\frac{\frac{1}{2}+x}{\frac{1}{2}-x}\right| + C$$

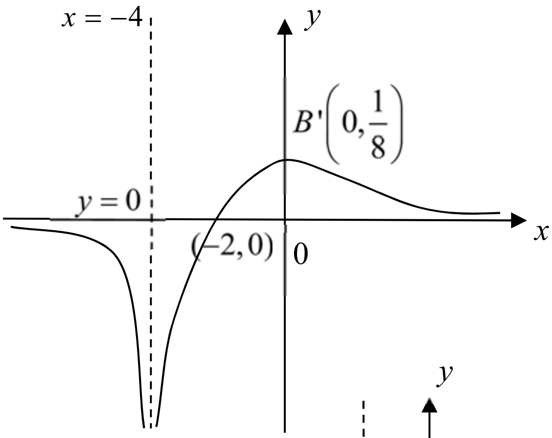
$$= \frac{1}{4}\ln\left|\frac{1+2x}{1-2x}\right| + C$$

2(b)
$$\int \sin^3 x\,dx = \int \sin x(1-\cos^2 x)\,dx$$

$$= \int \sin x - \sin x\cos^2 x\,dx$$

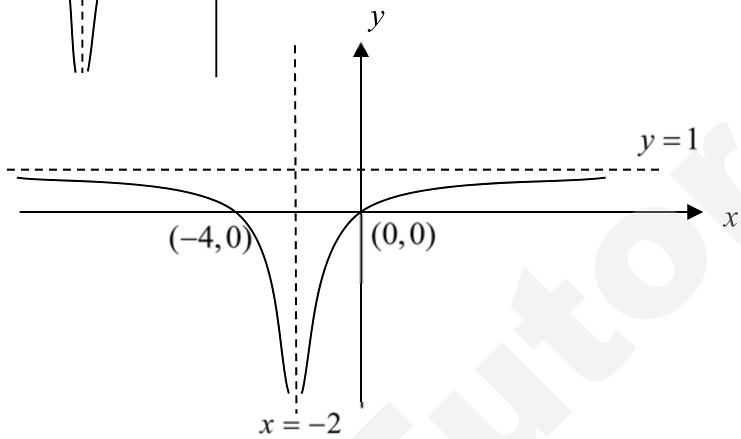
$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

3(i)



$y = \dfrac{1}{\mathrm{f}(x)}$

3(ii)



$y = \mathrm{f}'(x)$

4(i) $\sqrt{1 - x + x^2} = \left[1 - (x - x^2)\right]^{\frac{1}{2}}$

$$= \left(1 - \frac{1}{2}(x - x^2) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x - x^2)^2 + ...\right)$$

$$= 1 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{8}x^2 + ...$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 + ...$$

(ii)

$$\ln(a + bx) = \ln\left[a\left(1 + \frac{b}{a}x\right)\right]$$

$$= \ln a + \ln\left(1 + \frac{b}{a}x\right)$$

$$= \ln a + \left(\frac{b}{a}x - \frac{\left(\frac{b}{a}x\right)^2}{2}\right) + ...$$

$$= \ln a + \frac{b}{a}x - \frac{b^2}{2a^2}x^2 + ...$$

(iii)Comparing first two terms ,

$\ln a = 1 \Rightarrow a = \mathrm{e}$

$\frac{b}{a} = -\frac{1}{2} \Rightarrow b = -\frac{1}{2}\mathrm{e}$

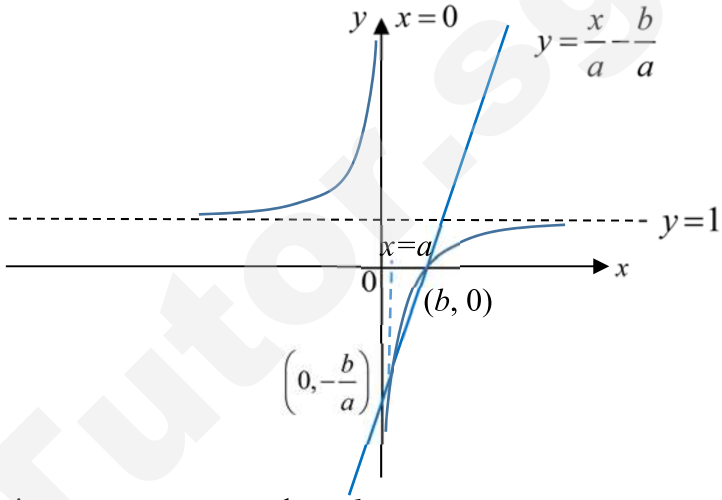
5 (i) $y = \frac{x-b}{x} = 1 - \frac{b}{x}$

When $y = 0, \; x = b$

$ay = x - b \Rightarrow y = \frac{x}{a} - \frac{b}{a}$

When $y = 0, \; x = b$

When $x = 0, \; y = -\frac{b}{a}$



(ii) By observation, the 2 graphs intersect at $x = a$ and $x = b$.

Hence, for $\frac{x-b}{x} > \frac{x-b}{a}$, $a < x < b$ or $x < 0$.

(iii) $\frac{x}{x+b} > \frac{x}{a} \Rightarrow \frac{(x+b)-b}{x+b} > \frac{(x+b)-b}{a}$

Replace x by $x+b$:

$a < x+b < b$ or $x+b < 0$

$a-b < x < 0$ or $x < -b$

6.(i) $3y^2 - 2xy + 3x^2 - 48 = 0$

Differentiate wrt x , $6y\frac{\mathrm{d}y}{\mathrm{d}x} - 2\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + 6x = 0$

$$6y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y + 6x = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y-6x}{6y-2x} = \frac{y-3x}{3y-x}$$

At stationary point , $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$,

$y = 3x$

Substitute into equation of C, $3(3x)^2-2x(3x)+3x^2-96=0$

$$24x^2=96$$
$$x^2=4$$
$$x=\pm 2$$

(ii) $6y\frac{dy}{dx}-2x\frac{dy}{dx}-2y+6x=0$

Differentiate wrt x, $(6y-2x)\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(6\frac{dy}{dx}-2\right)-2\frac{dy}{dx}+6=0$

When $x=-2$, $y=3x=-6$, $\frac{dy}{dx}=0$

Substitute : $\left(6(-6)+2(2)\right)\frac{d^2y}{dx^2}+6=0$

$$\frac{d^2y}{dx^2}=\frac{-6}{-32}=\frac{3}{16}>0$$

Hence the point is a minimum point.

(iii)Tangent // y-axis $\Rightarrow\frac{dy}{dx}$ is undefined: $3y-x=0$,

$$y=\frac{x}{3}$$

Substitute into equation of C, $3\left(\frac{x}{3}\right)^2-2x\left(\frac{x}{3}\right)+3x^2-96=0$

$$\frac{8}{3}x^2=96$$
$$x^2=36$$
$$x=\pm 6$$

7(i) $\frac{dx}{dt}=-t\sin t+\cos t$, $\frac{dy}{dt}=t\cos t+\sin t$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{t\cos t+\sin t}{-t\sin t+\cos t}$$

When the particle first crosses the y-axis , $x=0$

$$t\cos t=0$$

$t\neq 0$ (starting point) $\cos t=0$

$$t=\frac{\pi}{2}$$

$$y=\frac{\pi}{2}\sin\frac{\pi}{2}=\frac{\pi}{2}$$

Gradient of normal = $-\dfrac{-t\sin t+\cos t}{t\cos t+\sin t}=\dfrac{\pi}{2}$

equation of normal : $y-\dfrac{\pi}{2}=\dfrac{\pi}{2}(x-0)$

$$y=\frac{\pi}{2}x+\frac{\pi}{2}$$

(ii) When $y=t\sin t=0$,

$t\neq 0$ (starting point), $\sin t=0$,

$t=2\pi$ (crosses the x-axis for the second time)

$t\neq\pi$ (crosses the x-axis for the first time)

Speed = $\sqrt{\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2}$

$= \sqrt{\left(-t\sin t+\cos t\right)^2+\left(t\cos t+\sin t\right)^2}$

$= \sqrt{t^2\sin^2 t-2t\sin t\cos t+\cos^2 t+t^2\cos^2 t+2t\sin t\cos t+\sin^2 t}$

$= \sqrt{t^2\left(\sin^2 t+\cos^2 t\right)+\cos^2 t+\sin^2 t}$

$= \sqrt{t^2+1}$

$= \sqrt{4\pi^2+1}$

(iii) Since speed at time t is $\sqrt{t^2+1}$, $t^2\geq 0 \;\Rightarrow\; \sqrt{t^2+1}\geq 1$, so speed is never 0 and will never come to rest.

8(a) Required volume

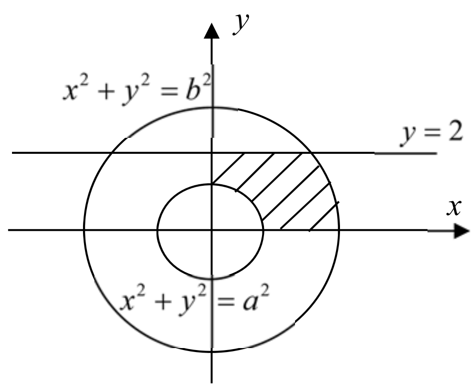
$=\pi\int_0^2 a^2-y^2\ \mathrm{d}y-\pi\int_0^b b^2-y^2\ \mathrm{d}y$

$=\pi\left(\left[a^2y-\dfrac{1}{3}y^3\right]_0^2-\left[b^2y-\dfrac{1}{3}y^3\right]_0^b\right)$

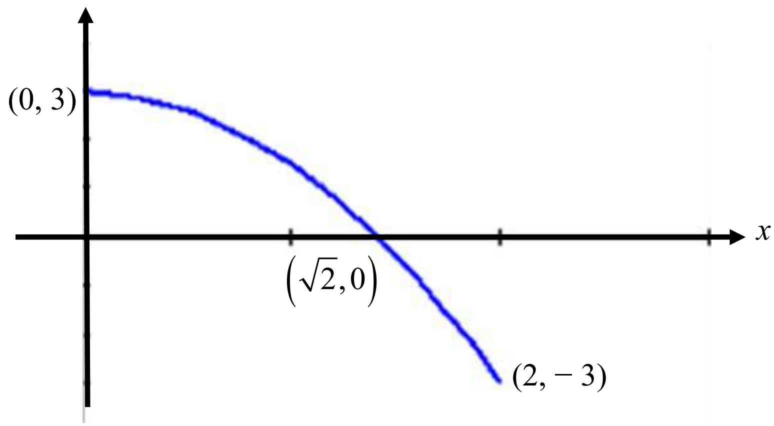
$=\pi\left(\left(2a^2-\dfrac{8}{3}\right)-\left(b^3-\dfrac{1}{3}b^3\right)\right)$

$=\pi\left(2a^2-\dfrac{8}{3}-\dfrac{2}{3}b^3\right)$

Hence, $p=2, q=-\dfrac{8}{3}, r=-\dfrac{2}{3}$



8(b)(i)



(b)(ii) Area bounded by curve and axes

$$=\int_0^3 x\ \mathrm{d}y$$

$$=\int_{\frac{\pi}{4}}^{0} 2\sin\theta\left(6\cos\left(2\theta+\frac{\pi}{2}\right)\mathrm{d}\theta\right)$$

$$=6\int_{\frac{\pi}{4}}^{0} 2\cos\left(2\theta+\frac{\pi}{2}\right)\sin\theta\mathrm{d}\theta$$

$$=6\int_{\frac{\pi}{4}}^{0} \sin\left(3\theta+\frac{\pi}{2}\right)-\sin\left(\theta+\frac{\pi}{2}\right)\mathrm{d}\theta$$

$$=6\left[-\frac{1}{3}\cos\left(3\theta+\frac{\pi}{2}\right)+\cos\left(\theta+\frac{\pi}{2}\right)\right]_{\frac{\pi}{4}}^{0}$$

$$=6\left[\left(-\frac{1}{3}\cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{2}\right)\right)-\left(-\frac{1}{3}\cos\left(\frac{5\pi}{4}\right)+\cos\left(\frac{3\pi}{4}\right)\right)\right]$$

$$=6\left[-\left(\frac{1}{3\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\right]$$

$$=2\sqrt{2}$$

Alternative Solution: Area bounded by curve and axes

$$=\int_0^{\sqrt{2}} y\ \mathrm{d}x$$

$$=\int_0^{\frac{\pi}{4}} 3\sin\left(2\theta+\frac{\pi}{2}\right)2\cos\theta\mathrm{d}\theta$$

$$=3\int_0^{\frac{\pi}{4}} 2\sin\left(2\theta+\frac{\pi}{2}\right)\cos\theta\mathrm{d}\theta$$

$$=3\int_0^{\frac{\pi}{4}} \sin\left(3\theta+\frac{\pi}{2}\right)+\sin\left(\theta+\frac{\pi}{2}\right)\mathrm{d}\theta$$

$$=3\left[-\frac{1}{3}\cos\left(3\theta+\frac{\pi}{2}\right)-\cos\left(\theta+\frac{\pi}{2}\right)\right]_0^{\frac{\pi}{4}}$$

$$= 3\left[\left(-\frac{1}{3}\cos\left(\frac{5\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right)\right) - \left(-\frac{1}{3}\cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right)\right)\right]$$
$$= 3\left[\frac{1}{3\sqrt{2}} + \frac{1}{\sqrt{2}}\right]$$
$$= 2\sqrt{2}$$

9(a) GP: a, ar, ar^2
AP: ….. , a, ar+6, ar^2 +7, …..

$$a(ar)(ar^2) = 1000$$
$$(ar)^3 = 1000 \Rightarrow ar = 10$$
$$(ar+6) - a = ar^2 + 7 - (ar+6)$$
$$ar^2 - 2ar + a - 5 = 0$$
$$10r - 20 + a - 5 = 0$$
$$10r + a = 25$$
$$10r + \frac{10}{r} = 25$$
$$10r^2 - 25r + 10 = 0$$
$$r = 2 \text{ or } \frac{1}{2}$$

Reject $r = 2$ since GP is convergent , $-1 < r < 1$
Hence $a = 20$

(b)(i)

	Amount (start of)	**Amount (end of)**
1st month	$1.005P$	$1.005P - x$
2nd month	$1.005(1.005P - x)$	$1.005(1.005P - x) - x$ $= 1.005^2P - 1.005x - x$
3rd month	$1.005(1.005^2P - 1.005x - x)$	$1.005(1.005^2P - 1.005x - x) - x$ $= 1.005^3P - 1.005^2x - 1.005x - x$

Hence, amount of money at the end of 20 years

$$1.005^{240}P - 1.005^{239}x - 1.005^{238}x - ...... - x = 0$$

$$1.005^{240}P = 1.005^{239}x + 1.005^{238}x + ... + x$$
$$1.005^{240}P = \frac{x\left[(1.005)^{240} - 1\right]}{1.005 - 1}$$
$$x = 0.005P\frac{1.005^{240}}{\left[(1.005)^{240} - 1\right]}$$

$k = 1.005$, $n = 240$

(ii) $r(1000000)\dfrac{(1+r)^{240}}{\left[(1+r)^{240}-1\right]}=10000$

Using GC , $r = 0.00877$
Hence the interest rate is 0.877 % per month

10(i)

$$\overrightarrow{OB}=\begin{pmatrix}16\\11\\0\end{pmatrix},\quad \overrightarrow{OC}=\begin{pmatrix}0\\11\\0\end{pmatrix},\quad \overrightarrow{OD}=\begin{pmatrix}3\\3\\6\end{pmatrix}\quad \overrightarrow{CD}=\begin{pmatrix}3\\-8\\6\end{pmatrix}$$

Vector parallel to $\overrightarrow{BC}$ is $\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$\begin{pmatrix}3\\-8\\6\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\6\\8\end{pmatrix}=2\begin{pmatrix}0\\3\\4\end{pmatrix}$$

$$\mathbf{r}\bullet\begin{pmatrix}0\\3\\4\end{pmatrix}=\begin{pmatrix}0\\11\\0\end{pmatrix}\bullet\begin{pmatrix}0\\3\\4\end{pmatrix}=33$$

Hence, cartesian equation is $3y+4z=33$ (shown).

(ii) **<u>Method 1:</u>** (Consider distance of plane from origin)

$$\mathbf{r}\bullet\begin{pmatrix}0\\3\\4\end{pmatrix}=33$$

$$\mathbf{r}\bullet\frac{\begin{pmatrix}0\\3\\4\end{pmatrix}}{\sqrt{3^2+4^2}}=\frac{33}{\sqrt{3^2+4^2}}$$

The 2 planes are

$$\mathbf{r}\bullet\frac{\begin{pmatrix}0\\3\\4\end{pmatrix}}{\sqrt{3^2+4^2}}=\frac{33}{\sqrt{3^2+4^2}}+10 \quad \text{and} \quad \mathbf{r}\bullet\frac{\begin{pmatrix}0\\3\\4\end{pmatrix}}{\sqrt{3^2+4^2}}=\frac{33}{\sqrt{3^2+4^2}}-10$$

i.e. $\mathbf{r}\bullet\begin{pmatrix}0\\3\\4\end{pmatrix}=83$ and $\mathbf{r}\bullet\begin{pmatrix}0\\3\\4\end{pmatrix}=-17$

i.e. $3y+4z=83$ and $3y+4z=-17$

Method 2:
(Obtain distance between 2 planes by considering length of projection of vector between a point on each plane)

Let equation of the plane be $\mathbf{r}\bullet\begin{pmatrix}0\\3\\4\end{pmatrix}=b$.

Let point H be a point on this plane with coordinates $\left(0,\frac{b}{3},0\right)$.

$$\overrightarrow{CH}=\begin{pmatrix}0\\ \frac{b}{3}-11\\ 0\end{pmatrix}$$

Distance between 2 planes = Length of projection of $\overrightarrow{CH}$ onto normal $=10$

$$\frac{\left|\begin{pmatrix}0\\ \frac{b}{3}-11\\ 0\end{pmatrix}\bullet\begin{pmatrix}0\\3\\4\end{pmatrix}\right|}{\sqrt{3^2+4^2}}=10$$

$|b-33|=50$

$b-33=50$ or $b-33=-50$

$b=83$ or $b=-17$

The 2 planes are $3y+4z=83$ and $3y+4z=-17$.

(iii) $\overrightarrow{OG}=\begin{pmatrix}3\\8\\6\end{pmatrix}$

$$\mathbf{r}=\begin{pmatrix}3\\8\\6\end{pmatrix}+\lambda\begin{pmatrix}0\\3\\4\end{pmatrix},\ \lambda\in\mathbb{R}$$

$$\begin{pmatrix}3\\8+3\lambda\\6+4\lambda\end{pmatrix}\bullet\begin{pmatrix}0\\3\\4\end{pmatrix}=33$$

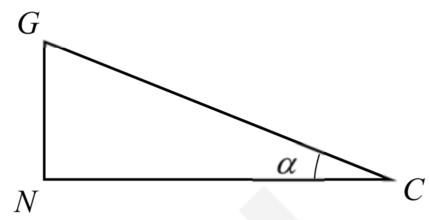
$48+25\lambda=33$

$\lambda=-\frac{3}{5}$

Coordinates of point is $\left(3,\frac{31}{5},\frac{18}{5}\right)$.

(iv) **Method 1:** (Use trigo ratio of right angle triangle CGN)

Let the point obtained in (iii) be N.



$$\overrightarrow{GN} = \begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 15 \\ 31 \\ 18 \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}, \quad \overrightarrow{CG} = \begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix}$$

$$\sin\alpha = \frac{GN}{CG} = \frac{3}{\sqrt{3^2+3^2+6^2}}$$

$$\alpha = 24.1^\circ$$

Method 2:

$$\overrightarrow{CG} = \begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = \sqrt{3^2+3^2+6^2}\sqrt{3^2+4^2}\cos\beta$$

$$\cos\beta = \frac{15}{5\sqrt{54}}$$

$$\beta = 65.905^\circ$$

$$\alpha = 90^\circ - 65.905^\circ = 24.1^\circ$$

11(i) Rate of change of population: $\frac{\mathrm{d}n}{\mathrm{d}t} = a - bn$

Since $\frac{\mathrm{d}n}{\mathrm{d}t} = 0$ when $n = 1$, $\quad 0 = a - b$. Hence, $a = b$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = a - an = a(1-n)$$

$$\int \frac{1}{1-n}\,\mathrm{d}n = \int a\,\mathrm{d}t$$

$$-\ln|1-n| = at + C$$

$$\ln|1-n| = -at - C$$

$$|1-n| = \mathrm{e}^{-at-C}$$

$$1-n = \pm\mathrm{e}^{-C}\mathrm{e}^{-at}$$

$$n = 1 + A\mathrm{e}^{-at}, A = \mp\mathrm{e}^{-C}$$

Sub $t = 0, n = 3$: $\quad 3 = 1 + A\mathrm{e}^0 \Rightarrow A = 2$

Sub $t = 1, n = 2$: $\quad 2 = 1 + 2\mathrm{e}^{-a} \Rightarrow \frac{1}{2} = \mathrm{e}^{-a} \Rightarrow -a = \ln\frac{1}{2}$

Hence, $n = 1 + 2\mathrm{e}^{\ln\frac{1}{2}t} = 1 + 2\left(\mathrm{e}^{\ln\frac{1}{2}}\right)^t = 1 + 2\left(\frac{1}{2}\right)^t = 1 + 2^{1-t}$

(ii) $\dfrac{d^2n}{dt^2} = 4 - 6t$

$\dfrac{dn}{dt} = 4t - 3t^2 + C$

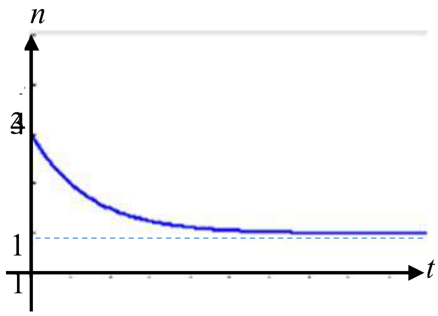
$n = 2t^2 - t^3 + Ct + D$

Sub $t = 0, n = 3$: $D = 3$

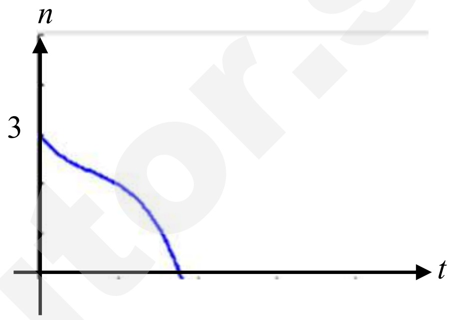
Sub $t = 1, n = 2$: $C = -2$

Hence, $n = 2t^2 - t^3 - 2t + 3$

(iii) Model I



Model II



Based on the graphs, the population decreases but approaches a constant of 1000 for Model I. Whereas for Model II, the microorganism would die off quickly. Hence the condition for model II is more harmful.

Pioneer Junior College
2018 Preliminary Examination H2 Mathematics Paper 2 (9758/02)
Section A: Pure Mathematics [40 marks]

1 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The point C lies on OA produced and is such that $OC = \lambda OA$, where $\lambda > 1$. The point D lies on OB, between O and B, such that $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{OB}$. It is given that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 8$ and $\angle AOB = 60°$.

(i) Show that $\overrightarrow{OD} = \frac{1}{4}\mathbf{b}$. [2]

(ii) Show that the vector equation of the line BC can be written as $\mathbf{r} = \lambda\mu\mathbf{a} + (1-\mu)\mathbf{b}$, where μ is a parameter. [2]

The point E lies on the line BC.

(iii) Find the values of μ, in terms of λ, such that the area of triangle ODE is $\sqrt{300}$. [4]

2 **Do not use a calculator in answering this question.**

(a) Solve the equation $z^2 = 5 - 12\mathrm{i}$, giving your answers in the form $x + \mathrm{i}y$. [4]

Hence, find the roots of the equation $w^4 - 10w^2 + 169 = 0$. [2]

(b) The complex number p has modulus r and argument θ, where $0 < \theta < \frac{1}{2}\pi$. State the argument of q, where $q = \frac{(-1+\mathrm{i})\,p^*}{p^2}$. Given that q^2 is real and negative, find the possible values of θ. [4]

3 The function f is defined by

$$\mathrm{f}(x) = \begin{cases} 2x^2 + 5, & x < 1, \\ |x - 8|, & x \geq 1. \end{cases}$$

(i) Sketch the graph of f and state the range of f. [3]

(ii) State whether the inverse function f^{-1} exists, justifying your answer. [2]

The function g is defined by

$$\mathrm{g}(x) = \sqrt{x} + 1, \qquad 0 < x \leq 16.$$

(iii) Find $\mathrm{g}^{-1}(4)$. [2]

(iv) State the range of g and find $\mathrm{fg}(x)$, stating the domain of fg. [3]

4 The rth term of a sequence is given by $u_r = \dfrac{1}{r!}$.

(i) Show that $u_r - u_{r+1} = \dfrac{1}{r! + (r-1)!}$. [2]

(ii) Hence find $\displaystyle\sum_{r=1}^{N} \frac{1}{r! + (r-1)!}$. [2]

(iii) Give a reason why the series in **(ii)** is convergent and state the sum to infinity. [2]

(iv) Use your answer to **(ii)** to find $\displaystyle\sum_{r=1}^{N+1} \frac{1}{r! + (r+1)!}$. [3]

(v) Deduce that $\displaystyle\sum_{r=1}^{N} \frac{1}{r!} < 2$. [3]

Section B: Statistics [60 marks]

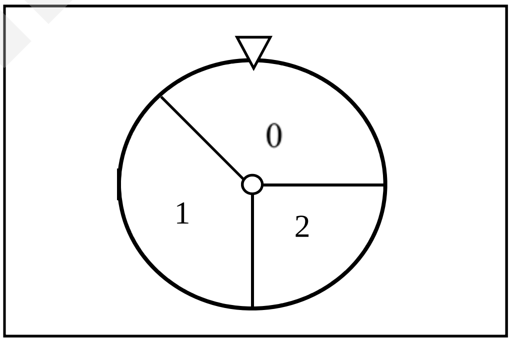
5 **(a)** Three boys and three girls are to be seated in two rows of three chairs each. How many ways can it be done if

(i) boys and girls must alternate, [2]

(ii) two particular girls must sit next to each other? [2]

(b) How many ways can six people be seated around two identical round tables such that there is at least one person at each table? [3]

6 A circular card is divided into 3 sectors scoring 0, 1, 2 and having angles 135°, 135°, 90° respectively. The card is mounted onto a wall at its centre such that it can rotate freely. A pointer is fixed on the wall just above the card, as shown in the diagram.



In a game, a participant will spin the card twice, and the random variable X is the product of the scores of the 2 spins.

(i) Tabulate the probability distribution of X. [3]

(ii) Show that $\mathrm{E}(X) = \dfrac{49}{64}$, and find $\mathrm{Var}(X)$. [2]

Dave intends to use the above setup in a fundraising carnival. For every game, the participant will be charged \$$y$ to play, and will be awarded a prize money of \$$5X$ based on the outcome of their spins. Find the least integer value of y such that on average, Dave can expect to earn at least \$1 from each participant. [2]

7 National Aeronautics and Space Administration (NASA) gives some information about the planets in the solar system. The distance from the Sun, x million km, and mean temperature, y Kelvin, of seven of them are as follows

Planet	Mercury	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
X	57.9	149.6	227.9	778.6	1433.5	2872.5	4495.1
y	440	288	208	163	133	78	73

(Source: https://nssdc.gsfc.nasa.gov/planetary/factsheet/)

(i) Draw the scatter diagram for these values, labelling the axes clearly. Explain whether your answer suggests that a linear model is appropriate. [2]

It is thought that the mean temperature y can be modelled by one of the formulae

$$y = a + \frac{b}{x} \text{ or } y = c + \frac{d}{\sqrt{x}}$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the product moment correlation coefficient between

(a) $\frac{1}{x}$ and y,

(b) $\frac{1}{\sqrt{x}}$ and y. [2]

(iii) Use your answers to part **(ii)** to explain which of $y = a + \frac{b}{x}$ and $y = c + \frac{d}{\sqrt{x}}$ is the better model. [1]

(iv) It is required to estimate the mean temperature of Venus, which is 108.2 million km from the Sun. Find the equation of a suitable regression line, and use it to find the required estimate. Comment on the reliability of your estimate. [4]

(v) Given that the conversion of Kelvin (T_K) to Celsius (T_C) follows the formula $T_K = T_C + 273$, re-write your equation from part **(iv)** so that it can be used to estimate the temperature, in Celsius, when the distance from the Sun is given. [2]

8 **In this question you should state clearly the values of the parameters of any normal distribution you use.**

The masses, in grams, of peaches produced by orchard A and orchard B have the distributions $N(145, 15^2)$ and $N(190, 20^2)$ respectively.

(i) Find the probability that the mass of a randomly chosen peach from A is less than 140 grams. [1]

(ii) Find the probability that of 3 randomly chosen peaches from A, only one will have a mass of less than 140 grams. [3]

(iii) The probability that the mass of a randomly chosen peach from A is at least k grams is at most 0.15. Find the range of values of k. [2]

(iv) Find the probability that the total mass of 4 randomly chosen peaches produced by A will differ from thrice the mass of a randomly chosen peach produced by B by at least 15 grams. [4]

State an assumption needed for your calculations in **(ii)** and **(iv)**. [1]

9 A swimming school administers a swimming assessment for 60 candidates each day, and the number of those candidates who pass the assessment is denoted by S.

(i) State, in context, two assumptions needed for S to be well modelled by a binomial distribution. [2]

Assume now that S has the distribution $\mathrm{B}(60, 0.7)$.

(ii) Find the probability that more than 40 candidates pass the assessment on a randomly chosen day. [1]

(iii) The school claims that the probability of at least m candidates passing the assessment each day is at least 90%. Find the greatest value of m. [3]

A "good" day is a day in which at least 45 candidates pass the assessment.

(iv) Find the probability that a day is good. [1]

(v) Find the probability that less than 50 candidates passed the assessment on a good day. [3]

(vi) Find the probability that there are at most 10 good days over a period of 50 days. [2]

10 A fruit juice seller claims that each bottle of fruit juice he sells contains, on average, 300 ml of fruit juice. A random sample of 35 bottles of fruit juice is selected and the amount of fruit juice, x ml, in each bottle is measured. The results are summarised by:

$$\sum(x-250)=1705\,,\ \sum(x-250)^2=83650\,.$$

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) Test, at the 5% level of significance, whether the fruit juice seller is overstating the average amount of fruit juice in each bottle. You should state your hypotheses and define any symbols you use. [5]

(iii) Explain why there is no need for the fruit juice seller to know anything about the distribution of the amount of fruit juice in each bottle. [1]

Improvements are made to the bottling process and the variance of the amount of fruit juice in each bottle is now known to be 15.7 ml^2. The fruit juice seller now claims that the amount of fruit juice in each bottle is more than 300ml. A new random sample of 15 bottles of fruit juice is chosen and the mean of this sample is y ml. A test at the 5% significance level indicates that the fruit juice seller's claim is valid for this improved process.

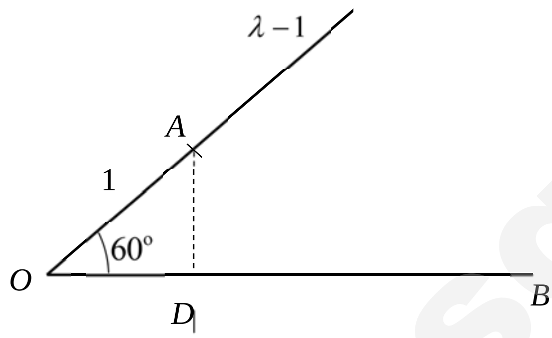
(iv) Find the least possible value of y, giving your answer correct to 2 decimal places. [3]

State any assumption you made in your calculation in **(iv)**. [1]

PJC J2 H2 Maths Prelim Paper 2 Solution:

1

(i) **Method 1:**



$$\overrightarrow{OD} = \left(4\cos 60^{\circ}\right)\hat{\mathbf{b}}$$
$$= \left(4\cos 60^{\circ}\right)\frac{\mathbf{b}}{|\mathbf{b}|}$$
$$= (2)\frac{\mathbf{b}}{8}$$
$$= \frac{1}{4}\mathbf{b} \text{ (shown)}$$

Method 2:

$$\overrightarrow{OD} = \left|\mathbf{a}\bullet\hat{\mathbf{b}}\right|\hat{\mathbf{b}}$$
$$= \left|\mathbf{a}\bullet\frac{\mathbf{b}}{|\mathbf{b}|}\right|\frac{\mathbf{b}}{|\mathbf{b}|}$$
$$= \frac{1}{|\mathbf{b}|^2}|\mathbf{a}\bullet\mathbf{b}|\mathbf{b}$$
$$= \frac{1}{|\mathbf{b}|^2}\left|\mathbf{a}\right|\left|\mathbf{b}\right|\left|\cos 60^{\circ}\right|\mathbf{b}$$
$$= \frac{1}{8}(4)\left(\frac{1}{2}\right)\mathbf{b}$$
$$= \frac{1}{4}\mathbf{b} \text{ (shown)}$$

Method 3:

$$\overrightarrow{AD}\bullet\overrightarrow{OB} = 0$$
$$\left(\overrightarrow{OD} - \overrightarrow{OA}\right)\bullet\overrightarrow{OB} = 0$$
$$\overrightarrow{OD}\bullet\overrightarrow{OB} - \overrightarrow{OA}\bullet\overrightarrow{OB} = 0$$
$$\left|\overrightarrow{OD}\right|\left|\overrightarrow{OB}\right| = \left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos 60^{\circ}$$
$$\left|\overrightarrow{OD}\right| = 4\cos 60^{\circ} = 2$$

Hence, $\overrightarrow{OD} = \frac{2}{8}\overrightarrow{OB} = \frac{1}{4}\mathbf{b}$ (Shown)

(ii) $\overrightarrow{BC} = \lambda\mathbf{a} - \mathbf{b}$

Equation of line BC: $\mathbf{r} = \mathbf{b} + \mu\left(\lambda\mathbf{a} - \mathbf{b}\right)$

$$\mathbf{r} = \lambda\mu\mathbf{a} + (1-\mu)\mathbf{b} \text{ (shown)}$$

(iii) Area of $\Delta ODE = \frac{1}{2}\left|\overrightarrow{OD}\times\overrightarrow{OE}\right| = \sqrt{300}$

$$\frac{1}{2}\left|\frac{1}{4}\mathbf{b}\times\left[\lambda\mu\mathbf{a}+(1-\mu)\mathbf{b}\right]\right| = \sqrt{300}$$

$$\frac{1}{8}\left|\mathbf{b}\times\lambda\mu\mathbf{a}\right| = \sqrt{300}$$

$$\lambda\left|\mathbf{b}\times\mu\mathbf{a}\right| = 8\sqrt{300} \text{ (since } \lambda > 0\text{)}$$

$$\lambda|\mu||\mathbf{b}||\mathbf{a}|\left|\sin 60^{\circ}\right| = 8\sqrt{300}$$

$$\lambda|\mu|(8)(4)\left(\frac{\sqrt{3}}{2}\right) = 8\sqrt{300}$$

$$|\mu| = \frac{5}{\lambda}$$

$$\mu = \frac{5}{\lambda} \quad \text{or} \quad \mu = -\frac{5}{\lambda}$$

2 Let $z = x + \mathrm{i}y$

$$(x+\mathrm{i}y)^2 = 5-12\mathrm{i}$$

$$x^2 - y^2 + \mathrm{i}2xy = 5-12\mathrm{i}$$

Comparing real and imaginary components,

Real: $x^2 - y^2 = 5$

Imaginary: $2xy = -12 \Rightarrow y = \frac{-6}{x}$

$$x^2 - \left(\frac{-6}{x}\right)^2 = 5$$

$$x^4 - 5x^2 - 36 = 0$$

$$x^2 = \frac{5\pm\sqrt{5^2-4(1)(-36)}}{2(1)} = \frac{5\pm 13}{2}$$

$x^2 = -4$ (rej since $x^2 \geq 0$) or $x^2 = 9$

$x = 3$ or $x = -3$

$y = -2$ or $y = 2$

Hence, $z = 3-2\mathrm{i}$, $z = -3+2\mathrm{i}$

$$w^4 - 10w^2 + 169 = 0$$

$$w^2 = \frac{10\pm\sqrt{10^2-4(1)(169)}}{2(1)} = \frac{10\pm 24\mathrm{i}}{2} = 5\pm 12\mathrm{i}$$

Since $w^4 - 10w^2 + 169 = 0$ is a polynomial with real coefficients, the roots occurs in conjugate pairs.

Hence, the roots are $w = 3-2\mathrm{i}$, $w = 3+2\mathrm{i}$, $w = -3-2\mathrm{i}$, $w = -3+2\mathrm{i}$

(a) $\arg(q)=\arg\left[\dfrac{(-1+\mathrm{i})\,p^{*}}{p^{2}}\right]=\arg(-1+\mathrm{i})+(-\theta)-(2\theta)=\dfrac{3}{4}\pi-3\theta$

Since q^2 is real,

$$\sin\left(\frac{3}{2}\pi-6\theta\right)=0$$

$$\frac{3}{2}\pi-6\theta=-\pi,0,\pi$$

$$6\theta=\frac{1}{2}\pi,\frac{3}{2}\pi,\frac{5}{2}\pi$$

$$\theta=\frac{1}{12}\pi,\frac{1}{4}\pi,\frac{5}{12}\pi$$

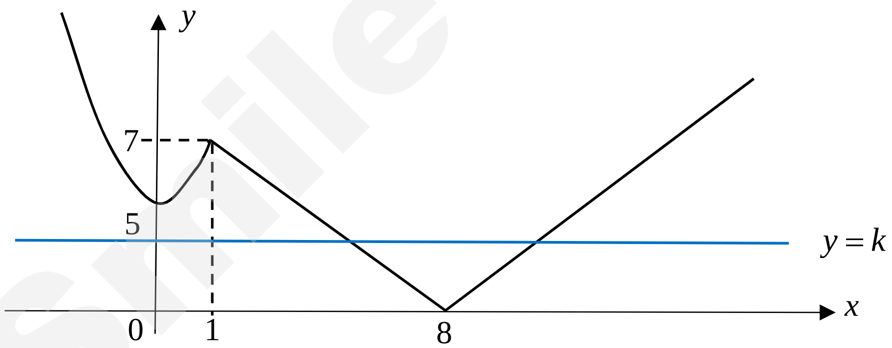
Since q^2 is negative,

$\cos\left(\dfrac{3}{2}\pi-6\theta\right)$ is negative $\Rightarrow\ \theta\neq\dfrac{1}{4}\pi$.

Hence $\theta=\dfrac{1}{12}\pi,\dfrac{5}{12}\pi$

3

(i)



$R_f=[0,\infty)$

(ii) Since there exists a horizontal line $y=k$ that cuts the graph of f at more than one point, f is not one-one. Hence, f^{-1} does not exist.

(ii) Let $g^{-1}(4)=x\ \Rightarrow g(x)=4$

$\sqrt{x}+1=4\ \Rightarrow\sqrt{x}=3\Rightarrow x=9$

Hence, $g^{-1}(4)=9$

(iii) $R_g=(1,5]$

Hence, $fg(x)=-\left[\left(\sqrt{x}+1\right)-8\right]=-\sqrt{x}+7$, $0<x\le 16$.

4

(i) $u_r - u_{r+1} = \frac{1}{r!} - \frac{1}{(r+1)!}$

$= \frac{1}{r!} - \frac{1}{(r+1)r!}$

$= \frac{(r+1)-1}{(r+1)r!}$

$= \frac{r}{(r+1)r!}$

$= \frac{1}{(r+1)(r-1)!}$

$= \frac{1}{r(r-1)!+(r-1)!}$

$= \frac{1}{r!+(r-1)!}$ (Shown)

(ii) $\sum_{r=1}^{N} \frac{1}{r!+(r-1)!} = \sum_{r=1}^{N} u_r - u_{r+1}$

$= u_1 - \cancel{u_2}$

$+ \quad \cancel{u_2} - \cancel{u_3}$

$+ \quad \cancel{u_3} - u_4$

…..

$+ \quad \cancel{u_{N-2}} - \cancel{u_{N-1}}$

$+ \quad \cancel{u_{N-1}} - \cancel{u_N}$

$+ \quad \cancel{u_N} - u_{N+1}$

$= u_1 - u_{N+1}$

$= 1 - \frac{1}{(N+1)!}$

(iii) As $N \to \infty$, $\frac{1}{(N+1)!} \to 0$ $\quad \sum_{r=1}^{N} \frac{1}{r!+(r-1)!} \to 1$ which is finite, hence $\sum_{r=1}^{N} \frac{1}{r!+(r-1)!}$ converges and the sum to infinity is 1

(iv) $$\sum_{r=1}^{N+1} \frac{1}{r!+(r+1)!} = \sum_{r=2}^{N+2} \frac{1}{r!+(r-1)!}$$

$$= \sum_{r=1}^{N+2} \frac{1}{r!+(r-1)!} - \frac{1}{2}$$

$$= 1 - \frac{1}{(N+3)!} - \frac{1}{2}$$

$$= \frac{1}{2} - \frac{1}{(N+3)!}$$

(v) Since $r! > (r-1)!$

$$\frac{1}{r!+r!} < \frac{1}{r!+(r-1)!}$$

$$\sum_{r=1}^{N} \frac{1}{r!+r!} < \sum_{r=1}^{N} \frac{1}{r!+(r-1)!}$$

$$\sum_{r=1}^{N} \frac{1}{2(r!)} < \sum_{r=1}^{N} \frac{1}{r!+(r-1)!}$$

$$\sum_{r=1}^{N} \frac{1}{r!} < 2\sum_{r=1}^{N} \frac{1}{r!+(r-1)!} = 2\left(1 - \frac{1}{(N+1)!}\right) < 2 \text{ since } \frac{1}{(N+1)!} > 0.$$

5 (a)(i) There are 2 possible cases:

Case 1	Case 2
GBG	BGB
BGB	GBG

For each case, there are $3! \times 3!$ ways

Total number of ways = $2 \times 3! \times 3! = 72$

(a)(ii) Group the 2 girls together: 2!

Number of ways to place the group of 2 girls: 4

No of ways to arrange the remaining 4 person: 4!

Number of ways = $2! \times 4 \times 4! = 192$

(b) Case 1 (table of 5, table of 1): Number of ways = ${}^6C_5 \times (5-1)! \times (1-1)! = 144$

Case 2 (table of 4, table of 2): Number of ways = ${}^6C_4 \times (4-1)! \times (2-1)! = 90$

Case 3 (table of 3, table of 3): Number of ways = $\dfrac{{}^6C_3 \times (3-1)! \times (3-1)!}{2} = 40$

(divide by 2 because the 2 tables are of identical size)

Total number of ways = $144 + 90 + 40 = 274$

6

(i) $\dfrac{90}{360}=\dfrac{1}{4}$, $\dfrac{135}{360}=\dfrac{3}{8}$

$\mathrm{P}(X=0)=\mathrm{P}(0,0)+\mathrm{P}(0,1)\times 2!+\mathrm{P}(0,2)\times 2!$

$=\left(\dfrac{3}{8}\right)^2+\left(\dfrac{3}{8}\right)\left(\dfrac{3}{8}+\dfrac{1}{4}\right)2!=\dfrac{39}{64}$

$\mathrm{P}(X=1)=\mathrm{P}(1,1)=\left(\dfrac{3}{8}\right)^2=\dfrac{9}{64}$

$\mathrm{P}(X=2)=\mathrm{P}(1,2)\times 2!=\left(\dfrac{3}{8}\right)\left(\dfrac{1}{4}\right)\times 2!=\dfrac{3}{16}$

$\mathrm{P}(X=4)=\mathrm{P}(2,2)=\left(\dfrac{1}{4}\right)^2=\dfrac{1}{16}$

X	0	1	2	4
$\mathrm{P}(X=x)$	$\dfrac{39}{64}$	$\dfrac{9}{64}$	$\dfrac{3}{16}$	$\dfrac{1}{16}$

(ii) $\mathrm{E}(X)=(0)\left(\dfrac{39}{64}\right)+(1)\left(\dfrac{9}{64}\right)+(2)\left(\dfrac{3}{16}\right)+(4)\left(\dfrac{1}{16}\right)=\dfrac{49}{64}$

$\mathrm{E}(X^2)=(0)^2\left(\dfrac{39}{64}\right)+(1)^2\left(\dfrac{9}{64}\right)+(2)^2\left(\dfrac{3}{16}\right)+(4)^2\left(\dfrac{1}{16}\right)=\dfrac{121}{64}$

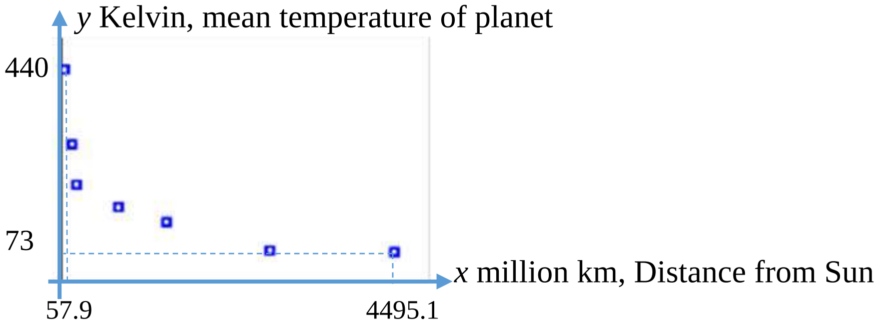
$\mathrm{Var}(X)=\mathrm{E}(X^2)-\left[\mathrm{E}(X)\right]^2=\dfrac{5343}{4096}$

(iii) $\mathrm{E}(5X)=3.8281$

Hence least integer value of y is 5.

7

(i)



From the diagram, it is observed that as x increases, y decreases by decreasing amounts. Hence a linear model is not appropriate.

(ii) (a) For $y = a + \frac{b}{x}$, $r = 0.968656 \approx 0.9687$

(b) For $y = c + \frac{d}{\sqrt{x}}$, $r = 0.991301 \approx 0.9913$

(iii) Since the r value of $y = a + \frac{b}{x}$ is nearer to 1 than the r value of $y = c + \frac{d}{\sqrt{x}}$, $y = a + \frac{b}{x}$ is a better model.

(iv) Using GC, $y = \frac{3052.7}{\sqrt{x}} + 33.963 \approx \frac{3050}{\sqrt{x}} + 34.0\,(1\text{dp})$

$y = \frac{3052.7}{\sqrt{108.2}} + 33.963 = 327.44 \approx 327$

Since r-value is near to 1 and this is an interpolation, the estimate is a reliable one.

(v) $y + 273 = \frac{20504.39}{x} + 107.07$

$y = \frac{20504.39}{x} - 165.93$

8(i) Let A be mass of a peach from Orchard A. $A \sim \text{N}(145, 15^2)$

$\text{P}(X < 140) = 0.36944 \approx 0.369$

(ii) $\text{P}(A < 140)\left[\text{P}(A \geq 140)\right]^2 \times \frac{3!}{2!} = 0.44067 \approx 0.440$

Alternatively

Let $Y \sim$ number of peaches from Orchard A with mass less than 140g out of 3

$Y \sim B(3,\ \text{P}(A < 140))$, i.e. $Y \sim B(3,\ 0.36944)$

$\text{P}(Y = 1) = 0.44067 \approx 0.440$

(iii) $\text{P}(A \geq k) \leq 0.15$

Consider $\text{P}(A \geq k) = 0.15$

From GC, $k = 160.55$

Hence for $\mathrm{P}(A \geq k) \leq 0.15$, $k \geq 160.55$
i.e. $k \geq 161$

(iv) Let $X = A_1 + \ldots + A_4 \sim \mathrm{N}(145 \times 4, 15^2 \times 4) \Rightarrow X \sim \mathrm{N}(580, 900)$

$3B \sim \mathrm{N}(190 \times 3, 20^2 \times 3^2) \Rightarrow 3B \sim \mathrm{N}(570, 3600)$

$X - 3B \sim \mathrm{N}(580 - 570, 900 + 3600) \Rightarrow X - 3B \sim \mathrm{N}(10, 4500)$

$\mathrm{P}(|X - 3B| \geq 15) = 1 - \mathrm{P}(|X - 3B| \leq 15) = 1 - \mathrm{P}(-15 < X - 3B < 15) = 0.824986 \approx 0.825$

Assume that the masses of all peaches are independent of one another.

9(i) 1) The assessment results of the candidates are independent of each other.
2) The probability of each candidate passing the assessment is constant.

(ii) $\mathrm{P}(S > 40) = 1 - \mathrm{P}(S \leq 40) = 0.66916 \approx 0.669$

(iii) $\mathrm{P}(S \geq m) \geq 0.9$

$1 - \mathrm{P}(S \leq m - 1) \geq 0.9$

$\mathrm{P}(S \leq m - 1) \leq 0.1$

When $m = 36$, $\mathrm{P}(S \leq m - 1) = 0.0362\ (< 0.1)$

When $m = 37$, $\mathrm{P}(S \leq m - 1) = 0.0632\ (< 0.1)$

When $m = 38$, $\mathrm{P}(S \leq m - 1) = 0.1041\ (> 0.1)$

When $m = 39$, $\mathrm{P}(S \leq m - 1) = 0.1618\ (> 0.1)$

Hence greatest value of m is 37.

(iv) $\mathrm{P}(S \geq 45) = 1 - \mathrm{P}(S \leq 44) = 0.24378 \approx 0.244$

(v)
$$\mathrm{P}(S < 50 \mid S \geq 45) = \frac{\mathrm{P}(45 \leq S < 50)}{\mathrm{P}(S \geq 45)}$$
$$= \frac{\mathrm{P}(S \leq 49) - \mathrm{P}(S \leq 44)}{1 - \mathrm{P}(S \leq 44)} = 0.94307 \approx 0.943$$

(vi) $X \sim$ Number of good days, out of 50.

$X \sim \mathrm{B}(50, 0.24378)$

$\mathrm{P}(X \leq 10) = 0.29608 \approx 0.296$

10(i) $\bar{x} = \dfrac{1705}{35} + 250 = 298.714 \approx 299$ (3s.f.)

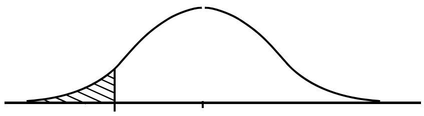
$$s^2 = \frac{1}{34}\left(83650 - \frac{1705^2}{35}\right) = 17.415966 \approx 17.4 \text{ (3 s.f.)}$$

(ii) Let X ~ amount of fruit juice in a bottle

$H_0 : \mu = 300$ vs $H_1 : \mu < 300$, where μ is the mean amount of fruit juice in a bottle.

Since n = 35 is large, by Central Limit Theorem,

$\overline{X} \sim N\left(300\ ,\ \frac{17.415966}{35}\right)$ approximately.



Level of significance: 5%

Critical region: $z < -1.64485$

$$\text{Test statistic: } z = \frac{\overline{x} - \text{"claimed value"}}{s/\sqrt{n}} = \frac{298.714 - 300}{\sqrt{\frac{17.415966}{35}}} = -1.82306 < -1.64485$$

From GC, p-value = 0.034147 <0.05

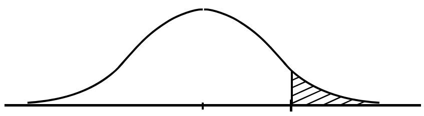
Since the p-value is less than the level of significance, we reject H_0 . There is sufficient evidence, at the 5% level, to indicate that the mean amount of fruit juice in each bottle is less than 300ml (OR: to indicate that the fruit juice seller overstated the mean amount of fruit juice in each bottle).

(iii) Central Limit Theorem states that sample means will have a normal distribution when sample size is big enough.

(iv) $H_0 : \mu = 300$ vs $H_1 : \mu > 300$

Assuming X is normal,

$\overline{X} \sim N\left(300\ ,\ \frac{15.7}{15}\right)$ exactly.



Level of significance: 5 %

Critical value: $z = 1.6449$

$$\text{Test statistic: } z = \frac{\overline{x} - \text{"claimed value"}}{s/\sqrt{n}} = \frac{y - 300}{\sqrt{\frac{15.7}{15}}}$$

Since the seller's claim is valid, we reject H_0 , i.e.

$$\frac{y - 300}{\sqrt{\frac{15.7}{15}}} > 1.6449$$

$y > 301.6828$

Hence, least value of y is 301.68.

(v) Assume that the amount of fruit juice in each bottle follows a normal distribution.

RAFFLES INSTITUTION
2018 YEAR 6 PRELIMINARY EXAMINATION

MATHEMATICS **9758/01**

Paper 1

Total Marks: 100 **3 hours**

Additional materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **9** printed pages.

1 Adult tickets for a parade are sold at three different prices, depending on the type of seats.

Children under the age of 12 and Senior Citizens aged 65 and above can enjoy 20% and 10% off the adult ticket prices respectively. Those who are between 12 and 64 years old (inclusive) will have to pay the full adult ticket price.

The number of tickets sold in each category for each group of people, together with the total cost of the tickets for each group, are given in the following table.

	Category 1 tickets (premier)	Category 2 tickets (sheltered)	Category 3 tickets (unsheltered)	Total cost
Adults (12 to 64 years old (inclusive))	150	120	200	\$18 000
Children (under 12 years old)	80	40	100	\$6 760
Senior Citizens (65 years old and above)	100	100	100	\$10 665

Write down and solve equations to find the price of an adult ticket for each of the ticket categories. [4]

2 Aden, a member of Raffles Art Club, is helping to design a graphic for the school's homecoming event. The main body of the graphic consists of N concentric circles C_1, C_2, $C_3, \ldots,$ C_N with radii r, $2r$, $3r, \ldots,$ Nr, where r is a constant and N is an even integer, as shown in Fig. 1.

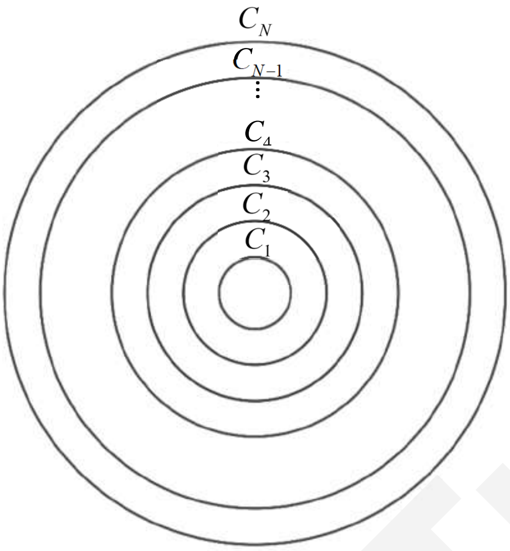


Fig. 1

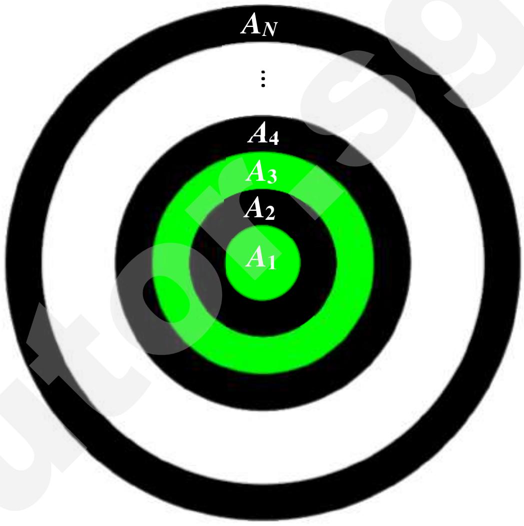


Fig. 2

The region enclosed by the circle C_1 is denoted by A_1 with area a_1 and the region between the circles C_{n-1} and C_n is denoted by A_n with area a_n for $n = 2, 3, 4, ..., N$, as shown in Fig. 2.

(i) Show that the sequence a_1, a_2, $a_3, \ldots,$ a_N is an arithmetic progression. [4]

With the help of a graphic software, Aden fills A_1, A_2, $A_3, \ldots,$ A_N with two of the school colours: green and black. A_n will be filled green if n is odd and will be filled black if n is even.

Aden wishes to create a better visual effect by having different intensities of green. He fills A_1 with the green colour that is the same as the school colour. He reduces the intensity of this green colour by ten percent to fill A_3; and reduces the intensity of the green colour used to fill A_3 by ten percent to fill A_5; and this process continues for all odd values of n. When he finishes filling up all the areas, Aden finds that the intensity of the green colour that fills A_{N-1} falls below one quarter of the intensity of the school colour for the first time.

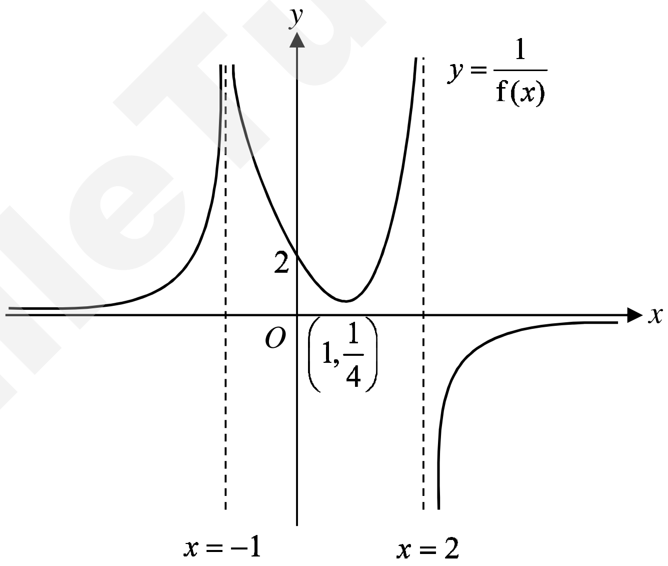
(ii) Find the value of N. [4]

[Turn over

3 **(a)** Find $\int_0^1 \frac{9x-4x^2}{9-4x^2}\,dx$, giving your answer in the form $p+q\ln 3+r\ln 5$ where p, q and r are rational numbers to be determined.

(b) Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.
Hence, or otherwise, find $\int \sin x \sin 2x \sin 3x\,dx$. [4]

4 **(a)** The diagram below shows the curve of $y=\frac{1}{f(x)}$ where $f(x)$ is a polynomial. The curve has a minimum point at $\left(1,\frac{1}{4}\right)$ and cuts the y-axis at $(0,2)$. The lines $x=-1$ and $x=2$ are the vertical asymptotes and the line $y=0$ is the horizontal asymptote to the curve.



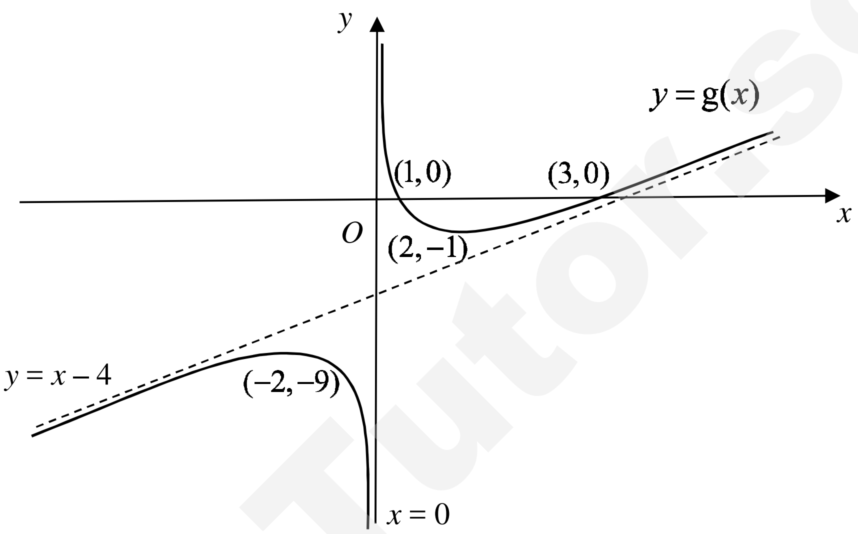
Sketch on separate diagrams, the graphs of

(i) $y=f(x)$,

(ii) $y=f(|x|)$,

labelling all relevant point(s).

4 **(b)** The diagram below shows the curve of $y = \mathrm{g}(x)$. The curve has a maximum point at $(-2,-9)$ and a minimum point at $(2,-1)$. The curve crosses the x-axis at $(1,0)$ and $(3,0)$. The line $x = 0$ is the vertical asymptote and the line $y = x-4$ is the oblique asymptote to the curve.

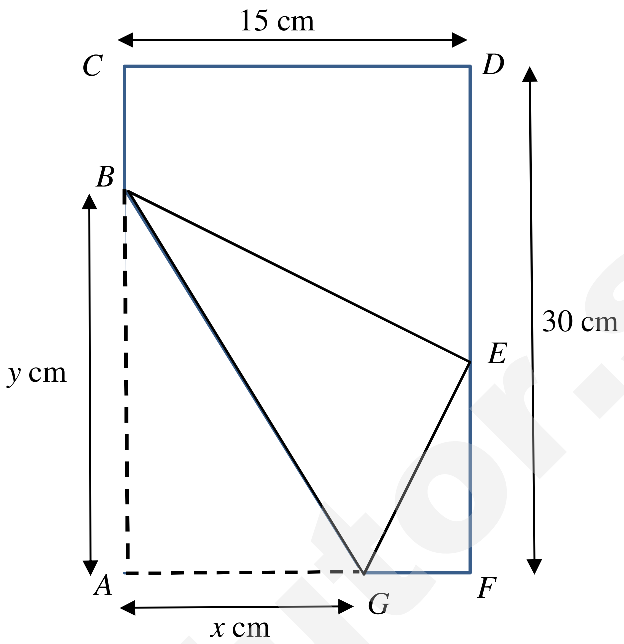


(i) Sketch the graph of $y = \mathrm{g}'(x)$, labelling all relevant point(s) and stating the equations of any asymptotes.

(ii) Find the area bounded by the graph of $y = \mathrm{g}'(x)$, the lines $x = 1$, $x = 2$ and the x-axis.

[Turn over

5



Tom has a rectangular piece of paper *ACDF* with length 30 cm and breadth 15 cm. He folds the lower left-hand corner, *A*, to reach the rightmost edge of the paper at *E*. After that he will cut out the triangle *EFG* and the trapezium *BCDE* to obtain a kite shaped figure *ABEG* with *AB* of length y cm and *AG* of length x cm.

(i) Find the length of *EF* in terms of x. [2]

(ii) Show that $y = \dfrac{x\sqrt{15}}{\sqrt{2x-15}}$. [2]

(iii) Using differentiation, find the exact values of x and y which give the minimum area of the kite *ABEG*. [5]

6 **(a)** It is given that $e^{y} = \tan\left(x + \dfrac{\pi}{3}\right)$.

(i) Show that $\dfrac{dy}{dx} = e^{y} + e^{-y}$.

(ii) Using differentiation, find the Maclaurin series for $\ln\left[\tan\left(x + \dfrac{\pi}{3}\right)\right]$ up to and including the term in x^2.

(b) Let $f(x) = \dfrac{3x^2 - 4x + 5}{(1+x)(1-x)^2}$.

(i) Express $f(x)$ in the form $\dfrac{A}{(1+x)} + \dfrac{B}{(1-x)} + \dfrac{C}{(1-x)^2}$ where A, B and C are constants to be determined. [2]

(ii) Hence find the expansion of $f(x)$ up to and including the term in x^4. [2]

(iii) Write down the coefficient of x^r in the expansion of $f(x)$ in terms of r. [1]

7 **Do not use a calculator in answering this question.**

(a) Find the roots of the equation $w^2(1-i) + 4w + (10+10i) = 0$, giving your answers in cartesian form $a + ib$. [3]

(b) It is given that $z = -\sqrt{3} + i$.

(i) Find an exact expression for z^5. Give your answer in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

(ii) Find the three smallest positive whole number values of n for which $\dfrac{z^n}{iz^*}$ is purely imaginary. [4]

(iii) Given that $\left|1 + \dfrac{p}{z^*}\right| = \sqrt{7}$, find exactly the possible values of the real number p. [3]

[Turn over

8 The function $\mathrm{f}(x)=a-(1+x)^{\frac{1}{3}}$ is defined for $0\le x\le 7$, where a is a positive real constant.

(i) Solve the inequality $\mathrm{f}(x)\ge a-1$. [2]

(ii) Show that the graph of $y=\mathrm{f}(x)$ has a negative gradient at all points on the graph. Find the range of f. [2]

Use $a=3$ in the rest of the question.

(iii) Find $\mathrm{f}^{-1}(x)$. [2]

(iv) Sketch on the same diagram the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$, showing clearly the geometrical relationship between the two graphs and the line $y=x$. [3]

(v) Show that the x-coordinate of the point of intersection of the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ satisfies the equation $x^3-9x^2+28x-26=0$.
Hence find the solution of the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [3]

9 **(a)** The variables x and y are related by the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{x}\mathrm{e}^{\sqrt{x}}$.
Using the substitution $u=\sqrt{x}$, show that $y=k\int u^2\mathrm{e}^u\,\mathrm{d}u$ where k is a constant to be determined. Hence find the general solution of the differential equation.[5]

(b) Ms Frugal bought a second-hand oven which has a broken timer. Each time she bakes she uses the timer on her handphone. One day, she decides to make a loaf of bread. She takes the fermented dough out of her refrigerator and checks that the internal temperature of the dough is $4°\mathrm{C}$. She puts it into the oven which has been pre-heated to a constant temperature of $180°\mathrm{C}$ and forgets to set the timer on her handphone. She also does not note the time. At 10.06 am, she realizes that the timer has not been set. She checks her dough which now has an internal temperature of $80°\mathrm{C}$. Twenty minutes later, she checks the dough again and the internal temperature has risen to $130°\mathrm{C}$.

Newton's Law of Cooling states that the rate of increase of the temperature $\theta°\mathrm{C}$ of an object after t minutes is directly proportional to the difference in the temperatures of the object and its surrounding.

The dough has to be baked at a constant oven temperature of $180°\mathrm{C}$ for one and a half hour to cook through. By forming a differential equation, find the time (to the nearest minute) that Ms Frugal should remove the dough from the oven. [8]

10 Two children Hansel and Gretel are participating in the junior category of the Festival of Lights competition. They decided to set up a structure consisting of a pyramid with a rectangular base *OABC* and vertex *V*. The main power supply switch is positioned at *O* with coordinates $(0, 0, 0)$ and relative to *O*, the coordinates of *A* and *C* are $(1, 2, 1)$ and $(-1, 1, -1)$ respectively. The vertex *V* is fixed at a height $\sqrt{8}$ metres above the rectangular base *OABC* such that it is equidistant to the points *O*, *A*, *B* and *C*. All the surfaces of the pyramid, excluding the base *OABC*, are completely covered with LED (light emitting diodes) light strips so that it illuminates in the dark.

(i) Point *D* is the foot of the perpendicular from *V* to the rectangular base *OABC*. Show that the coordinates of *D* is $(0, 1.5, 0)$. [1]

(ii) Find a vector perpendicular to the rectangular base *OABC*. Hence, find the position vector of *V*, given that its **k**-component is positive. [4]

Hansel and Gretel decide to install two different colour display schemes on the pyramid and the control switch is to be installed inside the pyramid at a point *E* with position vector $\alpha\mathbf{i} + 1.5\mathbf{j} - \alpha\mathbf{k}$, where α is a constant.

(iii) Show that *VE* is perpendicular to the rectangular base *OABC* and explain why $-2 < \alpha < 0$. [3]

(iv) Find the distance between *E* and the surface *OVA* in terms of α, simplifying your answer. [4]

(v) Given that the ratio of the distance between *E* and the rectangular base *OABC* to the distance between *E* and the surface *OVA* is $10 : \sqrt{105}$, find the value of α. [2]

********* End of Paper *********

2018 RI Prelim Exam Paper 1

1 Adult tickets for a parade are sold at three different prices, depending on the type of seats.

Children under the age of 12 and Senior Citizens aged 65 and above can enjoy 20% and 10% off the adult ticket prices respectively. Those who are between 12 and 64 years old (inclusive) will have to pay the full adult ticket price.

The number of tickets sold in each category for each group of people, together with the total cost of the tickets for each group, are given in the following table.

	Category 1 tickets (premier)	Category 2 tickets (sheltered)	Category 3 tickets (unsheltered)	Total cost
Adults (12 to 64 years old (inclusive))	150	120	200	\$18 000
Children (under 12 years old)	80	40	100	\$6 760
Senior Citizens (65 years old and above)	100	100	100	\$10 665

Write down and solve equations to find the price of an adult ticket for each of the ticket categories. [4]

1 **[4]**	Let the cost of an adult ticket for Category 1, 2 and 3 be \$$x$, \$$y$ and \$$z$ respectively. $150x+120y+200z=18000$ $80(0.8x)+40(0.8y)+100(0.8z)=6760$ $100(0.9x)+100(0.9y)+100(0.9z)=10665$ $x=50,\quad y=40,\quad z=28.5$ The cost of a Category 1, 2 and 3 adult ticket is \$50, \$40 and \$28.50 respectively.

2 Aden, a member of Raffles Art Club, is helping to design a graphic for the school's homecoming event. The main body of the graphic consists of N concentric circles C_1, C_2, $C_3, \ldots,$ C_N with radii $r,\ 2r,\ 3r, \ldots,\ Nr$, where r is a constant and N is an even integer, as shown in Fig. 1.

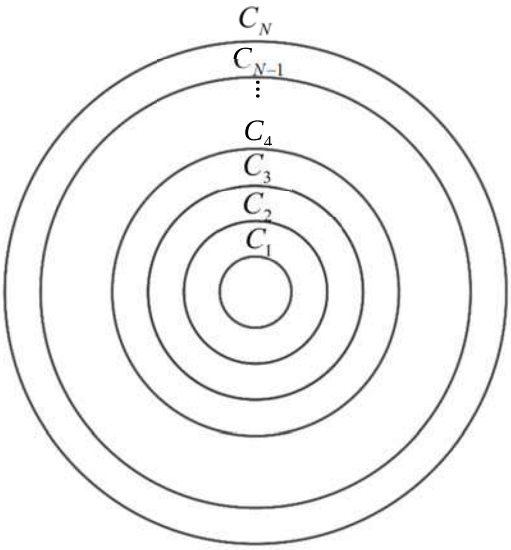


Fig. 1

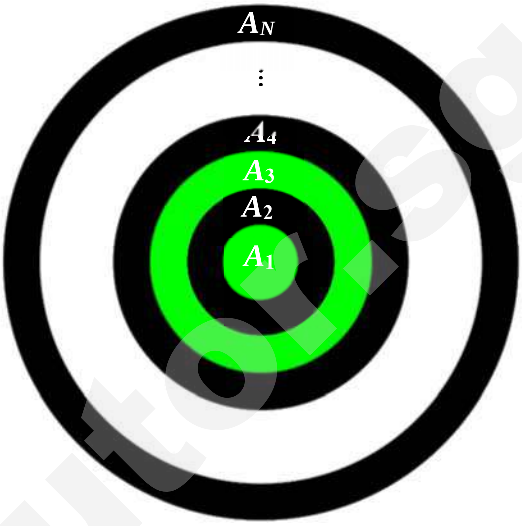


Fig. 2

The region enclosed by the circle C_1 is denoted by A_1 with area a_1 and the region between the circles C_{n-1} and C_n is denoted by A_n with area a_n for $n = 2,3,4,...,N$, as shown in Fig. 2.

(i) Show that the sequence $a_1,\ a_2,\ a_3, \ldots,\ a_N$ is an arithmetic progression. [4]

With the help of a graphic software, Aden fills $A_1,\ A_2,\ A_3, \ldots,\ A_N$ with two of the school colours: green and black. A_n will be filled green if n is odd and will be filled black if n is even.

Aden wishes to create a better visual effect by having different intensities of green. He fills A_1 with the green colour that is the same as the school colour. He reduces the intensity of this green colour by ten percent to fill A_3; and reduces the intensity of the green colour used to fill A_3 by ten percent to fill A_5; and this process continues for all odd values of n. When he finishes filling up all the areas, Aden finds that the intensity of the green colour that fills A_{N-1} falls below one quarter of the intensity of the school colour for the first time.

(ii) Find the value of N. [4]

2(i) [4]	$a_n = \pi(nr)^2 - \pi[(n-1)r]^2 \quad$ for $n = 2,3,...,N$ $= \pi[nr-(n-1)r][nr+(n-1)r]$ $= \pi(r)(2nr-r)$ $= \pi r^2(2n-1)$ $a_1 = \pi r^2 = \pi r^2(2\times1-1)$ and it follows the form $A_n = \pi r^2(2n-1)$ when	Read word problems carefully and clarify understanding of the symbols used.

$n = 1$.

Therefore, $A_n = \pi r^2(2n-1)$ for $n = 1, 2, 3, \ldots, N$.

The difference between any two consecutive terms (for $n = 2,3,\ldots,N$)

$$= a_n - a_{n-1}$$
$$= [\pi r^2(2n-1)] - \{\pi r^2[2(n-1]-1\}$$
$$= 2\pi r^2$$

Since $2\pi r^2$ is independent of n, it is a constant.

Since the difference between every two consecutive terms is constant, the sequence $a_1, a_2, a_3, \ldots, a_N$ is an arithmetic progression. (shown)

Candidates mixed up the A_i's (name of regions) with the a_i's (area of the regions), incorrectly labelling $\pi r^2(2n-1) = A_n - A_{n-1}$ as a constant.

Checking the **first few terms** of a sequence share a common difference to conclude that the sequence forms an arithmetic progression is **NOT** allowed

(ii) **[4]**

The intensity of green colour in area $A_1, A_3, A_5, \ldots, A_n, \ldots, A_{N-1}$, where n is an odd number, is:

$1,\ 0.9,\ 0.9^2\ \ldots,\ 0.9^{\frac{N-2}{2}}$, which is a geometric progression

$$0.9^{\frac{N-2}{2}} < 0.25$$
$$\frac{N-2}{2} > \frac{\ln 0.25}{\ln 0.9}$$
$$N > 28.315 \quad (5\text{s.f.})$$

Since N is an even integer, $N = 30$.

Alternative 1

Since $\frac{N-2}{2}$ is an integer and $\frac{N-2}{2} > \frac{\ln 0.25}{\ln 0.9} = 13.158 \quad (5\text{s.f.})$,

then $\frac{N-2}{2} = 14 \Rightarrow N = 30$.

Alternative 2

X	Y1
26	0.2824
27	0.2679
28	0.2542
29	0.2411
30	0.2288
31	0.217
32	0.2059
33	0.1953
34	0.1853
35	0.1758
36	0.1668

X=26

From GC, (N is an even integer)

when $N = 28$, $\quad 0.9^{\frac{N-2}{2}} = 0.2542 > 0.25$;

when $N = 30$, $\quad 0.9^{\frac{N-2}{2}} = 0.2288 < 0.25$.

$\therefore N = 30$

Candidate should take note of the alternative solutions.

Two common mistakes committed were failing to flip the inequality sign when dividing by ln(0.9), a negative number, and forgetting that N is an even number, thus concluding that the least value of N is 29 instead of 30.

	Alternative 3 Let the intensity of the green-colour regions form a geometric progression of the form $x,\ 0.9x,\ 0.9^2x,\ \ldots, 0.9^{n-1}x,\ \ldots$ where the intensity for region n is given by $0.9^{n-1}x$. We want $0.9^{n-1}x < 0.25x$ $n-1 > \dfrac{\ln 0.25}{\ln 0.9}$ $n > 14.158 \quad (5\text{s.f.})$ The least value of n is 15. Hence the first region that satisfies the condition is the 15th odd region, A_{29}, and N is 30.	

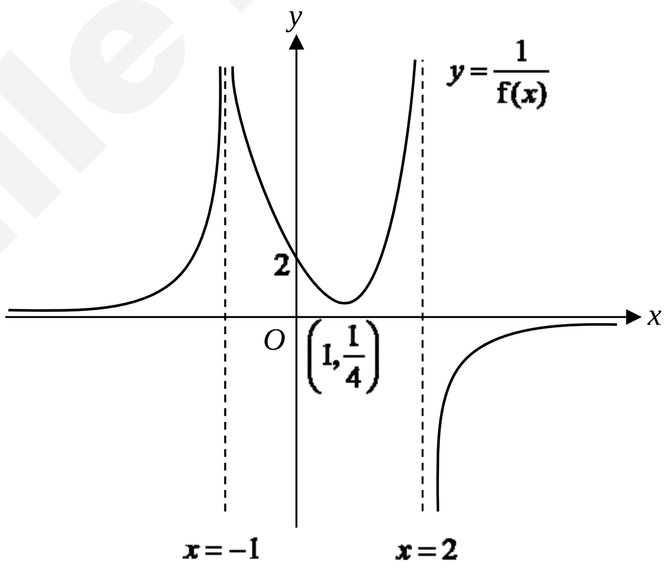
3 **(a)** Find $\int_0^1 \frac{9x-4x^2}{9-4x^2}\,dx$, giving your answer in the form $p+q\ln 3+r\ln 5$ where p, q and r are rational numbers to be determined. [5]

(b) Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.
Hence, or otherwise, find $\int \sin x \sin 2x \sin 3x\,dx$. [4]

3(a) **[5]**	$\int_0^1 \frac{9x-4x^2}{9-4x^2}\,dx$ $=\int_0^1 \left(1+\frac{9x-9}{9-4x^2}\right)dx$ $=1+9\int_0^1 \frac{x}{9-4x^2}\,dx - 9\int_0^1 \frac{1}{9-4x^2}\,dx$ $=1+9\left[-\frac{1}{8}\ln\left(9-4x^2\right)-\frac{1}{12}\ln\left(\frac{3+2x}{3-2x}\right)\right]_0^1$ $=1+\frac{9}{8}\ln\frac{9}{5}-\frac{3}{4}\ln 5$ $=1+\frac{9}{4}\ln 3-\frac{15}{8}\ln 5$	Pay attention to coefficients when integrating. 1st integral is of the form $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$ 2nd integral is of the form $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert + c$ **Check by differentiation.**
3(b) **[4]**	$\sin 3x = \sin(2x+x)$ $= \sin 2x\cos x + \cos 2x \sin x$ $= 2\sin x\cos^2 x + (1-2\sin^2 x)\sin x$ $= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$ $= 3\sin x - 4\sin^3 x$ $\int \sin x \sin 2x \sin 3x\,dx$ $=\int \sin x(2\sin x\cos x)(3\sin x - 4\sin^3 x)\,dx$ $=6\int \sin^3 x\cos x\,dx - 8\int \sin^5 x\cos x\,dx$ $=\frac{3}{2}\sin^4 x - \frac{4}{3}\sin^6 x + c$	Addition formula and double angle formula for sine and cosine. (MF 26) Don't forget to include the arbitrary constant c.

	Alternatively $\int \sin x \sin 2x \sin 3x \, dx$ $= -\frac{1}{2}\int (\cos 4x - \cos 2x) \sin 2x \, dx$ $= -\frac{1}{2}\int (\sin 2x \cos 4x - \sin 2x \cos 2x) \, dx$ $= -\frac{1}{2}\int (\sin 2x(2\cos^2 2x - 1) - \sin 2x \cos 2x) \, dx$ $= -\int \sin 2x \cos^2 2x \, dx + \frac{1}{2}\int \sin 2x \, dx + \frac{1}{4}\int \sin 4x \, dx$ $= -\int \sin 2x \cos^2 2x \, dx + \frac{1}{2}\int \sin 2x \, dx + \frac{1}{4}\int \sin 4x \, dx$ $= \frac{1}{6}\cos^3 2x - \frac{1}{4}\cos 2x - \frac{1}{16}\cos 4x + c$	

4 **(a)** The diagram below shows the curve of $y = \frac{1}{f(x)}$ where $f(x)$ is a polynomial. The curve has a minimum point at $\left(1, \frac{1}{4}\right)$ and cuts the y-axis at $(0, 2)$. The lines $x = -1$ and $x = 2$ are the vertical asymptotes and the line $y = 0$ is the horizontal asymptote to the curve.

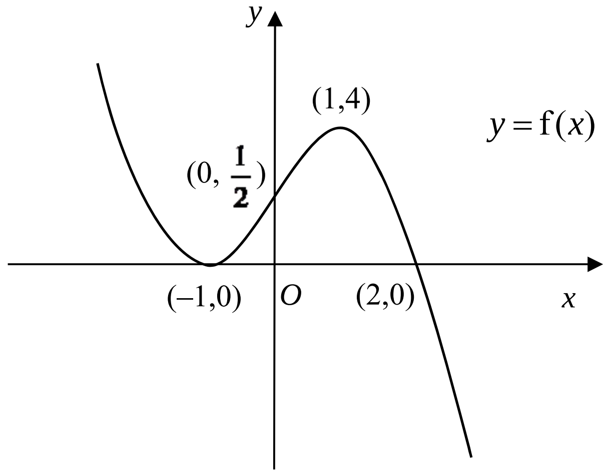
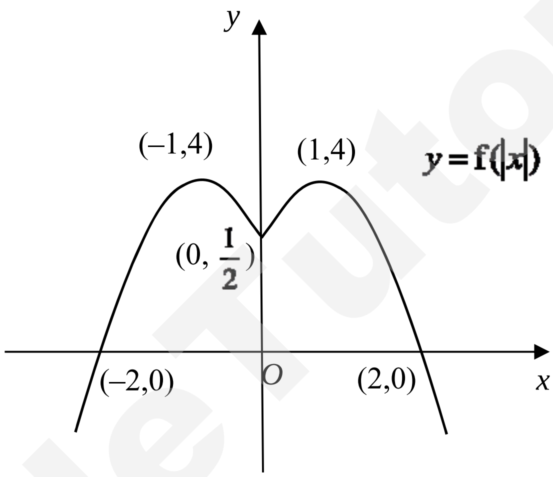


Sketch on separate diagrams, the graphs of

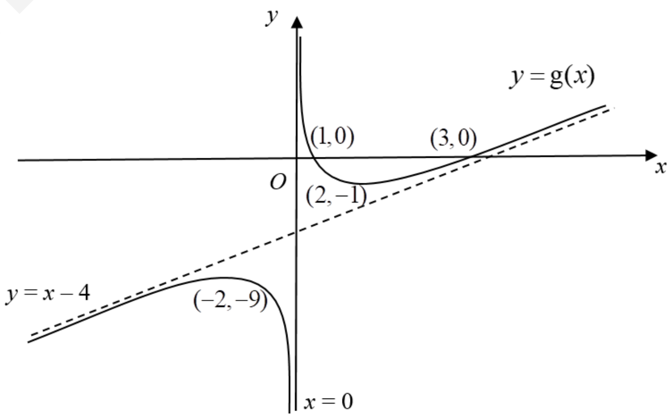
(i) $y = f(x)$,

(ii) $y = f(|x|)$,

Labelling all relevant point(s).

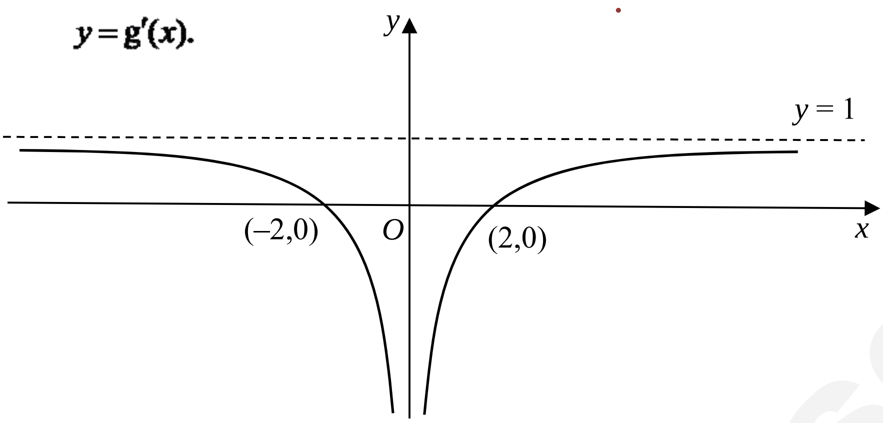
4a (i) [3]		The fact that $\mathrm{f}(x)$ is a polynomial means that the curve will be smooth. (no “holes” or “open circles” at the x-intercepts)
(ii) **[2]**		

(b) The diagram below shows the curve of $y = \mathrm{g}(x)$. The curve has a maximum point at $(-2,-9)$ and a minimum point at $(2,-1)$. The curve crosses the x-axis at $(1,0)$ and $(3,0)$. The line $x = 0$ is the vertical asymptote and the line $y = x - 4$ is the oblique asymptote to the curve.

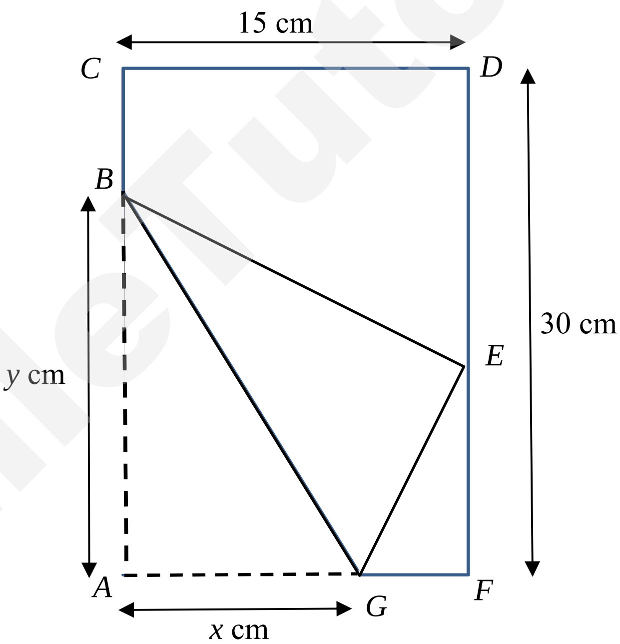


(i) Sketch the graph of $y = \mathrm{g}'(x)$, labelling all relevant point(s) and stating the equations of any asymptotes.

(ii) Find the area bounded by the graph of $y = \mathrm{g}'(x)$, the lines $x = 1$, $x = 2$ and the x-axis. [2]

(b) **[4]**	 Required area $= \int_1^2 g'(x)\,dx = -[g(x)]_1^2$ $= -[g(2) - g(1)]$ $= -[-1-0] = 1$	Key features of the derivative graph, such as the asymptotes and intercepts, must be included. The Fundamental Theorem of Calculus provides the link between the derivative function and integral. Note the negative sign for the required area.

5



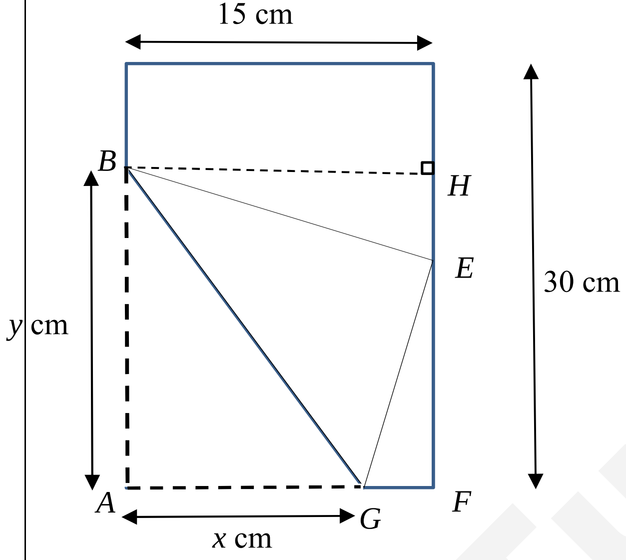
Tom has a rectangular piece of paper *ACDF* with length 30 cm and breadth 15 cm. He folds the lower left-hand corner, *A*, to reach the rightmost edge of the paper at *E*. After that he will cut out the triangle *EFG* and the trapezium *BCDE* to obtain a kite shaped figure *ABEG* with *AB* of length y cm and *AG* of length x cm.

(i) Find the length of *EF* in terms of x. [2]

(ii) Show that $y = \dfrac{x\sqrt{15}}{\sqrt{2x-15}}$. [2]

(iii) Using differentiation, find the exact values of x and y which give the minimum area of the kite *ABEG*. [5]

5(i)	Let the length of *EF* be w	Pythagoras

[2]	$w^2 = x^2 - (15-x)^2$ $w^2 = x^2 - (15^2 - 30x - x^2)$ $\Rightarrow w^2 = 30x - 15^2$ $\Rightarrow w = \sqrt{15}(2x-15)^{\frac{1}{2}}$	Theorem.
(ii) **[2]**	<u>Method 1 (similar triangles)</u>  Triangle *BHE* is similar to triangle *EFG* Therefore $\dfrac{BE}{EG} = \dfrac{BH}{EF} \Rightarrow \dfrac{y}{x} = \dfrac{15}{\sqrt{15}(2x-15)^{\frac{1}{2}}}$ $\Rightarrow y = \dfrac{x\sqrt{15}}{\sqrt{(2x-15)}}$ <u>Method 2 (Pythagoras Theorem)</u> $BH^2 + HE^2 = BE^2$ $15^2 + (y - EF)^2 = y^2$ $225 + (y - \sqrt{15}\sqrt{(2x-15)})^2 = y^2$ $225 + y^2 - 2y\sqrt{15}\sqrt{(2x-15)} + 15(2x-15) = y^2$ $-2y\sqrt{15}\sqrt{(2x-15)} + 30x = 0$ $y = \dfrac{x\sqrt{15}}{\sqrt{(2x-15)}}$	Candidates to note the alternative methods. Note that this is a "show" question so working or justification "by similar triangles …" needs to be explicitly stated.
(iii) **[5]**	Area of kite *ABEG* $= xy$ <u>Method 1 (find stationary point)</u> $\therefore \quad A = \dfrac{x^2\sqrt{15}}{\sqrt{2x-15}}$	

$$\Rightarrow A^2 = \frac{15x^4}{2x-15}$$

$$\Rightarrow 2A\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{60x^3(2x-15)-2(15x^4)}{(2x-15)^2}$$

For stationary value, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$

$$\Rightarrow 30x^3\left[2(2x-15)-x\right]=0$$

$$\Rightarrow \quad x=0 \text{ (reject)} \quad \text{or} \quad x=10$$

When $x=10$, $y=\frac{10\sqrt{15}}{\sqrt{5}}=10\sqrt{3}$

<u>Method 2 (find stationary point)</u>

$$\therefore \quad A=\frac{x^2\sqrt{15}}{\sqrt{2x-15}}=\sqrt{15}x^2(2x-15)^{-\frac{1}{2}}$$

$$\therefore \quad \frac{\mathrm{d}A}{\mathrm{d}x}=2\sqrt{15}x(2x-15)^{-\frac{1}{2}}+\sqrt{15}x^2\left[-\frac{1}{2}(2x-15)^{-\frac{3}{2}}(2)\right]$$

$$\Rightarrow \quad \frac{\mathrm{d}A}{\mathrm{d}x}=x\sqrt{15}(2x-15)^{-\frac{3}{2}}\left[2(2x-15)-x\right]$$

$$\Rightarrow \quad \frac{\mathrm{d}A}{\mathrm{d}x}=x\sqrt{15}(2x-15)^{-\frac{3}{2}}\left[3x-30\right]$$

For stationary value, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$

$$\Rightarrow \quad x\sqrt{15}(2x-15)^{-\frac{3}{2}}\left[3x-30\right]=0$$

$$\Rightarrow \quad x=0 \text{ (reject)} \quad \text{or} \quad x=10$$

When $x=10$, $y=\frac{10\sqrt{15}}{\sqrt{5}}=10\sqrt{3}$

Method 1 (Determine nature of stationary point)

Using Second Derivative Test

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=\sqrt{15}\left(-\frac{3}{2}\right)(2x-15)^{-\frac{5}{2}}\left[3x^2-30x\right]+\sqrt{15}(2x-15)^{-\frac{3}{2}}\left[6x-30\right]$$

$=10.3923>0$

Therefore $x=10$, $y=10\sqrt{3}$ gives the minimum area of the kite.

Method 2 (Determine nature of stationary point)

Using first derivative test

	$x=10^-$	$x=10$	$x=10^+$
x	+ve	+ve	+ve
$3(x-10)$	−ve	0	+ve
$\frac{\mathrm{d}A}{\mathrm{d}x}=x\sqrt{15}(2x-15)^{-\frac{3}{2}}[3x-30]$	−ve	0	+ve

We conclude that $x=10$, $y=10\sqrt{3}$ gives the minimum area of the kite.

	Marker Comments for Students (iii) Candidate should read the question carefully as many erroneously minimized y instead of the area. Many students also have the **misconception** that since the question asked for a minimum area, they do not have to verify it. **Do note that for any max/min questions, showing that it is a min/max is a MUST.** 2nd Derivative test might be easier here. For students who use the first derivative test, you need to identify the factor that resulted in the sign change. This table alone is not good enough.

	$x=10^-$	$x=10$	$x=10^+$
$\dfrac{dA}{dx}=x\sqrt{15}\left(2x-15\right)^{-\frac{3}{2}}\left[3x-30\right]$	$-$ve	0	+ve

6 **(a)** It is given that $e^y=\tan\left(x+\dfrac{\pi}{3}\right)$.

(i) Show that $\dfrac{dy}{dx}=e^y+e^{-y}$. [2]

(ii) Using differentiation, find the Maclaurin series for $\ln\left[\tan\left(x+\dfrac{\pi}{3}\right)\right]$ up to and including the term in x^2. [3]

(b) Let $f\left(x\right)=\dfrac{3x^2-4x+5}{\left(1+x\right)\left(1-x\right)^2}$.

(i) Express $f\left(x\right)$ in the form $\dfrac{A}{\left(1+x\right)}+\dfrac{B}{\left(1-x\right)}+\dfrac{C}{\left(1-x\right)^2}$ where A, B and C are constants to be determined. [2]

(ii) Hence find the expansion of $f\left(x\right)$ up to and including the term in x^4. [2]

(iii) Write down the coefficient of x^r in the expansion of $f\left(x\right)$ in terms of r. [1]

6a(i) **[2]**	$e^y=\tan\left(x+\dfrac{\pi}{3}\right)$ Differentiate with respect to x $e^y\dfrac{dy}{dx}=\sec^2\left(x+\dfrac{\pi}{3}\right)$ $e^y\dfrac{dy}{dx}=1+\tan^2\left(x+\dfrac{\pi}{3}\right)$ $e^y\dfrac{dy}{dx}=1+e^{2y}$ $\dfrac{dy}{dx}=e^y+e^{-y}$	Implicit differentiation/ chain rule $\dfrac{d}{dx}e^y=e^y\dfrac{dy}{dx}$ and $\dfrac{d}{dx}e^{-y}=-e^{-y}\dfrac{dy}{dx}$. Recall the trigo identity $\sec^2\theta=\tan^2\theta+1$

(ii) **[3]**	$\frac{dy}{dx}=e^{y}+e^{-y}$ $\frac{d^2y}{dx^2}=e^{y}\frac{dy}{dx}-e^{-y}\frac{dy}{dx}$ When $x=0$, $e^{y}=\tan\frac{\pi}{3}\Rightarrow y=\ln\left(\sqrt{3}\right)=\frac{1}{2}\ln 3$ $\frac{dy}{dx}=e^{\frac{1}{2}\ln 3}+e^{-\frac{1}{2}\ln 3}\Rightarrow\frac{dy}{dx}=\sqrt{3}+\frac{1}{\sqrt{3}}=\frac{4}{\sqrt{3}}$ $\frac{d^2y}{dx^2}=\frac{4}{\sqrt{3}}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)=\frac{8}{3}$ $\ln\left[\tan\left(x+\frac{\pi}{3}\right)\right]=\frac{1}{2}\ln 3+\frac{4}{\sqrt{3}}x+\frac{\frac{8}{3}}{2!}x^2+\cdots=\frac{1}{2}\ln 3+\frac{4}{\sqrt{3}}x+\frac{4}{3}x^2$	Note the significance of finding the expression in part (i) here to find $\frac{d^2y}{dx^2}$ Simplify all algebraic expressions.
b(i) **[2]**	$\frac{3x^2-4x+5}{(1+x)(1-x)^2}=\frac{A}{(1+x)}+\frac{B}{(1-x)}+\frac{C}{(1-x)^2}$ $3x^2-4x+5=A(1-x)^2+B(1+x)(1-x)+C(1+x)$ When $x=1$, $3-4+5=2C\Rightarrow C=2$ $x=-1$, $3+4+5=4A\Rightarrow A=3$ $x=0$, $5=A+B+C\Rightarrow B=0$	Candidates must show more care in getting the correct partial fractions.
(ii) **[2]**	$\frac{3}{(1+x)}+\frac{2}{(1-x)^2}=3(1+x)^{-1}+2(1-x)^{-2}$ $=3\left(1-x+x^2-x^3+x^4+\cdots\right)$ $+2\left(1+2x+\frac{-2(-3)}{2!}x^2+\frac{-2(-3)(-4)}{3!}(-x)^3+\frac{-2(-3)(-4)(-5)}{4!}(-x)^4+\cdots\right)$ $=3\left(1-x+x^2-x^3+x^4+\cdots\right)+2\left(1+2x+3x^2+4x^3+5x^4+\cdots\right)$ $=5+x+9x^2+5x^3+13x^4+\cdots$	Binomial series expansion from MF26.
(iii) **[1]**	The coefficient of x^r is $=3(-1)^r+2(r+1)$	Candidates to note general term.

7 **Do not use a calculator in answering this question.**

(a) Find the roots of the equation $w^2(1-i)+4w+(10+10i)=0$, giving your answers in cartesian form $a+ib$. [3]

(b) It is given that $z=-\sqrt{3}+i$.

(i) Find an exact expression for z^5. Give your answer in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [2]

(ii) Find the three smallest positive whole number values of n for which $\frac{z^n}{iz^*}$ is purely imaginary. [4]

(iii) Given that $\left|1+\frac{p}{z^*}\right|=\sqrt{7}$, find exactly the possible values of the real number p. [3]

7(a) **[3]**	$w^2(1-\mathrm{i})+4w+(10+10\mathrm{i})=0$ $w=\frac{-4\pm\sqrt{4^2-4(1-\mathrm{i})(10+10\mathrm{i})}}{2(1-\mathrm{i})}$ $=\frac{-4\pm\sqrt{16-40(1-\mathrm{i})(1+\mathrm{i})}}{2(1-\mathrm{i})}$ $=\frac{-4\pm\sqrt{16-40(2)}}{2(1-\mathrm{i})}$ $=\frac{-4\pm\sqrt{-64}}{2(1-\mathrm{i})}$ $=\frac{-4\pm 8\mathrm{i}}{2(1-\mathrm{i})}\times\frac{1+\mathrm{i}}{1+\mathrm{i}}$ $=(-1\pm 2\mathrm{i})(1+\mathrm{i})$ $w=(-1+2\mathrm{i})(1+\mathrm{i})$ or $w=(-1-2\mathrm{i})(1+\mathrm{i})$ $=-1-\mathrm{i}+2\mathrm{i}+2\mathrm{i}^2$ $\quad$ $=-1-\mathrm{i}-2\mathrm{i}-2\mathrm{i}^2$ $=-3+\mathrm{i}$ $\quad$ $=1-3\mathrm{i}$	Use quadratic formula. Recall Y6 CT2. The simplification of the intermediate answers $\frac{-4\pm 8\mathrm{i}}{2(1-\mathrm{i})}$ must be shown as the use of calculator is not allowed.
(b) **(i)** **[2]**	Given $z=-\sqrt{3}+\mathrm{i}$, $\lvert z\rvert=2$, $\arg(z)=\frac{5\pi}{6}$ Method 1 : $\lvert z^5\rvert=2^5=32$ $\arg(z^5)=5\arg(z)-4\pi$ $=5\left(\frac{5\pi}{6}\right)-4\pi$ $=\frac{25\pi}{6}-4\pi$ $=\frac{\pi}{6}$ $\therefore z^5=32\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ Method 2 : $z^5=\left(2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}\right)^5$ $=2^5\mathrm{e}^{\mathrm{i}\left(\frac{25\pi}{6}-4\pi\right)}$ $=32\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$	

(ii) **[4]**	$\frac{z^n}{\mathrm{i}z^*}=\frac{2^n\mathrm{e}^{\mathrm{i}\frac{5n\pi}{6}}}{\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}2\mathrm{e}^{-\mathrm{i}\frac{5\pi}{6}}}$ $=2^{n-1}\mathrm{e}^{\mathrm{i}\left(\frac{5n\pi}{6}-\frac{\pi}{2}+\frac{5\pi}{6}\right)}$ $=2^{n-1}\mathrm{e}^{\mathrm{i}\frac{(5n+2)\pi}{6}}$ Alternatively : Just determine $\arg\left(\frac{z^n}{\mathrm{i}z^*}\right)$ $n\arg(z)-\arg(\mathrm{i})-\arg(z^*)$ $=n\left(\frac{5\pi}{6}\right)-\frac{\pi}{2}+\frac{5\pi}{6}$ since $\arg(z^*)=-\arg(z)=-\frac{5\pi}{6}$ $=\left(\frac{5n+2}{6}\right)\pi$ For $\frac{z^n}{\mathrm{i}z^*}$ to be purely imaginary, $\cos\left(\frac{5n+2}{6}\right)\pi=0$ $\left(\frac{5n+2}{6}\right)\pi=(2k+1)\frac{\pi}{2}\quad,\ k\in\mathbb{Z}$ $5n+2=3(2k+1)$ $n=\frac{6k+1}{5}$ $\therefore$ The three smallest positive whole number values of n are $5,11,17$. **OR alternatively** $\frac{z^n}{\mathrm{i}z^*}=\frac{-\mathrm{i}z^{n+1}}{\lvert z\rvert^2}=-2^{n-1}\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{5(n+1)\pi}{6}}$ For $\frac{z^n}{\mathrm{i}z^*}$ to be purely imaginary, $\sin\left(\frac{5n+5}{6}\right)\pi=0\quad,\ k\in\mathbb{Z}$ $\left(\frac{5n+5}{6}\right)\pi=k\pi\quad,\ k\in\mathbb{Z}$ $5n=6k-5$ $n=\frac{6k}{5}-1$ $\therefore$ The three smallest positive whole number values of n are $5,11,17$.	Note that n is a positive integer.

(iii)	$\left\|1+\frac{p}{z^*}\right\| = \sqrt{7} \Leftrightarrow \left\|\frac{z^*+p}{z^*}\right\| = \sqrt{7}$ $\frac{\left\|-\sqrt{3}-\mathrm{i}+p\right\|}{\left\|z^*\right\|} = \sqrt{7}$ $\left\|\left(-\sqrt{3}+p\right)-\mathrm{i}\right\| = 2\sqrt{7}$ since $\left\|z^*\right\| = \left\|z\right\| = 2$ $\left(-\sqrt{3}+p\right)^2 + (-1)^2 = \left(2\sqrt{7}\right)^2$ $\left(-\sqrt{3}+p\right)^2 = 27$ $-\sqrt{3}+p = \pm\sqrt{27}$ $p = \sqrt{3} \pm 3\sqrt{3}$ $\therefore p = 4\sqrt{3}$ or $-2\sqrt{3}$	Simplify algebraic expressions
	Marker Comments for Students **(iii)** Candidates should understand why (a) $\left\|1+\frac{p}{z^*}\right\| \neq 1+\left\|\frac{p}{z^*}\right\|$ (b) $\left\|1+\frac{p}{z^*}\right\|^2 \neq \left(1+\frac{p}{z^*}\right)^2$ (c) $\left\|1+\frac{p}{z^*}\right\| = \sqrt{7}$ does not imply $1+\frac{p}{z^*} = \pm\sqrt{7}$	

8 The function $\mathrm{f}(x) = a-(1+x)^{\frac{1}{3}}$ is defined for $0 \le x \le 7$, where a is a positive real constant.

(i) Solve the inequality $\mathrm{f}(x) \ge a-1$. [2]

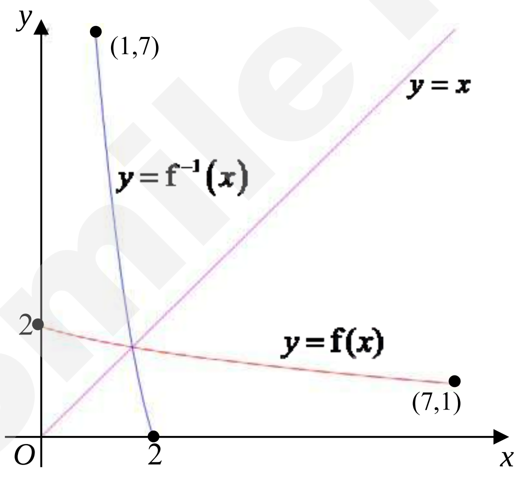
(ii) Show that the graph of $y = \mathrm{f}(x)$ has a negative gradient at all points on the graph. Find the range of f. [2]

Use $a = 3$ in the rest of the question.

(iii) Find $\mathrm{f}^{-1}(x)$. [2]

(iv) Sketch on the same diagram the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, showing clearly the geometrical relationship between the two graphs and the line $y = x$. [3]

(v) Show that the x-coordinate of the point of intersection of the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ satisfies the equation $x^3 - 9x^2 + 28x - 26 = 0$.
Hence find the solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

8(i) **[2]**	$a-(1+x)^{\frac{1}{3}} \geq a-1$ $(1+x)^{\frac{1}{3}} \leq 1$ $x \leq 0$ But $0 \leq x \leq 7$, Solution is $x = 0$	Note the restriction on solutions for *x* because of the domain of f.
(ii) **[2]**	$f'(x)=-\frac{1}{3}(1+x)^{-\frac{2}{3}}=-\frac{1}{3}\left(\frac{1}{\sqrt[3]{(1+x)}}\right)^2<0$ Thus, the graph of $y=f(x)$ has a negative gradient at all points on the graph Since f is continuous and decreasing over $0 \leq x \leq 7$, $f(7) \leq f(x) \leq f(0)$ $a-2 \leq f(x) \leq a-1$ $\therefore R_f=[a-2,a-1]$	
(iii) **[2]**	$y=3-(1+x)^{\frac{1}{3}}$ $1+x=(3-y)^3$ $x=(3-y)^3-1$ $f^{-1}(x)=(3-x)^3-1$	**Make x the subject**
(iv) **[3]**		Use the same scale along the x and y axis. Note the domain of f and indicate the end points. Draw a sketch so that the curves appear as reflections about y=x. Make sure the graphs do not "violate the vertical line test"
(v) **[3]**	From the graphs of $y = f(x)$ and $y=f^{-1}(x)$, we see that $f(x)=f^{-1}(x)$ exactly where $f^{-1}(x)=x$. $(3-x)^3-1=x$ $27-27x+9x^2-x^3-1=x$ $x^3-9x^2+28x-26=0$ (shown) From GC, $x = 1.62$ (3sf)	**Do not include complex solutions!**

9 **(a)** The variables x and y are related by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{x}e^{\sqrt{x}}$.

Using the substitution $u=\sqrt{x}$, show that $y=k\int u^2e^u\,\mathrm{d}u$ where k is a constant to be determined. Hence find the general solution of the differential equation.[5]

(b) Ms Frugal bought a second-hand oven which has a broken timer. Each time she bakes she uses the timer on her handphone. One day, she decides to make a loaf of bread. She takes the fermented dough out of her refrigerator and checks that the internal temperature of the dough is $4°\mathrm{C}$. She puts it into the oven which has been pre-heated to a constant temperature of $180°\mathrm{C}$ and forgets to set the timer on her handphone. She also does not note the time. At 10.06 am, she realizes that the timer has not been set. She checks her dough which now has an internal temperature of $80°\mathrm{C}$. Twenty minutes later, she checks the dough again and the internal temperature has risen to $130°\mathrm{C}$.

Newton's Law of Cooling states that the rate of increase of the temperature $\theta°\mathrm{C}$ of an object after t minutes is directly proportional to the difference in the temperatures of the object and its surrounding.

The dough has to be baked at a constant oven temperature of $180°\mathrm{C}$ for one and a half hour to cook through. By forming a differential equation, find the time (to the nearest minute) that Ms Frugal should remove the dough from the oven.[8]

9(a) **[5]**	$u=\sqrt{x}\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}\Rightarrow\mathrm{d}x=2u\,\mathrm{d}u$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{x}e^{\sqrt{x}}$ $y=\int\sqrt{x}e^{\sqrt{x}}\,\mathrm{d}x$ $=\int ue^u\cdot2u\,\mathrm{d}u$ $=2\int u^2e^u\,\mathrm{d}u$ $=2\left(u^2e^u-2\int ue^u\,\mathrm{d}u\right)$ $=2u^2e^u-4\left(ue^u-\int e^u\,\mathrm{d}u\right)$ $=2u^2e^u-4ue^u+4e^u+c$ $=2xe^{\sqrt{x}}-4\sqrt{x}e^{\sqrt{x}}+4e^{\sqrt{x}}+c$	Substitution technique for DE Integration by parts Final expression for y in terms of x
9(b) **[8]**	$\frac{\mathrm{d}\theta}{\mathrm{d}t}=k(180-\theta)$ where k is positive $\int\frac{1}{180-\theta}\,\mathrm{d}\theta=\int k\,\mathrm{d}t$ $-\ln(180-\theta)=kt+c,\quad$ where $\theta<180$ When $t=0,\theta=4$, $c=-\ln176$ $\therefore-\ln(180-\theta)=kt-\ln176$	Minus sign or $+c$ being omitted in $-\ln(180-\theta)=kt+c$ should be avoided. Absolute value for $\lvert180-\theta\rvert$ is not

	Let t_0 be the time elapsed from $4°\text{C}$ to $80°\text{C}$. When $t = t_0, \theta = 80$, $-\ln 100 = kt_0 - \ln 176 \Rightarrow kt_0 = \ln\dfrac{44}{25}$ ...(1) When $t = t_0 + 20, \theta = 130$, $-\ln 50 = k(t_0+20) - \ln 176 \Rightarrow k(t_0+20) = \ln\dfrac{88}{25}$ ...(2) By (1) and (2), $\dfrac{t_0}{t_0+20} = \dfrac{\ln\frac{44}{25}}{\ln\frac{88}{25}}$ $t_0 = 16.312\,(5\text{s.f})$ Remaining number of minutes in oven after 10.06am $= 90 - 16.312$ $= 73.688\,\text{mins}$ Thus the time that Ms Frugal should take out the dough is 11.20am. **Alternative** $\dfrac{d\theta}{dt} = k(180-\theta)$ where k is positive $\displaystyle\int \frac{1}{180-\theta}\,d\theta = \int k\,dt$ $-\ln(180-\theta) = kt + c,$ where $\theta < 180$ When $t = 0, \theta = 80$, $c = -\ln 100$ $\therefore -\ln(180-\theta) = kt - \ln 100$ When $t = 20, \theta = 130$, $-\ln 50 = k(20) - \ln 100$ $k = \dfrac{\ln 2}{20}$ $\therefore -\ln(180-\theta) = \left(\dfrac{\ln 2}{20}\right)t - \ln 100$ When $\theta = 4$, $-\ln 176 = \left(\dfrac{\ln 2}{20}\right)t - \ln 100$ $t = -16.312$ Remaining number of minutes in oven after 10.06am $= 90 - 16.312$ $= 73.688\,\text{mins}$ Thus the time that Ms Frugal should take out the dough is 11.20am.	necessary since $180-\theta > 0$ from the context of the question. The time elapsed between $4°C$ and $130°C$ should be $t_0 + 20$ where t_0 is the time taken for the temperature to increase from $4°C$ to $80°C$. The time required can be obtained without making θ the subject of the formula.

10 Two children Hansel and Gretel are participating in the junior category of the Festival of Lights competition. They decided to set up a structure consisting of a pyramid with a rectangular base *OABC* and vertex *V*. The main power supply switch is positioned at *O* with coordinates $(0,0,0)$ and relative to *O*, the coordinates of *A* and *C* are $(1,2,1)$ and $(-1,1,-1)$ respectively. The vertex *V* is fixed at a height $\sqrt{8}$ metres above the rectangular base *OABC* such that it is equidistant to the points *O*, *A*, *B* and *C*. All the surfaces of the pyramid, excluding the base *OABC*, are completely covered with LED (light emitting diodes) light strips so that it illuminates in the dark.

(i) Point *D* is the foot of the perpendicular from *V* to the rectangular base *OABC*. Show that the coordinates of *D* is $(0,1.5,0)$. [1]

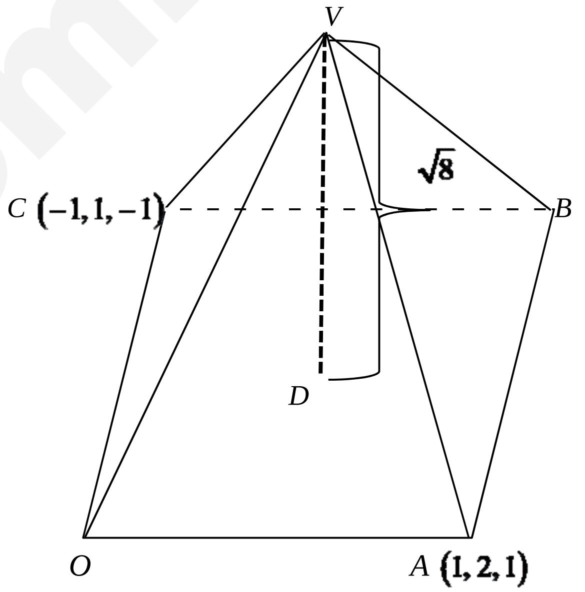
(ii) Find a vector perpendicular to the rectangular base *OABC*. Hence, find the position vector of *V*, given that its **k**-component is positive. [4]

Hansel and Gretel decide to install two different colour display schemes on the pyramid and the control switch is to be installed inside the pyramid at a point *E* with position vector $\alpha\mathbf{i}+1.5\mathbf{j}-\alpha\mathbf{k}$, where α is a constant.

(iii) Show that *VE* is perpendicular to the rectangular base *OABC* and explain why $-2<\alpha<0$. [3]

(iv) Find the distance between *E* and the surface *OVA* in terms of α, simplifying your answer. [4]

(v) Given that the ratio of the distance between *E* and the rectangular base *OABC* to the distance between *E* and the surface *OVA* is $10:\sqrt{105}$, find the value of α. [2]

10(i) **[1]**	 Since *V* is equidistant to the points *O*, *A*, *B* and *C*, and point *D* is the foot of perpendicular from *V* to the rectangular base *OABC*, by symmetry *D* is the centre of the rectangular *OABC*. In other	

	words, D is the midpoint of AC. $\overrightarrow{OD}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OC}\right)=\frac{1}{2}\left[\begin{pmatrix}1\\2\\1\end{pmatrix}+\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right]=\begin{pmatrix}0\\1.5\\0\end{pmatrix}$ $\therefore$ Co-ordinates of D is $(0,1.5,0)$.(shown)	Use the Ratio Theorem to find the midpoint easily.
(ii) **[4]**	A vector perpendicular to $OABC$ $=\overrightarrow{OA}\times\overrightarrow{OC}=\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}-1\\1\\-1\end{pmatrix}=\begin{pmatrix}-3\\0\\3\end{pmatrix}$ Since the **k**-component of V is positive, $\overrightarrow{DV}=\frac{\sqrt{8}}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\0\\2\end{pmatrix}$ $\overrightarrow{DV}=\overrightarrow{OV}-\overrightarrow{OD}$ $\overrightarrow{OV}=\begin{pmatrix}-2\\0\\2\end{pmatrix}+\begin{pmatrix}0\\1.5\\0\end{pmatrix}=\begin{pmatrix}-2\\1.5\\2\end{pmatrix}$	Cross 2 vectors parallel to the plane $OABC$ which are *non-parallel* to each other. Easiest to use the given vectors $\overrightarrow{OA}$ and $\overrightarrow{OC}$. Common mistakes include $\overrightarrow{VD}=\begin{pmatrix}0\\1.5\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $\left\lvert\overrightarrow{OV}\right\rvert=\sqrt{8}$ and $\overrightarrow{OV}=\begin{pmatrix}x\\y\\\sqrt{8}\end{pmatrix}$.
(iii) **[2]**	Method 1: $\overrightarrow{VE}=\overrightarrow{OE}-\overrightarrow{OV}=\begin{pmatrix}\alpha\\1.5\\-\alpha\end{pmatrix}-\begin{pmatrix}-2\\1.5\\2\end{pmatrix}=\begin{pmatrix}\alpha+2\\0\\-\alpha-2\end{pmatrix}=(\alpha+2)\begin{pmatrix}1\\0\\-1\end{pmatrix}$ Since $(\alpha+2)$ is a constant, $\overrightarrow{VE}$ is parallel to $\begin{pmatrix}1\\0\\-1\end{pmatrix}$, a vector normal to the plane $OABC$. Thus VE is perpendicular to the rectangular base $OABC$ (shown). Method 2: $\overrightarrow{VE}.\overrightarrow{OA}=(\alpha+2)\begin{pmatrix}1\\0\\-1\end{pmatrix}.\begin{pmatrix}1\\2\\1\end{pmatrix}=(\alpha+2)(1-1)=0$ $\overrightarrow{VE}.\overrightarrow{OC}=(\alpha+2)\begin{pmatrix}1\\0\\-1\end{pmatrix}.\begin{pmatrix}-1\\1\\-1\end{pmatrix}=(\alpha+2)(-1+1)=0$ Since $\overrightarrow{VE}$ is perpendicular to $\overrightarrow{OA}$ and $\overrightarrow{OC}$, which are two non-parallel vectors parallel to plane $OABC$, VE is perpendicular to the rectangular base $OABC$ (shown). Method 1:	To show 2 vectors are parallel, express one as a multiple of the other. [**Incorrect** to write $\overrightarrow{VE}=\begin{pmatrix}2\\0\\-2\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\-1\end{pmatrix}$ and conclude that the "direction vector" of $\overrightarrow{VE}$ is parallel to $\begin{pmatrix}1\\0\\-1\end{pmatrix}$]. Alternatively, perform 2 dot products. Insufficient to do only 1 dot product, e.g. $\begin{pmatrix}2\\-1\\0\end{pmatrix}\perp\begin{pmatrix}1\\2\\1\end{pmatrix}$ but $\begin{pmatrix}2\\-1\\0\end{pmatrix}$ is not perpendicular to plane $OABC$.

	$\overrightarrow{VE}=(\alpha+2)\begin{pmatrix}1\\0\\-1\end{pmatrix}=\left(\frac{\alpha+2}{2}\right)\overrightarrow{VD}$ Since E is between V & D, $0<\frac{\alpha+2}{2}<1 \Rightarrow -2<\alpha<0$. Method 2: Equation of line VD: $\mathbf{r}=\begin{pmatrix}0\\1.5\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \lambda\in\mathbb{R}$ We have $\overrightarrow{OV}$ and $\overrightarrow{OD}$ with $\lambda=-2$ and $\lambda=0$ respectively. $\overrightarrow{OE}=\begin{pmatrix}\alpha\\1.5\\-\alpha\end{pmatrix}=\begin{pmatrix}0\\1.5\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\-1\end{pmatrix}$ satisfies the equation of line VD with $\lambda=\alpha$. Since E is between V and D, $-2<\alpha<0$.	Or else show $\overrightarrow{VE}\times\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$.
(iv) **[4]**	Normal of plane $OVA=\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}-2\\3/2\\2\end{pmatrix}=\begin{pmatrix}2.5\\-4\\5.5\end{pmatrix}=\frac{1}{2}\begin{pmatrix}5\\-8\\11\end{pmatrix}$ Distance of E to plane $OVA=\frac{\left\lvert\overrightarrow{OE}.\begin{pmatrix}5\\-8\\11\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}5\\-8\\11\end{pmatrix}\right\rvert}=\frac{\left\lvert\begin{pmatrix}\alpha\\1.5\\-\alpha\end{pmatrix}.\begin{pmatrix}5\\-8\\11\end{pmatrix}\right\rvert}{\sqrt{210}}$ $=\frac{\lvert-6\alpha-12\rvert}{\sqrt{210}}$ $=\frac{6\alpha+12}{\sqrt{210}}$ since $-2<\alpha<0$	Be clear when to use dot product or cross product. Distance between a point and a plane can be found easily using $\frac{\lvert\overrightarrow{VE}.\overrightarrow{EF}\rvert}{\lvert\overrightarrow{EF}\rvert}$. Other methods such as $\left[\begin{pmatrix}\alpha\\1.5\\-\alpha\end{pmatrix}+\mu\begin{pmatrix}5\\-8\\11\end{pmatrix}\right].\begin{pmatrix}5\\-8\\11\end{pmatrix}=0$ or $\frac{\lvert\overrightarrow{VE}\times\overrightarrow{VM}\rvert}{\lvert\overrightarrow{VM}\rvert}$ where $\overrightarrow{OM}=\frac{1}{2}\overrightarrow{OA}$, work too.
(v) **[2]**	Distance between E and $OABC$ $=\lvert\overrightarrow{ED}\rvert=\left\lvert\alpha\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\rvert=-\alpha\sqrt{2}$ (since $\alpha<0$) Given $\sqrt{105}\left(-\alpha\sqrt{2}\right)=10\left(\frac{6\alpha+12}{\sqrt{210}}\right)$ $\Rightarrow -210\alpha=60\alpha+120 \Rightarrow \alpha=-\frac{4}{9}$	Need to consider that $-2<\alpha<0$, so $\lvert\overrightarrow{DE}\rvert=\sqrt{2\alpha^2}=-\alpha\sqrt{2}$.
	Marker Comments for Students Students should manage their time effectively to ensure sufficient time for a proper attempt for the question. Good to see many candidates drawing a diagram of the pyramid. Candidate should note the various methods for each part.	

RAFFLES INSTITUTION
2018 YEAR 6 PRELIMINARY EXAMINATION

MATHEMATICS **9758/02**

Paper 2

Total Marks: 100 **3 hours**

Additional materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **9** printed pages.

Pure Mathematics (40 marks)

1 When a solid turns into a gas without first becoming a liquid, the process is called sublimation. As a spherical mothball sublimes, its volume, in cm^3, decreases at a rate, in cm^3 per day, proportional to its surface area.

Show that the radius of the mothball decreases at a constant rate. [3]

[It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

2 The curve C has equation

$$y=\frac{(ax+1)(4x+b)}{2x+3}, \quad x\in\mathbb{R},\ x\neq-\frac{3}{2},$$

where a and b are constants. It is given that $y=2x-1$ is an asymptote of C.

(i) Find the values of a and b. [3]

(ii) Sketch C, labelling all relevant point(s) and stating the equations of any asymptotes. [4]

3 The position vectors of the points U and V with respect to the origin O are $\mathbf{u}$ and $\mathbf{v}$ respectively, where $\mathbf{u}$ and $\mathbf{v}$ are non-zero and non-parallel vectors.

(i) Show that $(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-2\mathbf{u})$ can be written as $k(\mathbf{u}\times\mathbf{v})$, where k is a constant to be determined, justifying your working. [2]

The vectors $\mathbf{u}$ and $\mathbf{v}$ are now given by

$$\mathbf{u}=\mathbf{i}-\mathbf{j}+2\mathbf{k} \quad\text{and}\quad \mathbf{v}=a\mathbf{i}+b\mathbf{j},$$

where a and b are constants.

(ii) If $a=2$ and $b=3$, find a unit vector parallel to $(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-2\mathbf{u})$. [2]

(iii) If $\mathbf{v}$ is perpendicular to the vector $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$, find $\mathbf{v}$ in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and b. Interpret the solution geometrically. [3]

4 A curve C has parametric equations

$$x = \cot\theta, \quad y = 2\cos^2\theta,$$

where $0 < \theta \le \frac{1}{2}\pi.$

(i) The point P on C has parameter $\theta = \frac{1}{6}\pi$. Show that the gradient of the tangent at P is $\frac{\sqrt{3}}{4}$. Hence find the equation of the normal, L, at P. [3]

(ii) Sketch C and L on the same diagram, stating the coordinates of the points where the line L cuts the curve C and the x-axis respectively. [2]

(iii) Find the exact area of the region bounded by the curve C, the positive y-axis and the line $y = \frac{3}{2}$. [3]

(iv) Show that the Cartesian equation of C is given by $y = \frac{2x^2}{a + x^b}$, where a and b are constants to be determined. [2]

(v) The region bounded by the curve C, the line L, the positive x-axis and the line $x = \frac{1}{\sqrt{3}}$ is rotated through 2π radians about the x-axis. Find the volume of the solid formed. [3]

[Turn over

5 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_r = \begin{cases} 0 & \text{for } r = 2, \\ \mathrm{f}(r-1) - 2\mathrm{f}(r-3) & \text{for } r \text{ even, } r \neq 2, \\ \mathrm{f}(r) & \text{for } r \text{ odd.} \end{cases}$$

(i) Use the method of differences to find $\sum_{r=1}^{2n} u_r$. [3]

It is given that $\mathrm{f}(r) = \ln\left(\frac{r+1}{r}\right)$.

(ii) Use your answer to part **(i)** to show that $\sum_{r=1}^{2n} u_r = -\ln 2 + 2\ln\left(1 + \frac{1}{2n-1}\right)$. Hence state the value of the sum to infinity. [2]

(iii) Find the smallest value of n for which $\sum_{r=1}^{2n} u_r$ is within 10^{-2} of the sum to infinity. [2]

(iv) By considering the graph of $y = \frac{1}{x}$ for $x > 0$, show, with the aid of a sketch, that $\frac{1}{2n} < u_{2n-1} < \frac{1}{2n-1}$, $n \in \mathbb{Z}^+$. [3]

Statistics (60 marks)

6 For events A and B it is given that $\mathrm{P}(A) = 0.5$ and $\mathrm{P}(B') = 0.35$.

(a) Given that events A and B are not independent, find the range of values of $\mathrm{P}(A \cup B)$. [2]

(b) Given that $\mathrm{P}(A \mid B') = 0.6$, find

(i) $\mathrm{P}(A \cup B)$, [3]

(ii) $\mathrm{P}(A' \cup B')$. [1]

7 The speed of the surface of a tennis court, affected by a variety of factors, including the physical makeup of the court, can be measured using the Court Pace Index (CPI). The CPI of a court varies with every match played on it. A higher mean CPI for a court indicates a faster court surface.

For the rest of this question, assume that data is collected under similar conditions.

At a tennis club, the CPI of the main court is a continuous random variable X. Based on past readings, the standard deviation of X is k and the expected value of X is 33.8.

(i) A random sample of 50 readings is to be taken. When $k = 1.5$, estimate the probability that the mean value of X, for this sample, will be at most 0.5 units from 34. [2]

In January 2018, the main court was resurfaced with a new physical makeup.

(ii) The club management claims that the new physical makeup of the resurfaced main court increases its mean CPI. A random sample of 60 readings were taken from the resurfaced main court after January 2018 and the mean CPI of this sample is found to be 34.3.

A test is carried out, at the 5% level of significance, to determine whether the resurfaced main court has a faster court surface.

Find the set of possible values of k for which the result of the test should be accepted. You should state your hypotheses clearly. [3]

(iii) The club management later obtained information that the value of k is 0.8 and that X follows a normal distribution. Another random sample of 30 readings were taken from the resurfaced main court after January 2018 and the mean CPI of this new sample is found to be $\bar{x}$.

A test is carried out, at the 5% level of significance, to determine whether the mean CPI of the resurfaced main court has changed.

Find the set of possible values of $\bar{x}$ for which the result of the test should not be accepted. You should state your hypotheses clearly. [3]

8 In this question you should state clearly the values of the parameters of any normal distribution you use.

The masses in grams of apples have the distribution $\mathrm{N}(80,\ 3^2)$ and the masses in grams of oranges have the distribution $\mathrm{N}(100,\ 5^2)$.

(i) Find the probability that the total mass of 2 randomly selected oranges is less than 205 grams. [2]

(ii) Find the probability that the total mass of 4 randomly selected apples is more than three times the mass of 1 randomly selected orange. [3]

During the Community Health Week Event, visitors who take part in the activities are given gift boxes. Each gift box contains 3 apples and 2 oranges that are individually machine wrapped. The mass of wrapper used for each fruit is dependent on the mass of the fruit resulting in the mass of each wrapped fruit being 7% more than the mass of the fruit. The fruits are packed in a gift box and the mass of an empty gift box is normally distributed with mean 50 grams and standard deviation 2 grams. During packing, parts of the gift box is cut and removed. This process reduces the mass of each empty gift box by 10%.

(iii) The probability that the total mass of a randomly selected gift box is less than k grams is 0.8. Find the value of k.

Find the probability that, at a particular collection point, the 25$^{\text{th}}$ gift box that is given is the 19$^{\text{th}}$ gift box whose total mass is less than k grams. [5]

9 The age of a species of trees can be determined by counting its rings, but that requires either cutting a tree down or extracting a sample from the tree's core. A forester attempts to find a relationship between a tree's age and its diameter instead. Based on past records, the forester found data for the diameter and the age (determined by the counting of its rings) of 8 trees of the same species that had been cut down. The results are given in the following table.

Diameter, D (inches)	1.8	6.6	9.9	10.8	12.8	13.2	15.4	16.5
Age, t (years)	5	8	16	12	22	28	30	40

(i) Draw a scatter diagram for these values, labelling the axes clearly. [2]

(ii) Find, correct to 4 decimal places, the product moment correlation coefficient between D and t. Comment on whether a linear model would be appropriate. [2]

(iii) It was suggested that the tree's age can be modelled by one of the formulae

$$t = a + bD^2, \qquad t = ae^{bD},$$

where a and b are constants.

Find, correct to 4 decimal places, the product moment correlation coefficient between

(A) t and D^2,

(B) $\ln t$ and D.

Explain which of $t = a + bD^2$ and $t = ae^{bD}$ best models the age of this species of trees based on the given data. [2]

(iv) The forester wants to estimate the diameter of a tree given its age. Using a suitable regression line, find the required estimate of a tree that is 50 years old. Comment on the reliability of your estimate.

Explain why neither the regression line of t on D^2 nor the regression line of $\ln t$ on D should be used. [4]

[Turn over

10 Amy has four identical star shaped ornaments and six identical heart shaped ornaments.

(a) Find the number of ways in which Amy can arrange the ten ornaments in a line. [1]

The star shaped ornaments are coloured Red, Green, Yellow and Purple, and the heart shaped ornaments are then coloured Red, Green, Yellow, Purple, Black and White.

(b) Find the number of ways in which Amy can arrange the ten ornaments in a line such that exactly five heart shaped ornaments are next to each other. [3]

(c) Amy then decides to arrange the ten ornaments in a circular manner. Find the number of ways in which all ornaments of the same colour are next to each other. [2]

(d) Amy randomly distributes the ten ornaments into one group of four and two groups of three.

(i) Show that the probability that the black and white ornaments are in the same group is $\frac{4}{15}$. [3]

(ii) Find the probability that there are at least two heart shaped ornaments in the group of four given that the black and white ornaments are in the same group. [3]

11 The salad bar at a restaurant has n bowls each containing a different ingredient. Each customer to the restaurant is allowed only one visit to the salad bar. Jon visits the salad bar and makes a selection. At each bowl, he can take some or none of the contents, and he does not return to the bowl again. As the salad bar may not serve the same ingredients everyday, Jon does not know what ingredients will be served at each visit. On average, the probability that he takes some of the contents from each bowl is 0.4. For each visit to the salad bar, the number of different ingredients Jon takes is the random variable X.

(i) State, in the context of this question, an assumption needed to model X by a binomial distribution, and explain why the assumption may not hold. [2]

Assume now that X has the distribution $\text{B}(n, 0.4)$.

(ii) Jon visits the salad bar on a randomly chosen day. The probability that Jon takes exactly 7 different ingredients is denoted by P. Determine the value of n that gives the greatest value of P. [2]

Jon's selection on any day is independent of his selection on any other day. During a promotional week, the salad bar serves 6 different ingredients (A, B, C, D, E, F) on Monday and Tuesday, 6 different ingredients (G, H, I, J, K, L) on Wednesday and Thursday, and all 12 different ingredients (A to L) on Friday, Saturday and Sunday.

(iii) If Jon visits the salad bar on Monday and Thursday of the promotional week, find the probability that he takes at least 3 different ingredients on each day. [2]

(iv) If Jon visits the salad bar on Saturday of the promotional week, find the probability that he takes at least 6 different ingredients. [1]

(v) Explain in context why the answer to part **(iv)** is greater than the answer to part **(iii)**. [1]

After the promotional week, the restaurant decides to charge customers \$3 for a visit to the salad bar, and an additional \$2 if they take more than $\frac{n}{2}$ different ingredients. Kai visits the salad bar after the promotional week and makes a selection. The number of different ingredients Kai takes is the random variable K. It is given that $\text{P}(K \le k) = \left(\frac{k}{n}\right)^2$, $k = 0, 1, 2, \ldots, n$.

(vi) If Kai visits the salad bar on a particular day after the promotional week when $n = 9$, find the expected amount Kai pays for that particular visit. [3]

Kai visits the salad bar on a randomly chosen day after the promotional week.

(vii) Show that $\text{P}(K = k) = \frac{2k-1}{n^2}$, $k = 1, 2, \ldots, n$. [1]

(viii) Find, in terms of n, the expected number of different ingredients Kai takes on that day. [You may use the result $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$.] [2]

******* **End of Paper** *******

2018 RI Prelim Exam Paper 2

Pure Mathematics (40 marks)

1 When a solid turns into a gas without first becoming a liquid, the process is called sublimation. As a spherical mothball sublimes, its volume, in cm^3, decreases at a rate, in cm^3 per day, proportional to its surface area.

Show that the radius of the mothball decreases at a constant rate. [3]

[It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

1 **[3]**	Let V and A be the volume and surface area of the spherical mothball respectively. $\frac{\mathrm{d}V}{\mathrm{d}t} = kA,\ k<0 \quad \cdots\cdots\cdots\cdots \quad (1)$ $V = \frac{4}{3}\pi r^3$ $\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = A\frac{\mathrm{d}r}{\mathrm{d}t} \quad \cdots\cdots\cdots \quad (2)$ $(1) = (2) \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = k < 0$ $\therefore$ The radius decreases at a constant rate (shown).	Indicate $k < 0$.

2 The curve C has equation

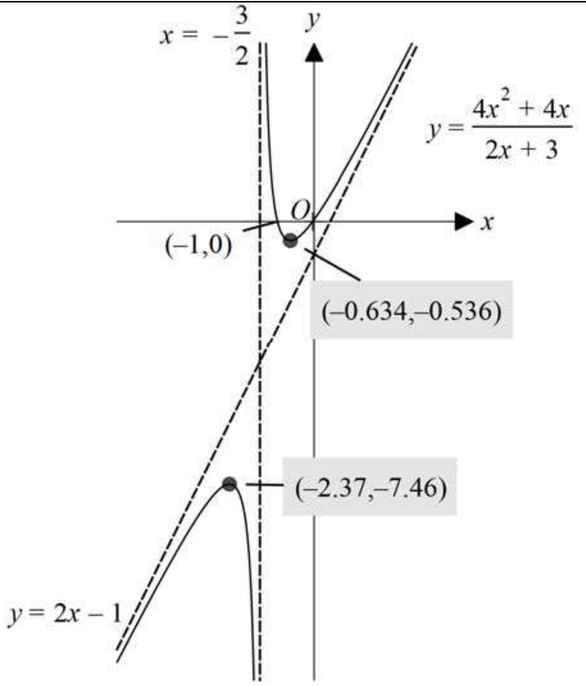
$$y = \frac{(ax+1)(4x+b)}{2x+3}, \quad x \in \mathbb{R},\ x \neq -\frac{3}{2},$$

where a and b are constants. It is given that $y = 2x-1$ is an asymptote of C.

(i) Find the values of a and b. [3]

(ii) Sketch C, labelling all relevant point(s) and stating the equations of any asymptotes. [4]

2(i) **[3]**	$y = \frac{(ax+1)(4x+b)}{2x+3} = (2x-1) + \frac{c}{2x+3}$ $\therefore \frac{(ax+1)(4x+b)}{2x+3} = \frac{(2x-1)(2x+3)+c}{2x+3}$ $(ax+1)(4x+b) = (2x-1)(2x+3)+c$ By comparing coefficients of $x^2 : a = 1,$ $x : ab + 4 = 6 - 2 \Rightarrow ab = 0 \Rightarrow b = 0$	Long division is a longer method.

(ii) **[4]**		Find the turning points using GC and indicate it on the graph !

3 The position vectors of the points U and V with respect to the origin O are $\mathbf{u}$ and $\mathbf{v}$ respectively, where $\mathbf{u}$ and $\mathbf{v}$ are non-zero and non-parallel vectors.

(i) Show that $(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-2\mathbf{u})$ can be written as $k(\mathbf{u}\times\mathbf{v})$, where k is a constant to be determined, justifying your working. [2]

The vectors $\mathbf{u}$ and $\mathbf{v}$ are now given by

$$\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j},$$

where a and b are constants.

(ii) If $a = 2$ and $b = 3$, find a unit vector parallel to $(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-2\mathbf{u})$. [2]

(iii) If $\mathbf{v}$ is perpendicular to the vector $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$, find $\mathbf{v}$ in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and b. Interpret the solution geometrically. [3]

3(i) **[2]**	$(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-2\mathbf{u}) = (\mathbf{u}\times\mathbf{v}) - 2(\mathbf{u}\times\mathbf{u}) + (\mathbf{v}\times\mathbf{v}) - 2(\mathbf{v}\times\mathbf{u})$ $= (\mathbf{u}\times\mathbf{v}) + 2(\mathbf{u}\times\mathbf{v})$ $(\because \mathbf{u}\times\mathbf{u} = \mathbf{0},\ \mathbf{v}\times\mathbf{v} = \mathbf{0},\ \mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}))$ $= 3(\mathbf{u}\times\mathbf{v}) \qquad \therefore k = 3$	This is a “show” question so explicit working/ reasoning should be shown. Important: $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v})$
(ii) **[2]**	$\mathbf{u}\times\mathbf{v} = \begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}2\\3\\0\end{pmatrix} = \begin{pmatrix}-6\\4\\5\end{pmatrix}$ A unit vector parallel to $(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-2\mathbf{u})$ is $= \pm\frac{1}{\sqrt{77}}\begin{pmatrix}-6\\4\\5\end{pmatrix}$.	Students must represent vectors appropriately. The “tilde” ~ underneath the letter should not be forgotten.

(iii) **[3]**	Since **v** is perpendicular to the vector $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$, $$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}=0$$ $$a-2b=0$$ $$a=2b$$ $$\mathbf{v}=\begin{pmatrix} 2b \\ b \\ 0 \end{pmatrix}=b\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$ The solution represents the position vector of any point on the line which passes through the origin and is parallel to the vector $2\mathbf{i}+\mathbf{j}$.	Explain/describe properly what **v** represents.

4 A curve C has parametric equations

$$x=\cot\theta,\ \ y=2\cos^2\theta,$$

where $0<\theta\le\frac{1}{2}\pi$.

(i) The point P on C has parameter $\theta=\frac{1}{6}\pi$. Show that the gradient of the tangent at P is $\frac{\sqrt{3}}{4}$. Hence find the equation of the normal, L, at P. [3]

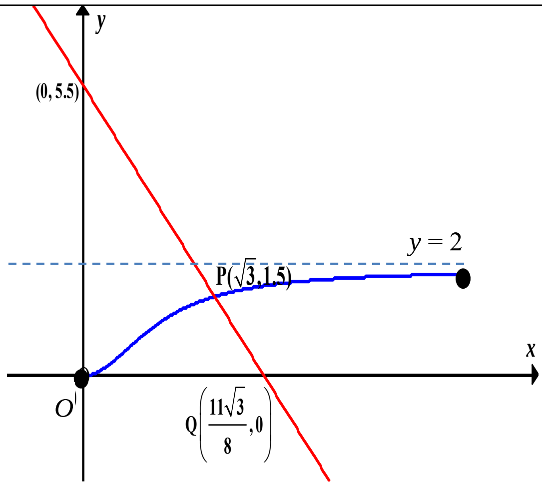
(ii) Sketch C and L on the same diagram, stating the coordinates of the points where the line L cuts the curve C and the x-axis respectively. [2]

(iii) Find the exact area of the region bounded by the curve C, the positive y-axis and the line $y=\frac{3}{2}$. [3]

(iv) Show that the Cartesian equation of C is given by $y=\frac{2x^2}{a+x^b}$, where a and b are constants to be determined.

(v) The region bounded by the curve C, the line L, the positive x-axis and the line $x=\frac{1}{\sqrt{3}}$ is rotated through 2π radians about the x-axis. Find the volume of the solid formed. [3]

4(i) **[3]**	$x=\cot\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta}=-\mathrm{cosec}^2\theta$ $y=2\cos^2\theta \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta}=-4\sin\theta\cos\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-4\sin\theta\cos\theta}{-\mathrm{cosec}^2\theta}=4\sin^3\theta\cos\theta$ When $\theta=\frac{\pi}{6}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=4\left(\frac{1}{2}\right)^3\left(\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{4}$ (shown) At P, $x=\sqrt{3}$ $y=\frac{3}{2}$. Equation of L, the normal at P, is $y-\frac{3}{2}=-\frac{4}{\sqrt{3}}\left(x-\sqrt{3}\right)$ or $y=-\frac{4}{\sqrt{3}}x+\frac{11}{2}$	In a show question, sufficient working must be demonstrated.

(ii) **[2]**		Label the graph clearly indicating the closed circle at the end points.
(iii) **[3]**	Area $= \int_0^{\frac{3}{2}} x\,\mathrm{d}y$ $y=0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \cot\theta \frac{\mathrm{d}y}{\mathrm{d}\theta}\mathrm{d}\theta$ $y=\frac{3}{2} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}\ (\because 0<\theta\le\frac{\pi}{2}) \Rightarrow \theta=\frac{\pi}{6}$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{\cos\theta}{\sin\theta}(4\cos\theta(-\sin\theta))\mathrm{d}\theta$ $= -4\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \cos^2\theta\,\mathrm{d}\theta$ $= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\mathrm{d}\theta$ $= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ $= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ Alternatively, use Cartesian form of the equation (found in **(iv)**) Area $= (\sqrt{3})(1.5) - \int_0^{\sqrt{3}} \frac{2x^2}{1+x^2}\,\mathrm{d}x$ $= \frac{3\sqrt{3}}{2} - \int_0^{\sqrt{3}} \frac{2(1+x^2-1)}{1+x^2}\,\mathrm{d}x$ $= \frac{3\sqrt{3}}{2} - \int_0^{\sqrt{3}} 2 - \frac{2}{1+x^2}\,\mathrm{d}x$ $= \frac{3\sqrt{3}}{2} - \left[2x - 2\tan^{-1}x\right]_0^{\sqrt{3}}$ $= \frac{3\sqrt{3}}{2} - \left[2\sqrt{3} - \frac{2\pi}{3}\right]$ $= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$	

(iv) **[2]**	$x=\cot\theta \Rightarrow x^2=\cot^2\theta$ $y=2\cos^2\theta \Rightarrow y=2-2\sin^2\theta \Rightarrow \operatorname{cosec}^2\theta=\dfrac{2}{2-y}$ $\operatorname{cosec}^2\theta=1+\cot^2\theta$ $\dfrac{2}{2-y}=1+x^2$ $2=2+2x^2-y-x^2y$ $y=\dfrac{2x^2}{1+x^2},\quad x>0 \qquad (\because 0<\theta\le\dfrac{\pi}{2}\Rightarrow x>0)$	
(v) **[2]**	Volume of solid $=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\pi y^2\,\mathrm{d}x+\frac{1}{3}\pi\left(\frac{3}{2}\right)^2\left(\frac{3\sqrt{3}}{8}\right)$ $=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\pi\left(\frac{2x^2}{1+x^2}\right)^2\mathrm{d}x+\frac{1}{3}\pi\left(\frac{3}{2}\right)^2\left(\frac{3\sqrt{3}}{8}\right)$ $=6.17$ units^3 Alternatively Volume of solid $=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\pi y^2\,\mathrm{d}x+\frac{1}{3}\pi\left(\frac{3}{2}\right)^2\left(\frac{3\sqrt{3}}{8}\right)$ $=\int_{\frac{\pi}{3}}^{\frac{\pi}{6}}\pi(2\cos^2\theta)^2\left(-\operatorname{cosec}^2\theta\,\mathrm{d}\theta\right)+\frac{1}{3}\pi\left(\frac{3}{2}\right)^2\left(\frac{3\sqrt{3}}{8}\right)$ $=6.17$ units^3	Read the question slowly/ clearly to identify the correct "region" to find the volume. Little or NO credit can be given if an incorrect region is used. Note alternative solution.

5 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_r=\begin{cases}0 & \text{for } r=2,\\ \mathrm{f}(r-1)-2\mathrm{f}(r-3) & \text{for } r \text{ even}, r\neq 2,\\ \mathrm{f}(r) & \text{for } r \text{ odd.}\end{cases}$$

(i) Use the method of differences to find $\sum_{r=1}^{2n}u_r$. [3]

It is given that $\mathrm{f}(r)=\ln\left(\frac{r+1}{r}\right)$.

(ii) Use your answer to part **(i)** to show that $\sum_{r=1}^{2n}u_r=-\ln 2+2\ln\left(1+\frac{1}{2n-1}\right)$. Hence state the value of the sum to infinity. [2]

(iii) Find the smallest value of n for which $\sum_{r=1}^{2n}u_r$ is within 10^{-2} of the sum to infinity. [2]

(iv) By considering the graph of $y=\frac{1}{x}$ for $x>0$, show, with the aid of a sketch, that $\frac{1}{2n}<u_{2n-1}<\frac{1}{2n-1},\ n\in\mathbb{Z}^+$. [3]

(i)
[3]

$$
\begin{aligned}
\sum_{r=1}^{2n} u_r = {} & \mathrm{f}(1) \\
& +0 \\
& +\mathrm{f}(3) \\
& +\mathrm{f}(3) - 2\mathrm{f}(1) \\
& +\mathrm{f}(5) \\
& +\mathrm{f}(5) - 2\mathrm{f}(3) \\
& +\mathrm{f}(7) \\
& +\mathrm{f}(7) - 2\mathrm{f}(5) \\
& +\ldots \\
& +\mathrm{f}(2n-3) \\
& +\mathrm{f}(2n-3) - 2\mathrm{f}(2n-5) \\
& +\mathrm{f}(2n-1) \\
& +\mathrm{f}(2n-1) - 2\mathrm{f}(2n-3) \\
= {} & -\mathrm{f}(1) + 2\mathrm{f}(2n-1)
\end{aligned}
$$

Consider odd and even values.

Candidates who are unable to make progress should review piecewise functions.

(ii)
[2]

$$
\begin{aligned}
\sum_{r=1}^{2n} u_r &= -\mathrm{f}(1) + 2\mathrm{f}(2n-1) \\
&= -\ln\left(\frac{2}{1}\right) + 2\ln\left(\frac{(2n-1)+1}{2n-1}\right) \\
&= -\ln 2 + 2\ln\left(1 + \frac{1}{2n-1}\right) \text{ (shown)}
\end{aligned}
$$

As $n \to \infty$, $\frac{1}{2n-1} \to 0$, $1+\frac{1}{2n-1} \to 1$, $\ln\left(1+\frac{1}{2n-1}\right) \to 0$, $\sum_{r=1}^{2n} u_r \to -\ln 2$.

Sum to infinity is $-\ln 2$.

While an explanation was not demanded, candidates should note correct reasoning here.

(iii)
[2]

$$\left| \sum_{r=1}^{2n} u_r - (-\ln 2) \right| < 10^{-2}$$

$$2\ln\left(1+\frac{1}{2n-1}\right) < 0.01 \quad \left(\because 1+\tfrac{1}{2n-1} > 1 \Rightarrow \ln\left(1+\tfrac{1}{2n-1}\right) > 0\right)$$

From GC, $n > 100.25$

Or alternatively

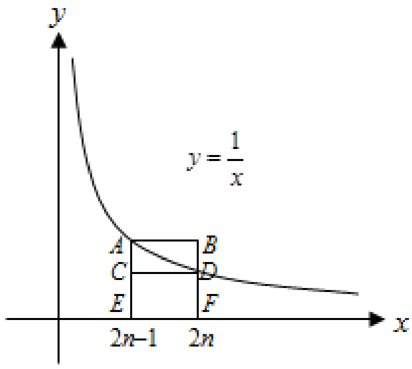
n	$2\ln\left(1+\frac{1}{2n-1}\right)$
100	$0.010025 > 0.01$
101	$0.0099256 < 0.01$

$\therefore$ Smallest value of $n = 101$

Candidates should note an absolute difference is required.

(iv)
[3]

From the graph of $y = \dfrac{1}{x}$ for $x > 0$,

<table>
<tr><td></td><td>

Area of rectangle CDFE $< \int_{2n-1}^{2n} \frac{1}{x}\,dx <$ Area of rectangle ABFE
$$(1)\left(\frac{1}{2n}\right) < [\ln x]_{2n-1}^{2n} < (1)\left(\frac{1}{2n-1}\right)$$
$$\frac{1}{2n} < \ln\frac{2n}{2n-1} < \frac{1}{2n-1}$$
$\ln\frac{2n}{2n-1} = \ln\frac{(2n-1)+1}{2n-1} = u_{2n-1}$ since $2n-1$ is odd,
$\therefore \frac{1}{2n} < u_{2n-1} < \frac{1}{2n-1}$ (shown).</td><td>Compare the integral with rectangle above and below

Candidates to note the connection between $y=\frac{1}{x}$ and $y=\ln x$</td></tr>
</table>

Statistics (60 marks)

6 For events A and B it is given that $\mathrm{P}(A)=0.5$ and $\mathrm{P}(B')=0.35$.

(a) Given that events A and B are not independent, find the range of values of $\mathrm{P}(A\cup B)$. [2]

(b) Given that $\mathrm{P}(A\mid B')=0.6$, find

(i) $\mathrm{P}(A\cup B)$, [3]

(ii) $\mathrm{P}(A'\cup B')$. [1]

<table>
<tr><td>(a)
[2]</td><td>$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$
$=0.5+0.65-\mathrm{P}(A\cap B)$
$=1.15-\mathrm{P}(A\cap B)$

Since $0\le \mathrm{P}(A\cup B)\le 1$,
$0.15\le \mathrm{P}(A\cap B)\le \min\{\mathrm{P}(A),\mathrm{P}(B)\} \Rightarrow 0.15\le \mathrm{P}(A\cap B)\le 0.5$

If A and B are independent,
$\mathrm{P}(A\cap B)=(0.5)(0.65)=0.325 \Rightarrow \mathrm{P}(A\cup B)=0.825$

$\therefore 0.65\le \mathrm{P}(A\cup B)\le 1$ and $\mathrm{P}(A\cup B)\ne 0.825$.

Range of $\mathrm{P}(A\cup B)$ is $[0.65, 1]\setminus\{0.825\}$.</td><td>Students to note that $\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)$ is true ONLY when A and B are independent.

A and B are NOT independent so $\mathrm{P}(A\cup B)$ cannot take value 0.825</td></tr>
<tr><td>(b)(i)
[3]</td><td>$\mathrm{P}(A'\mid B')=1-P(A\mid B')$
$=1-0.6$
$=0.4$</td><td>Candidates should use the Venn diagrams to analyze the relation of A</td></tr>
</table>

	$P(A'\cap B') = P(B')\,P(A'\mid B')$ $= (0.35)(0.4)$ $= 0.14$ $P(A\cup B) = 1 - P(A'\cap B')$ $= 1 - 0.14$ $= 0.86$ Alternatively, $P(A\cap B') = P(B')\,P(A\mid B')$ $= (0.35)(0.6)$ $= 0.21$ $P(A\cup B) = P(B) + P(A\cap B')$ $= (1-0.35) + 0.21$ $= 0.86$	and *B*. Candidates should understand the difference between $P(A\cap B)$ and $P(A\cup B)$.
(b)(ii) [1]	$P(A'\cup B') = P(A') + P(B') - P(A'\cap B')$ $= 0.5 + 0.35 - 0.14$ $= 0.71$ Alternatively, $P(A'\cup B') = 1 - P(A\cap B) = 1 - [P(A) - P(A\cap B')]$ $= 1 - [0.5 - 0.21]$ $= 0.71$	Do not confuse De Morgan's Laws, $P(A'\cup B') \neq P(A\cap B)$ $P(A'\cup B') \neq 1 - P(A\cup B)$.

7 The speed of the surface of a tennis court, affected by a variety of factors, including the physical makeup of the court, can be measured using the Court Pace Index (CPI). The CPI of a court varies with every match played on it. A higher mean CPI for a court indicates a faster court surface.

For the rest of this question, assume that data is collected under similar conditions.

At a tennis club, the CPI of the main court is a continuous random variable X. Based on past readings, the standard deviation of X is k and the expected value of X is 33.8.

(i) A random sample of 50 readings is to be taken. When $k = 1.5$, estimate the probability that the mean value of X, for this sample, will be at most 0.5 units from 34. [2

In January 2018, the main court was resurfaced with a new physical makeup.

(ii) The club management claims that the new physical makeup of the resurfaced main court increases its mean CPI. A random sample of 60 readings were taken from the resurfaced main court after January 2018 and the mean CPI of this sample is found to be 34.3.

A test is carried out, at the 5% level of significance, to determine whether the resurfaced main court has a faster court surface.

Find the set of possible values of k for which the result of the test should be accepted. You should state your hypotheses clearly. [3

(iii) The club management later obtained information that the value of k is 0.8 and that X follows a normal distribution. Another random sample of 30 readings were taken from the resurfaced main court after January 2018 and the mean CPI of this new sample is found to be $\bar{x}$.

A test is carried out, at the 5% level of significance, to determine whether the mean CPI of the resurfaced main court has changed.

Find the set of possible values of $\bar{x}$ for which the result of the test should not be accepted. You should state your hypotheses clearly. [3

(i) [2]

By Central Limit Theorem, since $n = 50$ is large, $\bar{X} \sim \mathrm{N}\left(33.8, \frac{1.5^2}{50}\right)$ approximately.

$\mathrm{P}(|\bar{X} - 34| < 0.5) = \mathrm{P}(33.5 < \bar{X} < 34.5) = 0.921 \text{ (3 s.f.)}$

Remarks:

- It is incorrect to claim that *X* is normally distributed by Central Limit Theorem. Central Limit theorem is used to find the approximate distribution of $\bar{X}$ (i.e. a random sample mean) when the distribution of *X* is either unknown or not normal. It is **not** for finding the distribution of *X*. It holds only when the sample size is large.

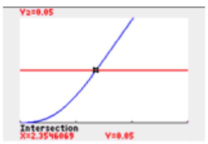
(ii) [3]

$H_o : \mu = 33.8 \qquad H_1 : \mu > 33.8$

Under H_0, since $n = 60$ is large, $\bar{X} \sim \mathrm{N}\left(\mu_0, \frac{k^2}{n}\right)$ approximately by Central Limit Theorem, with $\mu_0 = 33.8, \ n = 60$.

At 5% level of significance, to accept test, we reject H_0,

$$p\text{-value} \le 0.05$$
$$\mathrm{P}(\bar{X} \ge 34.3) \le 0.05$$

$\mathrm{P}(Z \ge \frac{34.3 - 33.8}{k/\sqrt{60}}) \le 0.05$ $\frac{34.3 - 33.8}{k/\sqrt{60}} \ge 1.6448536$ $k \le 2.3545 = 2.35 \text{ (3s.f.)}$	 From GC, $k \le 2.35 \text{ (3s.f.)}$

Hence, the set of values of *k* is $\{k \in \mathbb{R} : 0 < k \le 2.35\}$.

Remarks:

- In this hypothesis testing, we are testing on the **unknown** population mean μ. Thus the null and alternative hypothesis should **only** involve μ and the value that we are 'suggesting' that it takes. i.e $H_o : \mu = 33.8 \qquad H_1 : \mu > 33.8$

- To do the test, we need to use the distribution of $\bar{X}$ as we are using a sample result (namely the mean of a particular sample = $\bar{x}$) to determine if we are able to reject or not reject the hypothesis at a particular level of significance. As such, the distribution of $\bar{X}$ should only come **after** the hypothesis is stated.

- $\bar{X} \sim \mathrm{N}\left(\mu_0, \frac{\sigma^2}{n}\right)$ approximately via Central Limit Theorem or is exactly normal (if *X* is normal). Also, $\mathrm{E}(\bar{X}) = \mathrm{E}(X)$ and hence μ_0 should take the value of the population

	mean μ. However, since the value of μ is unknown, in hypothesis testing, we work under the assumption that μ takes on the value as stated in the null hypothesis and tries to find evidence against it. It is thus important to write the words 'Under H_0' and hence your working should read $H_o : \mu = 33.8$ $\quad H_1 : \mu > 33.8$ Under H_0, $\bar{X} \sim \mathrm{N}\left(\mu_0, \frac{\sigma^2}{n}\right)$ ….. where $\mu_0 = 33.8$, ... • Common error - wrong expression for p-value and confusion in algebraic manipulation: p-value refers to the chance of obtaining 'this kind' of sample result (see the notes for proper definition). The sample mean $\bar{x}$ based on an obtained sample was 34.3. Thus p-value= $\mathrm{P}(\bar{X} \geq 34.3)$ (i.e. the chance of sample mean being 34.3 or higher).Thus for (ii), it should read To reject H_o, p-value ≤ 0.05 $\mathrm{P}(\bar{X} \geq 34.3) \leq 0.05$ $\mathrm{P}(\mathrm{Z} \geq \frac{34.3-33.8}{k/\sqrt{60}}) \leq 0.05$ (standardisation using $\bar{X} \sim \mathrm{N}\left(33.8, \frac{k^2}{60}\right)$) $\frac{34.3-33.8}{k/\sqrt{60}} \geq 1.6448536$ (note the changes in inequality sign) $\quad k \leq 2.35$ (3s.f.)
(iii) **[3]**	$H_o : \mu = 33.8$ $\quad H_1 : \mu \neq 33.8$ Perform a 2-tail test at 5% level of significance. Under H_0, $\bar{X} \sim \mathrm{N}\left(\mu_0, \frac{\sigma^2}{n}\right)$ with $\mu_0 = 33.8$, $\sigma = 0.8$, $n = 30$. At 5% level of significance, for the test to be not accepted, do not reject $\mathrm{H_o}$. From GC p-value > 0.05 $\Rightarrow$ $\mathrm{P}(\bar{X} \geq \bar{x}) > 0.025$ and $\mathrm{P}(\bar{X} \leq \bar{x}) > 0.025$ $\Rightarrow$ $\bar{x} < 34.086 = 34.1$ (3.s.f.) and $\bar{x} > 33.514 = 33.5$ (3.s.f.) The set of values of $\bar{x}$ is $\{\bar{x} \in \mathbb{R} : 33.5 < \bar{x} < 34.1\}$. Alternatively, we can use symmetry about $\mu = 33.8$ to get the other value. Remarks: • $\bar{X} \sim \mathrm{N}\left(\mu_0, \frac{\sigma^2}{n}\right)$ exactly and **not** approximately • Incorrect expressions regarding p –value for 2 tail test eg. p-value is NOT $P(\bar{X} \neq \bar{x})$ or $P(\bar{X} < \bar{x}) + P(\bar{X} > \bar{x})$ • Inconsistent algebraic manipulation when working from expressions involving probability to expressions involving z-value.

8 In this question you should state clearly the values of the parameters of any normal distribution you use.

The masses in grams of apples have the distribution $N(80,\ 3^2)$ and the masses in grams of oranges have the distribution $N(100,\ 5^2)$.

(i) Find the probability that the total mass of 2 randomly selected oranges is less than 205 grams. [2]

(ii) Find the probability that the total mass of 4 randomly selected apples is more than three times the mass of 1 randomly selected orange. [3]

During the Community Health Week Event, visitors who take part in the activities are given gift boxes. Each gift box contains 3 apples and 2 oranges that are individually machine wrapped. The mass of wrapper used for each fruit is dependent on the mass of the fruit resulting in the mass of each wrapped fruit being 7% more than the mass of the fruit. The fruits are packed in a gift box and the mass of an empty gift box is normally distributed with mean 50 grams and standard deviation 2 grams. During packing, parts of the gift box is cut and removed. This process reduces the mass of each empty gift box by 10%.

(iii) The probability that the total mass of a randomly selected gift box is less than k grams is 0.8. Find the value of k.

Find the probability that, at a particular collection point, the 25th gift box that is given is the 19th gift box whose total mass is less than k grams. [5]

(i) **[2]**	Let X be the mass of an orange in grams. $X \sim N(100, 5^2)$ $X_1 + X_2 \sim N(2\times100, 2\times5^2)$ $X_1 + X_2 \sim N(200, 50)$ $P(X_1 + X_2 < 205) = 0.760$ (3 s.f.)	
(ii) **[3]**	Let Y be the mass of an apple in grams. $Y \sim N(80, 3^2)$ and $X \sim N(100, 5^2)$ $Y_1 + Y_2 + Y_3 + Y_4 - 3X \sim N(4\times80 - 3\times100, 4\times3^2 + 3^2\times5^2)$ $Y_1 + Y_2 + Y_3 + Y_4 - 3X \sim N(20, 261)$ $P(Y_1 + Y_2 + Y_3 + Y_4 > 3X) = P(Y_1 + Y_2 + Y_3 + Y_4 - 3X > 0)$ $= 0.892$	
(iii) **[5]**	Let B be the mass of an empty gift box. $B \sim N(50,\ 2^2)$ Let M be the total mass of a gift box. $M = 1.07(Y_1 + Y_2 + Y_3 + X_1 + X_2) + 0.9B$ $M \sim N(3\times1.07\times80 + 2\times1.07\times100 + 0.9\times50,$ $3\times1.07^2\times3^2 + 2\times1.07^2\times5^2 + 0.9^2\times2^2)$ $M \sim N(515.8,\ \ 91.3973)$ $P(M < k) = 0.8$ Using GC, $\therefore k = 523.8\text{(1d.p.)} = 524$ (3 s.f.) Required probability $=$ P(of the first 24 gift boxes given, 18 has total mass less than k g)$\times$	Calculate the variance carefully Common mistake: Forgotting to square the constants "1.07" and "0.9".

	P(25th gift box has total mass less than k g) $=\left[\binom{24}{18}(0.8)^{18}(0.2)^{6}\right](0.8)=0.124\,(3\text{s.f.})$	

9 The age of a species of trees can be determined by counting its rings, but that requires either cutting a tree down or extracting a sample from the tree's core. A forester attempts to find a relationship between a tree's age and its diameter instead. Based on past records, the forester found data for the diameter and the age (determined by the counting of its rings) of 8 trees of the same species that had been cut down. The results are given in the following table.

Diameter, D (inches)	1.8	6.6	9.9	10.8	12.8	13.2	15.4	16.5
Age, t (years)	5	8	16	12	22	28	30	40

(i) Draw a scatter diagram for these values, labelling the axes clearly. [2]

(ii) Find, correct to 4 decimal places, the product moment correlation coefficient between D and t. Comment on whether a linear model would be appropriate. [2]

(iii) It was suggested that the tree's age can be modelled by one of the formulae

$$t = a + bD^2, \qquad t = ae^{bD},$$

where a and b are constants.

Find, correct to 4 decimal places, the product moment correlation coefficient between

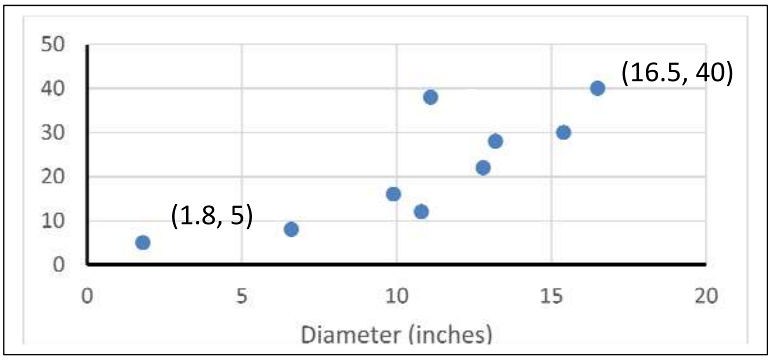
(A) t and D^2,

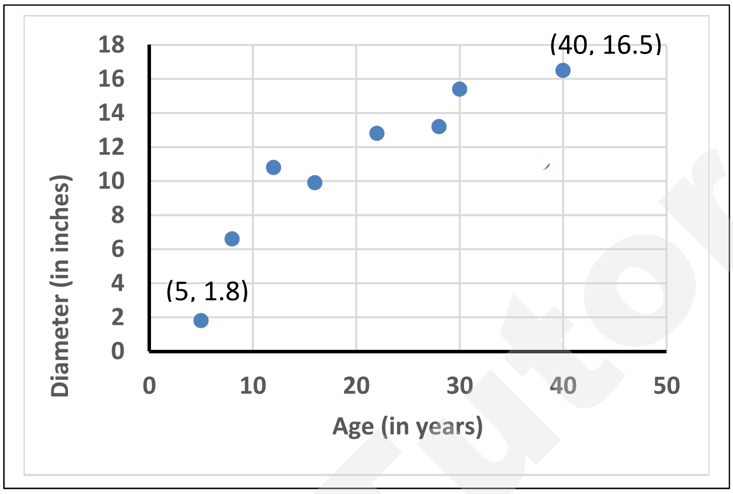
(B) $\ln t$ and D.

Explain which of $t = a + bD^2$ and $t = ae^{bD}$ best models the age of this species of trees based on the given data.

(iv) The forester wants to estimate the diameter of a tree given its age. Using a suitable regression line, find the required estimate of a tree that is 50 years old. Comment on the reliability of your estimate.

Explain why neither the regression line of t on D^2 nor the regression line of $\ln t$ on D should be used.

<table>
<tr>
<td>(i)
[2]</td>
<td>




</td>
<td>Use ruler to draw axes. Label axes, origin.

Ensure graph is or reasonable size.

Indicate the data range clearly.</td>
</tr>
<tr>
<td>(ii)
[2]</td>
<td>From GC, the product moment correlation coefficient between D and t is $r =$ 0.9146 (4 d.p.)

Since r is close to 1 and from the scatter diagram a positive linear correlation between D and t can be seen, a linear model is appropriate.

Alternatively Although r is close to 1, scatter diagram suggests that as t increases, D is increasing at a slower rate, so a non-linear model might be more appropriate.</td>
<td>4 decimal places is demanded in the question.</td>
</tr>
<tr>
<td>(iii)
[2]</td>
<td>From GC, the product moment correlation coefficient between D^2 and t is $r = 0.9689$ (4 d.p.)
From GC, the product moment correlation coefficient between D and $\ln t$ is $r = 0.9747$ (4 d.p.)

The $t = ae^{bD}$ provides a better model as its product moment correlation coefficient is closer to 1.</td>
<td>4 decimal places is demanded in the question.</td>
</tr>
<tr>
<td>(v)
[4]</td>
<td>From GC, the regression line of Diameter, D, on ln(age), $\ln t$, is $D = -7.642653677 + 6.595868978 \ln t$.
$t = 50 \Rightarrow \ln t = 3.9120$
$D = 18.161\ (5sf) = 18.2$ inches (1 d.p.)

The estimated diameter of a 50 year-old tree is 18.2 inches (1 d.p.)
As the value of $t = 50$ is outside the data range, the linear model may no longer be suitable. Hence the estimate is not reliable. To estimate D given t, we would need t, the age of the tree, to be the independent variable.</td>
<td>Use at least 5 s.f. for the intermediate working and express final answer in 3 s.f. for non-exact answer (unless specified otherwise).</td>
</tr>
</table>

10 Amy has four identical star shaped ornaments and six identical heart shaped ornaments.

(a) Find the number of ways in which Amy can arrange the ten ornaments in a line. [1]

The star shaped ornaments are coloured Red, Green, Yellow and Purple, and the heart shaped ornaments are then coloured Red, Green, Yellow, Purple, Black and White.

(b) Find the number of ways in which Amy can arrange the ten ornaments in a line such that exactly five heart shaped ornaments are next to each other. [3]

(c) Amy then decides to arrange the ten ornaments in a circular manner. Find the number of ways in which all ornaments of the same colour are next to each other. [2]

(d) Amy randomly distributes the ten ornaments into one group of four and two groups of three.

(i) Show that the probability that the black and white ornaments are in the same group is $\frac{4}{15}$. [3]

(ii) Find the probability that there are at least two heart shaped ornaments in the group of four given that the black and white ornaments are in the same group. [3]

(a) **[1]**	Number of ways $= \dfrac{10!}{4!6!}$ or ${}^{10}C_4 = 210$	Use GC carefully to find values.
(b) **[3]**	In order for exactly 5 hearts to be together, the remaining single heart must be separated from the group of 5 hearts. No. of ways to choose and arrange the 5 hearts in the box $=\binom{6}{5}\times 5!$ No. of ways to arrange the 4 stars $= 4!$ No. of ways to slot the group of 5 hearts and the single heart into separate slots $=\binom{5}{2}\times 2!$ Required number of ways = $4!\times\binom{6}{5}\times 5!\times\binom{5}{2}\times 2! = 345600$ Or $4!\times {}^6P_5 \times {}^5P_2 = 345600$ Alternative Solutions : Arrange group of 5 hearts with 4 stars, then slot in last heart : $\binom{6}{5}\times 5!\times 5!\times\binom{4}{1}$ Total (for group of 5 hearts + 4 stars + last heart) – 6 hearts together : $\left(\binom{6}{5}5!\right)6! - 6!5!2!$	**Explain your work carefully.** Incorrect answers with no explanation warrant little or no marks.

(c) **[2]**	2 Red, 2 Green, 2 Yellow, 2 Purple, 1 Black and 1 White Number of ways all the ornaments of the same colours are next to each other, arranged in a circular manner $=(6-1)!\times 2^4 = 1920$	Remember to arrange all 4 groups.
(d)(i) **[3]**	Number of ways to select the three groups = $\dfrac{\binom{10}{4}\binom{6}{3}\binom{3}{3}}{2!} = 2100$ Number of ways to select when black and white are in the same group = Number of ways when black and white are in group of 4 + number of ways when black and white are in a group of 3 $=\dfrac{\binom{8}{2}\binom{6}{3}\binom{3}{3}}{2!}+\binom{8}{4}\binom{4}{1}\binom{3}{3}=560$ Required probability = $\dfrac{560}{2100}=\dfrac{4}{15}$ Alternative solutions : $\dfrac{4}{10}\times\dfrac{3}{9}+2\left(\dfrac{3}{10}\times\dfrac{2}{9}\right)$ or $\dfrac{\binom{4}{2}+2\binom{3}{2}}{\binom{10}{2}}$ or $\dfrac{\binom{8}{2}}{\binom{10}{4}}+\dfrac{2\binom{8}{1}}{\binom{10}{3}}$	Candidates to know alternative solutions. In a "show question", more explicit working is required.
(d)(ii) **[3]**	Let A be the event that at least 2 heart shaped ornaments are in the group of 4. Let B be the event that the black and white ornaments are in the same group. $P(A\|B)=1-P(A'\|B)=\dfrac{n(B)-n(A'\cap B)}{n(B)}$ or $1-\dfrac{P(A'\cap B)}{P(B)}$ The cases for $A'\cap B$ are as follows: (see table below) $P(A\|B)=1-\dfrac{\frac{68}{2100}}{\frac{4}{15}}$ or $\dfrac{560-68}{560}$ $\left(\dfrac{492}{560}\right)$ $=\dfrac{123}{140}$ or 0.879 (3s.f, 0.87857(5s.f.))	Consider cases carefully

Group of 4	Group of 3	Group of 3	Number of Ways
3 stars with a Red or Green or Yellow or Purple heart	Black and white and 1 other	The remaining	$\binom{4}{3}\binom{4}{1}\binom{4}{1}\binom{3}{3}$ $=64$
4 stars	Black and white and 1 other	The remaining	$\binom{4}{4}\binom{4}{1}\binom{3}{3}=4$
			Total = 68

Alternatively, have to consider 4 cases.

Group of 4	Group of 3	Group of 3	Number of Ways
Black and white and 2 others	Any remaining 3	The remaining	$\binom{8}{2}\binom{6}{3}\binom{3}{3}/2$ $=280$
2 hearts (not black and white) + 2 stars	Black and white and 1 other	The remaining	$\binom{4}{2}\binom{4}{2}\binom{4}{1}\binom{3}{3}=144$
3 hearts (not black and white) + 1 stars			$\binom{4}{3}\binom{4}{1}\binom{4}{1}\binom{3}{3}=64$
4 hearts (not black and white)			$\binom{4}{4}\binom{4}{1}\binom{3}{3}=4$
			Total = 492

$$P(A\mid B)=\frac{492}{560}$$

$$=\frac{123}{140} \text{ or } 0.879 \text{ (3s.f, 0.87857(5s.f.))}$$

11 The salad bar at a restaurant has n separate bowls containing different ingredients. Each customer to the restaurant is allowed only one visit to the salad bar. Jon visits the salad bar and makes a selection. At each bowl, he can take some or none of the contents, and he does not return to the bowl again. As the salad bar may not serve the same ingredients everyday, Jon does not know what ingredients will be served at each visit. On average, the probability that he takes some of the contents from each bowl is 0.4. For each visit to the salad bar, the number of different ingredients Jon takes is the random variable X.

(i) State, in the context of this question, an assumption needed to model X by a binomial distribution, and explain why the assumption may not hold. [2]

Assume now that X has the distribution $\mathrm{B}(n, 0.4)$.

(ii) Jon visits the salad bar on a randomly chosen day. The probability that Jon takes exactly 7 different ingredients is denoted by P. Determine the value of n that gives the greatest value of P. [2]

Jon's selection on any day is independent of his selection on any other day. During a promotional week, the salad bar serves 6 different ingredients (A, B, C, D, E, F) on Monday and Tuesday, 6 different ingredients (G, H, I, J, K, L) on Wednesday and Thursday, and all 12 different ingredients (A to L) on Friday, Saturday and Sunday.

(iii) If Jon visits the salad bar on Monday and Thursday of the promotional week, find the probability that he takes at least 3 different ingredients on each day. [2]

(iv) If Jon visits the salad bar on Saturday of the promotional week, find the probability that he takes at least 6 different ingredients. [1]

(v) Explain in context why the answer to part **(iv)** is greater than the answer to part **(iii)**. [1]

After the promotional week, the restaurant decides to charge customers \$3 for a visit to the salad bar, and an additional \$2 if they take more than $\frac{n}{2}$ different ingredients. Kai visits the salad bar after the promotional week and makes a selection. The number of different ingredients Kai takes is the random variable *K*. It is given that $\mathrm{P}(K \le k) = \left(\frac{k}{n}\right)^2$, $k = 0, 1, 2, \ldots, n$.

(vi) If Kai visits the salad bar on a particular day after the promotional week when $n = 9$, find the expected amount Kai pays for that particular visit. [3]

Kai visits the salad bar on a randomly chosen day after the promotional week.

(vii) Show that $\mathrm{P}(K = k) = \frac{2k-1}{n^2}$, $k = 1, 2, \ldots, n$. [1]

(viii) Find, in terms of *n*, the expected number of different ingredients Kai takes on that day.

[You may use the result $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$.] [2]

<table>
<tr>
<td>(i)
[2]</td>
<td>The probability that Jon takes the ingredients from each bowl is constant at 0.4. However, the probability may not be constant as Jon may prefer certain ingredients and be more likely to take the ingredients from some bowls than others.
OR
Jon takes the ingredients from each bowl independently of other bowls. However, Jon may not take the ingredients independently as he may consider that some ingredients go well together while others do not.</td>
<td>Correct phrasing in context is required, eg Probabilities are NOT independent.</td>
</tr>
<tr>
<td>(ii)
[2]</td>
<td colspan="2">$X \sim \mathrm{B}(n, 0.4)$
$P = {}^{n}C_7 (0.4)^7 (0.6)^{n-7}$
Find n such that P has the greatest value.
<table>
<tr><th>n</th><th>P</th></tr>
<tr><td>16</td><td>0.18889</td></tr>
<tr><td>17</td><td>0.19267</td></tr>
<tr><td>18</td><td>0.18916</td></tr>
</table>
From GC, $n = 17$.

Alternatively,</td>
</tr>
</table>

	$P(n) \ge P(n-1)$ ${}^{n}C_7(0.4)^7(0.6)^{n-7} \ge {}^{n-1}C_7(0.4)^7(0.6)^{n-1-7}$ $\frac{n!}{7!(n-7)!}0.6 \ge \frac{(n-1)!}{7!(n-1-7)!}$ $\left(\frac{n}{n-7}\right)0.6 \ge 1$ $0.6n \ge n-7$ $n \le \frac{35}{2}$ P has greatest value when $n = 17$
(iii) **[2]**	When $n=6$, $X \sim \mathrm{B}(6,0.4)$. P(at least 3 different ingredients each on Mon and Thur) $=\left[\mathrm{P}(X \ge 3)\right]^2 = \left[1-\mathrm{P}(X \le 2)\right]^2 = 0.208$ (3s.f.)
(iv) **[1]**	When $n=12$, $X \sim \mathrm{B}(12,0.4)$. P(at least 6 different ingredients on Sat) $= \mathrm{P}(X \ge 6) = 1-\mathrm{P}(X \le 5) = 0.335$ (3s.f.)
(v) **[1]**	Part **(iv)** includes the cases in part **(iii)** as well as other cases. For example, the case of Jon taking exactly 2 different ingredients from A to F and exactly 4 different ingredients from G to L is included in part **(iv)** but not in part **(iii)**.
(vi) **[3]**	$\mathrm{P}(K > 4.5) = \mathrm{P}(K \ge 5) = 1-\mathrm{P}(K \le 4) = 1-\left(\frac{4}{9}\right)^2 = 1-\frac{16}{81} = \frac{65}{81}$ Expected amount Kai pays $= \$\left[3+2\left(\frac{65}{81}\right)\right] = \$4.60$ (to nearest cent) OR alternatively, $\$\left[3\left(\frac{16}{81}\right)+5\left(\frac{65}{81}\right)\right] = \$4.60$ (to nearest cent)
(vii) **[1]**	$\mathrm{P}(K=k) = \mathrm{P}(K \le k) - \mathrm{P}(K \le k-1), \quad k = 1,2,\ldots,n$ $=\left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2$ $=\frac{k^2-(k^2-2k+1)}{n^2}$ $=\frac{2k-1}{n^2}$ (shown)
(viii) **[2]**	Expected number of different ingredients Kai takes $=\sum_{k=0}^{n} k\ \mathrm{P}(K=k)$ $=\frac{1}{n^2}\sum_{k=1}^{n}\left(2k^2-k\right)$ $=\frac{1}{n^2}\left[\frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$ $=\frac{(n+1)(4n-1)}{6n}$

Name	( )	Class	

RIVER VALLEY HIGH SCHOOL
2018 Year 6 Preliminary Examination
Higher 2

MATHEMATICS

9758/01

Paper 1

13 September 2018

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Page

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, class and index number in the space at the top of this page.

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphic calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

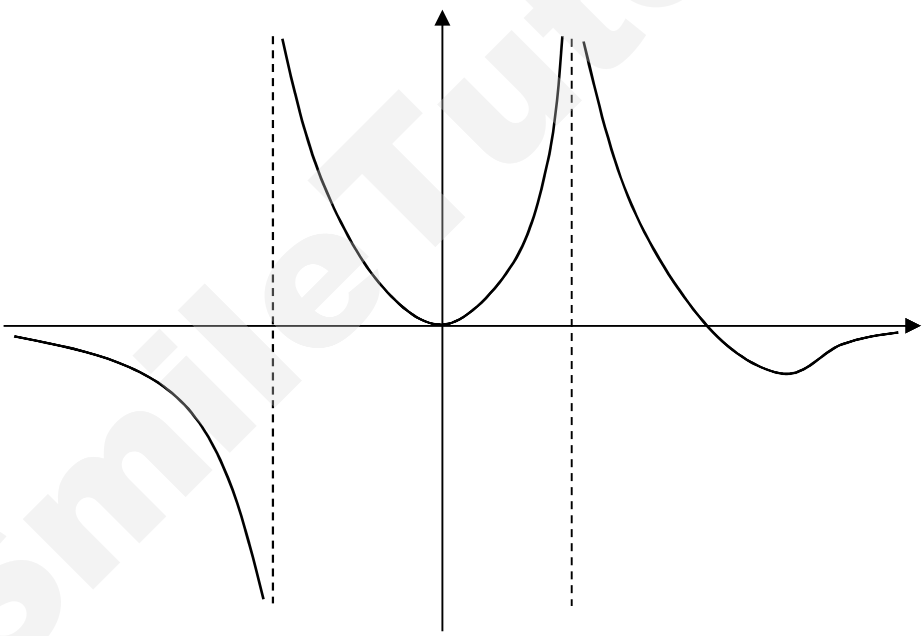
At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **8** printed pages.

1. Relative to the origin O, the points A, B, C and D have position vectors $\overrightarrow{OA} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OD} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$ respectively.

The vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular to each other. The vector $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ is a normal to the plane containing O, A and C. The point A lies on the line passing through the point D parallel to the vector $\overrightarrow{OB}$. Find the values of p, q and r. [6]

2. The diagrams show the graphs of $y=\left|\mathrm{f}(x)\right|$ and $y=\mathrm{f}'(x)$.



The graph of $y=\left|\mathrm{f}(x)\right|$ has stationary points at $(0,2)$ and $(5,2)$, and the equations of asymptotes are $x=-3$ and $x=2$. The equations of asymptotes of $y=\mathrm{f}'(x)$ are $x=-3$ and $x=2$.

On separate diagrams, sketch the graphs of

(i) $y=\mathrm{f}(x)$; [3]

(ii) $y=\dfrac{1}{\mathrm{f}'(x)}$. [3]

3. **In this question, all physical quantities are of appropriate units which you need not state.**

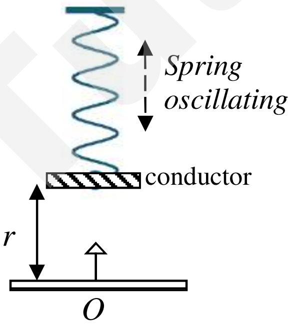
In a science experiment, charged particles from a fixed position O are bombarded towards a charged conductor. The electric potential energy (E) of each particle is given by

$$E = \frac{0.9qQ}{r}$$

where q is the charge of the particle, Q is the charge of conductor and r is the distance between them.

(i) Assuming q and Q are constants, express $\frac{\mathrm{d}E}{\mathrm{d}t}$ in terms of q, Q, r and $\frac{\mathrm{d}r}{\mathrm{d}t}$. [1]

It is further given that the conductor is attached to a spring that moves in an oscillating manner such that the distance between the conductor and O is given by $r = 0.8 + 0.6\cos\left(\frac{1}{2}t\right)$ as shown in the diagram below.



(ii) State the range of values of r. [1]

At a certain instant, a particle with a constant charge of 1.5 is of distance 1.1 units from the conductor, which has a constant charge of 4.0.

(iii) Find the exact time of this instant, and the rate at which the electrical potential energy of the particle is increasing at this instant. [5]

4. Relative to the origin O, the points A, B and C are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OC}=\mathbf{c}$ respectively, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors which are mutually non-parallel.

The plane Π and the line l have the following equations

$$\Pi : \mathbf{r}=\lambda\mathbf{a}+\mu\mathbf{b} \quad \text{where } \lambda,\mu\in\mathbb{R} \quad \text{and}$$

$$l : \mathbf{r}=\mathbf{a}+\beta\mathbf{c} \quad \text{where } \beta\in\mathbb{R}$$

respectively.

(i) Find the point(s) of intersection between Π and l given that the points O, A, B and C are

(a) coplanar,

(b) not coplanar. [3]

(ii) State the geometrical meaning of $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$. [1]

In the rest of the question, O, A, B and C are not coplanar.

(iii) The vector $\mathbf{p}$ is a unit vector in the direction of $\mathbf{a}\times\mathbf{b}$. State the geometrical meaning of $|\mathbf{c}\cdot\mathbf{p}|$. [1]

(iv) It is given that $\overrightarrow{OA}=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}0\\0\\2\end{pmatrix}$. Find the volume of the pyramid $OABC$. [3]

5. **(i)** By using the substitution $x=2\sin^2\theta$, find the exact value of $\int_0^1 \sqrt{2x-x^2}\,\mathrm{d}x$.

[5]

(ii) Hence, find the exact value of $\int_0^1 \frac{x(1-x)}{\sqrt{2x-x^2}}\,\mathrm{d}x$. [3]

6. A curve C has parametric equations

$$x = 1 - \sin 2t, \quad y = \cos t, \text{ for } 0 \le t < 2\pi.$$

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [2]

(ii) Find the stationary points of C. [2]

(iii) There are four points where the normals to C are parallel to the x-axis. Deduce that the four points form a rectangle. State the length and breadth of the rectangle. [5]

7. Given that $y = \mathrm{f}(x)$ where $\sqrt{1-x^2}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y}{2}$ and $\mathrm{f}(0) = 1$.

(i) Find the Maclaurin series for y up to and including the term in x^3. [5]

(ii) The Maclaurin series for $\mathrm{e}^{y'}$, up to and including the term in x, is $p + qx$, where y' denotes the first derivative of y. Find the exact values of p and q. [4]

8. **(a)** Given that $z^* = \dfrac{(1+a\mathrm{i})^3}{(1-\mathrm{i})^2}$, where a is a positive real number.

(i) Find a such that $|z| = 4$. [2]

(ii) Without the use of a calculator, express z in the form $r\mathrm{e}^{i\theta}$, where r and θ are exact values, $r > 0$ and $-\pi < \theta \le \pi$. [3]

(b) Given that $\mathrm{f}(z) = z^4 + 2z^3 + (\pi^2 + \mathrm{e}^2)z^2 + pz + q$, where $p \ge 0$ and $q \ge 0$, and that $\mathrm{i}\pi$ satisfies the equation $\mathrm{f}(z) = 0$.

(i) Find the exact values of p and q. [2]

(ii) Hence find the remaining roots of $\mathrm{f}(z) = 0$ in the form $\alpha + \beta\mathrm{i}$, where α and β are exact real numbers. [4]

9. A geometric series G has common ratio r and an arithmetic series A has first term a and common difference d, where a and d are non-zero. The first three terms of G are equal to the third, fourth and sixth terms of A respectively. The sum of the first four terms of A is -6.

(i) Show that $r^2 - 3r + 2 = 0$ and find the value of r. [2]

(ii) Deduce the values of a and d. [4]

(iii) Determine the least value of m such that the difference between the m^{th} terms of G and A is more than 10000. [2]

Let a_n denote the n^{th} term of A. The sequence x_n is defined as $x_n = (2k)^{a_n}$ where k is a constant.

(iv) Find the range of values of k such that the series $\sum_{n=1}^{\infty} x_n$ converges. [3]

10. A group of scientists have recently discovered the growth of a rare species of plant in a wildlife reserve area in central Africa that is well known for its severe climate changes at times. The scientists also discovered that the species of plant can be used in the cure for several human terminal illnesses.

Let N denote the number of the species of plant at time t years after its discovery.

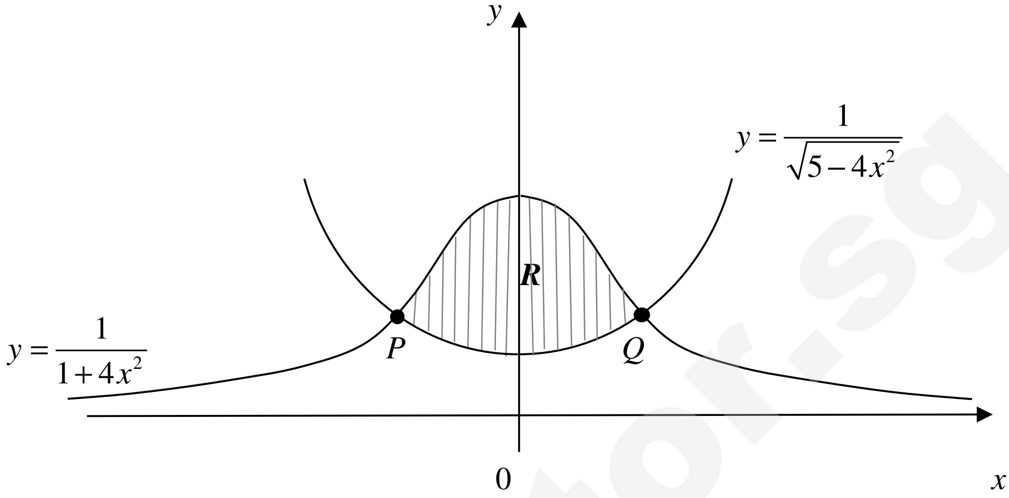
(i) Based on studies on other similar species of plant in other areas, the scientists believe that the rate of increase of N is proportional to the product of N and $(1000 - N)$. Given that they expect the number of plants to increase at a rate of 125 per year when there are 500 plants in the area, deduce a differential equation relating N and t. [2]

(ii) Given further that there are initially 200 of this rare species of plant in the area, find an expression for N in terms of t and sketch of the graph of N against t. [7]

(iii) The scientists target to have the number of the species of plants in the area to exceed 1200 by the end of 5 years. Determine if this target can be achieved. [1]

(iv) Based on the context given, provide two reasons why the proposed model of growth for this rare species of plant may not be suitable. [2]

11. Alice is tasked to design a new logo for a toy company. The logo consists of the finite region R bounded by the curves $y=\dfrac{1}{1+4x^2}$, $y=\dfrac{1}{\sqrt{5-4x^2}}$ and between the points P and Q as shown below.



(i) Find the x-coordinates of P and Q and show that the exact value of the area of R is given by

$$\frac{\pi}{4}-\sin^{-1}\left(\frac{1}{\sqrt{5}}\right).$$ [6]

Alice then decides to explore different regions bounded by various transformations of the above two curves.

(ii) Without evaluating the actual integrals, deduce the exact areas of the finite regions bounded by the following curves. Justify your answers.

(a) $y=\dfrac{1}{4x^2-8x+5}$ and $y=\dfrac{1}{\sqrt{1+8x-4x^2}}$ [2]

(b) $y=\dfrac{2}{1+4x^2}$ and $y=\dfrac{1}{\sqrt{\frac{5}{4}-x^2}}$ [2]

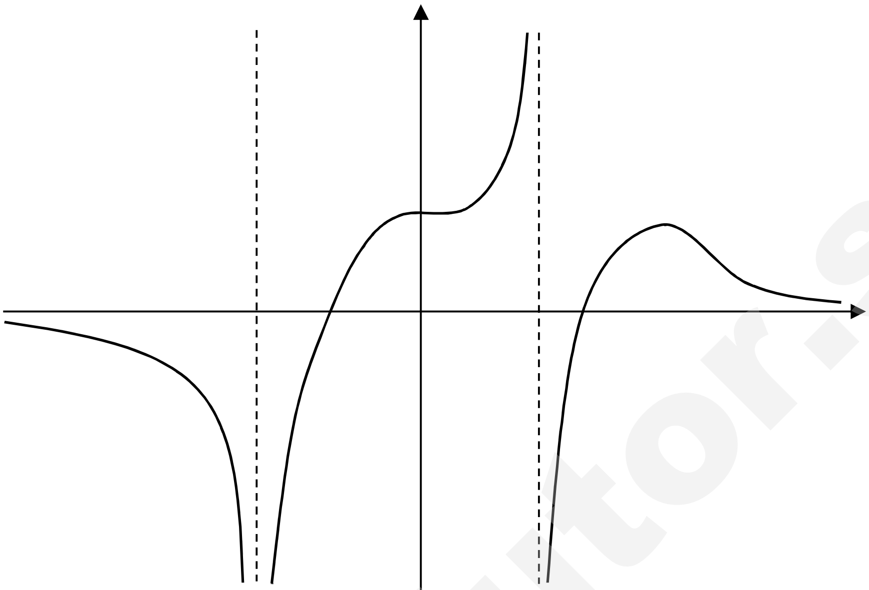
In designing a related 3-dimensional logo for the company, Alice considers rotating the above finite region R about the y-axis.

(iii) Find the volume of the 3-dimensional logo formed by rotating R 180° about the y-axis. [3]

END OF PAPER

Solutions to 2018 Y6 H2 Maths Prelim Exam P1

1.	Solution [6 marks]	
	Given that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular, $\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0$ $\Rightarrow p + 2q + r = 0$ ---- (1) Let $\mathbf{n} = \overrightarrow{OA} \times \overrightarrow{OC} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -r \\ r \\ p-q \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ ---- (2) From (2): $\begin{cases} -r = -k \\ r = k \\ p - q = 2k \end{cases} \Rightarrow p - q - 2r = 0$ ---- (3) Let l be the line passing through the point D parallel to the vector $\overrightarrow{OB}$. $l : \mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ where $\beta \in \Re$ Since A lies on l, $\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for some $\beta \in \Re$ ---- (4) From (4): $\begin{cases} p = \beta \\ q = -3 + 2\beta \Rightarrow 2p - q = 3 \text{ or } 2r - q = 3 \text{---- (5)} \\ r = \beta \end{cases}$ Solving (1), (3) and (5), $p = 1$, $q = -1$, $r = 1$.	

2.	Solution [6 marks]	
	(i)  	
	(ii)	

3.	Solution [7 marks]	
	(i) $E = \dfrac{0.9qQ}{r}$ diff wrt t, $\dfrac{dE}{dt} = \dfrac{-0.9qQ}{r^2}\dfrac{dr}{dt}$	
	(ii) $0.2 \le r \le 1.4$ (since $-1 \le \cos\left(\dfrac{1}{2}t\right) \le 1$)	
	(iii) $r = 0.8 + 0.6\cos\left(\dfrac{1}{2}t\right)$ Given: $Q = 4.0$, $q = 1.5$, $r = 1.1$. Subst $r = 1.1$ into eqn: $1.1 = 0.8 + 0.6\cos\left(\dfrac{1}{2}t\right)$ $\cos\left(\dfrac{1}{2}t\right) = \dfrac{1}{2}$ $\dfrac{1}{2}t = \dfrac{\pi}{3} + 2k\pi$ or $\dfrac{1}{2}t = \dfrac{5\pi}{3} + 2k\pi$, for $k = 0,1,2,3...$ $\therefore t = \dfrac{2\pi}{3} + 4k\pi$ or $\dfrac{10\pi}{3} + 4k\pi$ $\dfrac{dr}{dt} = -0.3\sin\left(\dfrac{1}{2}t\right)$ At the instant where the particle is 1.1 units from the conductor, $\sin\left(\dfrac{1}{2}t\right) = \dfrac{\sqrt{3}}{2}$ or $-\dfrac{\sqrt{3}}{2}$ $\therefore \dfrac{dE}{dt} = \dfrac{-0.9qQ}{r^2}\dfrac{dr}{dt}$ $= \left(\dfrac{-0.9(1.5)(4.0)}{1.1^2}\right)\left(-0.3\dfrac{\sqrt{3}}{2}\right)$ or $\left(\dfrac{-0.9(1.5)(4.0)}{1.1^2}\right)\left(0.3\dfrac{\sqrt{3}}{2}\right)$ $= \pm 1.15947$ $= \pm 1.16$ (to 3sf)	

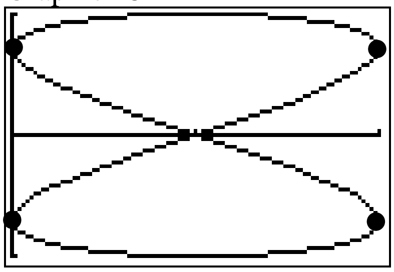
4.	Solution [8 marks]	
	(i)(a) Note that origin is on plane Π. When O, A, B and C are coplanar, the line l lies on the plane Π, therefore the line l and plane Π intersect along the line l. (Points of intersection are points on l.) (i)(b) When O, A, B and C are not coplanar the line l and plane Π intersect at the point A.	
	(ii) $\frac{1}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert$ gives the area of a triangle formed with two sides parallel and equal in magnitude to $\mathbf{a}$ and $\mathbf{b}$.	
	(iii) $\lvert\mathbf{c}\cdot\mathbf{p}\rvert$ gives the length of projection of $\mathbf{c}$ on $\mathbf{p}$ (<u>or</u> on $\mathbf{a}\times\mathbf{b}$, <u>or</u> on normal of Π).	
	(iv) Area of triangle OAB $=\frac{1}{2}\left\lvert\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}\right\rvert$ $=\frac{1}{2}\left\lvert\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\rvert$ $=\frac{\sqrt{3}}{2}$ units^2 Using (iii) result, height of pyramid = height of point C relative to plane OAB $=\lvert\mathbf{c}\cdot\mathbf{p}\rvert$ where $\mathbf{p}$ is a unit vector in the direction of $\mathbf{a}\times\mathbf{b}$. $=\left\lvert\begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\rvert=\frac{2}{\sqrt{3}}$ Volume of the pyramid $OABC$ $=\frac{1}{3}\times\frac{\sqrt{3}}{2}\times\frac{2}{\sqrt{3}}=\frac{1}{3}$ unit^3	

5.	Solution [8 marks]	
	(i) Note that $x = 2\sin^2\theta$ $\Rightarrow \dfrac{dx}{d\theta} = 2(2\sin\theta)\cos\theta = 4\sin\theta\cos\theta$. Also, when $x = 0$, $2\sin^2\theta = 0 \Rightarrow \theta = 0$; when $x = 1$, $2\sin^2\theta = 1 \Rightarrow \sin\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{\pi}{4}$ $\therefore \int_0^1 \sqrt{2x - x^2}\ \mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2\theta - (2\sin^2\theta)^2} \times 4\sin\theta\cos\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2\theta - 4\sin^4\theta} \times 2\sin 2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2\theta(1 - \sin^2\theta)} \times 2\sin 2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2\theta\cos^2\theta} \times 2\sin 2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} 2\sin\theta\cos\theta \times 2\sin 2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} 2\sin^2 2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} 1 - \cos 4\theta\ \mathrm{d}\theta$ $= \left[\theta - \dfrac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{4}}$ $= \dfrac{\pi}{4} - 0 = \dfrac{\pi}{4}$	
	(ii) $\int_0^1 \dfrac{x(1-x)}{\sqrt{2x - x^2}}\ \mathrm{d}x = \dfrac{1}{2}\int_0^1 \dfrac{x(2-2x)}{\sqrt{2x - x^2}}\ \mathrm{d}x$ $= \dfrac{1}{2}\int_0^1 (x) \times \dfrac{(2-2x)}{\sqrt{2x - x^2}}\ \mathrm{d}x$ $= \dfrac{1}{2}\left[\left[x \cdot 2\sqrt{2x - x^2}\right]_0^1 - \int_0^1 2\sqrt{2x - x^2} \cdot (1)\ \mathrm{d}x\right]$ $= \dfrac{1}{2}\left[2 - 0 - 2\int_0^1 \sqrt{2x - x^2}\ \mathrm{d}x\right]$ $= \dfrac{1}{2}\left[2 - 2 \times \dfrac{\pi}{4}\right] = 1 - \dfrac{\pi}{4}$	

	Alternatively, $\int_0^1 \frac{x(1-x)}{\sqrt{2x-x^2}}\,dx = \int_0^1 \sqrt{2x-x^2} - \frac{x}{\sqrt{2x-x^2}}\,dx$ $= \int_0^1 \sqrt{2x-x^2}\,dx + \frac{1}{2}\int_0^1 \frac{-2x+2}{\sqrt{2x-x^2}} - \frac{2}{\sqrt{2x-x^2}}\,dx$ $= \frac{\pi}{4} + \frac{1}{2}\int_0^1 \frac{-2x+2}{\sqrt{2x-x^2}}\,dx - \int_0^1 \frac{1}{\sqrt{1-(x-1)^2}}\,dx$ $= \frac{\pi}{4} + \frac{1}{2}\left[2\sqrt{2x-x^2}\right]_0^1 - \left[\sin^{-1}(x-1)\right]_0^1$ $= \frac{\pi}{4} + \frac{1}{2}(2) - \left(\sin^{-1}0 - \sin^{-1}(-1)\right)$ $= \frac{\pi}{4} + 1 - \left(0 - \left(-\frac{\pi}{2}\right)\right)$ $= 1 - \frac{\pi}{4}$	

6.	Solution [9 marks]	
	(i) $x = 1 - \sin 2t$ and $y = \cos t$ $\frac{dx}{dt} = -2\cos 2t$ and $\frac{dy}{dt} = -\sin t$ $\frac{dy}{dx} = \frac{\sin t}{2\cos 2t}$	
	(ii) For stationary points, let $\frac{dy}{dx}=0$ sin t =0 $t = 0, \pi$ *correspondingly* $x = 1,\ 1$ $y = 1,\ -1$ Thus the two stationary points are (1,1) and (1, –1).	

(iii)
Graph of C



Normal at point P is parallel to x-axis implies tangent at point P is vertical (grad is undefined).

Set denominator of $\frac{\mathrm{d}y}{\mathrm{d}x}$ to be 0

$\cos 2t = 0$

$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ or $\frac{7\pi}{2}$

$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

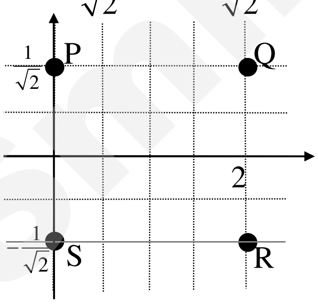
respectively,

$x = 0$, 2 , 0 or 2

$y = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}$

Thus the normal occur at four points

$P(0, \frac{1}{\sqrt{2}})$, $Q(2, \frac{1}{\sqrt{2}})$, $R(2, -\frac{1}{\sqrt{2}})$ or $S(0, -\frac{1}{\sqrt{2}})$



The points form a rectangle with length 2 and breadth $\frac{2}{\sqrt{2}}$.

7.	Solution [9 marks]	

(i)

$$\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{2}------(1)$$

Diff Implicitly wrt x:

$$\sqrt{1-x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\right)(-2x)=\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\sqrt{1-x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{-x}{\sqrt{1-x^2}}\right)=\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x}------(2)$$

$\times\sqrt{1-x^2}$ throughout,

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x}\sqrt{1-x^2}$$

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{y}{4}\quad\text{(from (1))}$$

Diff Implicitly wrt x:

$$(1-x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(-2x)-\left[x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\right]=\frac{1}{4}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$(1-x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{5}{4}\frac{\mathrm{d}y}{\mathrm{d}x}\ ---------(3)$$

Given that $x = 0$, $y = 1$, From (1), (2) and (3) respectively

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2},\qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{4},\qquad \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{5}{8}$$

$$\therefore\quad y=1+x(\frac{1}{2})+\frac{x^2}{2!}(\frac{1}{4})+\frac{x^3}{3!}(\frac{5}{8})+\ldots$$

$$=1+\frac{x}{2}+\frac{x^2}{8}+\frac{5x^3}{48}+\ldots\text{(up to the } x^3 \text{ term)}$$

<u>Alternatively</u>:

$$\left(1-x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{y^2}{4}$$

$$\left(1-x^2\right)2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)-2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{y}{2}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$4\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-4x\frac{\mathrm{d}y}{\mathrm{d}x}=y$$

$$4\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-8x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-4\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$4\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-12x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-5\frac{\mathrm{d}y}{\mathrm{d}x}=0\text{...... (the rest is the same as abv)}$$

	(ii) From (i)'s answer, diff term by term $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2} + \dfrac{x}{4} + \dfrac{5x^2}{16} + \ldots$(up to the x^2 term) Using the standard Maclaurin's formula for e^x, $\mathrm{e}^{y'} \approx \mathrm{e}^{\frac{1}{2}+\frac{x}{4}+\frac{5x^2}{16}}$ $= \mathrm{e}^{\frac{1}{2}} e^{\frac{x}{4}+\frac{5x^2}{16}}$ $\approx \mathrm{e}^{\frac{1}{2}}\left(1+\dfrac{x}{4}\right)$ (up to the x term) Hence $p = \sqrt{\mathrm{e}}$ and $q = \dfrac{\sqrt{\mathrm{e}}}{4}$	

8.	Solution [11 marks]	
	(a)(i) $\left\|z^*\right\| = 4$ since $\left\|z\right\| = 4$ $\left\|\dfrac{(1+ai)^3}{(1-i)^2}\right\| = 4$ $\dfrac{\left\|(1+ai)\right\|^3}{\left\|(1-i)\right\|^2} = 4$ $\dfrac{\left(\sqrt{1+a^2}\right)^3}{\left(\sqrt{2}\right)^2} = 4$ $\left(\sqrt{1+a^2}\right)^3 = 8$ $\sqrt{1+a^2} = 2$ $1+a^2 = 4$ $a = \pm\sqrt{3}$ $a = \sqrt{3}$ (reject -ve as a is given to be positive)	

	(a)(ii) $\arg z^* = \arg \dfrac{(1+ai)^3}{(1-i)^2}$ $= 3\arg(1+\sqrt{3}i) - 2\arg(1-i)$ $= 3(\frac{\pi}{3}) - 2(-\frac{\pi}{4})$ $= \frac{3\pi}{2}$ $= -\frac{\pi}{2}$ (principal value) $\arg z^* = -\arg z$ $\therefore \arg z = \frac{\pi}{2}$ (principal value) hence $z = 4e^{\frac{i\pi}{2}}$	
	(b) (i) since $i\pi$ is a root, $(i\pi)^4 + 2(i\pi)^3 + (\pi^2 + e^2)(i\pi)^2 + p(i\pi) + q = 0$ $\pi^4 + 2\pi^3(-i) + (\pi^2 + e^2)(-\pi^2) + i(p\pi) + q = 0$ $(q - e^2\pi^2) + i(p\pi - 2\pi^3) = 0$ Equating real and imaginary parts, $q = e^2\pi^2$ and $p = 2\pi^2$	

(b)(ii)
Now,

$$f(z)=z^4+2z^3+(\pi^2+e^2)z^2+2\pi^2 z+e^2\pi^2$$

As the polynomial f(z) has real coefficients, $-i\pi$ is also a root.

Thus $(z-i\pi)$ and $(z+i\pi)$ are factors of f(z)

i.e., $z^2+\pi^2$ is a factor

$f(z)=\left(z^2+\pi^2\right)\left(z^2+mz+e^2\right)$ where m is to be determined

Equating coeff of z^3 or z in f(z),

$m=2$

$$f(z)=\left(z^2+\pi^2\right)\left(z^2+2z+e^2\right)$$

for $f(z)=0$

$$\left(z^2+\pi^2\right)\left(z^2+2z+e^2\right)=0$$

$$z=\pm i\pi \quad \text{or} \quad z=\frac{-2\pm\sqrt{4-4e^2}}{2}$$
$$=-1\pm\sqrt{1-e^2}$$
$$=-1\pm i\sqrt{e^2-1}$$

($NOTE$: $1-e^2$ is negative)

Hence the remaining roots are

$-i\pi$, $-1+i\sqrt{e^2-1}$ or $-1+i\sqrt{e^2-1}$

9. Solution [11 marks]

(i)

Let y_3, y_4 and y_6 be the 3rd, 4th and 6th terms of the arithmetic series.

$y_3=a$, $y_4=ar$, $y_6=ar^2$

$y_6-y_4=2d=2(y_4-y_3)$

$\therefore ar^2-ar=2(ar-a)$

$r^2-3r+2=0$ (shown)

$r=2$ or $r=1$ Reject since $r=1$ implies $d=0$.

	(ii) $\frac{y_4}{y_3}=r=2$ $\Rightarrow y_4=2y_3$ $\Rightarrow a+3d=2(a+2d)$ $\Rightarrow a=-d$ ---- (1) OR $\frac{y_6}{y_4}=r=2$ $\Rightarrow y_6=2y_4$ $\Rightarrow a+5d=2(a+3d)$ $\Rightarrow a=-d$ ---- (1) $S_4=-6$ $\frac{4}{2}\left[2a+(4-1)d\right]=-6$ ---- (2) Sub $a=-d$ into (2): $2d=-6$ $d=-3$ and $a=3$ Alternatively $\frac{y_4}{y_3}=\frac{y_6}{y_4}\Rightarrow\frac{a+3d}{a+2d}=\frac{a+5d}{a+3d}$ $\Rightarrow$ ...... $d(a+d)=0$ i.e. $d=0$ (n.a.) or $d=-a$ ---- (1) Solving (1) and (2), $d=-3$ and $a=3$	
	(iii) Let g_m and a_m denote the m^{th} terms of G and A respectively. $\left\|g_m-a_m\right\|>10000$ ---- (3) Since $A:3,0,-3,-6,\ldots$ while $G:-3,-6,-12,-24,\ldots$, $a_m>g_m$ for all $m\in\mathbb{Z}^+$. $\left\|g_m-a_m\right\|>10000$ $a_m-g_m>10000$ $\left[3+(m-1)(-3)\right]-(-3)(2)^{m-1}>10000$ Using GC, when $m=12$, $a_m-g_m=6114$ <10000 when $m=13$, $a_m-g_m=12255$ >10000 Hence, the least value of $m=13$.	

	(iv) $a_n = 3 + (n-1)(-3)$ $x_n = (2k)^{a_n}$ $x_n = (2k)^{3+(n-1)(-3)}$ $x_n = (2k)^3\left[(2k)^{-3}\right]^{(n-1)}$ is a GP with first term $(2k)^3$ and common ratio $(2k)^{-3}$. The series $\sum_{n=1}^{\infty} x_n$ is the sum to infinity of the GP and it converges when $\left\|(2k)^{-3}\right\| < 1$ $\left\|\frac{1}{(2k)^3}\right\| < 1$ $(2k)^3 > 1$ or $(2k)^3 < -1$ $2k > 1$ or $2k < -1$ $k > \frac{1}{2}$ or $k < -\frac{1}{2}$	

10.	Solution [12 marks]	
	(i) Based on the proposed growth model by the scientists, $\frac{\mathrm{d}N}{\mathrm{d}t} = kN(1000-N)$, $k > 0$. Given that $\frac{\mathrm{d}N}{\mathrm{d}t} = 125$ when $N = 500$, $125 = k(500)(1000-500)$ $\Rightarrow k = \frac{125}{500 \times 500} = \frac{1}{2000}$ Thus, the required differential equation is $\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{1}{2000}N(1000-N)$	

(ii) Solving the D.E. $\dfrac{\mathrm{d}N}{\mathrm{d}t}=\dfrac{1}{2000}N(1000-N)$,

we have $\displaystyle\int \frac{1}{N(1000-N)}\,\mathrm{d}N = \int \frac{1}{2000}\,\mathrm{d}t$

$$\frac{1}{1000}\int \frac{1}{N}+\frac{1}{1000-N}\,\mathrm{d}N = \int \frac{1}{2000}\,\mathrm{d}t$$

$$\int \frac{1}{N}+\frac{1}{1000-N}\,\mathrm{d}N = \int \frac{1000}{2000}\,\mathrm{d}t = \int \frac{1}{2}\,\mathrm{d}t$$

Then, $\ln|N|-\ln|1000-N|=\dfrac{1}{2}t+c$

Simplifying, $\ln\left|\dfrac{N}{1000-N}\right|=\dfrac{1}{2}t+c$

$$\Rightarrow \left|\frac{N}{1000-N}\right| = e^{\frac{1}{2}t+c} \Rightarrow \frac{N}{1000-N} = \pm\, e^{\frac{1}{2}t+c}$$

$$\Rightarrow \frac{N}{1000-N} = Ae^{\frac{1}{2}t} \quad \text{where } A=\pm e^{c}$$

Given that when $t = 0$, $N = 200$, we have

$$A=\frac{200}{1000-200}=\frac{1}{4}$$

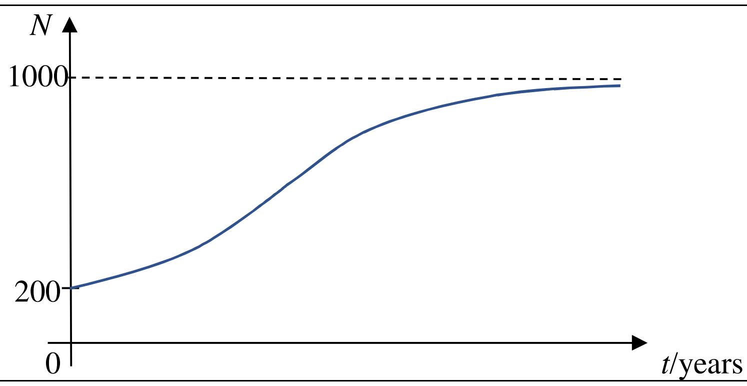
Then, $\dfrac{N}{1000-N}=\dfrac{1}{4}e^{\frac{1}{2}t}$

Making N the subject:

$$4N=(1000-N)e^{\frac{1}{2}t}$$

$$\Rightarrow (4+e^{\frac{1}{2}t})N=1000$$

$$\Rightarrow N=\frac{1000e^{\frac{1}{2}t}}{4+e^{\frac{1}{2}t}}=\frac{1000}{4e^{-\frac{1}{2}t}+1}$$



	(iii) Based on the graph, we can deduce that the value of N tends towards 1000 as t tends to infinity. Thus, it is not possible for the scientist to achieve their target of exceeding 1200 plants in the area after 5 years.	
	(iv) (1) With severe climate changes in the wildlife area, there may be harsh conditions to deter growth of the plant and thus causing decline in their numbers at times; or (2) With the news of their cure for terminal illnesses, it may attract further unwelcomed visitors to the area for harvesting of the plant for private profits, thus again causing significant decline in their numbers.	

11.	Solution [13 marks]	
	(i) For the x-coordinates of intersection points P and Q, we let $\frac{1}{1+4x^2}=\frac{1}{\sqrt{5-4x^2}}$ Using GC, we have $x=\frac{1}{2}$ for point P and $x=-\frac{1}{2}$ for point Q The area of region R $=\int_{-1/2}^{1/2} \frac{1}{1+4x^2}-\frac{1}{\sqrt{5-4x^2}}\,\mathrm{d}x$ $=\frac{1}{2}\int_{-1/2}^{1/2} \frac{2}{1+(2x)^2}\,\mathrm{d}x-\frac{1}{2}\int_{-1/2}^{1/2} \frac{2}{\sqrt{5-(2x)^2}}\,\mathrm{d}x$ $=\frac{1}{2}\left[\tan^{-1}(2x)\right]_{-\frac{1}{2}}^{\frac{1}{2}}-\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{\sqrt{5}}\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ $=\frac{1}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]-\frac{1}{2}\left[\sin^{-1}\frac{1}{\sqrt{5}}-\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right]$ $=\frac{\pi}{4}-\sin^{-1}\left(\frac{1}{\sqrt{5}}\right).$	

	(ii)(a) $y=\dfrac{1}{4x^2-8x+5}=\dfrac{1}{4(x-1)^2+1}$ and $y=\dfrac{1}{\sqrt{1+8x-4x^2}}=\dfrac{1}{\sqrt{5-4(x-1)^2}}$ Thus, the original curves are translated 1 unit in the positive x-direction. Thus, the exact area is still $\dfrac{\pi}{4}-\sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right)$ unit^2. (ii)(b) $y=\dfrac{2}{1+4x^2}=2\left(\dfrac{1}{1+4x^2}\right)$, and $y=\dfrac{1}{\sqrt{\frac{5}{4}-x^2}}=\dfrac{1}{\sqrt{\frac{5-4x^2}{4}}}=\dfrac{2}{\sqrt{5-4x^2}}=2\left(\dfrac{1}{\sqrt{5-4x^2}}\right)$ Thus, the original curves undergoes scaling by scale factor 2 parallel to the y-axis. The area of bounded region $= 2\times\displaystyle\int_{-1/2}^{1/2} \dfrac{1}{1+4x^2}-\dfrac{1}{\sqrt{5-4x^2}}\,\mathrm{d}x$ $= \dfrac{\pi}{2}-2\sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right)$ unit^2.	
	(iii) For $y=\dfrac{1}{\sqrt{5-4x^2}}$: when $x=0$, $y=\dfrac{1}{\sqrt{5}}$, and $x^2=\dfrac{5}{4}-\dfrac{1}{4y^2}$ For $y=\dfrac{1}{1+4x^2}$: when $x=0$, $y=1$, and $x^2=\dfrac{1}{4y}-\dfrac{1}{4}$ Also, $Q=\left(\dfrac{1}{2},\dfrac{1}{2}\right)$	

	Then the required volume $=\pi\int_{\frac{1}{\sqrt{5}}}^{\frac{1}{2}} \frac{5}{4}-\frac{1}{4}y^{-2}\ \mathrm{d}y \ + \ \pi\int_{\frac{1}{2}}^{1} \frac{1}{4y}-\frac{1}{4}\ \mathrm{d}y$ $=0.174\text{ unit}^3$	

Name	( )	Class	

RIVER VALLEY HIGH SCHOOL
2018 Year 6 Preliminary Examination
Higher 2

MATHEMATICS 9758/02

Paper 2

17 September 2018

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Page

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, class and index number in the space at the top of this page.

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphic calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages and **1** blank page.

Section A: Pure Mathematics [40 marks]

1. A curve C has equation $y=\dfrac{x^2-3x+3}{x-2}$ where $1\le x\le 3$ and $x\ne 2$.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and comment on the gradient of C. [3]

(ii) Sketch the graph of C. [2]

(iii) By adding a suitable graph to the sketch of C, solve the inequality

$$\frac{x^2-4x+5}{x-2}\ge 3\left(x-2\right)^2,\ \text{ for } 1\le x\le 3,\ x\ne 2. \qquad [3]$$

(iv) The function g is defined by $\mathrm{g}(x)=\dfrac{x^2-3x+3}{x-2}$ for $1\le x\le 3$ and $x\ne 2$ such that $\mathrm{g}(x)=\mathrm{g}(x+3)$. Sketch the graph of $y=\mathrm{g}(x)$ for $-1<x<5$. [2]

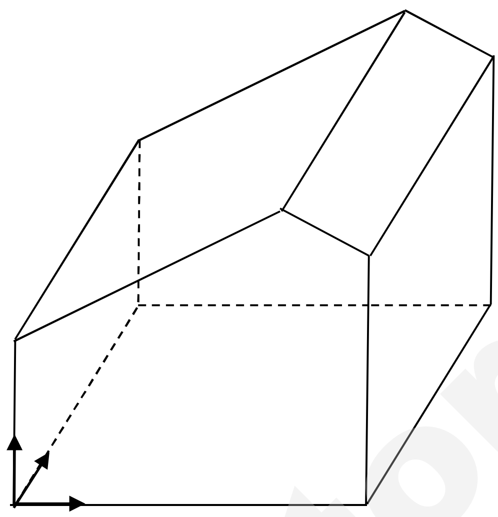
2. **(i)** Express $\dfrac{2(2r+3)}{(r+1)(r+2)(r+3)}$ as partial fractions. [2]

(ii) Hence find an expression for $\displaystyle\sum_{r=1}^{n}\frac{2r+3}{(r+1)(r+2)(r+3)}$ in terms of n. [4]

(iii) Using the result in part **(ii)**, show that the following series is convergent and find its sum to infinity.

$$\frac{21}{3\times4\times5}+\frac{27}{4\times5\times6}+\frac{33}{5\times6\times7}+\dots\dots\dots \qquad [4]$$

3. The diagram below shows the structure of an indoor farm to be built. Edges *OA*, *DC*, *IH* and *EF* are vertical, and edges *OA*, *AB*, *BC* and *CD* are equal to *EF*, *FG*, *GH* and *HI* respectively. Diagram is not drawn to scale.



Taking *O* as the origin and the vectors **i**, **j** and **k** as unit vectors along the *OD*, *OE* and *OA* respectively, the coordinates of *A* and *C* are (0, 0, 20) and (50, 0, 25) respectively. All lengths are measured in metres.

(i) Given that $-\mathbf{i} + 4\mathbf{k}$ is a vector perpendicular to the flat roof *ABGF*, and $\mathbf{i} + 2\mathbf{k}$ is perpendicular to the flat roof *BCHG*, find the obtuse angle between the two roofs. [2]

(ii) A humidity sensor is to be installed along the line *BG*, 50 metres away from *B*. Find the coordinates of the location of the sensor. [4]

(iii) A metal cable is anchored on the ground outside the farm in front of *OABCD*, at a point 25 metres away from *O* along *OD* and 10 metres in front of *OD*. The cable is attached to a point on *AB* to secure the roof *ABGF*. Find the length of the shortest cable needed. [3]

(iv) State an assumption for the above calculations to be valid. [1]

4. The function f is defined by $\mathrm{f}: x \mapsto \dfrac{1}{(x-2)^2}$ for $x < m$ where $m < 2$.

(i) Find the range of values of m such that both f^{-1} and f^2 exist. [4]

(ii) Suppose the range of values of m found in part **(i)** holds, find the functions f^{-1} and f^2, and state their domains and range. [6]

Section B: Statistics [60 marks]

5. Two fair six-sided dice are thrown. The random variable X is the smaller of the two scores if they are different, and their common value if they are the same.

(i) Show that $\mathrm{P}(X = 2) = \dfrac{1}{4}$ and find the probability distribution of X. [2]

(ii) Hence find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

A game is played with Ivan and Jon taking turns to throw the two dice each. Ivan throws the dice first and the player who first obtain the value of X equals to 2 wins the game. If Ivan wins the game, Jon pays him \$7. Otherwise, Ivan pays Jon \$10.

(iii) Find the player who has the higher expected gain. Justify your answer. [3]

6. **(a)** During a class reunion, 4 men and 6 women decide to stand in a row to take a class photograph. Find the number of ways that they can do so if

(i) there is no restriction, [1]

(ii) at least 2 men stand together, [2]

(iii) exactly 5 women stand together. [3]

(b) The 10 persons then sit at a round table for dinner. Find the probability that 2 particular women sit opposite each other. [2]

7. At a supermart, the option of using cashless payment for purchases is available to customers. On average, p% of the customers use cashless payment.

The manager of the supermart selects a sample of 30 customers for a survey.

(i) The probability of not more than 1 of customers surveyed use cashless payment for their purchases is 0.245. Write down an equation in terms of p. Find p, giving your answer correct to 3 decimal places. [3]

For the remaining parts of this question, use $p = 30$.

(ii) Find the probability that the last customer of the 30 selected for survey is the 10^{th} customer who uses a cashless payment for his purchase. [2]

Another manager of the supermart also selects a sample of 30 customers.

(iii) Given that there are a total of less than 16 customers using cashless payment for their purchases from both samples, find the probability that one of the sample has more than 12 customers using cashless payment for their purchases. [3]

(iv) State one assumption you have made in the calculation in part **(iii)**. [1]

8. Verde wants to investigate the time taken for different volumes of water to cool to room temperature. He prepared a few samples of different volume and heated the samples to their boiling point, and then recorded the time taken for the water to cool to room temperature. The results are given in the table.

Volume (x/cm^3)	100	200	300	400	500	600	700
Time (t/min)	14	23	47	83	a	172	293

(i) It is known that the regression line of t on x is $t = -65.1429 + 0.4318x$. Show that $a = 121$. [2]

(ii) Draw a scatter diagram for the data and find the product moment correlation coefficient between x and t. [2]

(iii) Comment whether the regression line is appropriate based on

(a) the scatter diagram; [1]

(b) the context. [1]

Verde considers using one of the following two models:

$$\text{A: } t = a + bx^2, \quad \text{B: } t = ae^{bx},$$

where $a, b \in \mathbb{R}$, for the relationship between x and t.

(iv) Explain which is the better model and find the equation of a suitable regression line for that model. [2]

(v) Estimate the time taken for 450 cm^3 of water to cool from boiling point to room temperature. Comment on the reliability of the estimate. [3]

9. A newspaper report claims that the mean starting monthly salary of a fresh university graduate is \$3500. However a human resource manager believes that this claim is incorrect. A random sample of 60 fresh university graduates is surveyed and their starting monthly salaries, \$$x$, are summarized by

$$\sum x = 207000, \quad \sum (x-3400)^2 = 5450000.$$

(i) Calculate the unbiased estimates of the population mean and variance. [2]

(ii) Test, at the 5% significance level, whether the human resource manager's belief is correct. [4]

(iii) State, in the context of the question, the meaning of the p-value found in part **(ii)**. [1]

A second sample of 50 fresh university graduates is surveyed and the sample mean and standard deviation of their starting salaries are found to be \$$\overline{y}$ and \$342 respectively.

(iv) Find the range of values of $\overline{y}$ such that this second sample would result in accepting the manager's belief at the 5% significance level. [3]

(v) Suppose the claim by the newspaper is correct and the standard deviation of the starting salaries of fresh university graduates is \$543, find the probability that a random sample of 50 fresh university graduates have its mean salary below \$3350. [2]

10. Gary likes to take part in duathlons where there is a total running distance of 15 km and a total cycling distance of 36 km. His timings, in minutes, for running and cycling are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean	Standard Deviation
Timing for running (min)	100	15
Timing for cycling (min)	μ	σ

(i) The probability that he completes his cycling in less than an hour is equal to the probability that he completes his cycling after 2 hours. Write down the value of μ. [1]

(ii) The probability that he completes his cycling under 80 minutes is 0.158655. Show that $\sigma = 10$. [2]

(iii) Find the probability that thrice the time needed for him to complete cycling is more than an hour from twice the time needed for him to complete the running. [2]

(iv) Find the probability that his timing for cycling is less than 80 minutes and his timing for running is less than 90 minutes. [2]

(v) Show that the probability of Gary finishing the duathlon under 170 minutes is 0.134. Give a reason why this probability is greater than the probability calculated in part **(iv)**. [2]

Gary's target timing for completing a duathlon is under 2 hours 50 minutes. Over the past two years, he has already completed 9 duathlons.

(vi) Find the probability that he achieves his target timing in at least 3 but fewer than 6 of the duathlons over the last two years. State an assumption needed for your calculation to be valid. [4]

END OF PAPER

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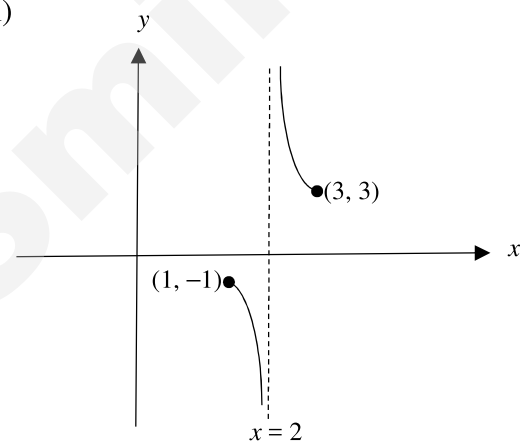
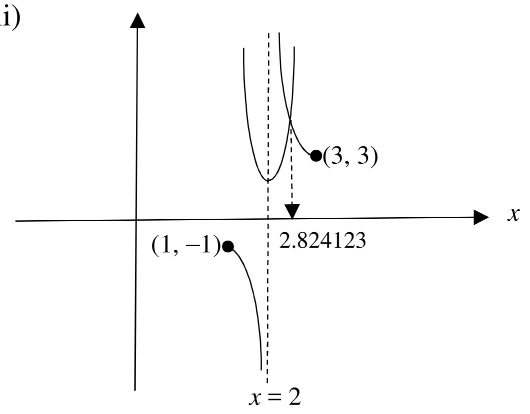
Solutions to 2018 Y6 H2 Maths Prelim Exam P2

1.	Solution [10 marks]	
	(i) $y=\dfrac{x^2-3x+3}{x-2}=x-1+\dfrac{1}{x-2}$ $\dfrac{dy}{dx}=1-\dfrac{1}{(x-2)^2}$ For $x=1$ or 3, $\dfrac{dy}{dx}=0$ $\Rightarrow C$ has stationary points at $x=1$ and $x=3$. For $1<x<2$ or $2<x<3$, $\dfrac{dy}{dx}<0$ $\Rightarrow C$ is decreasing on $1<x<2$ and $2<x<3$. Alternatively, Since $1\le x\le 3$, $0\le(x-2)^2\le 1$ $\therefore \dfrac{1}{(x-2)^2}\ge 1$ $\therefore \dfrac{dy}{dx}\le 0$ i.e. the gradient is always non-positive.	
	(ii)  	

(iii)



$$\frac{x^2-4x+5}{x-2} \geq 3(x-2)^2$$

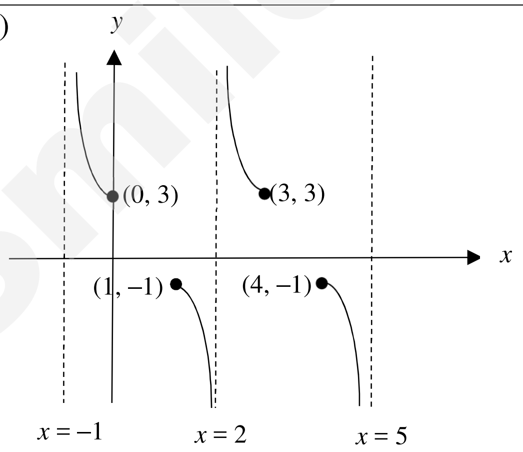
$$\frac{x^2-3x+3-x+2}{x-2} \geq 3(x-2)^2$$

$$\frac{x^2-3x+3}{x-2} \geq 3(x-2)^2+1$$

Sketch the addition graph $y=3(x-2)^2+1$.

Thus, $2 < x \leq 2.82$.

(iv)



2.	Solution [10 marks]	
	(i) Let $\dfrac{2(2r+3)}{(r+1)(r+2)(r+3)}=\dfrac{A}{r+1}+\dfrac{B}{r+2}+\dfrac{C}{r+3}$. By cover up rule, $A=\dfrac{2(1)}{(1)(2)}=1;\ B=\dfrac{2(-1)}{(-1)(1)}=2;\ C=\dfrac{2(-3)}{(-2)(-1)}=-3;$ $\therefore\dfrac{2(2r+3)}{(r+1)(r+2)(r+3)}=\dfrac{1}{r+1}+\dfrac{2}{r+2}-\dfrac{3}{r+3}$	
	(ii) $\displaystyle\sum_{r=1}^{n}\frac{2r+3}{(r+1)(r+2)(r+3)}=\frac{1}{2}\sum_{r=1}^{n}\frac{2(2r+3)}{(r+1)(r+2)(r+3)}$ $\displaystyle=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r+1}+\frac{2}{r+2}-\frac{3}{r+3}\right)$ $=\dfrac{1}{2}\left[\begin{array}{ccccc}\frac{1}{2}&+&\frac{2}{3}&-&\frac{3}{4}\\ \frac{1}{3}&+&\frac{2}{4}&-&\frac{3}{5}\\ \frac{1}{4}&+&\frac{2}{5}&-&\frac{3}{6}\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ \frac{1}{n-1}&+&\frac{2}{n}&-&\frac{3}{n+1}\\ \frac{1}{n}&+&\frac{2}{n+1}&-&\frac{3}{n+2}\\ \frac{1}{n+1}&+&\frac{2}{n+2}&-&\frac{3}{n+3}\end{array}\right]$ $=\dfrac{1}{2}\left[\dfrac{3}{2}-\dfrac{1}{n+2}-\dfrac{3}{n+3}\right]$ $=\dfrac{3}{4}-\dfrac{1}{2(n+2)}-\dfrac{3}{2(n+3)}$	

	(iii) $\frac{21}{3\times4\times5}+\frac{27}{4\times5\times6}+\frac{33}{5\times6\times7}+......$ $=3\left(\frac{7}{3\times4\times5}+\frac{9}{4\times5\times6}+\frac{11}{5\times6\times7}+....\right)$ $=3\sum_{r=2}^{\infty}\frac{2r+3}{(r+1)(r+2)(r+3)}$ $=3\left[\sum_{r=1}^{\infty}\frac{2r+3}{(r+1)(r+2)(r+3)}-\frac{2(1)+3}{(1+1)(1+2)(1+3)}\right]$ $=3\lim_{n\to\infty}\left[\frac{3}{4}-\frac{1}{2(n+2)}-\frac{3}{2(n+3)}\right]-\frac{5}{8}$ As $n\to\infty$, $\frac{3}{2(n+2)}\to0$ and $\frac{9}{2(n+3)}\to0$. Hence the series is convergent, and the sum = $3\left[\frac{3}{4}-0-0\right]-\frac{5}{8}=\frac{13}{8}$	

3.	Solution [10 marks]	
	(i) $\cos^{-1}\left(\frac{\begin{pmatrix}-1\\0\\4\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}}{\sqrt{17}\sqrt{5}}\right)=\cos^{-1}\frac{7}{\sqrt{17}\sqrt{5}}=40.6013°$ Angle between roofs = $180°-40.6013°=139.4°$	

(ii) Plane $ABGF$: $\mathbf{r}\cdot\begin{pmatrix}-1\\0\\4\end{pmatrix}=\begin{pmatrix}0\\0\\20\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\4\end{pmatrix}=80$

Plane $BCHG$: $\mathbf{r}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}50\\0\\25\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=100$

Taking intersection and using GC,

$l_{BG}:\mathbf{r}=\begin{pmatrix}40\\0\\30\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\0\end{pmatrix},\ \lambda\in\mathbb{R}$

When $y = 50$, $\lambda = 50$.
Thus coordinates of sensor is (40, 50, 30).

(iii) Coordinates of B is (40, 0, 30).

Cable is anchored to the ground at the point $P(25, -10, 0)$.

Required length = perpendicular distance from P to line AB.

$\overrightarrow{PA}=\begin{pmatrix}0\\0\\20\end{pmatrix}-\begin{pmatrix}25\\-10\\0\end{pmatrix}=\begin{pmatrix}-25\\10\\20\end{pmatrix}$

$\overrightarrow{AB}=\begin{pmatrix}40\\0\\30\end{pmatrix}-\begin{pmatrix}0\\0\\20\end{pmatrix}=\begin{pmatrix}40\\0\\10\end{pmatrix}$

Length of shortest cable

$=\left|\overrightarrow{PA}\times\begin{pmatrix}4\\0\\1\end{pmatrix}\right|=\left|\begin{pmatrix}-25\\10\\20\end{pmatrix}\times\begin{pmatrix}4\\0\\1\end{pmatrix}\right|$

$=\left|\dfrac{\begin{pmatrix}10\\105\\-40\end{pmatrix}}{\sqrt{17}}\right|=27.4$ metres

	(iv) The thickness of materials used for building is negligible.	

4.	Solution [10 marks]	
	(i) f^{-1} exists because f is one-one on $x < m < 2$. $D_f = (-\infty, m)$, $R_f = \left(0, \frac{1}{(m-2)^2}\right)$ For f^2 to exist, $R_f \subseteq D_f$. $0 < \frac{1}{(m-2)^2} \le m$ $m(m-2)^2 \ge 1$ $m^3 - 4m^2 + 4m - 1 \ge 0$ Solving with GC, $0.382 \le m \le 1$ or $m \ge 2.84$ (reject since $m < 2$)	
	(ii) $y = \frac{1}{(x-2)^2} \Rightarrow x = 2 \pm \frac{1}{\sqrt{y}}$ Since $x < m < 2$, $x = 2 - \frac{1}{\sqrt{y}}$. $f^{-1}(x) = 2 - \frac{1}{\sqrt{x}}$ $D_{f^{-1}} = \left(0, \frac{1}{(m-2)^2}\right)$ $R_{f^{-1}} = (-\infty, m)$ $f^2(x)$ $= f\left(\frac{1}{(x-2)^2}\right)$ $= \frac{1}{\left(\frac{1}{(x-2)^2} - 2\right)^2}$ $= \frac{(x-2)^4}{\left(1 - 2(x-2)^2\right)^2}$	

$D_{f^2} = (-\infty, m)$

$$(-\infty, m) \xrightarrow{f} \left(0, \frac{1}{(m-2)^2}\right) \xrightarrow{f} \left(\frac{1}{4}, \frac{(m-2)^4}{\left(1-2(m-2)^2\right)^2}\right) = R_{f^2}$$

5. Solution [7 marks]

(i) Using the following possibility table, we have:

2nd die (columns), **1st die** (rows)

	1	**2**	**3**	**4**	**5**	**6**
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

From the table, $P(X=2) = \frac{9}{36} = \frac{1}{4}$

Probability distribution of *X*:

x	1	2	3	4	5	6
$P(X=x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

	(ii) $\mathrm{E}(X)=\sum_{x=1}^{6} x\mathrm{P}(X=x)=\frac{91}{36}$ (or 2.53) $\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$ $=\left(\sum_{x=1}^{6} x^2\mathrm{P}(X=x)\right)-\left(\sum_{x=1}^{6} x\mathrm{P}(X=x)\right)^2$ $=\frac{301}{36}-\left(\frac{91}{36}\right)^2$ $=\frac{2555}{1296}$ (or 1.97)	
	(iii) P(Ivan wins) $=\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)+\ldots$ $=\frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^2}=\frac{4}{7}$ E(gain) for Ivan $=\frac{4}{7}\times 7+\frac{3}{7}\times(-10)=-\$0.29$ Since Ivan is expecting to lose 29 cents, Jon has a higher expected gain. <u>Alternatively</u>, P(Jon wins) $=\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)+\ldots$ $=\frac{\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)}{1-\left(\frac{3}{4}\right)^2}=\frac{3}{7}$ E(gain) for Jon $=\frac{3}{7}\times 10+\frac{4}{7}\times(-7)=\$0.29$ Since Jon is expecting to win 29 cents (i.e. Ivan is expected to lose 29 cents), he has a higher expected gain.	

6. Solution [8 marks]

(a) (i) Number of ways = 10! = 3 628 800 ways

(ii) $\uparrow W_1 \uparrow W_2 \uparrow W_3 \uparrow W_4 \uparrow W_5 \uparrow W_6 \uparrow$

$\uparrow$ Position for each of the men

Number of ways = 10! – No. of ways all men are separated

$= 10! - 6! \times \binom{7}{4} 4!$

= 3 628 800 – 604 800
= 3 024 000 ways

(iii)

$\uparrow M_1 \uparrow M_2 \uparrow M_3 \uparrow M_4 \uparrow$

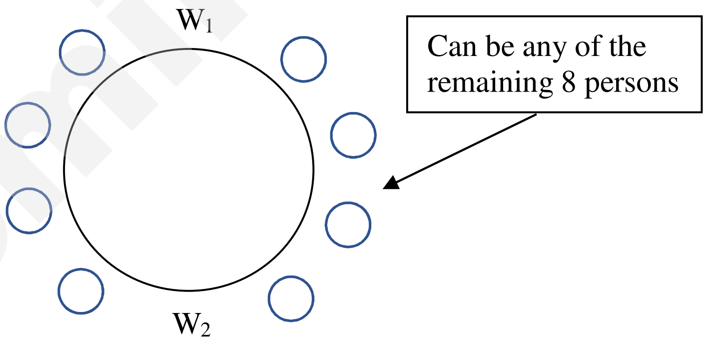
$\uparrow$ Position for the following "items"

$\boxed{W_1W_2W_3W_4W_5}$, W_6

Number of ways = $4! \times \binom{5}{2} 2! \times \overbrace{\binom{6}{5}}^{\text{the w group}} 5!$

= 345 600 ways

(b) Let particular women be W_1, W_2



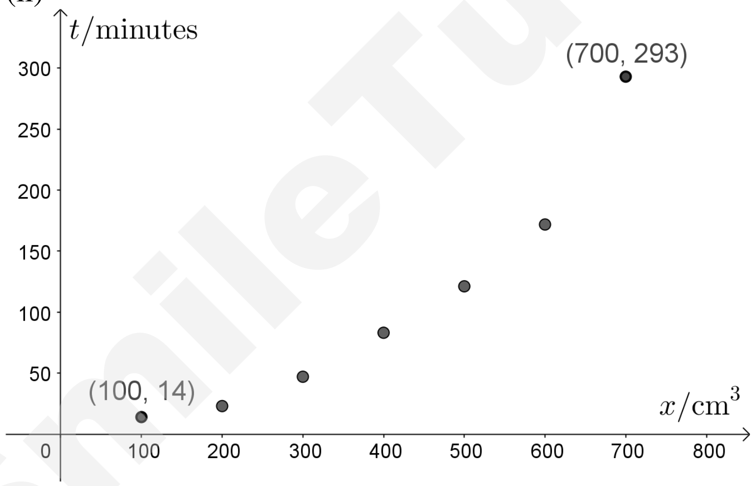
Fix W_1 and W_2, then arrange the remaining 8 persons.
$\Rightarrow$ No. of ways = 8!

$\therefore$ Required probability

$= \frac{8!}{(10-1)!} = \frac{1}{9}$

7.	Solution [10 marks]	
	(i) Let X be the r.v. "no of customers who uses cashless payment" out of 30. $X \sim \mathrm{B}\left(30, \frac{p}{100}\right)$ $\mathrm{P}(X \le 1) = 0.245$ $\mathrm{P}(X=0)+\mathrm{P}(X=1)=0.245$ $\binom{30}{0}\left(1-\frac{p}{100}\right)^{30}+\binom{30}{1}\left(\frac{p}{100}\right)\left(1-\frac{p}{100}\right)^{29}=0.245$ $\left(1-\frac{p}{100}\right)^{30}+0.3p\left(1-\frac{p}{100}\right)^{29}-0.245=0$ Using Solver in GC, $p = 8.811$	
	(ii) Let Y be the r.v. "no of customers who use cashless payment" out of 29. $Y \sim \mathrm{B}(29, 0.3)$ Required Probability $= \mathrm{P}(Y=9)\times 0.3$ $= 0.0471872337$ ≈ 0.0472	
	(iii) $X \sim \mathrm{B}(30, 0.3)$ Let W be the r.v. "no of customers who use cashless payment" out of 60. $W \sim \mathrm{B}(60, 0.3)$ Requires probability $= \frac{\mathrm{P}(X_1=13, X_2 \le 2)+\mathrm{P}(X_1=14, X_2 \le 1)+\mathrm{P}(X_1=15, X_2=0)}{\mathrm{P}(W<16)}$ $= \frac{2\times\left(\mathrm{P}(X=13)\mathrm{P}(X\le 2)+\mathrm{P}(X=14)\mathrm{P}(X\le 1)+\mathrm{P}(X=15)\mathrm{P}(X=0)\right)}{\mathrm{P}(W \le 15)}$ $= 0.0008312264247$ ≈ 0.000831	

	(iv) The assumption is that the people being surveyed are all different in both samples. (<u>OR</u> Samples are independently and identically distributed.)	

8.	Solution [10 marks]	
	(i) $\overline{x} = 400$ $\overline{t} = -65.1429 + 0.4318(400)$ $\overline{t} = 107.5771$ $14 + 23 + 47 + 83 + a + 172 + 293 = 7 \times \overline{t}$ $632 + a = 753.0397$ $a \approx 121$ (shown)	
	(ii)  $r = 0.9414677299 \approx 0.941$ (3 sig fig)	
	(iii) (a) Although the r value is 0.941 which is close to 1 suggesting that there is a strong linear correlation, the scatter diagram shows that the relationship is more of a curvilinear one, with t increasing increasingly as x increases. Thus, the regression line is not appropriate. (b) The regression line predicts that for volume less than a certain amount, it takes negative time for the water to cool to room temperature which is not possible in the context. Thus, the regression line is not appropriate.	

	(iv) Calculating the r- values for models A and B, Model A: r-value = 0.987 while Model B: $t = ae^{bx} \Rightarrow \ln t = \ln a + bx$, r-value = 0.994, Since the $\|r\|$-value for model B is closer to 1 compared to model A, model B is the better model. Regression line of ln t on x is $\ln t = 2.224858298 + 0.0050332105x$ $\ln t = 2.22 + 0.00503x$ (3 sig fig)	
	(v) When $x = 450$, $\ln t = 2.224858298 + 0.0050332105(450)$ $t = 89.10389273$ ≈ 89.1 minutes (3 sig fig) The estimate is reliable as the appropriate regression line is used (t being the dependent variable); r = 0.994 which is close to 1, suggesting a strong linear correlation between ln t and x; and the estimated point where x = 450 is within the data range of 100 to 700.	

9.	Solution [12 marks]	
	(i) $\bar{x} = \frac{207000}{60} = 3450$ $\sum(x - 3400) = 3000$ $s^2 = \frac{1}{59}\left[5450000 - \frac{3000^2}{60}\right] = 89830.50847 \approx 89800$	

	(ii) Let μ be the mean starting salaries of fresh graduates. Test H_0: $\mu = 3500$ (i.e. manager's belief is incorrect) Against H_1: $\mu \neq 3500$ (i.e. manager's belief is correct) Perform a 2-tailed test at 5% level of significance. Test Statistics: Under H_0, $\overline{X} \sim \mathrm{N}\left(3500, \frac{s^2}{60}\right)$ approximately $Z = \frac{\overline{X} - 3500}{s/\sqrt{60}} \sim \mathrm{N}(0,1)$ approximately Using GC, *p*-value = 0.196284 (>0.05) Do not reject H_0. There is insufficient evidence at 5% level of significance that the human resource manager's belief is correct.	
	(iii) *p*-value of 0.196284 is the smallest value of significance level for which the claim that the mean starting monthly salary of a fresh university graduate is \$3500 would be rejected. Or *p*-value of 0.196284 is the probability of obtaining a sample mean as extreme as the one obtained, assuming the claim that the mean starting monthly salary of a fresh university graduate is \$3500 is true.	

	(iv) Test H_0: $\mu = 3500$ Against H_1: $\mu \neq 3500$ Perform a 2-tailed test at 5% level of significance. $s^2 = \frac{50}{49} \times 342^2 = 119351.0204$ Test Statistics: Under H_0, $\overline{Y} \sim \mathrm{N}\left(3500, \frac{119351.0204}{50}\right)$ approximately $Z = \frac{\overline{Y} - 3500}{\sqrt{119024.4898/50}} \sim \mathrm{N}(0,1)$ approximately Since H_0 is rejected, $\frac{\overline{y} - 3500}{\sqrt{119351.0204/50}} < -1.95996$ $\overline{y} < 3404.241954$ Or $\frac{\overline{y} - 3500}{\sqrt{119351.0204/50}} > 1.95996$ $\overline{y} > 3595.758046$ Answer: $\overline{y} < 3400$ or $\overline{y} > 3600$ (to 3sf)	
	(v) $\overline{X} \sim \mathrm{N}\left(3500, \frac{543^2}{50}\right)$ approximately by CLT since sample size (50) is large. $\mathrm{P}(\overline{X} < 3350) = 0.0254$	

10.	Solution [13 marks]	
	(i) $\mu = \frac{60+120}{2} = 90$ minutes	

	(ii) Let C be Gary's timing for cycling. $C \sim N\left(90, \sigma^2\right)$ $Z = \dfrac{C-90}{\sigma} \sim N(0,1)$ $\mathrm{P}(C<80) = 0.158655$ $\mathrm{P}\left(Z < \dfrac{80-90}{\sigma}\right) = 0.158655$ $\dfrac{80-90}{\sigma} = -1.000001057$ $\sigma = \dfrac{-10}{-1.000001057} \approx 10$	
	(iii) Let R be Gary's timing for running. $R \sim N\left(100, 15^2\right)$ $3C - 2R \sim N(70, 1800)$ $\mathrm{P}(3C-2R>60) = 0.5931680976 \approx 0.593$ (3 sig fig)	
	(iv) Required probability $= \mathrm{P}(C<80) \times \mathrm{P}(R<90)$ $= 0.0400592579$ ≈ 0.0401 (3 sig fig)	
	(v) $C + R \sim N(190, 325)$ $\mathrm{P}(C+R<170) = 0.1336287896 \approx 0.134$ (3 sig fig)(shown) The event in part (iv) is but a proper subset of that in this part, hence the probability is here is greater than that in (iv).	

	(vi) Let X denote the number of duathlons which Gary manages to achieve his target timing, out of 9 duathlons. $X \sim B(9, 0.134)$ $\mathrm{P}(3 \le X < 6) = \mathrm{P}(X \le 5) - \mathrm{P}(X \le 2)$ $= 0.1080999944$ ≈ 0.108 (3 sig fig) The timings for each of the duathlons are independent.	

SERANGOON JUNIOR COLLEGE

2018 JC2 PRELIMINARY EXAMINATION

MATHEMATICS

Higher 2 **9758/1**

11 Sept 2018

3 hours

Additional materials: Writing paper

List of Formulae (MF 26)

TIME : 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and class on the cover page and on all the work you hand in.
Write in blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

Total marks for this paper is 100 marks.

This question paper consists of 6 printed pages (inclusive of this page) and 2 blank pages.

[TURN OVER]

Answer all questions [100 marks].

1	Find $\int \frac{x^2}{x^2+2x+5}\,dx$.	[3]
	Solution	
	$\int \frac{x^2}{x^2+2x+5}\,dx = \int 1 - \frac{2x+5}{x^2+2x+5}\,dx$	
	$= \int 1\,dx - \int \frac{2x+2}{x^2+2x+5}\,dx - 3\int \frac{1}{(x+1)^2+4}\,dx$	
	$= x - \ln\left(x^2+2x+5\right) - \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$	
2	The complex numbers z and w satisfy the equations	
	$zw^* + 2z = 15i$ and $2w + 3z = 11$.	
	Find the complex numbers z and w.	[6]
	Solution	
	$zw^* + 2z = 15i \cdots\cdots(1)$	
	$w = \frac{1}{2}(11-3z) \cdots\cdots(2)$	
	Subst (2) into (1) gives	
	$\frac{1}{2}z(11-3z^*) + 2z = 15i$	
	$\frac{11}{2}z - \frac{3}{2}\lvert z\rvert^2 + 2z = 15i$	
	$\frac{15}{2}z - \frac{3}{2}\lvert z\rvert^2 = 15i$	
	Let $z = x + iy$	
	$\frac{15}{2}(x+iy) = \frac{3}{2}(x^2+y^2) + 15i$	
	Comparing the real and Imaginary parts, we have	
	$\frac{15}{2}y = 15$ and $\frac{15}{2}x = \frac{3}{2}x^2 + \frac{3}{2}y^2 \cdots\cdots(3)$	
	$y = 2$	
	Subst into (3) gives	
	$x^2 - 5x + 4 = 0$	
	$(x-4)(x-1) = 0$	
	$\therefore x = 1$ or 4	
	When $z = 1+2i$, $w = \frac{1}{2}(11-3-6i) = 4-3i$	
	When $z = 4+2i$, $w = \frac{1}{2}(11-12-6i) = -\frac{1}{2} - 3i$	

Commented [DsK1]: A handful tried to split using partial fractions but the denominator is not able to be expressed in factors.

Many had careless mistakes here which cost them a lot of marks. Either a positive sign was inserted or with the positive sign, some wrote as 2x-5.

Commented [DsK2]: Many were found to have the inability to recognise $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C$.

Commented [DsK3]: Most students were able to obtain the first 2 expressions however they struggled for the last expression. Some continued with ln expressions. They will need more practice as inverse trigo types of integration.

A countable ones also forgot to insert the constant *C*.

Commented [LHMD4]: Careless mistake
Many were careless with their working and failed to obtain the correct complex numbers, often due to wrong inclusion or exclusion of negative signs.

Commented [LHMD5]: Misconception
Many rejected one of the answers (similarly for those who found the real parts for *w* first), often claiming the real part must be positive or they do not satisfy one of the equations.

Commented [LHMD6]: Careless mistake
•Some found the correct answer for *w* but wrote them to be the answers for *z* and hence found the wrong answers for *w* and vice versa.
•Some were able to obtain the real and imaginary parts of either *z* or *w* but were confused when they wrote down the answer for *z* or *w* and turned out to be wrong.

3	Without the use of a graphing calculator, find the range of values of x for which $\frac{18}{2-x} \geq x^2+4x+9$.	[3]
	Hence find the exact range of values of x for which $\frac{18}{2-e^x} \geq e^{2x}+4e^x+9$.	[3]

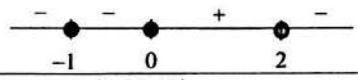
Solution

$\frac{18}{2-x} \geq x^2+4x+9$

$\frac{18+(x-2)(x^2+4x+9)}{2-x} \geq 0$

$\frac{18+x^3+2x^2+x-18}{2-x} \geq 0$

$\frac{x(x+1)^2}{2-x} \geq 0$



$x=-1$ or $0 \leq x < 2$

Replace x by e^x

So $e^x=-1$ or $0 \leq e^x < 2$

No solution or $(0 \leq e^x$ and $e^x < 2)$

If did not state the reason for why the answer to this inequality is $x < \ln 2$, "P" will be given.

No solution or $(x \in \mathbb{R}$ and $x < \ln 2)$

$\therefore x < \ln 2$

Commented [FTL7]: Some are still weak in algebraic calculation and cross multiply which is not acceptable

Commented [FTL8]: Many miss out the solution for $x=-1$

Commented [FTL9]: Many also miss out $e^x=-1$ (no solution)

Commented [FTL10]: Many also did not state the reason why $x < \ln 2$

4	**(i)** The curve with equation $\frac{y^2}{9}-x^2=1$ undergoes a two-step transformations to become curve C with equation $\frac{y^2}{9}-\frac{(x-1)^2}{4}=1$. State the two transformations involved for the curve $\frac{y^2}{9}-x^2=1$.	[2]
	(ii) Draw a sketch of the curve C, labelling clearly the equation(s) of its asymptote(s), intersection with the axes and the coordinates of any turning points.	[3]
	(iii) Show that the point $(-1,-3)$ lies on the line $y=mx+m-3$ for all real values of m.	[1]
	(iv) Hence using the diagram drawn in **(ii)**, find the range of values of k such that the equation $\frac{(kx+k-3)^2}{9}-\frac{(x-1)^2}{4}=1$ has 2 negative real roots.	[2]

Solution

[TURN OVER]

(i) Method 1

$$\frac{y^2}{9}-x^2=1\to\frac{y^2}{9}-\frac{x^2}{4}=1\to\frac{y^2}{9}-\frac{(x-1)^2}{4}=1$$

(1) Scaling parallel to the x axis by a scale factor of 2.

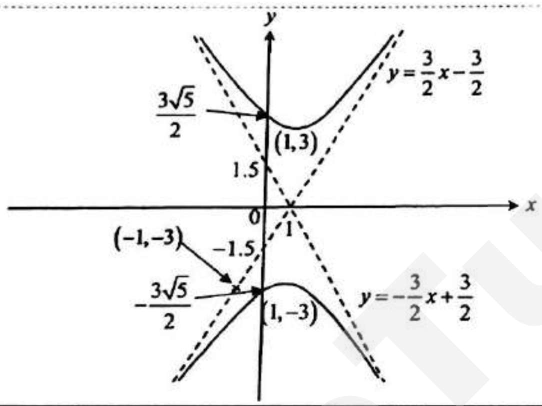
(2) Translation of 1 unit in the positive x direction.

Method 2

$$\frac{y^2}{9}-x^2=1\to\frac{y^2}{9}-\left(x-\frac{1}{2}\right)^2=1\to\frac{y^2}{9}-\frac{(x-1)^2}{4}=1$$

(1) Translation of 0.5 units in the positive x direction

(2) Scaling parallel to the x axis by a scale factor of 2.



(iii) When $x=-1$, RHS $=m(-1)+m-3=-3=$ LHS

Hence $(-1,-3)$ lies on the line $y=mx+m-3$ for all m.

(iv) From the sketch, the range of values of k are $k>\frac{3\sqrt{5}}{2}+3$ or $k<-\frac{3}{2}$

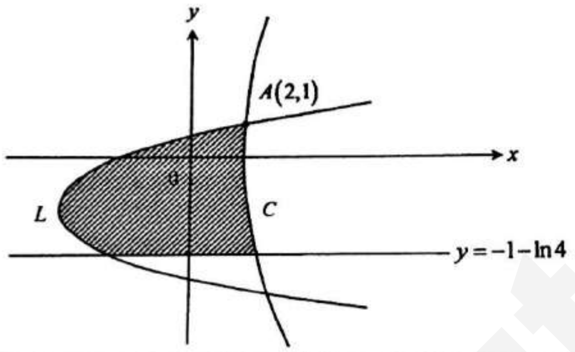
Commented [FTL11]: Weak students are still confused with the scale factor and direction of translation

Some students are also confused with the order of transformation

Commented [FTL12]: Needs to draw a big, clear and presentable sketch with labels.

Commented [FTL13]: Many are still unable to show this such as equating LHS = RHS at the beginning of solution.

Commented [FTL14]: Many are unable to find the range of values of k.

5	(i) Find $\int \ln(x+1)\,dx$ for $x > -1$. Show your working clearly.		[2]
	(ii) The curve C is defined by the parametric equations $x = 2t - 2\ln(t+1) + 2, \quad y = -2t - 2\ln(t+1) + 1 \qquad$ where $t > -1$.		
	Another curve L is defined by the equation $x = 2(y+1)^2 - 6$. The graphs of C and L intersect at the point $A(2,1)$ as shown in the diagram below.		
			
	Find the area of the shaded region bounded by C, L and the line $y = -1 - \ln 4$, express your answer in the form $\frac{62}{3} + 4A - 2A^2 - \frac{16A^3}{3}$, where A is an exact real constant.		[6]

Solution

(i) $\int \ln(1+x)\,dx = x\ln(1+x) - \int \frac{x}{1+x}\,dx$

$= x\ln(1+x) - \int 1 - \frac{1}{1+x}\,dx$

$= x\ln(1+x) - x + \ln(1+x) + c$

$= (x+1)\ln(1+x) - x + c$

Commented [LHMD15]: Misconception

Many gave $\frac{1}{1+x}$ as the answer, confusing it with differentiation of $\ln(1+x)$.

Commented [LHMD16]: Misconception

A few did by parts on $\int \frac{x}{1+x}\,dx$.

Commented [LHMD17]: Presentation

- Many did not simplify the answer to the simplest form.
- Some did not write the arbitrary constant c.

Misconception

Many left the answer with modulus i.e. $\ln|1+x|$ where $1+x > 0$

[TURN OVER]

	(ii) Area $= \int_{-1-\ln 4}^{1} x_C \, dy - \int_{-1-\ln 4}^{1} \left[2(y+1)^2 - 6\right] dy$	
	$= \int_{1}^{0} \left[2t+2-2\ln(1+t)\right]\left(-2-\frac{2}{1+t}\right) dt - \left[\frac{2(y+1)^3}{3} - 6y\right]_{-1-\ln 4}^{1}$	
	$= 4\int_{0}^{1} \left[t+1-\ln(1+t)\right]\left(1+\frac{1}{1+t}\right) dt - \left\{\frac{16}{3} - 6 - \left[\frac{-2(\ln 4)^3}{3} + 6 + 6\ln 4\right]\right\}$	
	$= 4\int_{0}^{1} t+2 \, dt - 4\int_{0}^{1} \ln(1+t) \, dt - 4\int_{0}^{1} \frac{\ln(1+t)}{1+t} dt - \left[-\frac{20}{3} + \frac{2(\ln 4)^3}{3} - 6\ln 4\right]$	
	$= \left[2(t+2)^2\right]_0^1 - 4\left[(1+t)\ln(1+t) - t\right]_0^1 - 2\left[(\ln(1+t))^2\right]_0^1 - \left[-\frac{20}{3} + \frac{2(\ln 4)^3}{3} - 6\ln 4\right]$	
	$= (18-8) - 4\left[2\ln 2 - 1\right] - 2(\ln 2)^2 + \frac{20}{3} - \frac{2(\ln 4)^3}{3} + 6\ln 4$	
	$= 14 - 8\ln 2 - 2(\ln 2)^2 + \frac{20}{3} - \frac{2(\ln 4)^3}{3} + 6\ln 4$	
	$= \frac{62}{3} + 4\ln 2 - 2(\ln 2)^2 - \frac{16(\ln 2)^3}{3}$	
6	**(a) (i)** The twelfth, eighth and fifth terms of an arithmetic progression are three consecutive terms of a converging geometric progression of positive terms with common ratio r. Find the value of r.	[3]
	(ii) Take the value of r to be $\frac{3}{4}$. If the difference between the sum of the first n terms of the geometric progression and its sum to infinity is less than 0.3% of the sum to infinity, find the least value of n.	[3]
	(b) A convergent geometric sequence of positive terms, G has first non-zero term a and common ratio r.	
	(i) The sum of the first n odd-numbered terms of G is equal to the sum of all terms after the $(2n-1)^{\text{th}}$ term of G. Show that $2r^{2n} + r^{2n-1} - 1 = 0$.	[2]
	(ii) In another sequence H, each term is the reciprocal of the corresponding term of G. If the nth term of G and H is denoted by u_n and v_n respectively, show that a new sequence whose nth term is $\ln\left(\frac{u_n}{v_n}\right)$, is an arithmetic progression.	[2]

Commented [LHMD18]: Many attempted to find Cartesian equation for the parametric but failed and even if they did find it, they could not use it as it is pretty complex.

Misconception

- Many found $\frac{dy}{dx}$, most likely thinking of integrating to obtain y but failed to do so as the expression is in terms of t.
- Some linked parametric equations to finding equation of tangent and went on the usual steps to finding the equation of tangent, which was useless.

Commented [LHMD19]: **Question interpretation**

- Some failed to see the area as a simple subtraction of two integrals and over-complicated their working, often integrating with respect to both x and y.
- Some interchanged the integrals i.e. $\int_{-1-\ln 4}^{1}\left[2(y+1)^2-6\right] dy - \int_{-1-\ln 4}^{1} x_C \, dy$

Commented [LHMD20]: **Misconception**

- Some used the expression for y instead of the one for x.
- Many either did not change the limits or obtained the wrong limits, with a few writing 0 as the lower limit and 1 as the upper limit.

Presentation

Many were very poor in the use of brackets or lack thereof i.e. $\int_{1}^{0} 2t+2-2\ln(1+t)\left(-2-\frac{2}{1+t}\right) dt$

Commented [LHMD21]: **Misconception**

Many stopped after substitution and did not integrate or integrated wrongly the term $4\int_{0}^{1}\frac{\ln(1+t)}{1+t} dt$, often giving $\ln(1+t)$ or forgetting to put back the constant 4.

	Solution	
	(ai) Let b and d be the first term and common difference of the arithmetic progression.	
	$ar^{n-1}=b+11d\cdots\cdots(1)$ $ar^{n}=b+7d\cdots\cdots(2)$ $ar^{n+1}=b+4d\cdots\cdots(3)$	
	Equation $(1)-(2)$ gives	
	$ar^{n-1}-ar^{n}=4d$	
	Equation $(2)-(3)$ gives	
	$ar^{n}-ar^{n+1}=3d$	
	Hence $4\left(ar^{n}-ar^{n+1}\right)=3\left(ar^{n-1}-ar^{n}\right)$	
	$4ar^{n+1}-7ar^{n}+3ar^{n-1}=0$	
	$ar^{n-1}\left(4r^{2}-7r+3\right)=0$	
	$(4r-3)(r-1)=0$ Since a and r are non-zero	
	$r=\frac{3}{4}$ Since it is a converging GP.	
	(ii) $\left\lvert\frac{a\left(1-r^{n}\right)}{1-r}-\frac{a}{1-r}\right\rvert<0.003\left(\frac{a}{1-r}\right)$ or $\frac{a}{1-r}-\frac{a\left(1-r^{n}\right)}{1-r}<0.003\left(\frac{a}{1-r}\right)$ Since $S_{\infty}>S_{n}$ as a and r are positive. "P" if missing reason.	
	$\left(\frac{3}{4}\right)^{n}<0.003$	
	$n>\frac{\ln(0.003)}{\ln\left(\frac{3}{4}\right)}$	
	$n>20.19$	
	Hence least n is 21.	
	(bi) $a+ar^{2}+ar^{4}+\ldots+a\left(r^{2}\right)^{n-1}=ar^{2n-1}+ar^{2n}+\ldots$	
	$\frac{a\left(1-r^{2n}\right)}{1-r^{2}}=\frac{ar^{2n-1}}{1-r}$ for sum of GP	
	Since $a\neq 0$, $r\neq 1$ $\frac{1-r^{2n}}{(1-r)(1+r)}=\frac{r^{2n-1}}{1-r}$	
	$1-r^{2n}=(1+r)r^{2n-1}$	

Commented [DsK22]: A number of students used the same first term for both progressions which is not correct as it is not stated in the qn. And the qn said consecutive terms and not the first, second or third terms.

Commented [DsK23]: Reason should be given for not considering other r or d values. r cannot be 1 because of converging and d cannot be equal to 0 because it d needs to be non-zero in order to generate a AP.

Commented [DsK24]: A big number of students were penalised P for omitting the modulus sign. Even if students did not include the modulus it must be substantiated with reason.

Commented [DsK25]: Signs of algebraic struggles were seen among many students.

Commented [DsK26]: Many did not revert the sign as ln(3/4) is a negative number and when they divide over they need to change the sign.

Commented [DsK27]: Many students had issues with this question due to question reading. For odd-numbered terms, many still wrote the common ratio to be r instead of the correct r^2. And the sum of the terms from the (2n-1)th term actually means it is another sum to infinity terms with first term ar^{2n-1} and common ratio r.

Many students wrote the first term as ar^{2n} because they misread the line " after the (2n-1)th term.

Commented [DsK28]: Since it is a show question, when they were not able to get the expression, many did some magic in order to get it. Some were found working backwards too.

[TURN OVER]

	$1-r^{2n}=r^{2n-1}+r^{2n}$ $2r^{2n}+r^{2n-1}-1=0$ (shown)	
	(ii) $\ln\left(\frac{u_n}{v_n}\right)-\ln\left(\frac{u_{n-1}}{v_{n-1}}\right)$	
	$=\ln u_n-\ln v_n-\ln u_{n-1}+\ln v_{n-1}$ $=\ln\left(\frac{u_n}{u_{n-1}}\right)+\ln\left(\frac{v_{n-1}}{v_n}\right)$	
	$=\ln r+\ln\left(\frac{1}{r}\right)^{-1}$ $=2\ln r$ (constant) Hence the new sequence is an arithmetic progression.	
7	**(a)** The function f is defined by	
	$f:x\mapsto x^2-2x-8, \quad x\in\mathbb{R},\ x>k.$	
	(i) State the least value of k such that f^{-1} exists and find f^{-1} in a similar form.	[3]
	(ii) Using the value of k found in (i), state the set of values of x such that $f^{-1}f(x)=ff^{-1}(x)$.	[1]
	(b) The functions g and h are defined by	
	$g:x\mapsto\sqrt{x+41}+a, \quad x\ge-41,\ a\in\mathbb{R},$	
	$h:x\mapsto x^2+10x-16, \quad x\in\mathbb{R},\ x<-7.$	
	(i) Find the exact value of x for which $h^{-1}(x)=h(x)$.	[3]
	(ii) Explain clearly why the composite function gh exists.	[1]
	(iii) Find gh in the form $bx+c$, where b is a real constant and c is in terms of a. Explain your answers clearly.	[2]
	(iv) State the exact range of gh in terms of a.	[1]
	Solution	
	(a)(i) Least value of k is 1.	
	$y=(x-1)^2-9$	
	$x=1\pm\sqrt{y+9}$	
	$\therefore x=1+\sqrt{y+9} \quad (\because x>1)$ "P" will be given if missing the reason	
	So $f^{-1}:x\mapsto1+\sqrt{x+9}, \quad x\in\mathbb{R},\ x>-9.$	
	(ii) $(1,\infty)$	
	(b)(i) $h^{-1}(x)=h(x)$	
	$\Rightarrow h(x)=x$	
	$x^2+10x-16=x$	
	$x^2+9x-16=0$	

Commented [DsK29]: Many struggled for this question. There were signs of misinterpretation also. Since students need to show that it is an AP, they need to manipulate to obtain a constant which is the common difference. But many were clueless on the manipulation. For those who managed to obtain the constant, they did not complete with an explanation.

Commented [K30]: Presentation
Question asked for **least** value of k not just the value of k.

Misconception
Some students looked at the y values instead of the x values when finding the least value of k.

Commented [K31]: Misconceptions
A number of students did not consider '±'.
Some others took both answers as the inverse function of f.

Commented [K32]: Presentation
A number of students did not state the reason for rejecting one of the answers.

Commented [K33]: Question Reading
Students did not define the inverse function in a similar form (meaning in a similar way as given in the question).

Misconception
$D_{f^{-1}}\ne D_f$ instead $D_{f^{-1}}=R_f$

Commented [K34]: Those who tried to solve this directly did not succeed and did not think of alternative solution.

Question Reading
Some did not read the word "exact" in the question and used GC to find the root.

	$x=\frac{-9\pm\sqrt{81-4(-16)}}{2}$	
	$x=-\frac{9}{2}+\frac{\sqrt{145}}{2}$ (Rejected $\because x<-7$) or $-\frac{9}{2}-\frac{\sqrt{145}}{2}$	
	(ii) $R_h=(-37,\infty)$ and $D_g=[-41,\infty)$	
	Since $R_h\subseteq D_g$, so gh exists.	
	(iii) $gh(x)=g\left(x^2+10x-16\right)$	
	$=\sqrt{x^2+10x+25}+a$	
	$=\lvert x+5\rvert+a$	
	$=-x+a-5\quad(\because\ x<-7)$	
	(iv) $(a+2,\infty)$	
8	A sequence $u_0,u_1,u_2...$ is such that $u_r=\frac{1}{r!}$ and $u_r=u_{r-2}+\frac{r+1-r^2}{r!}$, when $n\geq 2$.	
	(i) Show that $\sum_{r=2}^{n}\frac{r^2-r-1}{r!}=2-\frac{n+1}{n!}$.	[3]
	(ii) Hence find $\sum_{r=8}^{n+5}\frac{r^2-3r+1}{(r-1)!}$ in terms of n.	[3]
	Limit Comparison test states that for two series of the form $\sum_{r=k}^{n}a_r$ and $\sum_{r=k}^{n}b_r$ with $a_n,b_n\geq 0$ for all n, if $\lim_{n\to\infty}\frac{a_n}{b_n}>0$, then both $\sum_{r=k}^{\infty}a_r$ and $\sum_{r=k}^{\infty}b_r$ converges or both diverges.	
	(iii) Given that $\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}$ is convergent, using the test, explain why $\sum_{r=2}^{\infty}\frac{r-2}{(r-1)!}$ is convergent.	[2]
	(iv) Show that $e-2<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+...<2$.	[3]
	Solution	
	(i) $\sum_{r=2}^{n}\frac{r^2-r-1}{r!}=\sum_{r=2}^{n}\left(u_{r-2}-u_r\right)$	

Commented [K35]: Presentation
A number of students just stated the condition and did not write down what the range of h and the domain of g are.

Commented [K36]: Misconception
Many students thought that $\sqrt{x^2+10x+25}=x+5$, which is a misconception.

Commented [AA37]: 1.While it was easy obvious and easy to get this result, some students struggled to reach this step.
2.Some could not even recognise it as MOD.
3.Started the summation with r=1 and hence got expressions like u_{-1}.

[TURN OVER]

$$=\left[\begin{array}{l} u_0-u_2\\ +u_1-u_3\\ +u_2-u_4\\ +u_3-u_5\\ +\ldots\ldots\ldots\\ +u_{n-4}-u_{n-2}\\ +u_{n-3}-u_{n-1}\\ +u_{n-2}-u_n \end{array}\right.$$

$$=u_0+u_1-u_{n-1}-u_n$$

$$=2-\frac{1}{(n-1)!}-\frac{1}{n!}$$

$$=2-\left(\frac{n}{n!}+\frac{1}{n!}\right)$$

$$=2-\frac{n+1}{n!}$$

Commented [AA38]: Did not write this crucial step (or any other equivalent step) to justify the final answer.

(ii) $\displaystyle\sum_{r=8}^{n+5}\frac{r^2-3r+1}{(r-1)!}=\sum_{r=7}^{n+4}\frac{r^2-r-1}{r!}$ (Replace r by $r+1$)

$$=\sum_{r=2}^{n+4}\frac{r^2-r-1}{r!}-\sum_{r=2}^{6}\frac{r^2-r-1}{r!}$$

$$=2-\frac{n+5}{(n+4)!}-\left(2-\frac{7}{6!}\right)$$

Commented [AA39]: 1.Handful of students could not connect it with part (i) answer. 2.Made mistakes in the upper limit.

$$=\frac{7}{720}-\frac{n+5}{(n+4)!}$$

Commented [AA40]: Some did not simplify the final answer.

(iii) Consider $\displaystyle\lim_{n\to\infty}\left[\frac{\left(\frac{n-2}{(n-1)!}\right)}{\left(\frac{n^2-n-1}{n!}\right)}\right]=\lim_{n\to\infty}\left[\frac{n-2}{(n-1)!}\times\frac{n!}{n^2-n-1}\right]$

$$=\lim_{n\to\infty}\frac{n^2-2n}{n^2-n-1}=\lim_{n\to\infty}\left(1-\frac{n-1}{n^2-n-1}\right)=1$$

Commented [AA41]: While many could derive this limit as 1, some wrote it simply as >0.

Since $\displaystyle\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}$ is convergent,

So $\displaystyle\sum_{r=2}^{\infty}\frac{r-2}{(r-1)!}$ is convergent by the Limit Comparison Test.

(iv) Observe that $\displaystyle\sum_{r=2}^{\infty}\frac{r-1}{r!}=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots$

Commented [AA42]: Did not observe this relation.

and $\dfrac{r-1}{r!}<\dfrac{r^2-r-1}{r!}$ for all $r\geq 3$,

	$\therefore \sum_{r=3}^{n}\frac{r-1}{r!}<\sum_{r=3}^{n}\frac{r^2-r-1}{r!}$	
	but $\frac{1}{2}+\sum_{r=3}^{n}\frac{r-1}{r!}<\frac{1}{2}+\sum_{r=3}^{n}\frac{r^2-r-1}{r!}$	
	$\Rightarrow \sum_{r=2}^{n}\frac{r-1}{r!}<\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$	
	Method 1	
	Also, $\frac{1}{r!}<\frac{r-1}{r!}$ for all $r>2$ So $\sum_{r=3}^{n}\frac{1}{r!}<\sum_{r=3}^{n}\frac{r-1}{r!}$ $\Rightarrow \frac{1}{2!}+\sum_{r=3}^{n}\frac{1}{r!}<\frac{1}{2!}+\sum_{r=3}^{n}\frac{r-1}{r!}$ $\therefore \sum_{r=2}^{n}\frac{1}{r!}<\sum_{r=2}^{n}\frac{r-1}{r!}$	
	$\therefore \sum_{r=0}^{n}\frac{1}{r!}-\frac{1}{0!}-\frac{1}{1!}<\sum_{r=2}^{n}\frac{r-1}{r!}<\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$	
	$\therefore \lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{r!}-\frac{1}{0!}-\frac{1}{1!}\right)<\lim_{n\to\infty}\sum_{r=2}^{n}\frac{r-1}{r!}<\lim_{n\to\infty}\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$	
	$e-2<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots<2$	
	Method 2	
	Since $e=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots$ So $e-2=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots$	
	And $\sum_{r=2}^{\infty}\frac{r-1}{r!}<\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}$	
	So $e-2<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots<2$	
9	The position vectors of points A, B, C with respect to the origin O are given by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively. The non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy the equation $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$.	
	(i) By considering the plane OAB or otherwise, explain clearly why O, A, B and C lies on the same plane. Show that $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$.	[4]
	(ii) Show that the area of triangle ABC is given by $k\lvert\mathbf{a}\times\mathbf{b}\rvert$ where k is a constant to be determined.	[3]

Commented [AA43]: Majority of the students could not get this result.

Commented [AA44]: Proving the RHS inequality was found to be harder to students than the LHS.

Commented [CYF45]: Many students had the misconception that the normal vector of a plane is unique. The fact is there can be different parallel but distinct vectors that can be normal to the plane. So each of **a x b, b x c** and **c x a** can be a normal to the plane does not mean they need to be equal.

[TURN OVER]

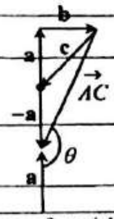
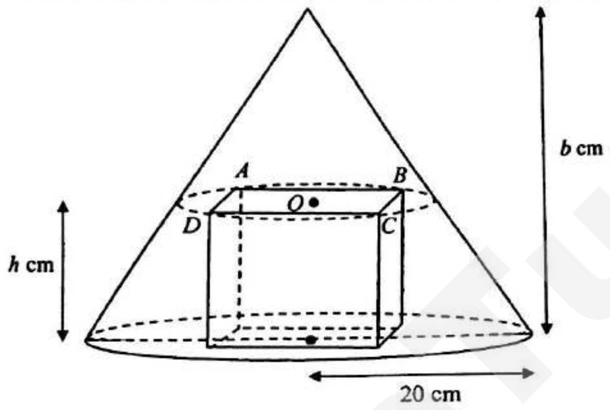
	(iii) If $\mathbf{b}$ is a unit vector and $\mathbf{a}$ is perpendicular to $\mathbf{b}$, find the length of projection of $\overrightarrow{AC}$ onto $\overrightarrow{OB}$. Given that the magnitude of $\overrightarrow{AC}$ is 2 units, deduce the angle between $\overrightarrow{AC}$ and $\overrightarrow{OA}$.	[5]
	Solution	
	(i) Eqn of plane OAB: $\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b} \quad \lambda, \mu \in \mathbb{R}$	
	Since $\mathbf{c} = -\mathbf{a} - \mathbf{b}$ where $\lambda = \mu = -1$, hence C lies on plane OAB. Thus O, A, B and C lies on the same plane.	
	OR $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{a}\times\mathbf{b})\cdot[-(\mathbf{a}+\mathbf{b})]$ $= -(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} - (\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}$ $= 0$ Thus O, A, B and C lies on the same plane.	
	Since $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$, $\mathbf{b} = -\mathbf{a} - \mathbf{c}$	
	$\mathbf{a}\times\mathbf{b} = \mathbf{a}\times(-\mathbf{a}-\mathbf{c})$	
	$= -\mathbf{a}\times\mathbf{a} - \mathbf{a}\times\mathbf{c}$	
	$= \mathbf{c}\times\mathbf{a}$ ($\because \mathbf{a}\times\mathbf{a} = \mathbf{0}$ & $\mathbf{c}\times\mathbf{a} = -\mathbf{a}\times\mathbf{c}$)	
	$\mathbf{b}\times\mathbf{c} = (-\mathbf{a}-\mathbf{c})\times\mathbf{c}$	
	$= -\mathbf{a}\times\mathbf{c} - \mathbf{c}\times\mathbf{c}$	
	$= \mathbf{c}\times\mathbf{a}$ ($\because \mathbf{c}\times\mathbf{c} = \mathbf{0}$ & $\mathbf{c}\times\mathbf{a} = -\mathbf{a}\times\mathbf{c}$)	
	$\therefore \mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$, shown	
	(ii) Area of triangle ABC	
	$= \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$	
	$= \frac{1}{2}\left\|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\right\|$	
	$= \frac{1}{2}\left\|\mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{a} + \mathbf{a}\times\mathbf{a}\right\|$	
	$= \frac{1}{2}\left\|\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{a}\times\mathbf{b}\right\|$	
	$= \frac{3}{2}\left\|\mathbf{a}\times\mathbf{b}\right\|$ where $k = \frac{3}{2}$ (Since $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$, $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$)	
	(iii) $\left\|\overrightarrow{AC}\cdot\hat{\mathbf{b}}\right\| = \left\|(\mathbf{c}-\mathbf{a})\cdot\mathbf{b}\right\|$	
	$= \left\|(-\mathbf{b}-2\mathbf{a})\cdot\mathbf{b}\right\|$	
	$= \left\|-\left\|\mathbf{b}\right\|^2 - 2\mathbf{a}\cdot\mathbf{b}\right\|$	
	$= 1 \quad (\because \mathbf{a}\cdot\mathbf{b} = 0, \left\|\mathbf{b}\right\| = 1)$	

Commented [CYF46]: Most students who did not draw any diagram could not solve this correctly.

Commented [CYF47]: Many students commented that since $\mathbf{c} = -\mathbf{a} - \mathbf{b}$, $\mathbf{c}$ is parallel to the vector **(a+b)** and used this as the only point to conclude O, A, B and C lies on the same plane. This is insufficient.

Commented [CYF48]: Many students were not aware that cross product is not commutative.
$-\mathbf{a}\times\mathbf{c} \neq -\mathbf{c}\times\mathbf{a}$
$-\mathbf{a}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$

Commented [CYF49]: At the start of part (iii), many students did not indicate the use of $\hat{\mathbf{b}}$. Instead, they used **b** instead, without stating "since **b** is an unit vector"

	Let θ be the angle between $\overrightarrow{AC}$ and $\mathbf{a}$. 	
	$\theta = 180^\circ - \sin^{-1}\left(\frac{1}{2}\right)$	
	$= 150^\circ$ or $\frac{5\pi}{6}$ rad	
10	The diagram below shows a rectangular container of variable height h cm, inscribed in a right circular cone with fixed height b cm and a radius of 20 cm. The four corners of the rectangular container's upper surface $ABCD$ is always in contact with the conical surface. The point O is at the centre of the rectangle $ABCD$. The rectangular container is made with material of negligible thickness.	
		
	(i) If the fixed angle BOC is θ, show that the storage volume of the rectangular container in the shape of a cuboid is given by $V = 800b^{-2}(b-h)^2 h\sin\theta$.	[3]
	(ii) Find the value of h such that the rectangular container has a maximum storage volume, leaving your answer in terms of b.	[5]
	The rectangular container is opened at its upper surface $ABCD$ and completely filled with a type of liquid perfume. A fragrance chemist placed the container in a room and allowed the liquid perfume to evaporate. It is known that the heat energy of the liquid perfume in the container, E joules, is related to the height of the container by the equation $$E = h - 3h^{-\frac{1}{2}}.$$	
	(iii) Given that the perfume evaporates at a rate of 0.08 c every hour and that the value of θ is $\frac{\pi}{6}$, calculate the rate of change of heat energy of the perfume when the height of the perfume in the rectangular container is	

Commented [CHD(H50]: Question Reading
Some students did not take note that 'b' is a fixed constant, likewise for θ.

[TURN OVER]

	$\frac{b}{5}$ cm, leaving your answer in the form $p+q\left(\frac{5}{b}\right)^{\frac{3}{2}}$, where p and q are exact constants to be determined.	[4]
	Solution	
	(i) Let the length of OB be r.	
	$V=4\left(r\sin\frac{\theta}{2}\right)\left(r\cos\frac{\theta}{2}\right)h$	
	$=2r^2h\sin\theta$	
	Method 2	
	$V=2\left[\frac{1}{2}r^2\sin\theta+\frac{1}{2}r^2\sin(\pi-\theta)\right]h$	
	$=\left[r^2\sin\theta+r^2\sin(\pi-\theta)\right]h$	
	$=2r^2h\sin\theta$ (since $\sin(\pi-\theta)=\sin\theta$)	
	Using similar triangle,	
	$\frac{r}{20}=\frac{b-h}{b}$	
	$r=\frac{20}{b}(b-h)$	
	$V=2\left(\frac{20}{b}\right)^2(b-h)^2h\sin\theta$	
	$=800b^{-2}(b-h)^2h\sin\theta$ (Shown)	
	(ii) $\frac{dV}{dh}=800b^{-2}(b-h)^2\sin\theta+800b^{-2}h\sin\theta(2)(b-h)(-1)$	
	$=800b^{-2}(b-h)(\sin\theta)(b-h-2h)$	
	$=800b^{-2}(\sin\theta)(b-h)(b-3h)$	
	Setting $\frac{dV}{dh}=0$,	
	$800b^{-2}(\sin\theta)(b-h)(b-3h)=0$	
	$(b-h)(b-3h)=0$	
	$h=b$ (rejected since $h\neq b$) or $h=\frac{b}{3}$	
	Method 1 (Using second derivative test)	
	$\frac{d^2V}{dh^2}=800b^{-2}(\sin\theta)\left[(b-h)(-3)+(b-3h)(-1)\right]$	
	$=800b^{-2}(\sin\theta)(6h-4b)$	

Commented [CHD(H51]: Unclear presentation or misconception
Some students assumed that the area of rectangle ABCD is four times of the area of triangle OBC without any explanation.
i.e. Area of rectangle ABCD $=4\left(\frac{1}{2}r^2\sin\theta\right)$

Commented [CHD(H52]: Common mistake
A significant group of students wrote $\frac{r}{20}=\frac{h}{b}$.
These students mistook h as the height of the smaller cone.

Commented [CHD(H53]: Students are to note that the two variables in this equation are V and h. Hence, we should go on to differentiate V with respect to h when finding the h that maximises the storage volume.

Commented [CHD(H54]: Quite a significant number of students made careless mistakes when finding dV/dh. Some students intended to differentiate with respect to h but ended up differentiating $\sin\theta$ (giving them $\cos\theta$).

Commented [CHD(H55]: Presentation
For those who arrived at this stage did not give a reason why h=b is rejected.

	When $h=\frac{b}{3}$, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=800b^{-2}(\sin\theta)(-2b)<0$	
	Hence, V is maximum when $h=\frac{b}{3}$.	
	Method 2 (Using first derivative test)	
	$\frac{\mathrm{d}V}{\mathrm{d}h}=800b^{-2}(\sin\theta)(b-h)(b-3h)$	
	When $h<\frac{b}{3}$, $800b^{-2}(\sin\theta)>0$, $(b-h)>0$ and $(b-3h)>0$, $\frac{\mathrm{d}V}{\mathrm{d}h}>0$	
	When $h>\frac{b}{3}$, $800b^{-2}(\sin\theta)>0$, $(b-h)>0$ and $(b-3h)<0$, $\frac{\mathrm{d}V}{\mathrm{d}h}<0$	
	Hence, V is maximum when $h=\frac{b}{3}$.	
	(iii) Since $\theta=\frac{\pi}{6}$, $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{400}{b^2}(b-h)(b-3h)$	
	$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$	
	When $h=\frac{b}{5}$,	
	$-0.08=\frac{400}{b^2}\left(\frac{4}{5}b\right)\left(\frac{2}{5}b\right)\times\frac{\mathrm{d}h}{\mathrm{d}t}$	
	$\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{1}{1600}$	
	$E=h-3h^{-\frac{1}{2}}$	
	$\frac{\mathrm{d}E}{\mathrm{d}h}=1+\frac{3}{2}h^{-\frac{3}{2}}$	
	$\frac{\mathrm{d}E}{\mathrm{d}t}=\frac{\mathrm{d}E}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$	
	$=\left(1+\frac{3}{2}h^{-\frac{3}{2}}\right)\left(-\frac{1}{1600}\right)$	
	$=\left[1+\frac{3}{2}\left(\frac{b}{5}\right)^{-\frac{3}{2}}\right]\left(-\frac{1}{1600}\right)$	
	$=-\frac{1}{1600}-\frac{3}{3200}\left(\frac{5}{b}\right)^{\frac{3}{2}}$	
	Hence, $p=-\frac{1}{1600}$, $q=-\frac{3}{3200}$	

Commented [CHD(H56]: Many students presented using a table. However, the part that is lacking is the explanations for the sign of dV/dh for the neighbouring values of h=(b/3).

Commented [CHD(H57]: <u>Question Reading</u> Since the perfume is evaporating and V represents the volume of the liquid perfume in the container, then dV/dt has to be a negative value. Therefore, dV/dt = -0.08.

Commented [CHD(H58]: <u>Common mistake</u> Quite a large number of students substituted h=(b/5) into this equation. This value should only be substituted in at a later stage. Some went on to find dV/db without knowing that b is a constant value.

Commented [CHD(H59]: <u>Question Reading</u> Some students have forgotten to write down the values of p and q.

[TURN OVER]

11	A charged particle is placed in a varying magnetic field. A researcher decides to fit a mathematical model for the path of the fast-moving charged particle under the influence of the magnetic field. The particle was observed for the first 1.5 seconds. The displacement of the particle measured with respect to the origin in the horizontal and vertical directions, at time t seconds, is denoted by the variables x and y respectively. It is given that when $t=0$, $x=-\frac{1}{32}$, $y=0$ and $\frac{dx}{dt}=3$. The variables are related by the differential equations $$(\cos t)\frac{dy}{dt}+y\sin t=4\cos^2 t-y^2 \text{ and } \frac{d^2x}{dt^2}=\cos 3t\cos t.$$	
	(i) Using the substitution $y=v\cos t$, show that $\frac{dv}{dt}=4-v^2$ and hence find y in terms of t.	[7]
	(ii) Show that $x=-\frac{1}{32}\cos 4t-\frac{1}{8}\cos 2t+3t+\frac{1}{8}$.	[4]
	(iii) Sketch the path travelled by the particle for the first 1.5 seconds, labelling the coordinates of the end points of the path. The evaluation of the y-intercept is not needed.	[2]
	Solution	
	(i) $y=v\cos t$	
	$\frac{dy}{dt}=-v\sin t+\frac{dv}{dt}\cos t$	
	$(\cos t)\frac{dy}{dt}+y\sin t=4\cos^2 t-y^2$	
	$(\cos t)\left(-v\sin t+\frac{dv}{dt}\cos t\right)+(v\cos t)\sin t=4\cos^2 t-v^2\cos^2 t$	
	$\frac{dv}{dt}(\cos^2 t)=(\cos^2 t)(4-v^2)$	
	$\frac{dv}{dt}=4-v^2$ (Shown)	
	$\int\frac{1}{4-v^2}\,dv=\int 1\,dt$	
	$\int\frac{1}{2^2-v^2}\,dv=\int 1\,dt$	
	$\frac{1}{2(2)}\ln\left\|\frac{2+v}{2-v}\right\|=t+d$	
	$\left\|\frac{2+v}{2-v}\right\|=e^{4t+4d}$	
	$\frac{2+v}{2-v}=\pm e^{4t+4d}$	

Commented [LT60]: Question Reading
Some did not provide a detail set of working to show how this expression was derived.

Commented [LT61]: Question Interpretation
Did not realise that one should sketch a curve with y against x.

Commented [LT62]: Question Reading
Did not indicate the end points coordinates.

Commented [LT63]: Misconception
A number did not perform product rule and treat either v or t as constants.

It should be noted that the whatever symbols used in the substitution should be taken as a variable unless stated otherwise by the question.

Commented [LT64]: Presentation
Some did not show sufficient working to show how it can be simplified to the answer provided.

Commented [LT65]: Misconception
There are still some who did not perform the variable separable correctly.

Commented [LT66]: It is unwise to substitute in the given information to find the value of the unknown constant at this stage.

Commented [LT67]: Misconception
Some forgot to add in the +/- sign after the removal of the modulus.

	$\frac{2+v}{2-v} = Ae^{4t}$, where $A = \pm e^{4d}$	
	$2 + v = 2Ae^{4t} - Ave^{4t}$	
	$v + Ave^{4t} = 2\left(Ae^{4t} - 1\right)$	
	$v = \frac{2\left(Ae^{4t}-1\right)}{Ae^{4t}+1}$	
	$\frac{y}{\cos t} = \frac{2\left(Ae^{4t}-1\right)}{Ae^{4t}+1}$	
	$y = \frac{2\left(Ae^{4t}-1\right)\cos t}{Ae^{4t}+1}$	
	When $t = 0,\ y = 0$	
	$0 = \frac{2(A-1)}{A+1}$	
	$A = 1$	
	$y = \frac{2\left(e^{4t}-1\right)\cos t}{e^{4t}+1}$	
	(ii) $\frac{d^2x}{dt^2} = \cos 3t \cos t$	
	$\frac{d^2x}{dt^2} = \frac{1}{2}(\cos 4t + \cos 2t)$	
	$\frac{dx}{dt} = \frac{1}{2}\int \cos 4t + \cos 2t \ dt$	
	$= \frac{1}{2}\left(\frac{1}{4}\sin 4t + \frac{1}{2}\sin 2t\right) + c$	
	$= \frac{1}{8}\sin 4t + \frac{1}{4}\sin 2t + c$	
	When $t = 0$, $\frac{dx}{dt} = 3$	
	$3 = \frac{1}{8}\sin 4(0) + \frac{1}{4}\sin 2(0) + c$	
	$c = 3$	
	$x = \int \frac{1}{8}\sin 4t + \frac{1}{4}\sin 2t + 3 \ dt$	
	$x = -\frac{1}{32}\cos 4t - \frac{1}{8}\cos 2t + 3t + k$	
	When $t = 0$, $x = -\frac{1}{32}$	
	$-\frac{1}{32} = -\frac{1}{32}\cos 4(0) - \frac{1}{8}\cos 2(0) + 3(0) + k$	
	$k = \frac{1}{8}$	

Commented [LT68]: Question Reading
Did not leave the answer as y in terms of t.

Commented [LT69]: Presentation
Some did not go on to evaluate the value of A.

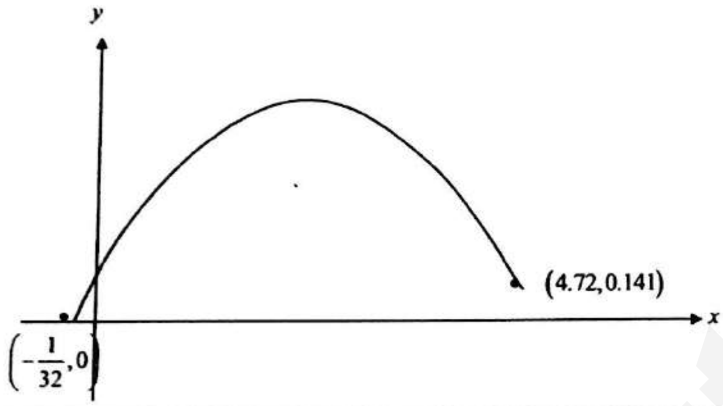
Commented [LT70]: Presentation
There are some who did a very long method to reduce the given expression to this expression.

Commented [LT71]: Presentation
Some did not show sufficient working to derive the value of c.

Commented [LT72]: Do not repeatedly use the same letter for 2 different integration constant since this is the constant obtained due to the second integral.

Commented [LT73]: Presentation of Answers
Insufficient working shown when showing the answer.

[TURN OVER]

	Hence, $x=-\frac{1}{32}\cos 4t-\frac{1}{8}\cos 2t+3t+\frac{1}{8}$	
	(iii) 	

Commented [LT74]: Accuracy
Ends points given are not indicated correctly.
Axes are not labelled correctly.
Some even gave the wrong shape when they could have just double check the answer using the G.C.

THE END

SERANGOON JUNIOR COLLEGE

2018 JC2 PRELIMINARY EXAMINATION

MATHEMATICS

Higher 2 **9758/1**

11 Sept 2018

3 hours

Additional materials: Writing paper

List of Formulae (MF 26)

**TIME : ** 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and class on the cover page and on all the work you hand in.
Write in blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

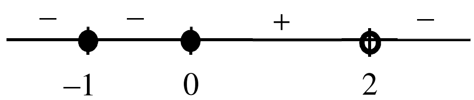
The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

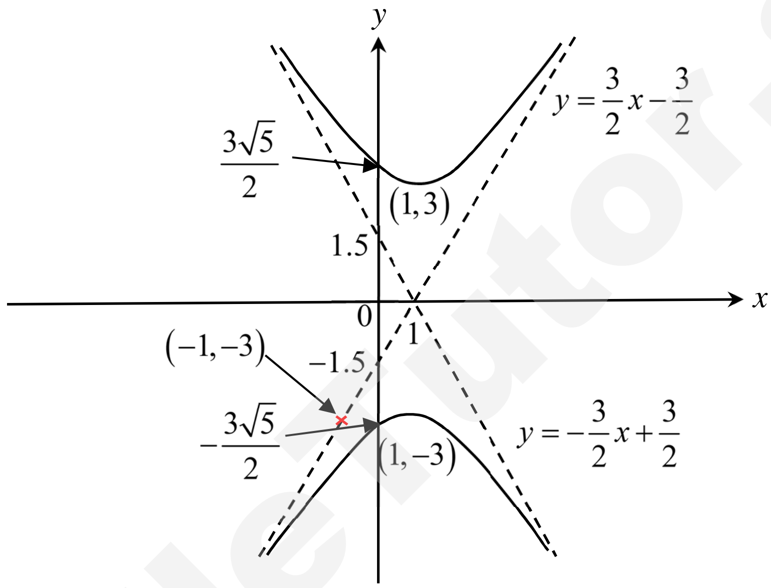
Total marks for this paper is 100 marks.

This question paper consists of 6 printed pages (inclusive of this page) and 2 blank pages.

Answer all questions [100 marks].

1	Find $\int \frac{x^2}{x^2+2x+5}\,dx$.	[3]
	Solution	
	$\int \frac{x^2}{x^2+2x+5}\,dx = \int 1 - \frac{2x+5}{x^2+2x+5}\,dx$	
	$= \int 1\,dx - \int \frac{2x+2}{x^2+2x+5}\,dx - 3\int \frac{1}{(x+1)^2+4}\,dx$	
	$= x - \ln\left(x^2+2x+5\right) - \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$	
2	The complex numbers z and w satisfy the equations	
	$zw^* + 2z = 15\text{i}$ and $2w + 3z = 11$.	
	Find the complex numbers z and w.	[6]
	Solution	
	$zw^* + 2z = 15\text{i} \cdots\cdots (1)$	
	$w = \frac{1}{2}(11-3z) \cdots\cdots (2)$	
	Subst (2) into (1) gives	
	$\frac{1}{2}z\left(11-3z^*\right) + 2z = 15\text{i}$	
	$\frac{11}{2}z - \frac{3}{2}\|z\|^2 + 2z = 15\text{i}$	
	$\frac{15}{2}z - \frac{3}{2}\|z\|^2 = 15\text{i}$	
	Let $z = x + \text{i}y$	
	$\frac{15}{2}(x+\text{i}y) = \frac{3}{2}\left(x^2+y^2\right) + 15\text{i}$	
	Comparing the real and Imaginary parts, we have	
	$\frac{15}{2}y = 15$ and $\frac{15}{2}x = \frac{3}{2}x^2 + \frac{3}{2}y^2 \cdots\cdots (3)$	
	$y = 2$	
	Subst into (3) gives	
	$x^2 - 5x + 4 = 0$	
	$(x-4)(x-1) = 0$	
	$\therefore x = 1$ or 4	
	When $z = 1+2\text{i}$, $w = \frac{1}{2}(11-3-6\text{i}) = 4-3\text{i}$	
	When $z = 4+2\text{i}$, $w = \frac{1}{2}(11-12-6\text{i}) = -\frac{1}{2}-3\text{i}$	

3	Without the use of a graphing calculator, find the range of values of x for which $\dfrac{18}{2-x} \geq x^2+4x+9$.	[3]
	Hence find the exact range of values of x for which $\dfrac{18}{2-e^x} \geq e^{2x}+4e^x+9$.	[3]
	Solution	
	$\dfrac{18}{2-x} \geq x^2+4x+9$	
	$\dfrac{18+(x-2)(x^2+4x+9)}{2-x} \geq 0$	
	$\dfrac{18+x^3+2x^2+x-18}{2-x} \geq 0$	
	$\dfrac{x(x+1)^2}{2-x} \geq 0$	
		
	$x=-1$ or $0 \leq x < 2$	
	Replace x by e^x	
	So $e^x=-1$ or $0 \leq e^x < 2$	
	No solution or $\left(0 \leq e^x \quad \text{and} \quad e^x < 2\right)$	
	No solution or $\left(x \in \mathbb{R} \quad \text{and} \quad x < \ln 2\right)$	
	$\therefore x < \ln 2$	
4	**(i)** The curve with equation $\dfrac{y^2}{9}-x^2=1$ undergoes a two-step transformations to become curve C with equation $\dfrac{y^2}{9}-\dfrac{(x-1)^2}{4}=1$. State the two transformations involved for the curve $\dfrac{y^2}{9}-x^2=1$.	[2]
	(ii) Draw a sketch of the curve C, labelling clearly the equation(s) of its asymptote(s), intersection with the axes and the coordinates of any turning points.	[3]
	(iii) Show that the point $(-1,-3)$ lies on the line $y=mx+m-3$ for all real values of m.	[1]
	(iv) Hence using the diagram drawn in **(ii)**, find the range of values of k such that the equation $\dfrac{(kx+k-3)^2}{9}-\dfrac{(x-1)^2}{4}=1$ has 2 negative real roots.	[2]
	Solution	
	(i) Method 1	

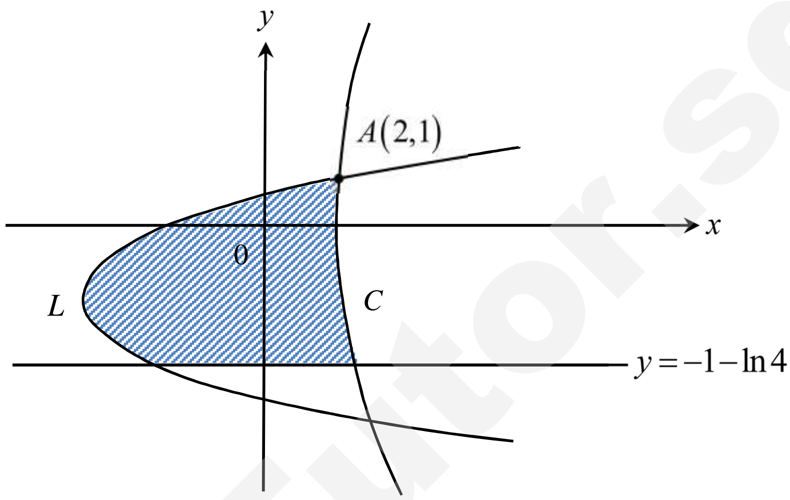
	$\frac{y^2}{9}-x^2=1 \to \frac{y^2}{9}-\frac{x^2}{4}=1 \to \frac{y^2}{9}-\frac{(x-1)^2}{4}=1$	
	(1) Scaling parallel to the x axis by a scale factor of 2.	
	(2) Translation of 1 unit in the positive x direction.	
	Method 2 $\frac{y^2}{9}-x^2=1 \to \frac{y^2}{9}-\left(x-\frac{1}{2}\right)^2=1 \to \frac{y^2}{9}-\frac{(x-1)^2}{4}=1$	
	(1) Translation of 0.5 units in the positive x direction	
	(2) Scaling parallel to the x axis by a scale factor of 2.	
		
	(iii) When $x=-1$, $\text{RHS}=m(-1)+m-3=-3=\text{LHS}$	
	Hence $(-1,-3)$ lies on the line $y=mx+m-3$ for all m.	
	(iv) From the sketch, the range of values of k are $k>\frac{3\sqrt{5}}{2}+3$ or $k<-\frac{3}{2}$	

5 **(i)** Find $\int \ln(x+1)\ \mathrm{d}x$ for $x > -1$. Show your working clearly. [2]

(ii) The curve C is defined by the parametric equations

$$x = 2t - 2\ln(t+1) + 2, \quad y = -2t - 2\ln(t+1) + 1 \qquad \text{where } t > -1.$$

Another curve L is defined by the equation $x = 2(y+1)^2 - 6$. The graphs of C and L intersect at the point $A(2,1)$ as shown in the diagram below.



Find the area of the shaded region bounded by C, L and the line $y = -1 - \ln 4$, express your answer in the form $\frac{62}{3} + 4A - 2A^2 - \frac{16A^3}{3}$, where A is an exact real constant. [6]

Solution

(i) $\int \ln(1+x)\ \mathrm{d}x = x\ln(1+x) - \int \frac{x}{1+x}\ \mathrm{d}x$

$= x\ln(1+x) - \int 1 - \frac{1}{1+x}\ \mathrm{d}x$

$= x\ln(1+x) - x + \ln(1+x) + c$

$= (x+1)\ln(1+x) - x + c$

(ii) Area = $\int_{-1-\ln 4}^{1} x_C\ \mathrm{d}y - \int_{-1-\ln 4}^{1} \left[2(y+1)^2 - 6\right] \mathrm{d}y$

$= \int_1^0 \left[2t + 2 - 2\ln(1+t)\right]\left(-2 - \frac{2}{1+t}\right) \mathrm{d}t - \left[\frac{2(y+1)^3}{3} - 6y\right]_{-1-\ln 4}^{1}$

$= 4\int_0^1 \left[t + 1 - \ln(1+t)\right]\left(1 + \frac{1}{1+t}\right) \mathrm{d}t - \left\{\frac{16}{3} - 6 - \left[\frac{-2(\ln 4)^3}{3} + 6 + 6\ln 4\right]\right\}$

$= 4\int_0^1 t + 2\ \mathrm{d}t - 4\int_0^1 \ln(1+t)\ \mathrm{d}t - 4\int_0^1 \frac{\ln(1+t)}{1+t}\ \mathrm{d}t - \left[-\frac{20}{3} + \frac{2(\ln 4)^3}{3} - 6\ln 4\right]$

	$=\left[2(t+2)^2\right]_0^1-4\left[(1+t)\ln(1+t)-t\right]_0^1-2\left[(\ln(1+t))^2\right]_0^1-\left[-\frac{20}{3}+\frac{2(\ln 4)^3}{3}-6\ln 4\right]$	
	$=(18-8)-4[2\ln 2-1]-2(\ln 2)^2+\frac{20}{3}-\frac{2(\ln 4)^3}{3}+6\ln 4$	
	$=14-8\ln 2-2(\ln 2)^2+\frac{20}{3}-\frac{2(\ln 4)^3}{3}+6\ln 4$	
	$=\frac{62}{3}+4\ln 2-2(\ln 2)^2-\frac{16(\ln 2)^3}{3}$	
6	**(a)** **(i)** The twelfth, eighth and fifth terms of an arithmetic progression are three consecutive terms of a converging geometric progression of positive terms with common ratio *r*. Find the value of *r*.	[3]
	(ii) Take the value of *r* to be $\frac{3}{4}$. If the difference between the sum of the first *n* terms of the geometric progression and its sum to infinity is less than 0.3% of the sum to infinity, find the least value of *n*.	[3]
	(b) A convergent geometric sequence of positive terms, *G* has first non-zero term *a* and common ratio *r*.	
	(i) The sum of the first *n* odd-numbered terms of *G* is equal to the sum of all terms after the $(2n-1)^{\text{th}}$ term of *G*. Show that $2r^{2n}+r^{2n-1}-1=0$.	[2]
	(ii) In another sequence *H*, each term is the reciprocal of the corresponding term of *G*. If the *n*th term of *G* and *H* is denoted by u_n and v_n respectively, show that a new sequence whose *n*th term is $\ln\left(\frac{u_n}{v_n}\right)$, is an arithmetic progression.	[2]
	Solution	
	(ai) Let *b* and *d* be the first term and common difference of the arithmetic progression.	
	$ar^{n-1}=b+11d\cdots\cdots(1)$ $ar^{n}=b+7d\cdots\cdots(2)$ $ar^{n+1}=b+4d\cdots\cdots(3)$	
	Equation $(1)-(2)$ gives	
	$ar^{n-1}-ar^{n}=4d$	
	Equation $(2)-(3)$ gives	
	$ar^{n}-ar^{n+1}=3d$	
	Hence $4\left(ar^{n}-ar^{n+1}\right)=3\left(ar^{n-1}-ar^{n}\right)$	
	$4ar^{n+1}-7ar^{n}+3ar^{n-1}=0$	

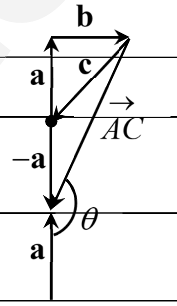
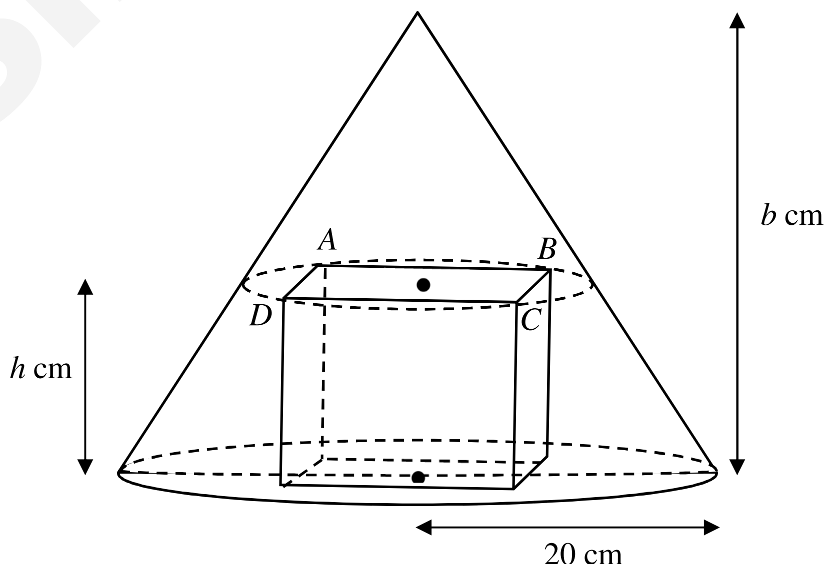
	$ar^{n-1}\left(4r^2-7r+3\right)=0$	
	$(4r-3)(r-1)=0$ Since a and r are non-zero	
	$r=\frac{3}{4}$ Since it is a converging GP.	
	(ii) $\left\lvert\frac{a\left(1-r^n\right)}{1-r}-\frac{a}{1-r}\right\rvert<0.003\left(\frac{a}{1-r}\right)$ or $\frac{a}{1-r}-\frac{a\left(1-r^n\right)}{1-r}<0.003\left(\frac{a}{1-r}\right)$ Since $S_\infty>S_n$ as a and r are positive.	
	$\left(\frac{3}{4}\right)^n<0.003$	
	$n>\frac{\ln(0.003)}{\ln\left(\frac{3}{4}\right)}$	
	$n>20.19$	
	Hence least n is 21.	
	(bi) $a+ar^2+ar^4+...+a\left(r^2\right)^{n-1}=ar^{2n-1}+ar^{2n}+...$	
	$\frac{a\left(1-r^{2n}\right)}{1-r^2}=\frac{ar^{2n-1}}{1-r}$ for sum of GP	
	Since $a\neq 0$, $r\neq 1$ $\frac{1-r^{2n}}{(1-r)(1+r)}=\frac{r^{2n-1}}{1-r}$	
	$1-r^{2n}=(1+r)r^{2n-1}$ $1-r^{2n}=r^{2n-1}+r^{2n}$ $2r^{2n}+r^{2n-1}-1=0(\text{shown})$	
	(ii) $\ln\left(\frac{u_n}{v_n}\right)-\ln\left(\frac{u_{n-1}}{v_{n-1}}\right)$	
	$=\ln u_n-\ln v_n-\ln u_{n-1}+\ln v_{n-1}$ $=\ln\left(\frac{u_n}{u_{n-1}}\right)+\ln\left(\frac{v_{n-1}}{v_n}\right)$	
	$=\ln r+\ln\left(\frac{1}{r}\right)^{-1}$ $=2\ln r$ (constant) Hence the new sequence is an arithmetic progression.	

7	**(a)** The function f is defined by	
	$\mathrm{f}: x \mapsto x^2 - 2x - 8, \qquad x \in \mathbb{R},\ \ x > k.$	
	(i) State the least value of *k* such that f^{-1} exists and find f^{-1} in a similar form.	[3]
	(ii) Using the value of *k* found in **(i)**, state the set of values of *x* such that $\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{ff}^{-1}(x)$.	[1]
	(b) The functions g and h are defined by	
	$\mathrm{g}: x \mapsto \sqrt{x+41} + a, \qquad x \geq -41,\ \ a \in \mathbb{R},$	
	$\mathrm{h}: x \mapsto x^2 + 10x - 16, \qquad x \in \mathbb{R},\ \ x < -7.$	
	(i) Find the exact value of *x* for which $\mathrm{h}^{-1}(x) = \mathrm{h}(x)$.	[3]
	(ii) Explain clearly why the composite function gh exists.	[1]
	(iii) Find gh in the form *bx* + *c*, where *b* is a real constant and *c* is in terms of *a*. Explain your answers clearly.	[2]
	(iv) State the exact range of gh in terms of *a*.	[1]
	Solution	
	(a)(i) Least value of *k* is 1.	
	$y = (x-1)^2 - 9$	
	$x = 1 \pm \sqrt{y+9}$	
	$\therefore x = 1 + \sqrt{y+9} \quad (\because x > 1)$	
	So $\mathrm{f}^{-1}: x \mapsto 1 + \sqrt{x+9}, \qquad x \in \mathbb{R},\ \ x > -9.$	
	(ii) $(1, \infty)$	
	(b)(i) $\mathrm{h}^{-1}(x) = \mathrm{h}(x)$	
	$\Rightarrow \mathrm{h}(x) = x$	
	$x^2 + 10x - 16 = x$	
	$x^2 + 9x - 16 = 0$	
	$x = \dfrac{-9 \pm \sqrt{81 - 4(-16)}}{2}$	
	$x = -\dfrac{9}{2} + \dfrac{\sqrt{145}}{2}$ (Rejected $\because x < -7$) or $-\dfrac{9}{2} - \dfrac{\sqrt{145}}{2}$	
	(ii) $R_{\mathrm{h}} = (-37, \infty)$ and $D_{\mathrm{g}} = [-41, \infty)$	
	Since $R_{\mathrm{h}} \subseteq D_{\mathrm{g}}$, so gh exists.	
	(iii) $\mathrm{gh}(x) = \mathrm{g}\left(x^2 + 10x - 16\right)$	
	$= \sqrt{x^2 + 10x + 25} + a$	
	$= \lvert x+5 \rvert + a$	
	$= -x + a - 5 \quad (\because \ x < -7)$	
	(iv) $(a+2, \infty)$	

8	A sequence $u_0, u_1, u_2...$ is such that $u_r = \frac{1}{r!}$ and $u_r = u_{r-2} + \frac{r+1-r^2}{r!}$, when $n \geq 2$.	
	(i) Show that $\sum_{r=2}^{n} \frac{r^2-r-1}{r!} = 2 - \frac{n+1}{n!}$.	[3]
	(ii) Hence find $\sum_{r=8}^{n+5} \frac{r^2-3r+1}{(r-1)!}$ in terms of *n*.	[3]
	Limit Comparison test states that for two series of the form $\sum_{r=k}^{n} a_r$ and $\sum_{r=k}^{n} b_r$ with $a_n, b_n \geq 0$ for all *n*, if $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, then both $\sum_{r=k}^{\infty} a_r$ and $\sum_{r=k}^{\infty} b_r$ converges or both diverges.	
	(iii) Given that $\sum_{r=2}^{\infty} \frac{r^2-r-1}{r!}$ is convergent, using the test, explain why $\sum_{r=2}^{\infty} \frac{r-2}{(r-1)!}$ is convergent.	[2]
	(iv) Show that $\mathrm{e} - 2 < \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... < 2$.	[3]
	Solution	
	(i) $\sum_{r=2}^{n} \frac{r^2-r-1}{r!} = \sum_{r=2}^{n} (u_{r-2} - u_r)$	
	$= \left[\begin{array}{l} u_0 - u_2 \\ +u_1 - u_3 \\ +u_2 - u_4 \\ +u_3 - u_5 \\ +\ldots\ldots\ldots \\ +u_{n-4} - u_{n-2} \\ +u_{n-3} - u_{n-1} \\ +u_{n-2} - u_n \end{array}\right]$	
	$= u_0 + u_1 - u_{n-1} - u_n$ $= 2 - \frac{1}{(n-1)!} - \frac{1}{n!}$ $= 2 - \frac{n+1}{n!}$	
	(ii) $\sum_{r=8}^{n+5} \frac{r^2-3r+1}{(r-1)!} = \sum_{r=7}^{n+4} \frac{r^2-r-1}{r!}$ (Replace *r* by *r* +1)	

	$=\sum_{r=2}^{n+4}\frac{r^2-r-1}{r!}-\sum_{r=2}^{6}\frac{r^2-r-1}{r!}$ $=2-\frac{n+5}{(n+4)!}-\left(2-\frac{7}{6!}\right)$	
	$=\frac{7}{720}-\frac{n+5}{(n+4)!}$	
	(iii) Consider $\lim_{n\to\infty}\left[\frac{\left(\frac{n-2}{(n-1)!}\right)}{\left(\frac{n^2-n-1}{n!}\right)}\right]=\lim_{n\to\infty}\left[\frac{n-2}{(n-1)!}\times\frac{n!}{n^2-n-1}\right]$	
	$=\lim_{n\to\infty}\frac{n^2-2n}{n^2-n-1}=\lim_{n\to\infty}\left(1-\frac{n-1}{n^2-n-1}\right)=1$	
	Since $\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}$ is convergent, So $\sum_{r=2}^{\infty}\frac{r-2}{(r-1)!}$ is convergent by the Limit Comparison Test.	
	(iv) Observe that $\sum_{r=2}^{\infty}\frac{r-1}{r!}=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots$ and $\frac{r-1}{r!}<\frac{r^2-r-1}{r!}$ for all $r\geq 3$,	
	$\therefore\sum_{r=3}^{n}\frac{r-1}{r!}<\sum_{r=3}^{n}\frac{r^2-r-1}{r!}$	
	but $\frac{1}{2}+\sum_{r=3}^{n}\frac{r-1}{r!}<\frac{1}{2}+\sum_{r=3}^{n}\frac{r^2-r-1}{r!}$	
	$\Rightarrow\sum_{r=2}^{n}\frac{r-1}{r!}<\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$	
	Method 1	
	Also, $\frac{1}{r!}<\frac{r-1}{r!}$ for all $r>2$ So $\sum_{r=3}^{n}\frac{1}{r!}<\sum_{r=3}^{n}\frac{r-1}{r!}$ $\Rightarrow\frac{1}{2!}+\sum_{r=3}^{n}\frac{1}{r!}<\frac{1}{2!}+\sum_{r=3}^{n}\frac{r-1}{r!}$ $\therefore\sum_{r=2}^{n}\frac{1}{r!}<\sum_{r=2}^{n}\frac{r-1}{r!}$	
	$\therefore\sum_{r=0}^{n}\frac{1}{r!}-\frac{1}{0!}-\frac{1}{1!}<\sum_{r=2}^{n}\frac{r-1}{r!}<\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$	
	$\therefore\lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{r!}-\frac{1}{0!}-\frac{1}{1!}\right)<\lim_{n\to\infty}\sum_{r=2}^{n}\frac{r-1}{r!}<\lim_{n\to\infty}\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$	

	$\mathrm{e}-2<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+...<2$	
	Method 2	
	Since $\mathrm{e}=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+...$ So $\mathrm{e}-2=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+...<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+...$	
	And $\sum_{r=2}^{\infty}\frac{r-1}{r!}<\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}$	
	So $\mathrm{e}-2<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+...<2$	
9	The position vectors of points A, B, C with respect to the origin O are given by $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ respectively. The non-zero vectors **a**, **b** and **c** satisfy the equation $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$.	
	(i) By considering the plane OAB or otherwise, explain clearly why O, A, B and C lies on the same plane. Show that $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$.	[4]
	(ii) Show that the area of triangle ABC is given by $k\lvert\mathbf{a}\times\mathbf{b}\rvert$ where k is a constant to be determined.	[3]
	(iii) If $\mathbf{b}$ is a unit vector and $\mathbf{a}$ is perpendicular to $\mathbf{b}$, find the length of projection of $\overrightarrow{AC}$ onto $\overrightarrow{OB}$. Given that the magnitude of $\overrightarrow{AC}$ is 2 units, deduce the angle between $\overrightarrow{AC}$ and $\overrightarrow{OA}$.	[5]
	Solution	
	(i) Eqn of plane OAB: $\mathbf{r}=\lambda\mathbf{a}+\mu\mathbf{b}$ $\quad\lambda,\mu\in\mathbb{R}$	
	Since $\mathbf{c}=-\mathbf{a}-\mathbf{b}$ where $\lambda=\mu=-1,$ hence C lies on plane OAB. Thus O, A, B and C lies on the same plane.	
	OR $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}=(\mathbf{a}\times\mathbf{b})\bullet\left[-(\mathbf{a}+\mathbf{b})\right]$ $=-(\mathbf{a}\times\mathbf{b})\bullet\mathbf{a}-(\mathbf{a}\times\mathbf{b})\bullet\mathbf{b}$ $=0$ Thus O, A, B and C lies on the same plane.	
	Since $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, $\mathbf{b}=-\mathbf{a}-\mathbf{c}$	
	$\mathbf{a}\times\mathbf{b}=\mathbf{a}\times(-\mathbf{a}-\mathbf{c})$	
	$=-\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{c}$	
	$=\mathbf{c}\times\mathbf{a}$ $(\because\mathbf{a}\times\mathbf{a}=\mathbf{0}\,\&\,\mathbf{c}\times\mathbf{a}=-\mathbf{a}\times\mathbf{c})$	
	$\mathbf{b}\times\mathbf{c}=(-\mathbf{a}-\mathbf{c})\times\mathbf{c}$	
	$=-\mathbf{a}\times\mathbf{c}-\mathbf{c}\times\mathbf{c}$	
	$=\mathbf{c}\times\mathbf{a}$ $(\because\mathbf{c}\times\mathbf{c}=\mathbf{0}\,\&\,\mathbf{c}\times\mathbf{a}=-\mathbf{a}\times\mathbf{c})$	
	$\therefore\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$, shown	
	(ii) Area of triangle ABC	

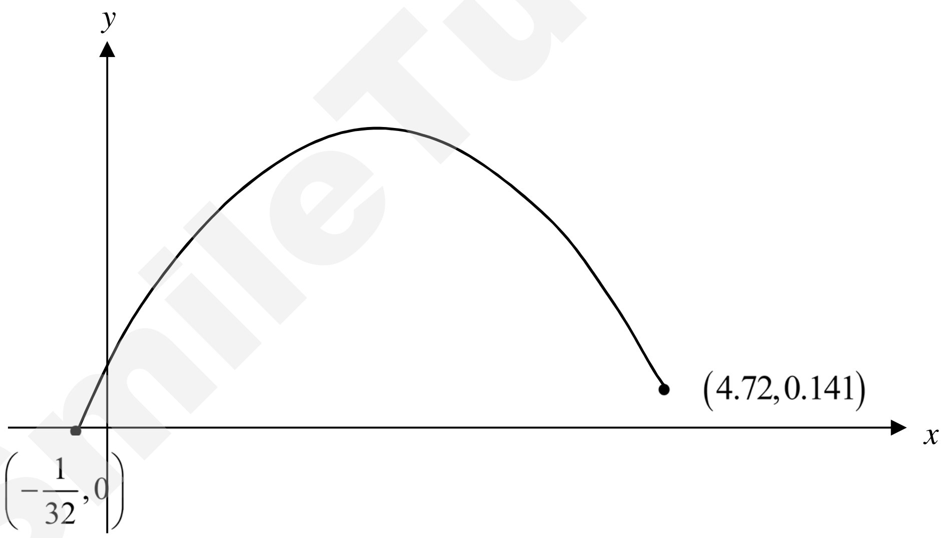
	$=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$	
	$=\frac{1}{2}\left\|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\right\|$	
	$=\frac{1}{2}\left\|\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{a}\right\|$	
	$=\frac{1}{2}\left\|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\right\|$	
	$=\frac{3}{2}\left\|\mathbf{a}\times\mathbf{b}\right\|$ where k = $\frac{3}{2}$ (Since $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$)	
	(iii) $\left\|\overrightarrow{AC}\bullet\hat{\mathbf{b}}\right\|=\left\|(\mathbf{c}-\mathbf{a})\bullet\mathbf{b}\right\|$	
	$=\left\|(-\mathbf{b}-2\mathbf{a})\bullet\mathbf{b}\right\|$	
	$=\left\|-\|\mathbf{b}\|^2-2\mathbf{a}\bullet\mathbf{b}\right\|$	
	$=1 \quad (\because \mathbf{a}\bullet\mathbf{b}=0, \|\mathbf{b}\|=1)$ 	
	Let θ be the angle between $\overrightarrow{AC}$ and **a** .	
	$\theta=180^\circ-\sin^{-1}\left(\frac{1}{2}\right)$	
	$=150^\circ$ or $\frac{5\pi}{6}$ rad	
10	The diagram below shows a rectangular container of variable height *h* cm, inscribed in a right circular cone with fixed height *b* cm and a radius of 20 cm. The four corners of the rectangular container's upper surface *ABCD* is always in contact with the conical surface. The point *O* is at the centre of the rectangle *ABCD*. The rectangular container is made with material of negligible thickness.	
		

	(i) If the fixed angle *BOC* is θ, show that the storage volume of the rectangular container in the shape of a cuboid is given by $V = 800b^{-2}(b-h)^2 h\sin\theta$.	[3]
	(ii) Find the value of *h* such that the rectangular container has a maximum storage volume, leaving your answer in terms of *b*.	[5]
	The rectangular container is opened at its upper surface *ABCD* and completely filled with a type of liquid perfume. A fragrance chemist placed the container in a room and allowed the liquid perfume to evaporate. It is known that the heat energy of the liquid perfume in the container, *E* joules, is related to the height of the container by the equation $$E = h - 3h^{-\frac{1}{2}}.$$	
	(iii) Given that the perfume evaporates at a rate of 0.08 c every hour and that the value of θ is $\frac{\pi}{6}$, calculate the rate of change of heat energy of the perfume when the height of the perfume in the rectangular container is $\frac{b}{5}$ cm, leaving your answer in the form $p+q\left(\frac{5}{b}\right)^{\frac{3}{2}}$, where *p* and *q* are exact constants to be determined.	[4]
	Solution	
	(i) Let the length of *OB* be *r*.	
	Method 1	
	$V = 4\left(r\sin\frac{\theta}{2}\right)\left(r\cos\frac{\theta}{2}\right)h$	
	$= 2r^2 h\sin\theta$	
	Method 2	
	$V = 2\left[\frac{1}{2}r^2\sin\theta + \frac{1}{2}r^2\sin(\pi-\theta)\right]h$	
	$= \left[r^2\sin\theta + r^2\sin(\pi-\theta)\right]h$	
	$= 2r^2 h\sin\theta$ (since $\sin(\pi-\theta) = \sin\theta$)	
	Using similar triangle,	
	$\frac{r}{20} = \frac{b-h}{b}$	
	$r = \frac{20}{b}(b-h)$	
	$V = 2\left(\frac{20}{b}\right)^2 (b-h)^2 h\sin\theta$	
	$= 800b^{-2}(b-h)^2 h\sin\theta$ (Shown)	
	(ii) $\frac{\mathrm{d}V}{\mathrm{d}h} = 800b^{-2}(b-h)^2\sin\theta + 800b^{-2}h\sin\theta(2)(b-h)(-1)$	

	$= 800b^{-2}(b-h)(\sin\theta)(b-h-2h)$	
	$= 800b^{-2}(\sin\theta)(b-h)(b-3h)$	
	Setting $\frac{dV}{dh} = 0$,	
	$800b^{-2}(\sin\theta)(b-h)(b-3h) = 0$	
	$(b-h)(b-3h) = 0$	
	$h = b$ (rejected since $h \neq b$) or $h = \frac{b}{3}$	
	<u>Method 1 (Using second derivative test)</u>	
	$\frac{d^2V}{dh^2} = 800b^{-2}(\sin\theta)\left[(b-h)(-3)+(b-3h)(-1)\right]$	
	$= 800b^{-2}(\sin\theta)(6h-4b)$	
	When $h = \frac{b}{3}$, $\frac{d^2V}{dh^2} = 800b^{-2}(\sin\theta)(-2b) < 0$	
	Hence, V is maximum when $h = \frac{b}{3}$.	
	<u>Method 2 (Using first derivative test)</u>	
	$\frac{dV}{dh} = 800b^{-2}(\sin\theta)(b-h)(b-3h)$	
	When $h < \frac{b}{3}$, $800b^{-2}(\sin\theta) > 0$, $(b-h) > 0$ and $(b-3h) > 0$, $\frac{dV}{dh} > 0$	
	When $h > \frac{b}{3}$, $800b^{-2}(\sin\theta) > 0$, $(b-h) > 0$ and $(b-3h) < 0$, $\frac{dV}{dh} < 0$	
	Hence, V is maximum when $h = \frac{b}{3}$.	
	(iii) Since $\theta = \frac{\pi}{6}$, $\frac{dV}{dh} = \frac{400}{b^2}(b-h)(b-3h)$	
	$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$	
	When $h = \frac{b}{5}$,	
	$-0.08 = \frac{400}{b^2}\left(\frac{4}{5}b\right)\left(\frac{2}{5}b\right) \times \frac{dh}{dt}$	
	$\frac{dh}{dt} = -\frac{1}{1600}$	
	$E = h - 3h^{-\frac{1}{2}}$	
	$\frac{dE}{dh} = 1 + \frac{3}{2}h^{-\frac{3}{2}}$	

	$\frac{dE}{dt} = \frac{dE}{dh} \times \frac{dh}{dt}$	
	$= \left(1 + \frac{3}{2}h^{-\frac{3}{2}}\right)\left(-\frac{1}{1600}\right)$	
	$= \left[1 + \frac{3}{2}\left(\frac{b}{5}\right)^{-\frac{3}{2}}\right]\left(-\frac{1}{1600}\right)$	
	$= -\frac{1}{1600} - \frac{3}{3200}\left(\frac{5}{b}\right)^{\frac{3}{2}}$	
	Hence, $p = -\frac{1}{1600}$, $q = -\frac{3}{3200}$	
11	A charged particle is placed in a varying magnetic field. A researcher decides to fit a mathematical model for the path of the fast-moving charged particle under the influence of the magnetic field. The particle was observed for the first 1.5 seconds. The displacement of the particle measured with respect to the origin in the horizontal and vertical directions, at time t seconds, is denoted by the variables x and y respectively. It is given that when $t = 0$, $x = -\frac{1}{32}$, $y = 0$ and $\frac{dx}{dt} = 3$. The variables are related by the differential equations $(\cos t)\frac{dy}{dt} + y\sin t = 4\cos^2 t - y^2$ and $\frac{d^2x}{dt^2} = \cos 3t \cos t$.	
	(i) Using the substitution $y = v\cos t$, show that $\frac{dv}{dt} = 4 - v^2$ and hence find y in terms of t.	[7]
	(ii) Show that $x = -\frac{1}{32}\cos 4t - \frac{1}{8}\cos 2t + 3t + \frac{1}{8}$.	[4]
	(iii) Sketch the path travelled by the particle for the first 1.5 seconds, labelling the coordinates of the end points of the path. The evaluation of the y-intercept is not needed.	[2]
	Solution	
	(i) $y = v\cos t$	
	$\frac{dy}{dt} = -v\sin t + \frac{dv}{dt}\cos t$	
	$(\cos t)\frac{dy}{dt} + y\sin t = 4\cos^2 t - y^2$	
	$(\cos t)\left(-v\sin t + \frac{dv}{dt}\cos t\right) + (v\cos t)\sin t = 4\cos^2 t - v^2\cos^2 t$	

	$\frac{\mathrm{d}v}{\mathrm{d}t}\left(\cos^2 t\right)=\left(\cos^2 t\right)\left(4-v^2\right)$	
	$\frac{\mathrm{d}v}{\mathrm{d}t}=4-v^2$ (Shown)	
	$\int \frac{1}{4-v^2}\ \mathrm{d}v=\int 1\ \mathrm{d}t$	
	$\int \frac{1}{2^2-v^2}\ \mathrm{d}v=\int 1\ \mathrm{d}t$	
	$\frac{1}{2(2)}\ln\left\lvert\frac{2+v}{2-v}\right\rvert=t+d$	
	$\left\lvert\frac{2+v}{2-v}\right\rvert=\mathrm{e}^{4t+4d}$	
	$\frac{2+v}{2-v}=\pm\mathrm{e}^{4t+4d}$	
	$\frac{2+v}{2-v}=A\mathrm{e}^{4t}$, where $A=\pm\mathrm{e}^{4d}$	
	$2+v=2A\mathrm{e}^{4t}-Av\mathrm{e}^{4t}$	
	$v+Av\mathrm{e}^{4t}=2\left(A\mathrm{e}^{4t}-1\right)$	
	$v=\frac{2\left(A\mathrm{e}^{4t}-1\right)}{A\mathrm{e}^{4t}+1}$	
	$\frac{y}{\cos t}=\frac{2\left(A\mathrm{e}^{4t}-1\right)}{A\mathrm{e}^{4t}+1}$	
	$y=\frac{2\left(A\mathrm{e}^{4t}-1\right)\cos t}{A\mathrm{e}^{4t}+1}$	
	When $t=0$, $y=0$	
	$0=\frac{2(A-1)}{A+1}$	
	$A=1$	
	$y=\frac{2\left(\mathrm{e}^{4t}-1\right)\cos t}{\mathrm{e}^{4t}+1}$	
	(ii) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\cos 3t\cos t$	
	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{1}{2}\left(\cos 4t+\cos 2t\right)$	
	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{2}\int \cos 4t+\cos 2t\ \mathrm{d}t$	
	$=\frac{1}{2}\left(\frac{1}{4}\sin 4t+\frac{1}{2}\sin 2t\right)+c$	
	$=\frac{1}{8}\sin 4t+\frac{1}{4}\sin 2t+c$	

	When $t=0$, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=3$	
	$3=\dfrac{1}{8}\sin 4(0)+\dfrac{1}{4}\sin 2(0)+c$	
	$c=3$	
	$x=\displaystyle\int \dfrac{1}{8}\sin 4t+\dfrac{1}{4}\sin 2t+3\ \mathrm{d}t$	
	$x=-\dfrac{1}{32}\cos 4t-\dfrac{1}{8}\cos 2t+3t+k$	
	When $t=0$, $x=-\dfrac{1}{32}$	
	$-\dfrac{1}{32}=-\dfrac{1}{32}\cos 4(0)-\dfrac{1}{8}\cos 2(0)+3(0)+k$	
	$k=\dfrac{1}{8}$	
	Hence, $x=-\dfrac{1}{32}\cos 4t-\dfrac{1}{8}\cos 2t+3t+\dfrac{1}{8}$	
	(iii) 	

THE END

SERANGOON JUNIOR COLLEGE

2018 JC2 PRELIMINARY EXAMINATION

MATHEMATICS

Higher 2 **9758/2**

17 Sept 2018

3 hours

Additional materials: Writing paper

List of Formulae (MF 26)

**TIME : ** 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and class on the cover page and on all the work you hand in.
Write in blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

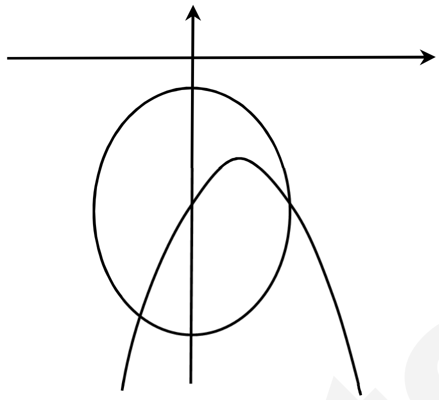
The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

Total marks for this paper is 100 marks

This question paper consists of 7 printed pages (inclusive of this page) and 1 blank page.

Section A: Pure Mathematics [40 marks].

1 R is the region enclosed by the line $y=-5$ and the curves $y=-x^2+2x-5$ and $\frac{x^2}{4}+\frac{(y+5)^2}{16}=1$ as shown in the diagram below.



Find the volume generated by region R when it is rotated 2π radians about the x-axis. Leave your answer correct to 2 decimal places. [3]

2 A curve has parametric equations

$$x=\tan\theta,\ \ y=2\sec\theta \ \text{ for } 0\le\theta<2\pi.$$

The equation of the tangent to the curve at the point P with parameter p is given by $y=(2\sin p)x+2\cos p$.

(i) The tangent at P meets the x and y axes at the points A and B respectively. Find the Cartesian equation of the locus of the mid-point of AB as p varies. [3]

(ii) The tangent at P meets the line $y=2x$ at the point S and the line $y=-2x$ at the point T. Show that the area of the triangle OST is independent of p, where O is the origin. [4]

3 Given that $\ln y=\mathrm{e}^{\tan^{-1}x}$, show that $\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(\ln y+1-2x\right)\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(i) Find the Maclaurin's series for y up to and including the term in x^3. [4]

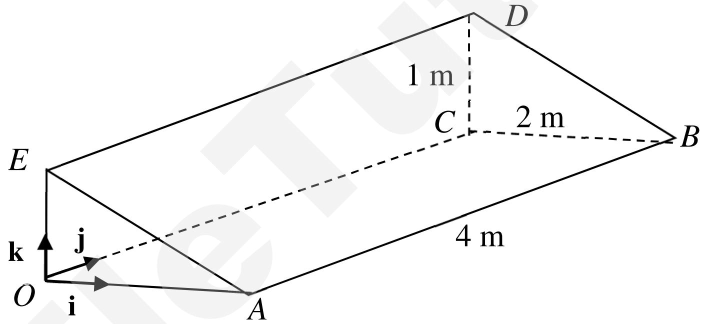
(ii) Deduce the Maclaurin's series for y, where $\ln y=\mathrm{e}^{\tan^{-1}x}+x-1$, up to and including the term in x^3. [3]

4 Given that the equation $\mathrm{f}(z) = 2z^3 + az^2 + (2a-4)z + (2a-8) = 0$ has no real solution, explain clearly why a is not a real number. [2]

(i) It is known that f has a factor $(2z+\mathrm{i})$, find a. Hence find all the roots of $\mathrm{f}(z) = 0$, showing your workings clearly. [4]

(ii) Deduce the roots of the equation $2 - \frac{a}{w} + \frac{(2a-4)}{w^2} - \frac{2a-8}{w^3} = 0$, where a takes the value obtained in **(i)**. [2]

5 A hollow metallic ramp, in the shape of a prism, is constructed for the marching contingent to march onto to reach an elevated platform from the ground during the national day parade.
The diagram below shows the prism with O as the origin of position vectors and the unit vectors **i**, **j** and **k** are parallel to OA, OC and OE respectively.
It is given that $OE = CD = 1$ m, $OA = CB = 2$ m and $OC = AB = ED = 4$ m.



A laser beam in the form of a line l has Cartesian equation $\frac{a+x}{2} = z,\ y = 1$, where $a \in \mathbb{R}$, is emitted onto the plane $ABDE$.

(i) Find, in terms of a, the coordinates of the point of intersection, M, of the laser beam and the plane $ABDE$. [4]

For the following parts of the question assume $a = 0$.

(ii) The laser beam is reflected about the plane $ABDE$. By finding the foot of perpendicular from $Q(0,1,0)$ to the plane $ABDE$, find the equation of the reflected beam. [5]

(iii) The path traced out by an ant crawling on the floor $OABC$ is given by $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \beta \in \mathbb{R}$. Let P be the point on the path, located under the ramp, whereby the ant is equidistant between the planes $ABDE$ and $OCDE$. Find the position vector of point P exactly. [4]

Section B: Statistics [60 marks]

6 A jackfruit farm produces 2 types of jackfruit pulps namely Grade A and Grade B. The pulps are randomly packed into boxes of 8.
The probability that a box contains at most 3 Grade A jackfruit pulps is 0.00245.
A fruit trading wholesaler places a monthly order of 1000 boxes of jackfruit pulps for 5 years. Find the approximate probability that the mean number of boxes that contain at most 3 Grade A jackfruit pulps in a month is more than 3.
[You may assume that there are 12 months in a year.] [4]

7 A player throws an unbiased six-sided die and the number shown on the top face is noted. If it is not a six, the score is the number shown on the top face. If it is a six, he throws the die a second time and the score is the total score obtained from his two throws. The player has at most two throws.

(i) Construct the probability distribution table on the score for the player. [2]

Given that the expected player's score is $\frac{49}{12}$,

(ii) find the exact value of the variance of the player's score, showing your workings clearly. [3]

8 The length of time that Somesong phones last in between charges has a normal distribution with mean 20 hours and standard deviation 2 hours. The length of time that Apfel phones last in between charges has a normal distribution with mean μ hours and standard deviation σ hours.

(a) The average length of time two randomly chosen Somesong phones and one randomly chosen Apfel phone will last in between charges is equally likely to be less than 19 hours or more than 23 hours, with the probability known to be 0.02275.

(i) Calculate the values of μ and σ. [3]

(ii) Find the probability that twice the length of time a Somesong phone lasts in between charges differs from 40 hours by at least 2 hours. [2]

(b) Six randomly chosen Somesong phones are examined. Find the probability that the sixth Somesong phone is the fourth Somesong phone that lasts more than 22 hours in between charges. [2]

9 A customer wishes to investigate the shelf life of the durian puffs produced by a baker before they turn bad when placed at room temperature and pressure. It is assumed that the shelf life of the durian puffs are independent of one another. Based on past records, the baker claims that the mean shelf life of the durian puffs is at least 8 hours. To test this claim, the customer recorded the shelf life of 55 randomly chosen durian puffs and found that its mean is 7.4 hours and standard deviation is 2.6 hours.

(i) Carry out a test, to determine whether there is any evidence to doubt the baker's claim at 5% significance level. [5]

(ii) Explain, in the context of the question, the meaning of 'at 5% significance level'. [1]

(iii) Suppose the population standard deviation is 3.2 hours now and the baker claims that the mean shelf life the durian puffs is 8 hours. A new sample of 10 durian puffs is taken. Using this sample, the customer conducts another test and found that the baker's claim is not rejected at the 5% significance level. Stating a necessary assumption for the test, find the set of values that the sample mean shelf life, $\overline{y}$, can take. [4]

10 **(a)** A group of twelve people consists of one pair of sisters, one set of three brothers, a family of three and 4 others.

(i) The twelve people are grouped into three groups of 4. Find the number of ways where the family of three are together. [2]

(ii) The twelve people are arranged randomly in a line. Find the number of ways that the sisters are not together and the brothers are all separated. [3]

(b) For events A and B, it is given that $\mathrm{P}(A)=\frac{11}{20}$, $\mathrm{P}(B)=\frac{2}{5}$ and $\mathrm{P}(A\cap B)=\frac{1}{4}$.

(i) Find the probability where event A occurs or event B occurs but not both. [1]

(ii) Find the probability where event B has occurred but not A. [1]

For a third event C, it is given that $\mathrm{P}(C)=\frac{1}{2}$ and that A and C are independent,

(iii) Find the range of values of $\mathrm{P}(A'\cap B'\cap C)$. [3]

11 In an attempt to reduce the operating costs of an airline company, an analyst collected data on the seat capacity and the total fuel use of the different types of airplanes used for a particular route, as shown in the table below.

Seat capacity, x (in hundreds)	4.07	3.84	4	3.31	2.48	2.92	3.1	3.57
Total fuel use, y (10^3 tonnes)	140	115	m	11	20	25	33	74

(a) **(i)** Given that the equation of the least squares regression line of y on x is

$$y = 79.695x - 205.86,$$

show that the value of m is 110, correct to the nearest whole number. [2]

(ii) Give an interpretation, in context, of the gradient of the least squares regression line given in **(a)(i)**. [1]

The analyst suspects that the fuel use for one of the points has been recorded wrongly.

(b) **(i)** Draw a scatter diagram for the data, labelling the axes clearly. On your diagram, circle the point for which the total fuel use has been recorded wrongly and label it as Q. [2]

(ii) Explain from your scatter diagram why the relationship between x and y should not be modelled by an equation of the form $y = ax + b$. [1]

For the following parts of this question, you should exclude the point Q for the subsequent calculation.

(iii) The analyst is presented with the following two models:

(A) $y = ae^x + b$

(B) $y = ax^3 + b$.

Explain clearly which of the two models is the more appropriate model. [2]

(iv) Using the more appropriate model from **(b)(iii)**, estimate the total fuel use when the seating capacity of an airplane is 331. Leave your answer to the nearest whole number. [2]

(v) Give two reasons why the estimation in part **(b)(iv)** is reliable. [2]

12 Based on past statistical data of high speed train journey, there is a 8% chance that a passenger with reservation made will not show up. In order to maximise revenue, a high speed train company accepts more reservations than the passenger capacity of its trains.

(i) State 2 assumptions needed such that the number of passengers who do not show up for a high speed train journey may be well modelled by a Binomial distribution. [2]

A train company operates a high speed train from Burong Lake to Huahin Cove which has a capacity of 280 passengers.

(ii) Find the probability that when 300 reservations are accepted, the high speed train journey is overbooked, i.e. there is not enough seats available for the passengers who show up. [2]

(iii) Find the maximum number of reservations that should be accepted in order to ensure that the probability of overbooking is less than 1%. [3]

The high speed train operates once daily throughout the year and 300 reservations are accepted for each journey.

(iv) Find the probability that no high speed train journey is overbooked in a week. [2]

(v) Two such journeys from Burong Lake to Huahin Cove with 300 reservations each are examined. If the total number of passengers who turned up for these two journeys is at most 550 passengers, find the probability that one of the journeys is overbooked by at most 2 passengers. [3]

THE END

SERANGOON JUNIOR COLLEGE

2018 JC2 PRELIMINARY EXAMINATION

MATHEMATICS

Higher 2 **9758/2**

17 Sept 2018

3 hours

Additional materials: Writing paper

List of Formulae (MF 26)

**TIME : ** 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and class on the cover page and on all the work you hand in.
Write in blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

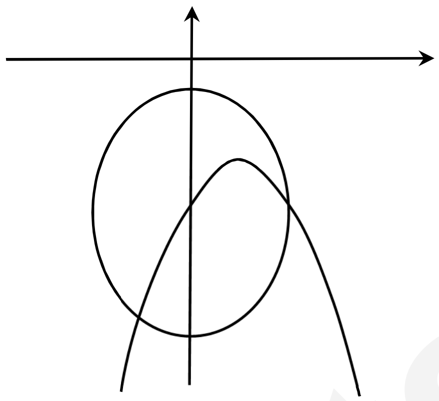
The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

Total marks for this paper is 100 marks

This question paper consists of 7 printed pages (inclusive of this page) and 1 blank page.

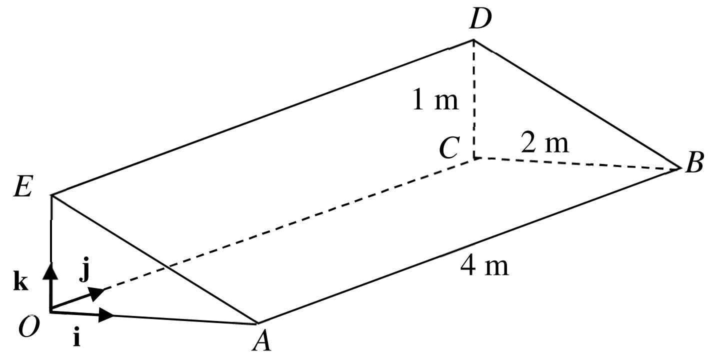
[TURN OVER]

Section A: Pure Mathematics [40 marks].

1	R is the region enclosed by the line $y=-5$ and the curves $y=-x^2+2x-5$ and $\frac{x^2}{4}+\frac{(y+5)^2}{16}=1$ as shown in the diagram below.	
		
	Find the volume generated by region R when it is rotated 2π radians about the x-axis. Leave your answer correct to 2 decimal places.	[3]
	Solution	
	Required volume $=\pi\int_{-2}^{-1.09}\left(-5-\sqrt{16-4x^2}\right)^2 \mathrm{d}x+\pi\int_{-1.09}^{0}\left(-x^2+2x-5\right)^2 \mathrm{d}x-\pi(25)(2)$ OR $=\pi\int_{-2}^{-1.09}\left(-5-\sqrt{16-4x^2}\right)^2\mathrm{d}x+\pi\int_{-1.09}^{0}\left(-x^2+2x-5\right)^2\mathrm{d}x-\pi\int_{-2}^{0}(-5)^2\mathrm{d}x$	
	= 146.91	
	Using Shell method $\mathrm{Vol}=2\pi\int_{-8.36}^{-5}-y\left(1-\sqrt{-y-4}+\sqrt{4-\frac{(y+5)^2}{4}}\right)\mathrm{d}y$	
	= 146.91	
2	A curve has parametric equations	
	$x=\tan\theta,\ y=2\sec\theta$ for $0\le\theta<2\pi$.	
	The equation of the tangent to the curve at the point P with parameter p is given by $y=(2\sin p)x+2\cos p$.	
	(i) The tangent at P meets the x and y axes at the points A and B respectively. Find the Cartesian equation of the locus of the mid-point of AB as p varies.	[3]
	(ii) The tangent at P meets the line $y=2x$ at the point S and the line $y=-2x$ at the point T. Show that the area of the triangle OST is independent of p, where O is the origin.	[4]
	Solution	
	(i) Coordinates of A: $(-\cot p, 0)$	
	Coordinates of B: $(0, 2\cos p)$	

	Midpoint of AB: $\left(-\frac{\cot p}{2}, \cos p\right)$	
	So let $x = -\frac{\cot p}{2}$ and $y = \cos p$	
	$\tan p = -\frac{1}{2x}$ and $\sec p = \frac{1}{y}$	
	Since $1 + \tan^2 p = \sec^2 p$	
	$1 + \frac{1}{4x^2} = \frac{1}{y^2}$	
	(ii) To find the coordinates of S: $2x = (2\sin p)x + 2\cos p$	
	$\therefore x = \frac{\cos p}{1-\sin p}$ and $y = \frac{2\cos p}{1-\sin p}$	
	To find the coordinates of T: $-2x = (2\sin p)x + 2\cos p$	
	$\therefore x = \frac{-\cos p}{1+\sin p}$ and $y = \frac{2\cos p}{1+\sin p}$	
	Method 1	
	Hence area of triangle OST $= \frac{1}{2}\sqrt{\left(\frac{\cos p}{1-\sin p}\right)^2 + \left(\frac{2\cos p}{1-\sin p}\right)^2}\sqrt{\left(\frac{-\cos p}{1+\sin p}\right)^2 + \left(\frac{2\cos p}{1+\sin p}\right)^2}\sin\left(2\tan^{-1}\frac{1}{2}\right)$	
	– is for using formula with the correct angle used.	
	$= \frac{1}{2}\frac{\cos^2 p}{(1-\sin p)(1+\sin p)}\sqrt{5}\sqrt{5}\left[2\sin\left(\tan^{-1}\frac{1}{2}\right)\cos\left(\tan^{-1}\frac{1}{2}\right)\right]$	
	$= \frac{5}{2}\left(\frac{\cos^2 p}{1-\sin^2 p}\right)\left(2 \times \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}}\right)$ $= 2$	
	Method 2	
	Hence area of triangle OST $= \frac{1}{2}\sqrt{\left(\frac{\cos p}{1-\sin p}\right)^2 + \left(\frac{2\cos p}{1-\sin p}\right)^2}\sqrt{\left(\frac{-\cos p}{1+\sin p}\right)^2 + \left(\frac{2\cos p}{1+\sin p}\right)^2}\sin\left(\tan^{-1}\frac{4}{3}\right)$	
	– is for using formula with the correct angle used.	
	$= \frac{1}{2}\frac{\cos^2 p}{(1-\sin p)(1+\sin p)}\sqrt{5}\sqrt{5}\sin\left(\tan^{-1}\frac{4}{3}\right)$	
	$= \frac{5}{2}\left(\frac{\cos^2 p}{1-\sin^2 p}\right)\left(\frac{4}{5}\right)$ $= 2$	
3	Given that $\ln y = e^{\tan^{-1} x}$, show that $(1+x^2)\frac{d^2 y}{dx^2} = (\ln y + 1 - 2x)\frac{dy}{dx}$.	[2]
	(i) Find the Maclaurin's series for y up to and including the term in x^3.	[4]

	(ii) Deduce the Maclaurin's series for *y*, where $\ln y = \mathrm{e}^{\tan^{-1}x} + x - 1$, up to and including the term in x^3.	[3]
	Solution	
	$\ln y = \mathrm{e}^{\tan^{-1}x}$	
	$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}\mathrm{e}^{\tan^{-1}x}$	
	$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = y\ln y$	
	$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = \ln y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x}$	
	$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(\ln y + 1 - 2x\right)\frac{\mathrm{d}y}{\mathrm{d}x}$	
	(i) $\left(1+x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(\ln y + 1 - 2x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} - 2\right)$	
	When $x = 0$,	
	$y = \mathrm{e}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\mathrm{e}$ and $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 3\mathrm{e}$	
	So the Maclaurin series of *y* is	
	$y = \mathrm{e} + \mathrm{e}x + \mathrm{e}x^2 + \frac{1}{2}\mathrm{e}x^3 + \ldots$	
	(ii) $y = \mathrm{e}^{\mathrm{e}^{\tan^{-1}x}+x-1} = \mathrm{e}^{\mathrm{e}^{\tan^{-1}x}} \bullet \mathrm{e}^x \bullet \mathrm{e}^{-1}$	
	$y = \mathrm{e}^{\mathrm{e}^{\tan^{-1}x}+x-1} = \left(\mathrm{e} + \mathrm{e}x + \mathrm{e}x^2 + \frac{\mathrm{e}x^3}{2} + \ldots\right)\bullet\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\bullet \mathrm{e}^{-1}$	
	$= 1 + x + x^2 + \frac{x^3}{2} + x + x^2 + x^3 + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^3}{6} + \ldots$	
	$= 1 + 2x + \frac{5}{2}x^2 + \frac{13}{6}x^3 + \ldots$	
4	Given that the equation $\mathrm{f}(z) = 2z^3 + az^2 + (2a-4)z + (2a-8) = 0$ has no real solution, explain clearly why *a* is not a real number.	[2]
	(i) It is known that f has a factor $(2z + \mathrm{i})$, find *a*. Hence find all the roots of $\mathrm{f}(z) = 0$, showing your workings clearly.	[4]
	(ii) Deduce the roots of the equation $2 - \frac{a}{w} + \frac{(2a-4)}{w^2} - \frac{2a-8}{w^3} = 0$, where *a* takes the value obtained in **(i)**.	[2]
	Solution	
	If all coefficients are real, then by conjugate root theorem, roots will occur in conjugate pairs. As f is a polynomial of degree 3, it would mean that it must	

	have either 3 real roots or it will be having 1 real root with a pair of conjugate roots.	
	But it is given that f(z) = 0 has no real solution and so it must therefore means that at least one of the coefficients is a complex number. Hence a is not a real number.	
	(i) $\mathrm{f}\left(-\frac{\mathrm{i}}{2}\right)=2\left(-\frac{\mathrm{i}}{2}\right)^3+a\left(-\frac{\mathrm{i}}{2}\right)^2+(2a-4)\left(-\frac{\mathrm{i}}{2}\right)+2a-8=0$	
	$\frac{\mathrm{i}}{4}-\frac{a}{4}-a\mathrm{i}+2\mathrm{i}+2a-8=0$	
	$8-\frac{9}{4}\mathrm{i}=a\left(\frac{7}{4}-\mathrm{i}\right)$	
	$=4+\mathrm{i}$	
	So $\mathrm{f}(z)=2z^3+(4+\mathrm{i})z^2+(4+2\mathrm{i})z+2\mathrm{i}=(2z+\mathrm{i})\left(z^2+2z+2\right)$	
	For $\mathrm{f}(z)=0\Rightarrow(2z+\mathrm{i})\left(z^2+2z+2\right)=0$	
	Consider $z^2+2z+2=0$,	
	$z=\frac{-2\pm\sqrt{4-8}}{2}$	
	$z=-1\pm\mathrm{i}$	
	Hence the roots are $-1+\mathrm{i}$, $-1-\mathrm{i}$ and $-\frac{\mathrm{i}}{2}$	
	(ii) Since $2z^3+az^2+(2a-4)z+2a-8=0$,	
	$\Rightarrow 2+\frac{a}{z}+\frac{2a-4}{z^2}+\frac{2a-8}{z^3}=0$	
	Replace z by $-w$	
	$\Rightarrow 2-\frac{a}{w}+\frac{2a-4}{w^2}-\frac{2a-8}{w^3}=0$	
	So the roots of this equations are $1-\mathrm{i}$, $1+\mathrm{i}$ and $\frac{\mathrm{i}}{2}$	
5	A hollow metallic ramp, in the shape of a prism, is constructed for the marching contingent to march onto to reach an elevated platform from the ground during the national day parade.	
	The diagram below shows the prism with O as the origin of position vectors and the unit vectors **i**, **j** and **k** are parallel to OA, OC and OE respectively. It is given that $OE = CD = 1$ m, $OA = CB = 2$ m and $OC = AB = ED = 4$ m.	
		

	A laser beam in the form of a line l has Cartesian equation $\frac{a+x}{2}=z,\ y=1$, where $a\in\cdots$, is emitted onto the plane *ABDE*.	
	(i) Find, in terms of *a*, the coordinates of the point of intersection, *M*, of the laser beam and the plane *ABDE*.	[4]
	For the following parts of the question assume $a=0$.	
	(ii) The laser beam is reflected about the plane *ABDE*. By finding the foot of perpendicular from $Q(0,1,0)$ to the plane *ABDE*, find the equation of the reflected beam.	[5]
	(iii) The path traced out by an ant crawling on the floor *OABC* is given by $\mathbf{r}=\begin{pmatrix}2\\0\\0\end{pmatrix}+\beta\begin{pmatrix}-1\\2\\0\end{pmatrix},\beta\in\cdots$. Let *P* be the point on the path, located under the ramp, whereby the ant is equidistant between the planes *ABDE* and *OCDE*. Find the position vector of point *P* exactly.	[4]
	Solution	
	(i) line l : $\mathbf{r}=\begin{pmatrix}-a\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\1\end{pmatrix}, a,\lambda\in\cdots$	
	normal vector of plane *ABDE*, $\underset{\sim}{n}=\begin{pmatrix}0\\1\\0\end{pmatrix}\times\left[\begin{pmatrix}2\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}\right]=\begin{pmatrix}-1\\0\\-2\end{pmatrix}=-\begin{pmatrix}1\\0\\2\end{pmatrix}$	
	plane *ABDE*: $\mathbf{r}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=2$	
	For point of intersection,	
	$\begin{pmatrix}-a+2\lambda\\1\\\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=2$	
	$\lambda=\frac{2+a}{4}$	
	$\overrightarrow{OM}=\begin{pmatrix}-a+\frac{2+a}{2}\\1\\\frac{2+a}{4}\end{pmatrix}=\begin{pmatrix}\frac{2-a}{4}\\1\\\frac{2+a}{4}\end{pmatrix}$ Coordinates are $\left(\frac{2-a}{2},1,\frac{2+a}{4}\right)$	

	(ii) As $a=0$, $\overrightarrow{OM}=\frac{1}{2}\begin{pmatrix}2\\2\\1\end{pmatrix}$	
	Let F be the foot of perpendicular from Q to the plane $ABDE$	
	Equation of line QF is $\mathbf{r}=\begin{pmatrix}0\\1\\0\end{pmatrix}+\mu\begin{pmatrix}1\\0\\2\end{pmatrix},\mu\in\cdots$	
	$\overrightarrow{OF}=\begin{pmatrix}\mu\\1\\2\mu\end{pmatrix}$ for some $\mu\in\cdots$ Missing out " for some μ" anywhere – P	
	$\begin{pmatrix}\mu\\1\\2\mu\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=2$	
	$\mu=\frac{2}{5}$	
	$\therefore\overrightarrow{OF}=\frac{1}{5}\begin{pmatrix}2\\5\\4\end{pmatrix}$	
	Let Q' be the point of reflection of point Q about plane $ABDE$.	
	$\overrightarrow{OQ}'=2\overrightarrow{OF}-\overrightarrow{OQ}=\begin{pmatrix}4/5\\2\\8/5\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}4/5\\1\\8/5\end{pmatrix}$	
	$\overrightarrow{MQ}'=\overrightarrow{OQ}'-\overrightarrow{OM}=\begin{pmatrix}4/5\\1\\8/5\end{pmatrix}-\begin{pmatrix}1\\1\\1/2\end{pmatrix}=\begin{pmatrix}-1/5\\0\\11/10\end{pmatrix}=1/10\begin{pmatrix}-2\\0\\11\end{pmatrix}$	
	Equation of line of reflection of the laser beam:	
	$\mathbf{r}=\begin{pmatrix}4/5\\1\\8/5\end{pmatrix}+\alpha\begin{pmatrix}-2\\0\\11\end{pmatrix}$, $\alpha\in\cdots$ **OR** $\mathbf{r}=\frac{1}{2}\begin{pmatrix}2\\2\\1\end{pmatrix}+\alpha\begin{pmatrix}-2\\0\\11\end{pmatrix}$, $\alpha\in\cdots$	
	(iii) Equation of line AC: $\mathbf{r}=\begin{pmatrix}2\\0\\0\end{pmatrix}+\beta\begin{pmatrix}-1\\2\\0\end{pmatrix},\beta\in\cdots$	

	$\overrightarrow{OP}=\begin{pmatrix}2-\beta\\2\beta\\0\end{pmatrix}$ for some β Missing " for some β" – P		
	$\left\lvert\dfrac{\overrightarrow{OP}\cdot\mathbf{n}_{OCDE}}{\lvert\mathbf{n}_{OCDE}\rvert}\right\rvert=\left\lvert\dfrac{\overrightarrow{EP}\cdot\mathbf{n}_{ABDE}}{\lvert\mathbf{n}_{ABDE}\rvert}\right\rvert$		
	$\left\lvert\dfrac{\begin{pmatrix}2-\beta\\2\beta\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{\left\lvert\begin{pmatrix}1\\0\\0\end{pmatrix}\right\rvert}\right\rvert=\left\lvert\dfrac{\begin{pmatrix}2-\beta\\2\beta\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}}{\left\lvert\begin{pmatrix}1\\0\\2\end{pmatrix}\right\rvert}\right\rvert$ OR $\left\lvert\dfrac{\begin{pmatrix}2-\beta\\2\beta\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}-0}{\left\lvert\begin{pmatrix}1\\0\\0\end{pmatrix}\right\rvert}\right\rvert=\left\lvert\dfrac{\begin{pmatrix}2-\beta\\2\beta\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}-2}{\left\lvert\begin{pmatrix}1\\0\\2\end{pmatrix}\right\rvert}\right\rvert$		
	$\lvert 2-\beta\rvert=\left\lvert\dfrac{2-\beta-2}{\sqrt{5}}\right\rvert$		
	$2-\beta=\pm\dfrac{\beta}{\sqrt{5}}$ OR $5(2-\beta)^2=\beta^2$ $2\sqrt{5}-\beta\sqrt{5}=\pm\beta$ $\left(2\sqrt{5}-\sqrt{5}\beta-\beta\right)\left(2\sqrt{5}-\sqrt{5}\beta+\beta\right)=0$		
	$\therefore\dfrac{2\sqrt{5}}{\sqrt{5}+1}$ or $\dfrac{2\sqrt{5}}{\sqrt{5}-1}$		
	$\dfrac{2\sqrt{5}\left(\sqrt{5}-1\right)}{4}$ or $\dfrac{2\sqrt{5}\left(\sqrt{5}+1\right)}{4}$		
	$\beta=\dfrac{5\pm\sqrt{5}}{2}$		
	$\overrightarrow{OP}=\dfrac{1}{2}\begin{pmatrix}-1-\sqrt{5}\\10+2\sqrt{5}\\0\end{pmatrix}$ (rejected since it gives a point outside of the ramp) Hence $\overrightarrow{OP}=\dfrac{1}{2}\begin{pmatrix}\sqrt{5}-1\\10-2\sqrt{5}\\0\end{pmatrix}$		

Section B: Statistics [60 marks]

6 A jackfruit farm produces 2 types of jackfruit pulps namely Grade A and Grade B. The pulps are randomly packed into boxes of 8.

The probability that a box contains at most 3 Grade A jackfruit pulps is 0.00245.

A fruit trading wholesaler places a monthly order of 1000 boxes of jackfruit pulps for 5 years. Find the approximate probability that the mean number of boxes that contain at most 3 Grade A jackfruit pulps in a month is more than 3.
[You may assume that there are 12 months in a year.] [4]

Solution

Let X be the r.v. "number of boxes that contain at most 3 Grade A jackfruit pulps in a box out of 1000 boxes"

$X \sim \mathrm{B}(1000, 0.00245)$

$\mathrm{E}(X) = 1000(0.00245) = 2.45$

$\mathrm{Var}(X) = 2.45(1-0.00245) = 2.444$

In 5 years, there are 60 months altogether.

Mean number of boxes with at most 3 Grade A jackfruit pulps is

$$\overline{X} = \frac{X_1 + X_2 + X_3 + ... + X_{60}}{60}$$

Since sample size is large, by Central Limit Theorem,

$\overline{X} \sim \mathrm{N}\left(2.45, \frac{2.444}{60}\right)$ approximately

From GC, $\mathrm{P}\left(\overline{X} > 3\right) = 0.00321$ (correct to 3 sig fig)

7 A player throws an unbiased six-sided die and the number shown on the top face is noted. If it is not a six, the score is the number shown on the top face. If it is a six, he throws the die a second time and the score is the total score obtained from his two throws. The player has at most two throws.

(i) Construct the probability distribution table on the score for the player. [2]

Given that the expected player's score is $\frac{49}{12}$,

(ii) find the exact value of the variance of the player's score, showing your workings clearly. [3]

Solution

(i) Let X be the random variable "total score of the player".

x	1	2	3	4	5	7	8	9	10	11	12
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

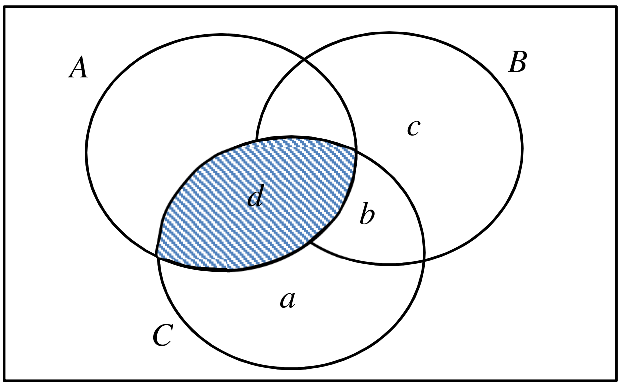
We can check that the total probability is 1.

(ii)

	$\mathrm{E}\left(X^2\right)=\sum x^2\ \mathrm{P}(X=x)$ $=(1)^2\left(\frac{1}{6}\right)+(2)^2\left(\frac{1}{6}\right)+(3)^2\left(\frac{1}{6}\right)+(4)^2\left(\frac{1}{6}\right)+(5)^2\left(\frac{1}{6}\right)$ $+(7)^2\left(\frac{1}{36}\right)+(8)^2\left(\frac{1}{36}\right)+(9)^2\left(\frac{1}{36}\right)+(10)^2\left(\frac{1}{36}\right)+(11)^2\left(\frac{1}{36}\right)+(12)^2\left(\frac{1}{36}\right)$	
	$=\frac{889}{36}$	
	$\mathrm{Var}(X)=\mathrm{E}\left(X^2\right)-\left[\mathrm{E}(X)\right]^2$ $=\frac{889}{36}-\left(\frac{49}{12}\right)^2$	
	$=\frac{385}{48}$	
8	The length of time that Somesong phones last in between charges has a normal distribution with mean 20 hours and standard deviation 2 hours. The length of time that Apfel phones last in between charges has a normal distribution with mean μ hours and standard deviation σ hours.	
	(a) The average length of time two randomly chosen Somesong phones and one randomly chosen Apfel phone will last in between charges is equally likely to be less than 19 hours or more than 23 hours, with the probability known to be 0.02275.	
	(i) Calculate the values of μ and σ.	[3]
	(ii) Find the probability that twice the length of time a Somesong phone lasts in between charges differs from 40 hours by at least 2 hours.	[2]
	(b) Six randomly chosen Somesong phones are examined. Find the probability that the sixth Somesong phone is the fourth Somesong phone that lasts more than 22 hours in between charges.	[2]
	Solution	
	(a) Let X denote the length of time that a randomly chosen Somesong phone lasts in between charges. $X \sim \mathrm{N}(20,4)$ Let Y denote the length of time that a randomly chosen Apfel phones lasts in between charges. $Y \sim \mathrm{N}\left(\mu,\sigma^2\right)$ Let $W=\frac{X_1+X_2+Y}{3}$ $W \sim \mathrm{N}\left(\frac{40+\mu}{3},\frac{8+\sigma^2}{9}\right)$	
	(i) By symmetry,	

	$\frac{40+\mu}{3}=\frac{19+23}{2}=21$ $\mu=23$	
	So, $W \sim \mathrm{N}\left(21, \frac{8+\sigma^2}{9}\right)$ $\mathrm{P}(W<19)=0.02275$ $\mathrm{P}\left(Z<\frac{-6}{\sqrt{8+\sigma^2}}\right)=0.02275$ $\frac{-6}{\sqrt{8+\sigma^2}}=-2$ $\sigma=1$	
	(ii) $2X \sim \mathrm{N}(40,16)$ $\mathrm{P}(\lvert 2X-40\rvert \geq 2)$ $=\mathrm{P}(2X \geq 42)+\mathrm{P}(2X \leq 38)$ $=0.617$	
	(b) $\mathrm{P}(X>22)=0.158655$	
	Required probability $=\binom{5}{3}(0.158655)^3(1-0.158655)^2(0.158655)$ $=0.00449$	
9	A customer wishes to investigate the shelf life of the durian puffs produced by a baker before they turn bad when placed at room temperature and pressure. It is assumed that the shelf life of the durian puffs are independent of one another. Based on past records, the baker claims that the mean shelf life of the durian puffs is at least 8 hours. To test this claim, the customer recorded the shelf life of 55 randomly chosen durian puffs and found that its mean is 7.4 hours and standard deviation is 2.6 hours.	
	(i) Carry out a test, to determine whether there is any evidence to doubt the baker's claim at 5% significance level.	[5]
	(ii) Explain, in the context of the question, the meaning of 'at 5% significance level'.	[1]
	(iii) Suppose the population standard deviation is 3.2 hours now and the baker claims that the mean shelf life the durian puffs is 8 hours. A new sample of 10 durian puffs is taken. Using this sample, the customer conducts another test and found that the baker's claim is not rejected at the 5% significance level. Stating a necessary assumption for the test, find the set of values that the sample mean shelf life, $\overline{y}$, can take.	[4]
	Solution	
	(i) Unbiased estimate of population variance = $\frac{55}{54}(2.6)^2=6.885185$	

	Let μ be the average shelf life of the durian puffs when placed at room temperature and pressure. Let X be the random variable "the shelf life of a randomly chosen durian puff".	
	To Test H_o: $\mu = 8$ Against H_1: $\mu < 8$	
	Left tailed z-test at 5% level of significance	
	Under H_o, since sample size = 55 is large, by Central Limit Theorem,	
	$\bar{X} \sim N\left(8, \frac{6.885185}{55}\right)$ approximately	
	From the G.C, p value = 0.0450 (Accept value of higher accuracy)	
	Since p-value = 0.0450 < 0.05, we reject H_o and conclude that there is sufficient evidence that the mean shelf life of the durian puff is less than 8 hours at 5% significance level.	
	(ii) There is a probability of 0.05 of concluding that the mean shelf life of the durian puffs is less than 8 hours when in fact it is 8 hours.	
	(iii) Assume that the shelf life of the durian puffs follows a normal distribution.	
	Let Y denote the shelf life of a randomly chosen durian puff.	
	$\bar{Y} \sim N\left(8, \frac{3.2^2}{10}\right)$	
	To Test H_o: $\mu = 8$ Against H_1: $\mu \neq 8$	
	2-tailed z-test at 5% level of significance	
	Since H_o is not rejected,	
	$-1.96 < \frac{\bar{x} - 8}{\left(\frac{3.2}{\sqrt{10}}\right)} < 1.96$	
	$6.02 < \bar{x} < 9.98$	
	Set of values of $\bar{x}$ is $\left\{\bar{x} \in \mathbb{R} : 6.02 < \bar{x} < 9.98\right\}$	
10	**(a)** A group of twelve people consists of one pair of sisters, one set of three brothers, a family of three and 4 others.	
	(i) The twelve people are grouped into three groups of 4. Find the number of ways where the family of three are together.	[2]
	(ii) The twelve people are arranged randomly in a line. Find the number of ways that the sisters are not together and the brothers are all separated.	[3]
	(b) For events A and B, it is given that $P(A) = \frac{11}{20}$, $P(B) = \frac{2}{5}$ and $P(A \cap B) = \frac{1}{4}$.	
	(i) Find the probability where event A occurs or event B occurs but not both.	[1]

	(ii) Find the probability where event B has occurred but not A.	[1]
	For a third event C, it is given that $\mathrm{P}(C)=\frac{1}{2}$ and that A and C are independent,	
	(iii) Find the range of values of $\mathrm{P}(A'\cap B'\cap C)$.	[3]
	Solution	
	(ai) Number of ways $=\frac{3!}{2!}\binom{9}{4}\binom{5}{4}$	
	$=1890$	
	– for selecting the 2 groups of 4 and arrange them i.e. $\binom{9}{4}\binom{5}{4}$	
	(ii) Number of ways = Only the Brothers separated – Brothers separated but sisters are together	
	$=9!\binom{10}{3}3!-2!8!\binom{9}{3}3!$	
	– Demonstrated the slotting method. (either slotting of the 3 brothers among the rest or slotting of the 2 sisters among the rest.) – Correct complementary method.	
	$=220631040$	
	(bi) Req Prob $=\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B)$	
	$=\mathrm{P}(A)+\mathrm{P}(B)-2\mathrm{P}(A\cap B)$	
	$=\frac{11}{20}+\frac{2}{5}-\frac{1}{2}$	
	$=\frac{9}{20}$	
	(ii) $\mathrm{P}(B)-\mathrm{P}(A\cap B)=\frac{2}{5}-\frac{1}{4}=\frac{3}{20}$	
	(iii) 	
	$\mathrm{P}(A\cap C)=\frac{11}{40}$	
	Largest $a=1-\mathrm{P}(A\cup B)=1-\frac{7}{10}=\frac{3}{10}$	

	As $a+b=\frac{9}{40}$,	
	To minimize $(a+b)$, b is to be as large as possible.	
	Since $b+c=\frac{3}{20}$, largest b is $\frac{3}{20}$	
	Hence min $a=\frac{3}{40}$	
	$\frac{3}{40} \le \mathrm{P}(A' \cap B' \cap C) \le \frac{3}{10}$	
11	In an attempt to reduce the operating costs of an airline company, an analyst collected data on the seat capacity and the total fuel use of the different types of airplanes used for a particular route, as shown in the table below.	

Seat capacity, x (in hundreds)	4.07	3.84	4	3.31	2.48	2.92	3.1	3.57
Total fuel use, y (10^3 tonnes)	140	115	m	11	20	25	33	74

	(a) **(i)** Given that the equation of the least squares regression line of y on x is $$y=79.695x-205.86,$$ show that the value of m is 110, correct to the nearest whole number.	[2]
	(ii) Give an interpretation, in context, of the gradient of the least squares regression line given in **(a)(i)**.	[1]
	The analyst suspects that the fuel use for one of the points has been recorded wrongly.	
	(b) **(i)** Draw a scatter diagram for the data, labelling the axes clearly. On your diagram, circle the point for which the total fuel use has been recorded wrongly and label it as Q.	[2]
	(ii) Explain from your scatter diagram why the relationship between x and y should not be modelled by an equation of the form $y=ax+b$.	[1]
	For the following parts of this question, you should exclude the point Q for the subsequent calculation.	
	(iii) The analyst is presented with the following two models: (A) $y=ae^x+b$ (B) $y=ax^3+b$. Explain clearly which of the two models is the more appropriate model.	[2]
	(iv) Using the more appropriate model from **(b)(iii)**, estimate the total fuel use when the seating capacity of an airplane is 331. Leave your answer to the nearest whole number.	[2]
	(v) Give two reasons why the estimation in part **(b)(iv)** is reliable.	[2]
	Solution	
	(a)(i) Given $y=79.695x-205.86$,	

$\bar{x} = \dfrac{4.07 + 3.84 + 4 + 3.31 + 2.48 + 2.92 + 3.1 + 3.57}{8}$

$= 3.41125$

$\bar{y} = \dfrac{140 + 115 + m + 11 + 20 + 25 + 33 + 74}{8}$

$= \dfrac{418 + m}{8}$

Substitute $\bar{x}$, $\bar{y}$ into the regression line,

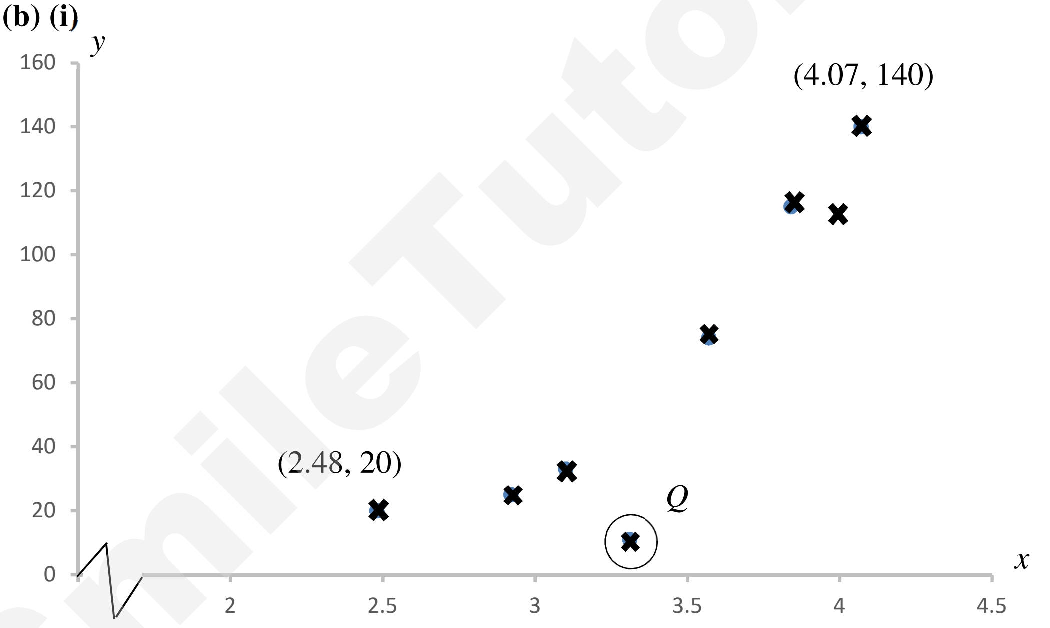
$\dfrac{418 + m}{8} = 79.695(3.41125) - 205.86$

$m = 109.997$

$m = 110$

(ii) It means that the total fuel use increases by 79.695×10^3 tonnes when the seat capacity increases by 100.

(b) (i)



(ii) The scatter diagram in (b)(i), excluding point Q, suggests that as x increases, y increases at an increasing rate so model A is not the most appropriate.

(iii) (A): $r = 0.98265$

(B): $r = 0.97787$

As $|r|$ for model (A) is the closest to 1, therefore, model A is the most appropriate model.

(iv) Least squares regression line using model A:

$$y = 2.6061e^x - 18.429$$

When $x = 3.31$, $y = 2.6061e^{3.31} - 18.429$

$= 52.939$

So total amount of fuel used is 52 939 tonnes.

(v) Since $r = 0.98265$ is close to 1, which suggests that there is a strong positive linear relationship between e^x and y, and $x = 3.31$ is between 2.48 and 4.07, so interpolation is reliable.

12	Based on past statistical data of high speed train journey, there is a 8% chance that a passenger with reservation made will not show up. In order to maximise revenue, a high speed train company accepts more reservations than the passenger capacity of its trains.	
	(i) State 2 assumptions needed such that the number of passengers who do not show up for a high speed train journey may be well modelled by a Binomial distribution.	[2]
	A train company operates a high speed train from Burong Lake to Huahin Cove which has a capacity of 280 passengers.	
	(ii) Find the probability that when 300 reservations are accepted, the high speed train journey is overbooked, i.e. there is not enough seats available for the passengers who show up.	[2]
	(iii) Find the maximum number of reservations that should be accepted in order to ensure that the probability of overbooking is less than 1%.	[3]
	The high speed train operates once daily throughout the year and 300 reservations are accepted for each journey.	
	(iv) Find the probability that no high speed train journey is overbooked in a week.	[2]
	(v) Two such journeys from Burong Lake to Huahin Cove with 300 reservations each are examined. If the total number of passengers who turned up for these two journeys is at most 550 passengers, find the probability that one of the journeys is overbooked by at most 2 passengers.	[3]
	Solution	
	(i) The event that a passenger shows up or fails to show up is independent of the other passengers. The probability that a passenger does not show up is a constant at 0.08 for every passenger.	
	(ii) Let X denote the number of passengers with reservations and show up out of 300 passengers. $X \sim \mathrm{B}(300\ ,\ 0.92)$	
	$\mathrm{P}(X > 280) = 1 - \mathrm{P}(X \le 280) = 0.16948$ $= 0.169$ (3 s.f.)	
	(iii) Let Y denote the number of passengers with reservations and show up out of n passengers. $Y \sim \mathrm{B}(n\ ,\ 0.92)$ $\mathrm{P}(Y > 280) < 0.01$	
	$1 - \mathrm{P}(Y \le 280) < 0.01$ $\mathrm{P}(Y \le 280) > 0.99$	
	Using GC, When $n = 292$, $\mathrm{P}(Y \le 280) = 0.9973 > 0.99$ When $n = 293$, $\mathrm{P}(Y \le 280) = 0.9944 > 0.99$ When $n = 294$, $\mathrm{P}(Y \le 280) = 0.9891 < 0.99$	
	Hence the maximum reservations that should be accepted is 293.	
	(iv) Let W denote the number of train journeys which is overbooked, out of 7. $W \sim \mathrm{B}(7\ ,\ 0.16948)$	

	$\mathrm{P}(W=0)=0.27255=0.273$ (3 s.f.)	
	(v) Let X denote the number of passengers with reservations and show up out of 300 passengers. $X \sim \mathrm{B}(300, 0.92)$ Let V denote the number of passengers with reservations and show up out of 600 passengers. $V \sim \mathrm{B}(600, 0.92)$	
	$\mathrm{P}(281 \le \text{one train journey} \le 282 \mid V \le 550)$	
	$=\dfrac{2\left[\mathrm{P}(X_1=281)\mathrm{P}(X_2 \le 269)+\mathrm{P}(X_1=282)\mathrm{P}(X_2 \le 268)\right]}{\mathrm{P}(V \le 550)}$ $=\dfrac{2\times 0.0068344}{0.40297}$	
	$=0.0339199$ $=0.0339$ (3 s.f.)	

THE END

2018 SAJC H2 Math Prelim Paper 1

1 Without the use of a calculator, solve the inequality
$$\frac{x^2+2x-3}{\left(x^2-2x+11\right)\left(x+1\right)} \geq 0.$$ [4]

2 The curve C has equation given by $y=\frac{x(x+3)}{x+2}$ where $x \neq -2$.

(i) Show that the gradient of C is always positive for $x \in \mathbb{R} \setminus \{-2\}$. [3]

(ii) Sketch the graph of C, indicating clearly the asymptotes and intercepts on the axes whenever applicable. [3]

3 **(i)** Show that $\mathrm{e}^{-r}-2\mathrm{e}^{-r+1}+\mathrm{e}^{-r+2}=\frac{(\mathrm{e}-1)^2}{\mathrm{e}^r}$. [1]

(ii) Hence find $\sum_{r=1}^{N}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r+1}}$ in terms of N. [4]

(iii) Using your result in part **(ii)**, find $\sum_{r=9}^{N+1}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r+1}}$ in terms of e. [2]

4 It is given that the function $y=\mathrm{f}\left(x\right)$ has the Maclaurin's series $1+x+bx^2+cx^3+\ldots$ and satisfies $\left(a-x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=y\left(1+2x-x^2\right)$, where a, b and c are real constants.

(i) Show that a = 1 and find the value of b and c. [7]

(ii) Given that the first four terms of this series are equal to the first four terms in the series expansion, in ascending powers of x, of $\frac{\mathrm{e}^x}{1-x^2}$, find the series expansion of $y=\frac{\mathrm{e}^{2x}}{1-x^2}$ up to and including the term in x^3. [3]

Do not use a graphic calculator in answering this question.

5 **(i)** By letting $z = a + b\mathrm{i}$, solve $z^2=\mathrm{i}$, giving your answers in exact form. [3]

(ii) Solve the equation $w^2+2w+\left(1-8\mathrm{i}\right)=0$, giving the roots in Cartesian form. [3]

(iii) Hence, solve $\left(1-8\mathrm{i}\right)z^2+2\mathrm{i}z-1=0$, giving your answers in Cartesian form. [3]

6 Relative to the origin O, the position vectors of points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. The point C with position vector $\mathbf{c}$ lies on the line segment AB such that $AC : CB$ is $\lambda : 1-\lambda$. It is given that $\mathbf{b}$ is a unit vector, $|\mathbf{a}| = \frac{4}{3}$ and the angle formed between OA and OB is $120°$.

(i) Find the value of λ such that the points O, A and C form a right angle AOC. [4]

(ii) Find $\mathbf{m}$, the position vector of M, the midpoint of AC, in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

A circle is drawn with AC as its diameter and O is a point on the circumference of the circle drawn.

(iii) Determine if OB is a tangent to the circle described above. [3]

(iv) Give a geometrical interpretation of $|\mathbf{b} \bullet \mathbf{m}|$. Hence, explain $(\mathbf{b} \bullet \mathbf{m})\mathbf{b}$ in terms of its magnitude and direction. [2]

7 A curve is defined by the parametric equations:

$$x = a\cos^3 t,\ y = a\sin^3 t,\ 0 \le t < 2\pi \text{ where } a \text{ is a positive constant.}$$

(i) Sketch the curve, showing clearly the coordinates of the points where $t = 0,\ \frac{\pi}{2},\ \pi$ and $\frac{3\pi}{2}$. [2]

(ii) Show that the equation of the normal at the point where $t = p$ is given by $y\sin p = x\cos p - a\cos 2p$. [4]

(iii) Find the value(s) of t when the normal at the point where $p = \frac{\pi}{3}$ meets the curve again. [3]

8 **(i)** Sketch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{y}{b} - \frac{x}{a} = 1$, where $0 < b < a$ on a single diagram, labelling clearly any intersection between the curve and the line. [2]

(ii) Show that the area of the region R, bounded by the curve with equation $x = -\sqrt{a^2 - \frac{a^2y^2}{b^2}}$ and the line $\frac{y}{b} - \frac{x}{a} = 1$, where $0 < b < a$ is given by

$$-\frac{1}{2}ab + \int_0^b \sqrt{a^2 - \frac{a^2y^2}{b^2}}\,\mathrm{d}y.$$

Hence, by substituting $y = b\cos\theta$, find the exact value of the area in terms of a and b. [5]

(iii) Find the exact volume of the solid obtained when R is rotated 4 right angles about the y-axis, giving your answer in the form of ka^2b, where k is a constant. [3]

Do not use a graphic calculator in answering this question.

9 **(i)** It is given that $z = 3e^{i\frac{\pi}{3}}$ is a root of the equation $z^2 - 3z + 9 = 0$. Find in similar form, the other root of the equation. [1]

(ii) Show that $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$. [1]

(iii) Let the root found in **(i)** to be w_1 and $w_2 = 3e^{i\frac{\pi}{9}}$. Hence, find the complex number $w_2 - w_1$, in the form $re^{i\theta}$, where $r > 0$ and $\theta \in (-\pi\ ,\ \pi]$. [3]

(iv) Let the point *A* represent the complex number $w_2 - w_1$ on the Argand diagram. A perpendicular line is drawn from the point *A* to the real axis. The intersection point between this line and the real axis is *B*. Show that the area of the triangle *OAB* is $9\sin^2\left(\frac{2\pi}{9}\right)\sin\left(\frac{7\pi}{9}\right)$ square units, where *O* is the origin. [3]

10 Albert and Betty each took a study loan of \$100 000 from a bank on 1 January 2014 and both graduated on 31 December 2017. The bank only starts charging an annual interest rate of 5% on the outstanding loan at the end of each year from 2018 onwards. Albert pays the bank \$*x* on the 11th day of every month starting in January 2018. Let *n* be the number of years after 2017 that repayment of the study loan has begun.

(i) Find an expression for the amount of outstanding loan at the end of *n* years. [4]

(ii) Find the minimum value of *x* if Albert wishes to complete his repayment of the loan at the end of 2027, giving your answer to the nearest dollar. [3]

An investment fund pays out a constant *r* % dividend per annum on 31 December every year based on the amount of funds held to maturity from 1 January till 30 December of the same year.

On 1 January 2017, Betty decided to invest \$50 000 in the fund to finance her repayment of her study loan using the annual dividend payout. Her repayment is once per year which she uses the full amount of the annual dividend payout. The schedule of repayment is fixed on 11 January of each year, starting with effect from 2018.

Find the minimum value of *r* such that Betty will be able to complete her repayment of the loan at the end of 2027. [5]

[Note that the principal sum of Betty's investment remains unchanged at \$50 000 throughout the repayment period]

11 Joe wants to grow vegetables at his house balcony and designs an irrigation device, which takes the form of a cylindrical water tank, with a fixed cross sectional area A cm^2, and a small opening at the bottom of it.

It is known that the rate of change of the volume, V cm^3 , of water at any time t hours, dispensed from the tank, is proportional to the square root of the height of the water, h cm, above the opening.

(i) Show that the rate of change of the height of the water in the tank at any time t hours, is given by $\dfrac{\mathrm{d}h}{\mathrm{d}t} = -\dfrac{k}{A}\sqrt{h},$ where k is a positive constant. [3]

(ii) Joe fills the water tank to an initial height of 81 cm with the opening closed. The water is then discharged from the opening and the height of the water level above the opening is decreasing at a rate of 0.3 cm/hr. By solving the differential equation in part **(i)**, find h in terms of t. [5]

(iii) Joe is planning for an overseas trip. He wants to make sure that he will be back before the water tank is empty. What is the maximum number of days, to the nearest integer, that Joe can be away? [2]

(iv) Given that the tank has a base radius of 20 cm, find the exact rate of change of the volume of water in the tank at the end of the fourth day. [4]

End of Paper

St Andrew's Junior College
2018 Preliminary Examination
H2 Mathematics Paper 1 (9758/01) Solutions

<table>
<tr><td>1</td><td>

$\dfrac{x^2+2x-3}{(x^2-2x+11)(x+1)} \geq 0$

$x^2-2x+11$

$= x^2-2x+\left(\dfrac{-2}{2}\right)^2-\left(\dfrac{-2}{2}\right)^2+11$

$=(x-1)^2-1+11$

$=(x-1)^2+10>0$ for all real values of x

Therefore, $\dfrac{x^2+2x-3}{(x^2-2x+11)(x+1)} \geq 0$

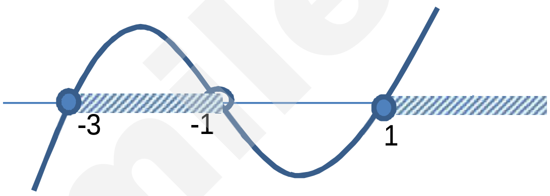
$\Rightarrow \dfrac{x^2+2x-3}{(x+1)} \geq 0$

$\dfrac{(x+3)(x-1)}{(x+1)} \geq 0, \quad x \neq -1$

Multiplying both sides by $(x+1)^2$

$(x+3)(x-1)(x+1) \geq 0$

Since $x \neq -1$,



Hence,
$-3 \leq x < -1$ or $x \geq 1$

</td></tr>
<tr><td>2 (i)</td><td>

$y=\dfrac{x^2+3x}{x+2}=x+1-\dfrac{2}{x+2}$

$\dfrac{\mathrm{dy}}{\mathrm{dx}}=1+\left(\dfrac{2}{(x+2)^2}\right)>0$ for all $x \in \mathbb{R}$

since $(x+2)^2>0$ for all x, $x \neq -2$.

Alternatively:

</td></tr>
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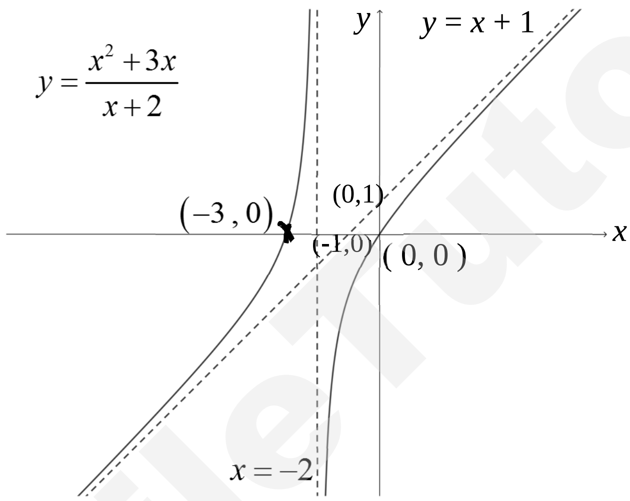
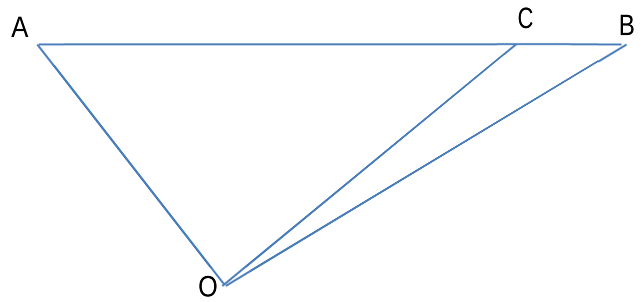
	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x+2)(2x+3)-(x^2+3x)(1)}{(x+2)^2}$ $=\dfrac{x^2+4x+6}{(x+2)^2}$ $=\dfrac{(x+2)^2+2}{(x+2)^2}>0$ Since $(x+2)^2\ge 0$ for all $x\in\mathbb{R}$, $(x+2)^2+2>0$ for all $x\in\mathbb{R}$ and $(x+2)^2>0$ for all $x\in\mathbb{R}\setminus\{2\}$, Hence C has a positive gradient for all $x\in\mathbb{R}$.
2 (ii)	 Oblique asymptote: $y = x + 1$ Vertical asymptote: $x=-2$
3 (i)	$\mathrm{e}^{-r}-2\mathrm{e}^{-r+1}+\mathrm{e}^{-r+2}$ $=\mathrm{e}^{-r}\left(1-2\mathrm{e}+\mathrm{e}^2\right)$ $=\dfrac{\left(\mathrm{e}-1\right)^2}{\mathrm{e}^r}$
3 (ii)	$\displaystyle\sum_{r=1}^{N}\frac{\left(\mathrm{e}-1\right)^2}{\mathrm{e}^{r+1}}$ $\displaystyle=\sum_{r=1}^{N}\frac{\left(\mathrm{e}-1\right)^2}{\mathrm{e}^r\mathrm{e}^1}$ $\displaystyle=\frac{1}{\mathrm{e}}\sum_{r=1}^{N}\frac{\left(\mathrm{e}-1\right)^2}{\mathrm{e}^r}$ $\displaystyle=\frac{1}{\mathrm{e}}\sum_{r=1}^{N}\left(\mathrm{e}^{-r}-2\mathrm{e}^{-r+1}+\mathrm{e}^{-r+2}\right)$

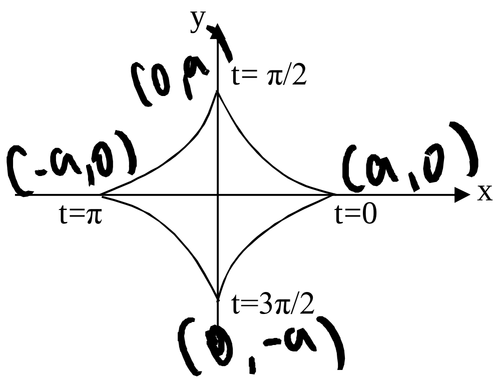
	$=\frac{1}{e}[e^{-1}-2e^{0}+e^{1}$ $+e^{-2}-2e^{-1}+e^{0}$ $+e^{-3}-2e^{-2}+e^{-1}$ $+e^{-4}-2e^{-3}+e^{-2}$ $+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $+e^{-N+2}-2e^{-N+3}+e^{-N+4}$ $+e^{-N+1}-2e^{-N+2}+e^{-N+3}$ $+e^{-N}-2e^{-N+1}+e^{-N+2}]$ $=\frac{1}{e}\left(e-1+e^{-N}-e^{-N+1}\right)$ $=1-\frac{1}{e}+e^{-N-1}-e^{-N}$
3 (iii)	$\sum_{r=9}^{N+1}\frac{(e-1)^2}{e^{r+1}}=\sum_{r=1}^{N+1}\frac{(e-1)^2}{e^{r+1}}-\sum_{r=1}^{8}\frac{(e-1)^2}{e^{r+1}}$ $=1-\frac{1}{e}+e^{-N-2}-e^{-N-1}-\left(1-\frac{1}{e}+e^{-9}-e^{-8}\right)$ $=e^{-N-2}-e^{-N-1}-\frac{1}{e^9}+\frac{1}{e^8}$
4 (i)	$y=f(x)$ $=1+x+bx^2+cx^3+\ldots$ $=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$ Comparing, $f(0)=1,\ f'(0)=1, \frac{f''(0)}{2!}=b,\ \frac{f'''(0)}{3!}=c$ When $x=0, y=1, \frac{dy}{dx}=1, \frac{d^2y}{dx^2}=2b, \frac{d^3y}{dx^3}=6c.$ $\left(a-x^2\right)\frac{dy}{dx}=y\left(1+2x-x^2\right)$ $\Rightarrow\left(a-0^2\right)(1)=(1)\left(1+2(0)-0^2\right)$ $\Rightarrow a=1$ (Shown) When $a=1$, we have $\left(1-x^2\right)\frac{dy}{dx}=y\left(1+2x-x^2\right)$ Differentiate w.r.t. x,

	$\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}=y\left(2-2x\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+2x-x^2\right)$ $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=y\left(2-2x\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+4x-x^2\right)\cdots(1)$ Differentiate w.r.t. x, $\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2y+\frac{\mathrm{d}y}{\mathrm{d}x}\left(2-2x\right)$ $+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(1+4x-x^2\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\left(4-2x\right)$ $\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-2y+\frac{\mathrm{d}y}{\mathrm{d}x}\left(6-4x\right)+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(1+6x-x^2\right)\cdots(2)$ substitute $x=0, y=1, \frac{\mathrm{d}y}{\mathrm{d}x}=1$ into (1) and (2), $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2(1)+1=3$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-2(1)+6(1)+3=7$ Hence, from the given expansion, $\frac{\mathrm{f}''(0)}{2!}=b$ $2b=3$ $\Rightarrow b=\frac{3}{2}$ $\frac{\mathrm{f}'''(0)}{3!}=c$ $\mathrm{f}'''(0)=6c=7$ $\Rightarrow c=\frac{7}{6}$ Hence, the expansion is $y=1+x+\frac{3}{2}x^2+\frac{7}{6}x^3+\ldots$

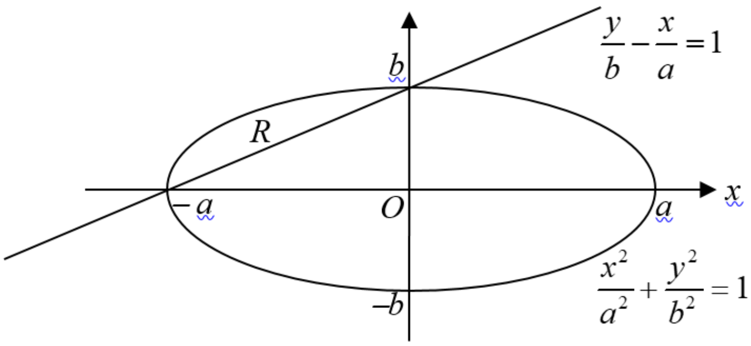
4 (ii)	$y=\frac{\left(e^{x}\right)^{2}}{1-x^{2}}$ $=\frac{e^{x}}{\left(1-x^{2}\right)}(e^{x})$ Using (i) and Standard Series of e^x $=\left(1+x+\frac{3}{2}x^{2}+\frac{7}{6}x^{3}+...\right)\left(1+x+\frac{x^{2}}{2}+\frac{x^{3}}{3!}\cdots\right)$ $\approx 1+x+\frac{x^{2}}{2}+\frac{x^{3}}{3!}+x+x^{2}+\frac{1}{2}x^{3}+\frac{3}{2}x^{2}+\frac{3}{2}x^{3}+\frac{7}{6}x^{3}$ $=1+2x+3x^{2}+\frac{10x^{3}}{3}$
5 (i)	Let $\left(a+bi\right)^{2}=i$ $a^{2}-b^{2}+2abi=i$ Comparing real and imaginary parts, $a^{2}-b^{2}=0$ and $2ab=1$ $a=b$ or $-b$ $\quad ab=\frac{1}{2}$ At $a=b$, $a^{2}=\frac{1}{2}$ $a=\pm\frac{1}{\sqrt{2}}$ Hence, $b=\pm\frac{1}{\sqrt{2}}$ For $a=-b$, $a^{2}=-\frac{1}{2}$ has no solutions since $a\in\mathbb{R}$ Hence the square roots are: $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$ or $-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$
5 (ii)	Given $w^{2}+2w+\left(1-8i\right)=0$, $w=\frac{-2\pm\sqrt{4-4\left(1-8i\right)}}{2}$ $=\frac{-2\pm2\sqrt{1-1+8i}}{2}$ $=-1\pm2\sqrt{2}\left(\sqrt{i}\right)$ For $w=-1\pm2\sqrt{2}\left(\sqrt{i}\right)$,

	At $\mathrm{i}=\left(\pm\frac{1}{\sqrt{2}}\pm\frac{1}{\sqrt{2}}\mathrm{i}\right)^2$, $\sqrt{\mathrm{i}}=\sqrt{\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}\right)^2}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}$ $w=-1\pm2\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}\right)$ $=-1\pm(2+2\mathrm{i})$ $=-1+2+2\mathrm{i}$ or $-1-2-2\mathrm{i}$ $=1+2\mathrm{i}$ or $-3-2\mathrm{i}$
5 (iii)	$(1-8\mathrm{i})z^2+2\mathrm{i}z-1=0$ $(1-8\mathrm{i})+\frac{2\mathrm{i}}{z}-\frac{1}{z^2}=0$ $-\frac{1}{z^2}+\frac{2\mathrm{i}}{z}+(1-8\mathrm{i})=0$ $\frac{\mathrm{i}^2}{z^2}+\frac{2\mathrm{i}}{z}+(1-8\mathrm{i})=0$ $\left(\frac{\mathrm{i}}{z}\right)^2+2\left(\frac{\mathrm{i}}{z}\right)+(1-8\mathrm{i})=0$ Comparing with $w^2+2w+(1-8\mathrm{i})=0$ in (ii), we replace w in (ii) as $\left(\frac{\mathrm{i}}{z}\right)$ in (iii) for both roots in (i), $\frac{\mathrm{i}}{z}=1+2\mathrm{i}$ and $\frac{\mathrm{i}}{z}=-3-2\mathrm{i}$ $z=\frac{\mathrm{i}}{1+2i}=\frac{(1-2\mathrm{i})\mathrm{i}}{1+2^2}=\frac{1}{5}(2+\mathrm{i})$ and $z=\frac{\mathrm{i}}{-3-2\mathrm{i}}=\frac{(-3+2\mathrm{i})\mathrm{i}}{3^2+2^2}=\frac{1}{13}(-2-3\mathrm{i})$
6 (i)	Since $AC:CB$ is $\lambda:1-\lambda$, 

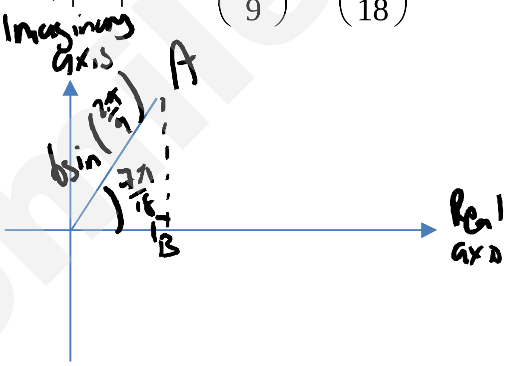
	By Ratio Theorem, $\mathbf{c}=\dfrac{\lambda\mathbf{b}+(1-\lambda)\mathbf{a}}{\lambda+1-\lambda}$ $\quad=\lambda\mathbf{b}+(1-\lambda)\mathbf{a}$ Since *OC* is perpendicular to *OA*, $\mathbf{c}\bullet\mathbf{a}=0$ $\left[\lambda\mathbf{b}+(1-\lambda)\mathbf{a}\right]\bullet\mathbf{a}=0$ $\lambda\mathbf{b}\bullet\mathbf{a}+(1-\lambda)\mathbf{a}\bullet\mathbf{a}=0$ $\lambda\left[\dfrac{4}{3}\times1\times\cos120°\right]+(1-\lambda)\lvert\mathbf{a}\rvert^2=0,\ \text{since } \lvert\mathbf{a}\rvert=\dfrac{4}{3}$ $\lambda\left[\dfrac{4}{3}\times\left(-\dfrac{1}{2}\right)\right]+(1-\lambda)\left(\dfrac{4}{3}\right)^2=0$ $\left(-\dfrac{2}{3}\right)\lambda+\dfrac{16}{9}(1-\lambda)=0$ $-\dfrac{2}{3}\lambda+\dfrac{16}{9}-\dfrac{16}{9}\lambda=0$ $\dfrac{22}{9}\lambda=\dfrac{16}{9}$ $\lambda=\dfrac{16}{22}=\dfrac{8}{11}$
6(ii)	$\mathbf{c}=\dfrac{3}{11}\mathbf{a}+\dfrac{8}{11}\mathbf{b}$ By Mid-point Theorem, find $\overrightarrow{OM}=\mathbf{m}$, where *M* is the mid-point of *AC*. $\overrightarrow{OM}=\mathbf{m}$ $\quad=\dfrac{\mathbf{c}+\mathbf{a}}{2}$ $\quad=\dfrac{1}{2}\left[\dfrac{3}{11}\mathbf{a}+\dfrac{8}{11}\mathbf{b}+\mathbf{a}\right]$ $\quad=\dfrac{1}{2}\left(\dfrac{14}{11}\mathbf{a}+\dfrac{8}{11}\mathbf{b}\right)$ $\quad=\dfrac{7}{11}\mathbf{a}+\dfrac{4}{11}\mathbf{b}$

6 (iii)	$\mathbf{m}\bullet\mathbf{b} = \left(\frac{7}{11}\mathbf{a}+\frac{4}{11}\mathbf{b}\right)\bullet\mathbf{b}$ $= \frac{7}{11}\mathbf{a}\bullet\mathbf{b}+\frac{4}{11}\mathbf{b}\bullet\mathbf{b}$ $= \frac{7}{11}\mathbf{a}\bullet\mathbf{b}+\frac{4}{11}\lvert\mathbf{b}\rvert^2$ $= \frac{7}{11}\left(-\frac{2}{3}\right)+\frac{4}{11}(1)^2$ $= -\frac{2}{33} \neq 0$ Since the vector **b** is not perpendicular to **m**, where *OM* is the radius of the circle, *OB* is not a tangent to the circle.
6 (iv)	$\lvert\mathbf{b}\bullet\mathbf{m}\rvert$ is the length of projection of **m** on **b**. $(\mathbf{b}\bullet\mathbf{m})\mathbf{b}$ is a vector with magnitude $\lvert\mathbf{b}\bullet\mathbf{m}\rvert$, which is the length of projection of **m** on **b**. Moreover, it is in the opposite direction of **b** as $\mathbf{m}\bullet\mathbf{b} = -\frac{2}{33} < 0$.
7 (i)	$x = a\cos^3 t$, $y = a\sin^3 t$ $t = 0,\quad x = a\cos^3 0 = a$, $y = a\sin^3 0 = 0$ $t = \frac{\pi}{2}, x = a\cos^3\frac{\pi}{2} = 0$, $y = a\sin^3\frac{\pi}{2} = a$ $t = \pi$, $x = a\cos^3\pi = -a$, $y = a\sin^3\pi = 0$ $t = \frac{3\pi}{2}, x = a\cos^3\frac{3\pi}{2} = 0$, $y = a\sin^3\frac{3\pi}{2} = -a$ 

7 (ii)	$\frac{dx}{dt}=3a\cos^2 t(-\sin t)$ $\frac{dy}{dt}=3a\sin^2 t(\cos t)$ $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{1}{\frac{dx}{dt}}$ $=\frac{3a\sin^2 t(\cos t)}{3a\cos^2 t(-\sin t)}$ $\frac{dy}{dx}=\frac{-\sin t}{\cos t}$ Gradient of normal $=\frac{\cos t}{\sin t}$ At $t = p$, the equation of the normal is: $y-a\sin^3 p=\frac{\cos p}{\sin p}(x-a\cos^3 p)$ $y\sin p-a\sin^4 p=x\cos p-a\cos^4 p$ $y\sin p=x\cos p-a\cos^4 p+a\sin^4 p$ $y\sin p=x\cos p+a(\sin^4 p-\cos^4 p)$ $=x\cos p+a(\sin^2 p-\cos^2 p)(\sin^2 p+\cos^2 p)$ $=x\cos p+a\left(-\cos 2p\right)(1)$ $=x\cos p-a\cos 2p$ (Shown)
7 (iii)	At $p=\frac{\pi}{3}$, $y\sin\frac{\pi}{3}=x\cos\frac{\pi}{3}-a(\cos\frac{2\pi}{3})$ $\frac{\sqrt{3}}{2}y=\frac{1}{2}x-a\left(-\frac{1}{2}\right)$ $\frac{\sqrt{3}}{2}y=\frac{1}{2}x+\frac{1}{2}a$ $\sqrt{3}y=x+a$ Since the normal meets the curve again, $\sqrt{3}(a\sin^3 t)=(a\cos^3 t)+a$ $\sqrt{3}\sin^3 t=\cos^3 t+1,\ a\neq 0$ Using GC, $t=2.32$ or 3.14 (3 s.f)

8 (i)	
8 (ii)	Area of R = Area of quadrant-Area of triangle $=-\int_0^b x\,\mathrm{d}y-\frac{1}{2}ab$ $=-\int_0^b -\sqrt{a^2-\frac{a^2y^2}{b^2}}\,\mathrm{d}y-\frac{1}{2}ab$ $=-\frac{1}{2}ab+\int_0^b \sqrt{a^2-\frac{a^2y^2}{b^2}}\,\mathrm{d}y$ $y=b\cos\theta$ $\frac{\mathrm{d}y}{\mathrm{d}\theta}=-b\sin\theta$ Area= $-\frac{1}{2}ab+\int_0^b \sqrt{a^2-\frac{a^2y^2}{b^2}}\,\mathrm{d}y$ $=-\frac{1}{2}ab+\int_{\frac{\pi}{2}}^0 \sqrt{a^2-\frac{a^2b^2\cos^2\theta}{b^2}}(-b\sin\theta)\,\mathrm{d}\theta$ $=-\frac{1}{2}ab+\int_{\frac{\pi}{2}}^0 -ab\sin^2\theta\ \mathrm{d}\theta$ $=-\frac{1}{2}ab-ab\int_{\frac{\pi}{2}}^0\left(\frac{1-\cos 2\theta}{2}\right)\ \mathrm{d}\theta$ $=-\frac{1}{2}ab-ab\left[\frac{1}{2}\theta-\frac{\sin 2\theta}{4}\right]_{\frac{\pi}{2}}^0$ $=-\frac{1}{2}ab-(ab)\left[-\left(\frac{\pi}{4}\right)\right]$ $=ab\left[\frac{\pi}{4}-\frac{1}{2}\right]\text{units}^2$

8 (iii)	Volume generated $=\pi\int_0^b x^2\,\mathrm{d}y-\frac{1}{3}\pi a^2 b$ $=\pi\int_0^b\left(a^2-\frac{a^2y^2}{b^2}\right)\mathrm{d}y-\frac{1}{3}\pi a^2 b$ $=\pi\left[a^2y-\frac{a^2y^3}{3b^2}\right]_0^b-\frac{1}{3}\pi a^2 b$ $=\pi\left[a^2b-\frac{a^2b^3}{3b^2}\right]-\frac{1}{3}\pi a^2 b$ $=\pi\left[a^2b-\frac{a^2b}{3}\right]-\frac{1}{3}\pi a^2 b$ $=\pi\left(\frac{2}{3}a^2b\right)-\frac{1}{3}\pi a^2 b$ $=\frac{1}{3}\pi a^2 b$ units^3
9(i)	Since $z^2-3z+9=0$ has all real coefficients, given that $z=3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ is a root of the equation, $z=3\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$ is the other root of the equation.
9(ii)	$\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}$ $=(\cos\theta+\mathrm{i}\sin\theta)-[\cos(-\theta)+\mathrm{i}\sin(-\theta)]$ $=(\cos\theta+\mathrm{i}\sin\theta)-(\cos\theta-\mathrm{i}\sin\theta)$ $=2\mathrm{i}\sin\theta$
9 (iii)	Since $w_1=3\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$, $w_2=3\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}$

	$w_2 - w_1$ $= 3e^{i\left(\frac{\pi}{9}\right)} - 3e^{i\left(-\frac{\pi}{3}\right)}$ $= 3e^{i\left(\frac{\pi}{9}\right)} - 3e^{i\left(-\frac{3\pi}{9}\right)}$ $= 3e^{i\left(\frac{2\pi}{9}-\frac{\pi}{9}\right)} - 3e^{i\left(-\frac{2\pi}{9}-\frac{\pi}{9}\right)}$ $= 3e^{i\left(-\frac{\pi}{9}\right)}\left[e^{i\left(\frac{2\pi}{9}\right)} - e^{i\left(-\frac{2\pi}{9}\right)}\right]$ $= 3e^{i\left(-\frac{\pi}{9}\right)}\left[2i\sin\left(\frac{2\pi}{9}\right)\right]$ $= 6\sin\left(\frac{2\pi}{9}\right)e^{i\left(-\frac{\pi}{9}+\frac{\pi}{2}\right)}$ $= 6\sin\left(\frac{2\pi}{9}\right)e^{i\left(\frac{7\pi}{18}\right)}$
9 (iv)	At point *B*, $\lvert OB\rvert = 6\sin\left(\frac{2\pi}{9}\right)\cos\left(\frac{7\pi}{18}\right)$  Hence, Area of triangle *OAB*

	$=\frac{1}{2}\lvert OB\rvert\lvert OA\rvert\sin\left(\frac{7\pi}{18}\right)$ $=\frac{1}{2}\left[6\sin\left(\frac{2\pi}{9}\right)\cos\left(\frac{7\pi}{18}\right)\right]\left[6\sin\left(\frac{2\pi}{9}\right)\right]\sin\left(\frac{7\pi}{18}\right)$ $=\frac{36}{2}\sin^2\left(\frac{2\pi}{9}\right)\sin\left(\frac{7\pi}{18}\right)\cos\left(\frac{7\pi}{18}\right)$ $=\frac{36}{2}\sin^2\left(\frac{2\pi}{9}\right)\left[\frac{\sin\left(\frac{14\pi}{18}\right)}{2}\right]$ $=9\sin^2\left(\frac{2\pi}{9}\right)\sin\left(\frac{7\pi}{9}\right)$
10 (i)	(table below) Amount of loan left at the end of n years $=100000(1.05)^n-12x(1.05)^n-12x(1.05)^{n-1}-\ldots-12x(1.05)$ $=100000(1.05)^n-12x(1.05)\left[1+1.05+1.05^2+\ldots+1.05^{n-1}\right]$ $=100000(1.05)^n-12x(1.05)\left[\frac{1.05^n-1}{1.05-1}\right]$ $=100000(1.05)^n-252x(1.05^n-1)$
10 (ii)	$(1.05)^{10}(100000)-252x(1.05^{10}-1)\le 0$ $x\left[252(1.05^{10}-1)\right]\ge(1.05)^{10}(100000)$ $x\ge 1027.81$

Year	n	Amount of loan at the beginning of nth year	Amount of loan at the end of nth year
2018	1	100000	$(100000-12x)(1.05)$
2019	2	$(100000-12x)(1.05)$	$\left[(100000-12x)(1.05)-12x\right](1.05)$ $=100000(1.05)^2-12x(1.05)^2$ $-12x(1.05)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
	n		$100000(1.05)^n-12x(1.05)^n$ $-12x(1.05)^{n-1}-\ldots$ $-12x(1.05)$

<table>
<tr><td></td><td>Hence, minimum value of x = \$1028 (to the nearest dollar)</td></tr>
<tr><td>10
(iii)</td><td>Amount of pay-out per year
$=50000\left(\frac{r}{100}\right)$
$=500r$</td></tr>
</table>

Year	n	Amt left on 1 Jan of nth year	Amt owed on 31 Dec of nth year
2018	1	$100000-500r$	$[100000-500r](1.05)$
2019	2	$[100000-500r](1.05)$ $-500r$	$[100000-500r](1.05)^2$ $-500r(1.05)$
…			
2027	10		$[100000-500r](1.05)^{10}$ $-500r(1.05)^9$ $-500r(1.05)^8 ...$ $-500r(1.05)^2-500r(1.05)$ $=100000(1.05)^{10}$ $-500r(1.05)[1+1.05+\cdots+1.05^9]$

Amount owed on 31 December of 10th year

$$=100000(1.05)^{10}-500r(1.05)^{10}-500r(1.05)^9$$
$$-500r(1.05)^8 ...-500r(1.05)^2-500r(1.05)$$
$$=100000(1.05)^{10}-500r(1.05)\left[1+1.05+\cdots+1.05^9\right]$$
$$=100000(1.05)^{10}-500r(1.05)\left[\frac{1(1.05^{10}-1)}{1.05-1}\right]$$
$$=100000(1.05)^{10}-500r(1.05)\left[\frac{(1.05^{10}-1)}{0.05}\right]$$
$$=100000(1.05)^{10}-10500r(1.05^{10}-1)$$

For her to have completed paying,

$$100000(1.05)^{10}-10500r(1.05^{10}-1)\le 0$$

$r \ge 24.668 = 24.7$ (to 3 sf)

<table>
<tr><td>11
(i)</td><td>$V = Ah$
$\frac{\mathrm{d}V}{\mathrm{d}t}=A\frac{\mathrm{d}h}{\mathrm{d}t}$

On the other hand,</td></tr>
</table>

	$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V_{in}}{\mathrm{d}t}-\frac{\mathrm{d}V_{out}}{\mathrm{d}t}$ $=0-k\sqrt{h}$, where k is a positive constant $=-k\sqrt{h}$ Equating the expressions, $-k\sqrt{h}=A\frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-k\sqrt{h}}{A}$
11 (ii)	$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-k\sqrt{h}}{A}$ $\int\frac{1}{\sqrt{h}}\,\mathrm{d}h=-\frac{k}{A}\int\mathrm{d}t$ $2\sqrt{h}=-\frac{k}{A}t+C$---(I) When $t=0$, $h=81$, $\therefore C=18$. $\Rightarrow 2\sqrt{h}=-\frac{k}{A}t+18$ When $t=0$, $\frac{\mathrm{d}h}{\mathrm{d}t}=-0.3$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-k\sqrt{h}}{A}$=-0.3 --- (II) $-\frac{k}{A}(9)=-0.3$ $\frac{k}{A}=\frac{1}{30}$ $\therefore 2\sqrt{h}=-\frac{1}{30}t+18$ $\sqrt{h}=-\frac{1}{60}t+9$ --- (III) $\Rightarrow h=(9-\frac{1}{60}t)^2$
11 (iii)	If Joe is to be back *before* the water tank to be emptied, $h\geq 0$ $(9-\frac{1}{60}t)^2\geq 0$ Equivalently, using (III),

	$\sqrt{h} = -\frac{1}{60}t + 9 \geq 0$ $-\frac{t}{60} \geq -90$ $\frac{t}{60} \leq 90$ $t \leq 540$ Number of days he can be away $\leq \frac{540}{24} = 22.5$ Maximum number of days = 22.
11 (iv)	At the end of 4th day, 96 hours have lapsed. $h = (9 - \frac{1}{60} \bullet 96)^2 = (7.4)^2$ **Method (1):** $A = 400\pi$ given that the radius is 20 cm. Using $\frac{k}{A} = \frac{1}{30}$ $k = \frac{400\pi}{30}$ $\Rightarrow k = \frac{40}{3}\pi$ $\frac{dV}{dt} = -k\sqrt{h}$ $= -\frac{40\pi}{3}(7.4)$ $= -\frac{296}{3}\pi$ cm^3/hour The water is being delivered at a rate of $-\frac{296}{3}\pi$ cm^3/hour. **Method (2):** $\frac{dV}{dt} = \frac{dV}{dh} \bullet \frac{dh}{dt}$ $\frac{dV}{dh} = A = 400\pi$ $\frac{dh}{dt} = \frac{-k\sqrt{h}}{A} = -\frac{1}{30}(7.4)$ $\frac{dV}{dt} = (400\pi)(-\frac{1}{30}(7.4)) = -\frac{296}{3}\pi$ cm^3/hour.

2018 SAJC H2 Math Prelim Paper 2

Section A: Pure Mathematics (40 marks)

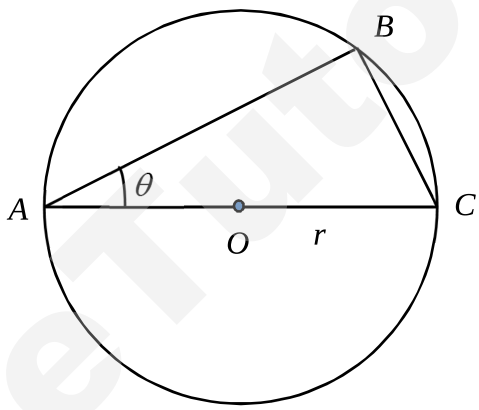
1 A curve C with equation $y = a(x-1) + \dfrac{b}{x+c}$, where a, b and c are real constants, undergoes in succession, the following transformations:

A: A reflection in the x-axis
B: A translation of 1 unit in the negative x-direction

The resulting curve with equation $y = \mathrm{f}(x)$ has the y-axis as one of its asymptotes.

Given that $\left(1, \dfrac{1}{6}\right)$ is a turning point of $y = \dfrac{1}{\mathrm{f}(x)}$, find the values of a, b and c. [5]

2



In the figure above, points A and C are fixed points on the circle which form the diameter passing through the centre O. The circle has a fixed radius r units. The variable point B moves along the circumference of the circle between A and C in the upper half of the circle. The chord AB makes an angle of θ radians with the diameter AC.

Use differentiation to find, the exact angle θ, such that S, the area of triangle ABC, is a maximum. [6]

3 **(i)** Using the R-formula, express $\sin\theta + m\cos\theta$, for $m > 0$, in the form $R\sin(\theta+\alpha)$ where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$ are constants to be determined in terms of m.

Hence show that $2(\sin\theta + m\cos\theta)\sin(\theta-\alpha) = R\left(\cos 2\alpha - \cos 2\theta\right)$. [3]

(ii) Given that $\alpha = \dfrac{\pi}{2}$, evaluate $\displaystyle\int \frac{\cos 2\theta}{(\sin\theta + m\cos\theta)\sin(\theta-\alpha)}\,\mathrm{d}\theta$ in terms of m and θ. [4]

4 **(i)** Solve the inequality $\ln(x-1) \le 0$. [2]

The function f is defined by

$$\mathrm{f} : x \mapsto |\ln(x-1)|, \qquad \text{for } x \in \mathbb{R},\ 1 < x \le 2.$$

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(iii) Sketch, on the same diagram, the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, giving the equations of any asymptotes and the coordinates of any points where the curves cross the axes. [3]

(iv) Hence solve the inequality $\mathrm{f}^{-1}(x) < \mathrm{f}(x)$. [1]

Another function g is defined by

$$\mathrm{g} : x \mapsto 1 + \frac{4}{4x^2+5} \text{ for } x \in \mathbb{R},\ x > 0.$$

(v) Using f^{-1}, find the exact value of a such that $\mathrm{fg}(a) = 3$. [2]

5 The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 11 \\ 6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and the line l_2 has equation

$$\frac{x-4}{2} = \frac{y}{-3} = \frac{z+1}{4}.$$

(i) Show that l_1 and l_2 are skew lines. [3]

The line l_1 contains points A and B with coordinates (11, 6, 0) and (1, 6, 5) respectively with respect to the origin O. The plane p_1 which is parallel to l_1 has equation

$$\mathbf{r} \bullet \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} = 240.$$

(ii) Find the position vector of the point P on p_1 which has the shortest distance to the line l_1 and is equidistant from the points A and B. [5]

A plane p_2 contains the point P and is parallel to both l_1 and l_2.

(iii) Hence, find the Cartesian equation of p_2. [3]

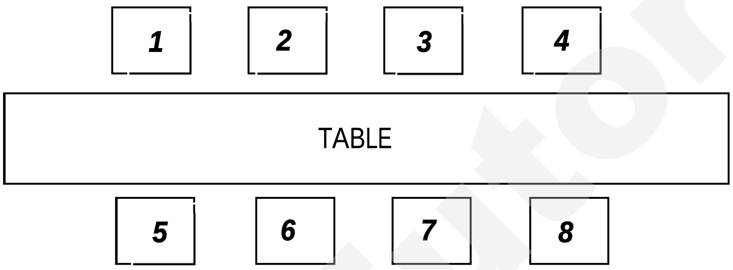
Section B: Statistics (60 marks)

6 A committee of eight people is to be chosen from 15 men and 7 women.

Find the number of ways in which the committee can be chosen if it consists of

at least 2 women. [3]

The chosen committee consists of 6 men (Allen, Ben, Calvin, Donald, Edwin and Felix) and 2 women (Gina and Hazel).
At a meeting, the committee members are seated at a rectangular table as shown in the diagram below, with seats labelled 1 to 8.



Find the number of possible seating arrangements if Gina and Hazel must be seated at any two of the corner seats labeled *1*, *4*, *5* or *8*. [2]

7 Oliver is practising for the upcoming target archery competition. During practices, Oliver shoots from distances ranging from 30m to 90m to the target. The probability, p, that he hits the bullseye is given by $p = \frac{2}{195}(95 - d)$, where d is the distance between the archer and the target in metres.

Each shot he made is assumed to be independent of any other shots made.

(i) Oliver shoots 18 arrows from a distance of 40 metres from the target. Find the probability that he hits the bullseye more than 6 times given that he hits the bullseye at most 10 times. [4]

(ii) Oliver shoots 18 arrows from a distance of x metres from the target. Find x such that Oliver has a 98% chance of hitting the bullseye at least twice. [3]

8 For two mutually exclusive events A and B, it is given that $\mathrm{P}(A) = 0.65$ and $\mathrm{P}(B \mid A') = \dfrac{2}{7}$.

(i) Show that $\mathrm{P}(B) = 0.1$. [2]

For a third event C, it is given that $\mathrm{P}(A \cap C) = 0.39$.

(ii) Find $\mathrm{P}(C' \mid A)$. [2]

It is given that B and C are independent and $\mathrm{P}(A' \cap B' \cap C) = 0.15$.

(iii) Find $\mathrm{P}(B \cap C)$. [2]

(iv) Hence or otherwise, determine whether the events A and C are independent. [1]

9 Connie and Sally play a game using two six-sided dice. One of the dice is fair and each face is labelled with a digit from '1' to '6' respectively. The other die is biased such that the score, denoted by Y, has a probability distribution given as follows:

$$\mathrm{P}(Y = y) = \begin{cases} \dfrac{1}{6} & \text{for} \quad y = 1, 3, 5 \\ \dfrac{1}{18}(y-1) & \text{for} \quad y = 2, 4, 6 \\ 0 & \text{otherwise.} \end{cases}$$

Connie throws the two dice. Sally pays Connie \$5 if the difference between the scores on the fair and biased dice is more than 3. Both players receive nothing if the scores on the fair and biased dice are identical. Connie pays Sally \$3 for all other outcomes. Let X be Sally's winnings after one game in dollars.

(i) Find Sally's expected winnings in one game, leaving your answer in exact form. [4]

(ii) Find the probability that Sally's total winnings in 50 independent games is at least \$65. [3]

10 **(a)** It is given that the regression line y on x for the following bivariate data is $y = 8 + 0.5x$.

x	20	22	24	26	28	30	32	34
y	16	21	a	24	22	24	27	20

Find a. [2]

(b) A botanist conducted an experiment to find out how the age of pine trees, x, in years, varies with their average height, y, in metres. The data collected were given below.

x (in years)	1	2	3	4	5	6	7	8	9	10
y (in metres)	2.74	3.38	3.75	4.08	4.30	4.48	4.51	4.68	4.72	4.75

(i) Draw a scatter diagram for the given data. [2]

The botanist felt that the data should be modelled by an equation of the form $y = a + bx$.

(ii) Give an interpretation, in this context, of the value of b. [1]

(iii) State, with a reason, which of the following models among A, B or C is the most appropriate for the given data.

A: $y = a - \dfrac{b}{x}$ B: $y = a + b\sqrt{x}$ C: $y = a + b\ln x$

Write down the equation of the least-squares regression line for the chosen model, stating clearly the values of a and b. [3]

(iv) Give two reasons why it would be reasonable to use your model to estimate the age of the pine tree when its height is 4.25 metres. [2]

11 A road named Spring Avenue has a speed limit of 40 km/h in a housing estate.

The residents were concerned that many vehicles travelled too fast along the road and they decided to set up a speed tracking device to monitor the speed of vehicles travelling along this road. The data generated from the device indicated that the mean speed of vehicles travelling through this road was 44.1 km/hour.

In an attempt to reduce the mean speed of vehicles travelling through Spring Avenue, life-size photographs of a police officer were put up next to the road. The speed, X km/hour of a sample of 100 randomly chosen vehicles was then measured and the following data obtained.

$$\sum x = 4327.0, \qquad \sum \left(x - \overline{x}\right)^2 = 925.71.$$

(i) Calculate the unbiased estimates of the population mean and variance of the speed of vehicles travelling along Spring Avenue. [2]

(ii) State an assumption that must be made about the sample in order to carry out a hypothesis test to investigate whether the desired reduction in mean speed had occurred. [1]

(iii) Given that the assumption that you stated in part **(ii)** is valid, carry out such a test, using the 5% level of significance. [5]

(iv) Explain what is meant by "5% level of significance" in the context of this question. [1]

(v) Subsequently, the residents detected that a measurement error has occurred when measuring the speed of the 100 randomly selected vehicles. To rectify the error, a multiplication of a positive constant k to each reading for the 100 randomly selected cars is recommended. Find the greatest possible value of k, to 3 significant figures, for the conclusion obtained in **(iii)** to remain the same. [3]

12 **(i)** Aquafresh mineral water is supplied in 1.5-litre bottles. The actual volume in millilitres, in a bottle may be modelled by a normal distribution with mean 1505 ml and standard deviation 10.2 ml.

(a) Calculate the probability that the volume of Aquafresh mineral water in a randomly selected bottle is more than 1480 ml. [1]

(b) The supplier requires that less than 10 per cent of bottles should contain less than 1480 ml of water.
Assuming that there has been no change in the value of the standard deviation, calculate the least mean volume in order to satisfy this requirement. Give your answer to one decimal place. [3]

(ii) Sparkling spring water is supplied in packs of six 0.5-litre bottles. The actual volume in a bottle may be modelled by a normal distribution with mean 508.5 ml and standard deviation 3.5 ml.

Find the probability that the volume of water in each of the 6 bottles from a randomly selected pack is more than 505 ml. [2]

(iii) Calculate the probability that the volume of 6 bottles of Sparkling spring water in a randomly selected pack differs from twice the volume of one randomly selected bottle of Aquafresh mineral water by less than 5.5 ml. [3]

(iv) The volume of tap water, V, used by a guest in a bathroom at a small hotel may be modelled by a random variable with mean 120 litres and standard deviation 65 litres. Give a numerical justification as to why V is unlikely to be normally distributed. [1]

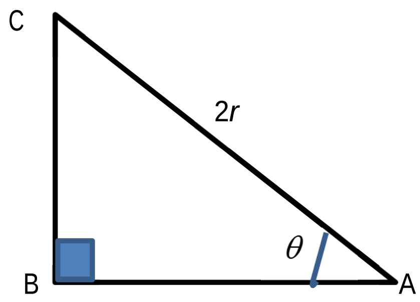
Explain why $\overline{V}$, the mean of a random sample of 30 observations of V, may be assumed to be approximately normally distributed and state its distribution. [2]

End of Paper

St Andrew's Junior College
2018 Preliminary Examination
H2 Mathematics Paper 2 (9758/02) Solutions

Section A

Qn	Solution
1	$y=a(x-1)+\dfrac{b}{x+c}$ $\downarrow$ A (replace y with $-y$) $-y=a(x-1)+\dfrac{b}{x+c}$ $y=-a(x-1)-\dfrac{b}{x+c}$ $\downarrow$ B (replace x with $x+1$) $y=-a(x+1-1)-\dfrac{b}{x+1+c}$ $y=-ax-\dfrac{b}{x+1+c}$ Since $x=0$ is a vertical asymptote, $1+c=0$ $c=-1$ Therefore, $y=-ax-\dfrac{b}{x}=\mathrm{f}(x)$. Since $\left(1,\dfrac{1}{6}\right)$ is a turning point on $y=\dfrac{1}{\mathrm{f}(x)}$, $(1,6)$ is a turning point on $y=\mathrm{f}(x)$. $-a-b=6$ --- (1) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-a+\dfrac{b}{x^2}$ Since when $x=1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$, $-a+b=0$ --- (2) Solving (1), (2) using a GC, $a=-3,\ b=-3$ Therefore, $a=-3,\ b=-3$ and $c=-1$
2	Given that AC is the diameter, $\angle ABC=\dfrac{\pi}{2}$.



$AB = 2r\cos\theta$, $BC = 2r\sin\theta$

Let S be the area of triangle ABC.

$$S = \frac{1}{2}(AB)(BC)$$

$$= \frac{1}{2}(2r\cos\theta)(2r\sin\theta)$$

$$= 2r^2\sin\theta\cos\theta$$

$$= r^2\sin 2\theta$$

$$\frac{dS}{d\theta} = r^2(2\cos 2\theta)$$

For stationary values of S, $\frac{dS}{d\theta} = 0$

Since $2r^2 \neq 0$,

$\cos 2\theta = 0$

Since $0 < \theta < \frac{\pi}{2} \Rightarrow 0 < 2\theta < \pi$

Hence,

$$2\theta = \frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

$$\frac{d^2S}{d\theta^2} = 2r^2(-2\sin 2\theta) = -4r^2\sin 2\theta$$

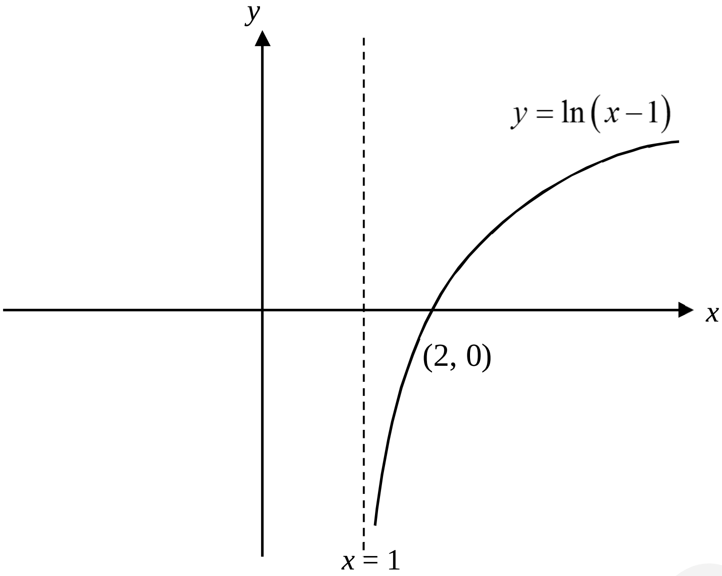
At $\theta = \frac{\pi}{4}$,

$$\frac{d^2S}{d\theta^2} = -4r^2\sin 2\left(\frac{\pi}{4}\right)$$

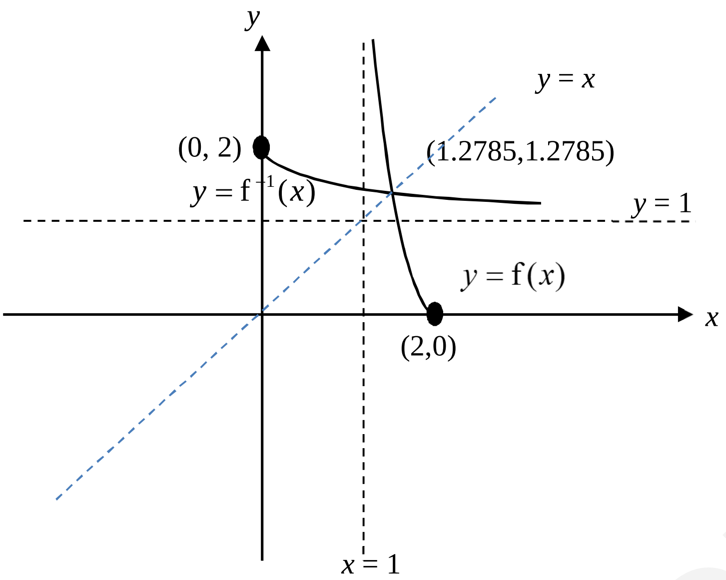
$$= -4r^2 < 0$$

Hence S is maximum when $\theta = \frac{\pi}{4}$.

3 (i)	$\sin\theta + m\cos\theta = \sqrt{1+m^2}\sin\left(\theta + \tan^{-1} m\right)$ $\therefore R = \sqrt{1+m^2}, \alpha = \tan^{-1} m$
	$2(\sin\theta + m\cos\theta)\sin(\theta-\alpha)$ $= R\left[2\sin(\theta+\alpha)\sin(\theta-\alpha)\right]$ $= R\left[-(\cos 2\theta - \cos 2\alpha)\right]$ $= R\left(\cos 2\alpha - \cos 2\theta\right)$
3 (ii)	$\int \frac{\cos 2\theta}{(\sin\theta + m\cos\theta)\sin(\theta-\alpha)}\, \mathrm{d}\theta$ $= 2\int \frac{\cos 2\theta}{2(\sin\theta + m\cos\theta)\sin(\theta-\alpha)}\, \mathrm{d}\theta$ $= \frac{2}{\sqrt{1+m^2}} \int \frac{\cos 2\theta}{(\cos(2\alpha) - \cos(2\theta))} \mathrm{d}\theta$ $= \frac{-2}{\sqrt{1+m^2}} \int \frac{\cos 2\theta}{(\cos 2\theta + 1)} \mathrm{d}\theta$ since $\cos(2\alpha) = -1$ when $\alpha = \frac{\pi}{2}$ $= \frac{-2}{\sqrt{1+m^2}} \int \left[\frac{2\cos^2\theta - 1}{2\cos^2\theta}\right] \mathrm{d}\theta$ $= \frac{-2}{\sqrt{1+m^2}} \int \left[1 - \frac{1}{2\cos^2\theta}\right] \mathrm{d}\theta$ $= \frac{-2}{\sqrt{1+m^2}} \left[\int 1 \mathrm{d}\theta - \frac{1}{2}\int \sec^2\theta\, \mathrm{d}\theta\right]$ $= \frac{-2}{\sqrt{1+m^2}} \left[\theta - \frac{1}{2}\tan\theta\right] + C,$ where C is an arbitrary constant

4(i)	 From the graph, the solution is $1<x\le 2$
	Alternative $\ln(x-1)\le 0$ $0<x-1\le 1$ $1<x\le 2$ Since $\ln(x-1)$ is defined for $x>1$, the solution is $1<x\le 2$.
4 (ii)	Let $y=\lvert\ln(x-1)\rvert$ Since $1<x\le 2$, $y=-\ln(x-1)$ $-y=\ln(x-1)$ $x-1=e^{-y}$ $x=1+e^{-y}$ $\mathrm{f}^{-1}(x)=1+e^{-x}$ $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=[0,\infty)$ Alternative: $y=\lvert\ln(x-1)\rvert$ $\pm y=\ln(x-1)$ $e^{\pm y}=x-1$ $e^{-y}=x-1$ (since $y\ge 0$ and $1<x\le 2$) $\therefore x=e^{-y}+1$ $\mathrm{f}^{-1}(x)=1+e^{-x}$ $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=[0,\infty)$

4 (iii)



4 (iv)

Point of intersection of $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}(x)$ is (1.28, 1.28)

$\mathrm{f}^{-1}(x) < \mathrm{f}(x)$

From the graph, $1 < x \le 1.27$.

4 (v)

$$\mathrm{fg}(a) = 3$$

$$\mathrm{g}(a) = \mathrm{f}^{-1}(3)$$

$$1 + \frac{4}{4a^2 + 5} = 1 + e^{-3}$$

$$\frac{4}{4a^2 + 5} = \frac{1}{e^3}$$

$$4e^3 = 4a^2 + 5$$

$$a^2 = \frac{4e^3 - 5}{4}$$

$a = \sqrt{\frac{4e^3 - 5}{4}}$ or $a = -\sqrt{\frac{4e^3 - 5}{4}}$ (reject since $D_{fg} = D_g = \mathbb{R}^+$)

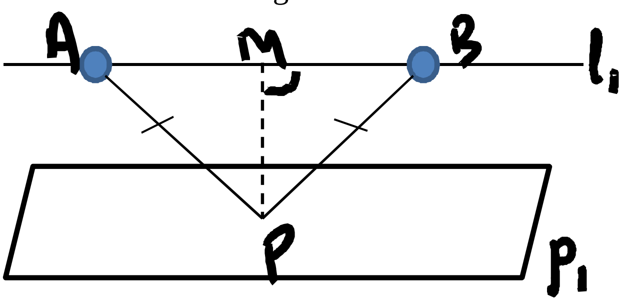
5 (i)

$$l_1 : \mathbf{r} = \begin{pmatrix} 11 \\ 6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

$$l_2 : \mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \mu \in \mathbb{R}$$

Condition 1:

$\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \ne k \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ for any real values of k.

	Hence lines l_1 and l_2 are not parallel to each other. **Condition 2:** Suppose both lines intersect, $\begin{pmatrix}11-2\lambda\\6\\\lambda\end{pmatrix}=\begin{pmatrix}4+2\mu\\-3\mu\\-1+4\mu\end{pmatrix}$ $\begin{cases}11-2\lambda=4+2\mu\\6=-3\mu\\\lambda=-1+4\mu\end{cases}$ $\Rightarrow\begin{cases}2\mu+2\lambda=7---(1)\\6=-3\mu---(2)\\-4\mu+\lambda=-1---(3)\end{cases}$ Using G.C. to solve (1) and (3), $\mu=\frac{9}{10},\ \lambda=\frac{13}{5}$ Checking with (2), $-3\mu=-3\left(\frac{9}{10}\right)$ $=-\frac{27}{10}\neq 6$ Hence, the two lines do not intersect at any unique points. Combining both conditions (1) and (2), lines l_1 and l_2 are skew lines.
5 (ii)	$\overrightarrow{OA}=\begin{pmatrix}11\\6\\0\end{pmatrix},\ \overrightarrow{OB}=\begin{pmatrix}1\\6\\5\end{pmatrix}$ Since P is on p_1 such that it is equidistant from A and B, ABP forms an isosceles triangle.  Let the mid-point of AB be M. By mid-point theorem,

$$\overrightarrow{OM} = \frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$$

$$= \frac{\begin{pmatrix}11\\6\\0\end{pmatrix}+\begin{pmatrix}1\\6\\5\end{pmatrix}}{2}$$

$$= \begin{pmatrix}6\\6\\\frac{5}{2}\end{pmatrix}$$

To find the foot of perpendicular, P, of M on p_1,

$$\overrightarrow{OP} = \begin{pmatrix}6\\6\\\frac{5}{2}\end{pmatrix} + s\begin{pmatrix}4\\3\\8\end{pmatrix}, \text{ for some } s \in \mathbb{R}$$

$$= \begin{pmatrix}6+4s\\6+3s\\\frac{5}{2}+8s\end{pmatrix}$$

Given $p_1 : \mathbf{r} \bullet \begin{pmatrix}4\\3\\8\end{pmatrix} = 240$

Since P lies on p_1,

$$\begin{pmatrix}6+4s\\6+3s\\\frac{5}{2}+8s\end{pmatrix} \bullet \begin{pmatrix}4\\3\\8\end{pmatrix} = 240$$

$$4(6+4s)+3(6+3s)+8\left(\frac{5}{2}+8s\right) = 240$$

$$24+16s+18+9s+20+64s = 240$$

$$62+89s = 240$$

$$89s = 178$$

$$s = \frac{178}{89} = 2$$

	Hence, $\overrightarrow{OP} = \begin{pmatrix} 6+4(2) \\ 6+3(2) \\ \frac{5}{2}+8(2) \end{pmatrix}$ $= \begin{pmatrix} 14 \\ 12 \\ \frac{37}{2} \end{pmatrix}$
5 (iii)	Normal vector of p_2 $= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ $= \begin{pmatrix} 0-(-3) \\ -(-8-2) \\ 6-0 \end{pmatrix}$ $= \begin{pmatrix} 3 \\ 10 \\ 6 \end{pmatrix}$ Since the plane p_2 contains the point P, $\mathbf{r} \bullet \begin{pmatrix} 3 \\ 10 \\ 6 \end{pmatrix} = \begin{pmatrix} 14 \\ 12 \\ \frac{37}{2} \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 10 \\ 6 \end{pmatrix}$ $= 273$ The Cartesian equation of p_2 is: $3x + 10y + 6z = 273$

Section B

6	Case 1: Committee only has 1 woman

	No. of ways = ${}^{15}C_7\,{}^{7}C_1 = 45045$ Case 2: Committee has no women No. of ways = ${}^{15}C_8 = 6435$ Hence, total number of ways = ${}^{22}C_8 - 45045 - 6435 = 268290$ <u>ALT</u> Case 1: Committee only has 2 women No. of ways = ${}^{15}C_6\,{}^{7}C_2 = 105105$ Case 2: Committee only has 3 women No. of ways = ${}^{15}C_5\,{}^{7}C_3 = 105105$ Case 3: Committee only has 4 women No. of ways = ${}^{15}C_4\,{}^{7}C_4 = 47775$ Case 4: Committee only has 5 women No. of ways = ${}^{15}C_3\,{}^{7}C_5 = 9555$ Case 5: Committee only has 6 women No. of ways = ${}^{15}C_2\,{}^{7}C_6 = 735$ Case 6: Committee only has 7 women No. of ways = ${}^{15}C_1\,{}^{7}C_7 = 15$ Hence, total number of ways = $105105 + 105105 + 47775 + 9555 + 735 + 15$ = 268290
	Number of ways to choose the 4 corner seats = ${}^{4}C_2$ Number of ways to arrange the remaining 6 committee members = 6! Number of ways to arrange the Gina and Hazel = 2! Hence, total number of ways = ${}^{4}C_2 \times 2! \times 6! = 8640$

7 (i)	Let W denote the number of shots that hits the bullseye 40m away from the target out of 18 shots. $W \sim B(18, \frac{2}{195}(95-40))$ $W \sim B(18, \frac{22}{39})$ $\mathrm{P}(W > 6 \mid W \le 10)$ $= \frac{\mathrm{P}([W > 6] \cap [W \le 10])}{\mathrm{P}(W \le 10)}$ $= \frac{\mathrm{P}(7 \le W \le 10)}{\mathrm{P}(W \le 10)}$ $= \frac{\mathrm{P}(W \le 10) - \mathrm{P}(W \le 6)}{\mathrm{P}(W \le 10)}$ $= \frac{0.51941}{0.56103}$ $= 0.926$ (to 3 s.f.)

7(ii)

Let Y denote the number of shots that hits the bullseye x m away from the target out of 18 shots.

$$Y \sim B\left(18, \frac{2}{195}(95-x)\right)$$

$$Y \sim B\left(18, \left(\frac{190}{195}-\frac{2}{195}x\right)\right)$$

$$\mathrm{P}(Y\geq 2)=0.98$$

$$1-\mathrm{P}(Y=0)-\mathrm{P}(Y=1)=0.98$$

$$\mathrm{P}(Y=0)+\mathrm{P}(Y=1)=0.02$$

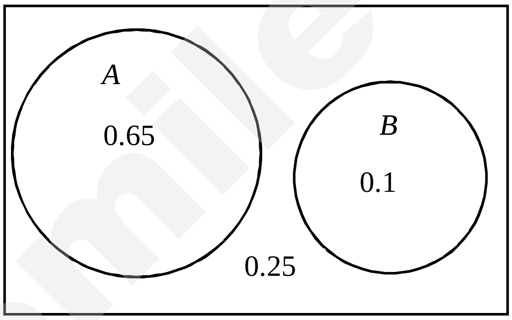
$$\left[1-\left(\frac{190}{195}-\frac{2}{195}x\right)\right]^{18}$$

$$+\left\{{}^{18}C_1\times\left(\frac{190}{195}-\frac{2}{195}x\right)^{1}\times\left[1-\left(\frac{190}{195}-\frac{2}{195}x\right)\right]^{17}\right\}=0.02$$

$$\left(\frac{1}{39}+\frac{2}{195}x\right)^{18}+\left[{}^{18}C_1\times\left(\frac{190}{195}-\frac{2}{195}x\right)^{1}\times\left(\frac{1}{39}+\frac{2}{195}x\right)^{17}\right]=0.02$$

Using the GC,
x = 67.3 m (to 3 s.f.)

8 (i)



$$\mathrm{P}(B\,|\,A')=\frac{2}{7}$$

$$\frac{\mathrm{P}(B\cap A')}{\mathrm{P}(A')}=\frac{2}{7}$$

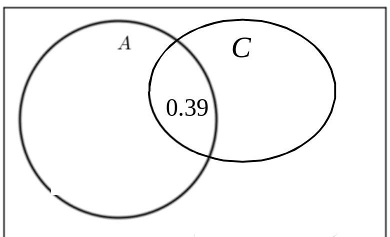
$$\mathrm{P}(B\cap A')=\frac{2}{7}(1-0.65)$$

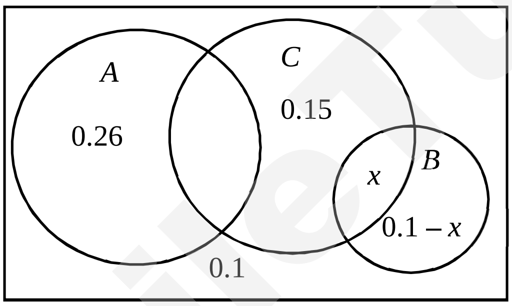
$$=0.1$$

$$\mathrm{P}(B)=\mathrm{P}(B\cap A')$$

$$=0.1 \qquad (\text{shown})$$

8 (ii)



<table>
<tr><td></td><td>$\mathrm{P}(C'|A)$
$= \dfrac{\mathrm{P}(A \cap C')}{\mathrm{P}(A)}$
$= \dfrac{\mathrm{P}(A) - \mathrm{P}(A \cap C)}{\mathrm{P}(A)}$
$= \dfrac{0.65 - 0.39}{0.65}$
$= 0.4$</td></tr>
<tr><td>8 (iii)</td><td>

Let $\mathrm{P}(B \cap C) = x$
When B and C are independent,
$\mathrm{P}(B \cap C) = \mathrm{P}(B) \times \mathrm{P}(C)$
$x = 0.1 \times (0.39 + 0.15 + x)$
$x = 0.054 + 0.1x$
$0.9x = 0.054$
$x = 0.06$
$\therefore \mathrm{P}(B \cap C) = 0.06$</td></tr>
<tr><td>8 (iv)</td><td>$\mathrm{P}(C) = 0.39 + 0.15 + 0.06 = 0.6$
Since $\mathrm{P}(A) \times \mathrm{P}(C) = (0.65)(0.6) = 0.39 = \mathrm{P}(A \cap C)$
(or $\mathrm{P}(C') = 1 - \mathrm{P}(C) = 0.4 = \mathrm{P}(C'|A)$)
A and C are independent.</td></tr>
<tr><td>9 (i)</td><td>Let X be the amount of winnings Sally gets after one game in dollars.</td></tr>
</table>

Let the random variables F be the scores on the fair dice.

$$\begin{aligned}
&\mathrm{P}(X=-5)\\
&=\mathrm{P}\left(\left|F-Y\right|>3\right)\\
&=\mathrm{P}\left[F=5,Y=1\right]+\mathrm{P}\left[F=6,Y=1\right]+\\
&\mathrm{P}\left[F=6,Y=2\right]+\mathrm{P}\left[F=1,Y=5\right]\\
&+\mathrm{P}\left[F=1,Y=6\right]+\mathrm{P}\left[F=2,Y=6\right]\\
&=\left(\frac{1}{6}\times\frac{1}{6}\right)+\left(\frac{1}{6}\times\frac{1}{6}\right)+\left(\frac{1}{6}\times\frac{1}{18}\right)\\
&+\left(\frac{1}{6}\times\frac{1}{6}\right)+\left(\frac{1}{6}\times\frac{5}{18}\right)+\left(\frac{1}{6}\times\frac{5}{18}\right)\\
&=\frac{5}{27}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{P}(X=0)\\
&=\mathrm{P}\left[F=1,Y=1\right]+\mathrm{P}\left[F=2,Y=2\right]+\\
&\mathrm{P}\left[F=3,Y=3\right]+\mathrm{P}\left[F=4,Y=4\right]\\
&+\mathrm{P}\left[F=5,Y=5\right]+\mathrm{P}\left[F=6,Y=6\right]\\
&=\frac{1}{6}\left(\frac{1}{6}+\frac{1}{18}+\frac{1}{6}+\frac{3}{18}+\frac{1}{6}+\frac{5}{18}\right)\\
&=\frac{1}{6}
\end{aligned}$$

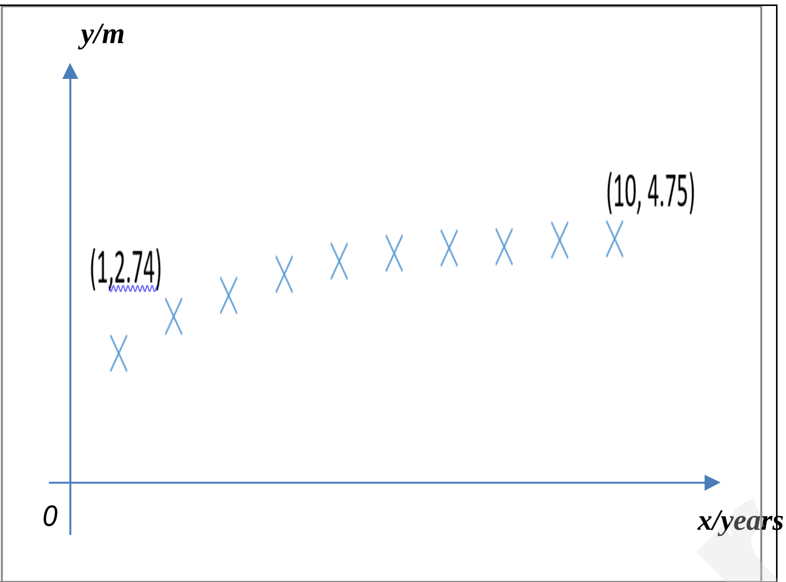
$\mathrm{P}(X = 3) = 1 - \mathrm{P}(X=0) - \mathrm{P}(X=-5)$

$$= 1-\frac{5}{27}-\frac{1}{6}=\frac{35}{54}$$

x	-5	0	3
$\mathrm{P}(X=x)$	$\frac{5}{27}$	$\frac{1}{6}$	$\frac{35}{54}$

$$\begin{aligned}
&\mathrm{E}(X)\\
&=\sum_{\text{all } x} x\mathrm{P}(X=x)\\
&=-5\left(\frac{5}{27}\right)+0\left(\frac{1}{6}\right)+(3)\left(\frac{35}{54}\right)\\
&=\frac{55}{54}
\end{aligned}$$

9(ii)	$\mathrm{E}(X^2)$ $=\sum_{\text{all } x} x^2\mathrm{P}(X=x)$ $=(-5)^2\left(\frac{5}{27}\right)+(0)^2\left(\frac{1}{6}\right)+(3)^2\left(\frac{35}{54}\right)$ $=\frac{565}{54}$ $\mathrm{Var}(X)$ $=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ $=\frac{565}{54}-\left(\frac{55}{54}\right)^2$ $=\frac{27485}{2916}$ Since the number of games of 50 is sufficiently large, by Central Limit Theorem, $X_1+X_2+\cdots+X_{50}\sim \mathrm{N}\left(50\left(\frac{55}{54}\right),50\left(\frac{27485}{2916}\right)\right)$ approximately $\mathrm{P}(X_1+X_2+\cdots+X_{50}\geq 65)$ $=0.25839$ $=0.258$ (to 3 sf)
10 (a)	$\overline{x}=\frac{20+22+24+...+34}{8}=27$ and $\overline{y}=\frac{16+21+a+24+22+24+27+20}{8}$ $=\frac{154+a}{8}$. Since $(\overline{x},\overline{y})$ lies on the regression line, $\frac{154+a}{8}=8+0.5(27)$ $a=18$
(b) (i)	



(ii)	For every increase of 1 year in the age of the trees, the height of the tree is expected to increase by *b* metres.
(iii)	Let the product moment correlation coefficient for this model be *r*. For A, $r_A = 0.958$, For B, $r_B = 0.972$ and for C, $r_C = 0.996$ Since $r_C = 0.996$ is closest to 1, the most appropriate model is Model C. Equation required is $y = 2.78 + 0.903\ln x$ $a = 2.78$ and $b = 0.903$
(iv)	Since $y = 4.25$ is within the given range of values [2.74, 4.75], the estimate obtained via interpolation is reliable. Since $r_C = 0.996$ is close to 1 which suggests a strong linear correlation between $\ln x$ and y , it would be reasonable to use Model C.
11 (i)	Let X be the random variable " speed of a vehicle travelling along Spring Avenue, measured in km/hour". The unbiased estimate of population mean = $\bar{x}$ = $\frac{\sum x}{100} = \frac{4327.0}{100} = 43.27$ The unbiased estimate of population variance,

	$s^2 = \dfrac{\sum (x-\bar{x})^2}{n-1} = \dfrac{925.71}{99} = 9.3506 = 9.35$ (to 3 sf)
(ii)	The 100 vehicles are selected independently.
(iii)	Let μ be the population mean speed Test H_0: $\mu = 44.1$ H_1: $\mu < 44.1$ Under H_0, $\bar{X} \sim N(44.1, \frac{9.3506}{100})$ approximately by Central Limit Theorem, since sample size 100 is sufficiently large. Carry out one-tailed z-test at 5% significance level. $\bar{x} = 43.27$ gives rise to test statistic z = -2.71 and p-value 0.00332. Since p-value $\le$ 0.05, we reject H_0 and conclude that at the 5% significance level, there is sufficient evidence that the mean speed of vehicles has been reduced after life sized police photos have been erected along Spring Avenue.
(iv)	5% significance level refers to the probability of 0.05 of wrongly concluding that the mean speed of vehicles is less than 44.1 km/hour when it is in fact 44.1 km/hour.
(v)	New unbiased estimate of population mean $= 43.27k$ New unbiased estimate of population variance = $\frac{30857}{3300}k^2$ Test H_0: $\mu = 44.1$ H_1: $\mu < 44.1$ Under H_0, $\bar{X} \sim N(44.1, \frac{\frac{30857}{3300}k^2}{100})$ approximately by Central Limit Theorem, since sample size 100 is sufficiently large. To have the same conclusion, i.e. to reject H_0, $z_{cal} \le -1.64485$ $\dfrac{(43.27k) - 44.1}{\left(k\sqrt{\frac{30857}{3300}}\right)/10} \le -1.64485$ $43.2k - 44.1 \le -\dfrac{1.64485\sqrt{\frac{30857}{3300}}}{10}k$ $k \le 1.01$ Since k > 0, $0 < k \le 1.01$ Greatest possible value of k is 1.01.

12 (i)(a)	Let X be the random variable " volume of Aquafresh mineral water in a bottle in ml" $X \sim \mathrm{N}(1505, 10.2^2)$ $\mathrm{P}(X > 1480) = 0.993$ from G.C.
(i)(b)	Let μ_1 be the new mean volume. $\mathrm{P}(X < 1480) < 0.10$ $\mathrm{P}(Z < \frac{1480-\mu_1}{10.2}) < 0.10$ $\frac{1480-\mu_1}{10.2} < -1.28155$ $\mu_1 > 1493.07$ The least mean volume is 1493.1 ml.
(ii)	Let Y be the random variable " volume of Sparkling spring water in a bottle in ml" $Y \sim \mathrm{N}(508.5, 3.5^2)$ P(the volume of water in each of the 6 bottles from a randomly selected pack is more than 505 ml) $= [\mathrm{P}(Y > 505)]^6$ $= [0.84134]^6$ = 0.355 (3 sig figures)
(iii)	Let $T = Y_1 + Y_2 + ... + Y_6 - 2X$ $T \sim N(41, 489.66)$ $P(\lvert T \rvert < 5.5) = P(-5.5 < T < 5.5) = 0.0365$
(iv)	Let V be the volume of water used by a guest in a bathroom in litres. $V \sim \mathrm{N}(120, 65^2)$ $\mathrm{P}(V < 0) = 0.0324$ which is impossible as the volume of water used cannot be negative.
	Since sample size 30 is sufficiently large, at least 20, Central Limit Theorem can be applied. $\therefore \overline{V} \sim N(120, \frac{65^2}{30})$ approximately.

H2

TAMPINES JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

MATHEMATICS 9758/01

Paper 1

Monday, 10 September 2018

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in, including the Cover Page.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

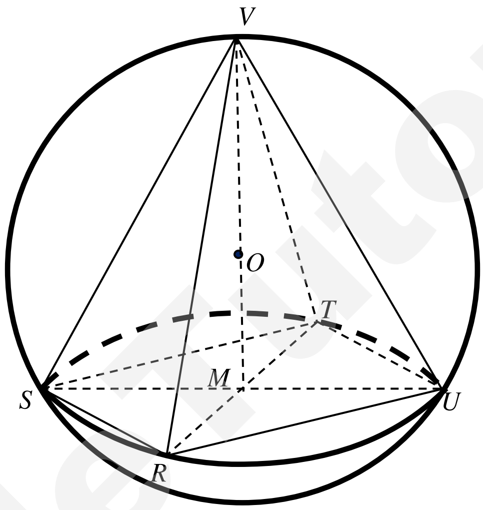
1 **(i)** Expand $\dfrac{1}{\sqrt[3]{8+12x}}$ in ascending powers of x up to and including the term in x^3. [3]

(ii) State the set of values of x for which the series expansion is valid. [1]

2 Without using a calculator, solve the inequality

$$\frac{3x^2+2x-2}{2x^2+3x-2} \le 1.$$ [5]

3



A right pyramid toy is placed inside a spherical container of fixed radius r. The toy has a square base $RSTU$ and vertical height VM of length $(r+h)$ where $0<h<r$. M is the point where the diagonals SU and RT of the square meet. The vertices R, S, T, U and V of the toy just touch the interior of the container with the vertical height VM passing through the centre O of the container.

(i) Show that the length of the side of the square base $RSTU$ is $\sqrt{2(r^2-h^2)}$. [2]

(ii) Hence, find the maximum volume of the toy in terms of r. [5]

[It is given that the volume of a pyramid is $\dfrac{1}{3}\times\text{base area}\times\text{height}$.]

4 **(i)** It is given that

$$y=\frac{x^2-3x+18}{x+10}, \qquad x\in\mathbb{R},\ x\neq -10.$$

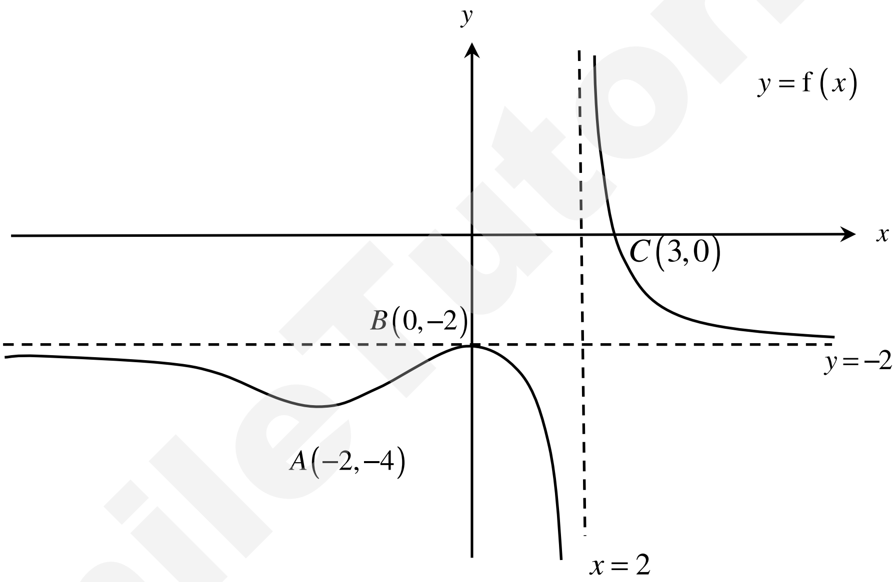
Without using a calculator, show that the range of values that y can take satisfies the inequality

$$y^2+46y-63\geq 0.$$

Find the range of y. [5]

(ii) Hence find the exact range of values of y for $e^{2y}+46e^{y}-63\geq 0$. [2]

5



The diagram shows the graph of $y=\mathrm{f}(x)$. The curve has turning points at $A(-2,-4)$ and $B(0,-2)$ and crosses the x-axis at point $C(3,0)$. The equations of the asymptotes of the curve are $x=2$ and $y=-2$.

On separate diagrams, sketch the graphs of

(i) $y=\mathrm{f}\left(\dfrac{1-x}{2}\right)$, [3]

(ii) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

(iii) $y=\mathrm{f}'(x)$, [3]

labelling, where applicable, the exact coordinates of the points corresponding to A, B and C and the equations of any asymptotes.

6 Referred to the origin O, points A and B have position vectors given by $\mathbf{a} = -p\mathbf{i} + 2p\mathbf{j} + 2p\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j}$ respectively, where $p > 0$.

(i) Given that $\mathbf{a}$ is a unit vector, find the exact value of p. [2]

(ii) Give a geometrical interpretation of $|\mathbf{a} \times \mathbf{b}|$. [1]

Point C lies on AB, between A and B, such that $AC : CB = 3 : 2$.

(iii) Find the position vector of C. [2]

(iv) Find the exact area of triangle OBC. [3]

7 **(i)** Prove that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(\operatorname{cosec} x^2 + \cot x^2\right)\right] = -2x \operatorname{cosec} x^2$. [3]

(ii) Find $\int x \cos x^2 \, \mathrm{d}x$. [2]

(iii) Hence find the exact value of $\displaystyle\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}} \left[x \cos x^2 \ln\left(\operatorname{cosec} x^2 + \cot x^2\right)\right] \mathrm{d}x$. [5]

8 **Do not use a calculator in answering this question.**

The complex numbers z and w are given by $1 - \mathrm{i}$ and $-1 + \mathrm{i}\sqrt{3}$ respectively.

(i) Express each of z and w in polar form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$. Give r and θ in exact form. [2]

(ii) Find zw and $\dfrac{z}{w}$ in exact polar form. [2]

(iii) Hence, by finding zw in exact cartesian form $x + \mathrm{i}y$, show that

$$\sin\left(\frac{5\pi}{12}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}.$$ [2]

(iv) Sketch the points A and B representing the complex numbers w and $\dfrac{z}{w}$ respectively on an Argand diagram. You should identify the modulus and argument of the points A and B. [2]

(v) Use part **(iii)** to find the exact area of triangle OAB, where O is the origin. [2]

9 **(i)** Show that $\dfrac{r}{(r+2)(r+3)(r+4)}$ can be expressed as $\dfrac{A}{r+2}+\dfrac{B}{r+3}+\dfrac{C}{r+4}$, where A, B and C are constants to be determined. [2]

The sum $\displaystyle\sum_{r=1}^{n}\frac{r}{(r+2)(r+3)(r+4)}$, is denoted by S_n.

(ii) Find an expression for S_n in terms of n. (There is no need to express your answer as a single algebraic fraction.) [3]

(iii) Explain why S_∞ is a convergent series, and write down its value. [2]

(iv) Find the smallest value of n for which $S_\infty - S_n < 0.05$. [2]

(v) Using results in parts **(ii)** and **(iii)**, show that

$$\sum_{r=1}^{\infty}\frac{r+1}{(r+5)^3}<\frac{3}{20}.$$ [4]

10 On 1 January 2018, Amy deposits \$200 into a bank account. On the first day of each subsequent month, she deposits \$20 more than in the previous month. Thus on 1 February, she deposits \$220 into the account and on 1 March, she deposits \$240 into the account, and so on. The account pays no interest.

(i) On what date will the value of Amy's account first exceed \$6000? [5]

On 1 January 2018, Benjamin deposits \$200 into a savings account which pays compound interest at a rate of 0.4% per month on the last day of each month. He puts a further \$200 into the account on the first day of each subsequent month.

(ii) Show that the value of Benjamin's savings account, in dollars, on the last day of the n^{th} month is given by

$$50200(1.004^n-1).$$ [2]

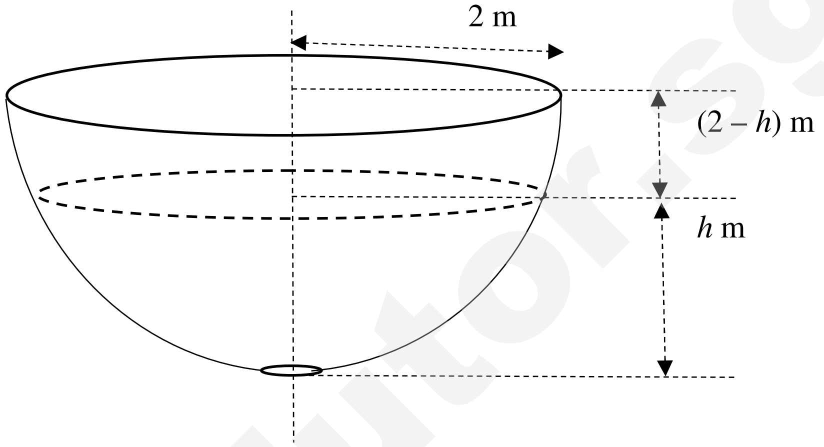
(iii) After how many complete months will the value of Benjamin's savings account first exceed \$6000? [3]

(iv) Benjamin would like the value of his savings account to be \$6000 on 2 December 2019. What interest rate per month, applied from January 2018, would achieve this? [3]

11 **Torricelli's Law** states that water will flow from an opening at the bottom of a tank with the same speed that it would attain falling from the surface of the water to the opening. One of the forms of Torricelli's Law is

$$A\frac{dh}{dt} = -k\sqrt{2gh}$$

where h is the height of the water in the tank, k is the area of the opening at the bottom of the tank, A is the horizontal cross-sectional area at height h, and g is the acceleration due to gravity.



A hemispherical water tank has a radius of 2 m. When the tank is full, a circular valve with a radius of 1 cm is opened at the bottom, as shown in the diagram.

Take $g = 10$ m/s^2.

(i) By expressing A in terms of h and finding the value of k, show that the rate of change of h metres, with respect to time, t seconds, satisfies the differential equation

$$(4h - h^2)\frac{dh}{dt} = -\frac{1}{10000}\sqrt{20h}\,. \qquad [4]$$

(ii) By solving the differential equation in part **(i)**, show that

$$t = \frac{400\sqrt{5}}{3}(ah^2\sqrt{h} - bh\sqrt{h} + 28\sqrt{2}),$$

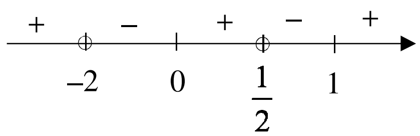
where a and b are constants to be determined. [7]

(iii) How long will it take for the tank to drain completely? Give your answer to the nearest second. [1]

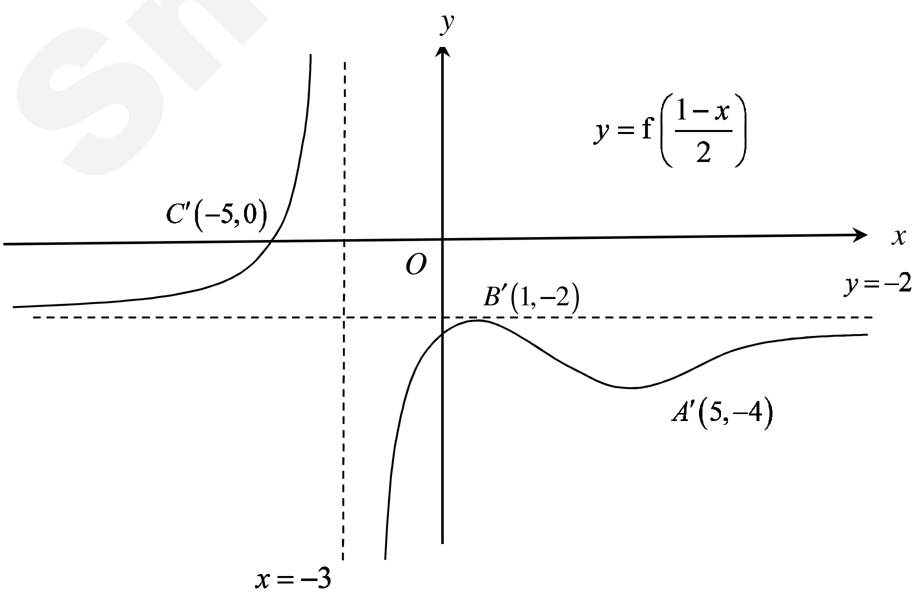
(iv) Sketch the graph of h against t. [2]

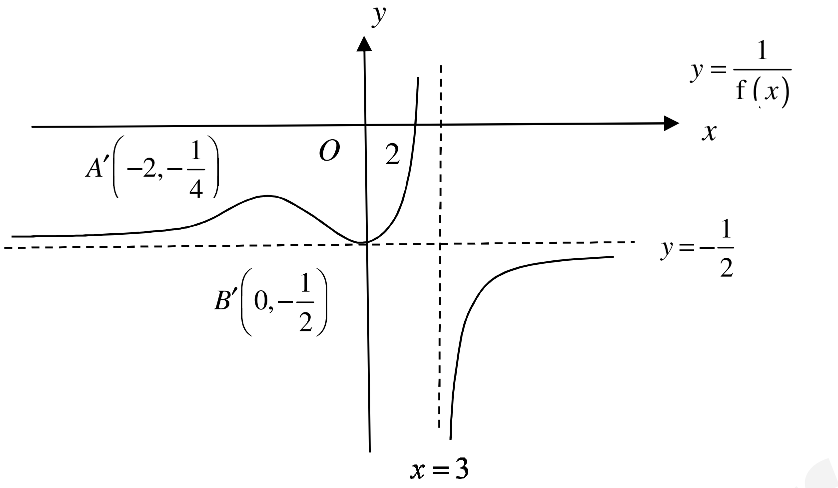
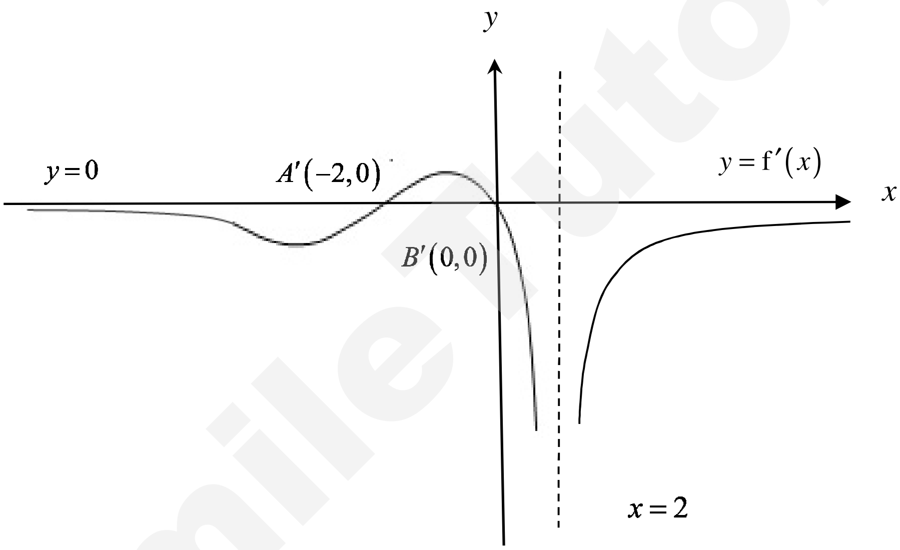
End of Paper

2018 JC2 Preliminary Examination
H2 Mathematics Paper 1
Solution

1(i)	$\frac{1}{\sqrt[3]{8+12x}}$ $=(8+12x)^{-\frac{1}{3}}$ $=(8)^{-\frac{1}{3}}\left(1+\frac{3}{2}x\right)^{-\frac{1}{3}}$ $=\frac{1}{2}\left(1+\left(-\frac{1}{3}\right)\left(\frac{3}{2}x\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}\left(\frac{3}{2}x\right)^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}\left(\frac{3}{2}x\right)^3+\cdots\right)$ $=\frac{1}{2}\left(1-\frac{1}{2}x+\frac{1}{2}x^2-\frac{7}{12}x^3+\cdots\right)$ $\approx\frac{1}{2}-\frac{1}{4}x+\frac{1}{4}x^2-\frac{7}{24}x^3$	
(ii)	$\left\{x\in\text{˜} : -\frac{2}{3}<x<\frac{2}{3}\right\}$	
2	$\frac{3x^2+2x-2}{2x^2+3x-2}\le 1$ $\frac{3x^2+2x-2}{2x^2+3x-2}-1\le 0$ $\frac{3x^2+2x-2-\left(2x^2+3x-2\right)}{2x^2+3x-2}\le 0$ $\frac{3x^2+2x-2-2x^2-3x+2}{(2x-1)(x+2)}\le 0$ $\frac{x^2-x}{(2x-1)(x+2)}\le 0$ $\frac{x(x-1)}{(2x-1)(x+2)}\le 0$  $-2<x\le 0$ or $\frac{1}{2}<x\le 1$	

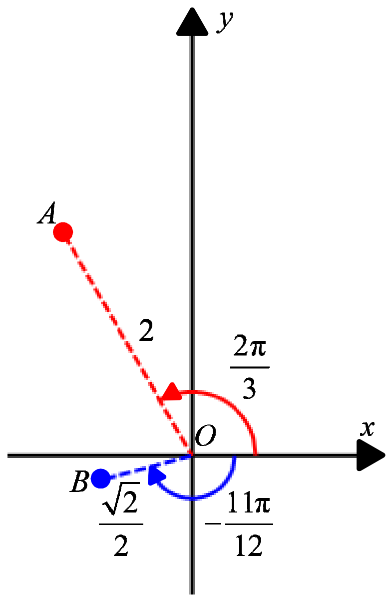
3(i)	$MU=\sqrt{r^2-h^2}$ $SU=2\sqrt{r^2-h^2}$ Length of square base $=\sin\left(\frac{\pi}{4}\right)2\sqrt{r^2-h^2}$ $=\frac{\sqrt{2}}{2}2\sqrt{r^2-h^2}$ $=\sqrt{2\left(r^2-h^2\right)}$ (shown)	
(ii)	Volume of toy, $V=\frac{2}{3}\left(r^2-h^2\right)\left(r+h\right)$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{2}{3}\left[\left(r^2-h^2\right)+\left(r+h\right)\left(-2h\right)\right]$ $=\frac{2}{3}\left(r^2-2rh-3h^2\right)$ For stationary point, $\frac{\mathrm{d}V}{\mathrm{d}h}=0$ $\frac{2}{3}\left(r^2-2rh-3h^2\right)=0$ $\left(r+h\right)\left(r-3h\right)=0$ $h=-r$ (reject since $h>0$) or $h=\frac{1}{3}r$ $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\frac{2}{3}(-2r-6h)$ When $h=\frac{1}{3}r$, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=-\frac{8}{3}r<0$ $h=\frac{1}{3}r$ gives maximum *V*. Maximum volume of toy, $V=\frac{2}{3}\left(r^2-\left(\frac{r}{3}\right)^2\right)\left(r+\left(\frac{r}{3}\right)\right)$ $=\frac{64r^3}{81}$ units^3	

4(i)	$y=\dfrac{x^2-3x+18}{x+10}$ $(x+10)y=x^2-3x+18$ $0=x^2-(3+y)x+(18-10y)$ For quadratic equations to have real roots, discriminant ≥ 0 $\left[-(3+y)\right]^2-4(1)(18-10y)\geq 0$ $9+6y+y^2-72+40y\geq 0$ $y^2+46y-63\geq 0$ Consider $y^2+46y-63=0$ $y=\dfrac{-46\pm\sqrt{46^2-4(-63)}}{2}$ $=-23\pm 4\sqrt{37}$ $\begin{array}{ccccc} + & & - & & + \\ & -23-4\sqrt{37} & & -23+4\sqrt{37} & \end{array}$ $y\leq -23-4\sqrt{37}$ or $y\geq -23+4\sqrt{37}$ Set of values is $\left\{y\in\mathbb{R} : y\leq -23-4\sqrt{37} \text{ or } y\geq -23+4\sqrt{37}\right\}$	
(ii)	$y^2+46y-63\geq 0$ Replace y with e^y $e^{2y}+46e^y-63\geq 0$ $e^y\leq -23-4\sqrt{37}$ or $e^y\geq -23+4\sqrt{37}$ $\left(\text{rej, } e^y>0\right)$ $y\geq \ln\left(-23+4\sqrt{37}\right)$	
5(i)		

(ii)		
(iii)		
6(i)	Since **a** is a unit vector, |**a**| = 1. $\sqrt{\left(p^2+4p^2+4p^2\right)}=1$ $\Rightarrow$ $\lvert 3p\rvert = 1$ (Accept: $9p^2=1$) $\Rightarrow$ $p=\frac{1}{3}$ (since $p>0$)	
(ii)	It is the length of projection of *OB* along *OA*.	
(iii)	By ratio theorem, $\overrightarrow{OC}=\frac{2\overrightarrow{OA}+3\overrightarrow{OB}}{5}=\frac{1}{5}\left[2\begin{pmatrix}-1/3\\2/3\\2/3\end{pmatrix}+3\begin{pmatrix}4\\3\\0\end{pmatrix}\right]$	

	$= \frac{1}{15}\begin{pmatrix}34\\31\\4\end{pmatrix}$	
(iv)	Area of triangle *OBC* $=\frac{1}{2}\left\|\begin{pmatrix}4\\3\\0\end{pmatrix}\times\frac{1}{15}\begin{pmatrix}34\\31\\4\end{pmatrix}\right\|$ $=\frac{1}{2}\frac{1}{15}\left\|\begin{pmatrix}12\\-16\\22\end{pmatrix}\right\|$ $=\frac{1}{15}\left\|\begin{pmatrix}6\\-8\\11\end{pmatrix}\right\|$ $=\frac{\sqrt{221}}{15}$	
7(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(\operatorname{cosec} x^2+\cot x^2\right)\right]$ $=\frac{2x\left(-\operatorname{cosec} x^2\cot x^2\right)+2x\left(-\operatorname{cosec}^2 x^2\right)}{\operatorname{cosec} x^2+\cot x^2}$ $=-2x\operatorname{cosec} x^2$	
(ii)	$\int x\cos x^2\,\mathrm{d}x$ $=\frac{1}{2}\int 2x\cos x^2\mathrm{d}x$ $=\frac{1}{2}\sin x^2+C$	
(iii)	$\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}}\left[\ln\left(\operatorname{cosec} x^2+\cot x^2\right)\cdot x\cos x^2\right]\mathrm{d}x$ $=\left[\ln\left(\operatorname{cosec} x^2+\cot x^2\right)\left(\frac{1}{2}\sin x^2\right)\right]_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}}-\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}}\left(\frac{1}{2}\sin x^2\right)\cdot\left(-2x\operatorname{cosec} x^2\right)\mathrm{d}x$ $=\ln\left(\operatorname{cosec}\frac{\pi}{2}+\cot\frac{\pi}{2}\right)\left(\frac{1}{2}\sin\frac{\pi}{2}\right)-\ln\left(\operatorname{cosec}\frac{\pi}{6}+\cot\frac{\pi}{6}\right)\left(\frac{1}{2}\sin\frac{\pi}{6}\right)+\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}}x\,\mathrm{d}x$ $==\ln\left(\frac{1}{1}+\frac{0}{1}\right)\left(\frac{1}{2}(1)\right)-\ln\left(\frac{1}{1/2}+\frac{\sqrt{3}/2}{1/2}\right)\left(\frac{1}{2}\left(\frac{1}{2}\right)\right)+\left[\frac{x^2}{2}\right]_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}}$	

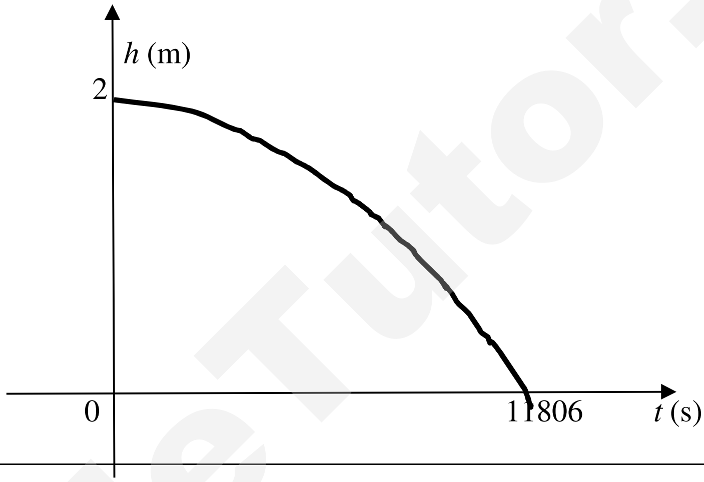
	$=-\frac{1}{4}\ln\left(2+\sqrt{3}\right)+\frac{\pi}{4}-\frac{\pi}{12}$ $=\frac{\pi}{6}-\frac{1}{4}\ln\left(2+\sqrt{3}\right)$	
8(i)	$z=1-\mathrm{i}=\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]$ $w=-1+\mathrm{i}\sqrt{3}=2\left[\cos\left(\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right]$	
(ii)	$zw=2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}+\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}+\frac{2\pi}{3}\right)\right]$ $=2\sqrt{2}\left[\cos\left(\frac{5\pi}{12}\right)+\mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right]$ $\frac{z}{w}=\frac{\sqrt{2}}{2}\left[\cos\left(-\frac{\pi}{4}-\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}-\frac{2\pi}{3}\right)\right]$ $=\frac{\sqrt{2}}{2}\left[\cos\left(-\frac{11\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{11\pi}{12}\right)\right]$	
(iii)	$zw=(1-\mathrm{i})\left(-1+\mathrm{i}\sqrt{3}\right)=\left(-1+\sqrt{3}\right)+\mathrm{i}\left(1+\sqrt{3}\right)$ $zw=\left(-1+\sqrt{3}\right)+\mathrm{i}\left(1+\sqrt{3}\right)$ $=2\sqrt{2}\left[\cos\left(\frac{5\pi}{12}\right)+\mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right]$ $=2\sqrt{2}\cos\left(\frac{5\pi}{12}\right)+\mathrm{i}2\sqrt{2}\sin\left(\frac{5\pi}{12}\right)$ Equating imaginary parts, $1+\sqrt{3}=2\sqrt{2}\sin\left(\frac{5\pi}{12}\right)$ $\Rightarrow\sin\left(\frac{5\pi}{12}\right)=\frac{1+\sqrt{3}}{2\sqrt{2}}$. (shown)	

(iv)		
(v)	$\angle AOB = 2\pi - \frac{2\pi}{3} - \frac{11\pi}{12}$ $= \frac{5\pi}{12}$ area of triangle $OAB = \frac{1}{2}(OA)(OB)\sin\angle AOB$ $= \frac{1}{2}(2)\left(\frac{\sqrt{2}}{2}\right)\sin\left(\frac{5\pi}{12}\right)$ $= \frac{\sqrt{2}}{2}\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)$ $= \frac{1+\sqrt{3}}{4}$ units^2	
9(i)	$\frac{r}{(r+2)(r+3)(r+4)} = \frac{A}{r+2} + \frac{B}{r+3} + \frac{C}{r+4}$ $r = A(r+3)(r+4) + B(r+2)(r+4) + C(r+2)(r+3)$ Sub $r=-2$: $A=-1$ Sub $r=-3$: $B=3$ Sub $r=-4$: $C=-2$ $\frac{r}{(r+2)(r+3)(r+4)} = -\frac{1}{r+2} + \frac{3}{r+3} - \frac{2}{r+4}$	

(ii)	$S_n = \sum_{r=1}^{n} \frac{r}{(r+2)(r+3)(r+4)}$ $= \sum_{r=1}^{n} \left[-\frac{1}{r+2} + \frac{3}{r+3} - \frac{2}{r+4} \right]$ $= [-\frac{1}{3} + \frac{3}{4} - \frac{2}{5}$ $-\frac{1}{4} + \frac{3}{5} - \frac{2}{6}$ $-\frac{1}{5} + \frac{3}{6} - \frac{2}{7}$ . . . $-\frac{1}{n} + \frac{3}{n+1} - \frac{2}{n+2}$ $-\frac{1}{n+1} + \frac{3}{n+2} - \frac{2}{n+3}$ $-\frac{1}{n+2} + \frac{3}{n+3} - \frac{2}{n+4}]$ $S_n = \frac{1}{6} + \frac{1}{n+3} - \frac{2}{n+4}$	
(iii)	As $n \to \infty, \frac{1}{n+3} \to 0$ and $\frac{2}{n+4} \to 0$ $\therefore$ S_∞ is a convergent series and $S_\infty = \frac{1}{6}$	
(iv)	$S_\infty - S_n < 0.05$ $\frac{1}{6} - \left(\frac{1}{6} + \frac{1}{n+3} - \frac{2}{n+4} \right) < 0.05$ $-\frac{1}{n+3} + \frac{2}{n+4} < 0.05$ $\frac{n+2}{(n+3)(n+4)} < 0.05$ Since $n \in$ ~ $^{+}$, Using G.C, Smallest value of n =15.	

(v)	$\sum_{r=1}^{\infty}\frac{r+1}{(r+5)^3}<\sum_{r=1}^{\infty}\frac{r+1}{(r+3)(r+4)(r+5)}$ $\left(\because (r+5)^2>(r+3)(r+4)\right)$ $\sum_{r=1}^{\infty}\frac{r+1}{(r+5)^3}<\sum_{r=2}^{\infty}\frac{r}{(r+2)(r+3)(r+4)}$ $\sum_{r=1}^{\infty}\frac{r+1}{(r+5)^3}<\sum_{r=1}^{\infty}\frac{r}{(r+2)(r+3)(r+4)}-\frac{1}{3(4)(5)}$ $\sum_{r=1}^{\infty}\frac{r+1}{(r+5)^3}<\frac{1}{6}-\frac{1}{60}$ $\therefore \sum_{r=1}^{\infty}\frac{r+1}{(r+5)^3}<\frac{3}{20}$ (shown)	
10 (i)	AP $a=200$ and $d=20$ $S_n>6000$ $\frac{n}{2}[2(200)+(n-1)20]>6000$ $200n+10n^2-10n>6000$ $n^2+19n-600>0$ $(n-16.77)(n+35.77)>0$ $n<-35.77$ (reject, n is positive integer) or $n>16.77$ Least $n=17$ Amy's account first exceeds \$6000 on 1 May 2019.	
(ii)	Amount at the end of first month = 200 (1.004) Amount at the end of second month = (200 (1.004) + 200) 1.004 = $200(1.004^2)+200(1.004)$ Amount at the end of n months = $200(1.004^n+1.004^{n-1}+...+1.004^2+1.004)$ = $200(1.004)\frac{1.004^n-1}{1.004-1}$ = $50200(1.004^n-1)$	
(iii)	$50200(1.004^n-1)>6000$ $1.004^n-1>\frac{30}{251}$ $n\ln(1.004)>\ln\left(\frac{281}{251}\right)$ $n>28.282$ Benjamin's account will exceed \$6000 after 29 months.	

(iv)	At the end of Nov 2019, $n=23$ $\frac{200r(r^{23}-1)}{r-1}+200>6000$ $\frac{r(r^{23}-1)}{r-1}>29$ Using GC, $r>1.01885$ Interest rate is 1.89% per month.	
11 (i)	Let r metre be the radius of the water surface when the height of water is h metre. By Pythagoras Theorem, $(2-h)^2+r^2=2^2$ $4-4h+h^2+r^2=4$ $r^2=4h-h^2$ $A=\pi r^2$ $=\pi(4h-h^2)$ $k=\pi(0.01)^2=\frac{\pi}{10000}$ Hence, $A\frac{\mathrm{d}h}{\mathrm{d}t}=-k\sqrt{2gh}$ where g = 10m/s^2 $\Rightarrow \pi(4h-h^2)\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{1}{10000}\pi\sqrt{20h}$ $\Rightarrow (4h-h^2)\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{1}{10000}\sqrt{20h}$	
(ii)	$(4h-h^2)\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{1}{10000}\sqrt{20h}$ $\int\frac{(4h-h^2)}{\sqrt{h}}\,\mathrm{d}h=\int-\frac{\sqrt{20}}{10000}\,\mathrm{d}t$ $\Rightarrow \int 4h^{\frac{1}{2}}-h^{\frac{3}{2}}\,\mathrm{d}h=\int-\frac{2\sqrt{5}}{10000}\,\mathrm{d}t$ $\Rightarrow \frac{8}{3}h^{\frac{3}{2}}-\frac{2}{5}h^{\frac{5}{2}}=-\frac{\sqrt{5}}{5000}t+C$ When $t=0$, $h=2$, $\frac{8}{3}(2)^{\frac{3}{2}}-\frac{2}{5}(2)^{\frac{5}{2}}=C$ $C=\frac{16\sqrt{2}}{3}-\frac{8\sqrt{2}}{5}=\frac{56\sqrt{2}}{15}$ $\therefore\ \frac{8}{3}h^{\frac{3}{2}}-\frac{2}{5}h^{\frac{5}{2}}=-\frac{\sqrt{5}}{5000}t+\frac{56\sqrt{2}}{15}$ Hence,	

	$t=\left(\frac{8}{3}h^{\frac{3}{2}}-\frac{2}{5}h^{\frac{5}{2}}-\frac{56\sqrt{2}}{15}\right)\left(\frac{-5000}{\sqrt{5}}\right)$ $=\frac{5000\sqrt{5}}{5(15)}\left(-40h^{\frac{3}{2}}+6h^{\frac{5}{2}}+56\sqrt{2}\right)$ $=\frac{400\sqrt{5}}{3}(3h^2\sqrt{h}-20h\sqrt{h}+28\sqrt{2})$ $\therefore\ a=3,\ b=20$	
(iii)	When $h=0$, $t=\frac{400\sqrt{5}}{3}(28\sqrt{2})=11805.8366\ \approx 11806$ (nearest second)	
(iv)		

H2

TAMPINES JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

MATHEMATICS 9758/02

Paper 2 **Wednesday, 12 September 2018**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in, including the Cover Page.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 Functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{-x^2+9}{x^2-4}, \quad x \in \mathbb{R},\ x \le 0,\ x \ne -2,$$

$$\mathrm{g}: x \mapsto -\frac{3x+1}{x+1}, \quad x \in \mathbb{R},\ x \ne -1.$$

(i) Find $\mathrm{f}^{-1}(x)$. [2]

(ii) Explain why the composite function fg does not exist. [2]

(iii) Find $\mathrm{gf}(x)$ and the range of gf. [3]

2 A curve C has parametric equations

$$x = \cos 2t, \quad y = \frac{1}{2}\sin 4t,$$

where $0 \le t \le \pi$.

(i) Find the equation of the normal to C at the point P with parameter p. [3]

(ii) The normal to C at the point where $t = \frac{\pi}{3}$ meets the curve again. Find the coordinates of the point of intersection. [3]

(iii) Find the cartesian equation of C. [2]

3 The line l has equation $\frac{x+9}{3} = \frac{y+5}{1}$, $z = 1$, and the plane p_1 has equation $-x + 2y + z = 6$.

(i) Find the acute angle between l and p_1. [3]

Referred to the origin O, the point A has position vector $2\mathbf{i} + \mathbf{j} - 6\mathbf{k}$.

(ii) Find the position vector of F, the foot of the perpendicular from A to p_1. [3]

(iii) Find the perpendicular distance from A to p_1, in exact form. [2]

(iv) Given that l is the line of intersection of the planes p_2 and p_3 with equations $x - 3y - z = a$ and $x + by + z = 7$ respectively, where a and b are real constants, find the values of a and b. [4]

4 **(i)** Express $\dfrac{18x}{\cos 9x^2 \cos\dfrac{\pi}{6} + \sin 9x^2 \sin\dfrac{\pi}{6}}$ in the form $18x\sec\left(ax^2+b\right)$, where a and b are constants to be determined. Show your workings clearly. [2]

(ii) It is given that $\mathrm{f}\left(x\right) = \dfrac{18x}{\cos 9x^2 \cos\dfrac{\pi}{6} + \sin 9x^2 \sin\dfrac{\pi}{6}}$. Use the substitution $\theta = 9x^2 - \dfrac{\pi}{6}$ to find the exact area bounded by the curve $y = \mathrm{f}\left(x\right)$, the x-axis and lines $x = \sqrt{\dfrac{\pi}{54}}$ and $x = \sqrt{\dfrac{\pi}{18}}$. [7]

(iii) The region bounded by the curve $y = \mathrm{f}\left(x\right)$, the x-axis and lines $x = -\sqrt{\dfrac{\pi}{18}}$ and $x = \sqrt{\dfrac{\pi}{18}}$ is rotated through 2π radians about the x-axis. Show that $\mathrm{f}\left(-x\right) = -\mathrm{f}\left(x\right)$. Hence, or otherwise, find the volume of the solid obtained, giving your answer correct to 2 decimal places. [4]

Section B: Probability and Statistics [60 marks]

5 An unbiased die has three faces painted red, two faces painted green and one face painted blue. A red face has a score of 1 point, a green face has a score of 2 points and a blue face has a score of 3 points.

Two such dice are thrown and the sum of their scores is denoted by X.

(i) Show that $\mathrm{P}(X = 4) = \dfrac{5}{18}$ and find the probability distribution of X. [3]

(ii) Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = \dfrac{10}{9}$. [2]

Suppose now a red face has a score of 3 points, a green face has a score of 2 points and a blue face has a score of 1 point.

(iii) Deduce the expectation and variance of the sum of scores obtained from a throw of two such dice. [2]

6 Find the number of ways in which the letters of the word DIGITISE can be arranged if

(i) there are no restrictions, [1]

(ii) G and S must not be next to each other, [2]

(iii) consonants (D, G, T, S) and vowels (I, E) must alternate, [3]

(iv) between any two Is there must be at least 2 other letters. [3]

7 A food processor produces large batches of jars of jam. The production manager wishes to take a random sample of the jars of jam produced in one day, for quality control purposes. He wishes to check that the mean mass of the jars of jam is 502 grams, as stated on the jars.

(i) State what it means for a sample to be random in this context. [1]

The masses, x grams, of a random sample of 50 jars of jam are summarised as follows.

$$n = 50 \qquad \sum(x-502) = -81 \qquad \sum(x-502)^2 = 1138$$

(ii) Calculate unbiased estimates of the population mean and variance of the mass of jars of jam. [2]

(iii) Test, at the 1% level of significance, the claim that the mean mass of jars of jam is 502 grams. You should state your hypothesis and define any symbols you use. [5]

(iv) Explain why there is no need for the production manager to know anything about the population distribution of the masses of the jars of jam. [2]

8 **(a)** Draw separate scatter diagrams, each with 6 points, all in the first quadrant, which represents the situation where the product moment correlation coefficient between x and y is

(i) between -0.9 and -0.5,

(ii) 1. [2]

(b) The age of students in years (x) and the median amount a month, in dollars (y), spent on tuition of a random sample of students are given in the table.

Age, x	6	7	9	11	13	15	18
Amount, y	155	170	211	230	248	260	265

(i) Draw a scatter diagram for these values, labelling the axes. [1]

It is thought that the median amount a month spent on tuition can be modelled by one of the formulae

$$y = a + bx \qquad \text{or} \qquad y = c + d\ln x$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

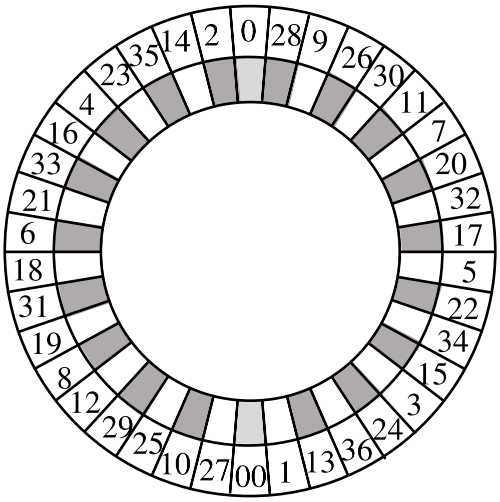
(A) x and y,

(B) $\ln x$ and y. [2]

(iii) Explain which of $y = a + bx$ and $y = c + d\ln x$ is the better model and find the equation of a suitable regression line for this model. [3]

(iv) Use the equation of your regression line to estimate the median amount a month spent on tuition by a student who is 16 years old. Comment on the reliability of your estimate. [2]

9



An American roulette wheel has 38 pockets numbered 00, 0, 1, 2, 3, …, 35 and 36. The 0 and 00 pockets are coloured "Green". 18 pockets are coloured "Black". The remaining pockets are coloured "Red". A typical round of roulette would involve spinning the roulette wheel in one direction and spinning a ball in the other direction, and the number of the pocket the ball eventually lands in would be the winning number and colour.

A player decides to play roulette for 10 rounds, and only makes the bet that "Black" will win for each round he plays in.

(i) State, in context, two assumptions needed for the number of wins with "Black" in 10 rounds to be well modelled by a binomial distribution. [2]

Assume now that the number of wins with "Black" in 10 rounds has a binomial distribution.

(ii) Find the expected number of rounds of roulette the player will win with "Black". [1]

(iii) Find the probability that the player will win at least 6 rounds of roulette. [2]

(iv) Find the probability that the player will win the fifth round of roulette given that the player has won two of the previous rounds. [2]

The player decides to visit the casino n times, playing 10 rounds of roulette each time. The player considers the casino visit "good" if he wins with "Black" for at least 6 rounds of roulette.

(v) Find the most probable number of "good" casino visits when $n = 20$. [3]

(vi) The player wants the probability that at least 5 casino visits are "good" in n casino visits to be more than 0.5. Find the range of values that n can take. [2]

10 **(a)** A and B are independent random variables with the distributions $\mathrm{N}(25, 20)$ and $\mathrm{N}(\mu, \sigma^2)$ respectively. It is known that $\mathrm{P}(B < 12) = \mathrm{P}(B > 19)$ and $\mathrm{P}(A > B) = 0.68$. State the value of μ and calculate the value of σ. [4]

(b) There are two vets Jerry and Mary who attend to customers with sick pets in a small vet clinic. The time taken for Jerry to attend to a customer follows a normal distribution with mean 10.1 minutes and standard deviation 0.8 minutes and the time taken for Mary to attend to a customer follows an independent normal distribution with mean 10.3 minutes and standard deviation 0.75 minutes.

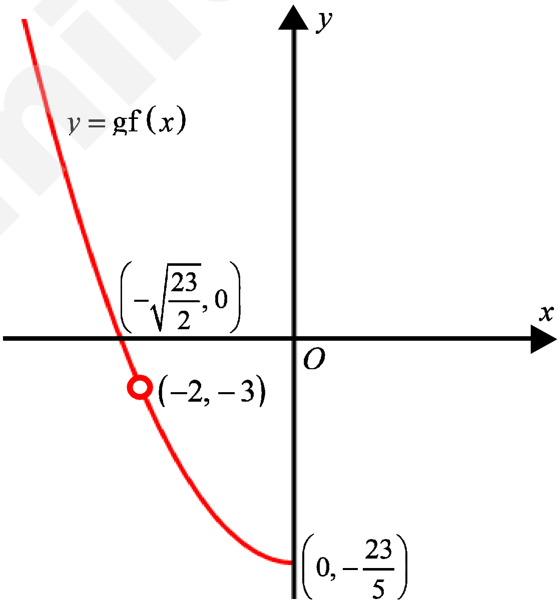
(i) Find the probability that among three randomly chosen customers attended to by Jerry, one took less than 10 minutes while the other two each took more than 10.5 minutes. [2]

(ii) Find the probability that the total time Jerry took to attend to two randomly chosen customers is less than twice the time Mary took to attend to one randomly chosen customer by at least 3 minutes. State the distribution you use and its parameters. [3]

(iii) Jerry and Mary each attended to two randomly chosen customers. Find the probability that the difference in the total time taken by Jerry and Mary to attend to their two customers is more than 2 minutes. State the parameters of any distribution you use. [3]

End of Paper

2018 JC2 Preliminary Examination
H2 Mathematics Paper 2
Solution

1(i)	Let $y=\mathrm{f}(x)=\dfrac{-x^2+9}{x^2-4}$ Then $y\left(x^2-4\right)=-x^2+9$ $\Rightarrow yx^2+x^2=4y+9$ $\Rightarrow x^2(y+1)=4y+9$ $\Rightarrow x^2=\dfrac{4y+9}{y+1}$ $\Rightarrow x=-\sqrt{\dfrac{4y+9}{y+1}}$ or $x=\sqrt{\dfrac{4y+9}{y+1}}$ (reject, $x\le 0$). Hence $\mathrm{f}^{-1}(x)=-\sqrt{\dfrac{4x+9}{x+1}}$	
(ii)	$\mathrm{R_g}=(-\infty,-3)\cup(-3,\infty)$, $\mathrm{D_f}=(-\infty,-2)\cup(-2,0]$ Since $\mathrm{R_g}\not\subseteq \mathrm{D_f}$, fg does not exist.	
(iii)	$\mathrm{gf}(x)=-\dfrac{3\left(\dfrac{-x^2+9}{x^2-4}\right)+1}{\left(\dfrac{-x^2+9}{x^2-4}\right)+1}=-\dfrac{3\left(-x^2+9\right)+\left(x^2-4\right)}{\left(-x^2+9\right)+\left(x^2-4\right)}=\dfrac{2x^2-23}{5}$ <u>Method 1: Graphical Method</u>  From the graph of $y=\mathrm{gf}(x)$, the range of gf can be seen to be	

	$\left[-\frac{23}{5},-3\right)\cup(-3,\infty)$. <u>Method 2: Analytical Method</u> $(-\infty,-2)\cup(-2,0]\xrightarrow{f}\left(-\infty,-\frac{9}{4}\right]\cup(-1,\infty)\xrightarrow{g}\left[-\frac{23}{5},-3\right)\cup(-3,\infty)$ Hence, the range of gf is $\left[-\frac{23}{5},-3\right)\cup(-3,\infty)$.	
2(i)	$x=\cos 2t$ $\quad\quad y=\frac{1}{2}\sin 4t$ $\frac{dx}{dt}=-2\sin 2t$ $\quad\quad \frac{dy}{dt}=\frac{1}{2}(4\cos 4t)=2\cos 4t$ $\frac{dy}{dx}=\frac{2\cos 4t}{-2\sin 2t}=-\frac{\cos 4t}{\sin 2t}$ At $t=p$, gradient of normal = $\frac{\sin 2p}{\cos 4p}$ Equation of normal at P is $y-\frac{1}{2}\sin 4p=\frac{\sin 2p}{\cos 4p}(x-\cos 2p)$ $y=\frac{\sin 2p}{\cos 4p}x-\frac{\sin 2p\cos 2p}{\cos 4p}+\frac{1}{2}\sin 4p$ $y=\frac{\sin 2p}{\cos 4p}x-\frac{1}{2}\tan 4p+\frac{1}{2}\sin 4p$	
(ii)	When $t=\frac{\pi}{3}$, equation of normal is $y=-\sqrt{3}x-\frac{1}{2}\sqrt{3}+\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)$ $y=-\sqrt{3}x-\frac{3}{4}\sqrt{3}$ Since the normal meets the curve, $\frac{1}{2}\sin 4t=-\sqrt{3}\cos 2t-\frac{3}{4}\sqrt{3}$ From GC, $t=\frac{\pi}{3}$ (reject since it is the given point) or 1.74796 Point of intersection is $(-0.938, 0.325)$.	
(iii)	$x^2=\cos^2 2t$ $y=\frac{1}{2}\sin 4t$ $=\frac{1}{2}(2\sin 2t\cos 2t)$	

<table>
<tr><td></td><td>$= \sin 2t \cos 2t$
$y^2 = \sin^2 2t \cos^2 2t$
$= \left(1-\cos^2 2t\right)\cos^2 2t$
$y^2 = \left(1-x^2\right)x^2$</td><td></td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>3(i)</td><td>$\theta = \sin^{-1}\left|\frac{\begin{pmatrix}3\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}}{\sqrt{10}\sqrt{6}}\right|$
$= \sin^{-1}\left|\frac{1}{2\sqrt{15}}\right|$
$= 7.4^\circ$ (1 d.p.)

<u>Accept:</u>
If student uses $\cos^{-1}$ and subtract their answer from 90°.</td><td></td></tr>
<tr><td>(ii)</td><td>$\overrightarrow{OF} = \begin{pmatrix}2-\lambda\\1+2\lambda\\-6+\lambda\end{pmatrix}$ for some $\lambda \in \mathbb{R}$.
Since F lies on p_1, $-(2-\lambda)+2(1+2\lambda)+(-6+\lambda)=6$
$6\lambda = 12$
$\lambda = 2$
$\therefore \overrightarrow{OF} = \begin{pmatrix}0\\5\\-4\end{pmatrix}$</td><td></td></tr>
<tr><td>(iii)</td><td>$\overrightarrow{AF} = \begin{pmatrix}0\\5\\-4\end{pmatrix} - \begin{pmatrix}2\\1\\-6\end{pmatrix} = \begin{pmatrix}-2\\4\\2\end{pmatrix}$
Perpendicular distance $= 2\left|\begin{pmatrix}-1\\2\\1\end{pmatrix}\right|$
$= 2\sqrt{6}$ units</td><td></td></tr>
<tr><td>(iv)</td><td>Since l is perpendicular to normal vector of p_3:
$\begin{pmatrix}3\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\b\\1\end{pmatrix} = 0$
$\therefore b = -3$
Since l lies in p_2:
$(-9)-3(-5)-(1) = a$
$\therefore a = 5$</td><td></td></tr>
<tr><td></td><td></td><td></td></tr>
</table>

4(i)	Using MF26, $\dfrac{18x}{\cos 9x^2 \cos\frac{\pi}{6}+\sin 9x^2 \sin\frac{\pi}{6}}$ $=\dfrac{18x}{\cos\left(9x^2-\frac{\pi}{6}\right)}$ $=18x\sec\left(9x^2-\frac{\pi}{6}\right)$ $\quad a=9, b=-\frac{\pi}{6}$ OR Using R-formula, $\dfrac{18x}{\cos 9x^2 \cos\frac{\pi}{6}+\sin 9x^2 \sin\frac{\pi}{6}}$ $=\dfrac{18x}{\frac{\sqrt{3}}{2}\cos 9x^2+\frac{1}{2}\sin 9x^2}$ $=\dfrac{36x}{\sqrt{3+1}\cos\left(9x^2-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)}$ $=18x\sec\left(9x^2-\frac{\pi}{6}\right)$	
(ii)	Area required $=\displaystyle\int_{\sqrt{\frac{\pi}{54}}}^{\sqrt{\frac{\pi}{18}}}\frac{18x}{\cos 9x^2\cos\frac{\pi}{6}+\sin 9x^2\sin\frac{\pi}{6}}\,\mathrm{d}x$ $=\displaystyle\int_{\sqrt{\frac{\pi}{54}}}^{\sqrt{\frac{\pi}{18}}}18x\sec\left(9x^2-\frac{\pi}{6}\right)\mathrm{d}x$ $=\displaystyle\int_0^{\frac{\pi}{3}}6\sqrt{\theta+\frac{\pi}{6}}\sec\theta\cdot\left(\frac{1}{6\left(\theta+\frac{\pi}{6}\right)^{\frac{1}{2}}}\right)\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{3}}\sec\theta\,\mathrm{d}\theta$ $=\left[\ln\left\|\sec\theta+\tan\theta\right\|\right]_0^{\frac{\pi}{3}}$ $=\ln\left\|\frac{1}{\frac{1}{2}}+\sqrt{3}\right\|-\ln\left\|\frac{1}{1}+0\right\|$ $=\ln(2+\sqrt{3})$ units^2 $\theta=9x^2-\frac{\pi}{6}$ $x=\dfrac{\sqrt{\theta+\frac{\pi}{6}}}{3}$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\dfrac{1}{2}\left(\dfrac{1}{3}\right)\left(\theta+\dfrac{\pi}{6}\right)^{-\frac{1}{2}}(1)$ $=\dfrac{1}{6\left(\theta+\frac{\pi}{6}\right)^{\frac{1}{2}}}$ When $x=\sqrt{\frac{\pi}{18}}$, $\theta=\frac{\pi}{3}$ When $x=\sqrt{\frac{\pi}{54}}$, $\theta=0$	

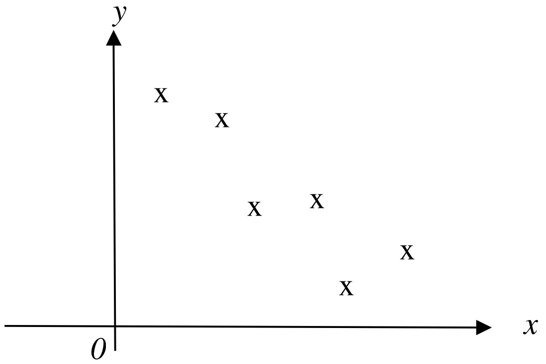
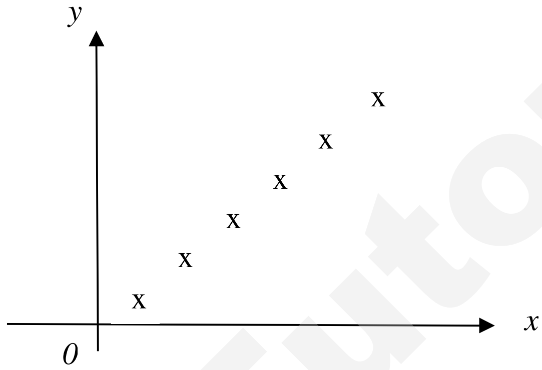
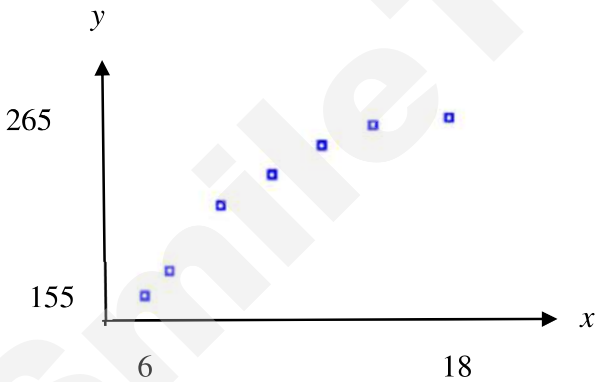
(iii)	$\mathrm{f}(-x)$ $=\dfrac{18(-x)}{\cos\left(9(-x)^2-\dfrac{\pi}{6}\right)}$ $=-\dfrac{18x}{\cos\left(9x^2-\dfrac{\pi}{6}\right)}$ $=-\mathrm{f}(x) \qquad (\text{shown})$ Volume required $=2\pi\displaystyle\int_0^{\sqrt{\frac{\pi}{18}}}\left(\dfrac{18x}{\cos\left(9x^2-\dfrac{\pi}{6}\right)}\right)^2 \mathrm{d}x$ $\approx 80.16 \text{ units}^3$	
5(i)	$\mathrm{P}(X=2)=\mathrm{P}(\text{red})\times\mathrm{P}(\text{red})=\dfrac{3}{6}\times\dfrac{3}{6}=\dfrac{1}{2}\times\dfrac{1}{2}=\dfrac{1}{4}$ $\mathrm{P}(X=3)=\mathrm{P}(\text{red})\times\mathrm{P}(\text{green})+\mathrm{P}(\text{green})\times\mathrm{P}(\text{red})$ $=\dfrac{3}{6}\times\dfrac{2}{6}+\dfrac{2}{6}\times\dfrac{3}{6}=\dfrac{1}{2}\times\dfrac{1}{3}\times2=\dfrac{1}{3}$ $\mathrm{P}(X=4)=\mathrm{P}(\text{red})\times\mathrm{P}(\text{blue})+\mathrm{P}(\text{blue})\times\mathrm{P}(\text{red})+\mathrm{P}(\text{green})\times\mathrm{P}(\text{green})$ $=\dfrac{3}{6}\times\dfrac{1}{6}+\dfrac{1}{6}\times\dfrac{3}{6}+\dfrac{2}{6}\times\dfrac{2}{6}=\dfrac{1}{2}\times\dfrac{1}{6}\times2+\dfrac{1}{3}\times\dfrac{1}{3}=\dfrac{5}{18}$ $\mathrm{P}(X=5)=\mathrm{P}(\text{green})\times\mathrm{P}(\text{blue})+\mathrm{P}(\text{blue})\times\mathrm{P}(\text{green})$ $=\dfrac{2}{6}\times\dfrac{1}{6}+\dfrac{1}{6}\times\dfrac{2}{6}=\dfrac{1}{3}\times\dfrac{1}{6}\times2=\dfrac{1}{9}$ $\mathrm{P}(X=6)=\mathrm{P}(\text{blue})\times\mathrm{P}(\text{blue})$ $=\dfrac{1}{6}\times\dfrac{1}{6}=\dfrac{1}{36}$	
(ii)	$\mathrm{E}(X)=\displaystyle\sum_{x=2}^{6}x\cdot\mathrm{P}(X=x)$ $=2\times\dfrac{1}{4}+3\times\dfrac{1}{3}+4\times\dfrac{5}{18}+5\times\dfrac{1}{9}+6\times\dfrac{1}{36}$ $=\dfrac{10}{3}$	

Table in 5(i):

x	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{18}$	$\frac{1}{9}$	$\frac{1}{36}$

	$\mathrm{E}\left(X^2\right)=\sum_{x=2}^{6} x^2 \cdot \mathrm{P}(X=x)$ $=2^2\times\frac{1}{4}+3^2\times\frac{1}{3}+4^2\times\frac{5}{18}+5^2\times\frac{1}{9}+6^2\times\frac{1}{36}$ $=\frac{110}{9}$ $\mathrm{Var}(X)=\mathrm{E}\left(X^2\right)-[\mathrm{E}(X)]^2$ $=\frac{110}{9}-\left(\frac{10}{3}\right)^2$ $=\frac{10}{9}$	
(iii)	Let Y be the sum of scores obtained from a throw of two dice where a red face has a score of 3 points, a green face has a score of 2 points and a blue face has a score of 1 point. Then $Y=8-X$. $\mathrm{E}(Y)=\mathrm{E}(8-X)=8-\mathrm{E}(X)=8-\frac{10}{3}=\frac{14}{3}$ $\mathrm{Var}(Y)=\mathrm{Var}(8-X)=\mathrm{Var}(8)+\mathrm{Var}(X)=\mathrm{Var}(X)=\frac{10}{9}$	
6(i)	Number of arrangements without restriction $=\frac{8!}{3!}=6720$	
(ii)	Number of arrangements with G and S grouped together $=\frac{7!}{3!}\times 2!=1680$ Number of arrangements with G and S not next to each other $=6720-1680$ = 5040	
(iii)	Two possible cases D_G_T_S_ _D_G_T_S Number of ways to arrange 4 consonants = 4! Number of ways to arrange 4 vowels (I, I, I, E) = 4 Total number of arrangements = $4!\times 4\times 2!$ = 192	
(iv)	Four possible cases I_ _ I _ _ _I I_ _ _ I _ _I _ I_ _ I _ _I I_ _ I _ _I _	

	Number of ways to arrange D, G, T, S, E = 5! = 120 Total number of arrangements = $5! \times 4 = 480$	
7(i)	A random sample is a sample drawn from the jars of jam produced in one day such that every jar of jam has an **equal chance** of being in the sample, and the selections of the jars of jam are made **independently.**	
(ii)	$\bar{x} = \frac{-81}{50} + 502 = 500.38$ $s^2 = \frac{1}{49}\left[1138 - \frac{(-81)^2}{50}\right] = 20.54653 = 20.5$ (3s.f)	
(iii)	Let X denote the mass of jars of jam, in grams. Let μ be the population mean mass of jars of jam, in grams. To test $H_0 : \mu = 502$ against $H_1 : \mu \neq 502$ at 1% level of significance. Since n = 50 is large, by Central Limit Theorem, $\bar{X} \sim N\left(502, \frac{20.54653}{50}\right)$ approximately under H_0 Test statistic: $Z = \frac{\bar{X} - 502}{\sqrt{\frac{20.54653}{50}}} \sim N(0,1)$ approximately under H_0 $z_{test} = \frac{500.38 - 502}{\sqrt{\frac{20.54653}{50}}} = -2.5271$ Using GC, $n = 50$, $\bar{x} = 500.38$, $s^2 = 20.54653$, $z_{test} = -2.5271$, p-value = 0.011499 Since p – value = 0.011499 > 0.01 , we do not reject H_0 and conclude that there is insufficient evidence, at the 1% significance level, that the mean mass of jars of jam differs from 502g.	
(iv)	Since n = 50, sample size is large, by Central Limit Theorem, the sample mean of mass of jars of jam follows a normal distribution approximately.	

8(a) (i)		
(ii)		
(b) (i)		
(ii) (A)	Correlation coefficient between x and y is 0.9483	
(B)	Correlation coefficient between $\ln x$ and y is 0.9849	
(iii)	$y = c + d\ln x$ is a better model. Scatter diagram shows a better fit. The correlation coefficient between $\ln x$ and y is closer to 1. As x increases, y increases at a decreasing rate. Equation of regression line of y on $\ln x$ is $y = -31.2643 + 106.56117\ln x$ $y = -31.3 + 107\ln x$	
(iv)	When $x = 16$, $y = -31.2643 + 106.56117\ln 16 = 264.1859$ Median amount a month spent on tuition by student who 16 years old is \$264. Estimate is reliable. Correlation coefficient is close to 1. $x = 16$ is within the range of values of x. Interpolation is a good practice.	

9(i)	Each round has the same probability of winning with "Black". Winning with "Black" in a round is independent of winning with "Black" in another round.	
(ii)	Let X be the number of wins with "Black" in 10 rounds. $X \sim \mathrm{B}\left(10, \frac{18}{38}\right)$ or $X \sim B\left(10, \frac{9}{19}\right)$ Expected number of wins $= \mathrm{E}(X) = 10 \times \frac{9}{19} = \frac{90}{19} = 4.74$ (3 s.f.)	
(iii)	Probability required $= 1 - \mathrm{P}(X \leq 5)$ $= 0.31412$ (5 s.f.) $= 0.314$ (3 s.f.)	
(iv)	Each round of roulette is independent of any other round of roulette. Hence, the probability that the player will win the fifth round of roulette given that the player has won two of the previous rounds $= \frac{9}{19}$. <u>Alternatively,</u> Probability required $= \mathrm{P}(\text{win fifth round} \mid \text{won two previous rounds})$ $= \frac{\mathrm{P}(\text{win fifth round and won two previous rounds})}{\mathrm{P}(\text{won two rounds out of four})}$ $= \frac{\binom{4}{2}\left(\frac{9}{19}\right)^2\left(1-\frac{9}{19}\right)^2 \times \frac{9}{19}}{\binom{4}{2}\left(\frac{9}{19}\right)^2\left(1-\frac{9}{19}\right)^2}$ $= \frac{9}{19}$	
(v)	Let Y be the number of "good" casino visits out of 20 visits. $Y \sim \mathrm{B}(20, 0.31412)$ Consider $\mathrm{P}(X = k)$, where $0 \leq k \leq 20$. When $k = 5$, $\mathrm{P}(X = 5) = 0.16582$ When $k = 6$, $\mathrm{P}(X = 6) = 0.18986$ When $k = 7$, $\mathrm{P}(X = 7) = 0.17390$ Hence, $\mathrm{P}(X = k)$ is the largest when $k = 6$. The most probable number of "good" casino visits when $n = 20$ is 6.	
(vi)	Let W be the number of casino visits that are "good" out of n casino visits. $W \sim \mathrm{B}(n, 0.31412)$ From GC, when $n = 14$, $\mathrm{P}(W \geq 5) = 1 - \mathrm{P}(W \leq 4) = 0.46206 < 0.5$ When $n = 15$, $\mathrm{P}(W \geq 5) = 0.53259 > 0.5$ Hence, $n \geq 15$	

10(a)	$A \sim N(25, 20)$ $B \sim N(\mu, \sigma^2)$ Since $P(B<12)=P(B>19)$, $\mu=\frac{12+19}{2}=15.5$ $B \sim N(15.5, \sigma^2)$ $A-B \sim N(9.5, 20+\sigma^2)$ $P(A>B)=0.68$ $P(A-B>0)=0.68$ $P(Z>-\frac{9.5}{\sqrt{20+\sigma^2}})=0.68$ $-\frac{9.5}{\sqrt{20+\sigma^2}}=-0.467699$ $\sigma=19.814=19.8$ (3 sf)	
(b)(i)	Let X and Y be the time taken by Jerry and Mary to attend to a customer in minutes respectively. $X \sim N(10.1, 0.8^2)$ $Y \sim N(10.3, 0.75^2)$ Required probability = $\binom{3}{1}\cdot P(X<10)\cdot\left[P(X>10.5)\right]^2$ $= 0.12859 = 0.129$ (3 sf)	
(ii)	$E\left(2Y-(X_1+X_2)\right)=2(10.3)-2(10.1)=0.4$ $\text{Var}\left(2Y-(X_1+X_2)\right)=2^2(0.75^2)+2(0.8^2)=3.53$ $2Y-(X_1+X_2) \sim N(0.4, 3.53)$ $P\left(2Y-(X_1+X_2)\geq 3\right)=0.0832$ (3 sf)	
(iii)	$(X_1+X_2)-(Y_1+Y_2) \sim N\left(2(10.1)-2(10.3),\ 2(0.8^2)+2(0.75^2)\right)$ $(X_1+X_2)-(Y_1+Y_2) \sim N(-0.4,\ 2.405)$ $P\left(\lvert(X_1+X_2)-(Y_1+Y_2)\rvert>2\right)=1-P\left(-2\leq(X_1+X_2)-(Y_1+Y_2)\leq 2\right)$ $= 0.21196$ $= 0.212$ (3 sf)	

VICTORIA JUNIOR COLLEGE

JC 2 PRELIMINARY EXAMINATION 2018

H2 Mathematics **9758/01**

Paper 1 **3 hours**

Additional Materials: Answer Paper
Graph Paper (Upon Request)
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 Abel, Mo and Paula train together for running. In a particular year, they spent time on three types of training programmes called "Circuit", "Intervals" and "Long Run".

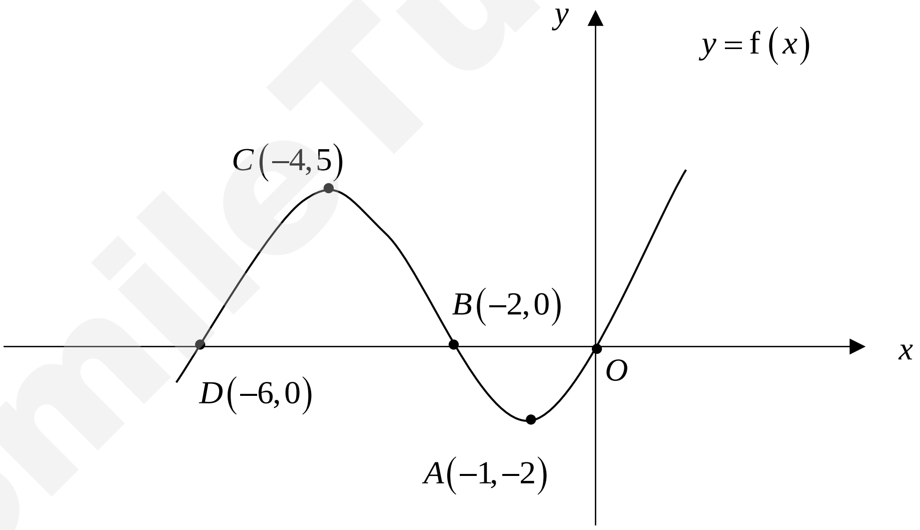
For each type of training programme, the duration of each training session is the same for all three runners, except that Mo requires 10% more training time for each "Interval" session compared to Abel and Paula.

The table below shows the number of training sessions each runner attended, as well as their total training time for the year.

	Number of training sessions in the year			Total training time (in minutes)
	"Circuit"	"Intervals"	"Long Run"	
Abel	23	40	61	12600
Mo	34	67	75	17600.5
Paula	33	53	87	17725

Find the total amount of time (in minutes) each runner spent on "Intervals" in the year.
[4]

2

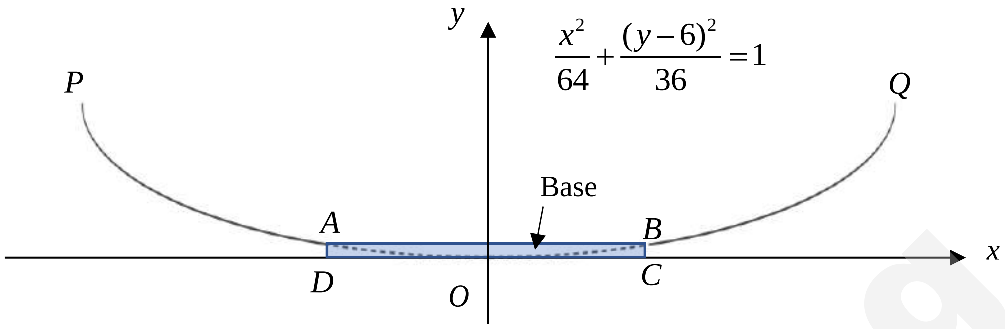


The diagram shows the graph of $y = \mathrm{f}(x)$. The curve passes through the points $A(-1,-2)$, $B(-2,0)$, $C(-4,5)$, $D(-6,0)$ and the origin O. Points A and C are the minimum point and maximum point respectively. On separate diagrams, sketch each of the following graphs indicating where possible, the coordinates of the stationary points, the points of intersections with the axes and the equations of any asymptotes.

(i) $y = \mathrm{f}\left(\dfrac{x}{2} + 2\right)$, [2]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

3



The diagram above shows the cross section of a bowl *PABQ* and a base *ABCD* of height h units. The curve $\frac{x^2}{64}+\frac{(y-6)^2}{36}=1,\ h \le y \le 6$, from point P to A and from point B to Q is rotated through π radians about the y-axis to form the curved surface of the bowl, while the rectangle *ABCD* is rotated through π radians about the y-axis to form the solid cylindrical base.

The bowl is assumed to have negligible thickness. Determine, correct to 3 decimal places, the greatest height of the base such that volume of the bowl is at least 790 units^3. [5]

4 A curve C has parametric equations

$$x = a\cos(2t),\ \ y = 2a\tan t,$$

where $0 \le t < \frac{\pi}{2}$ and a is a positive constant.

(i) Sketch C, showing clearly the coordinates of any point(s) of intersection with the axes and the equations of any asymptotes. [3]

(ii) It is given that $a = 3$. Find the equation of the tangent to C at the point where $t = \frac{\pi}{3}$. [3]

5 Using an algebraic method, solve the inequality $\frac{7x+6}{x+2} \le 3x-1$. [4]

Hence, find the set of values of x that satisfy $\frac{7e^{-x}+6}{e^{-x}+2} \le 3e^{-x}-1$. [2]

6 **Do not use a calculator in answering this question.**

(a) Express $\left(\dfrac{\sqrt{2}+\mathrm{i}\sqrt{2}}{1-\mathrm{i}\sqrt{3}}\right)^8$ in the cartesian form a + ib, where a and b are in a non-trigonometrical form. [3]

(b) Solve the simultaneous equations

$$2\mathrm{i}z^* - w = 1+7\mathrm{i} \text{ and } z+2w = 16-2\mathrm{i},$$

giving z and w in the cartesian form $p+\mathrm{i}q$.. [4]

7 Given that $y=\tan^{-1}(ax+1)$, where a is a constant, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=a\cos^2 y$. Use this result to find the Maclaurin series for y in terms of a, up to and including the term in x^3. [6]

Given that the first two terms in the series expansion of $\sqrt{9-x}$ are equal to the first two terms in the series expansion of $b\tan^{-1}(ax+1)$, find the exact values of a and b. [3]

8 **(a)** The straight line l has equation $\mathbf{r} = \mathbf{i} + 6\mathbf{j} - 3\mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$, $s \in \mathbb{R}$. The plane p has equation $\mathbf{r}\cdot(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = 6$. The line l intersects the plane p at a point C.

(i) Show that the position vector of C is $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. [2]

(ii) Find a vector equation of the line which lies in p, passes through C and is perpendicular to l. [3]

(b) Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.

(i) Show that $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$. [3]

It is further given that **a** is a unit vector, **b** has magnitude 3 and **a.b** = $\dfrac{1}{3}$.

(ii) Find the exact area of triangle OAB. [2]

(iii) Give the geometrical meaning of $|\mathbf{a}\cdot\mathbf{b}|$. [1]

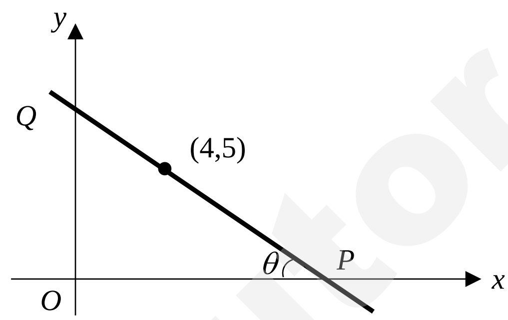
9 The curve C has equation $y = \dfrac{2x+5}{4-x^2}$.

(i) Sketch C, giving the coordinates of the axial intercepts, turning points and equations of any asymptotes. [4]

(ii) Find the numerical value of the area bounded by C and the line $y = 2$. [3]

(iii) Find the set of values of k such that $\left|\dfrac{2x+5}{x^2-4}\right| = k$ has 3 distinct negative real solutions. [3]

10 (a)

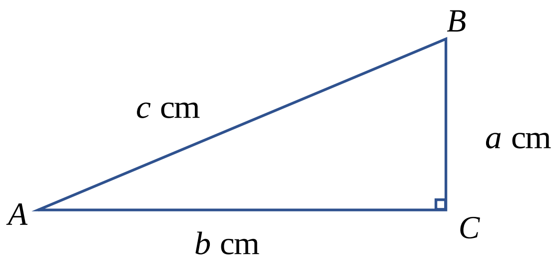


A line passes through the point $(4,5)$ and cuts the x-axis and y-axis at points P and Q respectively. It is given that angle $OPQ = \theta$, where $0 < \theta < \dfrac{\pi}{2}$ and O is the origin. You may assume that the same scale is used on both axes.

(i) Show that the equation of line PQ is given by $y = (4-x)\tan\theta + 5$. [3]

(ii) Hence or otherwise, show that $OP + OQ = 9 + 4\tan\theta + 5\cot\theta$. By differentiation, find the stationary value of $OP + OQ$ as θ varies. Determine the nature of this stationary value. [5]

(b)

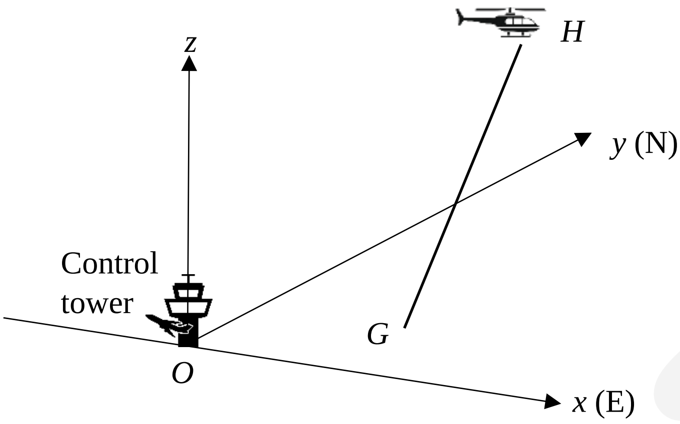


The diagram above shows a right angled triangle ABC. The lengths AB, BC and AC are denoted by c cm, a cm and b cm respectively. The triangle has a fixed area of 100 cm^2.

Express b in terms of a. [1]

Given that BC increases at a rate of 3 cm s^{-1}, find the rate of change of c when $a = 20$. [3]

11



The diagram illustrates the initial flight path of a helicopter H taking off from an airport. The origin O is taken to be at the base of the control tower. The x-axis is due east, the y-axis due north, and the z-axis is vertical. The units of distances are measured in kilometres.

The helicopter takes off from the point G on the ground. The position vector $\mathbf{r}$ of the helicopter t minutes after take-off is given by

$$\mathbf{r} = (1 + t)\mathbf{i} + (0.5 + 2t)\mathbf{j} + 2t\mathbf{k}.$$

(i) Write down the coordinates of G and describe the initial flight path. [3]

(ii) Find the acute angle that the helicopter's flight path makes with the horizontal. [3]

(iii) A mountain top is situated at the point M (5, 4.5, 3). Determine how long after take off the helicopter will be nearest to M. [2]

(iv) An eagle sets off from the mountain top to hunt for food. The position of the eagle satisfies the equation

$$\frac{x-2}{3} = \frac{z-2}{1}, \; y = 4.5.$$

Determine if the flight path of the helicopter will intersect the path traced out by the eagle, showing your reasoning clearly. [3]

(v) The helicopter enters a cloud at a height of 2 km. Given that the visibility on that day is 3.75 km, determine if the air traffic controller who is situated at 70 m above ground level, in the control tower, will be able to sight the helicopter as it enters the cloud. [2]

12 The logistic equation, sometimes called the *Verhulst model*, is a model of population growth. Letting N be the population size at any time, t (in years), this model is formalized by the differential equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right),$$

where r and K are real positive constants.

(i) By considering the value of $\frac{\mathrm{d}N}{\mathrm{d}t}$ when N approaches K, explain, in the context of the question, the significance of K. [2]

(ii) Solve the differential equation, and show that $N = \frac{K}{1+Be^{-rt}}$, where B is a constant. [5]

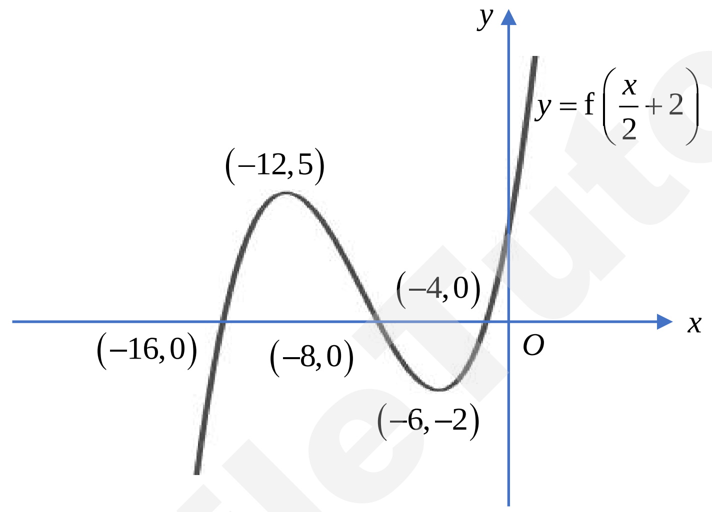
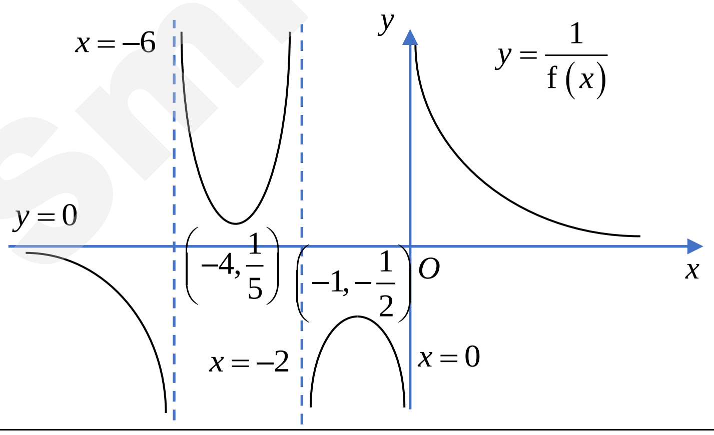
(iii) It is now given that the initial population in a small town is 10,000. The rate of population growth at that time is 100 per year. When the population is 15,000, the population growth rate is 75 per year. Use the *Verhulst model* to find N in terms of t.

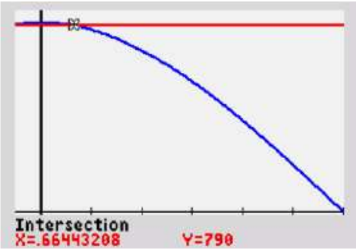
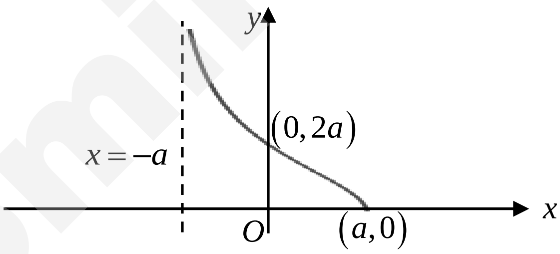
Hence, sketch the graph of N against t. [5]

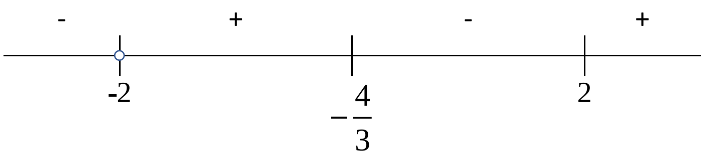
[End of Paper]

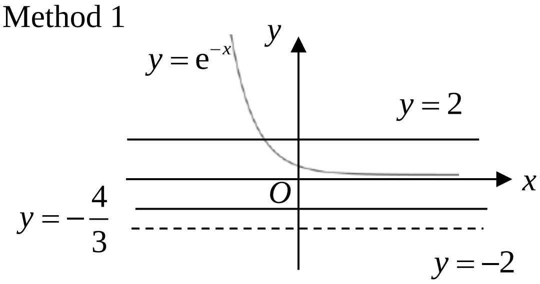
<table>
<tr><td>Qn</td><td>Solutions</td></tr>
<tr><td>1</td><td>Let c, i and l be the amount Abel and Paula spent on a “circuit”, “intervals” and “long run” session in minutes respectively.

$$\begin{aligned} 23c &+ 40i &+ 61l &= 12600 \\ 34c &+ 67(1.1)i &+ 75l &= 17600.5 \\ 33c &+ 53i &+ 87l &= 17725 \end{aligned}$$
By GC, $c = 90$, $i = 65$ and $l = 130$

Abel, Mo and Paula spent 2600 minutes, 4790.5 minutes and 3445 minutes on “Intervals” respectively in the year.</td></tr>
<tr><td>2i</td><td>
</td></tr>
<tr><td>ii</td><td>
</td></tr>
<tr><td>3</td><td>$$\frac{x^2}{64}+\frac{(y-6)^2}{36}=1$$
$$x^2=64\left(1-\frac{(y-6)^2}{36}\right)$$</td></tr>
</table>

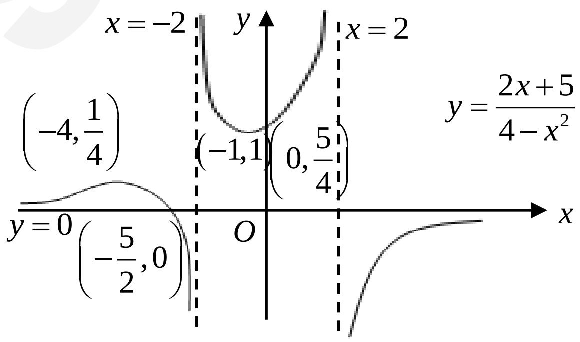
Qn	Solutions
	Volume of bowl $= \pi \int_h^6 64\left(1-\frac{(y-6)^2}{36}\right) dy$ $= 64\pi\left[y-\frac{(y-6)^3}{108}\right]_h^6$ $= 64\pi\left(6-h+\frac{(h-6)^3}{108}\right)$ $= 384\pi - 64\pi h + \frac{16(h-6)^3}{27}$ $384\pi - 64\pi h + \frac{16(h-6)^3}{27} \geq 790$  From the graph, $0 < x \leq 0.664$ Maximum height of the base is 0.664 units.
4i	 Optional: When x = 0, $a\cos 2t = 0 \Rightarrow t = \frac{\pi}{4}$ $y = 2a\tan\left(\frac{\pi}{4}\right) = 2a$ When y = 0, $2a\tan t = 0 \Rightarrow t = 0$ $x = a\cos 0 = a$ As $t \to \frac{\pi}{2}$, $x \to -a, y \to \infty$ $\therefore x = -a$ is an asymptote.
ii	$x = 3\cos(2t)$, $y = 6\tan t$

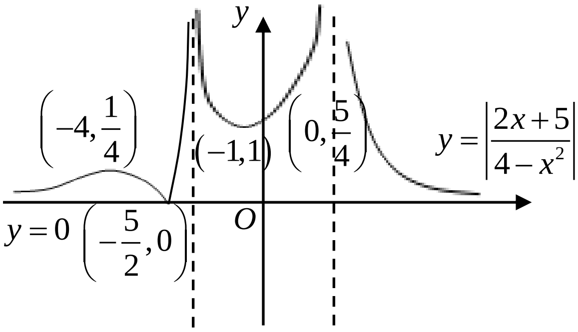
Qn	Solutions
	$\frac{dx}{dt}=-6\sin(2t), \quad \frac{dy}{dt}=6\sec^2 t$ $\frac{dy}{dx}=-\frac{\sec^2 t}{\sin(2t)}$ When $t=\frac{\pi}{3}$, $x=-\frac{3}{2}$, $y=6\sqrt{3}$, $\frac{dy}{dx}=-\frac{2^2}{\frac{\sqrt{3}}{2}}=-\frac{8}{\sqrt{3}}$ $y-6\sqrt{3}=-\frac{8}{\sqrt{3}}\left(x+\frac{3}{2}\right)$ $y=-\frac{8}{\sqrt{3}}x-4\sqrt{3}+6\sqrt{3}$ $y=-\frac{8}{\sqrt{3}}x+2\sqrt{3}$ Equation of tangent is $y=-\frac{8}{\sqrt{3}}x+2\sqrt{3}$
5	$\frac{7x+6}{x+2}\le 3x-1$ $\frac{(3x-1)(x+2)-7x-6}{x+2}\ge 0$ $\frac{3x^2+5x-2-7x-6}{x+2}\ge 0$ $\frac{3x^2-2x-8}{x+2}\ge 0$ $\frac{(x-2)(3x+4)}{x+2}\ge 0$  $-2<x\le-\frac{4}{3}$ or $x\ge 2$
	$\frac{7+6e^x}{1+2e^x}\le 3e^{-x}-1$ $\frac{7e^{-x}+6}{e^{-x}+2}\le 3e^{-x}-1$ Replace x with e^{-x}

Qn	Solutions
	Method 1  $\{x\in\mathbb{R}: x\le -\ln 2\}$ Method 2 $-2<\mathrm{e}^{-x}\le -\frac{4}{3}\left(\text{NA} \because \mathrm{e}^{-x}>0 \text{ for all } x\in\mathbb{R}\right)$ or $\mathrm{e}^{-x}\ge 2$ $-x\ge \ln 2$ $x\le -\ln 2$ $\{x\in\mathbb{R}: x\le -\ln 2\}$
6a	$\left(\frac{\sqrt{2}+\mathrm{i}\sqrt{2}}{1-\mathrm{i}\sqrt{3}}\right)^8=\left(\frac{2\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}}{2\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}\right)^8$ $=\left(\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}}\right)^8$ $=\mathrm{e}^{\frac{14\pi}{3}\mathrm{i}}$ $=\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$ $=-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$
b	$2\mathrm{i}z^*-w=1+7\mathrm{i}\Rightarrow w=2\mathrm{i}z^*-1-7\mathrm{i}$ $z+2w=16-2\mathrm{i}$ $z+2\left(2\mathrm{i}z^*-1-7\mathrm{i}\right)=16-2\mathrm{i}$ Let $z=a+b\mathrm{i},\quad a,b\in\mathbb{R}$ $a+b\mathrm{i}+2\left(2\mathrm{i}\left(a-b\mathrm{i}\right)-1-7\mathrm{i}\right)=16-2\mathrm{i}$ $a+b\mathrm{i}+4a\mathrm{i}+4b-2-14\mathrm{i}=16-2\mathrm{i}$ $a+4b+\mathrm{i}\left(4a+b\right)=18+12\mathrm{i}$ $a+4b=18--(1)$ $4a+b=12--(2)$

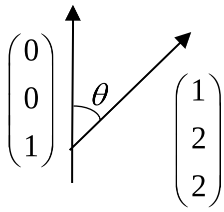
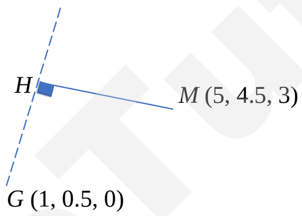
Qn	Solutions
	$(1)\times 4: 4a+16b=72--(3)$ $(3)-(2): 15b=60 \Rightarrow b=4$ $a=18-4(4)=2$ $z=2+4\mathrm{i}$ $w=2\mathrm{i}(2-4\mathrm{i})-1-7\mathrm{i}=7-3\mathrm{i}$
7	$y=\tan^{-1}(ax+1)$ $\tan y = ax+1$ $\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=a \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=a\cos^2 y$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2a\cos y\sin y\frac{\mathrm{d}y}{\mathrm{d}x}=-a\sin 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-2a\cos 2y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-a\sin 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ When x = 0, $y=\tan^{-1}(1)=\frac{\pi}{4}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=a\cos^2\left(\frac{\pi}{4}\right)=\frac{1}{2}a$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-a\sin\left(\frac{\pi}{2}\right)\left(\frac{1}{2}a\right)=-\frac{1}{2}a^2$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0-a\sin\left(\frac{\pi}{2}\right)\left(-\frac{a^2}{2}\right)=\frac{a^3}{2}$ $y=\frac{\pi}{4}+\frac{1}{2}ax+\left(-\frac{a^2}{2}\right)\frac{x^2}{2!}+\left(\frac{a^3}{2}\right)\frac{x^3}{3!}+\dots$ $=\frac{\pi}{4}+\frac{1}{2}ax-\frac{1}{4}a^2x^2+\frac{1}{12}a^3x^3+\dots$ $\sqrt{9-x}=3\left(1-\frac{x}{9}\right)^{\frac{1}{2}}$ $=3\left(1+\frac{1}{2}\left(-\frac{x}{9}\right)+\dots\right)$ $=3-\frac{x}{6}+\dots$ $b\tan^{-1}(ax+1)=\frac{\pi b}{4}+\frac{1}{2}abx+\dots$

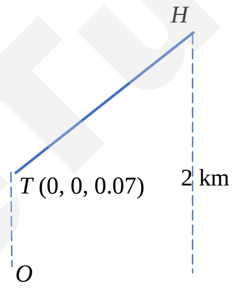
Qn	Solutions
	$\frac{\pi b}{4}=3 \Rightarrow b=\frac{12}{\pi}$ $\frac{1}{2}a\left(\frac{12}{\pi}\right)=-\frac{1}{6} \Rightarrow a=-\frac{\pi}{36}$
8ai	Equation of l is $\underset{\sim}{r}=\begin{pmatrix}1\\6\\-3\end{pmatrix}+s\begin{pmatrix}1\\-2\\2\end{pmatrix}, s\in\mathbb{R}$. Equation of p is $\underset{\sim}{r}\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix}=6$. Substituting equation of l into equation of p, $\left(\begin{pmatrix}1\\6\\-3\end{pmatrix}+s\begin{pmatrix}1\\-2\\2\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix}=6$ $-34+20s=6 \Rightarrow s=2$ $\overrightarrow{OC}=\begin{pmatrix}1\\6\\-3\end{pmatrix}+2\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix}$ $\therefore$ Position vector of C is $3\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}$ (shown).
ii	Since l is perpendicular to line, $\begin{pmatrix}1\\-2\\2\end{pmatrix}$ is perpendicular to the line. Since p is perpendicular to line, $\begin{pmatrix}2\\-3\\6\end{pmatrix}$ is perpendicular to the line. Direction vector of line = $\begin{pmatrix}1\\-2\\2\end{pmatrix}\times\begin{pmatrix}2\\-3\\6\end{pmatrix}=\begin{pmatrix}-6\\-2\\1\end{pmatrix}$ Equation of line is $\underset{\sim}{r}=\begin{pmatrix}3\\2\\1\end{pmatrix}+t\begin{pmatrix}-6\\-2\\1\end{pmatrix}, t\in\mathbb{R}$.
bi	Let θ be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Qn	Solutions
	$\lvert \underset{\sim}{a}\times\underset{\sim}{b}\rvert^2=\left(\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\sin\theta\right)^2$ $=\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2\sin^2\theta$ $=\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2-\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2\cos^2\theta$ $=\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2-\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)^2$ (shown) Alternatively $\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2-\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)^2=\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2-\left(\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta\right)^2$ $=\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2-\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2\cos^2\theta$ $=\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2\left(1-\cos^2\theta\right)$ $=\left(\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\sin\theta\right)^2$ $=\lvert\underset{\sim}{a}\times\underset{\sim}{b}\rvert^2$ (shown)
ii	Area of $\Delta OAB=\frac{1}{2}\lvert\underset{\sim}{a}\times\underset{\sim}{b}\rvert$ $=\frac{1}{2}\sqrt{\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2-\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)^2}$ $=\frac{1}{2}\sqrt{1^2(3)^2-\left(\frac{1}{3}\right)^2}$ $=\frac{1}{2}\sqrt{\frac{80}{9}}$ $=\frac{2\sqrt{5}}{3}$ units^2
iii	$\lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert$ is the length of projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.
9i	
ii	When y = 2, x = 0.82288 or -1.8229 Area formed $=\int_{-1.8229}^{0.82288}2-\frac{2x+5}{4-x^2}\,\mathrm{d}x$ $=1.95$ unit^2

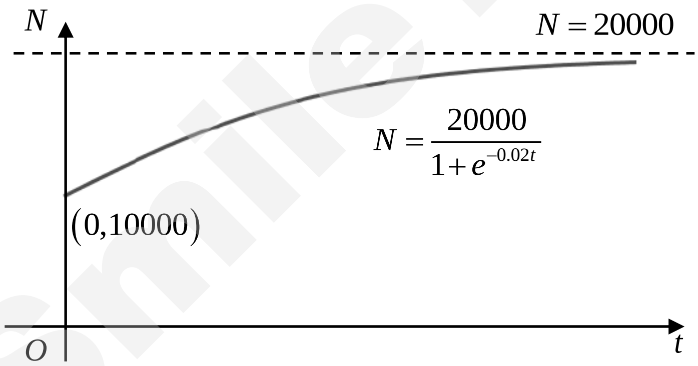
Qn	Solutions
iii	 From graph, $\left\{k\in\mathbb{R}:0<k<\frac{1}{4}\text{ or }1<k<\frac{5}{4}\right\}$.
10ai	Gradient of $PQ=\tan(\pi-\theta)=-\tan\theta$ $y-5=-\tan\theta(x-4)$ $y=(4-x)\tan\theta+5$
ii	When x = 0, $OQ=4\tan\theta+5$ When y = 0, $OP=4+5\cot\theta$ $OP+OQ=4\tan\theta+5\cot\theta+9$ Let $OP+OQ=s$ $\frac{\mathrm{d}s}{\mathrm{d}\theta}=4\sec^2\theta-5\operatorname{cosec}^2\theta$ At stationary value, $4\sec^2\theta-5\operatorname{cosec}^2\theta=0$ $4\tan^2\theta-5=0$ $\tan\theta=\frac{\sqrt{5}}{2}\left(\text{Since }0<\theta<\frac{\pi}{2}\right)$ $s=2\sqrt{5}+2\sqrt{5}+5=4\sqrt{5}+5$ $\frac{\mathrm{d}^2s}{\mathrm{d}\theta^2}=8\sec^2\theta\tan\theta+10\operatorname{cosec}^2\theta\cot\theta$ Since θ is in the first quadrant, $\frac{\mathrm{d}^2s}{\mathrm{d}\theta^2}>0$. The stationary value is a minimum value.
b	$\frac{1}{2}ab=100\Rightarrow b=\frac{200}{a}$ <u>Method 1</u>

Qn	Solutions
	$c^2 = a^2 + b^2$ $= a^2 + \frac{200^2}{a^2}$ $2c\frac{\mathrm{d}c}{\mathrm{d}t} = 2a\frac{\mathrm{d}a}{\mathrm{d}t} - \frac{2(200)^2}{a^3}\frac{\mathrm{d}a}{\mathrm{d}t}$ When $a = 20$, $c = \sqrt{20^2 + \frac{200^2}{20^2}} = \sqrt{500}$ $\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{1}{2\sqrt{500}}\left(2(20)(3) - \frac{2(200)^2}{20^3}(3)\right)$ $= \frac{90}{20\sqrt{5}}$ $= \frac{9}{2\sqrt{5}}$ (or 2.01) <u>Method 2</u> $c = \sqrt{a^2 + \frac{200^2}{a^2}}$ $\frac{\mathrm{d}c}{\mathrm{d}a} = \frac{1}{2}\left(a^2 + \frac{200^2}{a^2}\right)^{-\frac{1}{2}}\left(2a + 200^2\left(-2a^{-3}\right)\right)$ When $a = 20$, $\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{\mathrm{d}c}{\mathrm{d}a}.\frac{\mathrm{d}a}{\mathrm{d}t}$ $= \frac{1}{2}\left(20^2 + \frac{200^2}{20^2}\right)^{-\frac{1}{2}}\left(40 + 200^2\left(\frac{-2}{20^3}\right)\right)(3)$ $= 2.01$
11i	When $t = 0$, $\overrightarrow{OG} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$ The coordinates of G are (1, 0.5, 0). The flight path is a straight line starting from the point (1, 0.5, 0) moving in the direction of $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.
ii	Let θ be the angle the flight path makes with the vertical.

Qn	Solutions
	$\cos\theta = \dfrac{\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{3} = \dfrac{2}{3}$  $\theta = 48.190^\circ$ $(\text{or } 0.84107)$ Required angle = $90^\circ - 48.190^\circ = 41.8^\circ$ (or $\dfrac{\pi}{2} - 0.84107 = 0.730$) The acute angle the flight path makes with the horizontal is 41.8°.
iii	<u>Method 1</u> $\overrightarrow{OH} = \begin{pmatrix}1\\0.5\\0\end{pmatrix} + t\begin{pmatrix}1\\2\\2\end{pmatrix}$ $\overrightarrow{MH} = \begin{pmatrix}-4\\-4\\-3\end{pmatrix} + t\begin{pmatrix}1\\2\\2\end{pmatrix}$  $\left(\begin{pmatrix}-4\\-4\\-3\end{pmatrix} + t\begin{pmatrix}1\\2\\2\end{pmatrix}\right)\cdot\begin{pmatrix}1\\2\\2\end{pmatrix} = 0$ $-18 + 9t = 0 \Rightarrow t = 2$ <u>Method 2</u> $\overrightarrow{GM} = \begin{pmatrix}5\\4.5\\3\end{pmatrix} + \begin{pmatrix}1\\0.5\\0\end{pmatrix} = \begin{pmatrix}4\\4\\3\end{pmatrix}$ Projection of $\overrightarrow{GM}$ on $\overrightarrow{GH}$ $= \left[\begin{pmatrix}4\\4\\3\end{pmatrix}\cdot\dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\2\\2\end{pmatrix}\right]\dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\2\\2\end{pmatrix}$ $= 2\begin{pmatrix}1\\2\\2\end{pmatrix}$ The helicopter will be closest to M 2 min after takeoff.
iv	Equation for flight path of eagle is

Qn	Solutions
	$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 4.5 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, s \in \mathbb{R}$ Suppose the flight paths intersect, $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4.5 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ $1+t = 2+3s --- (1)$ $0.5+2t = 4.5 ---- (2)$ $2t = 2+s --- (3)$ From (2) and (3), t = 2 and s = 2. Substitute into (1): LHS = -4 and RHS = 1 The flight paths will not intersect.
v	When z = 2, t = 1. $\overrightarrow{OH} = \begin{pmatrix} 2 \\ 2.5 \\ 2 \end{pmatrix}$  Let top of control tower be T. $\overrightarrow{TH} = \begin{pmatrix} 2 \\ 2.5 \\ 1.93 \end{pmatrix}$ $TH = 3.7383$ The controller is able to sight the helicopter.
12i	When $N \to K$, $\frac{dN}{dt} \to 0$. K is the population size where there is no change in the population. OR K is the population size that the population will eventually approach.

Qn	Solutions
ii	$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1-\frac{N}{K}\right)$ $= rN\left(\frac{K-N}{K}\right)$ $\frac{K}{N(K-N)}\frac{\mathrm{d}N}{\mathrm{d}t} = r$ $\int \frac{1}{N}+\frac{1}{K-N}\,\mathrm{d}N = rt + C$ $\ln N - \ln\lvert K-N\rvert = rt + C$ $\frac{N}{\lvert K-N\rvert} = A\mathrm{e}^{rt}$ $\frac{N}{K-N} = D\mathrm{e}^{rt}$ $N = KD\mathrm{e}^{rt} - ND\mathrm{e}^{rt}$ $N + ND\mathrm{e}^{rt} = KD\mathrm{e}^{rt}$ $N = \frac{KD\mathrm{e}^{rt}}{1+D\mathrm{e}^{rt}}$ $= \frac{K}{\frac{1}{D}\mathrm{e}^{-rt}+1}$ Let $B = \frac{1}{D}$ $N = \frac{K}{1+B\mathrm{e}^{-rt}}$ Alternatively $\frac{1}{N(K-N)}\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{r}{K}$ $\int \frac{1}{\left(K/2\right)^2-\left(N-K/2\right)^2}\,\mathrm{d}N = \int \frac{r}{K}\,\mathrm{d}t$ $\frac{1}{2\left(\frac{K}{2}\right)}\ln\left\lvert\frac{\frac{K}{2}+\left(N-\frac{K}{2}\right)}{\frac{K}{2}-\left(N-\frac{K}{2}\right)}\right\rvert = \frac{r}{K}t + C$ $\frac{1}{K}\ln\frac{N}{\lvert K-N\rvert} = \frac{r}{K}t + C$

Qn	Solutions
	$\ln\dfrac{N}{\lvert K-N\rvert}=rt+CK$ $\dfrac{N}{\lvert K-N\rvert}=De^{rt}$
iii	When N = 10000, $\dfrac{\mathrm{d}N}{\mathrm{d}t}=10000r\left(1-\dfrac{10000}{K}\right)=100$ $100r-\dfrac{1000000r}{K}=1$ When N = 15000, $\dfrac{\mathrm{d}N}{\mathrm{d}t}=15000r\left(1-\dfrac{15000}{K}\right)=75$ $200r-\dfrac{3000000r}{K}=1$ By GC, $r=\dfrac{1}{50},\dfrac{r}{K}=10^{-6}\Rightarrow K=20000$ When t = 0, $N=\dfrac{20000}{1+B}=10000\Rightarrow B=1$ $N=\dfrac{20000}{1+e^{-0.02t}}$ 

VICTORIA JUNIOR COLLEGE

JC 2 PRELIMINARY EXAMINATION 2018

H2 Mathematics **9758/02**

Paper 2 **3 hours**

Additional Materials: Answer Paper
Graph Paper (Upon Request)
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages

[Turn over

Section A: Pure Mathematics [40 marks]

1 The function f is defined by

$$\mathrm{f}: x \mapsto \lambda + \frac{1}{1-x}\ ,\ x \in \mathbb{R}, x \neq 1,$$

where λ is a negative constant.

(i) Show that f^{-1} exists. [2]

(ii) Find the set of values of λ such that the equation $\mathrm{f}(x) = 2x$ has real solutions. [3]

The functions g and h are defined by

$$\mathrm{g}: x \mapsto \mathrm{f}(x),\ x < 0,$$

$$\mathrm{h}: x \mapsto \left(x - \left(\lambda + \frac{2}{3}\right)\right)^2, x \in \mathbb{R}.$$

(iii) Find the range of hg. [3]

2 **(i)** Show that $2\sin\left(\frac{1}{2}\right)\sin(n) = \cos\left(\frac{2n-1}{2}\right) - \cos\left(\frac{2n+1}{2}\right)$. [2]

(ii) By expressing $\sin(n)$ as $\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left(2\sin\left(\frac{1}{2}\right)\sin(n)\right)$,find

$\sin(1) + \sin(2) + \sin(3) + \ldots + \sin(N)$ in terms of N, where $N \in \mathbb{Z}^+$. [3]

(iii) Explain why the above series will not be equal to $\frac{1}{2}\cot\left(\frac{1}{2}\right)$ for all $N \in \mathbb{Z}^+$. [1]

(iv) Use your answer in part **(ii)** to find the numerical value of $\sum_{n=10}^{25} \sin(n)$. [2]

3 It is given that

$$\mathrm{f}(x) = \begin{cases} 2\sqrt{1+x^2} & \text{for } -1 \leq x \leq 1, \\ 1 & \text{for } 1 < x < 3, \end{cases}$$

and that $\mathrm{f}(x+4) = \mathrm{f}(x)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-1 \leq x < 7$. [4]

(ii) Using the substitution $x = \tan\theta$, show that $\int_3^5 \mathrm{f}(x)\,\mathrm{d}x = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\,\mathrm{d}\theta$.

By writing $\sec^3\theta$ as $\sec\theta\sec^2\theta$, find the value of $\int_3^5 \mathrm{f}(x)\,\mathrm{d}x$, leaving your answer in the form $a + \ln b - \ln c$, where a, b and c are constants to be determined exactly. [8]

4 Refinement is the process that mined material undergoes so as to recover the desired metals and separate all the undesirable minerals. Recovery rate measures the amount of metal extracted, as a percentage of the actual metal content. There are various refinement methods where the mined material undergoes numerous repetitions of separation reactions to reach the desired recovery rate. We will take the desired recovery rate to be 60% for this question.

(i) The following claim is made by a company using Method *A*:
Method *A* recovers 2% of the metal content after the first reaction. For each subsequent reaction, the amount of metal recovered is an additional t % compared to the amount recovered in the previous reaction. Hence, the amount of metal recovered during the first reaction is 2%, the amount of metal recovered during the second reaction is $(2 + t)$%, the amount of metal recovered during the third reaction is $(2 + 2t)$%, and so on.

Find the value of t such that the total percentage of metals recovered after 20 reactions reaches the desired recovery rate. [2]

Give a reason why the claim made by the company may not be realistic in the long run. [1]

(ii) Method *B* recovers 10% of the initial amount of the desired metals after the first reaction, but each subsequent reaction will recover r times as much as the previous reaction. If 10 reactions are required to reach the same desired recovery rate, show that r satisfies the equation $r^{10} - 6r + 5 = 0$. Explain why r cannot be 1, even though $r = 1$ is a root of this equation. Find the value of r. [4]

Each reaction for method *B* will require chemical *L*. At the start of the refinement process, there is 100 kg of chemical *L* and after each reaction, 30% of *L* will be used. 15 kg of chemical *L* will be added at the start of the next reaction.

(iii) Calculate the amount of chemical *L* left after 3 reactions. [2]

(iv) Express the amount of chemical *L* left after n reactions in the form $k\left(0.7\right)^{n-1}+m$, where k and m are constants to be determined. Hence, find the amount of chemical *L* left at the end of a reaction in the long run. [3]

Section B: Statistics [60 marks]

5 Twelve cards, numbered from 1 to 12 are arranged in a straight line. Find the number of ways this can be done if

(i) there are no restrictions, [1]

(ii) the cards numbered 2 and 3 are together, and all the six even numbered cards are adjacent. [3]

The twelve cards are arranged in a circle. Find the number of ways this can be done if all the cards numbered as multiples of 3 are separated. [2]

Six of the cards are selected at random, without replacement. Find the probability that at least two of the chosen cards are even numbered. [3]

6 A wildlife biologist wishes to test frogs for a genetic trait. He caught 1200 frogs from the wild and randomly packed them into boxes of 6. The number, x, of frogs found with the genetic trait in each box is recorded. The results obtained from 200 boxes are shown in the table below.

Number of frogs in box with genetic trait (x)	0	1	2	3	4	5	6
Number of occurrences	17	53	65	45	18	2	0

(i) Calculate, from the above data, the mean value of x. [1]

(ii) State, in context, two assumptions needed for the number of frogs carrying the genetic traits in a box to be well modelled by a binomial distribution. [2]

(iii) Assuming that X can be modelled by a binomial distribution having the same mean as the one calculated in part **(i)**, state the values for the binomial parameters n and p. [1]

Another 60 frogs are caught from the same place and packed randomly into ten boxes of 6. Find the probability that at least three of the boxes contain exactly 2 frogs with the genetic trait. [3]

7 A machine in a factory produces steel rods. The machine is reset periodically. After resetting, it should produce rods with mean length 27.0 cm. In order to test whether the mean length of rods produced differs significantly from 27.0 cm, a sample of fifty rods is taken and their lengths, x cm, are summarised as follows.

$$\sum(x-27)=12.0, \quad \sum(x-27)^2=36.4.$$

(i) Calculate unbiased estimates of the population mean and variance of the lengths of steel rods. [2]

(ii) Carry out an appropriate test at the 2.5% level of significance, explaining whether there is a need for the population distribution of the lengths of the steel rods to be known. [5]

(iii) Explain, in the context of the question, the meaning of 'at the 2.5% level of significance'. [1]

Another machine in the factory produces steel rods with mean length l cm and standard deviation 3 cm. In order to carry out a test to determine whether the mean length of rods produced is more than l cm at 5% significance level, a sample of fifty rods is taken and the sample mean is found to be 23.8 cm.

Given that the test concluded the population mean is more than l cm , find the set of possible values of l. [4]

8 The probability of obtaining a '6' on a biased cubical dice is thrice the probability of rolling every other number on the dice.

Show that probability of obtaining a '6' on the biased dice is $\frac{3}{8}$. [1]

This biased dice is put into a bag together with 3 fair cubical dice.

(i) One of the dice is chosen randomly from the bag and rolled once. Find the probability of obtaining a '6'. [2]

(ii) One of the dice is chosen randomly and is rolled n times.

(a) Find the probability that '6' is obtained on all the n rolls. [1]

(b) Given that '6' is obtained for all the n rolls, the probability that the biased dice is chosen is more than 0.95. By forming an inequality in terms of n, solve for the least value of n. [3]

(iii) One of the dice is chosen randomly from the bag and rolled once. The dice is then put back into the bag. Another dice is chosen randomly from the bag and rolled once.

For every '6' obtained, the player gains \$2, otherwise the player loses \$$k$. If the game is fair, that is, expected winnings of the player is \$0, show that the value of k is 0.56. [3]

Find the variance of the player's winnings. [2]

9 AppleC is a fruit farm that produces Cameo apples. It is known that 8.1% of the apples have a mass more than 125 g and 14.7% of the apples have a mass less than 90 g. It is also known that the masses of this batch of Cameo apples follows a normal distribution.

(i) Show that the mean and standard deviation of the masses for this batch of Cameo apples, correct to 3 significant figures, are 105 g and 14.3 g respectively. [4]

The Cameo apples are packed into bags of 8 apples.

(ii) Find the probability that the mass of a randomly chosen bag of Cameo apples is less than 845 g. State the distribution used and its parameters. [3]

A local distributor imports Cameo apples from AppleC. The distributor also imports Jazz apples from another supplier. It is known that the masses of bags of Jazz apples follow a normal distribution with mean 700 g with standard deviation of 35 g. The retail price of Cameo apples and Jazz apples are \$8.50 per kg and \$7.30 per kg respectively.

(iv) Find the probability that the cost of three bags of Jazz apples is more than twice the cost of a bag of Cameo apples by at least \$1. [4]

10 In an agricultural experiment, a certain fertilizer is applied at different rates to ten identical plots of land. Seeds of a type of grass are then sown and several weeks later, the mean height of the grass on each plot is measured. The results are shown in the table.

Rate of application of fertilizer, x g/m^2	10	20	30	40	50	60	70	80	90	100
Mean height of grass, y cm	6.2	11.4	13.2	14.8	15.8	17.0	19.4	19.4	20.6	20.8

(i) Draw the scatter diagram for these values, labelling the axes clearly. [1]

It is thought that the mean height of grass, y cm, can be modelled by one of the formulae

$$y = ax + b \quad \text{or} \quad y = c\ln x + d$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) x and y, **(b)** $\ln x$ and y. [2]

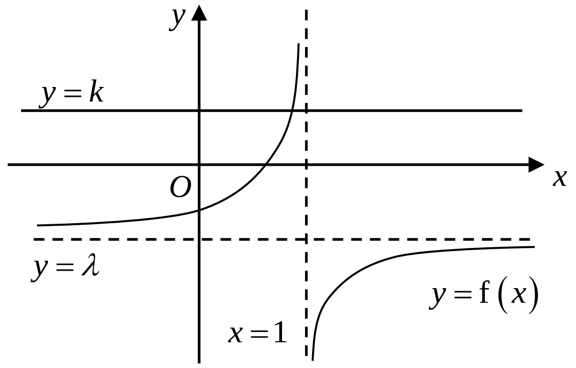
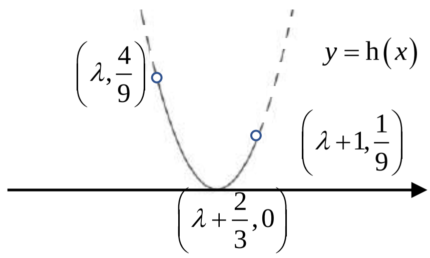
(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $y = ax + b$ or $y = c\ln x + d$ is the better model. [2]

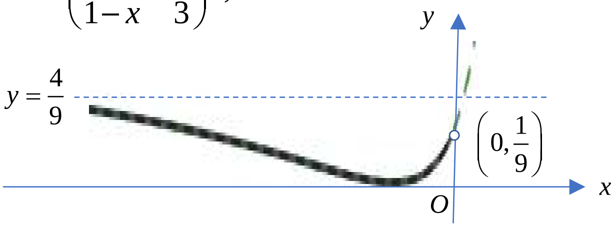
It is required to estimate the value of x for which $y = 17.2$.

(iv) Explain why neither the regression line of x on y nor the regression line of $\ln x$ on y should be used. [1]

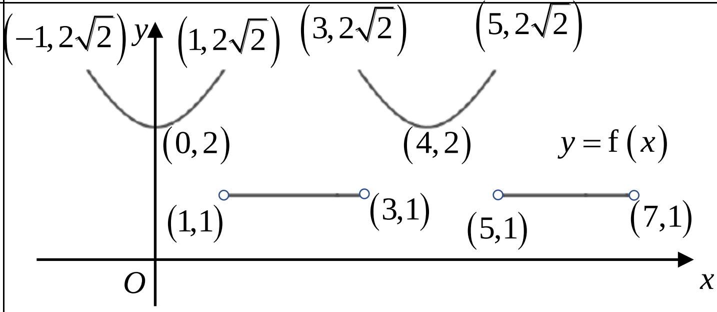
(v) Find the equation of a suitable regression line and use it to find the required estimate. [3]

[End of Paper]

Qn	Solution
1i	 Since <u>every</u> horizontal line $y = k$, $\underline{k \in \mathrm{R_f}}$, cuts the graph of $y=\mathrm{f}(x)$ exactly once, <u>f is one-one</u>. Hence, f^{-1} exists.
ii	$\lambda+\frac{1}{1-x}=2x$ $\lambda-\lambda x+1=2x-2x^2$ $2x^2-(\lambda+2)x+\lambda+1=0$ $\left[-(\lambda+2)\right]^2-4(2)(\lambda+1)\geq 0$ $\lambda^2+4\lambda+4-8\lambda-8\geq 0$ $\lambda^2-4\lambda-4\geq 0$ $\lambda \leq -0.82843$ or $\lambda \geq 4.8284$ Since $\lambda<0$, $\{\lambda \in \mathbb{R} : \lambda \leq -0.828\}$.
iii	Method A: $\mathrm{R_g} = (\lambda, \lambda+1)$ Using $\mathrm{R_g}$ as the domain of h,  (Observe that the coordinates of the minimum point of $\left(x-\left(\lambda+\frac{2}{3}\right)\right)^2$ are $\left(\lambda+\frac{2}{3}, 0\right)$.)

Qn	Solution
	$R_{hg} = \left[0, \frac{4}{9}\right)$. Method B: $hg: x \mapsto \left(\frac{1}{1-x} - \frac{2}{3}\right)^2, x < 0$  $R_{hg} = \left[0, \frac{4}{9}\right)$.
2i	$\cos\left(\frac{2n-1}{2}\right) - \cos\left(\frac{2n+1}{2}\right)$ $= -2\sin\left(\frac{1}{2}\left(\frac{2n-1}{2} + \frac{2n+1}{2}\right)\right)$ $\sin\left(\frac{1}{2}\left(\frac{2n-1}{2} - \frac{2n+1}{2}\right)\right)$ $= -2\sin\left(\frac{4n}{4}\right)\sin\left(-\frac{2}{4}\right)$ $= -2\sin(n)\left(-\sin\left(\frac{1}{2}\right)\right)$ $= 2\sin\left(\frac{1}{2}\right)\sin(n)$ Alternatively, $2\sin\left(\frac{1}{2}\right)\sin(n) = -(\cos P - \cos Q)$ $= 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$ Let $\frac{P+Q}{2} = n$ and $\frac{P-Q}{2} = \frac{1}{2}$. $P + Q = 2n$ $P - Q = 1$ $2P = 2n + 1 \Rightarrow P = \frac{2n+1}{2}$ $2Q = 2n - 1 \Rightarrow Q = \frac{2n-1}{2}$

Qn	Solution
	$2\sin\left(\frac{1}{2}\right)\sin(n)=\cos Q-\cos P$ $=\cos\left(\frac{2n-1}{2}\right)-\cos\left(\frac{2n+1}{2}\right)$
ii	$\sum_{n=1}^{N}\sin(n)$ $=\sum_{n=1}^{N}\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left(2\sin\left(\frac{1}{2}\right)\sin(n)\right)$ $=\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\sum_{n=1}^{N}\left(\cos\left(\frac{2n-1}{2}\right)-\cos\left(\frac{2n+1}{2}\right)\right)$ $=\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left(\begin{array}{lccc} & \cos\left(\frac{1}{2}\right) & - & \cancel{\cos\left(\frac{3}{2}\right)} \\ + & \cancel{\cos\left(\frac{3}{2}\right)} & - & \cancel{\cos\left(\frac{5}{2}\right)} \\ & & \vdots & \\ + & \cancel{\cos\left(\frac{2N-3}{2}\right)} & - & \cancel{\cos\left(\frac{2N-1}{2}\right)} \\ + & \cancel{\cos\left(\frac{2N-1}{2}\right)} & - & \cos\left(\frac{2N+1}{2}\right) \end{array}\right)$ $=\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left(\cos\left(\frac{1}{2}\right)-\cos\left(\frac{2N+1}{2}\right)\right)$ $=\frac{1}{2}\cot\left(\frac{1}{2}\right)-\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\cos\left(\frac{2N+1}{2}\right)$
iii	Since $\frac{2N+1}{2}$ is not equal to odd multiples of $\frac{\pi}{2}$ for all positive integers *N*, $\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\cos\left(\frac{2N+1}{2}\right)\neq 0.$ Hence, the series will not equal $\frac{1}{2}\cot\left(\frac{1}{2}\right)$ for all $N\in\mathbb{Z}^{+}$.
iv	$\sum_{n=10}^{25}\sin(n)$ $=\sum_{n=1}^{25}\sin(n)-\sum_{n=1}^{9}\sin(n)$ $=\frac{1}{2}\cot\left(\frac{1}{2}\right)-\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\cos\left(\frac{51}{2}\right)-\left(\frac{1}{2}\cot\left(\frac{1}{2}\right)-\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\cos\left(\frac{19}{2}\right)\right)$ $=-2.01$

Qn	Solution	
3i		
ii	$\int_3^5 \mathrm{f}(x)\,\mathrm{d}x$ $=\int_{-1}^{1} 2\sqrt{1+x^2}\,\mathrm{d}x$ $=2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1+\tan^2\theta}\sec^2\theta\,\mathrm{d}\theta$ $=2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \lvert\sec\theta\rvert\sec^2\theta\,\mathrm{d}\theta$	$x=\tan\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2\theta$ When $x=-1, \theta=-\frac{\pi}{4}$ When $x=1, \theta=\frac{\pi}{4}$
	Since $\sec\theta>0$ when $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$, $\lvert\sec\theta\rvert=\sec\theta$ $\int_3^5 \mathrm{f}(x)\,\mathrm{d}x=2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^3\theta\,\mathrm{d}\theta$	
	$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^3\theta\,\mathrm{d}\theta$ $=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^2\theta\sec\theta\,\mathrm{d}\theta$ $=\left[\tan\theta\sec\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}-\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec\theta\tan^2\theta\,\mathrm{d}\theta$ $=\left(\sqrt{2}+\sqrt{2}\right)-\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec\theta\left(\sec^2\theta-1\right)\mathrm{d}\theta$ $=2\sqrt{2}-\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^3\theta\,\mathrm{d}\theta+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec\theta\,\mathrm{d}\theta$ $2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^3\theta\,\mathrm{d}\theta=2\sqrt{2}+\left[\ln\lvert\sec\theta+\tan\theta\rvert\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$ $=2\sqrt{2}+\ln\left(\sqrt{2}+1\right)-\ln\left(\sqrt{2}-1\right)$ $\int_3^5 \mathrm{f}(x)\,\mathrm{d}x=2\sqrt{2}+\ln\left(\sqrt{2}+1\right)-\ln\left(\sqrt{2}-1\right)$	

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>4i</td><td>$\frac{20}{2}\left(2(2)+19t\right)=60$
$40+190t=60$
$t=\frac{2}{19}$
If method A’s claim is realistic, we will be able to recover more than 100% of the actual metal content.</td></tr>
<tr><td>ii</td><td>$\frac{10\left(1-r^{10}\right)}{1-r}=60$
$1-r^{10}=6-6r$
$r^{10}-6r+5=0$

If r = 1, we will recover 100% after 10 reactions, instead of 60%.

By GC, r = 0.879</td></tr>
<tr><td>iii</td><td><table><tr><th>n</th><th>Before reaction</th><th>After reaction</th></tr><tr><td>1</td><td>100</td><td>0.7(100)</td></tr><tr><td>2</td><td>0.7(100) + 15</td><td>$0.7^2(100) + 15(0.7)$</td></tr><tr><td>3</td><td>$0.7^2(100) + 15(0.7) +15$</td><td>$0.7^3(100) + 15(0.7)^2 +15(0.7)$</td></tr></table>
Amount left after 3 reactions = 52.15 kg</td></tr>
<tr><td>iv</td><td>Amount left after n reactions
$=(0.7)^n(100)+15\left(0.7+0.7^2+\ldots+0.7^{n-1}\right)$
$=(0.7)^n(100)+\frac{15(0.7)\left(1-0.7^{n-1}\right)}{1-0.7}$
$=(0.7)^n(100)+35\left(1-0.7^{n-1}\right)$
$=70(0.7)^{n-1}+35-35(0.7)^{n-1}$
$=35(0.7)^{n-1}+35$

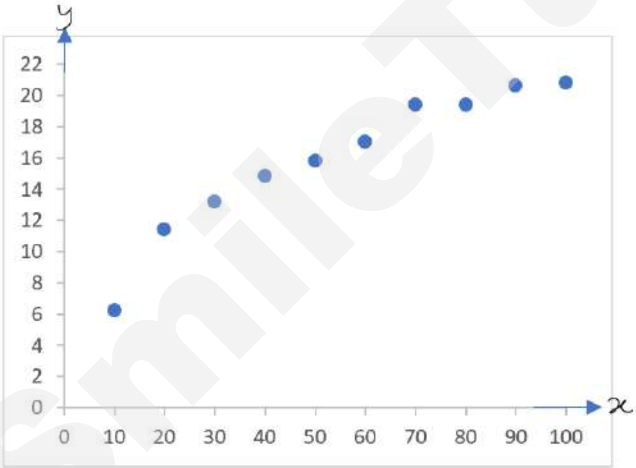
When $n\to\infty,\ 20(0.7)^{n-1}\to 0.$
Amount left at the end of a reaction in the long run is 35 kg.</td></tr>
<tr><td>5i</td><td>12! = 479001600</td></tr>
<tr><td>ii</td><td>$2!\times 5!\times 6!=172800$</td></tr>
<tr><td></td><td>$1\times 7!\times {}^{8}\mathrm{P}_4=8467200$</td></tr>
</table>

Qn	Solution
	$1-\dfrac{{}^{6}C_0 \times {}^{6}C_6 + {}^{6}C_1 \times {}^{6}C_5}{{}^{12}C_6} = \dfrac{887}{924}\left(\text{or } 0.960\right)$
6i	$\bar{x}=2$
ii	The probability that a frog carries the genetic trait is constant for all frogs **in a box.** The presence of the genetic trait in a frog is independent among the frogs **in a box**.
iii	$n=6,\ p=\dfrac{1}{3}$
	$X \sim \mathrm{B}\left(6, \dfrac{1}{3}\right)$ $\mathrm{P}\left(X=2\right)=0.32922$ Let Y be the number of boxes that contain exactly 2 frogs with the genetic trait. $Y \sim \mathrm{B}\left(10, 0.32922\right)$ $\mathrm{P}\left(Y>3\right)=1-\mathrm{P}\left(Y\le 2\right)=0.691$
7i	Let $y=x-27$ Unbiased estimate of population mean $=\bar{x}$ $=\bar{y}+27$ $=\dfrac{\sum y}{n}+27$ $=\dfrac{12}{50}+27$ $=27.24$ Unbiased estimate of population variance $=s^2$ $=\dfrac{1}{n-1}\left(\sum y^2-\dfrac{\left(\sum y\right)^2}{n}\right)$ $=\dfrac{1}{49}\left(36.4-\dfrac{12^2}{50}\right)$ $=\dfrac{33.52}{49}$ $=0.68408$ $=0.684 \quad (3\text{ s.f})$
ii	Let X cm be the length of a steel rod. Let μ cm be the population mean length of steel rods.

Qn	Solution
	$H_0 : \mu = 27$ $H_1 : \mu \neq 27$ Level of significance: 2.5% Test Statistic: since $n = 50$ is sufficiently large, by Central Limit Theorem, $\bar{X}$ is approximately normal. When H_0 is true, $Z = \dfrac{\bar{X} - 27}{\frac{S}{\sqrt{50}}} \sim N(0,1)$ approximately Computation: $\bar{x} = 27.24$, $s = \sqrt{0.68408} = 0.82709$, $p - \text{value} = 0.040185 = 0.0402$ (3 s.f) Conclusion : Since $p - \text{value} = 0.0402 > 0.025$, H_0 is not rejected at 2.5% level of significance. Hence there is insufficient evidence that the population mean length of steel rods produced differs from 27.0 cm. Since *n* = 50 is large enough, by Central Limit Theorem, sample mean lengths of steel rods is approximately normal. Hence, there is no need for the population distribution of lengths of steel rods to be known.
iii	There is a probability of 0.025 that the test will conclude that the population mean length of steel rods produced differs from 27.0 cm when it is actually 27.0 cm.
	$H_0 : \mu = l$ $H_1 : \mu > l$ Level of significance: 5% Test Statistic: since $n = 50$ is sufficiently large, by Central Limit Theorem, $\bar{X}$ is approximately normal. When H_0 is true, $Z = \dfrac{\bar{X} - l}{\frac{3}{\sqrt{50}}} \sim N(0,1)$ approximately . Computation: To reject H_0, $z \geq 1.6449$ $\dfrac{23.8 - l}{\frac{3}{\sqrt{50}}} \geq 1.6449$ $l \leq 23.102$ $\{l \in \mathbb{R}^+ : l \leq 23.1\}$

Qn	Solution
8	Let the probability of obtaining a '6' be k. $5\left(\frac{1}{3}k\right)+k=1$ $\frac{8}{3}k=1\Rightarrow k=\frac{3}{8}$
i	P(obtaining a '6') = $\frac{1}{4}\times\frac{3}{8}+\frac{3}{4}\times\frac{1}{6}$ $=0.21875\left(\text{or }\frac{7}{32}\right)$
iia	P(obtaining n '6's) = $\frac{1}{4}\times\left(\frac{3}{8}\right)^n+\frac{3}{4}\times\left(\frac{1}{6}\right)^n$
b	$\dfrac{\frac{1}{4}\times\left(\frac{3}{8}\right)^n}{\frac{1}{4}\times\left(\frac{3}{8}\right)^n+\frac{3}{4}\times\left(\frac{1}{6}\right)^n}>0.95$ <table><tr><td>n</td><td>Probability</td></tr><tr><td>4</td><td>0.89521</td></tr><tr><td>5</td><td>0.95055</td></tr></table> Least n is 5.
iii	Let \$$X$ be the winnings of the player. <table><tr><td>x</td><td>-2k</td><td>$2-k$</td><td>4</td></tr><tr><td>P($X=x$)</td><td>$\left(\frac{25}{32}\right)^2=\frac{625}{1024}$</td><td>$2\left(\frac{7}{32}\right)\left(\frac{25}{32}\right)=\frac{175}{512}$</td><td>$\left(\frac{7}{32}\right)^2=\frac{49}{1024}$</td></tr></table> $\mathrm{E}(X)=-2k\left(\frac{625}{1024}\right)+(2-k)\left(\frac{175}{512}\right)+4\left(\frac{49}{1024}\right)$ $0=-\frac{625k}{512}+\frac{175}{256}-\frac{175k}{512}+\frac{49}{256}$ $\frac{25}{16}k=\frac{7}{8}$ $k=0.56$ <table><tr><td>x</td><td>-1.12</td><td>1.44</td><td>4</td></tr><tr><td>P($X=x$)</td><td>$\frac{625}{1024}$</td><td>$\frac{175}{512}$</td><td>$\frac{49}{1024}$</td></tr></table> $\mathrm{Var}(X)=2.24$ The variance of the player's winnings is \$^{2}2.24.

Qn	Solution
	Alternatively, Let \$$A$ be the winnings from randomly choosing one dice from the bag and rolled once. Let \$$B$ be the winnings of the player. $B = A_1 + A_2$ $\mathrm{E}(B) = \mathrm{E}(A_1 + A_2)$ $= 2\mathrm{E}(A)$ Given that the expected winnings is \$0, $\mathrm{E}(B) = 0 \Rightarrow 2\mathrm{E}(A) = 0 \Rightarrow \mathrm{E}(A) = 0$ <table><tr><td>a</td><td>$-k$</td><td>2</td></tr><tr><td>P(A = a)</td><td>$\frac{25}{32}$</td><td>$\frac{7}{32}$</td></tr></table> $\frac{25}{32}(-k) + \frac{7}{32}(2) = 0$ $\Rightarrow k = 0.56$ (shown) $\mathrm{Var}(B) = \mathrm{Var}(A_1 + A_2)$ $= 2\mathrm{Var}(A)$ <table><tr><td>a</td><td>-0.56</td><td>2</td></tr><tr><td>P(A = a)</td><td>$\frac{25}{32}$</td><td>$\frac{7}{32}$</td></tr></table>By GC, Var(A) = 1.12 Var(B) = 2.24 The variance of the player's winnings is \$^{2}2.24.
9i	Let X g be the mass of a randomly chosen Cameo Apple. $X \sim \mathrm{N}(\mu, \sigma^2)$ <table><tr><td>$\mathrm{P}(X > 125) = 0.081$ $\mathrm{P}\left(Z > \frac{125-\mu}{\sigma}\right) = 0.081$ $\frac{125-\mu}{\sigma} = 1.3984$</td><td>$\mathrm{P}(X < 90) = 0.147$ $\mathrm{P}\left(Z > \frac{90-\mu}{\sigma}\right) = 0.147$ $\frac{90-\mu}{\sigma} = -1.0494$</td></tr></table>By GC, $\mu = 105.00, \quad \sigma = 14.298$ $\approx 105 \qquad \approx 14.3$
ii	Let C g be the mass of a randomly bag of Cameo apples.

Qn	Solution
	$C = X_1 + X_2 + \ldots + X_8$ $C \sim \mathrm{N}(840, 1635.92)$ $\mathrm{P}(C < 845) = 0.54919 \approx 0.549\ (3\text{sf})$
iii	Let J g be the mass of a randomly chosen bag of Jazz apples. $Y = \frac{7.30}{1000}(J_1 + J_2 + J_3) - \frac{8.50}{1000}(2C)$ $\mathrm{E}(Y) = \frac{7.30}{1000}(3)(700) - \frac{8.50}{1000}(2)(840)$ $= 1.05$ $\mathrm{Var}(Y)$ $= \left(\frac{7.30}{1000}\right)^2 (3)(35)^2 + \left[\frac{8.50}{1000}(2)\right]^2 (1635.92)$ $= 0.66862$ $Y \sim \mathrm{N}(1.05, 0.66862)$ $\mathrm{P}(Y > 1) = 0.524$
10 i	
iia	$r = 0.9541$ (4 d.p)
b	$r = 0.9942$ (4 d.p)
iii	From the scatter diagram, as x increases, y increases by decreasing amounts. In addition, the product moment correlation coefficient between $\ln x$ and y, 0.9942, is closer to 1 as compared to that between x and y, 0.9541. Hence $y = c + d \ln x$ is the better model.
iv	Since x is the independent variable, neither the regression line of x on y nor the regression line of $\ln x$ on y should be used to estimate the value of x when $y = 17.2$.
v	Equation of regression line of y on $\ln x$ is

Qn	Solution
	$y = 6.3074\ln x - 8.1904$ $y = 6.31\ln x - 8.19$ When y = 17.2, $17.2 = -8.1904 + 6.3074\ln x$ $x = 56.008$ $= 56.0$