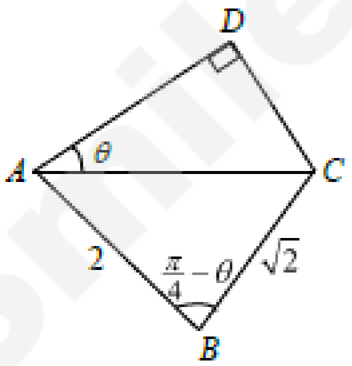
2017

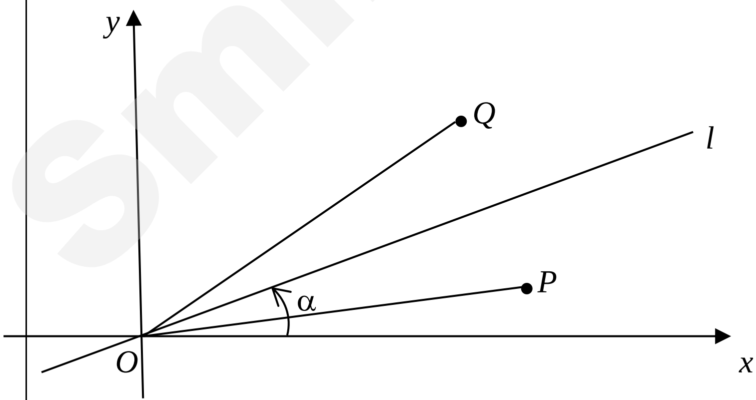
H2 Math

1.	Anderson Serangoon Junior College	
2.	Anglo Chinese Junior College	
3.	Catholic Junior College	
4.	Dunman High School	
5.	Hwa Chong Institution	
6.	Innova Junior College	
7.	Jurong Junior College	
8.	Meridian Junior College	
9.	Millennia Institute	
10.	Nanyang Junior College	
11.	National Junior College	
12.	Pioneer Junior College	
13.	Raffles Institution	
14.	River Valley High School	
15.	Serangoon Junior College	
16.	St. Andrew's Junior College	
17.	Tampines Junior College	
18.	Temasek Junior College	
19.	Victoria Junior College	
20.	Yishun Junior College	

ANDERSON JUNIOR COLLEGE
2017 Preliminary Examination
H2 Mathematics Paper 1 (9758/01)

Duration: 3 hours

1	Mr Tan invested a total of \$25,000 in a structured deposit account, bonds and an estate fund. He invested \$7,000 more in bonds than in estate fund. The projected annual interest rates for structured deposit account, bonds and estate fund are 2%, 3% and 4.5% respectively. Money that is not drawn out at the end of the year will be re-invested for the following year. Mr Tan plans to draw out his money from all investments at the end of the second year and estimates that he will receive a total of \$26,300. Find the amount of money Mr Tan invested in each investment, giving your answer to the nearest dollar. [5]
2	Show that the differential equation $$\frac{dy}{dx}+\frac{3xy}{1-3x^2}-x+1=0$$ may be reduced by means of the substitution $y=u\sqrt{1-3x^2}$ to $$\frac{du}{dx}=\frac{x-1}{\sqrt{1-3x^2}}.$$ Hence find the general solution for y in terms of x. [5]
3	 The diagram above shows a quadrilateral $ABCD$, where AB = 2, $BC=\sqrt{2}$, angle $ABC=\frac{\pi}{4}-\theta$ radians and angle $CAD=\theta$ radians. Show that $$AC=\sqrt{6-4\cos\theta-4\sin\theta}.$$ [2] Given that θ is small enough for θ^3 and higher powers of θ to be neglected, show that $$AD\approx a+b\theta+c\theta^2,$$ where a, b and c are constants to be determined. [5]

4	**(a)** Given that $\sum_{n=1}^{N}\frac{1}{4n^2-1}=\frac{1}{2}-\frac{1}{2(2N+1)}$, find $\sum_{n=1}^{2N}\frac{1}{4(n+1)^2-1}$. Deduce that $\sum_{n=1}^{2N}\frac{1}{(2n+3)^2}$ is less than $\frac{1}{6}$. [5] **(b)** The sum to n terms of a series is given by $S_n = n\ln 2-\frac{n^2-1}{\mathrm{e}}$. Find an expression for the n^{th} term of the series, in terms of n. Show that the terms of the series follow an arithmetic progression. [4]
5	A curve C has equation $y = \mathrm{f}(x)$. The equation of the tangent to the curve C at the point where $x = 0$ is given by $2x - ay = 3$ where a is a positive constant. It is also given that $y = \mathrm{f}(x)$ satisfies the equation $(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ and that the third term in the Maclaurin's expansion of f(x) is $\frac{1}{3}x^2$. Find the value of a. Hence, find the Maclaurin's series for f(x) in ascending powers of x, up to and including the term in x^3. [7]
6	The diagram below shows the line l that passes through the origin and makes an angle α with the positive real axis, where $0<\alpha<\frac{\pi}{2}$. Point P represents the complex number z_1 where $0<\arg z_1<\alpha$ and length of OP is r units. Point P is reflected in line l to produce point Q, which represents the complex number z_2.  Prove that arg z_1 + arg z_2 = 2α. [2] Deduce that $z_1z_2 = r^2(\cos 2\alpha + i\sin 2\alpha)$. [1] Let R be the point that represents the complex number z_1z_2. Given that $\alpha=\frac{\pi}{4}$, write down the cartesian equation of the locus of R as z_1 varies. [2]

7

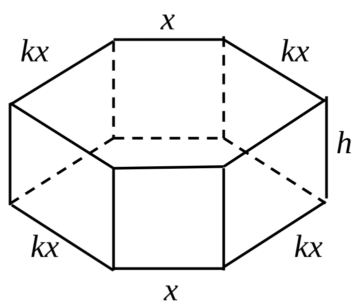


Fig 1

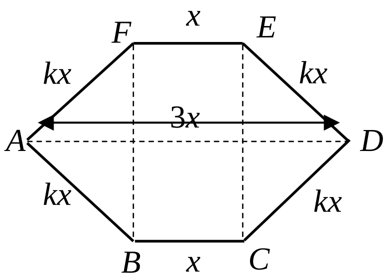


Fig 2

Figure 1 shows a solid metal hexagonal prism of height h cm. Figure 2 shows the hexagonal cross-section $ABCDEF$ of the prism where $AD = 3x$ cm, $BC = FE = x$ cm and the remaining 4 sides are of length kx cm each, where k is a constant.

Show that

$$S = 8x^2\sqrt{k^2-1} + 2xh(1+2k),$$ [3]

where S is the surface area of this solid hexagonal prism.

(a) If the volume of the prism is fixed at 400 cm^3, use differentiation to find, in terms of k, the exact value of x that gives a stationary value of S. [3]

Let $k = 2$.

(b) The prism is heated and it expands in such a way that, at time t seconds, the rate of increase of x is the same as the rate of increase of its height h. At the instant when $x = 3$, the prism's height is 8 cm and its surface area is increasing at a constant rate of 0.5 cm^2/s. Find the rate of change of the volume of the prism at this instant. [6]

8

The curve C has equation $y = \dfrac{4x^2 - kx + 2}{x-2}$, where k is a constant.

(i) Show that curve C has stationary points when $k < 9$. [3]

(ii) Sketch the graph of C for the case where $6 < k < 9$, clearly indicating any asymptotes and points of intersection with the axes. [4]

(iii) Describe a sequence of transformations which transforms the graph of $y = 2x + \dfrac{1}{x}$ to the graph of $y = \dfrac{4x^2 - 8x + 2}{x-2}$. [3]

(iv) By drawing a suitable graph on the same diagram as the graph of C, solve the inequality

$$\frac{4x^2 - 8x + 2}{x-2} > \frac{1}{x^2}.$$ [3]

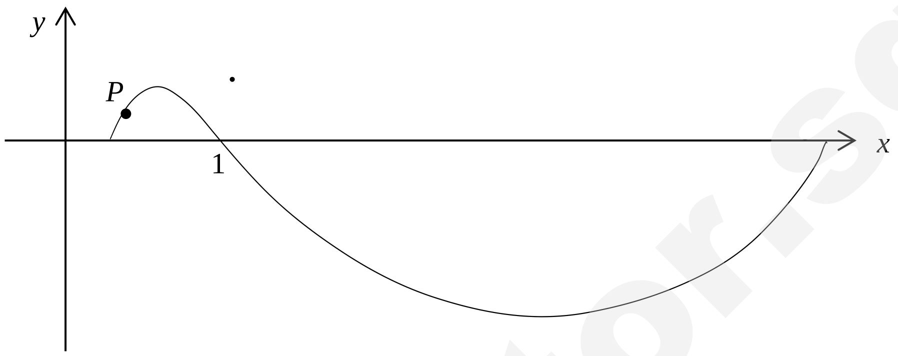
9	The position vectors of A, B and C with respect to the origin O are **a, b** and **c** respectively. It is given that $\overrightarrow{AC} = 4\overrightarrow{CB}$ and $\lvert \mathbf{a}+\mathbf{b} \rvert^2 = \lvert \mathbf{a} \rvert^2 + \lvert \mathbf{b} \rvert^2$. **(i)** By considering $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})$, show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. [2] **(ii)** Find the length of the projection of **c** on **a** in terms of $\lvert \mathbf{a} \rvert$. [3] **(iii)** Given that F is the foot of the perpendicular from C to OA and **f** denotes the position vector $\overrightarrow{OF}$, state the geometrical meaning of $\lvert \mathbf{c} \times \mathbf{f} \rvert$. [1] **(iv)** Two points X and Y move along line segments OA and AB respectively such that $$\overrightarrow{OX} = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + \frac{1}{2}\mathbf{k},$$ $$\overrightarrow{OY} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} - 2\mathbf{k},$$ where t is a real parameter, $0 \le t \le 2\pi$. By expressing the scalar product of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ in the form of $p\sin(qt)+r$ where p, q and r are real values to be determined, find the greatest value of the angle XOY. [5]
10	There are 25 toll stations, represented by T_1, T_2, T_3,……, T_{25} along a 2000 km stretch of highway. T_1 is located at the start of the highway and T_2 is located x km from T_1. Subsequently, the distance between two consecutive toll stations is 2 km more than the previous distance. Find the range of values x can take. [3] Use $x = 60$ for the rest of this question. Each toll station charges a fee based on the distance travelled from the previous toll station. The fee structure at each toll station is as follows: For the first 60 km, the fee per km will be 5 cents. For every additional 2 km, the fee per km will be 2% less than the previous fee per km. **(i)** Find, in terms of n, the amount of fees a driver will need to pay at T_n. [3] **(ii)** Find the total amount of fees a driver will need to pay, if he drives from T_1 to T_n. Leave your answer in terms of n. [4] More toll stations are built along the highway in the same manner, represented by T_{26}, T_{27}, T_{28},…….. beyond the 2000 km stretch. **(iii)** If a driver starts driving from T_1 and only has \$200, at which toll station will he not have sufficient money for the fees? [2]
11	**(i)** Show by integration that

$$\int e^{-2x} \sin x \ dx = -\frac{2}{5}e^{-2x}\sin x - \frac{1}{5}e^{-2x}\cos x + A$$

where A is an arbitrary constant. [3]

The diagram below shows a sketch of curve C, with parametric equations

$$x = e^{-t}\ ,\ y = e^{-t}\sin t\ ,\ -\pi \le t \le \pi .$$



Point P lies on C where $t = \frac{\pi}{2}$.

(ii) Find the equation of the normal at P. [3]

(iii) Find the exact area bounded by the curve C for $0 \le t \le \pi$, the line $x = 1$ and the normal at P. [5]

(iv) The normal at P cuts the curve C again at two points where $t = q$ and $t = r$. Find the values of q and r. [3]

End of paper

ANNEX B

AJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Equations and Inequalities	x = 13937.6 = 13938 (nearest dollars), y = 9031.2 ≈9031, z = 2031.2 ≈ 2031
2	Differential Equations	$y=-\frac{1}{3}\left(1-3x^2\right)-\frac{\sqrt{1-3x^2}}{\sqrt{3}}\sin^{-1}\left(\sqrt{3}x\right)+C\sqrt{1-3x^2}$
3	Binomial Expansion	$a=\sqrt{2}, b=-\sqrt{2}, c=-\frac{\sqrt{2}}{2}$
4	Sigma Notation and Method of Difference	(a) $\frac{1}{6}-\frac{1}{2\left(4N+3\right)}$ (b) $\ln 2-\frac{1}{e}\left(2n-1\right)$
5	Maclaurin series	$a=3;\ -1+\frac{2}{3}x+\frac{1}{3}x^2-\frac{5}{27}x^3+\ldots$
6	Complex numbers	$x=0, y>0$
7	Differentiation & Applications	(a) $x=\sqrt[3]{\frac{25\left(1+2k\right)}{2(k^2-1)}}$ (b) 1.02
8	Graphs and Transformation	(iii) A – Translate the graph by 2 units in the direction of x-axis B - Scaling, parallel to the y-axis by a scale factor of 2. C - Translate the graph by 8 units in the direction of y-axis <u>Alternately:</u> A – Translate the graph by 2 units in the direction of x-axis B - Translate the graph by 4 units in the direction of y-axis C - Scaling, parallel to the y-axis by a scale factor of 2 (iv) $0.805<x<1.69$ or $x>2$
9	Vectors	(ii) $\frac{1}{5}\lvert\mathbf{a}\rvert$ (iii) twice the area of the triangle COF (iv) 143.1°
10	AP and GP	(i) $7.9-4.9\left(0.98^{n-2}\right)$ (ii) $7.9n+245\left(0.98^{n-1}\right)-252.9$

		(iii) 45th toll station
11	Application of Integration	(ii) $y=-x+2e^{-\pi/2}$ (iii) $\frac{11}{10}e^{-\pi}-2e^{-\pi/2}+\frac{7}{10}$ (iv) $q=-1.92$ and $r=-1.01$
12	Q12 Topic	
13	Q13 Topic	

Anderson Junior College
Preliminary Examination 2017
H2 Mathematics Paper 1 (9758/01) solutions with comments

1	Let x, y and z be the amounts Mr Tan invested in structured deposit account, bonds and an estate fund respectively. $x+y+z = 25000$ --- (1) $y = z + 7000$ --- (2) $\left[(1.02x)\times1.02\right]+\left[(1.03y)\times1.03\right]+\left[(1.045z)\times1.045\right]=26300$ --- (3) Solving the 3 simultaneous equations : $x = 13937.6 = 13938$ (nearest dollars), $y = 9031.2 \approx 9031$, $z = 2031.2 \approx 2031$
2	Let $y=u\sqrt{1-3x^2}$ $\Rightarrow \quad \frac{dy}{dx}=\frac{du}{dx}\sqrt{1-3x^2}+u\left(\frac{1}{2}\right)\frac{-6x}{\sqrt{1-3x^2}}$ DE : $\quad \frac{\mathrm{d}y}{\mathrm{d}x}+\frac{3xy}{1-3x^2}-x+1=0$ $\Rightarrow \quad \frac{du}{dx}\sqrt{1-3x^2}-\frac{-3xu}{\sqrt{1-3x^2}}+\frac{3x}{1-3x^2}\left(u\sqrt{1-3x^2}\right)-x+1=0$ $\Rightarrow \quad \frac{du}{dx}\sqrt{1-3x^2}-\frac{3xu}{\sqrt{1-3x^2}}+\frac{3xu}{\sqrt{1-3x^2}}=x-1$ $\Rightarrow \quad \frac{du}{dx}\sqrt{1-3x^2}=x-1$ $\Rightarrow \quad \frac{du}{dx}=\frac{x}{\sqrt{1-3x^2}}-\frac{1}{\sqrt{1-3x^2}}$ $\Rightarrow \quad u=-\frac{1}{6}\int\frac{-6x}{\sqrt{1-3x^2}}dx-\int\frac{1}{\sqrt{1-3x^2}}\,dx$ $\Rightarrow \quad \frac{y}{\sqrt{1-3x^2}}=-\frac{1}{6}\left[2\sqrt{1-3x^2}\right]-\frac{\sin^{-1}\left(\sqrt{3}x\right)}{\sqrt{3}}+C$ $\Rightarrow \quad y=-\frac{1}{3}\left(1-3x^2\right)-\frac{\sqrt{1-3x^2}}{\sqrt{3}}\sin^{-1}\left(\sqrt{3}x\right)+C\sqrt{1-3x^2}$
3	Consider triangle ABC, $AC^2=4+2-2(2)\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)$ $=6-4\sqrt{2}\left(\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta\right)=6-4\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)$ $AC=\sqrt{6-4\cos\theta-4\sin\theta}$ (shown) Consider triangle ACD, $\cos\theta=\frac{AD}{AC}$ $AD=\cos\theta\sqrt{6-4\cos\theta-4\sin\theta}$

Since θ is small, $\sin\theta \approx \theta$, $\cos\theta \approx 1-\dfrac{\theta^2}{2}$,

$$AD \approx \left(1-\frac{\theta^2}{2}\right)\sqrt{6-4\left(1-\frac{\theta^2}{2}\right)-4\theta}$$

$$=\left(1-\frac{\theta^2}{2}\right)\left(2+2\theta^2-4\theta\right)^{\frac{1}{2}}$$

$$=\left(1-\frac{\theta^2}{2}\right)\sqrt{2}\left(1+\theta^2-2\theta\right)^{\frac{1}{2}}$$

$$=\sqrt{2}\left(1-\frac{\theta^2}{2}\right)\left(1+\frac{1}{2}(\theta^2-2\theta)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2}(\theta^2-2\theta)^2+\ldots\right)$$

$$=\sqrt{2}\left(1-\frac{\theta^2}{2}\right)\left(1+\frac{1}{2}\theta^2-\theta-\frac{1}{2}\theta^2+\ldots\right)$$

$$=\sqrt{2}\left(1-\frac{\theta^2}{2}\right)\left(1-\theta+\ldots\right)$$

$$=\sqrt{2}\left(1-\theta-\frac{\theta^2}{2}+\ldots\right)$$

$$\approx \sqrt{2}-\sqrt{2}\theta-\frac{\sqrt{2}}{2}\theta^2$$

4(a)

$$\sum_{n=1}^{2N}\frac{1}{4(n+1)^2-1}=\sum_{n=2}^{2N+1}\frac{1}{4n^2-1}$$

$$=\sum_{n=1}^{2N+1}\frac{1}{4n^2-1}-\frac{1}{3}$$

$$=\frac{1}{2}-\frac{1}{2\left[2(2N+1)+1\right]}-\frac{1}{3}$$

$$=\frac{1}{6}-\frac{1}{2(4N+3)}$$

$$\frac{1}{(2n+3)^2}=\frac{1}{4n^2+12n+9} \quad \text{and} \quad \frac{1}{4(n+1)^2-1}=\frac{1}{4n^2+8n+3}$$

$$\therefore \quad \frac{1}{(2n+3)^2}<\frac{1}{4(n+1)^2-1}$$

Alternative:

$$\frac{1}{(2n+3)^2}<\frac{1}{(2n+1)(2n+3)}=\frac{1}{4(n+1)^2-1}$$

Hence

$$\sum_{n=1}^{2N}\frac{1}{(2n+3)(2n+3)}<\sum_{n=1}^{2N}\frac{1}{4(n+1)^2-1}$$

$$\sum_{n=1}^{2N}\frac{1}{(2n+3)(2n+3)}<\frac{1}{6}-\frac{1}{2(4N+3)} \quad \left[\text{since } N>0 \text{ \& } \frac{1}{2(4N+3)}>0\right]$$

$$<\frac{1}{6}$$

4b	$T_n = S_n - S_{n-1} = n\ln 2 - \frac{n^2-1}{e} - \left[(n-1)\ln 2 - \frac{(n-1)^2-1}{e}\right]$ $= \left[n-(n-1)\right]\ln 2 - \frac{1}{e}\left[(n^2-1)-(n-1)^2+1\right]$ $= \ln 2 - \frac{1}{e}\left[n^2-1-n^2+2n-1+1\right]$ $= \ln 2 - \frac{1}{e}(2n-1)$ $T_n - T_{n-1} = \ln 2 - \frac{1}{e}(2n-1) - \left[\ln 2 - \frac{1}{e}(2(n-1)-1)\right]$ $= -\frac{2}{e}$ Since $-\frac{2}{e}$ is a constant, the terms follow an AP.
5	Curve C: $y = f(x)$ Tangent to C at x = 0 is $2x - ay = 3 \Rightarrow y = -\frac{3}{a} + \frac{2}{a}x$ Since the tangent to C at x = 0 is $y = f(0) + f'(0)x$, $\therefore f(0) = -\frac{3}{a}$ and $f'(0) = \frac{2}{a}$ The 3rd term of the series for $f(x)$ is $\frac{1}{3}x^2$ $\Rightarrow \frac{f''(0)}{2!}x^2 = \frac{1}{3}x^2$ $\Rightarrow f''(0) = \frac{2}{3}$ From $(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, When x = 0, we have $\frac{2}{3} + \left(-\frac{3}{a}\right)\left(\frac{2}{a}\right) = 0$ $\Rightarrow a^2 = 9$ $\Rightarrow a = 3$ (since $a > 0$)

$$(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Differentiate w.r.t. x:

$$(1+2x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)=0$$

When $x=0$, $y=-\frac{3}{3}=-1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{6}{9}=\frac{2}{3}$,

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+(2-1)\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2=0$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-\frac{2}{3}-\frac{4}{9}=-\frac{10}{9}$$

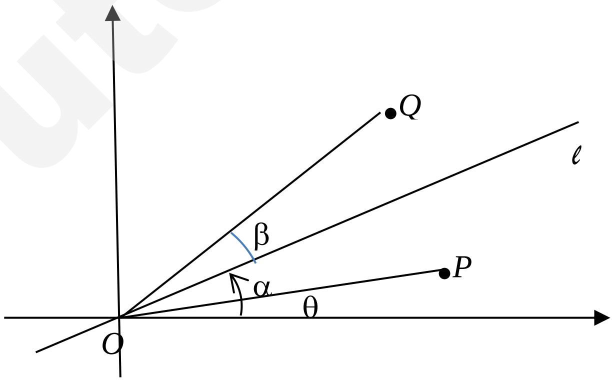
$$\therefore\ y=-1+\frac{2}{3}x+\frac{1}{3}x^2-\frac{10}{9(3!)}x^3+\ldots$$

$$=-1+\frac{2}{3}x+\frac{1}{3}x^2-\frac{5}{27}x^3+\ldots$$

6

$P\equiv z_1=re^{i\theta}$,

$|z_1|=r$ & arg (z_1) = θ



Let β be angle between lines OQ & l,

β = (α - θ) since line l **bisects** ∠POQ

arg z_1 + arg z_2

= θ+ (α + β)

= θ+ α + (α - θ)

= 2α

$|z_1z_2|=|z_1||z_2|=r^2$ AND arg (z_1 z_2) = arg z_1 + arg z_2 = 2α

Hence $z_1z_2=r^2(\cos 2\alpha+i\sin 2\alpha)$.

$\alpha=\frac{\pi}{4}$ $\Rightarrow$ $z_1z_2=r^2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=r^2i$ (Purely imaginary).

Cartesian equation of the locus of R is $x=0, y>0$

7

$FB = EC = 2\sqrt{(kx)^2-x^2}=2x\sqrt{k^2-1}$

Area of cross-section of prism

$=$ Area of $ABCD$ + Area of $AFED$

$=$ 2(Area of trapezium $ABCD$)

$$=2\left[\frac{1}{2}(x+3x)\sqrt{(kx)^2-x^2}\right]$$

$$=4x^2\sqrt{k^2-1}$$

Surface area of prism, $S=2\left(4x^2\sqrt{k^2-1}\right)+2xh+4kxh$

Hence $S=8x^2\sqrt{k^2-1}+2xh(1+2k)$ (shown) --- (2)

7a	Volume of prism $= 400 = \left(4x^2\sqrt{k^2-1}\right)h$ $h = \dfrac{100}{x^2\sqrt{k^2-1}}$ --- (1) (1) in (2): $S = 8x^2\sqrt{k^2-1} + 2(1+2k)\left(\dfrac{100}{x\sqrt{k^2-1}}\right)$ $\dfrac{dS}{dx} = 16x\sqrt{k^2-1} - \dfrac{200(1+2k)}{x^2\sqrt{k^2-1}}$ When $\dfrac{dS}{dx} = 0$, $x^3 = \dfrac{200(1+2k)}{16(k^2-1)} \Rightarrow x = \sqrt[3]{\dfrac{25(1+2k)}{2(k^2-1)}}$
7b	When $k = 2$, $S = 8x^2\sqrt{k^2-1} + 2xh(1+2k) = 8\sqrt{3}x^2 + 10xh$ and $V = \left(4x^2\sqrt{k^2-1}\right)h = 4\sqrt{3}x^2h$ Given that $\dfrac{dx}{dt} = \dfrac{dh}{dt}$ $\Rightarrow \dfrac{dh}{dx} = \dfrac{dh}{dt} \times \dfrac{dt}{dx} = 1$ <u>Method 1</u> $\dfrac{dS}{dx} = 8\sqrt{3}(2x) + 10h + 10x\dfrac{dh}{dx} = 16\sqrt{3}x + 10h + 10x$ --- (1) $\dfrac{dV}{dx} = 4\sqrt{3}\left(h.2x + x^2\dfrac{dh}{dx}\right) = 4\sqrt{3}\left(2xh + x^2\right)$ --- (2) When $x = 3$, $h = 8$, $\dfrac{dS}{dt} = 0.5$, $\dfrac{dS}{dx} = 16\sqrt{3}(3) + 10(8+3) = 48\sqrt{3} + 110$ $\dfrac{dV}{dx} = 4\sqrt{3}\left(2.3.8 + 3^2\right) = 228\sqrt{3}$ $\dfrac{dV}{dt} = \dfrac{dV}{dx} \times \dfrac{dx}{dS} \times \dfrac{dS}{dt}$ $= 228\sqrt{3} \times \dfrac{1}{48\sqrt{3}+110} \times 0.5$ $= 1.02$ (to 3 s.f.) <u>Method 2</u> $\dfrac{dS}{dt} = 8\sqrt{3}\left(2x\dfrac{dx}{dt}\right) + 10\left(h\dfrac{dx}{dt} + x\dfrac{dh}{dt}\right) = \left(16\sqrt{3}x + 10h + 10x\right)\dfrac{dx}{dt}$ --- (1) And $\dfrac{dV}{dt} = 4\sqrt{3}\left(h.2x\dfrac{dx}{dt} + x^2\dfrac{dh}{dt}\right) = 4\sqrt{3}\left(2xh + x^2\right)\dfrac{dx}{dt}$ --- (2)

	When $x=3,\ h=8,\ \dfrac{dS}{dt}=0.5,$ using eqn (1) to find $\dfrac{dx}{dt}$ $0.5=\left(16\sqrt{3}(3)+10(8)+10(3)\right)\dfrac{dx}{dt}$ $\dfrac{dx}{dt}=\dfrac{0.5}{48\sqrt{3}+110}\approx 0.0025888$ Sub into (2) to get $\dfrac{dV}{dt}$ $\dfrac{dV}{dt}=4\sqrt{3}\left(2.3.8+3^2\right)(0.0025888)=1.022343317\approx 1.02$ (to 3 sf)
8i)	$y=\dfrac{4x^2-kx+2}{x-2}$ By long division, $y=4x+8-k+\dfrac{18-2k}{x-2}$ $\dfrac{dy}{dx}=\dfrac{(x-2)(8x-k)-(4x^2-kx+2)(1)}{(x-2)^2}$ $=\dfrac{4x^2-16x+2k-2}{(x-2)^2}$ Let $\dfrac{dy}{dx}=0 \Rightarrow 4x^2-16x+2k-2=0$ $\Rightarrow 2x^2-8x+k-1=0$ $\Rightarrow x=\dfrac{8\pm\sqrt{64-4(2)(k-1)}}{4}=2\pm\sqrt{\dfrac{9-k}{2}}$ C has stationary point when $k\le 9$ However, when $k=9$, the value x=2 is undefined on the curve. In fact, the curve C is a straight line, $y=4x-1$. Hence C has stationary point when $k<9$. <u>Alternative Presentation 1:</u> Let $\dfrac{dy}{dx}=0 \Rightarrow 4x^2-16x+2k-2=0$ $\Rightarrow 2x^2-8x+k-1=0$ For $\dfrac{dy}{dx}$=0 to have real roots, "$b^2-4ac\ge 0$" $\Rightarrow 8^2-4(2)(k-1)\ge 0$ $\Rightarrow 64-8k+8\ge 0$ $\Rightarrow 8k\le 72$ $\Rightarrow k\le 9$ <u>Alternative Presentation 2:</u> $\dfrac{dy}{dx}=0 \Rightarrow 4x^2-16x+2k-2=0$ $\Rightarrow 2x^2-8x+k-1=0$ $\Rightarrow 2(x-2)^2+k-9=0$ $\Rightarrow 2(x-2)^2=9-k$ For $\dfrac{dy}{dx}=0$ to have roots x, $9-k\ge 0\Rightarrow k\le 9$ However, when $k=9$, the value x=2 is undefined on the curve. In fact, the curve C is a straight line, $y=4x-1$. Hence C has stationary point when $k<9$.

(ii)

$$y=\frac{4x^2-kx+2}{x-2}=4x+(8-k)+\frac{18-2k}{x-2}$$

Asymptotes of C are $y = 4x + 8 - k$ and $x = 2$

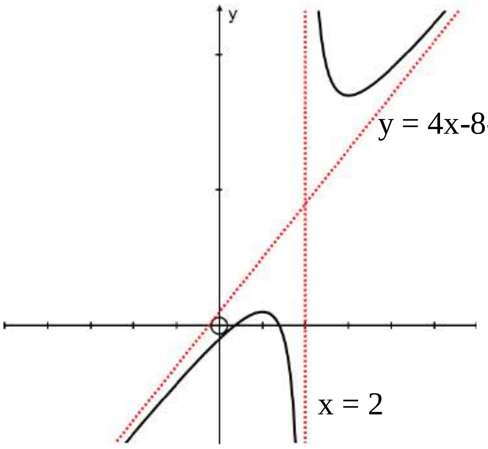
When $x=0$, $y==-1$.

When $y=0$, $4x^2-kx+2=0$

$$\Rightarrow x=\frac{k\pm\sqrt{k^2-32}}{8}$$

The axial intercepts are $(0,-1)$, $\left(\frac{k-\sqrt{k^2-32}}{8},0\right)$ and $\left(\frac{k+\sqrt{k^2-32}}{8},0\right)$.

ii)



(iii)

When k = 8, $y=4x+(8-8)+\frac{18-2(8)}{x-2}=4x+\frac{2}{x-2}$

$$y=2x+\frac{1}{x}\xrightarrow{A}y=2\left(2x+\frac{1}{x}\right)=4x+\frac{2}{x}$$

$$y=4x+\frac{2}{x}\xrightarrow{B}y=4(x-2)+\frac{2}{(x-2)}=y=4x-8+\frac{2}{(x-2)}$$

$$y=4x-8+\frac{2}{(x-2)}\xrightarrow{C}y=\left(4x-8+\frac{2}{(x-2)}\right)+8=4x+\frac{2}{(x-2)}$$

A – Translate the graph by 2 units in the direction of x-axis
B - Scaling, parallel to the y-axis by a scale factor of 2.
C - Translate the graph by 8 units in the direction of y-axis

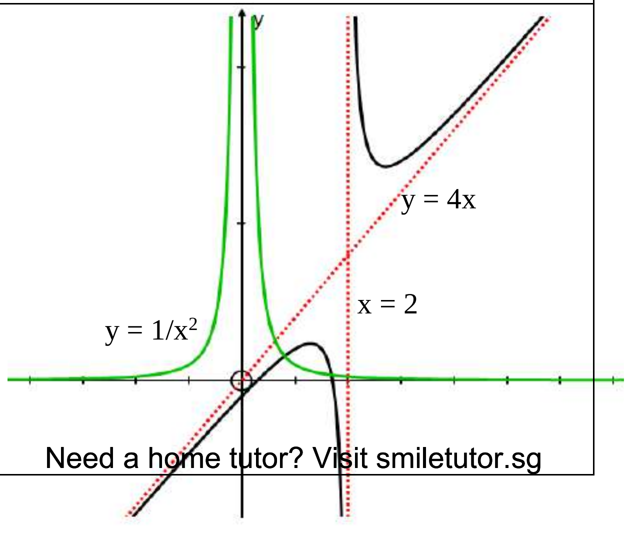
<u>Alternate Sequence of Transformations:</u>
A – Translate the graph by 2 units in the direction of x-axis
B - Translate the graph by 4 units in the direction of y-axis
C - Scaling, parallel to the y-axis by a scale factor of 2.

iv)

When $k=8$, $\frac{4x^2-kx+2}{x-2}>\frac{1}{x^2}$

$$\Rightarrow \frac{4x^2-8x+2}{x-2}>\frac{1}{x^2}$$

From G.C,
$0.805<x<1.69$ or $x>2$.



9(i)	$(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=\mathbf{a.a}+2\mathbf{a.b}+\mathbf{b.b}$ Since $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=\lvert\mathbf{a}+\mathbf{b}\rvert^2$ and given $\lvert\mathbf{a}+\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2$ $\therefore\ \mathbf{a.a}+2\mathbf{a.b}+\mathbf{b.b}=\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2$ $\lvert\mathbf{a}\rvert^2+2\mathbf{a.b}+\lvert\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2$ $2\mathbf{a.b}=0$ $\mathbf{a.b}=0$ $\therefore\ \mathbf{a}\perp\mathbf{b}$
ii)	Using ratio theorem, $\overrightarrow{OC}=\dfrac{4\mathbf{b}+\mathbf{a}}{5}=\dfrac{1}{5}\mathbf{a}+\dfrac{4}{5}\mathbf{b}$. **Length of projection of $\overrightarrow{\mathbf{OC}}$ onto $\overrightarrow{\mathbf{OA}}$** $=\dfrac{\lvert\overrightarrow{OC}\bullet\overrightarrow{OA}\rvert}{\lvert\overrightarrow{OA}\rvert}$ $=\dfrac{\left\lvert\left(\frac{1}{5}\mathbf{a}+\frac{4}{5}\mathbf{b}\right)\bullet\mathbf{a}\right\rvert}{\lvert\mathbf{a}\rvert}=\dfrac{\left\lvert\frac{1}{5}\mathbf{a}\bullet\mathbf{a}+\frac{4}{5}\mathbf{b}\bullet\mathbf{a}\right\rvert}{\lvert\mathbf{a}\rvert}$ $=\dfrac{\left\lvert\frac{1}{5}\lvert\mathbf{a}\rvert^2+\frac{4}{5}\mathbf{b}\bullet\mathbf{a}\right\rvert}{\lvert\mathbf{a}\rvert}\quad=\dfrac{1}{5}\lvert\mathbf{a}\rvert$ **(since $\mathbf{a}\perp\mathbf{b}$)**
iii)	$\lvert\mathbf{c}\times\mathbf{f}\rvert$ denotes twice the area of the triangle COF.
(iv)	$\overrightarrow{OX}\bullet\overrightarrow{OY}=\begin{pmatrix}\cos 3t\\ \sin 3t\\ \frac{1}{2}\end{pmatrix}\bullet\begin{pmatrix}\sin t\\ \cos t\\ -2\end{pmatrix}=\cos 3t\sin t+\sin 3t\cos t-1=\sin(4t)-1$ $\cos\measuredangle XOY=\dfrac{\overrightarrow{OX}\bullet\overrightarrow{OY}}{\lvert\overrightarrow{OX}\rvert\lvert\overrightarrow{OY}\rvert}=\dfrac{\sin 4t-1}{\sqrt{\cos^2 3t+\sin^2 3t+\frac{1}{4}}\sqrt{\sin^2 t+\cos^2 t+4}}$ $=\dfrac{\sin 4t-1}{\sqrt{\frac{5}{4}}\sqrt{5}}$ $=\dfrac{2}{5}(\sin 4t-1)$ Maximum $\measuredangle XOY$ occurs when is most negative. i.e. when $\sin 4t=-1$. At that value of *t*, $\cos\measuredangle XOY=\dfrac{2}{5}(-1-1)=-\dfrac{4}{5}$ $\therefore\measuredangle XOY=\cos^{-1}\left(-\dfrac{4}{5}\right)=143.1°$
10	$x+(x+2)+(x+2(2))\ +\ \ldots\ +(x+\ 23(2))\le 2000$ This is an AP with first term= *x*, common difference = 2, number of terms = 24 $\dfrac{24}{2}\left[2x+23(2)\right]\le 2000$ $0<x\le\dfrac{181}{3}$

10(i)

n	Amount paid at T_n
2	$60(0.05)$
3	$60(0.05)+2(0.05)(0.98)$
4	$60(0.05)+2(0.05)(0.98)+2(0.05)(0.98)^2$
	. .
n	$60(0.05)+2(0.05)(0.98)+2(0.05)(0.98)^2+...+2(0.05)(0.98)^{n-2}$

$$\text{Amount of fees at } T_n = 3+\frac{0.098\left(1-0.98^{n-2}\right)}{1-0.98}$$
$$=3+4.9\left(1-0.98^{n-2}\right)$$
$$=7.9-4.9\left(0.98^{n-2}\right)$$

ii

$$\sum_{r=2}^{n}\left[7.9-4.9\left(0.98^{r-2}\right)\right]$$
$$=\sum_{r=2}^{n}7.9-4.9\sum_{r=2}^{n}\left(0.98^{r-2}\right)$$
$$=7.9(n-1)-4.9\left[\frac{1\left(1-0.98^{n-1}\right)}{1-0.98}\right]$$
$$=7.9(n-1)-245\left(1-0.98^{n-1}\right)$$
$$=7.9n+245\left(0.98^{n-1}\right)-252.9$$

iii

Let $\mathrm{f}(n)=7.9n+245\left(0.98^{n-1}\right)-252.9$. Note that $\mathrm{f}(n)$ is increasing in n

Consider $7.9n+245\left(0.98^{n-1}\right)-252.9>200$

44	197.47
45	203.32
46	209.21

Using GC, $n\geq 45$

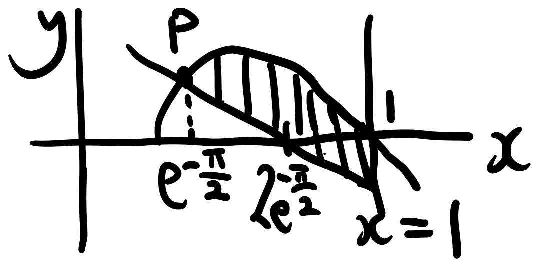
He will not have sufficient money at the 45th toll station.

11i

$$\int e^{-2x}\sin x \ \mathrm{d}x$$
$$=(-\cos x)\left(e^{-2x}\right)-\int(-\cos x)\left(-2e^{-2x}\right)\ \mathrm{d}x$$
$$=-e^{-2x}\cos x-2\left[(\sin x)\left(e^{-2x}\right)-\int \sin x\left(-2e^{-2x}\right)\ \mathrm{d}x\right]$$
$$=-e^{-2x}\cos x-2e^{-2x}\sin x-4\int e^{-2x}\sin x \ \mathrm{d}x$$
$$5\int e^{-2x}\sin x \ \mathrm{d}x=-e^{-2x}\cos x-2e^{-2x}\sin x+C$$
$$\int e^{-2x}\sin x \ \mathrm{d}x=-\frac{2}{5}e^{-2x}\sin x-\frac{1}{5}e^{-2x}\cos x+A$$

11ii

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-e^{-t} \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=-e^{-t}\sin t+e^{-t}\cos t$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-e^{-t}\sin t+e^{-t}\cos t}{-e^{-t}}=\sin t-\cos t$$

	At $t=\frac{\pi}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\sin\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{2}\right)=1-0=1$, so gradient of normal = -1 $x=e^{-\pi/2}$, $y=e^{-\pi/2}\sin\frac{\pi}{2}=e^{-\pi/2}$ Equation of normal: $y-e^{-\pi/2}=-1(x-e^{-\pi/2}) \Rightarrow y=-x+2e^{-\pi/2}$
11iii	 Area $=\int_{e^{-\pi/2}}^{1} e^{-t}\sin t-\left(-x+2e^{-\pi/2}\right)\mathrm{d}x$ $=\int_{\pi/2}^{0} e^{-t}\sin t(-e^{-t})\mathrm{d}t+\int_{e^{-\pi/2}}^{1} x-2e^{-\pi/2}\ \mathrm{d}x$ $=-\int_{\pi/2}^{0} e^{-2t}\sin t\ \mathrm{d}t+\left[\frac{x^2}{2}\right]_{e^{-\pi/2}}^{1}-\left[2e^{-\pi/2}x\right]_{e^{-\pi/2}}^{1}$ $=-\left[-\frac{2}{5}e^{-2x}\sin x-\frac{1}{5}e^{-2x}\cos x\right]_{\pi/2}^{0}+\left[\frac{1}{2}-\frac{e^{-\pi}}{2}\right]-\left[2e^{-\pi/2}\left(1-e^{-\pi/2}\right)\right]$ $=\frac{2}{5}e^{0}\sin 0+\frac{1}{5}e^{0}\cos 0-\frac{2}{5}e^{-\pi}\sin\left(\frac{\pi}{2}\right)-\frac{1}{5}e^{-\pi}\cos\frac{\pi}{2}+\frac{1}{2}-\frac{e^{-\pi}}{2}-2e^{-\pi/2}+2e^{-\pi}$ $=\frac{1}{5}-\frac{2}{5}e^{-\pi}+\frac{1}{2}-\frac{e^{-\pi}}{2}-2e^{-\pi/2}+2e^{-\pi}$ $=\frac{11}{10}e^{-\pi}-2e^{-\pi/2}+\frac{7}{10}$ ***Alternative:*** Area = $\int_{e^{-\pi/2}}^{1} e^{-t}\sin t-\frac{1}{2}(e^{-\pi/2})(2e^{-\pi/2}-e^{-\pi/2})+\frac{1}{2}(1-2e^{-\pi/2})^2$ [When x =1, y = 1+ $2e^{-\pi/2}$]
11iv	For normal to meet curve again, Substitute parametric eqns into $y=-x+2e^{-\pi/2}$ $e^{-t}\sin t=-e^{-t}+2e^{-\pi/2}$ $e^{-t}(\sin t+1)-2e^{-\pi/2}=0$ Using GC, $t=-1.92148,\ -1.0145,\ 1.5707$ (rej, this is $\frac{\pi}{2}$) So $q=-1.92$ and $r=-1.01$ (to 3 sf)

ANDERSON JUNIOR COLLEGE
2017 Preliminary Examination
H2 Mathematics Paper 2 (9758/02)

Duration: 3 hours

	Section A: Pure Mathematics [40 marks]
1	At the intensive care unit of a hospital, patients of a particular condition receive a certain treatment drug through an intravenous drip at a constant rate of 30mg per hour. Due to the limited capacity for absorption by the body, the drug is lost from a patient's body at a rate proportional to x , where x is the amount of drug (in mg) present in the body at time t (in hours). It is assumed that there is no presence of the drug in any patient prior to admission to the hospital. (i) Form a differential equation involving x and t and show that $x=\frac{30}{k}\left(1-\mathrm{e}^{-kt}\right)$ where k is a positive constant. [4] (ii) If there is more than 1000mg of drug present in a patient's body, it is considered an overdose. Suppose the drug continues to be administered, determine the range of values of k such that a patient will have an overdose. [2] For a particular patient, $k=\frac{1}{50}$. (iii) Find the time required for the amount of the drug present in the patient's body to be 200mg. [3]
2	The polynomial P(z) has real coefficients. The equation P(z) = 0 has a root $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0<\theta<\pi$. Write down a second root in terms of r and θ, and hence show that a quadratic factor of P(z) is $z^2-2rz\cos\theta+r^2$. [2] Let P(z) = $z^3+az^2+15z+18$ where a is a real number. One of the roots of the equation P(z) = 0 is $3\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$. By expressing P($z$) as a product of two factors with real coefficients, find a and the other roots of P(z) = 0. [4] Deduce the roots of the equation $18z^3+15z^2+az+1=0$. [2]

3	Planes Π_1 and Π_2 are defined by $$\Pi_1 : x - 2y + 2z = 7, \quad \Pi_2 : \mathbf{r} \bullet \begin{pmatrix} a \\ 3 \\ -1 \end{pmatrix} = 8.$$ where a is a constant. **(i)** The point P has position vector $-2\mathbf{i}+\mathbf{j}+\mathbf{k}$. Find the position vector of F, the foot of the perpendicular from P to plane Π_1. Hence, or otherwise, find the shortest distance from P to plane Π_1. [5] **(ii)** Line m passes through the point F and is parallel to both planes Π_1 and Π_2. Find the vector equation of line m. [2] **(iii)** It is given that the point $Q(1,-4,-1)$ lies on line m. Find the value of a. [3] **(iv)** Find the length of projection of $\overrightarrow{PQ}$ on the x-y plane. [3]
4	The function f is defined by $$\mathrm{f} : x \mapsto \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1} \quad \text{for} \quad x \in \mathbb{R}.$$ Sketch the graph of $y = \mathrm{f}(x)$ and state the range of f. [3] Another function h is defined by $$\mathrm{h} : x \mapsto \begin{cases} (x-1)^2 + 1 & \text{for} \quad x \le 1 \\ 1 - \dfrac{\lvert 1-x \rvert}{2} & \text{for} \quad 1 < x \le 4 \end{cases}$$ Sketch the graph of $y = \mathrm{h}(x)$ for $x \le 4$ and explain why the composite function $\mathrm{f}^{-1}\mathrm{h}$ exists. Hence find the exact value of $\left(\mathrm{f}^{-1}\mathrm{h}\right)^{-1}(3)$. [7]

Section B: Probability and Statistics [60 marks]

5 A vehicle insurance company classifies the drivers it insures as class A, B and C according to whether they are of low risk, medium risk or high risk with regard to having an accident. The company estimates that 30% of the drivers who are insured are class A and 50% are class B. The probability that a class A driver will have at least one accident in any 12 month period is 0.01, the corresponding probabilities for class B and C are 0.03 and 0.06 respectively.

(i) Find the probability that a randomly chosen driver will have at least one accident in a 12-month period. [2]

(ii) The company sold a policy to a driver and within 12 months, the driver had at least one accident. Find the probability that the driver is of class C. [2]

(iii) Three drivers insured by the company are chosen randomly. Find the probability that all three drivers are of class C and exactly one of them had at least one accident in a 12-month period. [3]

6 In an experiment to investigate the decay of organic material over time, a bag of leaf litter was allowed to sit for a 20-week period in a moderately forested area.

The table below shows the weight of the remaining leaf litter (y kg) when x number of weeks have passed.

x	1	2	4	6	8	9	11	15	20
y	60.9	51.8	34.7	26.2	14.0	12.3	8.2	3.1	1.4

(i) Draw a scatter diagram of these data. [1]

(ii) Find the equation of the regression line of y on x and calculate the corresponding estimated value of y when $x = 17$.
Comment on the suitability of the linear model for these data. [3]

The variable W is defined as $W = \ln y$.

(iii) Find the product moment correlation coefficient between W and x. [1]

(iv) It is given that the weight of the leaf litter in the bag was 75.0 kg initially. Using an appropriate regression line, estimate how long it takes for the weight of the leaf litter to drop to half its initial value, giving your answer to one decimal place. [3]
Give two reasons why you would expect this estimate to be reliable. [2]

7	**(a)** The Health Promotion Board of a certain country claims that the average number of hours of sleep of working adults is at most 6 hours per day. To investigate this claim, the editor of a magazine plans to conduct a survey on a sample of adults travelling to work by train. **(i)** Explain why this method of sampling will not give a random sample for the purpose of the investigation. [1] The editor of another magazine interviewed a random sample of 50 working adults and their number of hours of sleep per day, *x*, are summarised as follows: $\sum x = 320$, $\sum x^2 = 2308.5$ **(ii)** Test at the 5% level of significance whether there is any evidence to doubt the Health Promotion Board's claim. State with a reason, whether it is necessary to assume that the number of hours of sleep per day follows a normal distribution. [5] **(b)** The Health Promotion Board carried out their own survey on another random sample of 50 working adults. The sample yielded an average of 6.14 hours of sleep per day and a standard deviation of 2.1 hours. If the sample does not provide significant evidence at the 5% level of significance that the mean number of hours of sleep per day of working adults differs from μ_o hours, find the range of values of μ_o [4]
8	A biscuit manufacturer produces both cream and chocolate biscuits. Biscuits are chosen randomly and packed into boxes of 10. The number of cream biscuits in a box is denoted by *X*. **(a)** On average, the proportion of cream biscuits is *p*. Given that P(*X* = 1 or 2) = 0.15, write down an equation for the value of *p*. Hence find the value(s) of *p* numerically. [3] **(b)** It is given instead that the biscuit manufacturer produces 3 times as many cream biscuits as chocolate biscuits. **(i)** Find the most likely value of *X*. [2] **(ii)** A random sample of 18 boxes is taken. Find the probability that at least 3 but fewer than 7 boxes have equal numbers of cream and chocolate biscuits. [3] A box of biscuits is sold at \$10. The manufacturer gives a discount of \$2 per box to its premium customers. The mean and variance of the number of boxes sold per day to each type of customers (assuming independence) are as follows:

	Mean	Variance
Number of boxes sold at usual price	180	64
Number of boxes sold at discounted price	840	169

Find the approximate probability that the total amount collected per month from the sales of biscuits is not less than \$255,000, assuming that there are 30 days in a month. [4]

9	Four families arrive at Science Centre together. Mr and Mrs *A* brought their 2 children while Mr *B* brought his 2 children. Mr and Mrs *C* brought their 3 children while Mrs *D* brought her only child. All these 14 people have to go through a gate one at a time to enter the centre. **(i)** In how many different ways can they go through the gate if each family goes in one after another? [2] There are two experiments at the ***Science Magic Experience*** station. **(ii)** In one experiment, participants are to be in groups of twos or threes. In how many different ways can the 8 children from the four families be grouped among themselves? [3] **(iii)** In another experiment, the four families have to hold hands to form two separate circles of equal size to experience a science phenomenon. Each circle must have exactly four children and members of the same family must be in the same circle. Find the number of ways of arranging these 14 people in the two circles such that there is no more than one adult between any two children. [3]
10	Males and females visiting an amusement park have heights, in centimetres, which are normally distributed with means and standard deviations as shown in the following table: (see table below) It is found that 38.29% of the females have heights between 150 cm and 160 cm. **(i)** Show that $\sigma = 10.0$ cm, correct to 3 significant figures. [2] **(ii)** Find the probability that the height of a randomly chosen female is within 20 cm of three-quarter the height of a randomly chosen male. State an assumption that is necessary for the calculation to be valid. [4] The amount, $\$X$, a visitor has to pay for a popular ride in the park is \$10 if the visitor's height is at least 120 cm but less than 150 cm, and $\$m$ if the visitor's height is 150 cm and above. If the visitor's height is less than 120 cm, he/she does not need to pay for the ride. **(iii)** Assuming that a visitor purchasing a ticket for the ride is equally likely to be a male or female, find in terms of m, the probability distribution of X. [3] Given that the expected amount a visitor will pay for a ride is \$17.93, show that $m = 20.00$, correct to 2 decimal places. [1] **(iv)** Three visitors were randomly chosen. Find the probability that the total amount they paid for a ride together is more than \$40. [3]

	Mean (cm)	Standard deviation (cm)
Male	165	12
Female	155	σ

End of paper

ANNEX B

AJC H2 Math JC2 Preliminary Examination Paper 2

<table>
<tr><th>QN</th><th>Topic Set</th><th>Answers</th></tr>
<tr><td>1</td><td>Differential Equations</td><td>(ii) $0 < k < 0.03$
(iii) $t = 7.16\text{h}$ or $7\text{h } 9\text{min}$</td></tr>
<tr><td>2</td><td>Complex numbers</td><td>$a = 5$
$3e^{i\left(\frac{2\pi}{3}\right)}$, $3e^{i\left(-\frac{2\pi}{3}\right)}$ and $-2 = 2e^{i(\pi)}$
$z = \frac{1}{3}e^{i\left(-\frac{2\pi}{3}\right)}, \frac{1}{3}e^{i\left(\frac{2\pi}{3}\right)}, -\frac{1}{2}$</td></tr>
<tr><td>3</td><td>Vectors</td><td>(i) $\overrightarrow{OF} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$; 3
(ii) $\underset{\sim}{r} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 1+2a \\ 3+2a \end{pmatrix}$ where $\mu \in \mathbb{R}$
(iii) $5/2$
(iv) $\sqrt{34}$</td></tr>
<tr><td>4</td><td>Functions</td><td>$1-\sqrt{e^2+e}$</td></tr>
<tr><td>5</td><td>P&C, Probability</td><td>(i) 0.03
(ii) 0.4
(iii) 0.00127 (to 3sf)</td></tr>
<tr><td>6</td><td>Correlation & Linear Regression</td><td>(ii) $y = 49.7 - 3.09x$; - 2.85
(iii) - 0.998
(iv) 3.6 weeks</td></tr>
<tr><td>7</td><td>Hypothesis Testing</td><td>(a)(ii) do not reject H_0
(b) $5.55 < \mu_o < 6.73$</td></tr>
<tr><td>8</td><td>Binomial Distribution</td><td>(a) $5p(1-p)^8(2+7p) = 0.15$; $p = 0.0162$ or $p = 0.408$
(b)(i) 8 (ii) 0.0843 (iii) 0.798</td></tr>
<tr><td>9</td><td>P&C, Probability</td><td>(i) 829, 440
(ii) 385
(iii) 20736</td></tr>
<tr><td>10</td><td>Normal Distribution</td><td>(ii) 0.201
<table><tr><th>x (in $)</th><th>P(X = x)</th></tr><tr><td>0</td><td>0.00016056</td></tr><tr><td>10</td><td>0.20693</td></tr><tr><td>m</td><td>0.79291</td></tr></table></td></tr>
</table>

		(iii) 0.889
11	Q11 Topic	
12	Q12 Topic	
13	Q13 Topic	

Anderson Junior College
Preliminary Examination 2017
H2 Mathematics Paper 2 (9758/02)

1(i)	$\frac{dx}{dt}=30-kx, \quad k>0$ $\Rightarrow \int \frac{1}{30-kx}\,dx=\int dt$ $\Rightarrow -\frac{1}{k}\ln\lvert 30-kx\rvert=t+C$ $\Rightarrow \ln\lvert 30-kx\rvert=-kt-kC$ $\Rightarrow \lvert 30-kx\rvert=e^{-kt-kC}$ $\Rightarrow 30-kx=Ae^{-kt}, \quad$ where $A=\pm e^{-kC}$ $\Rightarrow x=\frac{1}{k}\left(30-Ae^{-kt}\right)$ At $t=0$, $x=0 \Rightarrow 0=\frac{1}{k}\left(30-Ae^{0}\right)\Rightarrow A=30$ $\Rightarrow x=\frac{1}{k}\left(30-30e^{-kt}\right)=\frac{30}{k}\left(1-e^{-kt}\right)$
1(ii)	For patient to have overdose, $x=\frac{30}{k}\left(1-e^{-kt}\right)>1000$ Since for $t>0$, $0<e^{-kt}<1$, so $0<1-e^{-kt}<1$ $\frac{30}{k}>\frac{30}{k}\left(1-e^{-kt}\right)>1000$ $0<k<\frac{30}{1000}=0.03$
(iii)	At $x=200$, $\quad 200=30(50)\left(1-e^{-\frac{t}{50}}\right)$ $1-e^{-\frac{t}{50}}=\frac{2}{15}$ $t=50\ln\left(\frac{15}{13}\right)$ Using GC, $t=7.16$h or 7h 9min
2	Second root is $re^{-i\theta}$. Quadratic factor of P(z) is $\left(z-re^{i\theta}\right)\left(z-re^{-i\theta}\right)$ $=z^2-\left(re^{i\theta}+re^{-i\theta}\right)z+\left(re^{i\theta}\right)\left(re^{-i\theta}\right)$ $=z^2-r\left(e^{i\theta}+e^{-i\theta}\right)z+r^2$ $=z^2-r\left(\cos\theta+i\sin\theta+\cos\theta-i\sin\theta\right)z+r^2$ $=z^2-\left(2r\cos\theta\right)z+r^2$ root of the equation is $3e^{i\left(\frac{2\pi}{3}\right)}$. So $r=3$ and $\theta=\frac{2\pi}{3}$. Quadratic factor is $z^2-2(3)\left(\cos\frac{2\pi}{3}\right)z+9 \;=\; z^2+3z+9$

	hence $z^3+az^2+15z+18=(z^2+3z+9)(z+2)$ By comparing z^2 term, $a=5$ The roots of the equation $z^3+az^2+15z+18=0$ are $3e^{i\left(\frac{2\pi}{3}\right)}$, $3e^{i\left(-\frac{2\pi}{3}\right)}$ and $-2=2e^{i(\pi)}$
	$18z^3+15z^2+az+1=0$ $z^3\left(18+15\left(\frac{1}{z}\right)+a\left(\frac{1}{z^2}\right)+\left(\frac{1}{z^3}\right)\right)=0$ Since $z\neq 0$, and let $w=\frac{1}{z}$ We have $w^3+aw^2-2w+18=0$ Hence $w=3e^{i\left(\frac{2\pi}{3}\right)},3e^{i\left(-\frac{2\pi}{3}\right)},-2$ $\frac{1}{z}=3e^{i\left(\frac{2\pi}{3}\right)},3e^{i\left(-\frac{2\pi}{3}\right)},-2$ Since $\left\lvert\frac{1}{z}\right\rvert=\frac{1}{\lvert z\rvert}$ and $\arg\left(\frac{1}{z}\right)=-\arg(z)$ So $z=\frac{1}{3}e^{i\left(-\frac{2\pi}{3}\right)},\frac{1}{3}e^{i\left(\frac{2\pi}{3}\right)},-\frac{1}{2}$ are the roots of $18z^3+15z^2+az+1=0$
3(i)	Equation of line through point *P* and perpendicular to π_1 is $\mathbf{r}=\begin{pmatrix}-2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\2\end{pmatrix},\quad \lambda\in\mathbb{R}$ Since *F* lies on plane π_1 , $(-2+\lambda)-2(1-2\lambda)+2(1+2\lambda)=7 \quad \Rightarrow \lambda=1$ $\vec{OF}=\begin{pmatrix}-2\\1\\1\end{pmatrix}+1\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}-1\\-1\\3\end{pmatrix}$ $\vec{PF}=\vec{OF}-\vec{OP}=\begin{pmatrix}-1\\-1\\3\end{pmatrix}-\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$ shortest distance from *P* to plane $\Pi_1=\left\lvert\vec{PF}\right\rvert=\sqrt{1^2+(-2)^2+2^2}=3$
3(ii)	Line *m* is parallel to both planes: $\begin{pmatrix}1\\-2\\2\end{pmatrix}\times\begin{pmatrix}a\\3\\-1\end{pmatrix}=\begin{pmatrix}2-6\\-(-1-2a)\\3+2a\end{pmatrix}=\begin{pmatrix}-4\\1+2a\\3+2a\end{pmatrix}$ Equation of this line $m: r=\begin{pmatrix}-1\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}-4\\1+2a\\3+2a\end{pmatrix}$ where $\mu\in\mathbb{R}$
3(iii)	$Q(1,-4,-1)$ lies on line *m*, $-1-4\mu=1$ --- (1) $-1+(1+2a)\mu=-4$ --- (2) $3+(3+2a)\mu=-1$ --- (3)

From (1) : $\mu = -\frac{1}{2}$
From (2) : $a = 5/2$
From (3) : $a = 5/2$.   Hence the value of a is 5/2

<u>Alternative method</u>

$$\overrightarrow{FQ} = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix}$$

Since line m contains F and is parallel to π_1 , line m lies on π_1.

Since line m is on π_1, Q is on π_1. hence $\overrightarrow{FQ}$ is // π_1 and $\perp n_1$

$$\begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} a \\ 3 \\ -1 \end{pmatrix} = 0$$

$$2a - 9 + 4 = 0$$

$$a = 5/2$$

(iv)

<u>Method 1 (dot product)</u>

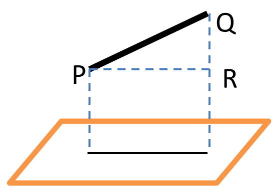
$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix}$$ and normal to the x-y plane = $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$QR = \left| \overrightarrow{PQ} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| = 2$$

length of projection of $\overrightarrow{PQ}$ on the x-y plane

= PR

$$= \sqrt{PQ^2 - 2^2} = \sqrt{(3^2 + 5^2 + 2^2 - 2^2} = \sqrt{34}$$



<u>Method 2 (cross product)</u>

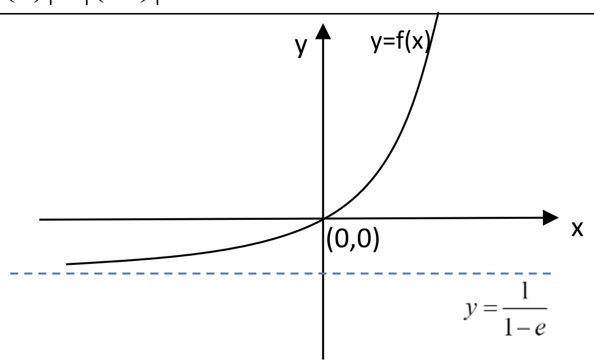
length of projection of $\overrightarrow{PQ}$ on the x-y plane

$$= PR = \left| \overrightarrow{PQ} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| = \left| \begin{pmatrix} -5 \\ -3 \\ 0 \end{pmatrix} \right| = \sqrt{5^2 + 3^2} = \sqrt{34}$$

4

Soln:

$$\mathrm{R_f} = \left(\frac{1}{1-e}, \infty \right)$$



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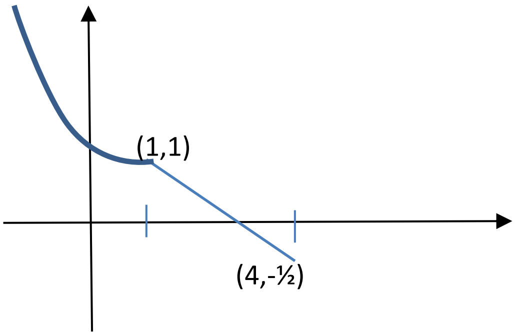


$$R_h = \left[-\frac{1}{2}, \infty\right)$$

$$D_{f^{-1}} = R_f = \left(\frac{1}{1-e}, \infty\right) = (-0.582, \infty)$$

Hence $R_h \subseteq D_{f^{-1}}$, so $f^{-1}h$ exists .

Let $\left(f^{-1}h\right)^{-1}(3) = k$

$\Rightarrow f^{-1}h(k) = 3$

$\Rightarrow h(k) = f(3)$

$\Rightarrow h(k) = \dfrac{e^3-1}{e-1} = e^2+e+1$

Since $e^2+e+1 > 1$,
hence $h(x) = (x-1)^2+1$

$(k-1)^2+1 = e^2+e+1$

$\Rightarrow k = 1 \pm \sqrt{e^2+e}$

Since x <1, hence the exact value of $\left(f^{-1}h\right)^{-1}(3) = 1-\sqrt{e^2+e}$.

Alternative method: use $f^{-1}(x)$

Let $\left(f^{-1}h\right)^{-1}(3) = k$

$\Rightarrow f^{-1}h(k) = 3$

$\Rightarrow \ln\left[1+(e-1)h(k)\right] = 3$

$\Rightarrow 1+(e-1)h(k) = e^3$

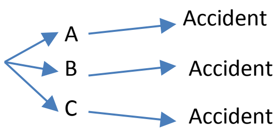
$\Rightarrow h(k) = \dfrac{e^3-1}{e-1} = e^2+e+1$

5(i)

Required probability

$= (0.3\times0.01)+(0.5\times0.03)+(0.2\times0.06)$

= 0.03



5(ii)

$$\text{P(class C/accident)} = \frac{P(\text{accident} \cap \text{class C})}{P(\text{accident})}$$

$$= \frac{0.2\times0.06}{0.03} = 0.4$$

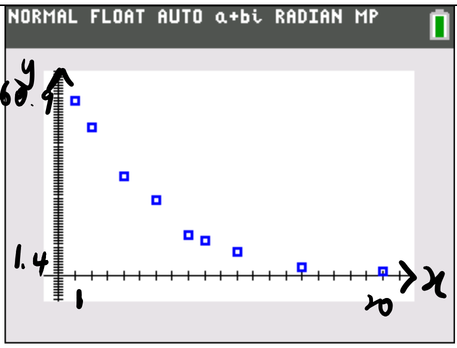
5(iii)

P(all three drivers are of class C and exactly one have accident)

$= (0.2\times0.94)^2 \times 0.2\times0.06\times\dfrac{3!}{2!}$

= 0.00127 (to 3sf)

6(i)



6(ii)	Regression line of y on x is $y = 49.7 - 3.09x$ When x = 17, y = -2.8466…= - 2.85 The linear model is not suitable since 1) the negative value of y is impossible or 2) the scatter diagram shows a curved relationship between the two variables.
6(iii)	Product moment correlation coefficient between W and x = -0.997837…= - 0.998
6(iv)	Since x is the controlled variable, we use the regression line of $\ln y$ on x : $\ln y = 4.3549 - 0.20532x$ [from GC] When $y = \frac{1}{2}(75)$, we have $\ln\frac{75}{2} = 4.3549 - 0.20532x$ $\Rightarrow x = 3.5581... = 3.6$ The weight will drop to half its original value in 3.6 weeks. The estimate is reliable since 1) The product moment correlation coefficient between $\ln y$ and x is -0.998 which is very close to -1, showing a strong negative linear correlation between $\ln y$ and x. 2) The estimate is an interpolation, because $y = \frac{1}{2}(75)$ is in the data range $1.4 \le y \le 60.9$.
7a(i)	Only working adults travelling by train will have a chance of being selected. Those who do not travel by train will have no chance of being chosen. Hence not every working adult in the country has an equal chance to be selected – therefore the sample is not a random sample.
7a(ii)	Let X hours be the number of hours of sleep of a randomly chosen adult and μ be the mean of X. To test $H_o : \mu = 6$ vs $H_1 : \mu > 6$ Since sample size is large, by CLT, $\overline{X} \sim N\left(6, \frac{\sigma^2}{50}\right)$ Since population variance σ^2 is unknown, it is replaced by s^2 Under H_o, test statistic $Z = \frac{\overline{X} - 6}{\frac{s}{\sqrt{50}}} \sim N(0,1)$ We use a one-tailed test at 5% level of significance, that is, reject H_o if p-value < 0.05 Sample readings: $\overline{x} = \frac{320}{50} = 6.4$, $s^2 = \frac{1}{49}\left(2308.5 - \frac{(320)^2}{50}\right) = 5.31633$ From GC, p-value = 0.109967 = 0.110 > 0.05 $\Rightarrow$ we **do not reject** H_o. Hence we conclude that there is **insufficient** evidence at the 5% level of significance to doubt the Health Promotion Board's claim. It is not necessary to assume that the number of hours of sleep follow a normal distribution because since the sample size is large, by the Central Limit Theorem, the sample mean follows a normal distribution approximately.

7b

To test $H_o : \mu = \mu_o$ vs $H_1 : \mu \neq \mu_o$

$s^2 = \frac{50}{49}(2.1)^2$, $\bar{x} = 6.14$,

Since H_o is not rejected at the 5% level,

$$-1.95996 < \frac{\bar{x} - \mu_o}{\frac{s}{\sqrt{n}}} < 1.95996$$

$$\Rightarrow -1.95996 < \frac{6.14 - \mu_o}{\frac{2.1}{\sqrt{49}}} < 1.95996$$

$$\Rightarrow 6.14 - 1.95996\frac{2.1}{\sqrt{49}} < \mu_o < 6.14 + 1.95996\frac{2.1}{\sqrt{49}}$$

$$\Rightarrow 5.552012 < \mu_o < 6.727988$$

$$\Rightarrow 5.55 < \mu_o < 6.73$$

8 (a)

Let X be the number of cream biscuits per box. $X \sim B(10, p)$

$P(X = 1 \text{ or } 2) = 0.15$

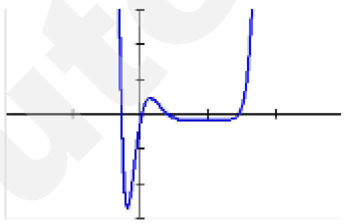
$P(X = 1) + P(X = 2) = 0.15$

$^{10}C_1 p^1(1-p)^9 + {}^{10}C_2 p^2(1-p)^8 = 0.15$

$10p(1-p)^9 + 45p^2(1-p)^8 = 0.15$

$5p(1-p)^8[2(1-p)+9p] = 0.15$

$5p(1-p)^8(2+7p) = 0.15$



Draw $y = 5p(1-p)^8(2+7p) - 0.15$

From G.C.,

$p = 0.0162$ or $p = 0.408$

(other values are 1.45 or -0.288 need to be rejected)

8 (b) (i)

$X \sim B(10, \frac{3}{4})$. Let $Y_1 = P(X = x)$.

X	Y_1
0	9.5E-7
1	2.9E-5
2	3.9E-4
3	.00309
4	.01622
5	.0584
6	.146
7	.25028
8	.28157
9	.18771
10	.05631

From G.C.,

since $P(X = 8)$ is the highest,

The most likely no. of cream biscuits = 8

(ii)

Let Y denote the random variable: Number of boxes with X = 5.

$Y \sim B(18, p)$ where $p = P(X=5) = 0.058399$

$P(3 \leq Y < 7) = P(Y \leq 6) - P(Y \leq 2)$

$= 0.0843$ (3 s.f.)

(iii)

Let U = no. of boxes sold at Usual price

Let D = no. of boxes sold at Discounted price

Let W: Total income per day.

$W = 10U + 8D$

$E(W) = 10E(U) + 8E(D) = 180 \times \$10 + 840 \times \$8 = \$8520$

$Var(W) = 10^2 Var(U) + 8^2 Var(D) = 64 \times 10^2 + 169 \times 8^2 = 17216$

Let $T = W_1 + W_2 + \ldots + W_{30}$

Since n = 30 is large, by Central Limit Theorem,

$T \sim N(30 \times 8520, 30 \times 17216) = N(255600, \sqrt{516480}^2)$ approximately

$P(T \geq \$255000) = 0.798$

9(i)

family	A		B		C		D	
	Adult	kids	Adult	kids	Adult	kids	Adult	kids
	2	2	1	2	1	3	1	1

4 family units, No. of ways = $4!\times4!\times3!\times5!\times2!$ = 829, 440

9(ii)

Case 1: 3,3,2 No. of ways = $\dfrac{{}^8C_3\times{}^5C_3\times{}^2C_2}{2!}$ = 280

Case 2: 2,2,2,2 No. of ways = $\dfrac{{}^8C_2\times{}^6C_2\times{}^4C_2\times{}^2C_2}{4!}$ = 105

Total no. of ways = 280 + 105 = 385

9(iii)

There is only 1 way to divide the 8 children and the adults into 2 circles to satisfy all conditions. Family A and B (3 adults & 4 kids) must be in 1 circle and Family C & D are in another circle.
Arrange the children in 1 circle : (4-1)!
Slot in adults in between children : ${}^4C_3\times3!$

No. of ways = $[(4-1)!\times{}^4C_3\times3!]\times[(4-1)!\times{}^4C_3\times3!]$
= 20736

10(i)

Let M denote the random variable: Height of a male visitor in cm. $M\sim N(165,12^2)$
Let F denote the random variable: Height of a female visitor in cm. $F\sim N(155,\sigma^2)$

$$P(150<F<160)=0.3829$$
$$P\left(-\frac{5}{\sigma}<Z<\frac{5}{\sigma}\right)=0.3829$$
$$P\left(Z<-\frac{5}{\sigma}\right)=\frac{1-0.3829}{2}=0.30855$$

From G.C. $-\dfrac{5}{\sigma}=-0.4999646$

$\Rightarrow \sigma = 10.0$ cm (3 sig. fig.)

(ii)

$$\frac{3}{4}M-F\sim N\left(\frac{3}{4}\times165-155,\ \frac{9}{16}\times12^2+10^2\right)=N(-31.25,181)$$
$$P\left(\left|\frac{3}{4}M-F\right|\le20\right)=P\left(-20\le\frac{3}{4}M-F\le20\right)$$
$$=0.201\quad(3\text{ sig. fig.})$$

Assumption: The heights of all male and female visitors are independent of one another.

(iii)

Probability Distribution of X:

x **(in \$)**	**P(X = x)**
0	$\frac{1}{2}P(M<120)+\frac{1}{2}P(F<120)=0.00016056$
10	$\frac{1}{2}P(120\le M<150)+\frac{1}{2}P(120\le F<150)=0.20693$
m	$\frac{1}{2}P(M\ge150)+\frac{1}{2}P(F\ge150)=0.79291$

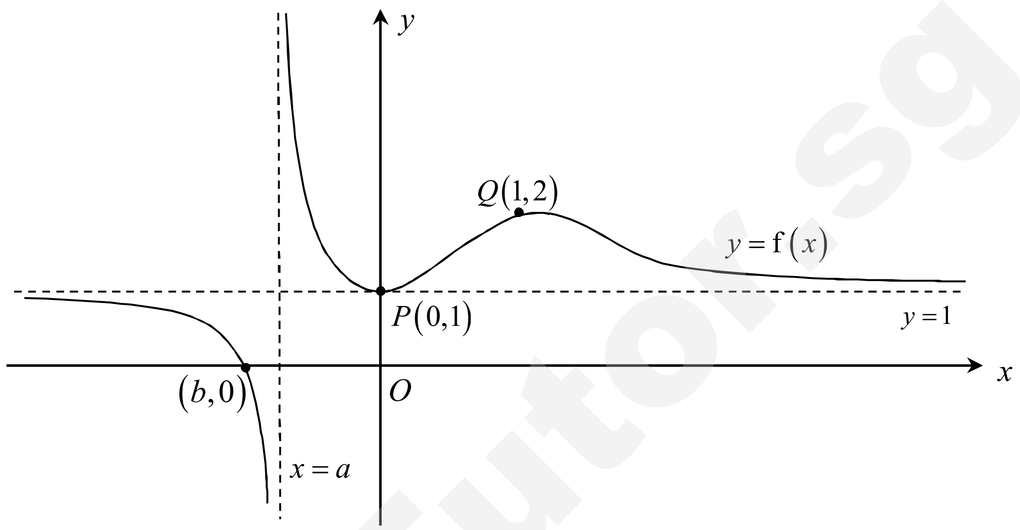
Given E(X) = 17.93 = 0 (0.00016056)+10(0.20693)+m(0.79291)
$\Rightarrow$ m = 20.00 (shown)

(iv)

$P(X_1+X_2+X_3>40)$ = P(20, 20, 20) + 3. P(20, 20, 10)
= 0.889

1 The graph of $y=\dfrac{x-1}{ax^2+bx+c}$, where a, b and c are non-zero constants, has a turning point at $(-1,1)$, and an asymptote with equation $x=-\dfrac{1}{3}$. Find the values of a, b and c. [5]

2 The diagram below shows the graph of $y=\mathrm{f}(x)$.



The graph passes through the point $(b,0)$ and has turning points at $P(0,1)$ and $Q(1,2)$. The lines $y=1$ and $x=a$, where $b<a<-\dfrac{1}{2}$, are asymptotes to the curve.

On separate diagrams, sketch the graphs of

(i) $y=\mathrm{f}\left(\dfrac{x-1}{2}\right)$, [3]

(ii) $y=\mathrm{f}'(x)$, [3]

labelling, in terms of a and b where applicable, the exact coordinates of the points corresponding to P and Q, and the equations of any asymptotes.

3 Solve the inequality $\dfrac{1}{x+a}\le\dfrac{2a}{x^2-a^2}$, leaving your answer in terms of a, where a is a positive real number. [3]

Hence or otherwise, find $\displaystyle\int_{2a}^{4a}\left|\frac{1}{x+a}-\frac{2a}{x^2-a^2}\right|\mathrm{d}x$ exactly. [4]

4 **(i)** Expand $(k+x)^n$, in ascending powers of x, up to and including the term in x^2, where k is a non-zero real constant and n is a negative integer. [3]

(ii) State the range of values of x for which the expansion is valid. [1]

(iii) In the expansion of $\left(k+y+3y^2\right)^{-3}$, the coefficient of y^2 is 2. By using the expansion in **(i)**, find the value of k. [3]

5 The points O, A and B are on a plane such that relative to the point O, the points A and B have non-parallel position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.
The point C with position vector $\mathbf{c}$ is on the plane OAB such that OC bisects the angle AOB.
Show that $\left(\dfrac{\mathbf{a}}{|\mathbf{a}|}-\dfrac{\mathbf{b}}{|\mathbf{b}|}\right)\cdot\mathbf{c}=0$. [2]

The lines AB and OC intersect at P. By first verifying that $\overrightarrow{OC}$ is parallel to $\dfrac{\mathbf{a}}{|\mathbf{a}|}+\dfrac{\mathbf{b}}{|\mathbf{b}|}$, show that the ratio of $AP:PB=|\mathbf{a}|:|\mathbf{b}|$. [6]

6 It is given that $\mathrm{e}^y=(1+\sin x)^2$.

(i) Show that

$$\mathrm{e}^y\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]=2(\cos 2x-\sin x).$$

By repeated differentiation, find the series expansion of y in ascending powers of x, up to and including the term in x^3, simplifying your answer. [5]

(ii) Show how you can use the standard series expansion(s) to verify that the terms up to x^3 for your series expansion of y in **(i)** are correct. [3]

7 **(a)** Given that $2z+1=|w|$ and $2w-z=4+8\mathrm{i}$, solve for w and z. [5]

(b) Find the exact values of x and y, where $x, y\in\mathbb{R}$, such that $2\mathrm{e}^{-\left(\frac{3+x+\mathrm{i}y}{\mathrm{i}}\right)}=1-\mathrm{i}$. [4]

8 The curve C and the line L have equations $y=x^2$ and $y=\dfrac{1}{2}x-2$ respectively.

(i) The point A on C and the point B on L are such that they have the same x-coordinate. Find the coordinates of A and B that gives the shortest distance AB. [3]

(ii) The point P on C and the point Q on L are such that they have the same y-coordinate. Find the coordinates of P and Q that gives the shortest distance PQ. [3]

(iii) Find the exact area of the polygon formed by joining the points found in **(i)** and **(ii)**. [2]

(iv) A variable point on the curve C with coordinates (s, s^2) starts from the origin O and moves along the curve with s increasing at a rate of 2 units/s. Find the rate of change of the area bounded by the curve, the y-axis and the line $y=s^2$, at the instant when $s=\sqrt{2}$. [4]

9 **(a)** By writing

$$\sin\left(x+\frac{1}{4}\right)\pi-\sin\left(x-\frac{3}{4}\right)\pi$$

in terms of a single trigonometric function, find $\displaystyle\sum_{x=1}^{n}\cos\left(x-\frac{1}{4}\right)\pi$, leaving your answer in terms of n. [4]

(b) The function f is defined by

$$\mathrm{f}:x\mapsto\sin\left(x+\frac{1}{4}\right)\pi-\sin\left(x-\frac{3}{4}\right)\pi,\ x\in\mathbb{R},\ a\le x\le 1.$$

(i) State the range of f and sketch the curve when $a=-1$, labelling the exact coordinates of the points where the curve crosses the x- and y- axes. [3]

(ii) State the least value of a such that f^{-1} exists, and define f^{-1} in similar form. [3]

The function g is defined by

$$\mathrm{g}:x\mapsto\frac{2x}{1-x},\ x\in\mathbb{R},\ x\ge\frac{13}{5}.$$

Given that fg exists, find the greatest value of a, and the corresponding range of fg. [3]

10 Abbie and Benny each take a \$50 000 study loan for their 3-year undergraduate program, disbursed on the first day of the program. The terms of the loan are such that during the 3-year period of their studies, interest is charged at 0.1% of the outstanding amount at the end of each month. Upon graduation, interest is charged at 0.375% of the outstanding amount at the end of each month.

(a) Since the interest rate is lower during her studies, Abbie decides that she will make a constant payment at the beginning of each month from the start of the program for its entire duration.

(i) Find the amount, correct to the nearest cent, Abbie needs to pay at the beginning of each month so that the outstanding amount after interest is charged remains at \$50 000 at the end of every month. [2]

(ii) After graduating, Abbie intends to increase her payment to a constant \$$k$ at the beginning of every month. Show that the outstanding amount Abbie owes the bank at the end of n months after graduation, and after interest is charged, is

$$\$\left[1.00375^n\left(50000\right)-\frac{803}{3}k\left(1.00375^n-1\right)\right].$$ [2]

(iii) Abbie plans to repay her loan within 10 years after graduation. Determine if she can do this with a monthly instalment of \$500, justifying your answer. [1]

Find the amount she needs to pay so that she fully repays her loan at the end of exactly 10 years after graduation, leaving your answer to the nearest cent. [2]

(b) Benny wishes to begin his loan repayment only after graduation. Like Abbie, he aims to repay the loan at the end of exactly 10 years after graduation.

Leaving your answer to the nearest cent, find

(i) the constant amount Benny needs to pay each month in order to do this, [3]

(ii) the amount of interest Benny pays altogether. [2]

11 **(i)** Show that for any real constant k,

$$\int t^2\,\mathrm{e}^{-kt}\,\mathrm{d}t=-\mathrm{e}^{-kt}\left(\frac{a}{k}t^2+\frac{b}{k^2}t+\frac{c}{k^3}\right)+D,$$

where D is an arbitrary constant, and a, b, and c are constants to be determined. [3]

On the day of the launch of a new mobile game, there were 100,000 players. After t months, the number of players on the game is x, in hundred thousands, where x and t are continuous quantities. It is known that, on average, one player recruits 0.75 players into the game per month, while the number of players who leave the game per month is proportional to t^2.

(ii) Write down a differential equation relating x and t. [1]

(iii) Using the substitution $x=u\,\mathrm{e}^{\frac{3}{4}t}$, show that the differential equation in **(ii)** can be reduced to

$$\frac{\mathrm{d}u}{\mathrm{d}t}=-pt^2\,\mathrm{e}^{-\frac{3}{4}t},$$

where p is a positive constant.

Hence solve the differential equation in **(ii)**, leaving your answer in terms of p. [5]

(iv) For $p=\frac{1}{3}$, find the maximum number of players on the game, and determine if there will be a time when there are no players on the game. [2]

(v) Find the range of values of p such that the game will have no more players after some time. [2]

ANNEX B

ACJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Graphs and Transformation	$a=3,\ b=7$ and $c=2$.
2	Graphs and Transformation	(i) $P(1, 1), Q(3, 2), x = 2a + 1, y = 1$; (ii) $P(0, 0), Q(1, 0), x = a, y = 0$.
3	Integration techniques	$x<-a$ or $a<x\le 3a$; $\ln\frac{75}{64}$.
4	Binomial Expansion	(i) $k^n\left(1+\frac{n}{k}x+\frac{(n)(n-1)}{2k^2}x^2+\ldots\right)$; (ii) $-\lvert k\rvert<x<\lvert k\rvert$; (iii) 0.642.
5	Vectors	
6	Maclaurin series	(i) $y=2x-x^2+\frac{1}{3}x^3+\ldots$;
7	Complex numbers	(a) $z=2,\ \ w=3+4\mathrm{i}$; (b) $x=-\frac{\pi}{4}-3,\ y=\frac{1}{2}\ln 2$.
8	Differentiation & Applications	(i) A$\left(\frac{1}{4},\frac{1}{16}\right)$ & B$\left(\frac{1}{4},-\frac{15}{8}\right)$; (ii) P$\left(\frac{1}{4},\frac{1}{16}\right)$ & Q$\left(\frac{33}{8},\frac{1}{16}\right)$; (iii) $\frac{961}{256}$; (iv) 8 units^2/s
9	Functions	(a) $\frac{1}{2}\sin\left(n+\frac{1}{4}\right)\pi-\frac{1}{2\sqrt{2}}$; (b)(i) $R_{\mathrm{f}}=[-2,2]$, $(-\frac{1}{4},0)$, $(\frac{3}{4},0)$, $(0,\sqrt{2})$; (b)(ii) $a=\frac{1}{4}$, $\mathrm{f}^{-1}:x\mapsto\frac{1}{\pi}\cos^{-1}\left(\frac{x}{2}\right)+\frac{1}{4}$, $x\in\left[-\sqrt{2},2\right]$; (b)(iii) greatest value of a is $-\frac{13}{4}$, $R_{\mathrm{fg}}=\left[-2,\sqrt{2}\right)$.
10	AP and GP	(a)(i) \$49.95; (iii) No, \$516.26 per month; (b)(i) \$535.17 per month; (ii) \$14220.43
11	Differential Equations	(i) $-\mathrm{e}^{-kt}\left(\frac{1}{k}t^2+\frac{2}{k^2}t+\frac{2}{k^3}\right)+D$ (ii) $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{3}{4}x-pt^2$

		(iii) $x = p\left(\frac{4}{3}t^2 + \frac{32}{9}t + \frac{128}{27}\right) + D\,e^{\frac{3}{4}t}$; (iv) max no of players on the game = 365 000; yes, $x = 0$ when $t = 4.35$ months; (v) $p > \frac{27}{128} = 0.211$.

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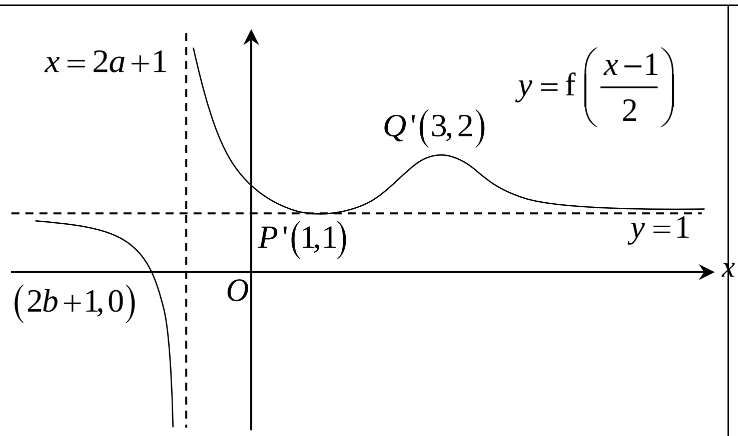
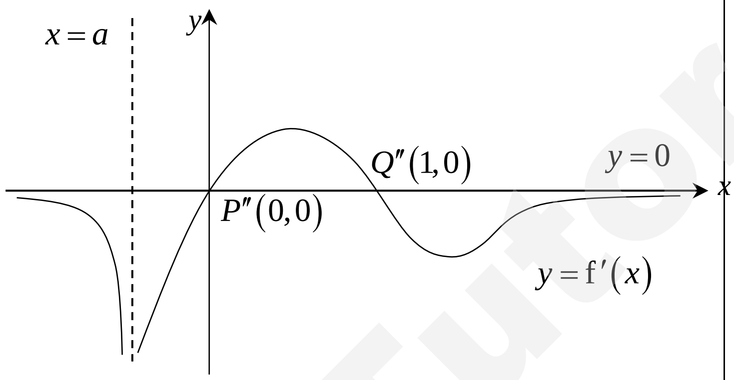
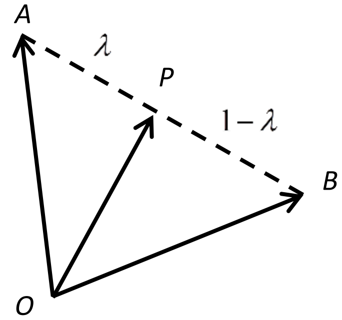
Preliminary Examination Paper 1 Markers Report

Qns	Solutions	Remarks
1	Passes through $\left(-1,1\right)$: $1=\dfrac{-2}{a-b+c} \;\Rightarrow\; a-b+c=-2$ ……………………..(1) Turning point at $\left(-1,1\right)$: $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right\|_{x=-1}=0$ now $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\left(ax^2+bx+c\right)-\left(x-1\right)\left(2ax+b\right)}{\left(ax^2+bx+c\right)^2}$ Hence $\dfrac{\left(a-b+c\right)-\left(-2\right)\left(-2a+b\right)}{\left(a-b+c\right)^2}=0$ $\Rightarrow\; \left(a-b+c\right)-\left(-2\right)\left(-2a+b\right)=0$ $\Rightarrow\; -3a+b+c=0$…………………………………(2) When $x=-\frac{1}{3}$, $ax^2+bx+c=0$: Hence $\dfrac{a}{9}-\dfrac{b}{3}+c=0$………………………………...(3) Solving (1), (2) and (3) simultaneously, we get $a=3$, $b=7$ and $c=2$. y	Some students forgot that the turning point (-1,1) lies on the curve and failed to substitute the point into the given equation to get an essential equation required for solving the unknowns. Some students made mistakes when differentiating using the product or quotient rule, or incorrectly rewrote y as $y=(x-1)(ax^2+bx+c)$ instead of $y=(x-1)(ax^2+bx+c)^{-1}$ which also resulted in an incorrect derivative. Some students did not know how to handle the information given on the asymptote. Some completed the square or did long division (both not necessary) and came up with an incorrect equation/conclusion. Some wrongly assumed that since $x=-\frac{1}{3}$ is an asymptote, therefore, $\to ax^2+bx+c=\left(x+\dfrac{1}{3}\right)\left(x-c\right)$ $\to ax^2+bx+c=\left(3x+1\right)\left(x-c\right)$ $\to ax^2+bx+c=\left(x+\dfrac{1}{3}\right)^2$ $\to ax^2+bx+c=\left(3x+1\right)^2$ which made assumptions on the values of *a*; those who assumed *a*=3 might have obtained the same final answer because *a* happened to be 3 in this case, but the method was incorrect.

2(i)		Almost the whole cohort gets either full marks or 1 mark (shape of the curve) for this question. Students has difficulty in handling $\frac{x-1}{2}$. Most students failed to read it as $\frac{x}{2}-\frac{1}{2}$. Thus the common mistake majority did was a translation of 1 unit in the positive x – direction followed by a stretching of factor 2 parallel to the x – axis.
2(ii)		About 80% of the students are able to identify the asymptotes $x=a$, $y=0$ and the x-axis intercepts $P''(0,0)$, $Q''(1,0)$. Of these students, about 70% got full marks as some students couldn't get the shape of the curve. Most students remembered to write the points in coordinates.
3	$\frac{1}{x+a}\le\frac{2a}{x^2-a^2}\Rightarrow\frac{1}{x+a}-\frac{2a}{x^2-a^2}\le 0$ $\Rightarrow\frac{x-3a}{(x+a)(x-a)}\le 0$  $\therefore x<-a$ or $a<x\le 3a$ $\int\frac{1}{x+a}-\frac{2a}{x^2-a^2}\,\mathrm{d}x=\ln(x+a)-\ln\left(\frac{x-a}{x+a}\right)=\ln\frac{(x+a)^2}{x-a}$ OR $\int\frac{x-3a}{(x+a)(x-a)}\,\mathrm{d}x=\int\frac{2}{x+a}-\frac{1}{x-a}\,\mathrm{d}x=\ln\frac{(x+a)^2}{x-a}$ $\int_{2a}^{4a}\left\lvert\frac{1}{x+a}-\frac{2a}{x^2-a^2}\right\rvert\mathrm{d}x$ $=\int_{2a}^{3a}-\left(\frac{1}{x+a}-\frac{2a}{x^2-a^2}\right)\mathrm{d}x+\int_{3a}^{4a}\frac{1}{x+a}-\frac{2a}{x^2-a^2}\,\mathrm{d}x$	Many students still are unfamiliar with the basics of solving inequalities and lack the basic skills of factorisation: (1) Do not know how to find the lowest common multiple of the denominators. Many gave $(x^2-a^2)(x+a)$ as the denominator instead of $(x-a)(x+a)$. Those who did so made a mess out of the numerator and could not factorise the numerator properly. (2) Many did not even know how to factorise x^2-a^2. (3) Many insisted on removing the denominator and change the inequality to an inequality involving polynomial only. However they could not do it properly and made a mess out of the polynomial and could not factorise. (4) For those using graphical method, they attempted to draw the graph of $y=\frac{1}{x+a}-\frac{2a}{x^2-a^2}$ and did not do it properly. They most likely just copied the graph from G.C. without drawing the horizontal asymptote. (5) Whether by using the sign test with number line or using the graphical method, students still could not obtain the answer correctly, giving the wrong range of values of x.

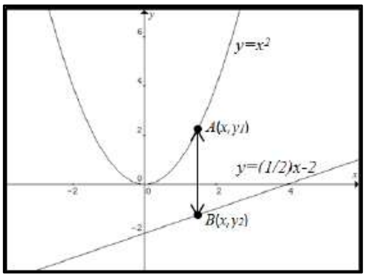
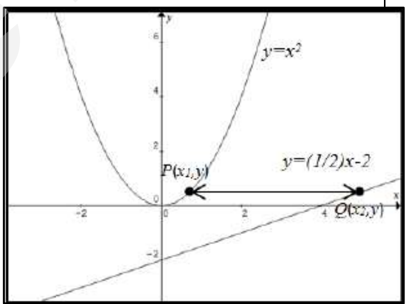
	$\int_{2a}^{3a} -\left(\frac{1}{x+a}-\frac{2a}{x^2-a^2}\right)\mathrm{d}x+\int_{3a}^{4a}\frac{1}{x+a}-\frac{2a}{x^2-a^2}\,\mathrm{d}x$ $=-\left[\ln\frac{(x+a)^2}{x-a}\right]_{2a}^{3a}+\left[\ln\frac{(x+a)^2}{x-a}\right]_{3a}^{4a}$ $=-\left(\ln\frac{16a^2}{2a}-\ln\frac{9a^2}{a}\right)+\left(\ln\frac{25a^2}{3a}-\ln\frac{16a^2}{2a}\right)$ $=-\ln\frac{8}{9}+\ln\frac{25}{24}=\ln\left(\frac{25}{24}\times\frac{9}{8}\right)=\ln\frac{75}{64}$	Even some of the values for the x-intercept and vertical asymptotes, $x=-a, x=a, x=3a$ were incorrect particularly, $x=3a.$ Even for those who did almost everything correct included $x=-a, x=a$ as part of the answer. For integration, very few students use Partial Fractions but used the formula in MF26 to integrate directly and most people applied the formula correctly. Most people could carry out the integration properly but could not obtain the final simplified answer $\ln\frac{75}{64}$. There were quite a number of students who apply the formula $\int\frac{1}{x^2-a^2}\,dx=\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+c$ to $\int\left\|\frac{1}{x^2-a^2}\right\|dx=\left\|\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)\right\|+c$. Some even carried forward the polynomial obtained in the earlier portion for the question on inequality to replace fractions $\frac{1}{x+a}-\frac{2a}{x^2-a^2}$ as the integrand.
4(i)	$(k+x)^n=k^n\left(1+\frac{x}{k}\right)^n$ $=k^n\left(1+n\left(\frac{x}{k}\right)+\frac{(n)(n-1)}{2!}\left(\frac{x}{k}\right)^2+...\right)$ $=k^n\left(1+\frac{n}{k}x+\frac{(n)(n-1)}{2k^2}x^2+...\right)$	(i) Most candidates knew more or less what to do, although mistakes were common; the most common were $(k+x)^n=k\left(1+\frac{x}{k}\right)^n$ or $(k+x)^n=\left(\frac{1}{k}\right)^n(1+kx)^n$ $=\left(\frac{1}{k}\right)^n\left(1+nkx+\frac{(n)(n-1)}{2!}(kx)^2+...\right)$ Some left answer as $(k+x)^n=k^n\left(1+n\left(\frac{x}{k}\right)+\frac{(n)(n-1)}{2!}\left(\frac{x}{k}\right)^2+...\right)$ Did not simplify $\left(\frac{x}{k}\right)^2=\frac{x^2}{k^2}$ No marks was awarded for $(k+x)^n\cong\left(k^n+nk^{n-1}x+\frac{(n)(n-1)}{2}k^{n-2}x^2\right)$. And $(k+x)^n=\left(k^n+\binom{n}{1}k^{n-1}x+\binom{n}{2}k^{n-2}x^2+.....\right)$

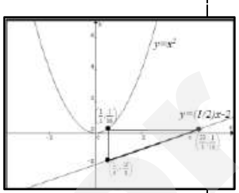
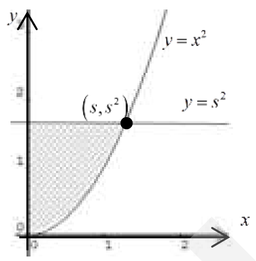
		$= \left(k^n + nk^{n-1}x + \frac{(n)(n-1)}{2}k^{n-2}x^2\right)$
4(ii)	$\left\|\frac{x}{k}\right\| < 1 \Rightarrow \|x\| < \|k\|$ $\therefore -\|k\| < x < \|k\|$	Very badly done . Do not know how to proceed after $\left\|\frac{x}{k}\right\| < 1$ and left answers like $\|x\| < \|k\|$ or $-k < x < k$ or $-1 < x < 1$ Candidates who used Maclaurin series to find the binomial expansion of $(k+x)^n$ have problems finding region of validity. Gave answers like $\|x\| < 1$ or $x \in R$
4(iii)	Let $x = y + 3y^2$ and $n = -3$: $(k + y + 3y^2)^{-3}$ $= k^{-3}\left(1 + \frac{(-3)}{k}(y+3y^2) + \frac{(-3)(-4)}{2k^2}(y+3y^2)^2 + ...\right)$ $= k^{-3}\left(1 - \frac{3}{k}y - \frac{9}{k}y^2 + \frac{6}{k^2}y^2 + ...\right)$ $\Rightarrow k^{-3}\left(-\frac{9}{k} + \frac{6}{k^2}\right) = 2 \Rightarrow 2k^5 + 9k - 6 = 0$ $\therefore k = 0.642$ (to 3 sf)	Surprisingly quite a number of students do not know how to solve $-\frac{9}{k^4} + \frac{6}{k^5} = 2$ or $2k^5 + 9k - 6 = 0$
5	$\frac{\overrightarrow{OC}\cdot\overrightarrow{OA}}{\left\|\overrightarrow{OC}\right\|\left\|\overrightarrow{OA}\right\|} = \frac{\overrightarrow{OC}\cdot\overrightarrow{OB}}{\left\|\overrightarrow{OC}\right\|\left\|\overrightarrow{OB}\right\|}$ $\frac{\mathbf{c}\cdot\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{c}\cdot\mathbf{b}}{\|\mathbf{b}\|} \Rightarrow \frac{\mathbf{c}\cdot\mathbf{a}}{\|\mathbf{a}\|} - \frac{\mathbf{c}\cdot\mathbf{b}}{\|\mathbf{b}\|} = 0 \Rightarrow \mathbf{c}\cdot\left(\frac{\mathbf{a}}{\|\mathbf{a}\|} - \frac{\mathbf{b}}{\|\mathbf{b}\|}\right) = 0$ Alternatively $\left(\frac{\mathbf{a}}{\|\mathbf{a}\|} - \frac{\mathbf{b}}{\|\mathbf{b}\|}\right)\cdot\mathbf{c} = \frac{\mathbf{a}\cdot\mathbf{c}}{\|\mathbf{a}\|} - \frac{\mathbf{b}\cdot\mathbf{c}}{\|\mathbf{b}\|}$ $= \frac{\|\mathbf{a}\|\|\mathbf{c}\|\cos\theta}{\|\mathbf{a}\|} - \frac{\|\mathbf{b}\|\|\mathbf{c}\|\cos\theta}{\|\mathbf{b}\|} = 0$	This question was not well done with a significant number of students not attempting the question at all. Among those who attempted the questions, very few students managed to show that $AP : PB = \|\mathbf{a}\| : \|\mathbf{b}\|$. Many students **wrongly assumed** that $\|\mathbf{a}\| = \|\mathbf{b}\|$. Students need to know that for this question, ⇨ *OC* bisecting angle *AOB* **doesn't mean** that *AP=PB*. ⇨ $\overrightarrow{OP}$ & $\overrightarrow{OC}$ may **NOT** be perpendicular to $\overrightarrow{AB}$. ⇨ $\mathbf{c}$ **may not** be parallel to $\frac{\mathbf{a}}{\|\mathbf{a}\|} + \frac{\mathbf{b}}{\|\mathbf{b}\|}$ since $\|\mathbf{a}\|$ **may not** be equal to $\|\mathbf{b}\|$. ⇨ $\mathbf{a} + \mathbf{b} \neq \frac{\mathbf{a}}{\|\mathbf{a}\|} + \frac{\mathbf{b}}{\|\mathbf{b}\|}$
	$\left(\frac{\mathbf{a}}{\|\mathbf{a}\|} + \frac{\mathbf{b}}{\|\mathbf{b}\|}\right)\cdot\left(\frac{\mathbf{a}}{\|\mathbf{a}\|} - \frac{\mathbf{b}}{\|\mathbf{b}\|}\right) = \left(\frac{\mathbf{a}\cdot\mathbf{a}}{\|\mathbf{a}\|^2} - \frac{\mathbf{b}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\right)$ $= \left(\frac{\|\mathbf{a}\|^2}{\|\mathbf{a}\|^2} - \frac{\|\mathbf{b}\|^2}{\|\mathbf{b}\|^2}\right) = 1 - 1 = 0$	
	P is on $l_{AB} \Rightarrow \overrightarrow{OP} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = \lambda\mathbf{b} + (1-\lambda)\mathbf{a}$ P is on $l_{OC} \Rightarrow \overrightarrow{OP} = \mu\overrightarrow{OC} = \mu\left(\frac{\mathbf{a}}{\|\mathbf{a}\|} + \frac{\mathbf{b}}{\|\mathbf{b}\|}\right)$ Equating $\lambda\mathbf{b} + (1-\lambda)\mathbf{a} = \mu\left(\frac{\mathbf{a}}{\|\mathbf{a}\|} + \frac{\mathbf{b}}{\|\mathbf{b}\|}\right)$ Comparing coefficients of **a** and **b**	

	$\lambda=\frac{\mu}{\lvert\mathbf{b}\rvert}$ and $1-\lambda=\frac{\mu}{\lvert\mathbf{a}\rvert}$ Note that $AP:PB=\lambda:1-\lambda$, therefore $AP:PB=\frac{\mu}{\lvert\mathbf{b}\rvert}:\frac{\mu}{\lvert\mathbf{a}\rvert}=\lvert\mathbf{a}\rvert:\lvert\mathbf{b}\rvert$. 	$\Rightarrow \frac{\mathbf{a}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert^2}\neq\frac{\mathbf{a}^2}{\lvert\mathbf{a}\rvert^2}$ There was also poor usage of notation. For example many students wrote "a" instead of "$\underset{\sim}{a}$" and also $\overrightarrow{AB}$ instead of $\frac{\overrightarrow{AB}}{\lvert\overrightarrow{AB}\rvert}$.
6(i)	$e^y=(1+\sin x)^2$ Differentiating w.r.t. x, $e^y\frac{dy}{dx}=2(1+\sin x)\cos x$ $e^y\frac{dy}{dx}=2\cos x+\sin 2x$ Differentiating w.r.t. x again, $e^y\frac{d^2y}{dx^2}+\frac{dy}{dx}e^y\frac{dy}{dx}=-2\sin x+2\cos 2x$ $e^y\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=2(\cos 2x-\sin x)$ (shown) Differentiating w.r.t. x: $e^y\left[\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}\right]+\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]e^y\frac{dy}{dx}=2(-2\sin 2x-\cos x)$ Substituting $x=0$, $y=0$; $\frac{dy}{dx}=2$; $\frac{d^2y}{dx^2}=-2$; $\frac{d^3y}{dx^3}=2$ $\Rightarrow y=0+2x+\frac{-2}{2!}x^2+\frac{2}{3!}x^3+\ldots$ $\therefore y=2x-x^2+\frac{1}{3}x^3+\ldots$	Most students can do the proof in the first part quite well although some have longer methods. Shorter method is to differentiate implicitly to get $e^y\frac{dy}{dx}=2(1+\sin x)\cos x$ Most students could differentiate correctly $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2$ to get $\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}$ A few fail to use the product rule to differentiate and got this part wrong.
6(ii)	**<u>Method 1</u>**:	Common mistake made is to assume x is a small angle and use the small angle approximation. Correct approximation is

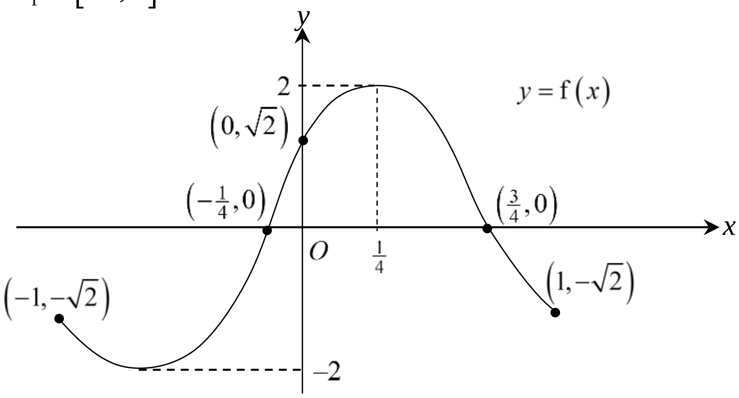
	$e^y = (1+\sin x)^2$ $\Rightarrow y = \ln(1+\sin x)^2$ $= 2\ln(1+\sin x)$ $= 2\ln\left(1+\left(x-\frac{x^3}{3!}\right)+...\right)$ $= 2\left(\left(x-\frac{x^3}{3!}\right)-\frac{\left(x-\frac{x^3}{3!}\right)^2}{2}+\frac{\left(x-\frac{x^3}{3!}\right)^3}{3}+...\right)$ $= 2\left(x-\frac{x^3}{6}-\frac{x^2}{2}+\frac{x^3}{3}+...\right)$ $= 2x-x^2+\frac{1}{3}x^3+...$ which is same as the expansion for y found in (i), up to and including the term in $x^3 \Rightarrow$ verified. **<u>Method 2</u>**: $\text{RHS} = (1+\sin x)^2$ $= \left(1+x-\frac{x^3}{3!}\right)^2$ $= 1+x-\frac{x^3}{6}+x+x^2-\frac{x^3}{6}+...$ $= 1+2x+x^2-\frac{x^3}{3}+...$ $\text{LHS} = e^y$ $= e^{\left(2x-x^2+\frac{1}{3}x^3+...\right)}$ (using expansion for y in (i)) $= 1+\left(2x-x^2+\frac{1}{3}x^3\right)+\frac{\left(2x-x^2+\frac{1}{3}x^3\right)^2}{2!}+\frac{\left(2x-x^2+\frac{1}{3}x^3\right)^3}{3!}+...$ $= 1+2x-x^2+\frac{1}{3}x^3+\frac{4x^2-2x^3-2x^3}{2}+\frac{8x^3}{6}+...$ $= 1+2x+x^2-\frac{1}{3}x^3+...$ LHS = RHS $\Rightarrow$ verified.	$\sin x = x-\frac{x^3}{3!}$. In some answers, detailed workings were not shown clearly.

7(a)	$2z+1=\|w\|$ ....(1) $2w-z=4+8\mathrm{i}$.......(2) $2z+1=$ a positive real number $\Rightarrow$ Let $z=x$ and $w=a+b\mathrm{i}$ From (2): $2(a+bi)-x=4+8\mathrm{i}$ $\Rightarrow$ Comparing Re and Im parts, $2a-x=4$ $2b=8\Rightarrow b=4$ From (1): $2x+1=\sqrt{a^2+b^2}$....(3) Substitute $b=4$ and $x=2a-4$ into (3): $2(2a-4)+1=\sqrt{a^2+16}\Rightarrow(4a-7)^2=a^2+16$ $16a^2-56a+49=a^2+16\Rightarrow 15a^2-56a+33=0$ $\Rightarrow a=\dfrac{11}{15}$ or $a=3$ $\Rightarrow x=-\dfrac{98}{15}$ or $x=2$ but $2z+1=$ a positive real number $\Rightarrow$ when $x=-\dfrac{98}{15}$, $2z+1=2\left(-\dfrac{98}{15}\right)+1<0$ $\Rightarrow$ reject $x=-\dfrac{98}{15}$ and $a=\dfrac{11}{15}$ $\Rightarrow x=2,\quad a=3, b=4$ $\Rightarrow z=2,\quad w=3+4\mathrm{i}$	Many students failed to see that z is a real number from eqn (1), resulting in solving simultaneous eqns with many unknown, which most failed to simplify and continue to solve correctly. Some common mistakes: 1. $\|w\|=w$ 2. $\|w\|=\pm w$ 3. $\|w\|=\sqrt{a^2+(ib)^2}=\sqrt{a^2-b^2}$
7(b)	$2\mathrm{e}^{-\left(\frac{3+x+\mathrm{i}y}{\mathrm{i}}\right)}=1-\mathrm{i}$ $2\mathrm{e}^{3\mathrm{i}+x\mathrm{i}-y}=\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}$ $2\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}(3+x)}=\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}$ $\Rightarrow$ By comparing modulus and args: $2\mathrm{e}^{-y}=\sqrt{2}$ and $\qquad 3+x=-\dfrac{\pi}{4}$ $-y=\ln\left(\dfrac{\sqrt{2}}{2}\right)\qquad\Rightarrow x=-\dfrac{\pi}{4}-3$ $\Rightarrow y=-\ln\left(\dfrac{\sqrt{2}}{2}\right)$ (or $\ln\sqrt{2}$ or $\dfrac{1}{2}\ln 2$)	It's a surprise to see that many students didn't write $1-\mathrm{i}$ in $r\mathrm{e}^{\mathrm{i}\theta}$ form to solve the problem. Even if some did it, they made a mistake in the value of $\theta=\dfrac{3}{4}\pi$ or $\dfrac{1}{4}\pi$. In general, students have good idea how to manipulate $-\left(\frac{3+x+\mathrm{i}y}{\mathrm{i}}\right)$ to get $-y+3\mathrm{i}+x\mathrm{i}$ and they also have clear idea of comparing the modulus and argument terms.

8(i)	Let V be the distance AB.  $V = y_1 - y_2$ $= x^2 - \left(\frac{1}{2}x - 2\right)$ $= x^2 - \frac{1}{2}x + 2$ $\frac{\mathrm{d}V}{\mathrm{d}x} = 2x - \frac{1}{2}$ when $\frac{\mathrm{d}V}{\mathrm{d}x} = 0, \quad x = \frac{1}{4}$ $\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = 2 > 0 \Rightarrow$ min. value when $x = \frac{1}{4}$ when $x = \frac{1}{4}$, $y = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$ $y = \frac{1}{2}\left(\frac{1}{4}\right) - 2 = -\frac{15}{8}$ $\therefore$ coords on C (Pt A):$\left(\frac{1}{4}, \frac{1}{16}\right)$ & coords on L (Pt B):$\left(\frac{1}{4}, -\frac{15}{8}\right)$.	For many, distance was not even considered, instead look at gradients of L and C. Those who used distance, some were penalised for not checking nature of stationary value. Many students made slips in simple calculations such as $2x - \frac{1}{2} \Rightarrow x = 1$, $y^{-\frac{1}{2}} = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$ etc.
8(ii)	Let H be the distance PQ.  $H = x_2 - x_1 = 2(y+2) - \sqrt{y}$ $\frac{\mathrm{d}H}{\mathrm{d}y} = 2 - \frac{1}{2}y^{-\frac{1}{2}}$ when $\frac{\mathrm{d}H}{\mathrm{d}y} = 0$, $2 - \frac{1}{2}y^{-\frac{1}{2}} = 0 \Rightarrow 2 = \frac{1}{2}y^{-\frac{1}{2}}$ $\Rightarrow y = 4^{-2} = \frac{1}{16}$ $\frac{\mathrm{d}^2H}{\mathrm{d}y^2} = \frac{1}{4}y^{-\frac{3}{2}}$ $\Rightarrow$ when $y = \frac{1}{16}, \frac{\mathrm{d}^2H}{\mathrm{d}y^2} = \frac{1}{4}\left(\frac{1}{16}\right)^{-\frac{3}{2}} = 16 > 0$ $\Rightarrow$ min. value when $y = \frac{1}{16}$	

	when $y=\dfrac{1}{16}$, $x=\sqrt{\dfrac{1}{16}}=\dfrac{1}{4}$ $x=2\left(\dfrac{1}{16}\right)+2=\dfrac{33}{8}$ $\therefore$ coords on C (Pt P):$\left(\dfrac{1}{4},\dfrac{1}{16}\right)$ & coords on L (Pt Q):$\left(\dfrac{33}{8},\dfrac{1}{16}\right)$.	
8(iii)	Area of polygon = Area of triangle Minimum distance $AB=\dfrac{1}{16}-\left(-\dfrac{15}{8}\right)=\dfrac{31}{16}$ Minimum distance $PQ=\dfrac{33}{8}-\left(\dfrac{1}{4}\right)=\dfrac{31}{8}$  $\therefore$ Area of polygon $=\dfrac{1}{2}\times\dfrac{31}{16}\times\dfrac{31}{8}=\dfrac{961}{256}$ sq units	
8(iv)	 $\dfrac{ds}{dt}=2$ Method 1: Area $=A=\int_0^{s^2} x\ dy=\int_0^{s^2}\sqrt{y}\ dy=\left[\dfrac{y^{\frac{3}{2}}}{3/2}\right]_0^{s^2}=\dfrac{2}{3}s^3$ $\dfrac{dA}{dt}=\dfrac{dA}{ds}\times\dfrac{ds}{dt}=2s^2\times 2=4s^2$ $\therefore$ when $s=\sqrt{2}$, $\dfrac{dA}{dt}=(4)\left(\sqrt{2}\right)^2=8$ units^2/s	Well answered except those who treated area bounded as a constant instead of a variable, hence were clueless as to how to get $\dfrac{dA}{ds}$. When finding area, confused by the variable point, many students did not use definite integral.

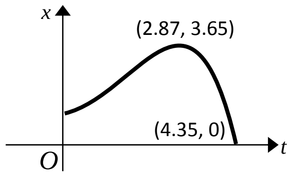
	Method 2: Area = A = Area of rectangle – Area bounded by curve, x-axis and $x = s$ $= s \times s^2 - \int_0^s y \ \mathrm{d}x = s^3 - \int_0^s x^2 \ \mathrm{d}x = s^3 - \left[\frac{x^3}{3}\right]_0^s = \frac{2}{3}s^3$ $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}s} \times \frac{\mathrm{d}s}{\mathrm{d}t} = 2s^2 \times 2 = 4s^2$ $\therefore$ when $s = \sqrt{2}, \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 4\left(\sqrt{2}\right)^2 = 8$ units^2/s	
9(a)	By factor formula, $\sin\left(x+\tfrac{1}{4}\right)\pi - \sin\left(x-\tfrac{3}{4}\right)\pi = 2\cos\left[\tfrac{1}{2}\left(2x-\tfrac{1}{2}\right)\pi\right]\sin\left(\tfrac{1}{2}\pi\right)$ $= 2\cos\left(x-\tfrac{1}{4}\right)\pi.$ Hence $\sum_{x=1}^{n} 2\cos\left(x-\tfrac{1}{4}\right)\pi$ $= \sum_{x=1}^{n}\left[\sin\left(x+\tfrac{1}{4}\right)\pi - \sin\left(x-\tfrac{3}{4}\right)\pi\right]$ $=\left[\sin\tfrac{5}{4}\pi - \sin\tfrac{1}{4}\pi\right] + \left[\sin\tfrac{9}{4}\pi - \sin\tfrac{5}{4}\pi\right] + \ldots$ $+\left[\sin\left(n-\tfrac{3}{4}\right)\pi - \sin\left(n-\tfrac{7}{4}\right)\pi\right] + \left[\sin\left(n+\tfrac{1}{4}\right)\pi - \sin\left(n-\tfrac{3}{4}\right)\pi\right]$ $= \sin\left(n+\tfrac{1}{4}\right)\pi - \sin\tfrac{1}{4}\pi$ $= \sin\left(n+\tfrac{1}{4}\right)\pi - \tfrac{1}{\sqrt{2}}$ Therefore, $\sum_{x=1}^{n} \cos\left(x-\tfrac{1}{4}\right)\pi = \tfrac{1}{2}\sin\left(n+\tfrac{1}{4}\right)\pi - \tfrac{1}{2\sqrt{2}}$.	Many students expanded each term using compound angle formula then tried to collapse the terms back into one trig function, mostly without success. The **most common error** was to first factorise π out of the expression then use factor formula: $\sin\left(x+\tfrac{1}{4}\right)\pi - \sin\left(x-\tfrac{3}{4}\right)\pi$ $= \pi\left[\sin\left(x+\tfrac{1}{4}\right) - \sin\left(x-\tfrac{3}{4}\right)\right]$ which is ridiculous. Students need to realise that this is a 1-mark question which should not require page-long working. Those who couldn't do the first part naturally were not able to do this part accurately. Amongst those who did, some evaluated the value of each trigo expression and hence could not see which terms cancelled out using the method of difference: $\sum_{x=1}^{n}\left[\sin\left(x+\tfrac{1}{4}\right)\pi - \sin\left(x-\tfrac{3}{4}\right)\pi\right]$ $=\left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] + \left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] + \ldots$ $+\left[\sin\left(n-\tfrac{3}{4}\right)\pi - \sin\left(n-\tfrac{7}{4}\right)\pi\right] + \left[\sin\left(n+\tfrac{1}{4}\right)\pi - \sin\left(n-\tfrac{3}{4}\right)\pi\right]$

9(b)(i)	$R_\mathrm{f} = [-2, 2]$  	Biggest problem for the plot is students keying in to G.C. wrongly. Plotting $Y = \sin\left(X + \frac{1}{4}\right)\pi - \sin\left(X - \frac{3}{4}\right)\pi$ instead of $Y = \sin\left[\left(X + \frac{1}{4}\right)\pi\right] - \sin\left[\left(X - \frac{3}{4}\right)\pi\right]$ Students should be careful, using brackets when appropriate. Once the graph is correctly plotted in the G.C. with the correct domain, they should notice that one full period is plotted, and that the range is easily read off the G.C.
(b)(ii)	Least value of a is $\frac{1}{4}$. Let $y = 2\cos\left(x - \frac{1}{4}\right)\pi$. Then $\cos^{-1}\left(\frac{y}{2}\right) = \left(x - \frac{1}{4}\right)\pi \quad \Rightarrow \quad x = \dfrac{\cos^{-1}\left(\frac{y}{2}\right)}{\pi} + \dfrac{1}{4}$. $\therefore \quad \mathrm{f}^{-1} : x \mapsto \dfrac{1}{\pi}\cos^{-1}\left(\dfrac{x}{2}\right) + \dfrac{1}{4}, \qquad x \in \left[-\sqrt{2}, 2\right]$	If graph is correctly sketched, least value of a is easily found. Method mark for making x the subject of $y = 2\cos\left(x - \frac{1}{4}\right)\pi$ is awarded for any attempt to find the inverse function, regardless of whether students' graphs are sketched correctly. Many students were careless in either not quoting the domain of f^{-1} or, for those who did, quoted it forgetting that domain of f is now restricted so that its inverse exsits.
(b)(iii)	fg exists $\quad \Rightarrow \quad R_\mathrm{g} \subseteq D_\mathrm{f}$ now $R_\mathrm{g} = \left[-\dfrac{13}{4}, -2\right)$ and $D_\mathrm{f} = [a, 1]$ since fg exists, $a \le -\dfrac{13}{4}$. Hence the greatest value of a is $-\frac{13}{4}$. $R_\mathrm{fg} = \mathrm{f}\left(R_\mathrm{g}\right) = \mathrm{f}\left[-\dfrac{13}{4}, -2\right) = \left[-2, \sqrt{2}\right]$.	Students were not tenacious enough to find R_g properly, perhaps discouraged from the earlier parts. g is a straight forward function that can be sketched with the G.C., bearing in mind that there is a horizontal asymptote at $y = -2$.
10(a)(i)	After one month, if she pays \$$x$ at the beginning of the month, she will owe the bank	Many students were confused about the interest rate, and hence multiplied by 1.1 or

	$\$(50000-x)\times(1.001)$ Hence $(50000-x)\times(1.001)=50000 \quad \Rightarrow \quad x=49.95$ Abbie needs to pay \$49.95 (to the nearest cent) a month.	1.01. Some merely took 0.1% of \$50,000.
(a)(ii)	One month after graduating, she owes $(50000-k)\times(1.00375)$. n months after graduating, she will owe $1.00375^n(50000-k)-1.00375^{n-1}k-\ldots-1.00375k$ $=1.00375^n(50000)-k\left(1.00375^n+1.00375^{n-1}+\ldots+1.00375\right)$ $=1.00375^n(50000)-k\left[\dfrac{1.00375\left(1.00375^n-1\right)}{1.00375-1}\right]$ $=1.00375^n(50000)-\dfrac{803}{3}k\left(1.00375^n-1\right)$ (shown).	While many students were able to deduce that this was the sum of a GP, a common mistake was thinking that the last/first term of the GP was 1 instead of 1.00375.
(a)(iii)	Sub n = 120, and k = 500: $1.00375^{120}(50000)-\dfrac{803}{3}(500)\left(1.00375^{120}-1\right)=2467.11>0$. No, she cannot. A monthly payment of \$500 is not enough. When n = 120, $1.00375^{120}(50000)-\dfrac{803}{3}k\left(1.00375^{120}-1\right)=0$ $\Rightarrow \quad k=516.26$ (nearest cent) She needs to pay \$516.26 per month.	Many students did not realise n was in months, and used n = 10.
(b)(i)	Oustanding amount upon graduation $=1.001^{36}(50000)$ $=51831.86$ Using Abbie's formula, but with a starting outstanding amount of \$51831.86, $1.00375^{120}(51831.86)-\dfrac{803}{3}k\left(1.00375^{120}-1\right)=0$ $\Rightarrow \quad k=535.17$ (nearest cent) He needs to pay \$535.17 per month.	Some students used 1.00375^{36}. Some took the 35^{th} power. Many students did not realise they could use the same formula as (a)(iii) but with a different starting amount. As with the previous parts, some interpreted the interest rate wrongly and used 1.1 or 1.01, and some thought n was in years.
(b)(ii)	$120\times535.17-50000=14220.43$ (to 2 d.p.) He paid \$14220.43 in interest altogether.	Some students had very involved ways of calculating the interest, including summing the GP all over again. Many students did not subtract

		50,000.
11(i)	$\int t^2\,\mathrm{e}^{-kt}\,\mathrm{d}t = -\frac{1}{k}\mathrm{e}^{-kt}(t^2) - \int -\frac{1}{k}\mathrm{e}^{-kt}(2t)\,\mathrm{d}t$ $= -\frac{1}{k}t^2\,\mathrm{e}^{-kt} + \frac{2}{k}\left[-\frac{1}{k}\mathrm{e}^{-kt}(t) - \int -\frac{1}{k}\mathrm{e}^{-kt}(1)\,\mathrm{d}t\right]$ $= -\frac{1}{k}t^2\,\mathrm{e}^{-kt} - \frac{2}{k^2}t\,\mathrm{e}^{-kt} - \frac{2}{k^3}\mathrm{e}^{-kt} + D$ $= -\mathrm{e}^{-kt}\left(\frac{1}{k}t^2 + \frac{2}{k^2}t + \frac{2}{k^3}\right) + D$	Some students were careless in the first step and could only be awarded the subsequent method mark if they proceeded to integrate by parts a second time. Some students integrated the terms incorrectly or made wrong choices for the terms. Students should remember that the aim of integration by parts is to obtain a simpler integral which can then be integrated (unless it requires the "loop" technique which is not the case for this question) and realise that something is wrong if they ended up with one which looks even more complicated. Few students left this part blank or did not proceed to do integration by parts a second time. Quite a number of students did not put the final expression in the required form and lost marks. Students are reminded to take note of the requirements of the questions.
(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3}{4}x - pt^2$	Majority could not get this expression or even gave an expression for x in terms of t instead ($\frac{\mathrm{d}x}{\mathrm{d}t}$ was not even seen) which should not be the case since the question asked for a "differential equation". Some students also made mistakes in the unit for x (in hundred thousands) or missed out the "x" in the "0.75x" term (or incorrectly wrote it as 0.75t) or missed out the constant of proportionality "p".

(iii)	$x = u e^{\frac{3}{4}t} \Rightarrow \dfrac{dx}{dt} = \dfrac{3}{4} u e^{\frac{3}{4}t} + e^{\frac{3}{4}t} \dfrac{du}{dt}$ $\dfrac{3}{4} u e^{\frac{3}{4}t} + e^{\frac{3}{4}t} \dfrac{du}{dt} = \dfrac{3}{4} u e^{\frac{3}{4}t} - pt^2 \Rightarrow \dfrac{du}{dt} = -pt^2 e^{-\frac{3}{4}t}$ $u = p e^{-\frac{3}{4}t} \left(\dfrac{1}{\frac{3}{4}} t^2 + \dfrac{2}{\left(\frac{3}{4}\right)^2} t + \dfrac{2}{\left(\frac{3}{4}\right)^3} \right) + D$ $= p e^{-\frac{3}{4}t} \left(\dfrac{4}{3} t^2 + \dfrac{32}{9} t + \dfrac{128}{27} \right) + D$ $\Rightarrow \dfrac{x}{e^{\frac{3}{4}t}} = p e^{-\frac{3}{4}t} \left(\dfrac{4}{3} t^2 + \dfrac{32}{9} t + \dfrac{128}{27} \right) + D$ $\therefore x = p \left(\dfrac{4}{3} t^2 + \dfrac{32}{9} t + \dfrac{128}{27} \right) + D e^{\frac{3}{4}t}$ When $t = 0, x = 1$, $1 = p \left(\dfrac{128}{27} \right) + D \Rightarrow D = 1 - \dfrac{128}{27} p$ $x = p \left(\dfrac{4}{3} t^2 + \dfrac{32}{9} t + \dfrac{128}{27} \right) + \left(1 - \dfrac{128}{27} p \right) e^{\frac{3}{4}t}$	Students would not be able to show the given differential equation if the expression in (i) was incorrect. Some students were not able to correctly differentiate $u e^{\frac{3}{4}t}$. Students should read the question carefully and if they are not able to show the required DE, students should still proceed to solve the given DE, and not solve their own incorrect DE, which was what many students did. Many students incorrectly used $k = -\frac{3}{4}$ and were penalised. A few students failed to see the link to part (i) and redid the integration without using the results obtained in (i). Many students failed to substitute "*x*" back into the solution and of those who did, majority forgot the arbitrary constant D or forgot to multiply $e^{\frac{3}{4}t}$ to D – some even labelled $D e^{\frac{3}{4}t}$ as another constant $E = D e^{\frac{3}{4}t}$ which is incorrect since it now contains the variable *t* and is not just a product of constants. Many also failed to sub in the initial conditions, which was required to obtain the arbitrary constant in terms of *p*. Some did so in the next part but no credit was awarded since it was the requirement in (iii). Some students used the wrong units or failed to show the link from *x* to *u* when using the initial conditions.
(iv)	When $p = \frac{1}{3}$,	Parts (iv) and (v) were badly

	$x=\frac{1}{3}\left(\frac{4}{3}t^2+\frac{32}{9}t+\frac{128}{27}\right)+\left(-\frac{47}{81}\right)e^{\frac{3}{4}t}$  Maximum number of players on the game = 365 000. Yes, x = 0 when t = 4.35 months.	done as students were not likely to obtain the answers to these parts if their expression for x was incorrectly in (iii) – only a handful of students could obtain the correct final expression for x in (iii). Students were expected to use the GC (graph) for this part and not expected to differentiate, solve the equation, etc to find the maximum value or the t-value which gave 0 players - only 2 marks are awarded for the two required answers and students can get the hint from the marks allocation that they were not expected to manually find these answers on their own.
(v)	For $x=0$ after some time, $1-\frac{128}{27}p<0 \Rightarrow p>\frac{27}{128}=0.211$	This required students to see that the coefficient of the exponential term, $\left(1-\frac{128}{27}p\right)e^{\frac{3}{4}t}$, in the expression for x found in (iii) had to be negative in order for there to be no players after some time. However, as mentioned in (iii), only a handful of students had the exponential term in their solution for x, thus this part was not well done.

Section A: Pure Mathematics [40 marks]

1 Given that $1+\mathrm{i}$ is a root of the equation $z^3-4(1+\mathrm{i})z^2+(-2+9\mathrm{i})z+5-\mathrm{i}=0$, find the other roots of the equation. [4]

2 A curve C has parametric equations

$$x=\cos t$$
$$y=\frac{1}{2}\sin 2t$$

where $\frac{\pi}{2}\le t\le\frac{3\pi}{2}$.

(i) Find the equation of the normal to C at the point P with parameter p. [2]

The normal to C at the point when $t=\frac{2\pi}{3}$ cuts the curve again. Find the coordinates of the point of intersection. [2]

(ii) Sketch C, clearly labelling the coordinates of the points where the curve crosses the x- and y- axes. [1]

(iii) Find the cartesian equation of C. [2]

The region bounded by C is rotated through π radians about the x-axis. Find the exact volume of the solid formed. [3]

3

(i) Find $\displaystyle\int \frac{x}{\left(1+x^2\right)^2}\,\mathrm{d}x$. [2]

(ii) By using the substitution $x=\tan\theta$, show that

$$\int \frac{1}{\left(1+x^2\right)^2}\,\mathrm{d}x=k\left(\frac{x}{1+x^2}+\tan^{-1}x\right)+c,$$

where c is an arbitrary constant, and k is a constant to be determined. [5]

(iii) Hence find $\displaystyle\int \frac{x^2}{\left(1+x^2\right)^2}\,\mathrm{d}x$. [3]

(iv) Using all of the above, find $\displaystyle\int \frac{x^2+2x+5}{\left(1+x^2\right)^2}\,\mathrm{d}x$, simplifying your answer. [2]

4 **(a)** **(i)** The unit vector $\mathbf{d}$ makes angles of $60°$ with both the x- and y-axes, and θ with the z-axis, where $0°\le\theta\le 90°$. Show that $\mathbf{d}$ is parallel to $\mathbf{i}+\mathbf{j}+\sqrt{2}\mathbf{k}$. [3]

(ii) The line m is parallel to $\mathbf{d}$ and passes through the point with coordinates $(2,-1,0)$. Find the coordinates of the point on m that is closest to the point with coordinates $(3,2,0)$. [3]

(b) The plane p_1 has equation $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+2\mathbf{k})=5$, and the line l has equation $\frac{x-a}{2}=\frac{y-1}{b}=-\frac{z}{2}$, where a and b are constants.

Given that l lies on p_1, show that $b=1$ and find the value of a. [2]

(i) The plane p_2 contains l and is perpendicular to p_1. Find the equation of p_2 in the form $\mathbf{r}\cdot\mathbf{n}=c$, where c is a constant to be determined. [3]

	(ii) The variable point $P(x, y, z)$ is equidistant from p_1 and p_2. Find the cartesian equation(s) of the locus of P. [3]

Section B: Statistics [60 marks]

5	A group of 12 students consists of 5 bowlers, 4 canoeists and 3 footballers. **(i)** The group sits at a round table with 12 seats. In how many different ways can they sit so that all the players of the same sport sit together? [2] **(ii)** The group stands in a line. In how many different ways can they stand so that *either* the bowlers are all next to one another *or* the canoeists are all next to one another *or* both? [2] **(iii)** Find the number of ways in which a delegation of 8 can be selected from this group if it must include at least 1 student from each of the 3 sports. [2]
6	Alex and his friend stand randomly in a queue with 3 other people. The random variable X is the number of people standing between Alex and his friend. **(i)** Show that $\mathrm{P}(X = 2) = 0.2$. [2] **(ii)** Tabulate the probability distribution of X. [2] **(iii)** Find $\mathrm{E}(X)$ and $\mathrm{E}(X-1)^2$. Hence find $\mathrm{Var}\,(X)$. [3]
7	It has been suggested that the optimal pH value for shampoo should be 5.5, to match the pH level of healthy scalp. Any pH value that is too low or too high may have undesirable effects on the user's hair and scalp. A shampoo manufacturer wants to investigate if the pH level of his shampoo is at the optimal value, by carrying out a hypothesis test at the 10% significance level. He measures the pH value, x, of n randomly chosen bottles of shampoo, where n is large. **(a)** In the case where $n = 30$, it is found that $\sum x = 178.2$ and $\sum x^2 = 1238.622$. **(i)** Find unbiased estimates of the population mean and variance, and carry out the test at the 10% significance level. [6] **(ii)** Explain if it is necessary for the manufacturer to assume that the pH value of a bottle of shampoo follows a normal distribution. [1] **(b)** In the case where n is unknown, assume that the sample mean is the same as that found in **(a)**. **(i)** State the critical region for the test. [1] **(ii)** Given that n is large and that the population variance is found to be 6.5, find the greatest value of n that will result in a favourable outcome for the manufacturer at the 10% significance level. [3]
8	A swim school takes in both male and female primary school students for competitive swimming lessons. The school assesses its students' progress each year by recording the time, t seconds, each student takes to swim a 50-metre lap in breaststroke, and the number of months, m, that he or she has been at the school. The records for 8 randomly chosen students are shown in the following table. (table below) **(i)** Labelling the axes clearly, draw a scatter diagram for the data and explain, in context, why a linear model would not be suitable to predict the time taken by a student to swim a lap of breaststroke given the number of months that he or she has been at the school. [2]

m	6	7	10	12	15	19	21	24
t	92.32	87.11	66.12	59.41	53.94	43.82	42.07	41.45

	It is desired to fit a model of the form $\ln(t-C)=a+bm$, where C is a suitable constant. The product moment correlation coefficient r between m and $\ln(t-C)$ for some possible values of C are shown in the table below.

C	36	37	38	39
r	–0.992114		–0.992681	–0.992192

(ii) Calculate the value of r for $C=37$, giving your answer correct to 6 decimal places. [1]

(iii) Use the table and your answer to **(ii)** to choose the most appropriate value for C. Explain your choice. [2]

For the remainder of this question, use the value of C that you have chosen in **(iii)**.

(iv) Find the equation of the least squares regression line of $\ln(t-C)$ on m. Give an interpretation of C in the context of the question. [2]

(v) Another student who has been swimming at the school for 9 months clocked a time of 60.33 seconds for a lap of breaststroke. Using your regression line, comment on the student's swimming ability. [2]

(vi) Suggest an improvement to the data collection process so that the results could provide a fairer gauge of the expected outcome for the students in the first 2 years of lessons. [1]

9 **(i)** A procedure for accepting or rejecting a large batch of manufactured articles is such that an inspector first selects and examines a random sample of 10 articles from the batch. If the sample contains at least 2 defective articles, the batch is rejected.
It is known that the proportion of articles that are defective is 0.065. Show that the probability that a batch of articles is accepted is 0.866, correct to three significant figures. [1]

To confirm the decision, another inspector follows the same procedure with another random sample of 10 articles from the batch. If the conclusion of both inspectors are the same, the batch will be accepted or rejected as the case may be. Otherwise, one of the inspectors will select a further random sample of 10 from the same batch to examine. The batch is then rejected if there are at least 2 defective articles. Otherwise, it is accepted. Find

(a) the probability that a batch is eventually accepted, [3]

(b) the expected number of articles examined per batch. [4]

(ii) In order to cut labour cost, an alternative procedure is introduced. A random sample of 10 articles is taken from the batch and if the sample contains not more than 1 defective article then the batch is accepted. If the sample contains more than 2 defective articles, the batch is rejected. If the sample contains exactly 2 defective articles, a second sample of 10 articles is taken and if this contains no defective article then the batch is accepted. Otherwise, the batch is rejected. Given that the proportion of defective articles in the batch is p, show that the probability that the batch is accepted is A where

$$A=(1+9p)(1-p)^9+45p^2(1-p)^{18}.$$ [2]

If the probability that, of 100 batches inspected, more than 80 of them will be accepted is 0.98, find the value of p. [3]

10 **(a)** An examination taken by a large number of students is marked out of a total score of 100. It is found that the mean is 73 marks and that the standard deviation is 15 marks.

(i) Give a reason why the normal distribution is not a good model for the distribution of marks for the examination. [1]

(ii) The marks for a random sample of 50 students is recorded. Find the probability that the mean mark of this sample lies between 70 and 75. [2]

	(b)	The interquartile range of a distribution is the difference between the upper and lower quartile values for the distribution. The lower quartile value, l, of a distribution X, is such that $\mathrm{P}(X < l) = 0.25$. The upper quartile value, u, of the same distribution is such that $\mathrm{P}(X < u) = 0.75$. The marks of another examination is known to follow a normal distribution. If a student who scores 51 marks is at the 80th percentile, and the interquartile range is found to be 10.8 marks, find the mean mark and the standard deviation of the marks scored by students who took the examination. [5]
	(c)	In a third examination, the marks scored by students are normally distributed with a mean of 52 marks and a standard deviation of 13 marks. **(i)** If 50 is the passing mark and 289 students are expected to pass, how many candidates are there? [2] **(ii)** Find the smallest integer value of m such that more than 90% of the candidates will score within m marks of the mean. [3]

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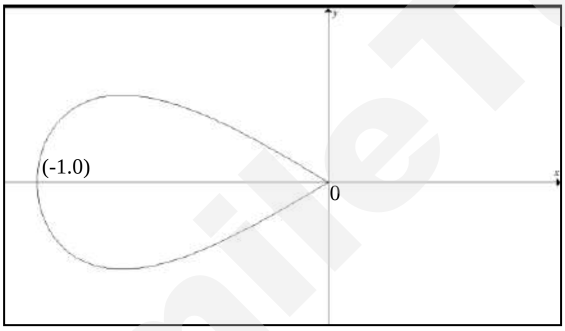
Preliminary Examination Paper 2 Markers Report

<table>
<tr><th>Qns</th><th>Solutions</th><th>Remarks</th></tr>
<tr><td>1</td><td>$z^3-4(1+\mathrm{i})z^2+(-2+9\mathrm{i})z+5-\mathrm{i}=0$
$\left(z-(1+\mathrm{i})\right)\left(Az^2+Bz+C\right)=0$
By comparing coefficients,
$z^3: A=1$
$z^0: -(1+\mathrm{i})C=5-\mathrm{i}$
$\Rightarrow C=\dfrac{5-\mathrm{i}}{-(1+\mathrm{i})}=-2+3\mathrm{i}$
$z^2: B-(1+\mathrm{i})=-4(1+\mathrm{i})$
$\Rightarrow B=-3(1+\mathrm{i})$
$\Rightarrow \left(z-(1+\mathrm{i})\right)\left(z^2-3(1+\mathrm{i})z-2+3\mathrm{i}\right)=0$
Solving $\left(z^2-3(1+\mathrm{i})z-2+3\mathrm{i}\right)=0$:
$z=\dfrac{-\left(-3(1+\mathrm{i})\right)\pm\sqrt{\left(-3(1+\mathrm{i})\right)^2-4(1)\left(-2+3\mathrm{i}\right)}}{2(1)}$
$=\dfrac{3+3\mathrm{i}\pm\sqrt{8+6\mathrm{i}}}{2}$
$=\dfrac{3+3\mathrm{i}\pm(3+\mathrm{i})}{2}=3+2\mathrm{i}$ or i
$\therefore$ other 2 roots are $z=3+2\mathrm{i}$ or $z=\mathrm{i}$</td><td>Quite a large number of students say that $1-\mathrm{i}$ is another root, which is wrong because not all the coefficients are real. Students who did this gets a 0.

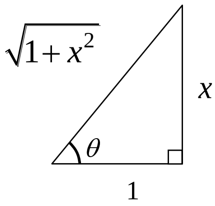
When comparing coefficients, many students use $a+ib, c+id$ as the two other roots which resulted in unnecessarily tedious and complicated working.

About half who used the quadratic formula had problem evaluating $\sqrt{8+6\mathrm{i}}$, which can be done using GC.</td></tr>
<tr><td>2(i)</td><td>$x=\cos t$
$y=\dfrac{1}{2}\sin 2t$
$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\dfrac{\cos 2t}{-\sin t}$
$\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=p}=\dfrac{\cos 2p}{-\sin p}\Rightarrow$ gradient of normal $=\dfrac{\sin p}{\cos 2p}$
$\Rightarrow$ equation of normal at $\left(\cos p, \dfrac{1}{2}\sin 2p\right)$:
$y-\dfrac{1}{2}\sin 2p=\dfrac{\sin p}{\cos 2p}\left(x-\cos p\right)$
$y=\dfrac{\sin p}{\cos 2p}x+\dfrac{1}{2}\left(\sin 2p-\tan 2p\right)$</td><td>Generally students were able to write down the eqn of normal at point with parameter p.
However, some wrote $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}p}\times\dfrac{\mathrm{d}p}{\mathrm{d}x}$. Although no mark is deducted here, students should realize that p in most cases is a constant (though not specified by question) and $\dfrac{\mathrm{d}y}{\mathrm{d}p}=0$.
A minority wrote the eqn of normal as $y-\dfrac{1}{2}\sin 2p=\dfrac{\sin t}{\cos 2t}\left(x-\cos p\right)$ without putting $t=p$.

Many careless mistakes in evaluating the cosine and sine values when $t=\dfrac{2\pi}{3}$, resulting in wrong eqns of</td></tr>
</table>

	$\Rightarrow$ equation of normal at $t=\dfrac{2\pi}{3}$: $y=\dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}x+\dfrac{1}{2}\left(-\dfrac{\sqrt{3}}{2}-\sqrt{3}\right)\Rightarrow y=-\sqrt{3}x-\dfrac{1}{4}\left(3\sqrt{3}\right)$....(1) To find point of intersection of normal and C (when the normal cuts C again), Substitute $x=\cos t$ and $y=\dfrac{1}{2}\sin 2t$ into (1): $\dfrac{1}{2}\sin 2t=-\sqrt{3}\left(\cos t\right)-\dfrac{1}{4}\left(3\sqrt{3}\right)$ $\dfrac{1}{2}\sin 2t+\sqrt{3}\left(\cos t\right)+\dfrac{1}{4}\left(3\sqrt{3}\right)=0$ From GC, $t=2.094395$ (corresponds to $t=\dfrac{2\pi}{3}$) or $t=3.495928$ $\Rightarrow$ point normal meets C again: $\left(\cos\left(3.495928\right),\dfrac{1}{2}\sin\left(2\left(3.495928\right)\right)\right)=\left(-0.938,0.325\right)$	normal, such as $y=-\sqrt{3}x-\dfrac{\sqrt{3}}{4}$, $y=\sqrt{3}x-\dfrac{3\sqrt{3}}{4}$ etc Many did not understand that the question is asking for point of intersection **between the curve and the normal at** $t=\dfrac{2\pi}{3}$ and simply sub $t=\dfrac{2\pi}{3}$ to find the point. Those who correctly sub $x=\cos t$ and $y=\dfrac{1}{2}\sin 2t$ into (1) often did not use GC to solve the eqn, and simply stopped at this step.
2(ii)		Many did not note the range of values of t and sketched 2 loops. A number of students did not give the coordinates of the x-intercept.
2(iii)	Method 1: $x=\cos t\Rightarrow x^2=\cos^2 t$ $y=\dfrac{1}{2}\sin 2t\Rightarrow y=\sin t\cos t$ $\Rightarrow y^2=\sin^2 t\cos^2 t=\left(1-\cos^2 t\right)\cos^2 t=\left(1-x^2\right)x^2$ $\therefore$ Cartesian equation: $y^2=\left(1-x^2\right)x^2$ Method 2: $x=\cos t\Rightarrow\cos t=\dfrac{x}{1},\sin t=\dfrac{\pm\sqrt{1-x^2}}{1}\quad\left(\because\dfrac{\pi}{2}\le t\le\dfrac{3\pi}{2}\right)$ $y=\dfrac{1}{2}\sin 2t\Rightarrow y=\sin t\cos t=\pm\sqrt{1-x^2}\left(x\right)$ $\therefore$ Cartesian equation: $y=\pm x\sqrt{1-x^2}$	Many simply wrote the eqn as $y=\sin 2(\cos^{-1}x)$ and did not go on to simplify. Those who used method 2 often omitted the negative sign.

	Method 3: $x=\cos t \Rightarrow x^2=\cos^2 t \Rightarrow \cos 2t = 2\cos^2 t - 1 = 2x^2-1$ $y=\dfrac{1}{2}\sin 2t \Rightarrow \sin 2t = 2y$ Using $\sin^2 2t+\cos^2 2t=1$, $(2y)^2+\left(2x^2-1\right)^2=1$ $\therefore$ Cartesian equation: $4y^2+\left(2x^2-1\right)^2=1$	
	Method 1: $\int_{-1}^{0}\pi y^2\,\mathrm{d}x$ $=\pi\int_{-1}^{0}\left(1-x^2\right)x^2\,\mathrm{d}x$ $=\pi\int_{-1}^{0}x^2-x^4\,\mathrm{d}x=\pi\left[\dfrac{x^3}{3}-\dfrac{x^5}{5}\right]_{-1}^{0}=\dfrac{2}{15}\pi$ units^3 Method 2 (not advised): $x=\cos t\Rightarrow\dfrac{\mathrm{d}x}{\mathrm{d}t}=-\sin t$ when $x=0,\ t=\dfrac{\pi}{2},\dfrac{3\pi}{2}$ (can use either) when $x=-1,\ t=\pi$ $\int_{-1}^{0}\pi y^2\,\mathrm{d}x$ $=\pi\int_{\pi}^{\frac{3\pi}{2}}\left(\dfrac{1}{2}\sin 2t\right)^2(-\sin t)\ \mathrm{d}t$ $=-\pi\int_{\pi}^{\frac{3\pi}{2}}(\sin t\cos t)^2(\sin t)\ \mathrm{d}t$ $=-\pi\int_{\pi}^{\frac{3\pi}{2}}\sin^2 t\cos^2 t(\sin t)\ \mathrm{d}t$ $=-\pi\int_{\pi}^{\frac{3\pi}{2}}\left(1-\cos^2 t\right)\cos^2 t(\sin t)\ \mathrm{d}t$ $=-\pi\int_{\pi}^{\frac{3\pi}{2}}\left(\cos^2 t-\cos^4 t\right)(\sin t)\ \mathrm{d}t$ $=-\pi\left(-\int_{\pi}^{\frac{3\pi}{2}}(\cos t)^2(-\sin t)\ \mathrm{d}t+\int_{\pi}^{\frac{3\pi}{2}}(\cos t)^4(-\sin t)\ \mathrm{d}t\right)$ $=-\pi\left(-\left[\dfrac{(\cos t)^3}{3}\right]_{\pi}^{\frac{3\pi}{2}}+\left[\dfrac{(\cos t)^5}{5}\right]_{\pi}^{\frac{3\pi}{2}}\right)$ $=-\pi\left(-0-\dfrac{1}{3}+0+\dfrac{1}{5}\right)=\dfrac{2}{15}\pi$ units^3	Many did not realize that method 1 is the desired method and were stucked with method 2 as they did not know how to integrate the integrand. For method 2, common mistakes include wrong limits, or writing volume as $2\int_{-1}^{0}\pi y^2\mathrm{d}x$.
3(i)	$\int\dfrac{x}{(1+x^2)^2}\,\mathrm{d}x=\dfrac{1}{2}\int\dfrac{2x}{(1+x^2)^2}\,\mathrm{d}x=-\dfrac{1}{2(1+x^2)}+c$	This is a simple question. No one should be getting this wrong.

3(ii)	$x=\tan\theta \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2\theta$ $\displaystyle\int \frac{1}{(1+x^2)^2}\,\mathrm{d}x=\int \frac{1}{\left(1+\tan^2\theta\right)^2}\sec^2\theta\,\mathrm{d}\theta$ $\displaystyle=\int \frac{1}{\sec^2\theta}\,\mathrm{d}\theta=\int\cos^2\theta\,\mathrm{d}\theta$ $\displaystyle=\int \frac{\cos 2\theta+1}{2}\,\mathrm{d}\theta=\frac{1}{2}\left(\frac{\sin 2\theta}{2}+\theta\right)$ $=\frac{1}{2}\left(\sin\theta\cos\theta+\theta\right)+c$ $=\frac{1}{2}\left(\frac{x}{1+x^2}+\tan^{-1}x\right)+c$ $x=\tan\theta$ $\sin\theta=\frac{x}{\sqrt{1+x^2}}$ $\cos\theta=\frac{1}{\sqrt{1+x^2}}$ 	(ii) was done better than (i) in general. A significant minority did not know that $1+\tan^2\theta=\sec^2\theta$ though, and either got stuck or used very long methods to get to a integrand they could work with. As this is a **show** question, students have to present the way they substitute the variable x back into the integral clearly, either using the triangle or with identities. This was quite poorly done though a lot of leeway was given in the awarding of marks.
3(iii)	$\displaystyle\int \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x=\int\frac{x^2+1-1}{(1+x^2)^2}\,\mathrm{d}x=\int \frac{1}{1+x^2}-\frac{1}{\left(1+x^2\right)^2}\,\mathrm{d}x$ $=\tan^{-1}x-\frac{1}{2}\left(\frac{x}{1+x^2}+\tan^{-1}x\right)+c$ $=\frac{1}{2}\left(\tan^{-1}x-\frac{x}{1+x^2}\right)+c$	There were many different methods available here, the splitting (shown on the left). Other easy methods include: (1) using the substitution provided in (ii). (2) by parts with parts $\frac{x}{\left(1+x^2\right)^2}$ and x and using (i). A long method uses the parts $\frac{1}{\left(1+x^2\right)^2}$ and x^2. Many careless mistakes surfaced in this part (although they were prevalent throughout the question as well), such as confusing $\frac{1}{\left(1+x^2\right)^2}$ with $\frac{1}{1+x^2}$ or $\frac{1}{\left(1+x\right)^2}$.
3(iv)	$\displaystyle\int\frac{x^2+2x+5}{(1+x^2)^2}\,\mathrm{d}x=\int\frac{x^2}{(1+x^2)^2}+\frac{2x}{(1+x^2)^2}+\frac{5}{(1+x^2)^2}\,\mathrm{d}x$ $=\frac{1}{2}\left(\tan^{-1}x-\frac{x}{1+x^2}\right)+2\left(-\frac{1}{2(1+x^2)}\right)+5\left(\frac{1}{2}\left(\frac{x}{1+x^2}+\tan^{-1}x\right)\right)+c$ $=3\tan^{-1}x+\frac{2x-1}{1+x^2}+c$	This was generally well done, as students could use (i)-(iii). Working mark was given even if their integrals were wrong, as long as they were based on their answers in the earlier part. The simplification of the answer was not done by a significant minority.
4(a) (i)	$\mathbf{d}=\cos 60^\circ\mathbf{i}+\cos 60^\circ\mathbf{j}+\cos\gamma\mathbf{k}$ $\cos^2 60^\circ+\cos^2 60^\circ+\cos^2\gamma=1$ $\Rightarrow \cos^2\gamma=1-\frac{1}{4}-\frac{1}{4}=\frac{1}{2}$ $\Rightarrow \cos\gamma=\frac{1}{\sqrt{2}}\ (\because \gamma \text{ is acute})$	This part was rather poorly done, though most students can apply the geometrical definition of the scalar product and get 1 or 2 marks. Common errors include: (1) Not reading that **d** is a **unit** vector.

	$\mathbf{d}=\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k} \,//\, \mathbf{i}+\mathbf{j}+\sqrt{2}\mathbf{k}$	(2) poor presentation with regard to the treatment of vectors and scalars, for e.g. **d** = 0.5. In addition, the **<u>show</u>**ing part needs to be worked on. Students have to present steps logically and quote relevant information from the question as part of their reasoning.
(a)(ii)	$m:\mathbf{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}$ $\left(\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}-\begin{pmatrix}3\\2\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}=0\Rightarrow\left(\begin{pmatrix}-1\\-3\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}=0$ $\therefore(-1-3)+\lambda(1^2+1^2+\sqrt{2}^2)=0\Rightarrow\lambda=1$ Therefore position vector of point is $\begin{pmatrix}2\\-1\\0\end{pmatrix}+\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}=\begin{pmatrix}3\\0\\\sqrt{2}\end{pmatrix}$ Coordinates = $\left(3,0,\sqrt{2}\right)$ <u>OR</u> $\overrightarrow{AN}=\left(\overrightarrow{AP}\cdot\mathbf{d}\right)\mathbf{d}=\frac{\left(\begin{pmatrix}3\\2\\0\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}}{\sqrt{1+1+2}}\frac{\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}}{\sqrt{1+1+2}}=\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}$ $\therefore\overrightarrow{ON}=\overrightarrow{OA}+\overrightarrow{AN}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}=\begin{pmatrix}3\\0\\\sqrt{2}\end{pmatrix}$ Coordinates = $\left(3,0,\sqrt{2}\right)$	This is a simple part. No one should be getting this wrong. There were still students who upon not being able to show (a)(i), decided that (a)(ii) was not doable and had no attempt on it. A variety of methods were applied, though the easiest one is shown first on the left. Students who applied the vector of the projection with modulus sign instead of brackets could arrive at the answer as well, but they were not awarded the full marks due to a conceptual error. Of those who could do this part, around 50% of them lost the answer mark for not expressing in coordinates form.
4(b)	$l:\frac{x-a}{2}=\frac{y-1}{b}=-\frac{z}{2}\Rightarrow l:\mathbf{r}=\begin{pmatrix}a\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\b\\-2\end{pmatrix}$ $\begin{pmatrix}2\\b\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=0\Rightarrow 2+2b-4=0\Rightarrow b=1$ $\begin{pmatrix}a\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=5\Rightarrow a+2=5\Rightarrow a=3$	This was generally well-done, though a minority wrote $\begin{pmatrix}a+2\lambda\\1+b\lambda\\-2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=5$ $\Rightarrow a+2\lambda+2+2b\lambda-4\lambda=5$ but obviously did not understand why $2\lambda+2b\lambda-4\lambda=0$.
(b)(i)	p_2 perpendicular to p_1 $\Rightarrow\mathbf{n}_1 \,//\, p_2$ $p_2:\mathbf{r}=\begin{pmatrix}3\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}+\mu\begin{pmatrix}1\\2\\2\end{pmatrix}$	Some used longer method where they solved $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}=0$ and $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=0$ Some remembered that the direction vector of line of intersection is $\mathbf{n}_1\times\mathbf{n}_2$ and

	$\begin{pmatrix}2\\1\\-2\end{pmatrix}\times\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}6\\-6\\3\end{pmatrix}//\begin{pmatrix}2\\-2\\1\end{pmatrix}$ $p_2:\mathbf{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}3\\1\\0\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix}=4$	wrote $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}2\\1\\-2\end{pmatrix}$ but failed to include $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=0$ as another condition. A significant minority made careless mistakes while computing the vector product. They should remind themselves how to check for correctness of the vector product.
(b)(ii)	$\frac{1}{\sqrt{9}}\left\|\begin{pmatrix}x-3\\y-1\\z\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}\right\|=\frac{1}{\sqrt{9}}\left\|\begin{pmatrix}x-3\\y-1\\z\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}\right\|$ $\|x-3+2(y-1)+2z\|=\|2(x-3)-2(y-1)+z\|$ $\Rightarrow x+2y+2z-5=2x-2y+z-4$ $\Rightarrow x-4y-z=-1$ *or* $\Rightarrow x+2y+2z-5=-(2x-2y+z-4)$ $\Rightarrow 3x+3z=9\Rightarrow x+z=3$	Only less than 30 students are able to do this part. A handful gave good solutions, obtaining **n** as $\begin{pmatrix}1\\2\\2\end{pmatrix}+\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}3\\0\\3\end{pmatrix}$ or $\begin{pmatrix}1\\2\\2\end{pmatrix}-\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}-1\\4\\1\end{pmatrix}$.
5 (i)	Number of ways = $(3-1)!\cdot5!\cdot4!\cdot3!$ = 34 560	Generally well done
5 (ii)	Number of ways = N(5 bowlers together) + N(4 canoeists together) – N(5 bowlers together & 4 canoeists together) = $8!\cdot5!+9!\cdot4!-5!\cdot5!\cdot4!$ = 4 838 400 + 8 709 120 – 345 600 = 13 201 920	Most students added the three numbers instead of subtracting the case for intersection: $8!\cdot5!+9!\cdot4!+5!\cdot5!\cdot4!$. If students had drawn a venn diagram, the correct operation would have been clearer.
5 (iii)	Number of ways = N(Total) – N(0 bowlers) – N(0 canoeists) – N(0 footballers) = ${}^{12}C_8-0-{}^{8}C_8-{}^{9}C_8=485$	Very badly done, although there is a question in Tutorial 20 Q9. Many did ${}^{5}C_1*{}^{4}C_1*{}^{3}C_1*{}^{9}C_1$ which is a gross overcount.
6 (i)	$\mathrm{P}(X=2)=\mathrm{P}(A{*}{*}F{*},{*}A{*}{*}F)=2\left(\frac{2\times3!}{5!}\right)=\frac{1}{5}=0.2$ (shown)	6(i) and (ii) were very crucial parts to this question. Students who were unable to start finding the pdf of X, or did it wrongly, would not have been able to answer (iii). Some students lost marks for (i) because they lacked sufficient elaboration, e.g. writing simply 4/20 or 2/10 without justifying how they arrived at these numbers. They would have gotten the
6 (ii)	$\mathrm{P}(X=0)=\mathrm{P}(AF{*}{*}{*},{*}AF{*}{*},{*}{*}AF{*},{*}{*}{*}AF)=4\left(\frac{2\times3!}{5!}\right)=\frac{2}{5}$ $=0.4$ $\mathrm{P}(X=1)=\mathrm{P}(A{*}F{*}{*},{*}A{*}F{*},{*}{*}A{*}F)=3\left(\frac{2\times3!}{5!}\right)=\frac{3}{10}=0.3$ $\mathrm{P}(X=3)=\mathrm{P}(A{*}{*}{*}F)=\left(\frac{2\times3!}{5!}\right)=\frac{1}{10}=0.1$ $\begin{array}{\|c\|c\|c\|c\|c\|}\hline x & 0 & 1 & 2 & 3\\\hline\end{array}$	

<table>
<tr>
<td></td>
<td><table><tr><td>$\mathrm{P}(X=x)$</td><td>0.4</td><td>0.3</td><td>0.2</td><td>0.1</td></tr></table></td>
<td>mark if they had drawn some diagram of how there are 4 ways of arranging Alex and his friend 2 persons apart (ignoring the arrangement of the other 3 people).

A significant number of students assumed X was a binomial random variable.

Students are also reminded to present sufficient working for the other probabilities in the table.</td>
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<tr>
<td>6 (iii)</td>
<td>$\mathrm{E}(X)=\sum_{all\ x} x\mathrm{P}(X=x)=0(0.4)+1(0.3)+2(0.2)+3(0.1)=1$
$\mathrm{E}(X-1)^2=\sum_{all\ x}(x-1)^2\mathrm{P}(X=x)=1(0.4)+0(0.3)+1(0.2)+4(0.1)=1$
$\mathrm{Var}(X)=\mathrm{E}(X-\mu)^2=\mathrm{E}(X-1)^2=1$</td>
<td>This part was generally well done. Most of the errors came from the formula for $\mathrm{E}(X-1)^2$. A variety of methods were seen for calculating $\mathrm{Var}(X)$, but very few students figured out the shortest method: by the definition of $\mathrm{Var}(X)$, which is $\mathrm{E}[(X-\mu)^2]$.

One way for students to check if their answer for $\mathrm{Var}(X)$ is correct is to know that variance cannot be a negative number.</td>
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<tr>
<td>7(a)
(i)</td>
<td>Let X be the random variable "pH value of a randomly chosen bottle of shampoo".
Unbiased estimate of population mean
$=\bar{x}$
$=\dfrac{178.2}{30}$
$=\underline{5.94}$
Unbiased estimate of population variance
$=s^2$
$=\dfrac{1}{29}\left(1238.622-\dfrac{178.2^2}{30}\right)$
$=6.21083$
$=\underline{6.21}$ (3 s.f.)

To test $\mathrm{H_0}:\ \mu=5.5$
against $\mathrm{H_1}:\ \mu\neq 5.5$ at 10% significance level
Under $\mathrm{H_0}$, since $n=30>20$ is large,
$\bar{X}\sim \mathrm{N}\left(5.5,\dfrac{6.21083}{30}\right)$ approx. by Central Limit Theorem
Test statistic $Z=\dfrac{\bar{X}-5.5}{\sqrt{\dfrac{6.21083}{30}}}\sim \mathrm{N}(0,1)$ approx.</td>
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Value of test statistic $z=\dfrac{5.94-5.5}{\sqrt{\dfrac{6.21083}{30}}}=0.967$ (3 s.f.)

Either Since $-1.64<0.967<1.64$, z lies <u>outside</u> the critical region
$\Rightarrow$ Do <u>not</u> reject H_0
Or p-value $=0.334>0.1$ $\Rightarrow$ Do <u>not</u> reject H_0

$\therefore$ There is insufficient evidence at 10% significance level to conclude that the mean pH value of the shampoo is not 5.5.

<u>Comments</u>
The best solutions for this question are a result of careful attention to the way students phrase their working and calculate the required values. If students take some time to understand the rationale for writing things a certain way, they would be able to appreciate the principles behind a statistical hypothesis test.

Students are encouraged to spell out “unbiased estimate of” rather than just writing $\bar{x}$ or s^2 . Some students even wrote “pop. mean/variance” or μ and σ^2 instead of the unbiased estimates.

The correct alternative hypothesis has been hinted in the question (“…too high or too low…”). Presentation wise, a number of students wrote subscripts on μ , which is not necessary.

Many students are still writing the wrong mean in the distribution for $\bar{X}$. The phrase “Under H_0 ” implies that we’re assuming that the population mean $\mu=5.5$, therefore $E(\bar{X})=5.5$. Students should also be aware of whether CLT is used.

An alarming number of students attempted to write down the formula of the p-value, and then seemed to calculate the p-value using normalcdf instead of the Z-test. Students should only attempt to do this if they’re very sure of the correct formula for the p-value in the respective tests; otherwise, they’re better off using the Z-test function in the GC and letting it do its work.

Some students keyed in the wrong σ into the GC, which resulted in an extremely low p-value.

The final part of comparing p-value to significance level and the conclusion was also horribly done. Students generally made some permutation of the following mistakes:
1. Dividing the p-value by 2, or using the p-value for the one-tail test
2. Comparing p-value to 0.05 instead of 0.1
3. Comparing wrongly (e.g. $0.334<0.1$)
4. Mixing up the results of the test (e.g. $0.334>0.1$, hence reject H_0)
5. Mixing up “sufficient/insufficient evidence” and “ H_0 / H_1 is true/not true”.
In particular, students should learn that the purpose of the test is to use the evidence to try and prove that H_1 is true, and hence the final conclusion must reflect this (i.e. is there sufficient evidence to conclude that H_1 is true?).
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<td>(a)(ii)</td>
<td>It is not necessary to assume X is normally distributed. As the sample size is large, by Central Limit Theorem, $\bar{X}$ is approximately normally distributed.</td>
<td>This question has highlighted a fundamental conceptual error that many students have about CLT: that CLT allows us to approximate X as a normal distribution. It therefore results in answers</td>
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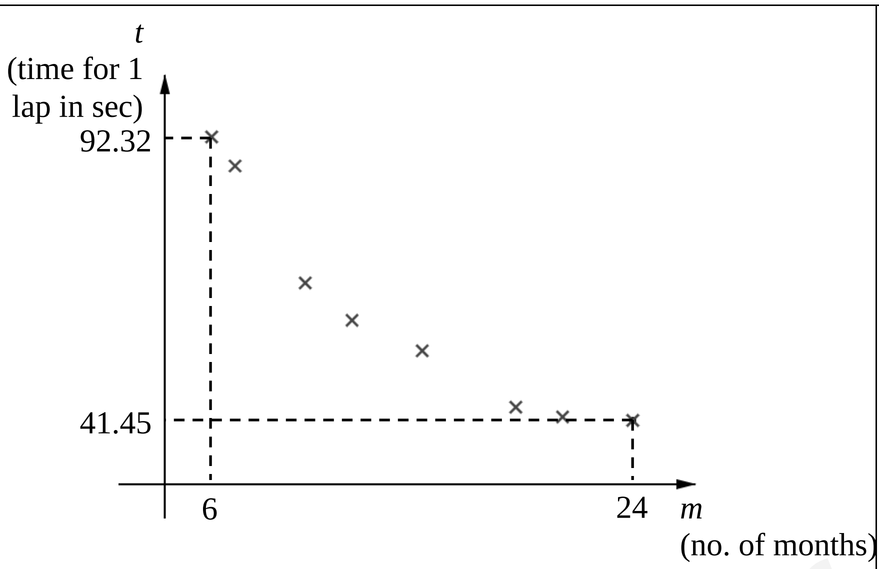
		ranging from "No, CLT says X is normal" to "Yes, since CLT says X is normal". Because it is very easy for students to simply give the correct answer "No" with a superficial explanation, the marking of this part is very much stricter. Many students simply said "It is not necessary, since n is large, **it** is approximately normal by CLT". These are important concepts that need to be corrected so students can have a better picture of how CLT is used.
(b)(i)	Critical region of the test is $z < -1.64485$ or $z > 1.64485$ $\Rightarrow$ $\underline{z < -1.64 \text{ or } z > 1.64}$ (3 s.f.)	The phrases "critical value" and "critical region" are added into the new syllabus, so students must know and distinguish between them. A number of students gave just the critical values. Also, critical region is usually expressed in terms of the test statistic (in our case, z). Finally, there are also students who gave the non-critical region as the critical region. One way to rectify this is to reinforce the fact that the critical region is also known as the rejection region (i.e. rejection of H_0).
(b)(ii)	Value of test statistic $z = \dfrac{5.94-5.5}{\sqrt{\frac{6.5}{n}}} = \dfrac{0.44\sqrt{n}}{\sqrt{6.5}}$ For a favourable outcome at 10% significance level, do not reject H_0 $\Rightarrow$ z lies <u>outside</u> the critical region $\Rightarrow -1.64485 < \dfrac{0.44\sqrt{n}}{\sqrt{6.5}} < 1.64485$ $\Rightarrow \dfrac{-1.64485\sqrt{6.5}}{0.44} < \sqrt{n} < \dfrac{1.64485\sqrt{6.5}}{0.44}$ $\Rightarrow n < \left(\dfrac{1.64485\sqrt{6.5}}{0.44}\right)^2$ $\Rightarrow n < 90.837$ Hence largest $n = \underline{90}$	Students who are careless with reading the questions would have used either $\frac{178.2}{n}$ as the sample mean or 6.2108 as s^2. Some students were confused about what the "favourable outcome" meant about the rejection of H_0. This involves understanding the context of the problem. A significant portion of students only wrote down $z < 1.64485$ and not the full non-critical region. Credit was only given if the correct inequality with the p-value was given earlier; the assumption is that with the correct inequality, students would be able to use invNorm to find the correct critical value. Otherwise, the

		full region should be written down. It is actually also possible to obtain the correct answer with $z > -1.64485$, but the earlier inequality would have been more appropriate since the test statistic here is positive. Students were also generally very careless with solving inequalities.

8(i)	 A linear model would imply that in the long run, the time taken to swim a lap would be <u>negative</u>, which is unrealistic. *(Note: Extrapolation is not accepted as a reason, as the question isn't looking for a reason based on the data obtained.)*	3 important points to note for scatter diagram: 1) axes *t* and *m* labelled 2) extreme values labelled 3) 8 points in total Acceptable answers include: - negative time - zero time
8(ii)	Using GC, for $C=37$, $r=\underline{-0.992555}$	**R**: 6 d.p.
8(iii)	The most appropriate value for C is <u>38</u>, as the magnitude of its corresponding value of r is closest to 1.	Acceptable answers include: - $\lvert r \rvert \approx 1$ - $r \approx -1$ Quite a number of scripts had "closet" instead of "closest"!
8(iv)	From GC, least squares regression line of $\ln(t-38)$ on m is $\ln(t-38)=5.01236-0.16349m$ $\Rightarrow \underline{\ln(t-38)=5.01-0.163m}$ (3 s.f.) $C=38$ is the <u>fastest time</u> that a student can expect to complete a lap of breaststroke <u>after spending a long time</u> at the swim school. (Making t the subject in the equation of the regression line gives us $t=38+e^{5.01-0.163m}$, so as $m\to\infty$, $t\to 38$.)	**R**: use C =38 **R**: $\ln(t-38)$ on m 3 s.f. for final answer Please note that C is NOT the gradient; C is NOT the *y*-intercept Acceptable answers include: - fastest time after a long period - shortest time after a long period
8(v)	When $m=9$, $t=38+e^{5.01236-0.16349(9)}$ $=72.50$ (2 d.p.) A timing of 60.33 seconds is well below the expected timing of 72.50 seconds. Therefore, we can say that the student is <u>exceptionally strong</u> in his/her swimming ability.	Acceptable answers include: - very strong - very talented - way above average
8(vi)	The 8 randomly selected students might have been of different genders and ages. To make the results fairer, data could be collected separately based on <u>genders</u> and <u>age ranges</u>.	The following may not give **fairer results**: - increase sample size - increase frequency - group by ability (beginner, intermediate, advanced) is subjective

9 (a)	Let X be the random variable 'number of defective articles in sample of 10'. $X \sim \mathrm{B}(10, 0.065)$ $\mathrm{P}(\text{accepting a batch}) = \mathrm{P}(X \le 1) = 0.86563 = 0.866$	Although most people are able to do this part, there are quite a number of students who doesn't know how to do this basic question. Or some calculated this manually instead of using Binomial distribution.
(i)	P(batch eventually accepted) $= (0.86563)^2 + 2(0.86563)(1-0.86563)(0.86563)$ $= 0.95069$ $= 0.951$	Most students who got this wrong did not multiply by 2 for the second case. Some did not understand the question and interpret it as a geometric series question.
(ii)	Let N be the number of articles examined per batch. $N = \begin{cases} 20 & \text{if both findings agree} \\ 30 & \text{otherwise} \end{cases}$ $\mathrm{P}(N=20) = (0.86563)^2 + (1-0.86563)^2 = 0.76737$ $\mathrm{P}(N=30) = 1 - 0.76737 = 0.23263$ $\therefore\ \mathrm{E}(N) = 20(0.76737) + 30(0.23263) = 22.3$	About 30% have no clue how to do this part. 40% of those who attempted missed out some cases, such as RR or did not multiply by 2 to account for AR and RA.
9 (b)	Let Y be the random variable 'number of defective articles in a sample of 10'. $Y \sim \mathrm{B}(10, p)$ $A = \mathrm{P}(Y \le 1) + \mathrm{P}(Y=2)\cdot\mathrm{P}(Y=0)$ $= {}^{10}C_0 p^0 (1-p)^{10} + {}^{10}C_1 p^1 (1-p)^9 + {}^{10}C_2 p^2 (1-p)^8 \cdot {}^{10}C_0 p^0 (1-p)^{10}$ $= (1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^{18}$ $= (1+9p)(1-p)^9 + 45p^2(1-p)^{18}$ (shown)	Except for some who did not interpret the question properly, this part is quite well done for those who attempted it. Except for those who did not use the formula and thus left out ${}^{10}C_1$ or ${}^{10}C_2$.
9 (b)	Let W be the random variable 'number of acceptable batches, out of 100 inspected'. $W \sim \mathrm{B}(100, A)$ $\mathrm{P}(W > 80) = 0.98 \Rightarrow \mathrm{P}(W \le 80) = 0.02$ By GC, $A = 0.876235$ $\therefore\ A = (1+9p)(1-p)^9 + 45p^2(1-p)^{18} = 0.87624$ By GC, $p = 0.08$	There are a good number students who have problem dealing with complement. $\mathrm{P}(W>80) = 0.98 \Rightarrow 1 - \mathrm{P}(W \le 79) = 0.98$ A large number of students applied (CLT) erroneously or normal approximation to this qn, and took invNorm. Students should also be advised not to use table to solve for A as A is not an integer value.
10 (a) (i)	Let X be the random variable 'marks of an examination'. By GC, $\mathrm{P}(X > 100) = 0.0359$ if $X \sim \mathrm{N}(73, 15^2)$	Common wrong answers are: The

	i.e., there are 3.59% of the students scoring more than the maximum mark of 100, which is impossible.	students marks are not independent of one another / the mean should be around 50 / mark is a discrete random variable / mark cannot take negative values or values above 100. Students need to understand that normal distribution is a model to help analyze the data and can be applied as long the population is large and the values that it cannot take have negligible probabilities.
10 (a) (ii)	Since $n = 50 \geq 20$ is large, by Central Limit Theorem, $\overline{X} \sim \text{N}(73, \frac{15^2}{50})$ approximately. $\therefore\ \text{P}(70 < \overline{X} < 75) = 0.748$	Majority assumed X is normal and then applied CLT for $\overline{X}$. This questions shows that most people do not understand the meaning of $\overline{X}$.
10 (b)	Let Y be the random variable 'marks of a school examination'. $Y \sim \text{N}(\mu, \sigma^2)$ $\text{P}(Y < 51) = 0.8$ $\text{P}(Z < \frac{51-\mu}{\sigma}) = 0.8$ $\frac{51-\mu}{\sigma} = 0.84162$ $\mu + 0.84162\sigma = 51$ $\text{P}(\mu - 5.4 < Y < \mu + 5.4) = 0.5$ $\text{P}(\frac{-5.4}{\sigma} < Z < \frac{5.4}{\sigma}) = 0.5$ $\text{P}(Z < -\frac{5.4}{\sigma}) = 0.25$ $-\frac{5.4}{\sigma} = -0.67449$ $\therefore\quad \sigma = 8.01$ $\therefore\quad \mu = 51 - 0.84162(8.0061) = 44.3$	Quite a number had problem with 80th percentile: P(Y > 51) = 0.8 & P(Y = 51) are WRONG! Standardisation should be $\boxed{Z = \frac{X-\mu}{\sigma}}$ Note that $\text{InvNorm }(0.8) = 0.84162$ $\text{InvNorm }(0.8) \neq 0.8$ Note the interquartile range and its related probability: $\text{P}(Y < u) - \text{P}(Y < l) = 0.5$ where $u - l = 10.8$ $\text{P}(Y < u) - \text{P}(Y < l) = 10.8$ is WRONG!
10 (c) (i)	Let M be the random variable 'marks of another school examination'. $M \sim \text{N}(52, 13^2)$ $\text{P}(50 < M) = 0.56113$ $\text{Number of passes} = (\text{total candidature}) \times 0.56113 = 289$ $\therefore\ \text{total candidature} = 289 \div 0.56113 = 515$	

10 (c) (ii)	$P(\|M-52\|<m)>0.9 \Rightarrow P(52-m<M<52+m)>0.9$ where $M \sim N(52,13^2)$ $\Rightarrow P(M<52-m)<0.05$ $\Rightarrow \quad 52-m<30.6$ $\Rightarrow \quad m>21.4$ $\therefore$ Smallest integral value of $m=22$	**P**: Missing first step **R**: *m* marks from mean, 90%, more than, etc. As 52 & 13 are given, there is no need for standardisation. The preferred method is InvNorm(0.05, 52, 13). Trial and error using GC table is not advisable.

1 A parabola, P with equation $(y-a)^2 = ax$, where a is a constant, undergoes in succession, the following transformations:

A: A translation of 2 units in the positive x-direction

B: A scaling parallel to the y-axis by a factor of $\frac{1}{3}$

The resulting curve, Q passes through the point with coordinates $\left(2, \frac{4}{3}\right)$.

(i) Show that $a = 4$. [3]

(ii) Find the range of values of k for which the line $y = kx$ does not meet P. [3]

2 The region bounded by the curve $y = \frac{1}{\sqrt{x}-2}$, the x-axis and the lines $x = 9$ and $x = 16$ is rotated through 2π radians about the x-axis. Use the substitution $t = \sqrt{x}$ to find the exact volume of the solid obtained. [6]

3 **(i)** Express $\frac{r+1}{(r+2)!}$ in the form $\frac{A}{(r+1)!} + \frac{B}{(r+2)!}$, where A and B are integers to be found. [2]

(ii) Find $\sum_{r=1}^{n} \frac{r+1}{3(r+2)!}$. [3]

(iii) Hence, evaluate $\sum_{r=0}^{\infty} \frac{r+1}{3(r+2)!}$. [2]

4 Kumar wishes to purchase a gift priced at \$280 for his mother.

Starting from January 2017,

- Kumar saves \$100 in his piggy bank on the 1st day of each month;
- Kumar donates 30% of his money in his piggy bank to charity on the 15th day of each month and
- Kumar's father puts an additional \$20 in Kumar's piggy bank on the 25th day of each month.

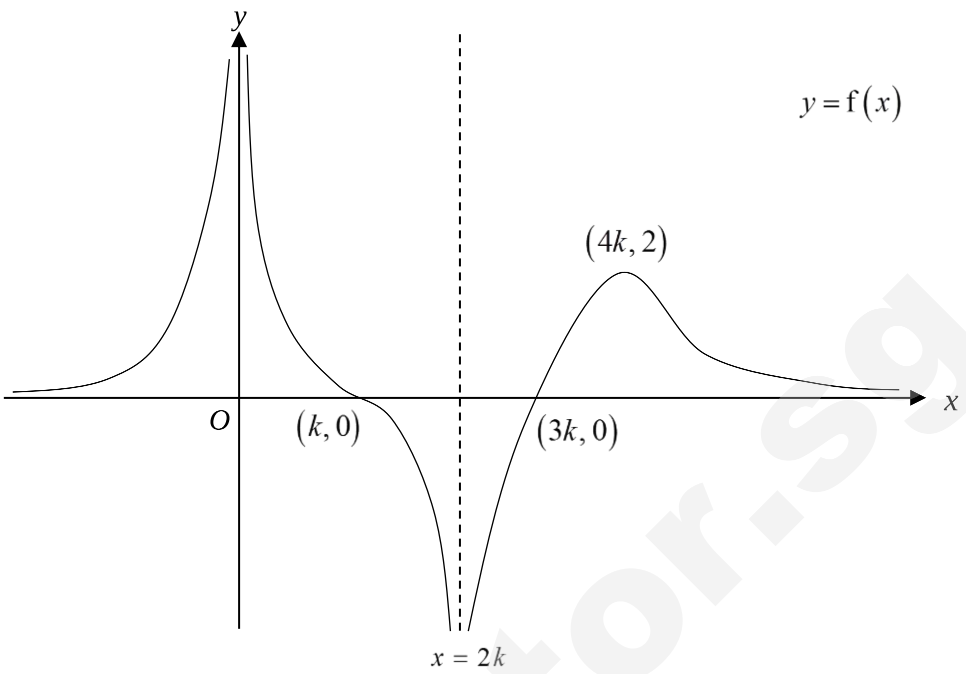
(i) Find the amount of money in Kumar's piggy bank at the end of March 2017. [2]

(ii) Show that the amount of money in Kumar's piggy bank at the end of n months is

$$300\left(1-0.7^n\right).$$ [3]

(iii) At the end of which month will Kumar first be able to purchase the gift for his mother? [2]

5 The diagram below shows the sketch of the graph of $y = \mathrm{f}(x)$ for $k > 0$. The curve passes through the points with coordinates $(k, 0)$ and $(3k, 0)$, and has a maximum point with coordinates $(4k, 2)$. The asymptotes are $x = 0$, $x = 2k$ and $y = 0$.



Sketch on separate diagrams, the graphs of

(i) $y=\mathrm{f}(-x-k)$, [2]

(ii) $y=\mathrm{f}'(x)$, [2]

(iii) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

showing clearly, in terms of k, the equations of any asymptote(s), the coordinates of any turning point(s) and any points where the curve crosses the x- and y-axes.

6 A straight line passes through the point with coordinates $(4,\ 3)$, cuts the positive x-axis at point P and the positive y-axis at point Q. It is given that $\angle PQO=\theta$, where $0<\theta<\dfrac{\pi}{2}$ and O is the origin.

(i) Show that the equation of line PQ is given by $y=(4-x)\cot\theta+3$. [2]

(ii) By finding an expression for $OP+OQ$, show that as θ varies, the stationary value of $OP+OQ$ is $a+b\sqrt{3}$, where a and b are constants to be determined. [5]

7 A curve C has parametric equations

$$x=\frac{4}{t+1} \quad \text{and} \quad y=t^2-3, \quad t\neq -1.$$

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [2]

(ii) Find the equation of the normal to C at P where $x=-2$. [3]

(iii) Find the other values of t where the normal at P meets the curve C again. [3]

8 The curve C has equation

$$y=\frac{2x^2-3x+5}{x-5}.$$

(i) Express y in the form $px+q+\frac{r}{x-5}$ where p, q and r are constants to be found. [3]

(ii) Sketch C, stating the equations of any asymptotes, the coordinates of any stationary points and any points where the curve crosses the x- and y-axes. [4]

(iii) By sketching another suitable curve on the same diagram in part **(ii)**, state the number of roots of the equation

$$\left(2x^2-3x+5\right)^2=5x\left(x-5\right)^2.$$ [3]

9 **(a)** Given that the first two terms in the series expansion of $\sqrt{4-x}$ are equal to the first two terms in the series expansion of $p+\ln\left(q-x\right)$, find the constants p and q. [5]

(b)(i) Given that $y=\tan^{-1}\left(ax+1\right)$ where a is a constant, show that $\frac{\mathrm{d}y}{\mathrm{d}x}=a\cos^2 y$. Use this result to find the Maclaurin series for y in terms of a, up to and including the term in x^3. [5]

(ii) Hence, or otherwise, find the series expansion of $\frac{1}{1+\left(4x+1\right)^2}$ up to and including the term in x^2.

[3]

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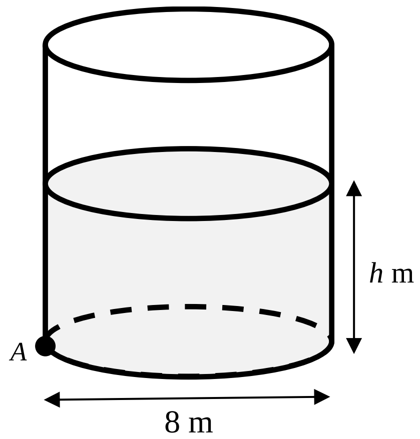


The figure above shows a cylindrical water tank with base diameter 8 metres. Water is flowing into the tank at a constant rate of $0.36\pi\,\text{m}^3$/min. At time t minutes, the depth of water in the tank is h metres. However, the tank has a small hole at point A located at the bottom of the tank. Water is leaking from point A at a rate of $0.8\pi h$ m^3/min.

(i) Show that the depth, h metres, of the water in the tank at time, t minutes satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{400}(9-20h).$$ [3]

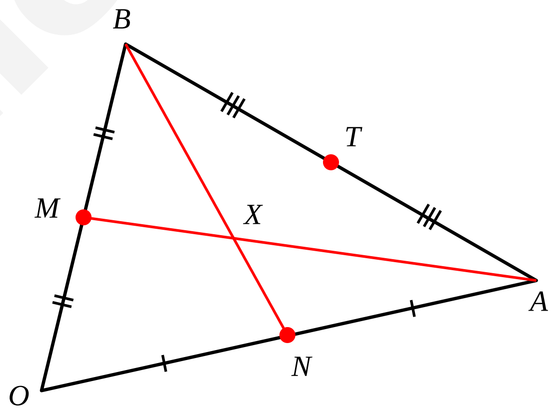
(ii) Given that $h = 0.4$ when $t = 0$, find the particular solution of the above differential equation in the form $h = \mathrm{f}(t)$. [6]

(iii) Explain whether the tank will be emptied. [1]

(iv) Sketch the part of the curve with the equation found in part **(ii)**, which is relevant in this context. [2]

11 A median of a triangle is a line segment joining a vertex to the midpoint of the opposite side.

For the triangle shown below, O, A and B are vertices, where O is the origin, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The midpoints of OB, OA and AB are M, N and T respectively.



It is given that X is the point of intersection between the medians of triangle OAB from vertices A and B.

(i) Show that $\overrightarrow{OX} = \frac{1}{3}(\mathbf{a}+\mathbf{b})$. [4]

(ii) Prove that X also lies on OT, the median of triangle OAB from vertex O. [2]

The **centroid** of triangle OAB is the common point of intersection X between all three medians of the triangle.

Ray tracing is a technique in computer graphics rendering used to realistically capture the lighting effect in a scene being modelled. Starting from a chosen viewpoint, different rays are being traced backwards towards different parts of an object in the scene and reflected off the object. For each ray, if it reflects off the object and intersects a light source, then the part of the object at which the ray is reflected off would be made to appear brighter.

In a particular scene depicting a dolphin jumping out of the ocean, a ray is being traced back from a chosen viewpoint at V to the **centroid** X of a particular triangular facet defined by the vertices comprising the origin O, $A(5,\ 4,\ 6)$ and $B(-2,\ 2,\ 3)$, and then reflected off the facet at X, as shown in Figure 1.

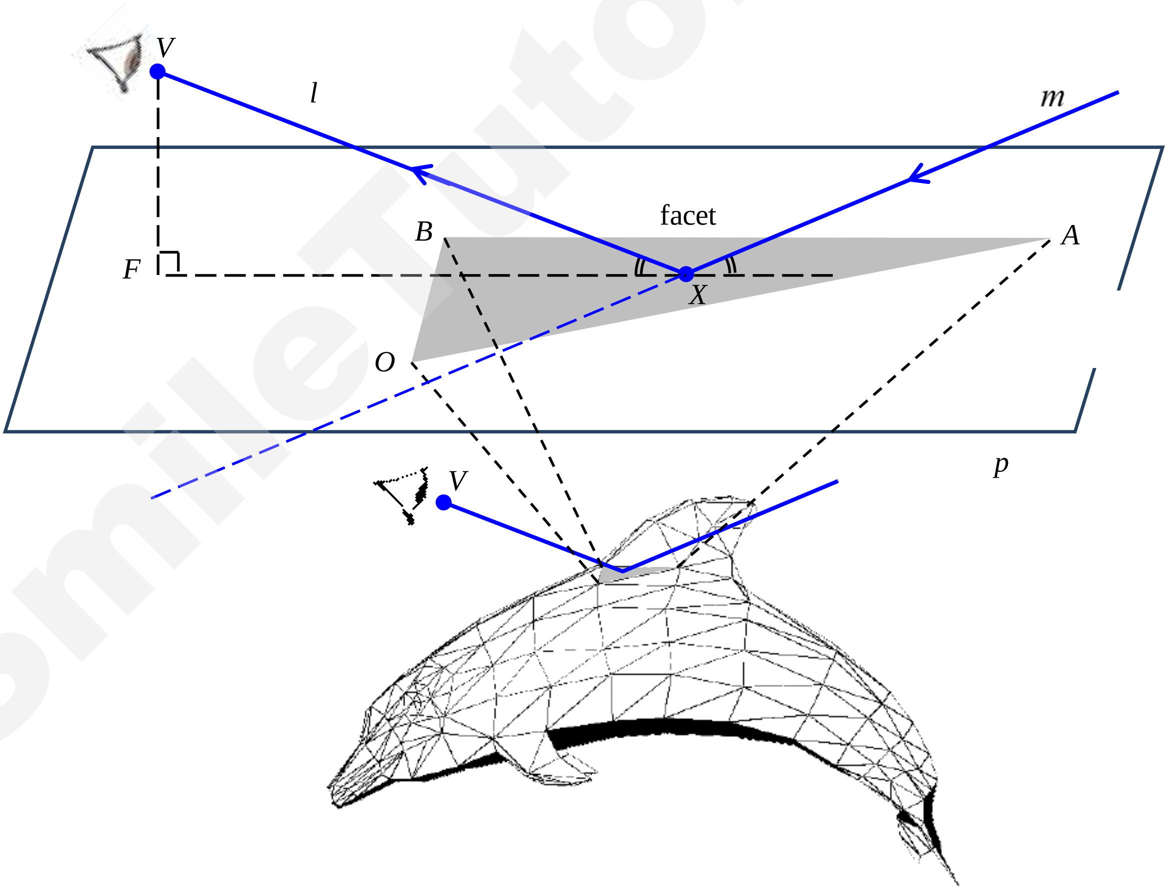


Figure 1

(iii) Show that the plane p which contains the triangular facet OAB can be represented by the cartesian equation $-3y+2z=0$. [2]

(iv) Given $V(1,-68,-37)$, determine the coordinates of the foot of perpendicular F from V to plane p. [4]

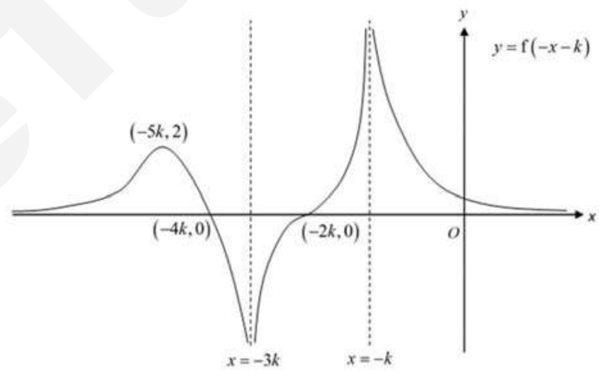
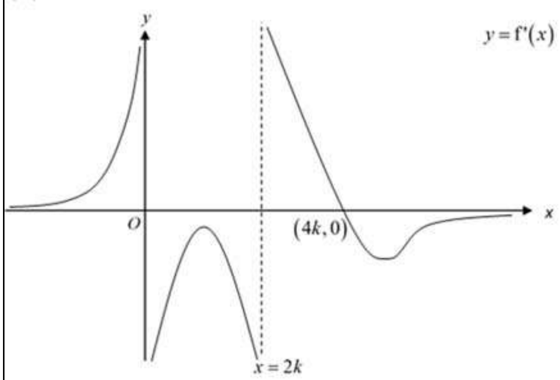
The reflected ray travels along a line m such that:

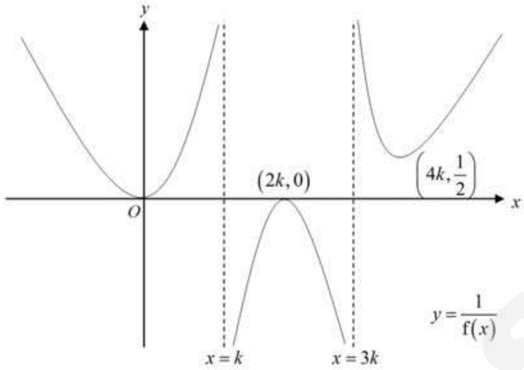
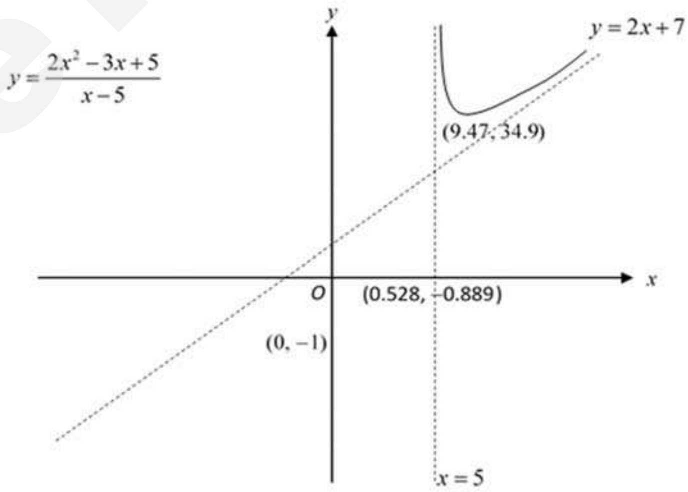
- both line VX (denoted by l) and line m lie in a plane that is perpendicular to plane p, and
- the angle between line l and plane p equals the angle between line m and plane p.

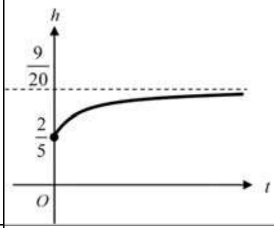
(v) By first finding two suitable points lying on line *m*, or otherwise, find a cartesian equation for line *m*. [5]

ANNEX B

CJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Graphs and Transformation	(ii) $k < -\frac{1}{4}$
2	Application of Integration	$\pi(2\ln 2 + 2)$
3	Sigma Notation and Method of Difference	(i) $\frac{1}{(r+1)!} - \frac{1}{(r+2)!}$ (ii) $\frac{1}{3}\left[\frac{1}{2} - \frac{1}{(n+2)!}\right]$ (iii) $\frac{1}{3}$
4	AP and GP	(i) \$197.10 (iii) August 2017
5	Graphs and Transformation	(i)  (ii)  (iii)

		
6	Differentiation & Applications	(ii) $7+4\sqrt{3}$
7	Differentiation & Applications	(i) $-\dfrac{t(t+1)^2}{2}$ (ii) $y=-\dfrac{1}{6}x+\dfrac{17}{3}$ or $6y+x=34$ (iii) $t=-3$ (given) or -0.915 or 2.91
8	Graphs and Transformation	(i) $p=2,\ q=7,\ r=40$ (ii)  (iii)2
9	Maclaurin series	(a) $p=2-\ln 4$ (b)(i) $\dfrac{\pi}{4}+\dfrac{1}{2}ax-\dfrac{1}{4}a^2x^2+\dfrac{1}{12}a^3x^3+\ldots$ (ii) $\dfrac{1}{2}-2x+4x^2$

10	Differential Equations	(ii) $h=\frac{1}{20}\left(9-\mathrm{e}^{-\frac{1}{20}t}\right)$ (iv) 
11	Vectors	(iv) $(1,-38,-57)$ (v) $l_m: x=1,\ y-2=\frac{z-3}{8}$

Q1. Transformations, Conics and Inequalities		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Determine the transformations on the graph of $y=\mathrm{f}(x)$ as represented by $y=\mathrm{f}(x)+a$ and $ay=\mathrm{f}(x)$.	**(i)** $(y-a)^2=ax$ $\downarrow A$ $(y-a)^2=a(x-2)$ $\downarrow B$ $(3y-a)^2=a(x-2)$ Since resulting curve passes through point $\left(2,\frac{4}{3}\right)$, $(4-a)^2=a(2-2)$ $(4-a)^2=0$ $a=4$ (shown)	Most candidates were able to answer this part correctly. Some forgot that scaling to a variable is achieved by dividing the variable by the scaling factor.
Applying the concept no real roots $\Rightarrow b^2-4ac<0$	**(ii)** **Method ①:** Parabola: $(y-4)^2=4x$ — ① Line: $y=kx$ — ② Substitute ② into ①: $(kx-4)^2=4x$ $k^2x^2-8kx+16=4x$ $k^2x^2+(-8k-4)x+16=0$ For the line not to meet the parabola, $b^2-4ac<0$	Many presented satisfactory answers. Some students failed to link the intersection of linear/quadratic curves to solving simultaneous and subsequently quadratic equations, and that the number of common points can be inferred from the sign of the determinant. Some also had algebraic slips when handling inequalities. They need to practise more.

	$$\left(-8k-4\right)^2-4k^2\left(16\right)<0$$ $$64k^2+64k+16-64k^2<0$$ $$64k+16<0$$ $$k<-\frac{1}{4}$$ **Method ②:** Parabola: $\left(y-4\right)^2=4x$ — ① Line: $y=kx \Rightarrow x=\frac{k}{y}$ — ② Substitute ② into ①: $$\left(y-4\right)^2=\frac{4y}{k}$$ $$ky^2-8ky+16k=4y$$ $$ky^2+\left(-8k-4\right)y+16k=0$$ For the line not to meet the parabola, $b^2-4ac<0$ $$\left(-8k-4\right)^2-4k\left(16k\right)<0$$ $$64k^2+64k+16-64k^2<0$$ $$64k+16<0$$ $$k<-\frac{1}{4}$$	

Q2. Definite Integral		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Find the volume of solid formed by revolution. Perform integration by a given substitution.	**Method ①:** Volume $= \pi \int_9^{16} \left(\frac{1}{\sqrt{x}-2}\right)^2 \text{ d}x$ $= \pi \int_3^4 \left(\frac{1}{t-2}\right)^2 (2t)\text{ d}t$ $= \pi \int_3^4 \frac{2t}{t^2-4t+4} \text{ d}t$ $= \pi \int_3^4 \frac{2t-4}{t^2-4t+4} + \frac{4}{(t-2)^2} \text{ d}t$ $= \pi \left[\ln\lvert t^2-4t+4\rvert\right]_3^4 + \pi \int_3^4 4(t-2)^{-2} \text{ d}t$ $= \pi \left[\ln\lvert t^2-4t+4\rvert + 4\frac{(t-2)^{-1}}{-1}\right]_3^4$ $= \pi \left[\ln\lvert t^2-4t+4\rvert - \frac{4}{(t-2)}\right]_3^4$ $= \pi\left[(\ln 4 - 2) - (\ln 1 - 4)\right]$ $= \pi(\ln 4 + 2)$ units^3 Substitution: $t = \sqrt{x}$ $t^2 = x$ $2t = \frac{\text{d}x}{\text{d}t}$ When $x = 9, t = 3$ When $x = 16, t = 4$	Most candidates were able to setup the correct integral for the volume of revolution. However, many failed make the correct substitution of dx by $\frac{dx}{dt}dt$ and thus $2tdt$. Another group of students forgot to change the upper and lower limits to the respective values of t when the variable was changed. Many students were also stuck at the integration of $\frac{4}{(t-2)^2}$ as it is not very easy for those who don't practise much to identify the fraction as a power function of power -2.

	Method ②: Volume $= \pi \int_9^{16} \left(\frac{1}{\sqrt{x}-2} \right)^2 \, dx$ $= \pi \int_3^4 \left(\frac{1}{t-2} \right)^2 (2t) \, dt$ $= \pi \int_3^4 \frac{2t}{(t-2)^2} \, dt$ $= \pi \int_3^4 \frac{2}{(t-2)} + \frac{4}{(t-2)^2} \, dt$ $= \pi \left[2\ln\lvert t-2 \rvert \right]_3^4 + \pi \int_3^4 4(t-2)^{-2} \, dt$ $= \pi \left[2\ln\lvert t-2 \rvert + 4\frac{(t-2)^{-1}}{-1} \right]_3^4$ $= \pi \left[2\ln\lvert t-2 \rvert - \frac{4}{(t-2)} \right]_3^4$ $= \pi \left[(2\ln 2 - 2) - (2\ln 1 - 4) \right]$ $= \pi (2\ln 2 + 2)$ units^3 <u>Substitution:</u> $t = \sqrt{x}$ $t^2 = x$ $2t = \frac{dx}{dt}$ When $x = 9, t = 3$ When $x = 16, t = 4$	

Q3. Sigma Notation		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Apply concept of factorial	**(i)** **<u>Method ①:</u>** $\dfrac{r+1}{(r+2)!}=\dfrac{A}{(r+1)!}+\dfrac{B}{(r+2)!}$ $r+1=\dfrac{A(r+2)!}{(r+1)!}+\dfrac{B(r+2)!}{(r+2)!}$ $r+1=A(r+2)+B$ When $r=-1,\ A+B=0$ — ① When $r=0,\ 2A+B=1$ — ② Solving, $A=1$ and $B=-1$ $\therefore \dfrac{1}{(r+2)r!}=\dfrac{1}{(r+1)!}-\dfrac{1}{(r+2)!}$ **<u>Method ②:</u>** $\dfrac{r+1}{(r+2)!}=\dfrac{A}{(r+1)!}+\dfrac{B}{(r+2)!}$ When $r=1,\ \dfrac{A}{2}+\dfrac{B}{6}=\dfrac{2}{6}$ $3A+B=2$ — ① When $r=0,\ A+\dfrac{B}{2}=\dfrac{1}{2}$ $2A+B=1$ — ② Solving, $A=1$ and $B=-1$ $\therefore \dfrac{1}{(r+2)r!}=\dfrac{1}{(r+1)!}-\dfrac{1}{(r+2)!}$	Most students were able to get the values of A and B correctly. There were a variety of methods used to get the correct answers.

Apply summation of series by the method of differences.	**(ii)** $\displaystyle\sum_{r=1}^{n}\frac{r+1}{3(r+2)!}=\frac{1}{3}\sum_{r=1}^{n}\frac{r+1}{(r+2)!}$ $\displaystyle=\frac{1}{3}\sum_{r=1}^{n}\left[\frac{1}{(r+1)!}-\frac{1}{(r+2)!}\right]$ $\displaystyle=\frac{1}{3}\left[\frac{1}{2!}-\cancel{\frac{1}{3!}}\right.$ $\displaystyle+\cancel{\frac{1}{3!}}-\cancel{\frac{1}{4!}}$ $+\cancel{\dots}\ \cancel{\dots}$ $\displaystyle+\cancel{\frac{1}{n!}}-\cancel{\frac{1}{(n+1)!}}$ $\displaystyle\left.+\cancel{\frac{1}{(n+1)!}}-\frac{1}{(n+2)!}\right]$ $\displaystyle=\frac{1}{3}\left[\frac{1}{2}-\frac{1}{(n+2)!}\right]$	Most students were able to get this part correct.
Understand convergence of a series and the sum to infinity.	**(iii)** From (ii), $\displaystyle\sum_{r=1}^{n}\frac{r+1}{3(r+2)!}=\frac{1}{3}\left[\frac{1}{2}-\frac{1}{(n+2)!}\right]$ As $n\to\infty$, $\displaystyle\frac{1}{(n+2)!}\to 0$, thus $\displaystyle\sum_{r=1}^{n}\frac{r+1}{3(r+2)!}\to\frac{1}{6}$ $\displaystyle\sum_{r=0}^{\infty}\frac{r+1}{3(r+2)!}=\frac{1}{3(2)(1)}+\sum_{r=1}^{\infty}\frac{r+1}{3(r+2)!}$ $\displaystyle=\frac{1}{6}+\frac{1}{6}$. $\displaystyle=\frac{1}{3}$	Most students were able to get the sum to infinity correct but failed to realized that the starting value of r had change.

Q4. Geometric Progression

Assessment Objectives

Determine sum of a finite geometric series

Solution

(i)

		Amount of \$ Kumar has @ the …		
		Beginning	Middle	End
Jan 2017	1	100	$0.7(100)$	$0.7(100)+20$
Feb 2017	2	$100+0.7(100)+20$	$0.7[100+0.7(100)+20]$	$0.7(100)+0.7^2(100)+0.7(20)+20$
Mar 2017	3	$100+0.7(100)+0.7^2(100)+0.7(20)+20$	$0.7[100+0.7(100)+0.7^2(100)+0.7(20)+20]$	$0.7(100)+0.7^2(100)+0.7^3(100)+0.7^2(20)+0.7(20)+20$
		…	…	…
	n			$0.7(100)+0.7^2(100)+0.7^3(100)+...+0.7^n(100)+0.7^{n-1}(20)+...+0.7^2(20)+0.7(20)+20$

Amount of money Kumar has at the end of March 2015

$$=\$\left[0.7(100)+0.7^2(100)+0.7^3(100)+0.7^2(20)+0.7(20)+20\right]$$

$$=\$197.10$$

(ii) Amount of money Kumar has at the end of n months

$$=0.7(100)+0.7^2(100)+0.7^3(100)+...+0.7^n(100)$$
$$+0.7^{n-1}(20)+...+0.7^2(20)+0.7(20)+20$$

$$=100\left(0.7+0.7^2+...+0.7^n\right)+20\left(1+0.7+0.7^2+...+0.7^{n-1}\right)$$

$$=100\left[\frac{0.7\left(1-0.7^n\right)}{1-0.7}\right]+20\left[\frac{1-0.7^n}{1-0.7}\right]$$

$$=\frac{700}{3}\left(1-0.7^n\right)+\frac{200}{3}\left(1-0.7^n\right)$$

$$=300\left(1-0.7^n\right)\text{ (shown)}$$

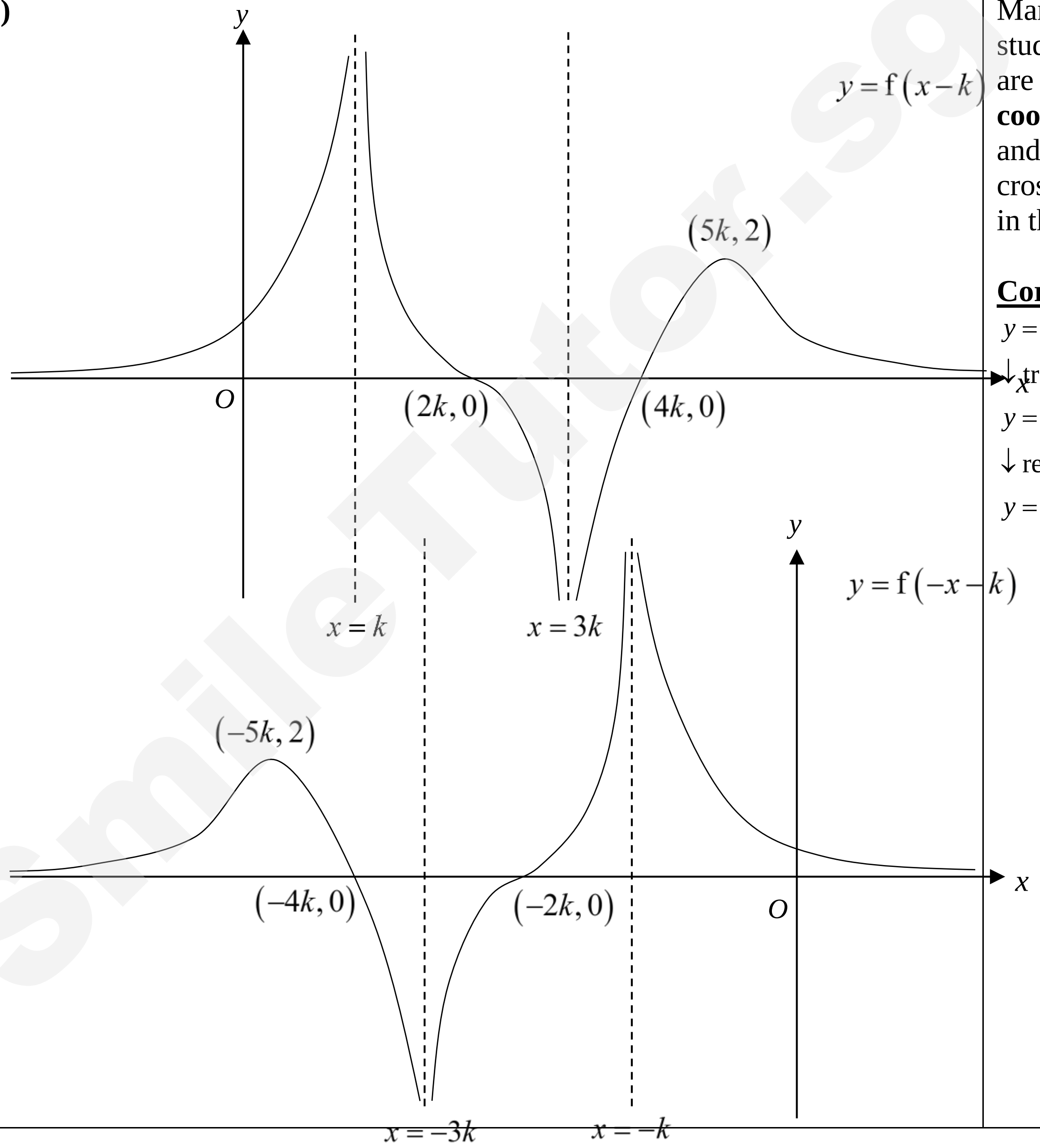
Examiner's Feedback

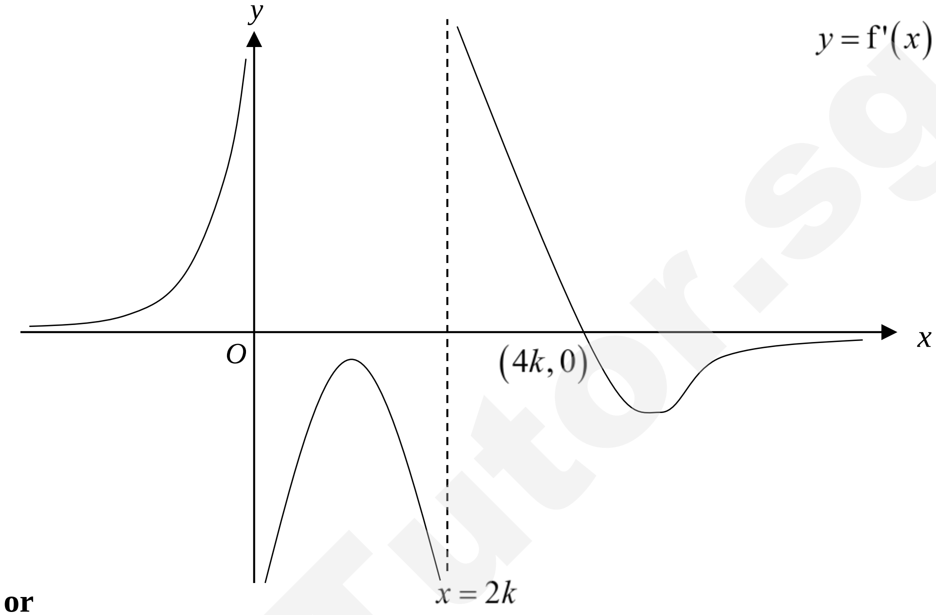
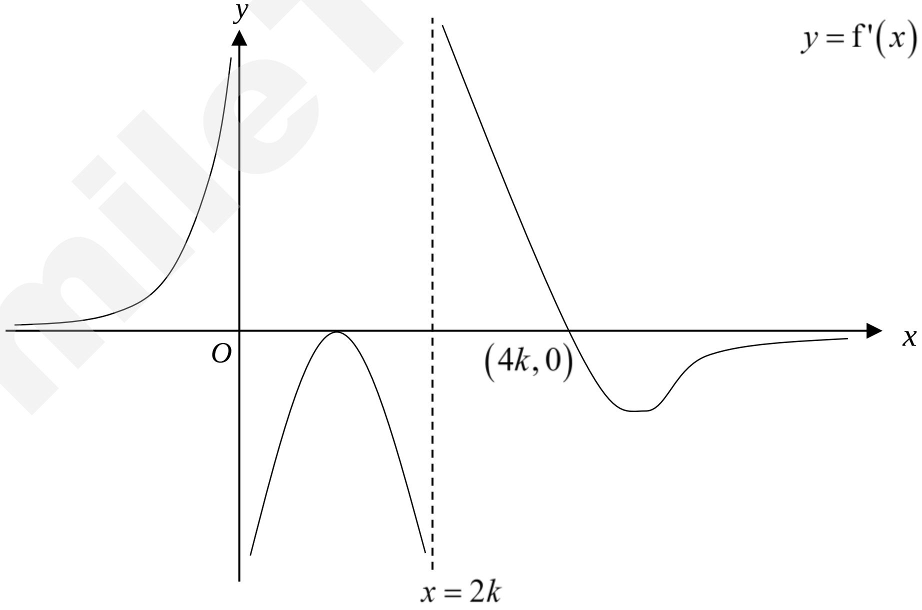
(i) Most students were able to get the value correct.

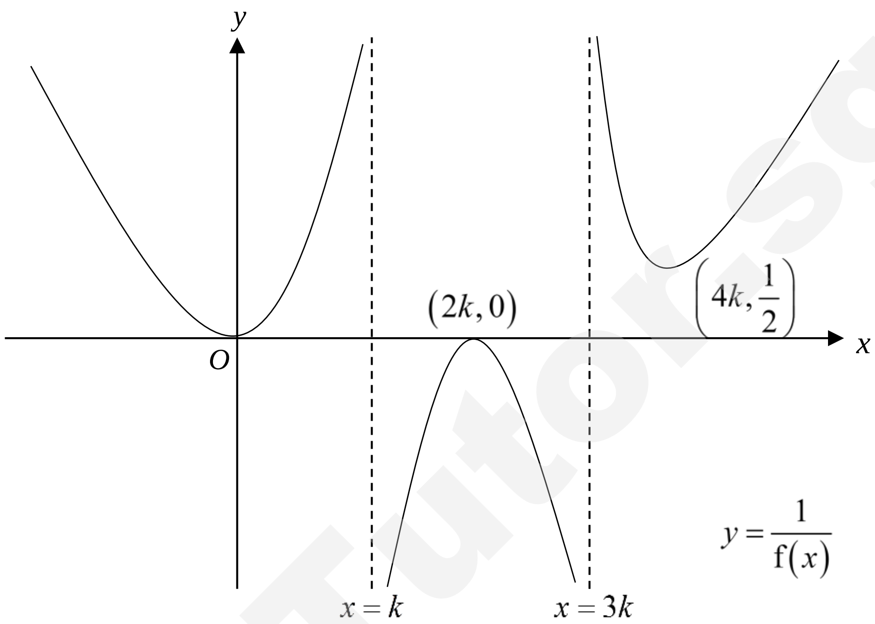
(ii) There were a many methods presented by students. However, because it is a show question, their working must be clear. Credit were not given to students who just state that $a=90, r=0.7$ unless the explanation on why $a=90$ is clear.

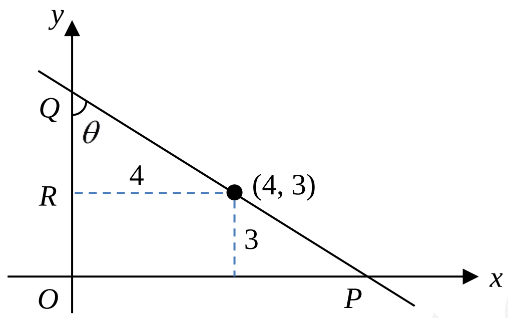
Solve inequality	**(iii)**	$300\left(1-0.7^{n}\right)\geq 280$ $1-0.7^{n}\geq\frac{14}{15}$ $0.7^{n}\leq\frac{1}{15}$ $n\geq 7.59$ Kumar will first be able to purchase the gift for his mother at the 8^{th} month. (or August 2017)	Quite badly done by students wo did not use the GC table method with many students not realized that $\ln 0.7<0$ and hence there is a need to change the inequality sign when dividing by $\ln 0.7$ on both sides of the unequalities.

Q5. Transformations

Assessment Objectives	Solution	Examiner's Feedback
Apply effect of transformation on the graph of $y=\mathrm{f}(x)$ as represented by $y=\mathrm{f}(-x-k)$.	**(i)** 	Many good answers. However, students should take note that they are supposed to state the **coordinates** of any turning point(s) and any points where the curve crosses the *x*- and *y*-axes as given in the question. **<u>Correct sequence:</u>** $y=\mathrm{f}(x)$ $\downarrow$ translate k units in the +ve x-direction $y=\mathrm{f}(x-k)$ $\downarrow$ reflection in the $y-$axis $y=\mathrm{f}(-x-k)$

Relate the graph of $y = \mathrm{f}\,'(x)$ to the graph of $y = \mathrm{f}(x)$.	**(ii)**  **or** 	Many good answers. However, students should take note that they are supposed to state the **coordinates** of any turning point(s) and any points where the curve crosses the *x*- and *y*-axes as given in the question. There were a number of students who mixed up the sketching of $y = \mathrm{f}\,'(x)$ and $y = \dfrac{1}{\mathrm{f}(x)}$.

Relate the graph of $y=\dfrac{1}{\mathrm{f}(x)}$ to the graph of $y=\mathrm{f}(x)$.	**(iii)** 	Many good answers. However, students should take note that they are supposed to state the **coordinates** of any turning point(s) and any points where the curve crosses the *x*- and *y*-axes as given in the question. There were a number of students who mixed up the sketching of $y=\mathrm{f}'(x)$ and $y=\dfrac{1}{\mathrm{f}(x)}$.

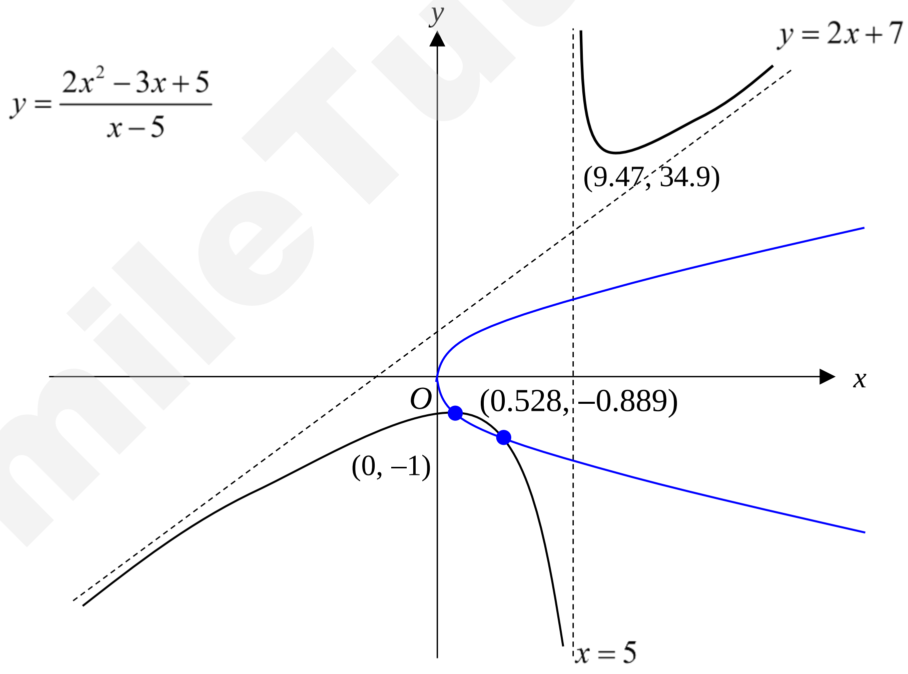
Q6. Application of Differentiation		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Use of trigonometric ratio to express gradient and y-intercept in terms of θ.	**(i)**  Gradient $= -\dfrac{1}{\frac{OP}{OQ}} = -\dfrac{1}{\tan\theta} = -\cot\theta$ $\tan\theta = \dfrac{4}{QR} \Rightarrow QR = 4\cot\theta$ $y-\text{intercept} = 3+4\cot\theta$ Equation of line PQ is $y = -(\cot\theta)x + 3 + 4\cot\theta$ $y = (4-x)\cot\theta + 3$ (shown)	Poorly attempted. Many students could identify that they needed to find gradient but did not realized that gradient in this question is in fact negative. Students who attempted to 'work backwards' but did not show sufficient and accurate working were penalized.

Find axial intercepts using equation of line. Find stationary value using first derivative.	**(ii)**	When $x=0,\ y=4\cot\theta+3$ When $y=0,\ 0=(4-x)\cot\theta+3$ $x=4-\dfrac{-3}{\cot\theta}=4+3\tan\theta$ $OP+OQ=4+3\tan\theta+4\cot\theta+3$ $=7+3\tan\theta+4\cot\theta$ Let $L=OP+OQ$ $\dfrac{dL}{d\theta}=3\sec^2\theta-4\operatorname{cosec}^2\theta$ $\dfrac{dL}{d\theta}=0\Rightarrow 3\sec^2\theta=4\operatorname{cosec}^2\theta$ $\dfrac{3}{\cos^2\theta}=\dfrac{4}{\sin^2\theta}$ $\tan^2\theta=\dfrac{4}{3}$ $\tan\theta=\dfrac{2}{\sqrt{3}}\left(\text{rej.}-\dfrac{2}{\sqrt{3}}\because 0<\theta<\dfrac{\pi}{2}\right)$ Stationary value of $OP+OQ=7+3\left(\dfrac{2}{\sqrt{3}}\right)+4\left(\dfrac{\sqrt{3}}{2}\right)$ $=7+4\sqrt{3}$	Many students were unable to find the x-coordinate of point P. For students who found the expression for $OP+OQ$, they were unable to differentiate the expression. Students should know the following: (1) $\dfrac{d}{d\theta}(\tan\theta)=\sec^2\theta$ (2) $\dfrac{d}{d\theta}(\cot\theta)=-\operatorname{cosec}^2\theta$

Q7. Parametric Equations

Assessment Objectives	Solution	Examiner's Feedback
Find first derivative of a function defined parametrically.	**(i)** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\dfrac{2t}{\frac{-4}{(t+1)^2}}=-\dfrac{t(t+1)^2}{2}$	Most students able to get the correct answers. Those who were unable to do made careless mistake in dx/dt.
Find equation of normal.	**(ii)** When $x=-2$, $\dfrac{4}{t+1}=-2$ $t=-3$ $y=6$ Gradient of normal $=\dfrac{2}{-3(-3+1)^2}=-\dfrac{1}{6}$ Equation of normal at $P(-2,6)$ is $y-6=-\dfrac{1}{6}(x+2)$ $y=-\dfrac{1}{6}x+\dfrac{17}{3}$ or $6y+x=34$	Most students got the correct concept to solve for the eqn of normal but lost the accuracy mark due of the wrong expression in part (i).
Find *t*-values at points of intersection of a Cartesian line and a parametric curve.	**(iii)** $t^2-3=-\dfrac{1}{6}\left(\dfrac{4}{t+1}\right)+\dfrac{17}{3}$ $6(t+1)(t^2-3)=-4+34(t+1)$ $3(t+1)(t^2-3)=17t+15$ $3t^3+3t^2-9t-9=17t+15$ $3t^3+3t^2-26t-24=0$ Using GC, $t=-3$ (given) or $t=-0.915$ (3 s.f.) or $t=2.91$ (3 s.f.)	Many students attempt to convert the parametric eqn of the curve to cartesian form first then solve for the *x* values, then solve for the *t* values which lead to a longer method. Please note that this method may not work for all questions as it may be hard/impossible to convert to cartesian form. Also, many students didn't make use of their **GC** to solve and hence wasted their time to solve algebraically.

Q8. Graphing Techniques

Assessment Objectives	Solution	Examiner's Feedback
Perform long division	**(i)** $y=\dfrac{2x^2-3x+5}{x-5}=2x+7+\dfrac{40}{x-5}$ $\therefore p=2$ $q=7$ $r=40$ $$\begin{array}{r} 2x+7 \\ x-5\overline{)2x^2-3x+5} \\ -\underline{(2x^2-10x)} \\ 7x+5 \\ -\underline{(7x-35)} \\ 40 \end{array}$$	This part was generally well done.
Identify characteristic of asymptotes, turning points and axial intercepts. Use of a G.C. to graph a given function.	**(ii)** Asymptotes: $y=2x+7$ and $x=5$ 	Candidates need to use a ruler for axis and asymptotes to provide an accurate sketch. Many students did not write down intercept in coordinate form.

Understand that the number of intersections is equivalent to the number of roots in an equation.	**(ii)**	$\left(2x^2-3x+5\right)^2=5x\left(x-5\right)^2$ $\left(\dfrac{2x^2-3x+5}{x-5}\right)^2=5x$ $y^2=5x$ $y=\pm\sqrt{5x}$ Sketch $y=\pm\sqrt{5x}$ in part **(ii).** From the diagram, there are 2 points of intersections. Hence, there are 2 roots.	Many students did not include $y=-\sqrt{5x}$. Some students drew a sketch of $y=\pm\sqrt{5x}$ which did not touch the origin.

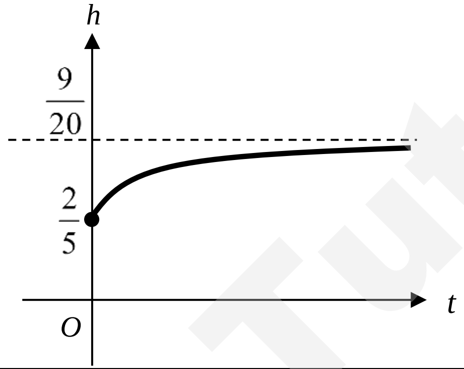
Q9. Maclaurin's Series		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Use of series expansion formula in MF26.	**(a)** **Method ①:** $\sqrt{4-x}=4^{\frac{1}{2}}\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$ $=2\left(1-\frac{x}{8}+\ldots\right)$ $=2-\frac{x}{4}+\ldots$ $p+\ln(q-x)=p+\ln\left[(q)\left(1-\frac{x}{q}\right)\right]$ $=p+\ln q+\ln\left(1-\frac{x}{q}\right)$ $=(p+\ln q)-\frac{x}{q}+\ldots$ Comparing, $2=p+\ln q$ and $-\frac{x}{4}=-\frac{x}{q}$ $2=p+\ln 4$ $q=4$ $p=2-\ln 4$ **Method ②:** Let $\mathrm{f}(x)=\sqrt{4-x}\Rightarrow \mathrm{f}'(x)=\frac{-1}{2\sqrt{4-x}}$, $\therefore \mathrm{f}(0)=2\ \&\ \mathrm{f}'(0)=-\frac{1}{4}$ Let $\mathrm{g}(x)=p+\ln(q-x)\Rightarrow \mathrm{g}'(x)=\frac{-1}{q-x}$, $\therefore \mathrm{g}(0)=p+\ln q\ \&\ \mathrm{f}'(0)=-\frac{1}{q}$ Comparing, $q=4$ and $p=2-\ln 4$	Quite a majority of the students attempted this question successfully with a variety of methods. The most successful method being the use of repeated derivatives to form equations in p and q. Common errors include erroneous use of the standard series expansions and also not knowing how to convert the expressions into the standard form required in their use. A significant number of students also made arithmetic errors on the rules of logarithms, resulting in many marks lost.

Implicit differentiation involving trigonometric expressions. Use of formula given in MF26 to find Maclaurin series.	**(i)** $y=\tan^{-1}(ax+1)$ $\tan y = ax+1$ $\sec^2 y\dfrac{\mathrm{d}y}{\mathrm{d}x}=a$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=a\cos^2 y$ (shown) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2a\cos y(-\sin y)\dfrac{\mathrm{d}y}{\mathrm{d}x}=-a\sin 2y\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^2}=-2a\cos 2y\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2-a\sin 2y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ When $x=0$, $y=\tan^{-1}(1)=\dfrac{\pi}{4}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=a\left(\cos\dfrac{\pi}{4}\right)^2=\dfrac{1}{2}a$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-a\left(\sin\dfrac{\pi}{2}\right)\left(\dfrac{1}{2}a\right)=-\dfrac{1}{2}a^2$ $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=-2a\left(\cos\dfrac{\pi}{2}\right)\left(\dfrac{1}{2}a\right)^2-a\left(\sin\dfrac{\pi}{2}\right)\left(-\dfrac{1}{2}a^2\right)=\dfrac{1}{2}a^3$ $\tan^{-1}(ax+1)=\dfrac{\pi}{4}+\dfrac{1}{2}ax+\dfrac{-\frac{1}{2}a^2}{2!}x^2+\dfrac{\frac{1}{2}a^3}{3!}x^3+\ldots$ $=\dfrac{\pi}{4}+\dfrac{1}{2}ax-\dfrac{1}{4}a^2x^2+\dfrac{1}{12}a^3x^3+\ldots$	Most students performed badly for this question as they are unclear about the process of implicit differentiation, often omitting the multiplication of the first derivative. Students who attempted direct differentiation are rarely successful due to the complexity of the equations. Most students who are successful with the repeated differentiation ended up with the correct expression, except a few who made arithmetic errors on the coefficients.

Use of chain rule and formula given in MF26 to differentiate $\tan^{-1}(4x+1)$, and make use of expression found in **(i)**.	**(ii)** **<u>Method ①</u> (HENCE: direct differentiation using MF26)** $\frac{d}{dx}\left[\tan^{-1}(4x+1)\right]=\frac{4}{1+(4x+1)^2}$ $\frac{1}{1+(4x+1)^2}=\frac{1}{4}\frac{d}{dx}\left[\tan^{-1}(4x+1)\right]$ $=\frac{1}{4}\frac{d}{dx}\left[\frac{\pi}{4}+2x-4x^2+\frac{16}{3}x^3+...\right]$ $=\frac{1}{4}\left[2-8x+16x^2+...\right]$ $=\frac{1}{2}-2x+4x^2+...$ **<u>Method ②</u> (OTHERWISE: binomial expansion)** $\frac{1}{1+(4x+1)^2}=\left[1+(16x^2+8x+1)\right]^{-1}$ $=2^{-1}\left[1+(4x+8x^2)\right]^{-1}$ $=\frac{1}{2}\left[1-(4x+8x^2)+\frac{(-1)(-2)}{2!}(4x)^2+...\right]$ $=\frac{1}{2}\left[1-4x-8x^2+16x^2+...\right]$ $=\frac{1}{2}-2x+4x^2+...$ **<u>Method ③</u> (OTHERWISE: repeated differentiation)** $f(x)=\frac{1}{1+(4x+1)^2}\Rightarrow f'(x)=\frac{-8(4x+1)}{\left[1+(4x+1)^2\right]^2}\Rightarrow f(0)=\frac{1}{2}\ \&\ f'(0)=-2$ $\Rightarrow f''(x)=\frac{-32\left[1+(4x+1)^2\right]^2+128(4x+1)^2\left[1+(4x+1)^2\right]}{\left[1+(4x+1)^2\right]^4}\Rightarrow f''(0)=8$ $\therefore f(x)=\frac{1}{2}-2x+4x^2+\dots$	Most students were unable to see the link necessary for the "hence" method and adopted the otherwise methods. The most successful methods were those which involved repeated differentiation as it does not depend on the previous answers. Many students attempted to use the series expansion for $(1+x)^n$ using $(4x+1)^2$ in place of x, but failing to realize that all powers of $(4x+1)$ will result in terms which have to be include (i.e. constant, x and x^2).

Q10. Differential Equations		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Use of chain rule Formulate differential equation from a problem situation	**(i)** $V=\pi\left(4^2\right)h$ $\frac{\mathrm{d}V}{\mathrm{d}h}=16\pi$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V_{\text{in}}}{\mathrm{d}t}-\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}t}$ $=0.36\pi-0.8\pi h$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}h}\cdot\frac{\mathrm{d}h}{\mathrm{d}t}$ $0.36\pi-0.8\pi h=16\pi\cdot\frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{0.36\pi-0.8\pi h}{16\pi}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{400}\left(9-20h\right)$ (shown)	This part is usually well done. Some candidates introduced t, representing time, in an attempt to establish an equation of h in terms of t. Followed by wrong differentiation of h with respect to t .This approach earn no mark.

Solve differential equations to find particular solution.	**(ii)** $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{400}(9-20h)$ $\int \frac{1}{9-20h}\,\mathrm{d}h=\frac{1}{400}\int 1\,\mathrm{d}t$ $-\frac{1}{20}\int \frac{-20}{9-20h}\,\mathrm{d}h=\frac{1}{400}\int 1\,\mathrm{d}t$ $-\frac{1}{20}\ln\lvert 9-20h\rvert=\frac{1}{400}(t+A)$ $\ln\lvert 9-20h\rvert=-\frac{1}{20}(t+A)$ $\lvert 9-20h\rvert=\mathrm{e}^{-\frac{1}{20}(t+A)}$ $9-20h=\pm\mathrm{e}^{-\frac{1}{20}(t+A)}$ $9-20h=\pm\mathrm{e}^{-\frac{1}{20}t}\cdot\mathrm{e}^{-\frac{1}{20}A}$ $9-20h=B\mathrm{e}^{-\frac{1}{20}t}$ where $B=\pm\mathrm{e}^{-\frac{1}{20}A}$ When $t=0$, $h=0.4$, $9-20(0.4)=B\mathrm{e}^{-\frac{1}{20}(0)}$ $B=1$ $\therefore\ 9-20h=\mathrm{e}^{-\frac{1}{20}t}$ $20h=9-\mathrm{e}^{-\frac{1}{20}t}$ $h=\frac{1}{20}\left(9-\mathrm{e}^{-\frac{1}{20}t}\right)$	Careless mistakes in writing the numbers are unusually frequent in this part and resulted in marks loss. Generally well done.

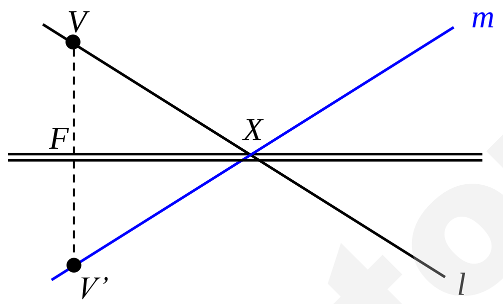
Interpret a differential equation and its solution in terms of a problem situation.	**(iii)** If the tap is on indefinitely, the tank will not be empty. In the long run, there will be $\frac{9}{20}$ m of water in the tank.	No marks awarded to candidates who attempted to explain in words without clear reference to the mathematical equation obtained earlier.
Interpret a differential equation and sketch a graph.	**(iv)** $h=\frac{1}{20}\left(9-\mathrm{e}^{-\frac{1}{20}t}\right)$ 	Many candidates often overlooked the presence of a horizontal asymptote. Lacks proper labelling of axes or asymptote.

Q11. Vectors		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Solve a two-dimensional vector geometry problem involving abstract vectors, by : • Using the collinearity theorem to formulate expressions for the position vector of an unknown point, in terms of two non-zero non-parallel base vectors, and • Compare and equate corresponding coefficients of respective base vectors in a vector equation, to solve the problem.	**(i)** $\overrightarrow{AM}=-\mathbf{a}+\frac{1}{2}\mathbf{b}$ $\overrightarrow{BN}=-\mathbf{b}+\frac{1}{2}\mathbf{a}$ $\overrightarrow{OX}=\mathbf{a}+\overrightarrow{AX} \qquad =\mathbf{b}+\overrightarrow{BX}$ $=\mathbf{a}+\lambda(-\mathbf{a}+\frac{1}{2}\mathbf{b}) \ =\mathbf{b}+\mu(-\mathbf{b}+\frac{1}{2}\mathbf{a})$ for some scalars $\lambda,\ \mu$ $(1-\lambda)\mathbf{a}+\frac{\lambda}{2}\mathbf{b} \ =\frac{\mu}{2}\mathbf{a}+(1-\mu)\mathbf{b}$ $\therefore \begin{cases}1-\lambda=\frac{\mu}{2}\\ \frac{\lambda}{2}=1-\mu\end{cases}$ Solving, $\lambda=\mu=\frac{2}{3}$ $\therefore\ \overrightarrow{OX}=\mathbf{a}+\frac{2}{3}(-\mathbf{a}+\frac{1}{2}\mathbf{b}) \ =\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$ (shown) **<u>Method ①:</u> (Using Equation of Lines)** $l_{AM}:\mathbf{r}=\mathbf{a}+\lambda\left(-\mathbf{a}+\frac{1}{2}\mathbf{b}\right),\ \lambda\in\mathbb{R}$ $l_{BN}:\mathbf{r}=\mathbf{b}+\mu\left(-\mathbf{b}+\frac{1}{2}\mathbf{a}\right),\ \mu\in\mathbb{R}$ Since X lies on both lines, $\mathbf{a}+\lambda\left(-\mathbf{a}+\frac{1}{2}\mathbf{b}\right)=\mathbf{b}+\mu\left(-\mathbf{b}+\frac{1}{2}\mathbf{a}\right)$ $\begin{cases}1-\lambda=\frac{\mu}{2}\\ \frac{\lambda}{2}=1-\mu\end{cases}$ Solving, $\lambda=\mu=\frac{2}{3}$ $\therefore\ \overrightarrow{OX}=\mathbf{a}+\frac{2}{3}(-\mathbf{a}+\frac{1}{2}\mathbf{b}) \ =\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$ (shown)	Most students were able to obtain at least 2 out of 4 marks by using ratio theorem to find $\overrightarrow{AM}$ and $\overrightarrow{BN}$, but some were unsure how to continue. Since **a** and **b** are non-parallel, we can compare the coefficients of the 2 vectors to obtain the 2 equations.

	<u>Method ②:</u> (Using Ratio Theorem) Using triangle OAM, $\overrightarrow{OX}=\dfrac{\lambda(\mathbf{a})+(1-\lambda)\left(\frac{1}{2}\mathbf{b}\right)}{\lambda+(1-\lambda)}=\lambda\mathbf{a}+\dfrac{(1-\lambda)}{2}\mathbf{b}$ Using triangle ONB, $\overrightarrow{OX}=\dfrac{\mu\left(\frac{1}{2}\mathbf{a}\right)+(1-\mu)\mathbf{b}}{\mu+(1-\mu)}=\dfrac{1}{2}\mu\mathbf{a}+(1-\mu)\mathbf{b}$ $\lambda\mathbf{a}+\dfrac{(1-\lambda)}{2}\mathbf{b}=\dfrac{1}{2}\mu\mathbf{a}+(1-\mu)\mathbf{b}$ $\begin{cases}\lambda=\dfrac{\mu}{2}\\ \dfrac{1-\lambda}{2}=1-\mu\end{cases}$ Solving, $\lambda=\dfrac{1}{3},\ \mu=\dfrac{2}{3}$ $\overrightarrow{OX}=\dfrac{1}{3}\mathbf{a}+\dfrac{1}{3}\mathbf{b}$ (shown)	
Apply the midpoint theorem. Apply the collinearity theorem to determine whether three distinct points are collinear.	**(ii)** $\overrightarrow{OT}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$ using the midpoint theorem $\overrightarrow{OX}=\frac{1}{3}(\mathbf{a}+\mathbf{b})$ $=\frac{2}{3}\left[\frac{1}{2}(\mathbf{a}+\mathbf{b})\right]=\frac{2}{3}\overrightarrow{OT}$ Since $\overrightarrow{OX}=k\overrightarrow{OT}$ for some scalar k where $0<k<1$, $\overrightarrow{OX}$ is parallel to $\overrightarrow{OT}$ with a common point O, hence X lies on OT.	Most students gave an incomplete proof for X lying on OT. It is essential to show that $\overrightarrow{OX}$ is a scalar multiple of $\overrightarrow{OT}$ and hence the 2 vectors are parallel.

Find a normal vector for a plane given three non-collinear points on the plane. Formulate a vector equation of a plane in scalar product form, using a point on the plane and a normal vector to the plane.	**(iii)** $\overrightarrow{OA}=\begin{pmatrix}5\\4\\6\end{pmatrix},\ \overrightarrow{OB}=\begin{pmatrix}-2\\2\\3\end{pmatrix}$ $\overrightarrow{OA}\times\overrightarrow{OB}=\begin{pmatrix}5\\4\\6\end{pmatrix}\times\begin{pmatrix}-2\\2\\3\end{pmatrix}=\begin{pmatrix}(4)(3)-(6)(2)\\(6)(-2)-(5)(3)\\(5)(2)-(4)(-2)\end{pmatrix}$ $=\begin{pmatrix}0\\-27\\18\end{pmatrix}=9\begin{pmatrix}0\\-3\\2\end{pmatrix}$ Since $\begin{pmatrix}0\\-3\\2\end{pmatrix}$ is perpendicular to the plane, and origin O is on the plane, it is represented by $\mathbf{r}\cdot\begin{pmatrix}0\\-3\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-3\\2\end{pmatrix}=0.$ $\therefore\ -3y+2z=0$ (shown)	Most students were able to obtain the normal of the plane. Since O is on the plane, the most direct method is to cross $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Find the foot of the perpendicular from a given point to a given plane, by: • Formulate an equation for the perpendicular line passing through the point, and • Find the point of intersection between this perpendicular line and the plane.	**(iv)** Line *VF*, $l_{VF}: \mathbf{r} = \begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$. Since *F* is on l_{VF}, $\overrightarrow{OF} = \begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$, for some $\lambda \in \mathbb{R}$. Since *F* is on *p*, $\overrightarrow{OF} \cdot \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = 0$. $\Rightarrow \left[\begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} \right] \cdot \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = 0$ $130 + 13\lambda = 0$ $\lambda = -10$ $\overrightarrow{OF} = \begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} + (-10) \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -38 \\ -57 \end{pmatrix}$ The coordinates of *F* is $(1, -38, -57)$.	Some students had the **misconception** that $\overrightarrow{VF} = \begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$ and that since $\overrightarrow{VF}$ is parallel to the normal of plane, $\overrightarrow{VF} \cdot \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = 0$. The 2 vectors are parallel, **not** perpendicular. A number of students were careless in solving for the value of λ.

Given a line and a plane that intersects at a point, construct a vector equation for the reflection of a line in a plane, by : • Locating the point of intersection between the line and the plane, • Finding the point of reflection of another point on the line in the plane, and • Constructing a vector equation of the reflected line containing these two points. Convert a vector equation for the line into a Cartesian equation.	**(v)** $\overrightarrow{OX}=\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB})=\begin{pmatrix}1\\2\\3\end{pmatrix}$  Let V' be the reflection of V in plane p. $\overrightarrow{OF}=\dfrac{\overrightarrow{OV}+\overrightarrow{OV'}}{2}$ $\quad\left[\text{or use } \overrightarrow{VF}=\overrightarrow{FV'}\right]$ $\overrightarrow{OV'}=2\overrightarrow{OF}-\overrightarrow{OV}$ $\overrightarrow{OV'}=2\begin{pmatrix}1\\-38\\-57\end{pmatrix}-\begin{pmatrix}1\\-68\\-37\end{pmatrix}=\begin{pmatrix}1\\-8\\-77\end{pmatrix}$ $\overrightarrow{V'X}=\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}1\\-8\\-77\end{pmatrix}=\begin{pmatrix}0\\10\\80\end{pmatrix}=10\begin{pmatrix}0\\1\\8\end{pmatrix}$ Line $m:\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+k\begin{pmatrix}0\\1\\8\end{pmatrix},\ k\in\mathbb{R}$ Line $m:\ x=1,\ y-2=\dfrac{z-3}{8}$	Some students failed to notice that X is the centroid of triangle OAB although it was mentioned in the question. Common mistakes include using $\overrightarrow{OX}=\dfrac{\overrightarrow{OV}+\overrightarrow{OV'}}{2}$ instead of $\overrightarrow{OF}$. The above mistake could have been avoided if the student had **drawn a diagram**. The question asked for a cartesian equation of line m, hence students were penalized for giving the vector equation form as the final answer.

1	The curve with equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is a cubic polynomial, has a maximum point with coordinates $\left(-2, \frac{34}{3}\right)$ and a minimum point with coordinates $\left(3, -\frac{19}{2}\right)$. Find the equation of the curve. [4]
2	Referred to the origin O, the points A, B, P and Q have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{p}$ and $\mathbf{q}$ respectively, such that $\lvert\mathbf{a}\rvert = 2$, $\mathbf{b}$ is a unit vector, and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{4}$. **(i)** Give a geometrical interpretation of $\lvert\mathbf{b}\cdot\mathbf{a}\rvert$. [1] **(ii)** Find $\lvert\mathbf{a}\times\mathbf{b}\rvert$, leaving your answer in exact form. [2] It is also given that $\mathbf{p} = 3\mathbf{a}+(\mu+2)\mathbf{b}$ and $\mathbf{q} = (\mu+3)\mathbf{a}+\mu\mathbf{b}$, where $\mu \in \mathbb{R}$. **(iii)** Show that $\mathbf{p}\times\mathbf{q} = \left(\mu^2+2\mu+6\right)(\mathbf{b}\times\mathbf{a})$. [3] **(iv)** Hence find the smallest area of the triangle OPQ as μ varies. [3]
3	The function f is defined by $\mathrm{f}: x \mapsto \frac{1}{3}\tan\left(\frac{x}{3}\right)$ for $x \in \mathbb{R}$, $0 \le x < \frac{3\pi}{2}$. **(i)** Sketch the graph of $y = \mathrm{f}(x)$, indicating clearly the vertical asymptote. [2] **(ii)** State the equation of the line of reflection between the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, and hence sketch the graph of $y = \mathrm{f}^{-1}(x)$ on the same diagram, indicating clearly the horizontal asymptote. [2] The solutions to the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ are $x = 0$ and $x = \alpha$, where $0 < \alpha < \frac{3\pi}{2}$. **(iii)** Using the diagram drawn, find, in terms of α, the area of the region bounded by the curves $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$. [5] Another function g is defined by $\mathrm{g}: x \mapsto \mathrm{e}^x$ for $x \in \mathbb{R}$, $x \ge -2$. **(iv)** Show that the composite function gf exists and define gf in a similar form. [3]
4	**(a)** The complex numbers z and w satisfy the simultaneous equations $z + w^* + 5\mathrm{i} = 10$ and $\lvert w\rvert^2 = z + 18 + \mathrm{i}$. Find z and w. [4]

(b) **(i)** It is given that $2+\mathrm{i}$ is a root of the equation $z^2-5z+7+\mathrm{i}=0$. Find the second root of the equation in cartesian form, showing your working clearly. [2]

(ii) Hence find the roots of the equation $-\mathrm{i}w^2+5w+7\mathrm{i}-1=0.$ [2]

(c) The complex number z is given by $z=-a+a\mathrm{i}$, where a is a positive real number.

(i) It is given that $w=-\dfrac{\sqrt{2}z^*}{z^4}$. Express w in the form $r\mathrm{e}^{\mathrm{i}\theta}$, in terms of a, where $r>0$ and $-\pi<\theta\le\pi$. [4]

(ii) Find the two smallest positive whole number values of n such that $\mathrm{Re}\,(w^n)=0.$ [3]

5 A planning committee of 12 students consisting of one male and one female student from each of the 6 Arts stream classes (Class A to Class F) in a junior college is to be formed for the Humanities Seminar. There are 10 male and 10 female students in Class A.

(i) How many ways can the representatives from Class A be chosen? [1]

The committee meets for their first planning meeting and is seated at a round table.

(ii) How many ways can the committee be seated if all the members need to be seated together with the member from the same class? [2]

At the seminar, the committee members are to be seated in a row of 14 seats in the theatre together with the Principal and the Guest of Honour. The chairperson and the secretary are selected from the committee and they are both from Class F.

(iii) How many ways can this be done if the Principal and the Guest of Honour occupy the middle seats and the committee members are seated together with the member from the same class except for the chairperson and the secretary? [4]

6 The table below shows the petrol mileage, y km/L and the weight, x kg in thousands for various car models in the year 1995.

x	3.5	3	2.75	2.5	2.25	2	1.75	1.5	1.25
y	7.5	8.0	8.5	8.7	10.0	k	13.5	16.8	18.0

(i) The equation of the regression line of y on x is $y=22.51355-4.908387x$. Show that $k=11.0$. [2]

(ii) Draw a scatter diagram to illustrate the data. [1]

	(iii) With reference to the scatter diagram and context of the question, explain why model (C) below is the most appropriate for modelling the data as compared to the other 2 models. (A) $y = a + bx$, where a is positive and b is negative, (B) $y = a + b\ln x$, where a is positive and b is negative, (C) $y = a + \frac{b}{x}$, where a and b are positive. [1] **(iv)** Calculate the least squares estimates of a and b for model (C). [1] **(v)** Predict the weight of the car if the petrol mileage is 12 km/L. Comment on the reliability of your prediction. [2] **(vi)** Suppose there was an error in recording the y values and all the y values must be increased by a constant M km/L, state any change you would expect in the values of **(a)** $\bar{y}$, [1] **(b)** standard deviation of y and [1] **(c)** the correlation coefficient. [1]
7	**(a)** The random variable X follows a binomial distribution $\mathrm{B}(10, p)$. **(i)** Given that X has two modes, $X = 4$ and $X = 5$, find the exact value of p. [2] **(ii)** Given instead that $\mathrm{P}(X \le 9) = \frac{1023}{1024}$, find the exact value of p. [2] **(b)** The random variable Y follows a binomial distribution $\mathrm{B}(500, 0.5)$. A sample of 30 independent values of Y is recorded. **(i)** Find the probability that all the values recorded are less than or equal to 256. [2] **(ii)** The mean of the 30 values is calculated. Estimate the probability that this sample mean is less than or equal to 256, stating clearly the approximation used. [3] **(iii)** Explain why the probability found in part **(ii)** is larger than that found in part **(i)**. [1]
8	A trading card game has rectangular cards of nominal size 64 mm wide and 89 mm long. However, due to the limited precision of the machine used to cut the cards to size, the widths of the trading cards follow a normal distribution with mean 64 mm and standard deviation 0.3 mm. The lengths of the trading cards follow an independent normal distribution with mean 89 mm and standard deviation 0.45 mm. The perimeter of the trading cards is twice the sum of its length and width.

	(i) Trading cards with length 90 mm and above are called "tall" cards. Find the percentage of trading cards that are "tall". [1] **(ii)** Write down the distribution of the perimeter of the trading cards, in mm, and find the perimeter that is exceeded by 8% of the trading cards. [4] A brand of rectangular card sleeves are manufactured for the trading cards and the widths of the card sleeves follow a normal distribution with mean 66 mm and standard deviation 0.45 mm, whereas the lengths of the card sleeves follow an independent normal distribution with mean 91 mm and standard deviation 0.675 mm. For a card sleeve to fit the trading card nicely, the dimensions of the sleeves must be larger than the dimensions of the trading card, but there should only be a maximum allowance of 1.2 mm on each side. **(iii)** Find the probability that a randomly chosen card sleeve fits a randomly chosen trading card nicely, stating clearly the parameters of any distribution used. [5]
9	A computer hard drive manufacturer claims that the mean usage hours before failure of their R series hard drives is 50 thousand hours. A technology columnist wishes to investigate this claim and collected the usage hours, t thousand hours for each of the 50 randomly chosen hard drives which were submitted to the local service centre for drive failures. The data is summarized as follows. $n = 50 \qquad \Sigma t = 2384.5 \qquad \Sigma t^2 = 115885.23$ The technology columnist wants to use hypothesis testing to test whether the mean usage hours before failure of a hard drive is different from what the manufacturer has stated. **(i)** Explain whether it is necessary for the columnist to know about the distribution of the usage hours before failure of the drives in order to carry out a hypothesis test. [1] **(ii)** Find the unbiased estimates of the population mean and variance and carry out the test at 1% level of significance for the columnist. [6] The columnist published the data and the results of the hypothesis testing in an online article. **(iii)** Suggest a reason why the test result might not be useful to a reader of the article who is deciding whether to buy an R series hard drive from the manufacturer. [1] **(iv)** State an alternative hypothesis that is more relevant to the decision making process and explain whether the result will differ from the earlier test carried out by the columnist at 1% level of significance. [2] **(v)** State a necessary assumption that was made for all the tests carried out. [1]

10	A box contains 2 red balls, 3 green balls and x blue balls, where $x \in \mathbb{Z}, x \geq 5$. A game is played where the contestant picks 5 balls from the box without replacement. The total score, S, for the contestant is the sum of the number of green balls chosen and thrice the number of red balls chosen. The blue balls will not contribute any points, unless all 5 balls are blue. If all the 5 balls are blue, the score will be 25 points.

(i) Show that $\mathrm{P}(S=6)=\dfrac{20x(x^2-3x+14)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$. [2]

(ii) Given that $\mathrm{P}(S=6)=\dfrac{5}{63}$, calculate x. [2]

(iii) Complete the probability distribution table for S. [4]

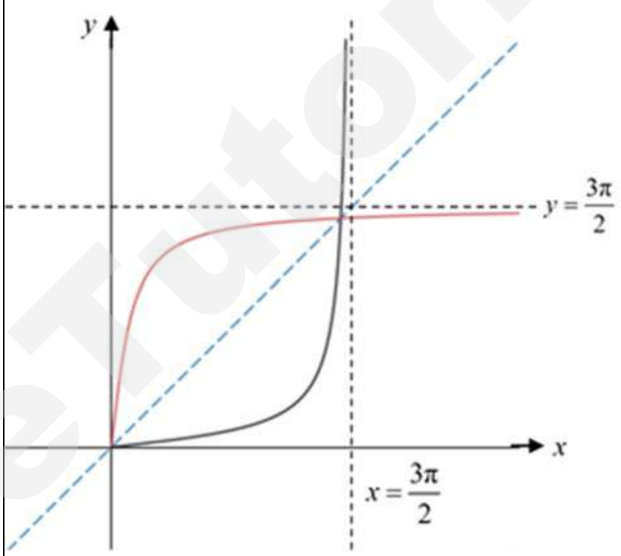
s	1	2	3	4	5	6	7	8	9	25
$\mathrm{P}(S=s)$		$\frac{5}{42}$	$\frac{5}{63}$		$\frac{5}{21}$	$\frac{5}{63}$		$\frac{5}{84}$	$\frac{1}{252}$	

(iv) Evaluate $\mathrm{E}(S)$ and find the probability that S is more than $\mathrm{E}(S)$. [2]

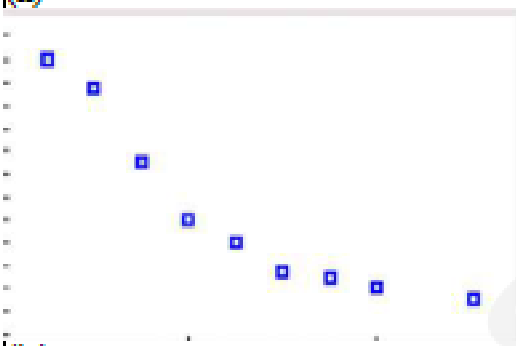
(v) Find the probability that there are no green balls drawn given that S is more than $\mathrm{E}(S)$. [2]

ANNEX B

CJC H2 Math JC2 Preliminary Examination Paper 2

<table>
<tr><th>QN</th><th>Topic Set</th><th>Answers</th></tr>
<tr><td>1</td><td>Equations and Inequalities</td><td>$y=\frac{1}{3}x^3-\frac{1}{2}x^2-6x+4$</td></tr>
<tr><td>2</td><td>Vectors</td><td>(ii) $\sqrt{2}$
(iv) $\frac{5\sqrt{2}}{2}\text{unit}^2$</td></tr>
<tr><td>3</td><td>Functions</td><td>(i)


(ii) $y=x$
(iii) $\alpha^2+2\ln\left[\cos\left(\frac{\alpha}{3}\right)\right]$
(iv) $\mathrm{gf}: x \mapsto e^{\frac{1}{3}\tan\left(\frac{x}{3}\right)}$ for $x\in\mathbb{R}$, $0\le x<\frac{3\pi}{2}$</td></tr>
<tr><td>4</td><td>Complex numbers</td><td>(a) $w=3+4\mathrm{i}$, $z=7-\mathrm{i}$
$w=-4+4\mathrm{i}$, $z=14-\mathrm{i}$
(b)(i) $3-\mathrm{i}$
(ii) $w=1-2\mathrm{i}$, $w=-1-3\mathrm{i}$
(c)(i) $\frac{1}{2a^3}e^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}$
(ii) 2, 6</td></tr>
<tr><td>5</td><td>P&C, Probability</td><td>(i) 100
(ii)7680
(iii)92160</td></tr>
</table>

6	Correlation & Linear Regression	(ii)  (iv) $a = 0.257$, $b = 22.8$ (v) 1860kg (vi)(a) $\bar{y}$ will be increased by a. (b) remain unchanged. (c) remain unchanged.
7	Binomial Distribution	(a)(i) $p = \frac{5}{11}$ (ii) $p = \frac{1}{2}$ (b)(i) $= 0.0000514$ (ii) 0.998
8	Normal Distribution	(i) 1.31%. (ii) $t = 307.51$mm (iii)0.525
9	Hypothesis Testing	(ii) $\bar{t} = 47.69$ thousand hours $s^2 = 44.3$ (iv)Yes
10	DRV	(ii) $x = 5$ (iv) $\frac{127}{252}$ or 0.504 (v) $\frac{11}{127}$

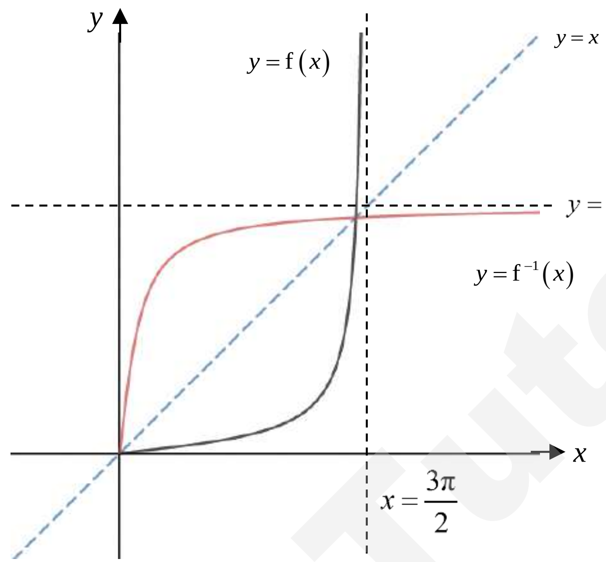
Q1. System of Linear Equations		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Formulate a system of linear equations. Solve a system of linear equations using a G.C.	**(i)** $y = ax^3 + bx^2 + cx + d$ Curve passes through $\left(-2, \frac{34}{3}\right)$: $a(-2)^3 + b(-2)^2 + c(-2) + d = \frac{34}{3}$ $-8a + 4b - 2c + d = \frac{34}{3}$ — ① Curve passes through $\left(3, -\frac{19}{2}\right)$: $a(3)^3 + b(3)^2 + c(3) + d = -\frac{19}{2}$ $27a + 9b + 3c + d = -\frac{19}{2}$ — ② $\frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 2bx + c$ Curve has maximum point $\left(-2, \frac{34}{3}\right)$: $3a(-2)^2 + 2b(-2) + c = 0$ $12a - 4b + c = 0$ — ③ Curve has minimum point $\left(3, -\frac{19}{2}\right)$: $3a(3)^2 + 2b(3) + c = 0$ $27a + 6b + c = 0$ — ④	Most common mistake: - Some students assumed the coeff of x^3 is 1, eg, $y = x^3 + bx^2 + cx + d$ Some attempt to form ONLY 2 or 3 equations to solve for 4 unknowns; note that at least 4 eqns are needed to solve for 4 unknowns. A few students left their eqn as $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 4$ instead of $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 4$

	Solving, $a=\frac{1}{3}, b=-\frac{1}{2}, c=-6, d=4$ $\therefore\ y=\frac{1}{3}x^3-\frac{1}{2}x^2-6x+4$	

Q2. Vectors		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Concept of geometrical interpretation.	**(i)** Length of projection of $\mathbf{a}$ on to $\mathbf{b}$	Generally OK, but many gave the answer as length of projection of $\mathbf{b}$ onto $\mathbf{a}$.
Concept of unit vector and cross product formula.	**(ii)** $\lvert\mathbf{a}\times\mathbf{b}\rvert = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$ $= (2)(1)\sin\frac{\pi}{4}$ $= \sqrt{2}$	Many students mixed up the definition of dot and cross product, although $\sin\frac{\pi}{4}$ is the same as $\cos\frac{\pi}{4}$ which some students ended up with the correct final answer, but they still get penalized as they are using the wrong definition.
Expansion of cross product.	**(iii)** $\mathbf{p}\times\mathbf{q}$ $= \left[3\mathbf{a}+(\mu+2)\mathbf{b}\right]\times\left[(\mu+3)\mathbf{a}+\mu\mathbf{b}\right]$ $=3(\mu+3)(\mathbf{a}\times\mathbf{a})+3\mu(\mathbf{a}\times\mathbf{b})+(\mu^2+5\mu+6)(\mathbf{b}\times\mathbf{a})+\mu(\mu+2)(\mathbf{b}\times\mathbf{b})$ $=(-3\mu+\mu^2+5\mu+6)(\mathbf{b}\times\mathbf{a})$ $[\because \mathbf{a}\times\mathbf{a}=\mathbf{0}$ and $\mathbf{b}\times\mathbf{b}=\mathbf{0}]$ $=(\mu^2+2\mu+6)(\mathbf{b}\times\mathbf{a})$	Common mistakes: - $\mathbf{a}\times\mathbf{a}=\lvert\mathbf{a}\rvert^2$ - The third term in the expansion was $(\mu^2+5\mu+6)(\mathbf{a}\times\mathbf{b})$ instead of $(\mu^2+5\mu+6)(\mathbf{b}\times\mathbf{a})$; note that the direction of cross product is important. - $\underset{\sim}{a}\times\underset{\sim}{a}=\underset{\sim}{0}$ → null vector not $\underset{\sim}{a}\times\underset{\sim}{a}=0$
Finding stationary value.	**(iv)** Area $OPQ = \frac{1}{2}\lvert(\mu^2+2\mu+6)\rvert\lvert(\mathbf{b}\times\mathbf{a})\rvert$ $=\frac{1}{2}\lvert(\mu^2+2\mu+6)\rvert\sqrt{2}$ $=\frac{\sqrt{2}}{2}\lvert(\mu+1)^2+5\rvert$	Common mistake: $\frac{1}{2}(\mu^2+2\mu+6)\mathbf{b}\times\mathbf{a}$ Note that the above expression is a vector, not magnitude.

	Smallest Area $OPQ = \ = \dfrac{5\sqrt{2}}{2}\,\text{unit}^2$	

Q3. Functions & Definite Integrals

Assessment Objectives	Solution	Examiner's Feedback
Understand the relationship between a function and its inverse.	**(i)** 	Many students did not fully extend the curve past $x=\frac{3\pi}{2}$ and/or $y=\frac{3\pi}{2}$
	(ii) $y=x$	
Find area bounded by 2 curves.	**(iii)** <u>**Method ①:**</u> Area $=2\int_0^{\alpha} x-\frac{1}{3}\tan\left(\frac{x}{3}\right)\mathrm{d}x$ $=2\left[\frac{x^2}{2}-\ln\left\|\sec\left(\frac{x}{3}\right)\right\|\right]_0^{\alpha}$ $=2\left[\frac{\alpha^2}{2}-\ln\left\|\sec\left(\frac{\alpha}{3}\right)\right\|-0+0\right]$ $=\alpha^2+2\ln\left[\cos\left(\frac{\alpha}{3}\right)\right]$	Most students did not use this method, opting for the more tedious alternative. Students can use area of triangle formula $\frac{1}{2}\alpha(\alpha)$ instead of $\int_0^{\alpha} x\,\mathrm{d}x$ (Some used $\frac{1}{2}\alpha\left(\frac{1}{3}\tan\left(\frac{\alpha}{3}\right)\right)$ which is not simplified

		Method ②: $$\text{Area } =\int_0^{\alpha} 3\tan^{-1}3x-\frac{1}{3}\tan\left(\frac{x}{3}\right)\mathrm{d}x$$ $$=\left[3x\tan^{-1}3x\right]_0^{\alpha}-\int_0^{\alpha}3x\frac{3}{1+(3x)^2}\,\mathrm{d}x-\left[\ln\left(\sec\frac{x}{3}\right)\right]_0^{\alpha}$$ $$=3\alpha\tan^{-1}3\alpha-\frac{1}{2}\int_0^{\alpha}\frac{18x}{1+9x^2}\,\mathrm{d}x-\ln\left(\sec\frac{\alpha}{3}\right)+\ln 1$$ $$=3\alpha\tan^{-1}3\alpha-\frac{1}{2}\left[\ln\left(1+9x^2\right)\right]_0^{\alpha}-\ln\left(\sec\frac{\alpha}{3}\right)$$ $$=3\alpha\tan^{-1}3\alpha-\frac{1}{2}\ln\left(1+9x^2\right)-\ln\left(\sec\frac{\alpha}{3}\right)$$	Note the two answers are equal. Most students did this method, not utilizing the symmetry of the curves.
Determine if the composite function exists. Find the rule and domain of a composite function.	**(iv)**	$R_f=[0,\infty)$ $D_g=[-2,\infty)$ Since $R_f\subseteq D_g$, gf exists. $$\mathrm{gf}(x)=\mathrm{g}\left[\frac{1}{3}\tan\left(\frac{x}{3}\right)\right]$$ $$=\mathrm{e}^{\frac{1}{3}\tan\left(\frac{x}{3}\right)}$$ $$D_{gf}=D_f=\left[0,\frac{3\pi}{2}\right)$$ $\mathrm{gf}: x\mapsto \mathrm{e}^{\frac{1}{3}\tan\left(\frac{x}{3}\right)}$ for $x\in\mathbb{R}$, $0\le x<\frac{3\pi}{2}$.	Common mistakes: $D_{gf}=D_g=[-2,\infty)$ Many students did not put in similar form

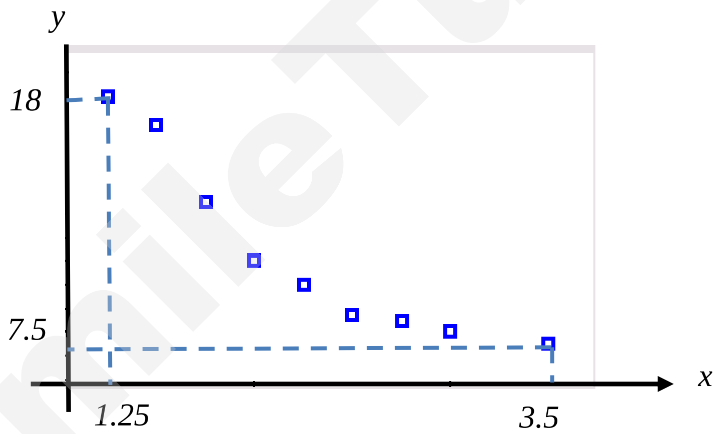
Q4. Complex Numbers		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Solving simultaneous equations involving complex numbers.	**(a)** $z=10-w^*-5\text{i}$ $\lvert w\rvert^2=10-w^*-5\text{i}+18+\text{i}$ $\lvert w\rvert^2+w^*=28-4\text{i}$ Let $w=a+b\text{i}$, $a^2+b^2+a-b\text{i}=28-4\text{i}$ By comparing, $b=4$, $a^2+(4)^2+a=28$ $a^2+a-12=0$ $(a+4)(a-3)=0$ $a=-4$ or $a=3$ $\therefore w=3+4\text{i}$ or $w=-4+4\text{i}$ When $w=3+4\text{i}$, $z=10-(3-4\text{i})-5\text{i}$ $=7-\text{i}$ When $w=-4+4\text{i}$, $z=10-(-4+4\text{i})-5\text{i}$ $=14-9\text{i}$	Most students were able to do this question except for the occasional slips in algebraic manipulation. A number of students mistook $\lvert w\rvert^2$ for w^2. Presentation for simultaneous equation is unclear.
	(b)(i) $z^2-5z+7+\text{i}=[z-(2+\text{i})][z-k]$ By comparing coefficient of z: $z^2-5z+7+\text{i}=[z-(2+\text{i})][z-k]$ $-5=-k-(2+\text{i})$ $k=3-\text{i}$ The second root is $3-\text{i}$	Generally well done.

	(ii) $-\mathrm{i}w^2+5w+7\mathrm{i}-1=0$ $-w^2-5\mathrm{i}w+7+\mathrm{i}=0$ $(\mathrm{i}w)^2-5(\mathrm{i}w)+7+\mathrm{i}=0$ $\mathrm{i}w=2+\mathrm{i}$ or $\mathrm{i}w=3-\mathrm{i}$ $w=1-2\mathrm{i}$ or $w=-1-3\mathrm{i}$	Badly done. Most students fail to identify the term to replace.
Property of modulus and argument.	**(c)(i)** **<u>Method ①:</u>** $z=-a+a\mathrm{i}$ $=a\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$ $w=\left(\mathrm{e}^{\mathrm{i}(\pi)}\right)\dfrac{\sqrt{2}a\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}}{4a^4\mathrm{e}^{\mathrm{i}(3\pi)}}$ $=\dfrac{1}{2a^3}\mathrm{e}^{\mathrm{i}\left(-\frac{11\pi}{4}+2\pi\right)}$ $=\dfrac{1}{2a^3}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}$	Badly done. Most students prefer to simplify the denominator but had problems with the algebraic manipulation. Most students got the argument wrong as they left their answer as $-\dfrac{1}{2a^3}\mathrm{e}^{\mathrm{i}\alpha}$, or they mistook $\arg(-\sqrt{2}z^*)=-\sqrt{2}\arg(z^*)$. Another common mistake was that many students left the argument of z as $\frac{\pi}{4}$.
Concepts of argument to find purely imaginary roots.	**(ii)** If $\mathrm{Re}\,(w^n)=0$, $n\left(-\dfrac{3\pi}{4}\right)=\dfrac{\pi}{2}+k\pi$, where $k\in\mathbb{Z}$ $n=-\dfrac{2}{3}(1+2k)$, where $k\in\mathbb{Z}$ Three smallest positive whole number values of n are 2, 6.	Most students got the method marks but lost the last mark as their argument was wrong.

Q5. Permutations and Combinations

Assessment Objectives	Solution	Examiner's Feedback
Solving simple counting problems involving multiplication principle	**(i)** No. of ways $= {}^{10}C_1 \times {}^{10}C_1 = 100$	Generally well done except for some who did ${}^{10}C_1 + {}^{10}C_1$ instead.
Solving counting problems involving circular arrangements	**(ii)** No. of ways $= (6-1)! \times (2!)^6 = 7680$	Some students used $(6-1)! \times 2!$ or $(6-1)! \times 6(2)!$ instead. Since there are 6 couples, and for each couple there are 2! ways to arrange them, we do $2! \times 2! \times 2! \times 2! \times 2! \times 2!$, which is different from $6(2)!$.
Solving complex arrangement problems involving restrictions	**(iii)** **<u>Method ①:</u>** (1) Arrange P and GoH $= 2!$ (2) Choose side where CP and Sec are on (left or right) $= {}^2C_1$ (3) Choose 2 classes to be seated with CP and Sec $= {}^5C_2$ (4) Arrange the 2 classes and the people within each class $= 2! \times (2!)^2$ (5) Slot in CP and Sec $= {}^3C_2 \times 2!$ (6) Arrange the 3 other classes and the people within each class $= 3! \times (2!)^3$ No. of ways $= 2! \times 2 \times {}^5C_2 \times 2! \times (2!)^2 \times {}^3C_2 \times 2! \times 3! \times (2!)^3 = 92160$ **<u>Method ②:</u> (Arrange the 5 classes at one go)** No. of ways $= 2! \times 2 \times 5! \times (2!)^5 \times {}^3C_2 \times 2! = 92160$ **<u>Method ③:</u> (Complement)** No. of ways $=$ n(CP/S on 1 side but may be tog) $-$ n(CP/S tog) $= \left\{2 \times \left[{}^5C_2 \times 3! \times 2^3\right] \times 2 \times (2)^2 \times 4!\right\} - \left[6! \times (2)^6 \times 2\right] = 92160$	Most students were able to get 1 or 2 marks for this part. Key is for the GoH and P to be seated in the middle, and for the CP and S to be separated, they must be on the same side. Otherwise with 5 students on one side and 7 students on the other side, GoH and P will not be in the middle. So the remaining 5 classes will be split to 3-2, with CP and S joining the side with 2 classes. Hence, using the 2 classes, we will choose 2 out of 3 slots for CP and S. Bear in mind that the 2 classes can either be on the left, or on the right.

Q6. Correlation and Linear Regression

Assessment Objectives	Solution	Examiner's Feedback
Concept of $\left(\bar{x}, \bar{y}\right)$ lies on the regression line.	**(i)** $\bar{y} = 22.51355 - 4.908387\bar{x}$ $\left(\frac{91+k}{9}\right) = 22.51355 - 4.908387\left(\frac{20.5}{9}\right)$ $k = 11.0$	Poorly attempted. Many students simply substituted x = 2 into the equation of the regression line and hoped that the resulting y, i.e. k will be 2. They failed to understand that the point x = 2 may not pass through the regression line. Students must understand the concept that $\left(\bar{x}, \bar{y}\right)$ lies on the regression line.
Scatter diagram	**(ii)** 	Most students handled this part accurately. There were students who carelessly wrote the x-intercepts as y-intercepts and vice versa. Others thought that 1.25 > 3.5 and 7.5 > 18. All these could have been avoided if students made an effort to check their scatter diagram before proceeding.

Concept of linearization of non-linear model.	**(iii)**	As x increases, y decreases at a decreasing rate and tends towards a limit.	Poorly attempted. Students merely says that the graph of model (C) is similar to the graph in the scatter diagram. This warrants no marks. Students are reminded that they need to describe the shape of the graph. Students are advised against describing the gradient of the scatter diagram as it is prone to careless mistakes. In this question, as x increases, the gradient actually increases because it becomes less negative.
Use of GC to find the regression line.	**(iv)**	$a = 0.257$ $b = 22.8$	Many students failed to leave their final answer in 3 s.f.
Estimation and its reliability.	**(v)**	$y = 0.25681 + \frac{22.837}{x}$ $12 = 0.25681 + \frac{22.837}{x}$ $x = 1.94$ (3 s.f.) The weight of the car is 1940kg. The prediction is reliable as $y = 12$ is within the data range of y and the $\|r\|$-value is close to 1.	Many students left their answer as x = 1.94. They did not conclude that the weight of the car is 1940kg or 1.94 kg in thousands. Many students failed to mention that the $\|r\|$-value is close to 1 when stating that the prediction is reliable.
Concept of mean and standard deviation	**(vi)**	(a) $\bar{y}$ will be increased by a. (b) Standard deviation of y remain unchanged. (c) Correlation coefficient remain unchanged.	Well-attempted by students.

Q7. Binomial Distribution and Sampling Distribution		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Setting up and solving equations using the formula for a Binomial random variable	**(a)(i)** $\mathrm{P}(X=4)=\mathrm{P}(X=5)$ $\frac{10!}{4!6!}p^4(1-p)^6=\frac{10!}{5!5!}p^5(1-p)^5$ $5(1-p)=6p$ $p=\frac{5}{11}$	Most students could identify that the probabilities for the outcomes of 4 and 5 should be equal and wrote the expressions according to the formula. Some failed to solve the equation due to inadequate skills in algebra.
Setting up and solving equations using the formula for a Binomial random variable	**(ii)** $\mathrm{P}(X\le 9)=\frac{1023}{1024}$ $\mathrm{P}(X=10)=\frac{1}{1024}$ $p^{10}=\left(\frac{1}{2}\right)^{10}$ $p=\frac{1}{2}$	Many students did not realise that the complementary case is simply 10. Once this hurdle was overcome most were able to find the final answer.
Solving simple problems based on random samples from a binomial random variable	**(b)(i)** $\mathrm{P}(Y\le 256)=0.719485301$ P(all 100 values are less than or equal to 256) $=0.719485301^{30}$ $=0.0000514$	Many students immediately dived into the irrelevant routine of using CLT to find the sampling distribution once they saw the conditions given, without analyzing the question carefully. Majority of the students left the first probability as the answer. Their understanding of the term "sample" may be in question.

Applying Central Limit Theorem for the sampling distribution of a random sample from a binomial random variable	**(ii)** $\mathrm{E}(Y)=500(0.5)=250$, and $\mathrm{Var}(Y)=500(0.5)(0.5)=125$ Since the sample size is sufficiently large, $\bar{Y} \sim \mathrm{N}\left(250,\frac{125}{30}\right)$ approximately by CLT $\mathrm{P}\left(\bar{Y}\le 256\right)=0.998$	Most were able to follow the routine to write down the expectation and variance of Y. However, half of them did not show clear understanding of sampling distribution and central limit theorem in their subsequent presentation of the solution. The most common mistake is that quoting CLT to write down $Y \sim N(250,125)$, which is WRONG! It is the mean of samples of large size may be considered as normally distributed approximately, not the individual observation. Other common mistakes include forgetting to divide the variance by sample size, or using a wrong notation for the random variable of sample mean.
Making comparison between probabilities that are calculated based on the context	**(iii)** The probability in part **(ii)** included cases where some of the values can be larger than 256, but the final average is still at most 256.	Many students were able to give the correct reason though the phrasing can still be improved. For example many casually wrote "probability in (i) is a subset of probability in (ii)", which showed understanding but failed to make sense mathematically when it is the collection of "events/outcomes" in one being subset of the other.

Q8. Normal Distribution

Assessment Objectives	Solution	Examiner's Feedback
Practical application of Normal distribution to obtain percentages of items with certain properties	**(i)** Let L be the random variable denoting length of a trading card in mm. $L \sim \mathrm{N}(89,\ 0.2025)$ $\mathrm{P}(L > 90) = 0.0131$, hence the percentage is 1.31%.	Badly done by students. Mistakes: 1) Confusing CRV and DRV by writing $\mathrm{P}(L \le 90) = \mathrm{P}(L \le 89)$ 2) Not answering in %
Practical application of Normal distribution to obtain the critical value of a certain property satisfied by a given percentage of items	**(ii)** Let T be the random variable denoting the perimeter of a trading card, in mm. $T \sim \mathrm{N}\left(2(64)+2(89),\ 2^2(0.3^2)+2^2(0.45^2)\right)$ $\sim \mathrm{N}\left(306,\ 1.17\right)$ $\mathrm{P}(T > t) = 0.08$ $\mathrm{P}(T < t) = 0.92$ Hence $t = 307.51\,\mathrm{mm}$	Badly done by students. Most common mistake: Taking invNorm with RHS area 0.2.
Practical application of Normal distribution to obtain probability of randomly selected items fulfilling certain independent physical properties	**(iii)** Let X and Y be random variable denoting the width and length of a card sleeve subtracting away the width and length of a trading card respectively in mm. Hence $X \sim \mathrm{N}\left(66-64,\ (0.45^2)+(0.3^2)\right)$ $\sim \mathrm{N}\left(2,\ 0.2925\right)$ and $Y \sim \mathrm{N}\left(91-89,\ (0.45^2)+(0.675^2)\right)$ $\sim \mathrm{N}\left(2,\ 0.658125\right)$ $\mathrm{P}(\text{width fits nicely}) = \mathrm{P}(0 < X \le 2.4) = 0.7701200999$ $\mathrm{P}(\text{length fits nicely}) = \mathrm{P}(0 < Y \le 2.4) = 0.6821730404$ $\mathrm{P}(\text{sleeve fits nicely}) = \mathrm{P}(\text{width fits nicely})\,\mathrm{P}(\text{length fits nicely})$ $= 0.7701200999 \times 0.6821730404$ $= 0.525$	Badly done. Better students were able to find the new mean and new variance but mistakes were made when calculate the probabilities, Mistakes: 1) $\mathrm{P}(X \le 2.4)\,\mathrm{P}(Y \le 2.4)$ 2) $\mathrm{P}(X \le 1.2)\,\mathrm{P}(Y \le 1.2)$ 3) $\mathrm{P}(0 \le X \le 1.2)\,\mathrm{P}(0 \le Y \le 1.2)$ Most students left this part blank.

Q9. Hypothesis Testing		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Provide reasoning to support the application of Central Limit Theorem in Hypothesis testing	**(i)** It is not necessary as the <u>sample size is sufficiently large</u> for <u>Central Limit Theorem to apply</u>.	This part is poorly attempted with many students discussing about the PARAMETERS of the distribution rather than the fact of whether it is a normal distribution. Some students simply stated that it is necessary as the hypothesis testing requires the use of a normal distribution, showing clearly their lack of understanding for the Central Limit Theorem and Sampling Distributions in general. There are also quite a number of students who either left out the fact that the sample size is large or that Central Limit Theorem is applicable, resulting in an incomplete explanation.
Conduct a z-test for a practical situation	**(ii)** $\bar{t} = 2384.5/50 = 47.69$ thousand hours $s^2 = \frac{1}{50-1}\left(115885.23 - \frac{2384.5^2}{50}\right) = 44.25357143 = 44.3$ $\mathrm{H}_0 : \mu = 50$ $\mathrm{H}_1 : \mu \neq 50$ Under H_0, since *n* is large, by C.L.T. $\bar{T} \sim \mathrm{N}\left(50,\ \frac{44.25357143}{50}\right)$ appx p-value = 0.01407 OR test statistic = – 2.4554	Most candidates are successful with the unbiased estimates, but some left the answer as 47.7 for population mean, not realizing that it is an exact decimal. Quite a number of students also quoted the wrong formula for population variance or left their answers as the value before dividing by 49. While most students with the correct estimates were successful

		Since p-value = 0.01407 > 0.01, we do not reject H_0 and conclude that there is insufficient evidence at 1% level of significance to claim that the mean number of hours before failure is not 50 thousand hour.	with the testing, there is also a significant number of students who lost all marks by simply stating an incorrect p-value based on their wrong parameters. P-values calculated based on 3 s.f. values of the parameters or a combination of 5 s.f. and 3 s.f. values were accepted. Correct p-values based on erroneous presentation of the sampling distributions were not penalized due to benefit of doubt given. Students who attempted to use critical values were less successful as they applied modulus to the the test statistic without doing so for the critical value resulting in erroneous comparisons. Students should state clearly the rejection region when using critical values. Most conclusions were not given in context, did not mention level of significance, or were not phrase in terms of the alternate hypothesis. Many students phrased the conclusion wrongly as "having sufficient evidence to claim that the mean is 50 thousand hours".
Provide reasoning to support the choice of alternate hypothesis based on the context of the practical situation	**(iii)**	The reader would be more interested to test whether the mean is actually lower than the stated value which is not beneficial to them.	Most students simply stated that the test did not indicate whether the mean is more of less, but did not make any reference to why these

			cases would matter to the reader. Students who manage to make reference to higher mean being beneficial and/or lower mean being not beneficial, were given credit based on benefit of doubt.
Identifying alternate hypothesis relevant to the context and provide reasoning on the effect of the alternate hypothesis on the resulting p-value and final conclusion of a z-test	**(iv)**	$H_1 : \mu < 50$ Yes, the result will differ as the p-value will be halved when switching to a one-tail test.	Most students were successful with stating the correct alternate hypothesis, except some who used the left tail test in (ii). However, not all were able to provide an explanation to support the change in conclusion, especially those who used critical values in (ii). Students who did not identify that the p-value is exactly half of the value found in (ii) will have to state the actual value, simply mentioning that the p-value is smaller is in sufficient as 0.011 is also a smaller p-value, but it will not result in a change in the conclusion. Most students who re-did the test were most successful for this part, but many went on to write the full conclusion, which is not required by the question.
Identifying implicit assumptions made in a hypothesis test	**(v)**	We need to assume that the usage hours before failure for hard drives are independent for all hard drives.	Many students were able to state the assumption needed, but some did not exhibit any understanding of the situation.

Q10. Probability and Discrete Random Variables		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Formulating an expression for the probability density function of a discrete random variable	**(i)** $\mathrm{P}(S=6)$ $=\mathrm{P}(RRBBB)+\mathrm{P}(RGGGB)$ $=\dfrac{{}^5C_3(2!)\left[x(x-1)(x-2)\right]}{(x+5)(x+4)(x+3)(x+2)(x+1)}+\dfrac{{}^5C_3x(2)(2)(3!)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$ $=\dfrac{20x\left[x^2-3x+2\right]}{(x+5)(x+4)(x+3)(x+2)(x+1)}+\dfrac{20x(12)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$ $=\dfrac{20x(x^2-3x+14)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$	This part is usually well done. Lack of essential working is not acceptable.
Solving for an unknown parameter based on the expression for the probability density function of the discrete random variable	**(ii)** $\dfrac{5}{63}=\dfrac{20x(x^2-3x+14)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$ $5(x+5)(x+4)(x+3)(x+2)(x+1)-1260x(x^2-3x+14)=0$ Solving, the only integer root is $x=5$	This part is well done.
Completing the probability distribution table of a discrete random variable	**(iii)** see table below $\mathrm{P}(S=1)=\mathrm{P}(\text{GBBBB})$ $=\left(\dfrac{3}{10}\right)\left(\dfrac{5}{9}\right)\left(\dfrac{4}{8}\right)\left(\dfrac{3}{7}\right)\left(\dfrac{2}{6}\right)\left(\dfrac{5!}{4!}\right)$ $=\dfrac{5}{84}$ or $\dfrac{15}{252}$ Note: $\dfrac{5!}{4!}$ is for arranging GBBBB with 4 repeated "B"s.	$\mathrm{P}(S=4)$ and $\mathrm{P}(S=7)$ proved to be quite challenging for most candidates.

s	1	4	7	25
$\mathrm{P}(S=s)$	$\frac{5}{84}$ or $\frac{15}{252}$	$\frac{5}{21}$ or $\frac{60}{252}$	$\frac{5}{42}$ or $\frac{30}{252}$	$\frac{1}{252}$

		$\mathrm{P}(S=4)=\mathrm{P}(\mathrm{RGBBB})$ $=\left(\frac{2}{10}\right)\left(\frac{3}{9}\right)\left(\frac{5}{8}\right)\left(\frac{4}{7}\right)\left(\frac{3}{6}\right)\left(\frac{5!}{3!}\right)$ $=\frac{5}{21}$ or $\frac{60}{252}$ Note: $\frac{5!}{3!}$ is for arranging RGBBB with 3 repeated "B"s. $\mathrm{P}(S=7)=\mathrm{P}(\mathrm{RRGBB})$ $=\left(\frac{2}{10}\right)\left(\frac{1}{9}\right)\left(\frac{3}{8}\right)\left(\frac{5}{7}\right)\left(\frac{4}{6}\right)\left(\frac{5!}{2!2!}\right)$ $=\frac{5}{42}$ or $\frac{30}{252}$ Note: $\frac{5!}{2!2!}$ is for arranging RRGBB with 2 repeated "R"s and 2 repeated "B"s. $\mathrm{P}(S=25)=\mathrm{P}(\mathrm{BBBBB})$ $=\left(\frac{5}{10}\right)\left(\frac{4}{9}\right)\left(\frac{3}{8}\right)\left(\frac{2}{7}\right)\left(\frac{1}{6}\right)$ $=\frac{1}{252}$	
Calculating expectation of a discrete random variable and probabilities based on the probability distribution table	**(iv)**	$\mathrm{E}(S)=4.60$ or $\frac{1159}{252}$ $\mathrm{P}(S>4.60)=\frac{127}{252}$ or 0.504	This part is poorly done as a result of errors from (iii).

Solving for conditional probabilities based on a discrete random variable	**(v)** $\mathrm{P}(\text{no}\, G \mid S > 4.60)$ $= \dfrac{\mathrm{P}(RRBBB)+\mathrm{P}(BBBBB)}{\frac{127}{252}}$ $= \dfrac{\frac{20(5)(4)(3)}{(10)(9)(8)(7)(6)}+\frac{1}{252}}{\frac{127}{252}}$ $= \dfrac{\frac{10}{252}+\frac{1}{252}}{\frac{127}{252}}$ $= \dfrac{11}{127}$	Common omission or error pertaining to the case $\mathrm{P}(RRBBB)$, led to wrong answer. This part is poorly done.

H2 Mathematics 2017 Prelim Exam Paper 1 Question
Answer all questions [100 marks].

1

Given that $\sum_{k=1}^{n} k!(k^2+1)=(n+1)!\,n$, find $\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)$. [3]

2

A geometric sequence $T_1,\ T_2,\ T_3,\ \ldots$ has a common ratio of e. Another sequence $U_1,\ U_2,\ U_3,\ \ldots$ is such that $U_1=1$ and

$$U_r=\ln T_r-3 \quad \text{for all } r\geq 1.$$

(i) Prove that the sequence $U_1,\ U_2,\ U_3,\ \ldots$ is arithmetic. [2]

A third sequence $W_1,\ W_2,\ W_3,\ \ldots$ is such that $W_1=\frac{1}{2}$ and $W_{r+1}=W_r+U_r$ for all $r\geq 1$.

(ii) By considering $\sum_{r=1}^{n-1}(W_{r+1}-W_r)$, show that $W_n=\frac{1}{2}(n^2-n+1)$. [3]

3

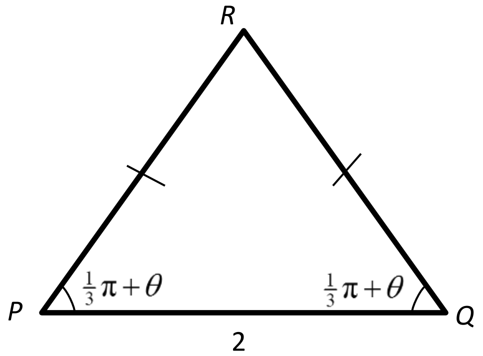
Using an algebraic method, find the set of values of x that satisfies the inequality

$$2-x\leq\frac{x}{2-x}.$$

[3]

Hence solve $2-x^2\leq\frac{x^2}{2-x^2}$. [2]

4

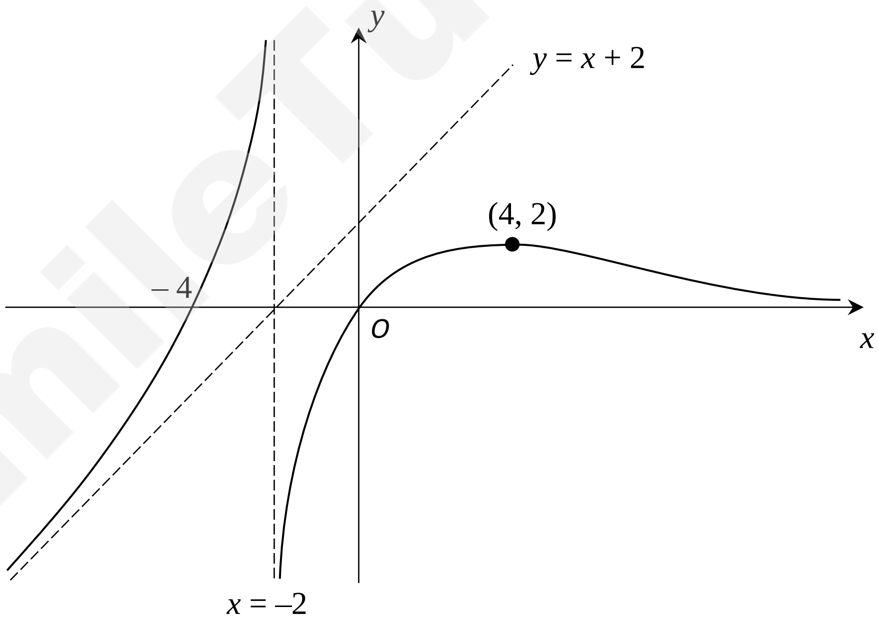


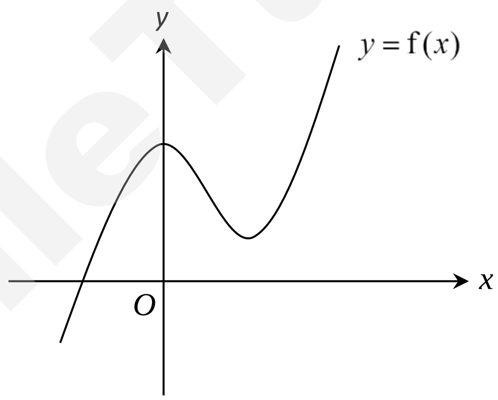
In the isosceles triangle PQR, PQ = 2 and the angle QPR = angle $PQR=\left(\frac{1}{3}\pi+\theta\right)$ radians. The area of triangle PQR is denoted by A.

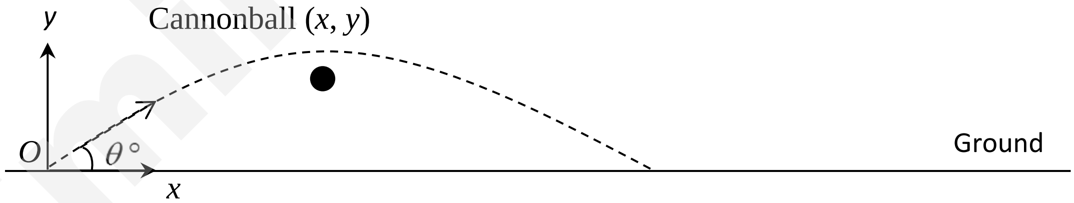
Given that θ is a sufficiently small angle, show that

$$A=\frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}\approx a+b\theta+c\theta^2,$$

for constants a, b and c to be determined in exact form. [5]

5	**(a)** Given that $\operatorname{cosec} y = x$ for $0 < y < \frac{1}{2}\pi$, find $\dfrac{dy}{dx}$ in terms of y. Deduce that $\dfrac{d}{dx}(\operatorname{cosec}^{-1}x) = -\dfrac{1}{x\sqrt{(x^2-1)}}$ for $x > 1$. [3] **(b)** The function f is such that $f(x)$ and $f'(x)$ exist for all real x. Sketch a possible graph of f which illustrates that the following statement is not necessarily true: “If the equation $f'(x) = 0$ has exactly one root $x = 0$ and $f''(0) > 0$, then $f(x) \to \infty$ as $x \to \pm\infty$.” [2]
6	**(a)** State a sequence of transformations that transform the graph of $x^2 + \frac{1}{3}(y-2)^2 = 1$ to the graph of $(x-2)^2 + y^2 = 1$. [3] **(b)** The diagram below shows the curve $y = f(x)$. It has a maximum point at $(4, 2)$ and intersects the x-axis at $(-4, 0)$ and the origin. The curve has asymptotes $x = -2$, $y = 0$ and $y = x + 2$.  Sketch on separate diagrams, the graphs of **(i)** $y = f'(x)$, [3] **(ii)** $y = \dfrac{1}{f(x)}$, [3] including the coordinates of the points where the graphs cross the axes, the turning points and the equations of any asymptotes, where appropriate.
7	**(i)** Express $\sin x + \sqrt{3}\cos x$ as $R\sin(x+\alpha)$, where $R > 0$ and α is an acute angle. [1]

	The function f is defined by $$\mathrm{f}: x \mapsto \sin x + \sqrt{3}\cos x, \quad x \in \mathbb{R},\ -\tfrac{1}{3}\pi \le x \le \tfrac{1}{6}\pi.$$ **(ii)** Sketch the graph of $y = \mathrm{f}(x)$. [2] **(iii)** Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. On the same diagram as in part **(ii)**, sketch the graph of $y = \mathrm{f}^{-1}(x)$, indicating the equation of the line of symmetry. [4] **(iv)** Using integration, find the area of the region bounded by the graph of f^{-1} and the axes. [3] The function g is defined by $$\mathrm{g}: x \mapsto \lvert \ln(x+2) \rvert, \text{ for } x \in \mathbb{R},\ x > -2.$$ **(v)** Show that the composite function gf^{-1} exists, and find the range of gf^{-1}. [3]
8	**Do not use a graphic calculator in answering this question.** **(a)**  It is given that $\mathrm{f}(x)$ is a cubic polynomial with real coefficients. The diagram shows the curve with equation $y = \mathrm{f}(x)$. What can be said about all the roots of the equation $\mathrm{f}(x) = 0$? [2] **(b)** The equation $2z^2 - (7+6\mathrm{i})z + 11 + \mathrm{i}c = 0$, where c is a non-zero real number, has a root $z = 3 + 4\mathrm{i}$. Show that $c = -2$. Determine the other root of the equation in cartesian form. Hence find the roots of the equation $2w^2 + (-6+7\mathrm{i})w - 11 + 2\mathrm{i} = 0$. [6] **(c)** The complex number z is given by $z = 1 + \mathrm{e}^{\mathrm{i}\alpha}$. **(i)** Show that z can be expressed as $2\cos\left(\tfrac{1}{2}\alpha\right)\mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\alpha\right)}$. [2]

	(ii) Given $\alpha = \frac{1}{3}\pi$ and $w = -1-\sqrt{3}\mathrm{i}$, find the exact modulus and argument of $\left(\dfrac{z}{w^3}\right)^*$. [5]
9	The line l_1 passes through the point A, whose position vector is $3\mathbf{i}+7\mathbf{j}+3\mathbf{k}$, and is parallel to the vector $3\mathbf{i}+4\mathbf{j}+\mathbf{k}$. The line l_2 is given by the cartesian equation $x-2=\dfrac{3-y}{2}=\dfrac{z-5}{2}$. The plane p_1 contains l_1 and is parallel to l_2. Another plane p_2 also contains l_1 and is perpendicular to p_1. **(i)** Find the cartesian equation of p_1. [3] **(ii)** Find the distance of l_2 to p_1. [2] **(iii)** Find the equation of p_2 in the scalar product form. [2] A particle P moves along a straight line c which lies in the plane p_2 and c passes through a point $(5, \frac{1}{2}, -3)$. P hits the plane p_1 at A and rebounds to move along another straight line d in p_2. The angle between d and l_1 is the same as the angle between c and l_1. **(iv)** Find the direction cosines of d. [6] **(v)** Another particle, Q, is placed at the point $(\frac{25}{2}, \frac{21}{2}, -\frac{1}{2})$. Find the shortest distance PQ as P moves along d. [3]
10	 The diagram shows the trajectory of a cannonball fired off from an origin O with an initial speed of v ms^{-1} and at an angle of $\theta°$ above the ground. At time t seconds, the position of the cannonball can be modelled by the parametric equations $x=(v\cos\theta)t,\ y=(v\sin\theta)t-5t^2,$ where x m is the horizontal distance of the cannonball with respect to O and y m is the vertical distance of the cannonball with respect to ground level. **(i)** Find the horizontal distance, d m, that a cannonball would have travelled by the time it hits the ground. Leave your answer in terms of v and θ. [4] **Use $v=200$ to answer the remaining parts of the question.** An approaching target is travelling at a constant speed of 10 ms^{-1} along the ground. A cannonball is fired towards the target when it is 3000 m away. You may assume the height of the moving target is negligible.

(ii) Show that in order to hit the target, the possible angles at which the cannonball should be fired are $22.7°$ and $69.5°$. [2]

(iii) Explain at which angle the cannonball should be fired in order to hit the target earlier. [2]

(iv) Given that $\theta = 22.7,$ find the angle that the tangent to the trajectory makes with the horizontal when $x = 370.$ [4]

11 **For this question, you may leave your answers to the nearest dollar.**

(a) Mr Foo invested $25,000 in three different stocks *A*, *B* and *C*. After a year, the value of the stocks *A* and *B* grew by 2% and 6% respectively, while the value of stock *C* fell by 2%. Mr Foo did not gain or lose any money. Let *a*, *b* and *c* denote the amount of money he invested in stocks *A*, *B* and *C* respectively.

(i) Find expressions for *a* and *b*, in terms of *c*. [2]

(ii) Find the values between which *c* must lie. [2]

(b) Mr Lee is interested in growing his savings amount of $55,000 and is considering the Singapore Savings Bonds. He is able to enjoy a higher average return per year when he invests over a longer period of time as shown in the following table.

Number of years invested	1	2	3	4	5	6	7	8
Average return per year, %	1.04	1.21	1.35	1.48	1.60	1.71	1.82	1.92

For example, if Mr Lee invests for two years, he is able to enjoy compound interest at a rate of 1.21% per year.

(i) Calculate the compound interest earned by Mr Lee if he were to invest $55,000 in this bond for a period of five years. [2]

A bank offers a dual-savings account with the following scheme:

“ For every $1,000 deposited into the normal savings account, an individual can deposit $10,000 into the special savings account to enjoy a higher interest rate. The annual compound interest rates for the normal savings account and the special savings account are 0.19% and 1.8% respectively.”

Mr Lee is interested in setting up this dual-savings account and considers an *n*-year investment plan as such:

At the start of each year, he will place $1,000 in the normal savings account and $10,000 in the special savings account.

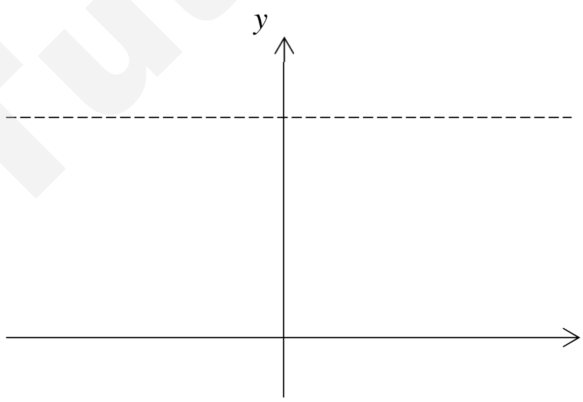
(ii) Find the respective amount of money in the normal savings account and special savings account at the end of *n* years. Leave your answers in terms of *n*. [4]

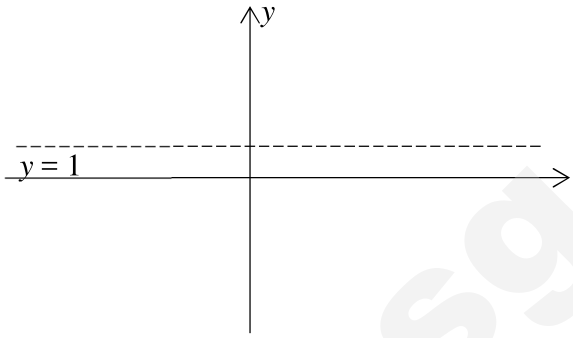
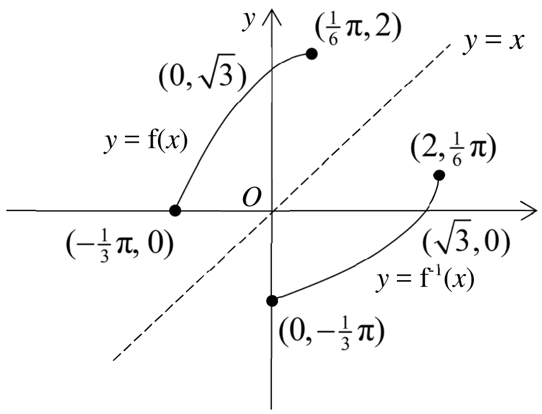
	(iii) Find the least value of n such that the compound interest earned in dual-savings account is more than the compound interest earned in part **(i)**. [2]

End Of Paper –

ANNEX B

DHS H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Sigma Notation and Method of Difference	$(n+1)!n-2$
2	Sigma Notation and Method of Difference	--
3	Equations and Inequalities	$\{1\le x<2 \text{ or } x\ge 4\}$; $x\le -2$ or $-\sqrt{2}<x\le -1$ or $1\le x<\sqrt{2}$ or $x\ge 2$
4	Maclaurin series	$\sqrt{3}+4\theta+\left(4\sqrt{3}\right)\theta^2$
5	Differentiation & Applications	(a) $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\operatorname{cosec} y \cot y}$ (b) 
6	Graphs and Transformation	(a) 1. Translate 2 units in the positive x-direction 2. Translate 2 units in the negative y-direction 3. Scale by a factor of $\frac{1}{\sqrt{3}}$ parallel to the y-direction Alternative 2. Scale by a factor of $\frac{1}{\sqrt{3}}$ parallel to the y-direction 3. Translate $\frac{2}{\sqrt{3}}$ units in the negative y-direction

		(b)(i)  (b) (ii)
7	Functions	(i) $\sin x + \sqrt{3}\cos x = 2\sin(x + \frac{1}{3}\pi)$ (ii)  (iii) $\mathrm{f}^{-1}(x) = -\frac{1}{3}\pi + \sin^{-1}(\frac{1}{2}x)$; $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = [0, 2]$ (iv) 1 (v) $\mathrm{R}_{\mathrm{gf}^{-1}} = [0,\ 0.926]$

8	Complex numbers	(a) Since the curve shows only one x-intercept, it means that there is only one real root in the equation $f(x)=0$. Since the equation has all real coefficients, then the two other roots must be non-real and they are a conjugate pair. (b) $\frac{1}{2}-\mathrm{i}$; $4-3\mathrm{i}$ and $-1-\frac{1}{2}\mathrm{i}$. (c) (ii) $\frac{\sqrt{3}}{8}$; $-\frac{\pi}{6}$
9	Vectors	(i) $2x-y-2z=-7$ (ii) $\frac{2}{3}$ (iii) $\mathbf{r}.\begin{pmatrix}-7\\8\\-11\end{pmatrix}=2$ (iv) $\frac{16}{\sqrt{329}},\frac{3}{\sqrt{329}}$ and $-\frac{8}{\sqrt{329}}$ or $-\frac{16}{\sqrt{329}},-\frac{3}{\sqrt{329}}$ and $\frac{8}{\sqrt{329}}$. (v) $\frac{1}{2\sqrt{329}}\left\|\begin{pmatrix}-35\\40\\-55\end{pmatrix}\right\|=2.11$ (3 s.f.)
10	Differentiation & Applications	(i) $\frac{v^2\sin\theta\cos\theta}{5}$ (ii) – (iii) 22.7° (iv) 17.2° (to 1dp)
11	AP and GP	(a)(i) $a=37500-2c, b=c-12500$ (ii) between 12500 and 18750 (b)(i) 4543 (to the nearest dollar) (ii) Normal savings account: $527315.79(1.0019^n-1)$ Special savings account: $565555.56\left(1.018^n-1\right)$ (iii) 7

H2 Mathematics 2017 Prelim Exam Paper 1 Solution

<table>
<tr><td>1</td><td>
<u>Method 1</u>
Consider replace k by $(k-1)$:
$$\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)=\sum_{k-1=1}^{k-1=n-1}(k-1+1)!((k-1)^2+2(k-1)+2)$$
$$=\sum_{k=2}^{n}k!(k^2+1)$$
$$=\sum_{k=1}^{n}k!(k^2+1)-1!(1^2+1)$$
$$=(n+1)!n-2$$
<u>Method 2</u>
$$\sum_{k=1}^{n}k!\left(k^2+1\right)=\sum_{k=0}^{n-1}(k+1)!\left((k+1)^2+1\right)$$
$$=\sum_{k=0}^{n-1}(k+1)!\left(k^2+2k+2\right)$$
$$=(n+1)!n$$
$$\sum_{k=1}^{n-1}(k+1)!\left(k^2+2k+2\right)=\sum_{k=0}^{n-1}(k+1)!\left(k^2+2k+2\right)$$
$$+(0+1)!\left(0^2+2(0)+2\right)$$
$$=(n+1)!n+2$$
</td></tr>
<tr><td>2</td><td>
(i) To prove AP, consider
$U_{r+1}-U_r$
$=(\ln T_{r+1}-3)-(\ln T_r-3)$
$=\ln\left(\frac{T_{r+1}}{T_r}\right)$
$=\ln \mathrm{e}$
$=1$
Since difference is a constant, the sequence is arithmetic. (Proven)
$$\sum_{r=1}^{n-1}(W_{r+1}-W_r)=\sum_{r=1}^{n-1}U_r$$
</td></tr>
</table>

$$\text{LHS} = \sum_{r=1}^{n-1}(W_{r+1} - W_r)$$

$= W_2 - W_1$

$+ \cancel{W_3} - \cancel{W_2}$

$+ \cancel{W_4} - \cancel{W_3}$

$\vdots$

$+ W_n - \cancel{W_{n-1}}$

$= W_n - W_1$

$= W_n - \frac{1}{2}$

$$\text{RHS} = \sum_{r=1}^{n-1} U_r$$

$= U_1 + U_2 + \ldots + U_{n-1}$

$= \dfrac{n-1}{2}(2(1) + (n-2)1)$

$= \dfrac{n(n-1)}{2}$

Thus, $W_n - \dfrac{1}{2} = \dfrac{n(n-1)}{2}$

$\therefore W_n = \dfrac{1}{2}\left(n^2 - n + 1\right)$ (shown)

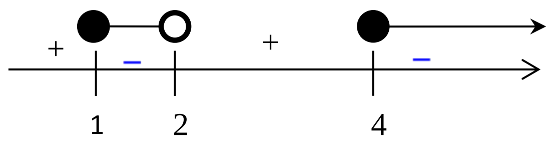
3 (i) $2 - x \le \dfrac{x}{2-x}$

$2 - x - \dfrac{x}{2-x} \le 0$

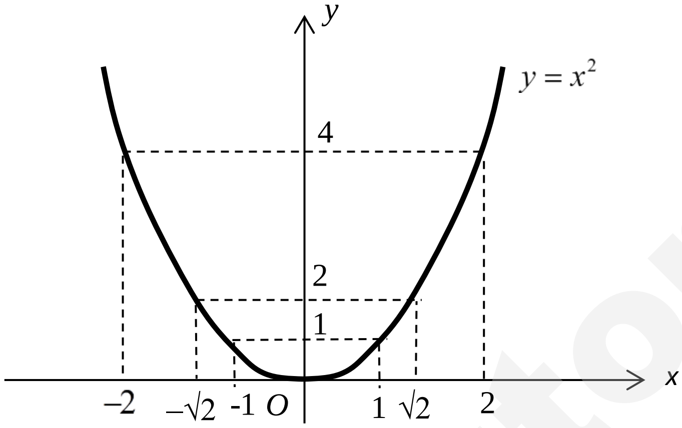
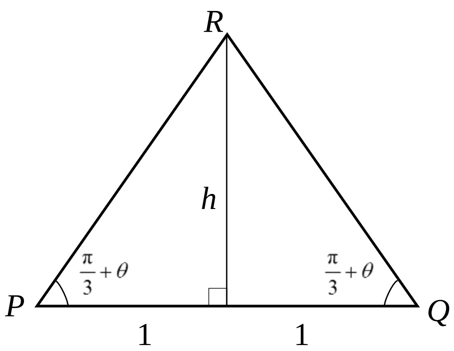
$\dfrac{(2-x)^2 - x}{2-x} \le 0$

$\dfrac{x^2 - 5x + 4}{2-x} \le 0$

$\dfrac{(x-4)(x-1)}{2-x} \le 0$



Set of values of x: $\{1 \le x < 2 \text{ **or** } x \ge 4\}$

	(ii) Let $y = x^2$. $2 - x^2 \le \frac{x^2}{2 - x^2} \Rightarrow 2 - y \le \frac{y}{2 - y}$ $1 \le y < 2$ or $y \ge 4$ Method 1: Using $y = x^2$ graph  The range of values of x is $x \le -2$ or $-\sqrt{2} < x \le -1$ or $1 \le x < \sqrt{2}$ or $x \ge 2$ Method 2: Using definition of $\lvert x \rvert$ Since $x^2 = \lvert x \rvert^2$ For $1 \le \lvert x \rvert^2 < 2$ $\Rightarrow 1 \le \lvert x \rvert < \sqrt{2} \Rightarrow -\sqrt{2} \le x < -1$ **or** $1 \le x < \sqrt{2}$ For $\lvert x \rvert^2 \ge 4$ $\lvert x \rvert \ge 2 \Rightarrow x \le -2$ **or** $x \ge 2$ Hence, the range of values of x is $x \le -2$ or $-\sqrt{2} < x \le -1$ or $1 \le x < \sqrt{2}$ or $x \ge 2$
4	

	$h=\tan\left(\frac{\pi}{3}+\theta\right)$ $A=\frac{1}{2}(2)\tan\left(\frac{\pi}{3}+\theta\right)=\tan\left(\frac{\pi}{3}+\theta\right)$ $=\frac{\tan\left(\frac{\pi}{3}\right)+\tan\theta}{1-\tan\left(\frac{\pi}{3}\right)\tan\theta}=\frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}$ (shown) $\approx\frac{\sqrt{3}+\theta}{1-\theta\sqrt{3}}$ $=\left(\sqrt{3}+\theta\right)\left(1-\theta\sqrt{3}\right)^{-1}$ $\approx\left(\sqrt{3}+\theta\right)\left(1+\theta\sqrt{3}+3\theta^2\right)$ $=\sqrt{3}+4\theta+\left(4\sqrt{3}\right)\theta^2$
5	$\operatorname{cosec} y=x$ Diff wrt x: $-\operatorname{cosec} y\ \cot y\frac{\mathrm{d}y}{\mathrm{d}x}=1$ $\therefore\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\operatorname{cosec} y\ \cot y}$ Using $\cot^2 y+1\equiv\operatorname{cosec}^2 y$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\operatorname{cosec} y\sqrt{(\operatorname{cosec} y)^2-1}}$ [since $0<y<\frac{\pi}{2}\Rightarrow\tan y>0$ $\Rightarrow\cot y>0$ $\Rightarrow\cot y=\sqrt{(\operatorname{cosec} y)^2-1}$] $=-\frac{1}{x\sqrt{x^2-1}}$ (shown) Since $y=\operatorname{cosec}^{-1}x$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}$

Alternative

$\operatorname{cosec} y = x$

Diff wrt x:

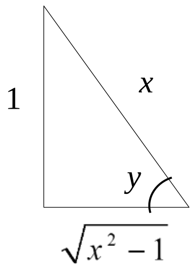
$$-\operatorname{cosec} y \cot y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\operatorname{cosec} y \cot y}$$

Since $\operatorname{cosec} y = x$,

$$\therefore \frac{1}{\sin y} = x$$

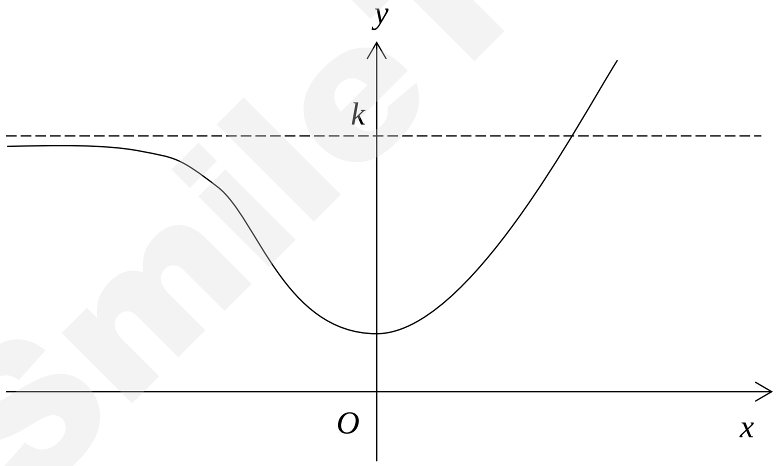
$$\therefore \sin y = \frac{1}{x}$$



By constructing the right angle triangle, $\tan y = \dfrac{1}{\sqrt{x^2-1}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\operatorname{cosec} y \cot y} = -\frac{\tan y}{\operatorname{cosec} y} = -\frac{1}{x\sqrt{x^2-1}} \text{ (shown)}$$

(b)



6 (a) $x^2 + \frac{1}{3}(y-2)^2 = 1$

$\downarrow$ Replace x by $x-2$

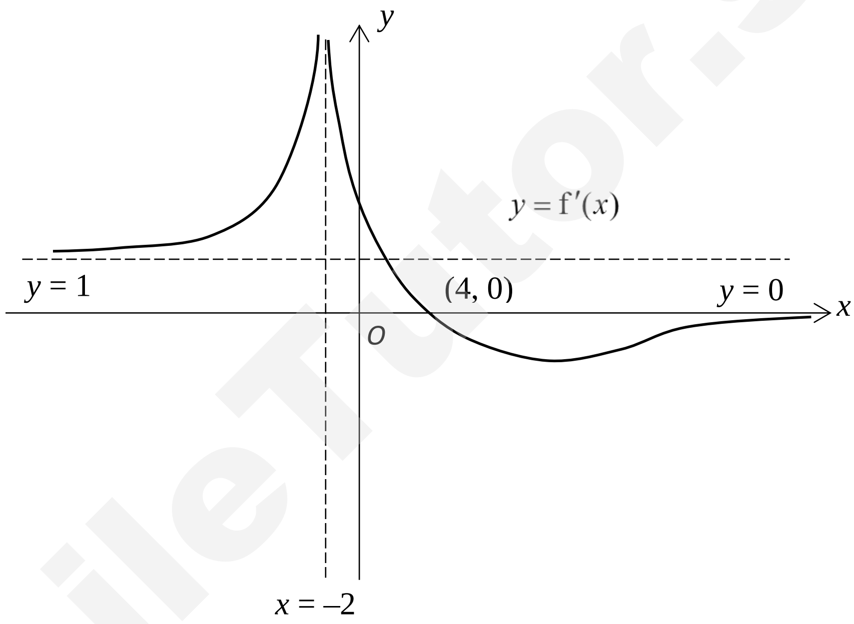
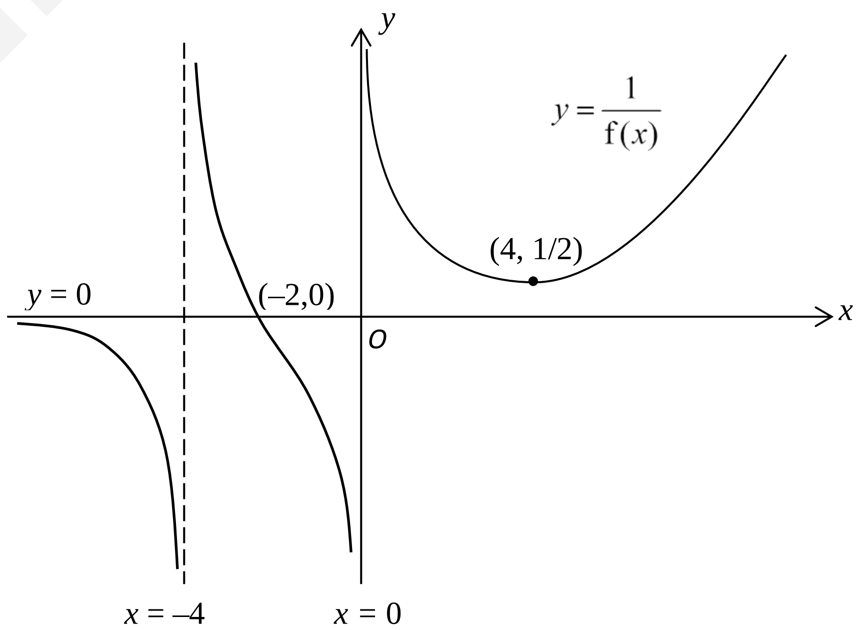
$(x-2)^2 + \frac{1}{3}(y-2)^2 = 1$

$\downarrow$ Replace y by $y+2$

$(x-2)^2 + \frac{1}{3}(y)^2 = 1$

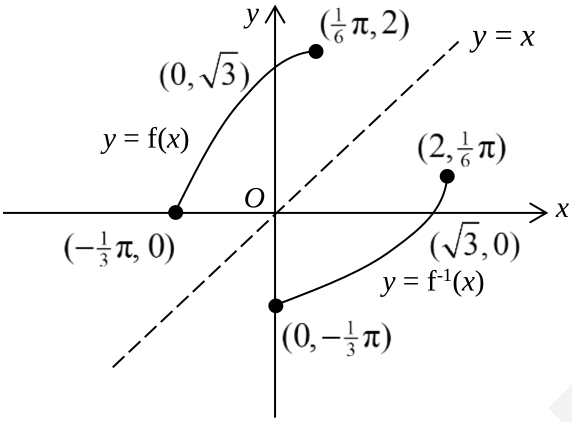
$\downarrow$ Replace y by $\sqrt{3}y$

$(x-2)^2 + y^2 = 1$

	1. Translate 2 units in the positive x-direction 2. Translate 2 units in the negative y-direction 3. Scale by a factor of $\dfrac{1}{\sqrt{3}}$ parallel to the y-direction <u>Alternative:</u> 2. Scale by a factor of $\dfrac{1}{\sqrt{3}}$ parallel to the y-direction 3. Translate $\dfrac{2}{\sqrt{3}}$ units in the negative y-direction (b) (i)  (b) (ii) 

7	(i) Using R formula, $\sin x+\sqrt{3}\cos x=2\sin(x+\frac{1}{3}\pi)$

(ii)



(iii) To find f^{-1}:

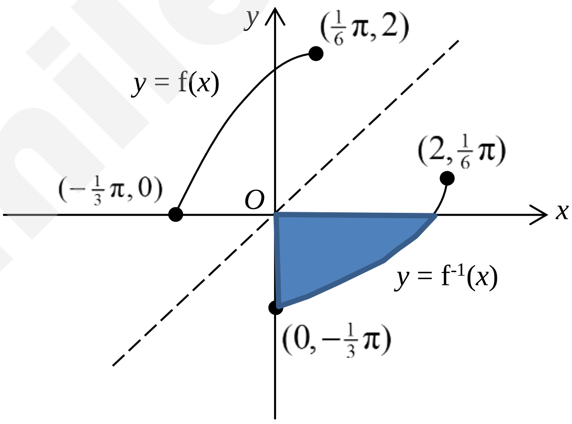
Let $y=2\sin(x+\frac{1}{3}\pi)$

$\therefore x=-\frac{1}{3}\pi+\sin^{-1}(\frac{1}{2}y)$

$f^{-1}(x)=-\frac{1}{3}\pi+\sin^{-1}(\frac{1}{2}x)$

$D_{f^{-1}}=R_f=[0,2]$

(iv) For the area bounded by the graph of f^{-1} and the axes:



By symmetry,

Area

$$=\int_{-\frac{\pi}{3}}^{0} f(x)\ dx \ =\int_{-\frac{\pi}{3}}^{0}(\sin x+\sqrt{3}\cos x)\ dx$$

$$=\left[-\cos x+\sqrt{3}\sin x\right]_{-\frac{\pi}{3}}^{0}=(-1+0)-\left(-\frac{1}{2}-\frac{3}{2}\right)=1$$

(v) gf^{-1} exists if $R_{f^{-1}} \subseteq D_g$.

Since

$R_{f^{-1}} = [-\frac{1}{3}\pi, \frac{1}{6}\pi]$

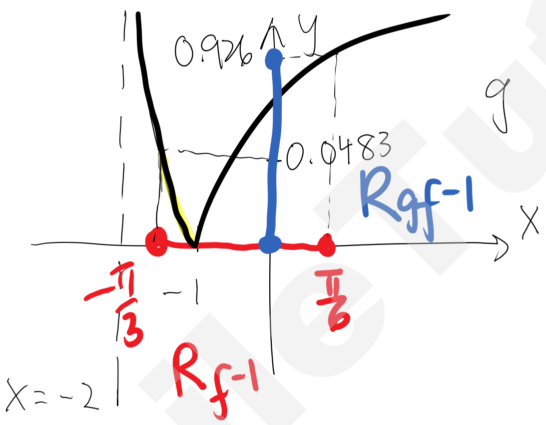
$D_g = (-2, \infty)$,

Ie. $R_{f^{-1}} \subseteq D_g \Rightarrow gf^{-1}$ exists

To find the range of gf^{-1}:

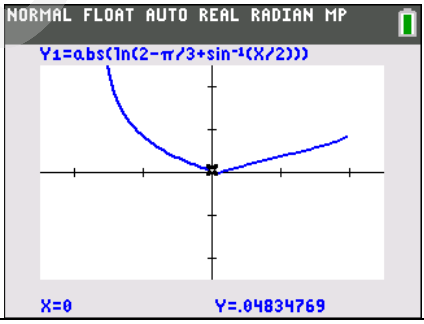
Method 1 (two stage mapping method)

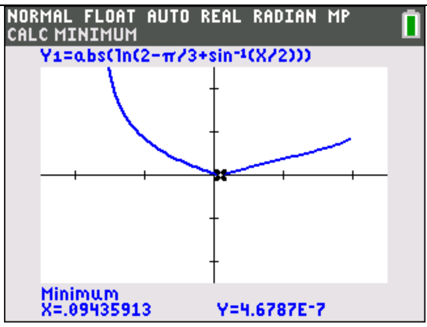
$$\begin{array}{ccccc} D_{f^{-1}} & \xrightarrow{f^{-1}} & R_{f^{-1}} & \xrightarrow{g} & R_{gf^{-1}} \\ [0, 2] & & [-\frac{1}{3}\pi, \frac{1}{6}\pi] & & ? \end{array}$$

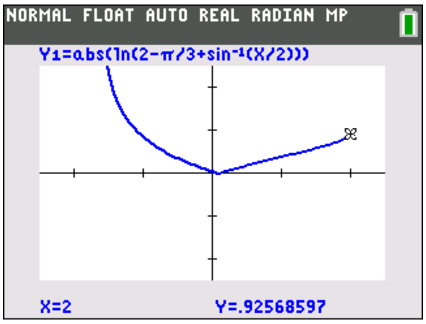


$R_{gf^{-1}} = [0,\ 0.926]$

Method 2 (find gf^{-1}) (need to use GC to see shape)







$\mathrm{gf}^{-1}(x) = \left|\ln(2 - \tfrac{1}{3}\pi + \sin^{-1}(\tfrac{1}{2}x))\right|$

$D_{\mathrm{gf}^{-1}} = D_{\mathrm{f}^{-1}} = [0, 2]$

$R_{\mathrm{gf}^{-1}} = [0, 0.926]$

8 (a) Since the curve shows only one x-intercept, it means that there is only one real root in the equation $\mathrm{f}(x) = 0$.

Since the equation has all real coefficients, then the two other roots must be non-real and they are conjugate pair.

(b) Since $z = 3 + 4\mathrm{i}$ is a root of $2z^2 - (7 + 6\mathrm{i})z + 11 + \mathrm{i}c = 0$,

$2(3 + 4\mathrm{i})^2 - (7 + 6\mathrm{i})(3 + 4\mathrm{i}) + 11 + \mathrm{i}c = 0$

$2(9 + 24\mathrm{i} - 16) - (21 + 28\mathrm{i} + 18\mathrm{i} - 24) + 11 + \mathrm{i}c = 0$

Comparing the Im - part,

$2 + c = 0$

$\therefore c = -2$ (shown)

Since $z = 3 + 4\mathrm{i}$ is a root of $2z^2 - (7 + 6\mathrm{i})z + 11 - 2\mathrm{i} = 0$,

$2z^2 - (7 + 6\mathrm{i})z + 11 - 2\mathrm{i} = \left[z - (3 + 4\mathrm{i})\right](2z - a)$, where $a \in \mathbb{C}$

Comparing the coefficient of constant term,

$11 - 2\mathrm{i} = a(3 + 4\mathrm{i})$

$$a = \frac{11 - 2\mathrm{i}}{3 + 4\mathrm{i}} = \frac{(11 - 2\mathrm{i})(3 - 4\mathrm{i})}{25} = \frac{25 - 50\mathrm{i}}{25} = 1 - 2\mathrm{i}$$

$$2z - (1 - 2\mathrm{i}) = 0 \Rightarrow z = \frac{1}{2} - \mathrm{i}$$

Therefore, the other root is $\dfrac{1}{2} - \mathrm{i}$.

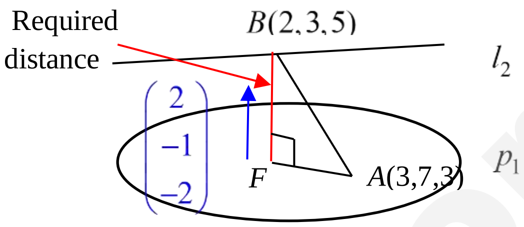
	Replace z by $\mathrm{i}w$ $2(\mathrm{i}w)^2-\left(7+6\mathrm{i}\right)(\mathrm{i}w)+11-2\mathrm{i}=0$ $-2w^2-\left(-6+7\mathrm{i}\right)w+11-2\mathrm{i}=0$ $2w^2+\left(-6+7\mathrm{i}\right)w-11+2\mathrm{i}=0$ $\mathrm{i}w=3+4\mathrm{i} \Rightarrow w=4-3\mathrm{i}$ or $\mathrm{i}w=\frac{1}{2}-\mathrm{i} \Rightarrow w=-1-\frac{1}{2}\mathrm{i}$ $\therefore$ The roots of the equation are $4-3\mathrm{i}$ and $-1-\frac{1}{2}\mathrm{i}$. Alternative Method: $2z^2-(7+6\mathrm{i})z+11-2\mathrm{i}=0$ Let the other root be $a+b\mathrm{i}$. Sum of the roots $=3+4\mathrm{i}+a+b\mathrm{i}=\frac{7+6\mathrm{i}}{2}=\frac{7}{2}+3\mathrm{i}$ Comparing real and imaginary parts: $a+3=\frac{7}{2}\Rightarrow a=\frac{1}{2}$ $4+b=3\Rightarrow b=-1$ The other root is $\frac{1}{2}-\mathrm{i}$ (c) (i) $z=1+\mathrm{e}^{\mathrm{i}\alpha}$ $=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}(\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}+\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}})$ $=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\left[2\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\right)\right]$ $=2\cos\frac{\alpha}{2}\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}$ (shown) Alternative Method: $z=1+\mathrm{e}^{\mathrm{i}\alpha}$ $=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}(\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}+\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}})$ $=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\left(\cos\left(-\frac{\alpha}{2}\right)+\mathrm{i}\sin\left(-\frac{\alpha}{2}\right)+\cos\frac{\alpha}{2}+\mathrm{i}\sin\frac{\alpha}{2}\right)$ $=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\left[\cos\frac{\alpha}{2}-\mathrm{i}\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}+\mathrm{i}\sin\frac{\alpha}{2}\right]$ $=2\cos\frac{\alpha}{2}\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}$ (shown)

	Alternative Method: $z = 1 + e^{i\alpha}$ $= 1 + \cos\alpha + i\sin\alpha$ $= 1 + 2\cos^2\frac{\alpha}{2} - 1 + i\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$ $= 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$ $= 2\cos\frac{\alpha}{2} e^{i\frac{\alpha}{2}}$ (shown) (ii) $\left\lvert\left(\frac{z}{w^3}\right)^*\right\rvert = \left\lvert\left(\frac{z}{w^3}\right)\right\rvert = \frac{\lvert z\rvert}{\lvert w\rvert^3} = \frac{\left\lvert 2\cos\frac{\pi}{6}\right\rvert}{\left(\sqrt{1+3}\right)^3} = \frac{2\left(\frac{\sqrt{3}}{2}\right)}{(2)^3} = \frac{\sqrt{3}}{8}$ $\arg\left(\frac{z}{w^3}\right)^* = -\arg\left(\frac{z}{w^3}\right) = -[\arg(z) - 3\arg(w)]$ $= -\frac{\alpha}{2} + 3\left(-\frac{2\pi}{3}\right) = -\frac{\pi}{6} - 2\pi$ $\therefore \arg\left(\frac{z}{w^3}\right)^* = -\frac{\pi}{6}$
9	(i) A vector equation of l_1 is $\mathbf{r} = \begin{pmatrix}3\\7\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\1\end{pmatrix}, \lambda \in \mathbb{R}$ Let $\mu = x - 2 = \frac{3-y}{2} = \frac{z-5}{2}$. $\therefore x = 2 + \mu, y = 3 - 2\mu, z = 5 + 2\mu$ Then a vector equation of l_2 is $\mathbf{r} = \begin{pmatrix}2\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\2\end{pmatrix}, \lambda \in \mathbb{R}$ A vector perpendicular to p_1 is $\begin{pmatrix}3\\4\\1\end{pmatrix} \times \begin{pmatrix}1\\-2\\2\end{pmatrix} = \begin{pmatrix}10\\-5\\-10\end{pmatrix} = 5\begin{pmatrix}2\\-1\\-2\end{pmatrix} // \begin{pmatrix}2\\-1\\-2\end{pmatrix}$

Eqn of p_1: $\mathbf{r}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}3\\7\\3\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-7$

Cartesian eqn : $2x-y-2z=-7$

(ii) Distance of a line // to a plane is the distance between a point on this line to the plane



Distance of l_2 to p_1 $= BF = \dfrac{\left|\overrightarrow{AB}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right|}=\dfrac{1}{3}\left|\begin{pmatrix}-1\\-4\\2\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right|=\dfrac{2}{3}$

Alternative :

Equation of line BF: $\mathbf{r}=\begin{pmatrix}2\\3\\5\end{pmatrix}+\alpha\begin{pmatrix}2\\-1\\-2\end{pmatrix}$

$\overrightarrow{OF}=\begin{pmatrix}2\\3\\5\end{pmatrix}+\alpha\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ for some $\alpha\in\mathbb{R}$

As F lies in p_1,

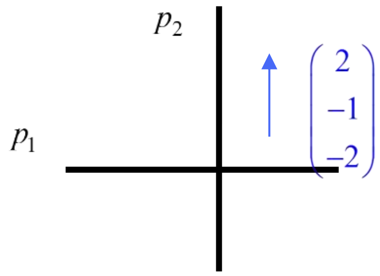
$$\left[\begin{pmatrix}2\\3\\5\end{pmatrix}+\alpha\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-7$$

$$-9+9\alpha=-7$$

$$\therefore\alpha=\frac{2}{9}$$

Distance of l_2 to p_1 $=BF=\left|\overrightarrow{OF}-\overrightarrow{OB}\right| = \dfrac{2}{9}\left|\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right|=\dfrac{2}{9}\times 3=\dfrac{2}{3}$

(iii)



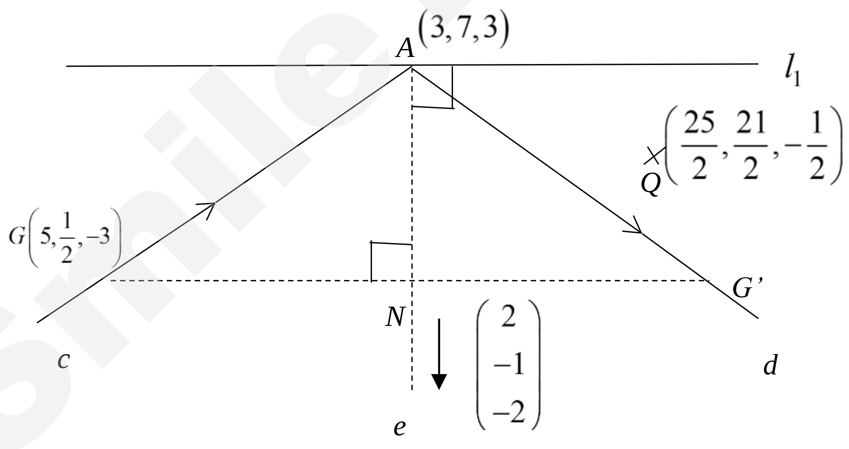
A vector perpendicular to p_2

$$=\begin{pmatrix}3\\4\\1\end{pmatrix}\times\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}-7\\8\\-11\end{pmatrix}$$

Equation of p_2

$$\mathbf{r}.\begin{pmatrix}-7\\8\\-11\end{pmatrix}=\begin{pmatrix}3\\7\\3\end{pmatrix}.\begin{pmatrix}-7\\8\\-11\end{pmatrix}=2$$

(iv)



line d is a reflection of line c in the line e which passes through A, is perpendicular to l_1 and p_1 and lying in p_2.

Eqn of line e : $\mathbf{r}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}$

Foot of perpendicular, N, from $\left(5,\frac{1}{2},-3\right)$ to line e

$$\left[\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}-\begin{pmatrix}5\\0.5\\-3\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=0$$

$$\left[\begin{pmatrix}-2\\6.5\\6\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=0$$

$$-4-6.5-12+\mu(4+1+4)=0$$

$$\mu=\frac{45}{18}=\frac{5}{2}$$

$$\overrightarrow{ON}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}8\\4.5\\-2\end{pmatrix}$$

$$\overrightarrow{ON}=\frac{1}{2}\left(\overrightarrow{OG}+\overrightarrow{OG'}\right)$$

$$\overrightarrow{OG'}=2\begin{pmatrix}8\\4.5\\-2\end{pmatrix}-\begin{pmatrix}5\\0.5\\-3\end{pmatrix}=\begin{pmatrix}11\\8.5\\-1\end{pmatrix}$$

$$\overrightarrow{AG'}=\begin{pmatrix}11\\8.5\\-1\end{pmatrix}-\begin{pmatrix}3\\7\\3\end{pmatrix}=\begin{pmatrix}8\\1.5\\-4\end{pmatrix}//\begin{pmatrix}16\\3\\-8\end{pmatrix}$$

Direction cosines of line d are $\frac{16}{\sqrt{329}},\frac{3}{\sqrt{329}}$ and $-\frac{8}{\sqrt{329}}$

or $-\frac{16}{\sqrt{329}},-\frac{3}{\sqrt{329}}$ and $\frac{8}{\sqrt{329}}$.

[**Alternative** to find $\overrightarrow{ON}$ - intersection of 2 lines]

Eqn of line e : $\mathbf{r}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}$

Eqn of line GN: $\mathbf{r}=\begin{pmatrix}5\\0.5\\-3\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix}$

At N,

	$\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}5\\0.5\\-3\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix}$ $2\mu-3\alpha=2$ $\mu+4\alpha=6.5$ $2\mu+\alpha=6$ Solving, $\mu=2.5,\alpha=1$ $\overrightarrow{ON}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}8\\4.5\\-2\end{pmatrix}$ (v) Shortest distance from Q to line d $=\left\lvert\frac{\overrightarrow{AQ}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}}{\sqrt{329}}\right\rvert$ $=\frac{1}{\sqrt{329}}\left\lvert\left[\begin{pmatrix}\frac{25}{2}\\\frac{21}{2}\\-\frac{1}{2}\end{pmatrix}-\begin{pmatrix}3\\7\\3\end{pmatrix}\right]\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right\rvert$ $=\frac{1}{\sqrt{329}}\left\lvert\begin{pmatrix}\frac{19}{2}\\\frac{7}{2}\\-\frac{7}{2}\end{pmatrix}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right\rvert=\frac{1}{2\sqrt{329}}\left\lvert\begin{pmatrix}19\\7\\-7\end{pmatrix}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right\rvert$ $=\frac{1}{2\sqrt{329}}\left\lvert\begin{pmatrix}-35\\40\\-55\end{pmatrix}\right\rvert=2.11$ (3 s.f.)
10	(i) To determine range of cannonball, we consider $y = 0$: $0=(v\sin\theta)t-5t^2$ $0=t\left[v\sin\theta-5t\right]$ $\therefore t=0$ (rejected) or $v\sin\theta-5t=0$ $\therefore t=\frac{v\sin\theta}{5}$

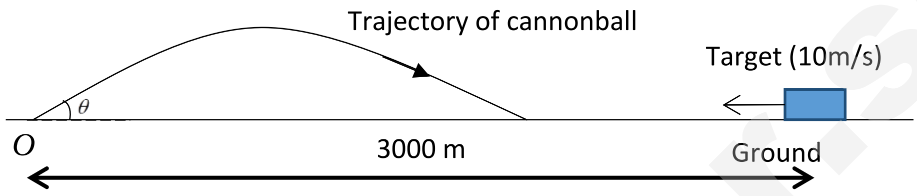
When $t = \dfrac{v\sin\theta}{5}$,

$x = (v\cos\theta)t$

$= (v\cos\theta)\dfrac{v\sin\theta}{5}$

$= \dfrac{v^2\sin\theta\cos\theta}{5} \quad \therefore d = \dfrac{v^2\sin\theta\cos\theta}{5}$

(ii)



Time taken for cannonball to hit the ground = time taken for the target to reach the point of impact of the cannonball.

$\dfrac{v\sin\theta}{5} = \dfrac{3000-d}{10}$

$2v\sin\theta = 3000 - \dfrac{v^2\sin\theta\cos\theta}{5}$

$\dfrac{(200)^2\sin\theta\cos\theta}{5} + 400\sin\theta = 3000$

Possible angles are $22.7°$ (to 1 dp) or $69.5°$ (to 1 dp). (shown)

(iii) Since $t = \dfrac{v\sin\theta}{5}$ when cannon hits target and $\dfrac{v\sin 22.7°}{5} < \dfrac{v\sin 69.5°}{5}$

Therefore to hit target earlier, cannonball should be fired at $22.7°$.

(iv) $x = (200\cos 22.7°)t \qquad y = (200\sin 22.7°)t - 5t^2$

$\dfrac{dx}{dt} = 184.51 \qquad \dfrac{dy}{dt} = 77.181 - 10t$

$\therefore \dfrac{dy}{dx} = \dfrac{77.181-10t}{184.51}$

When $x = 370$, $184.51t = 370 \Rightarrow t = 2.0053$

$\therefore \dfrac{dy}{dx} = \dfrac{77.181-10(2.0053)}{184.51} = 0.30962$

Let the required angle be α.

$\tan\alpha = 0.30962 \Rightarrow \alpha = 17.2°$ (to 1dp)

11	(a)(i) $a+b+c=25000$ ----------(1) $0.02a+0.06b-0.02c=0$ ----------(2) [or $1.02a+1.06b+0.98c=25000$] Solving SLE, $a=37500-2c$ $b=c-12500$ (ii) Since a and b must both be positive, it implies that c must lie between 12500 and 18750. (b)(i) Since Mr Lee invested in a period of five years, the average return per year will be 1.6%. Total amount of interest earned $=(1.016)^5(55000)-55000$ $=4543$ (to the nearest dollar) (ii) Amount of money in the normal savings account at the end of n years $=1000(1.0019+1.0019^2+1.0019^3+...+1.0019^n)$ $=1000(1.0019)\left(\frac{1.0019^n-1}{1.0019-1}\right)$ $=527315.79(1.0019^n-1)$ Amount of money in the special savings account at the end of n years $=10000(1.018)\left(\frac{1.018^n-1}{1.018-1}\right)$ $=565555.56\left(1.018^n-1\right)$ (iii) Total interest earned from dual-savings account $=527315.79(1.0019^n-1)+565555.56\left(1.018^n-1\right)-11000n$ $527315.79(1.0019^n-1)+565555.56\left(1.018^n-1\right)-11000n>4543$ From GC, $n\geq 7$ Least value of n is 7.

1	**(i)** Find $\frac{\mathrm{d}}{\mathrm{d}x}\tan^2 x$. Hence evaluate $\int_0^{\frac{1}{4}\pi} \sec^2 x \tan x \, \mathrm{e}^{\tan^2 x} \, \mathrm{d}x$, leaving your answer in exact form. [3] **(ii)** By expressing $1+72x-32x^3$ as $1+mx(9-4x^2)$ where m is a constant, find $\int \frac{1+72x-32x^3}{\sqrt{(9-4x^2)}} \, \mathrm{d}x$. [2]
2	The curve C with equation $y=\frac{x^2+(a-1)x-a-1}{x-1}$, where a is a constant, has the oblique asymptote $y=x+1$. **(i)** Show that a = 1. Hence sketch C, giving the equations of any asymptotes and the exact coordinates of any points of intersection with the axes. [3] **(ii)** The region bounded by C for $x>1$ and the lines $y=x+1$, $y=2$ and $y=4$ is rotated through 2π radians about the line $x=1$. By considering a translation of C, or otherwise, find the volume of revolution formed. [5]
3	The variables y and x satisfy the differential equation $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-\ln x}{x\ln x+2x^2}.$$ **(i)** Show that the substitution $u=\frac{\ln x}{x}$ reduces the differential equation to $\frac{\mathrm{d}u}{\mathrm{d}y}=u+2$. Given that $y=0$ when $x=1$, show that $y=\ln\left(\frac{\ln x}{2x}+1\right)$. [6] The curve C has equation $y=\ln\left(\frac{\ln x}{2x}+1\right)$. It is given that C has a maximum point and two asymptotes $y=a$ and $x=b$. **(ii)** Find the exact coordinates of the maximum point. [2] **(iii)** Explain why $a=0$. [You may assume that as $x\to\infty$, $\frac{\ln x}{x}\to 0$.] [1] **(iv)** Determine the value of b, giving your answer correct to 4 decimal places. [2] **(v)** Sketch C. [2]

4 Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel. The point C lies on OB produced such that $3OC = 5OB$. It is given that $|\mathbf{a}| = 2|\mathbf{b}|$ and $\cos\angle AOB = -\frac{1}{4}$.

(a) **(i)** Show that a vector equation of the line AC is $\mathbf{r} = \mathbf{a} + \lambda(3\mathbf{a} - 5\mathbf{b})$, where λ is a real parameter. [2]

The line l lies in the plane containing O, A and B.

(ii) Explain why the direction vector of l can be expressed as $s\mathbf{a} + t\mathbf{b}$, where s and t are real numbers. [1]

Given that l is perpendicular to AB, show that $t = 3s$. [4]

Given further that l passes through B, write down a vector equation of l, in a similar form as part **(i)**. [1]

(iii) Find the position vector of the point of intersection of line AC and l, in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(b) Explain why, for any constant k, $|(\mathbf{a} + k\,\mathbf{b}) \times \mathbf{b}|$ gives the area of the parallelogram with sides OA and OB. Find the area of the parallelogram, leaving your answer in terms of $|\mathbf{a}|$. [4]

5 A new game has been designed for a particular casino using two fair die. In each round of the game, a player places a bet of \$2 before proceeding to roll the two die. The player's score is the sum of the results from both die. For the scores in the following table, the player keeps his bet and receives a payout as indicated.

Score	Payout
9 or 10	\$1
2 or 4	\$5
11	\$8

For any other scores, the player loses his bet.

Let X be the random variable denoting the winnings of the casino from each round of the game.

(i) Show that $\mathrm{E}(X) = \dfrac{1}{12}$ and find $\mathrm{Var}(X)$. [4]

(ii) $\bar{X}$ is the mean winnings of the casino from n rounds of this game. Find $\mathrm{P}(\bar{X} > 0)$ when $n = 30$ and $n = 50\,000$. Make a comparison of these probabilities and comment in context of the question. [3]

6		The students in a college are separated into two groups of comparable sizes, Group X and Group Y. The marks for their Mathematics examination are normally distributed with means and variances as shown in the following table.

	Mean	Variance
Group X	55	20
Group Y	34	25

(i) Explain why it may not be appropriate for the mark of a randomly chosen student from the college population to be modelled by a normal distribution. [1]

(ii) In order to pass the examination, students from Group Y must obtain at least d marks. Find, correct to 1 decimal place, the maximum value of d if at least 60% of them pass. [3]

(iii) Find the probability that the total marks of 4 students from Group Y is less than three times the mark of a student from Group X. State clearly the mean and variance of the distribution you use in your calculation. [3]

(iv) The marks of 40 students, with 20 each randomly selected from Group X and Group Y, are used to compute a new mean mark , $\bar{M}$. Given that $\mathrm{P}(|\bar{M}-44.5|<k)=0.9545$, find the value of k. [4]

State a necessary assumption for your calculations to hold in parts **(iii)** and **(iv)**. [1]

7 The company Snatch provides a ride-hailing service comprising taxis and private cars in Singapore. Snatch claims that the mean waiting time for a passenger from the booking time to the time of the vehicle's arrival is 7 minutes.

To test whether the claim is true, a random sample of 30 passengers' waiting times is obtained. The standard deviation of the sample is 2 minutes. A hypothesis test conducted concludes that there is sufficient evidence at the 1% significance level to reject the claim.

(i) State appropriate hypotheses and the distribution of the test statistic used. [3]

(ii) Find the range of values of the sample mean waiting time, $\bar{t}$. [3]

(iii) A hypothesis test is conducted at the 1% significance level whether the mean waiting time of passengers is more than 7 minutes. Using the existing sample, deduce the conclusion of this test if the sample mean waiting time is more than 7 minutes. [2]

8 A retail manager of a large electrical appliances store wants to investigate the relationship between the monthly advertising expenditure, x hundred dollars, and the monthly sales of their refrigerators, y thousand dollars. The table below shows the results of the investigation.

x	5	8	12	16	18	20	23
y	12.5	12.9	13.6	14.8	17.0	19.3	25.1

(i) The manager concludes that an increase in monthly advertising expenditure will result in an increase in the monthly sales of refrigerators. State, with a reason, whether you agree with his conclusion. [1]

(ii) Draw a scatter diagram to illustrate the above data. Explain why a linear model is not likely to be appropriate. [2]

It is thought that the monthly sales y thousand dollars can be modelled by one of the formulae

$$y = a + b\mathrm{e}^{\sqrt{x}} \quad \text{or} \quad y = a + bx^2$$

where a and b are constants.

(iii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(A) $\mathrm{e}^{\sqrt{x}}$ and y,

(B) x^2 and y.

Explain which of $y = a + b\mathrm{e}^{\sqrt{x}}$ or $y = a + bx^2$ is the better model. [2]

Assume that the better model in part **(iii)** holds for part **(iv)**.

(iv) The manager forgot to record the monthly advertising expenditure when the monthly sales of refrigerators was $11300. Combining this with the above data set, it is found that a = 10.876 and b = 0.09906 for the model. Find the monthly advertising expenditure that the manager forgot to record, leaving your answer to the nearest hundred. [3]

9 A sample of 5 people is chosen from a village of large population.

(i) The number of people in the sample who are underweight is denoted by X. State, in context, the assumption required for X to be well modelled by a binomial distribution. [1]

(ii) On average, the proportion of people in the village who are underweight is p. It is known that the mode of X is 2. Use this information to show that $\frac{1}{3} < p < \frac{1}{2}$. [3]

1000 samples of 5 people are chosen at random from the village and the results are shown in the table below.

x	0	1	2	3	4	5
Number of groups	93	252	349	220	75	11

(iii) Using the above results, find $\bar{x}$. Hence estimate the value of p. [2]

You may now use your estimate in part **(iii)** as the value of p.

(iv) Two random samples of 5 people are chosen. Find the probability that the first sample has at least 4 people who are underweight and has more people who are underweight than the second sample. [3]

10 **(a)** The word DISTRIBUTION has 12 letters.

(i) Find the number of different arrangements of the 12 letters that can be made. [1]

(ii) Find the number of different arrangements which can be made if there are exactly 8 letters between the two Ts. [3]

One of the Is is removed from the word and the remaining letters are arranged randomly.

(iii) Find the probability that no adjacent letters are the same. [4]

(b) The insurance company Adiva classifies 10% of their car policy holders as 'low risk', 60% as 'average risk' and 30% as 'high risk'. Its statistical database has shown that of those classified as 'low risk', 'average risk' and 'high risk', 1%, 15% and 25% are involved in at least one accident respectively.

Find the probability that

(i) a randomly chosen policy holder is not involved in any accident if the holder is classified as 'average risk', [1]

(ii) a randomly chosen policy holder is not involved in any accident, [2]

(iii) a randomly chosen policy holder is classified as 'low risk' if the holder is involved in at least one accident. [2]

It is known that the cost of repairing a car when it meets with an accident has the following probability distribution.

Cost incurred (in thousand dollars)	5	10	50	100
Probability	0.75	0.15	0.08	0.02

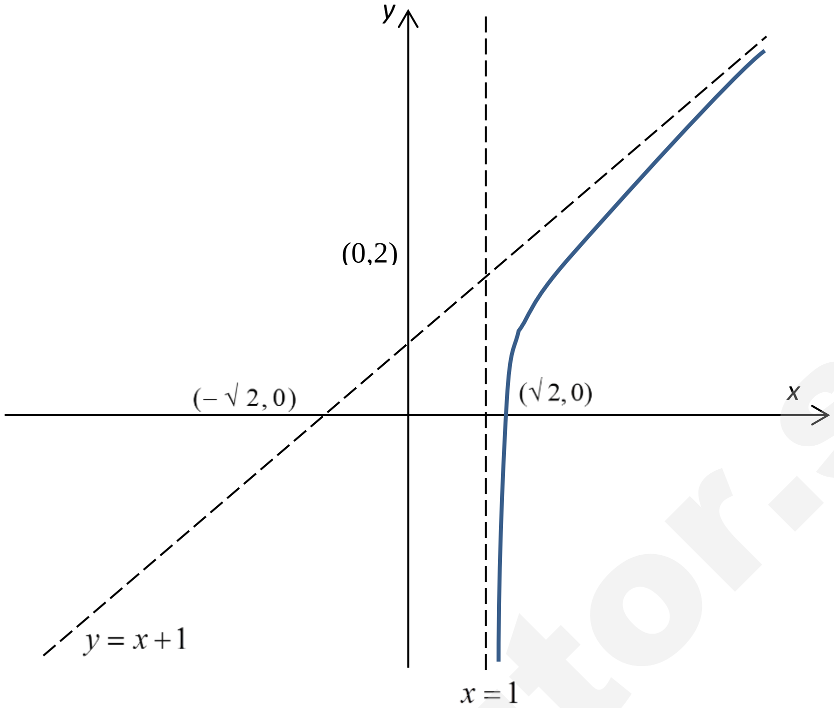
It is known that a 'low risk' policy holder will not be involved in more than one accident in a year. You may assume that there will be no cost incurred by the company in insuring a holder whose car is not involved in any accident.

(iv) Construct the probability distribution table of the cost incurred by Adiva in insuring a 'low risk' policy holder assuming that the cost of repairing a car is independent of a 'low risk' policy holder meeting an accident. [1]

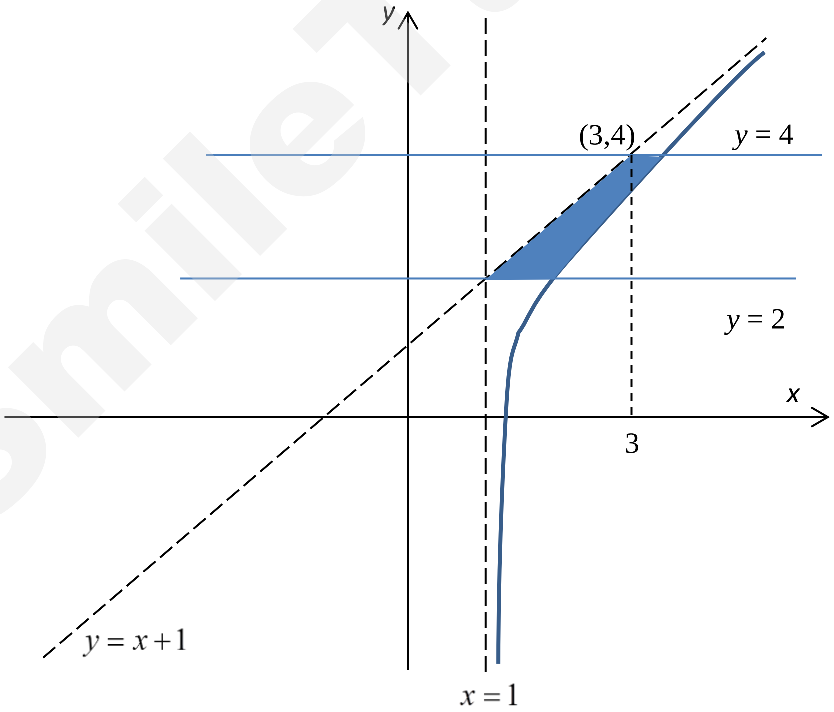
(v) In order to have an expected profit of \$200 from each policy holder, find the amount that Adiva should charge a 'low risk' policy holder when he renews his annual policy. [2]

H2 Mathematics 2017 Preliminary Exam Paper 2 Solution

1	(i) $\int_0^{\frac{\pi}{4}} \sec^2 x \tan x \, \mathrm{e}^{\tan^2 x} \, \mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sec^2 x \tan x \, \mathrm{e}^{\tan^2 x} \, \mathrm{d}x$ $= \frac{1}{2}\left[\mathrm{e}^{\tan^2 x}\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{2}\left(\mathrm{e}^{\tan^2 \frac{\pi}{4}} - \mathrm{e}^{\tan^2 0}\right)$ $= \frac{1}{2}(\mathrm{e}-1)$ (ii) $\int \frac{1+72x-32x^3}{\sqrt{9-4x^2}} \, \mathrm{d}x = \int \frac{1+8x\left(9-4x^2\right)}{\sqrt{9-4x^2}} \, \mathrm{d}x$ $= \int \frac{1}{\sqrt{9-4x^2}} + 8x\left(9-4x^2\right)^{\frac{1}{2}} \, \mathrm{d}x$ $= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) - \frac{2}{3}\left(9-4x^2\right)^{\frac{3}{2}} + C$
2	(i) $y = \frac{x^2+(a-1)x-a-1}{x-1}$ $= \frac{(x+a)(x-1)-1}{x-1}$ $= (x+a) - \frac{1}{x-1}$ Given that oblique asymptote is $y = x+1$, $\therefore a=1$ (shown) <u>Alternative</u> Let $\frac{x^2+(a-1)x-a-1}{x-1} = (x+1) + \frac{b}{x-1}$ $\Rightarrow x^2+(a-1)x-a-1 = x^2-1+b$ Comparing coeff of x: $a-1=0$ $\therefore a=1$ (shown) and $b=-1$ $\therefore y = (x+1) - \frac{1}{x-1} = \frac{x^2-2}{x-1}$ HA: $x=1$ OA: $y=x+1$(given)



(ii)



$$y=\frac{x^2-2}{x-1}\xrightarrow{\text{replace } x \text{ with } (x+1)} y=\frac{(x+1)^2-2}{x}$$

	$y=\dfrac{(x+1)^2-2}{x}$ $xy=x^2+2x-1$ $x^2+(2-y)x-1=0$ $x=\dfrac{-(2-y)\pm\sqrt{(2-y)^2+4(1)(1)}}{2}$ $\therefore x=\dfrac{(y-2)+\sqrt{y^2-4y+8}}{2}$ **(reject -ve root)** Volume = $\pi\int_2^4\left(\dfrac{(y-2)+\sqrt{y^2-4y+8}}{2}\right)^2 \mathrm{d}y-\dfrac{1}{3}\pi(2)^2(2)$ = 9.75 units^3 (3 s.f)
3	(i) $u=\dfrac{\ln x}{x}\Rightarrow\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1-\ln x}{x^2}$ $\dfrac{\mathrm{d}u}{\mathrm{d}y}=\dfrac{\mathrm{d}u}{\mathrm{d}x}\times\dfrac{\mathrm{d}x}{\mathrm{d}y}=\dfrac{1-\ln x}{x^2}\times\dfrac{x\ln x+2x^2}{1-\ln x}=\dfrac{\ln x+2x}{x}$ $\dfrac{\mathrm{d}u}{\mathrm{d}y}=u+2$ (shown) $\dfrac{1}{u+2}\dfrac{\mathrm{d}u}{\mathrm{d}y}=1\Rightarrow\ln\|u+2\|=y+c$, c is an arbitrary constant $\|u+2\|=\mathrm{e}^{y+c}=\mathrm{e}^c\mathrm{e}^y$ $u+2=A\mathrm{e}^y$, A is an arbitrary constant $\dfrac{\ln x}{x}+2=A\mathrm{e}^y$ $y=0, x=1$: $A=2$ $\dfrac{\ln x}{2x}+1=\mathrm{e}^y$ $y=\ln\left(\dfrac{\ln x}{2x}+1\right)$ (shown) <u>Alternative</u> $\dfrac{1}{u+2}\dfrac{\mathrm{d}u}{\mathrm{d}y}=1\Rightarrow\ln\|u+2\|=y+c$, c is an arbitrary constant With the boundary condition $u=0, y=0,$ we see that $u+2>0$ Thus $\ln\|u+2\|=y+c$ and $c=\ln 2$ (ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1-\ln x}{x\ln x+2x^2}$

When $\frac{dy}{dx}=0$, $1-\ln x=0 \quad \Rightarrow x=e$, $y=\ln\left(\frac{1}{2e}+1\right)$

Therefore the maximum point is $\left(e, \ln\left(\frac{1}{2e}+1\right)\right)$.

(iii) $y=\ln\left(\frac{\ln x}{2x}+1\right)$

When $x\to\infty$, $\frac{\ln x}{2x}\to 0$.

$y=\ln\left(\frac{\ln x}{2x}+1\right)\to \ln 1=0$.

Thus $a=0$ (shown)

(iv) For $y\to-\infty$, $\frac{\ln x}{2x}+1\to 0$

$$\ln x+2x\to 0$$
$$x\to 0.4263$$

$\therefore b=0.4263$

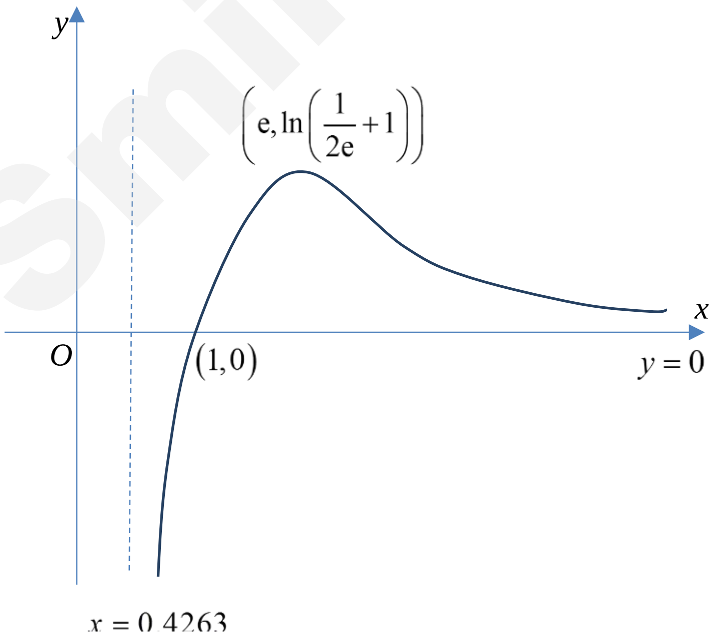
<u>Alternative</u>

When y is undefined, $\frac{dy}{dx}=\frac{1-\ln x}{x\ln x+2x^2}$ is undefined.

Thus $x\ln x+2x^2=0$.

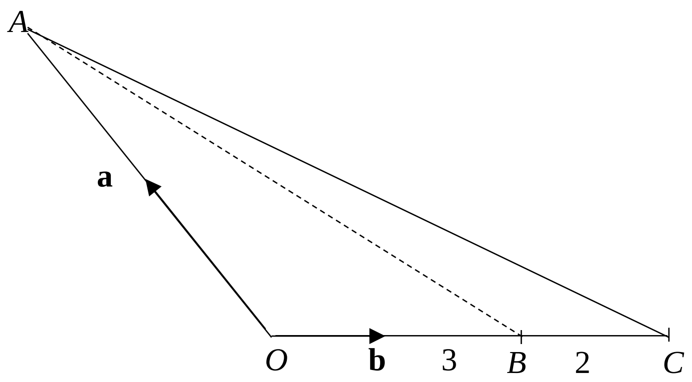
Since $x>0$ for $\ln x$ to be defined, $\ln x+2x=0$.

(v)



4

(a)



(i) $\overrightarrow{OC}=\frac{5}{3}\mathbf{b}\quad \therefore \overrightarrow{AC}=\frac{5}{3}\mathbf{b}-\mathbf{a}=\frac{1}{3}(5\mathbf{b}-3\mathbf{a})\,//\,5\mathbf{b}-3\mathbf{a}$

Equation of line AC: $\mathbf{r}=\mathbf{a}+\lambda(3\mathbf{a}-5\mathbf{b}),\quad \lambda\in\mathbb{R}$

(ii) Since l lies on the plane containing O, A and B, its direction vector is coplanar with a and b, thus it will be a linear combination of a and b, i.e. $s\mathbf{a}+t\mathbf{b}$ is a direction vector for l.

(iii) Let intersection point be D.

At D,

$\mathbf{a}+\lambda(3\mathbf{a}-5\mathbf{b})=\mathbf{b}+\mu(\mathbf{a}+3\mathbf{b})$

Since a and b are non-zero, non-parallel vectors,

$1+3\lambda=\mu-----(1)$

$-5\lambda=1+3\mu----(2)$

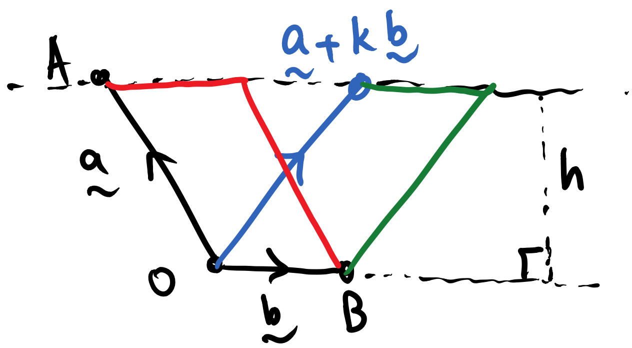
Solving,

$-5\lambda=1+3(1+3\lambda)$

$14\lambda=-4$

$\therefore \lambda=-\frac{2}{7},\mu=\frac{1}{7}\qquad\qquad \therefore \overrightarrow{OD}=\frac{1}{7}\mathbf{a}+\frac{10}{7}\mathbf{b}$

(b)



<table>
<tr><td></td><td>
<u>Method 1</u>
Since the base length (OB) and perpendicular height remain the same, the area of parallelograms formed by different k remains the same as the area of the parallelogram with sides OA and OB.

<u>Method 2</u>
$\left|(\mathbf{a}+k\mathbf{b})\times\mathbf{b}\right| = |\mathbf{a}\times\mathbf{b}+k\mathbf{b}\times\mathbf{b}| = |\mathbf{a}\times\mathbf{b}+\mathbf{0}| = |\mathbf{a}\times\mathbf{b}|$

Area of parallelogram
$=\left|\mathbf{a}\times\mathbf{b}\right| = |\mathbf{a}||\mathbf{b}|\sin\theta$
$=\left|\mathbf{a}\right|\left(\frac{1}{2}\left|\mathbf{a}\right|\right)\sqrt{1-\left(-\frac{1}{4}\right)^2}$
$=\frac{\sqrt{15}}{8}\left|\mathbf{a}\right|^2$
</td></tr>
<tr><td>5</td><td>
(i)
<table>
<tr><td>x</td><td>- 8</td><td>- 5</td><td>- 1</td><td>2</td></tr>
<tr><td>P(X = x)</td><td>$\frac{2}{36}=\frac{1}{18}$</td><td>$\frac{4}{36}=\frac{2}{18}$</td><td>$\frac{7}{36}$</td><td>$\frac{23}{36}$</td></tr>
</table>

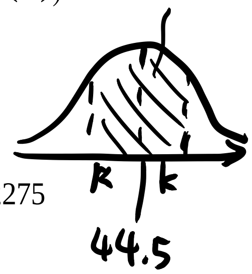
$\mathrm{E}(X)=\frac{46}{36}-\frac{7}{36}-\frac{20}{36}-\frac{16}{36}=\frac{1}{12}$

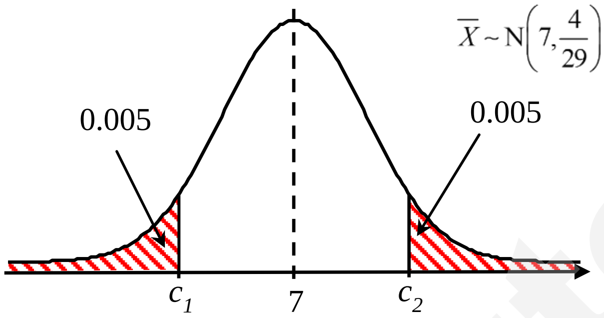
$\mathrm{Var}(X)=\mathrm{E}(X^2)-\left(\mathrm{E}(X)\right)^2=\frac{92}{36}+\frac{7}{36}+\frac{50}{18}+\frac{64}{18}-\frac{1}{12^2}$
$=\frac{1307}{144}$ or 9.08 (to 3sf)

(ii) Since n is large, $\overline{X}\sim \mathrm{N}\left(\frac{1}{12},\frac{1307}{144n}\right)$ approximately by Central Limit Theorem.

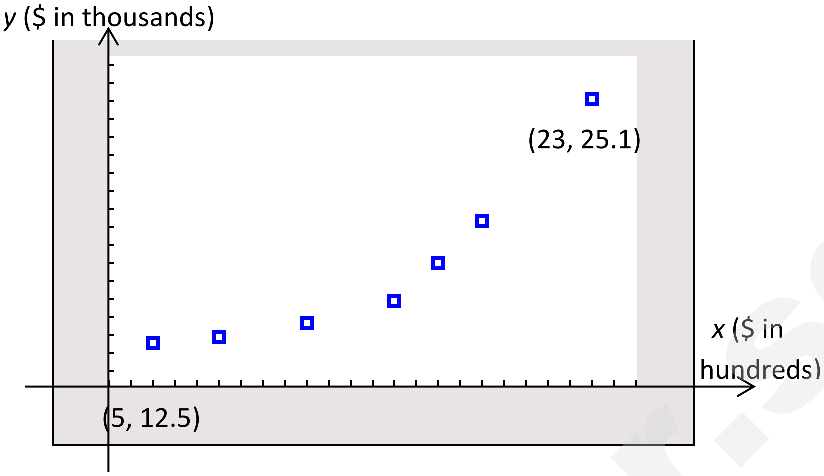
For n = 30, $\mathrm{P}\left(\overline{X}>0\right)=0.560$ (to 3sf)
For n = 50000, $\mathrm{P}\left(\overline{X}>0\right)=1.00$ (to 3sf)

The more rounds this game is played, the higher the chance of casino receiving a positive average winnings. In other words, it is almost certain that casino will win in the long run.
</td></tr>
<tr><td>6</td><td>
(i) The distribution may become <u>bimodal</u> when the data for both groups are combined
(ii) Let Y be the score of a random student from Group Y. $Y\sim \mathrm{N}(34,\ 25)$
$\mathrm{P}(Y\ge d)\ge 0.6$
$\mathrm{P}(Y\le d)\le 0.4$
</td></tr>
</table>

	When $P(Y \le d_c) = 0.4,\ d_c = 32.733$. Thus $d < 32.733$. The maximum mark is 32.7 (iii) $E\left(\sum_1^4 Y_i - 3X\right) = 4E(Y) - 3E(X) = -29$ $\text{Var}\left(\sum_1^4 Y_i - 3X\right) = 4\text{Var}(Y) + 9\text{Var}(X) = 280$ $\therefore \sum_1^4 Y_i - 3X \sim N(-29, 280)$ $P(\sum_1^4 Y_i < 3X) = P(\sum_1^4 Y_i - 3X < 0) = 0.958$ (to 3sf) (iv) $\overline{M} = \dfrac{\sum_{i=1}^{20} X_i + \sum_{i=1}^{20} Y_i}{40}$ $E(\overline{M}) = \dfrac{20E(X) + 20E(Y)}{40} = \dfrac{1}{2}(E(X) + E(Y)) = 44.5$ Let $\sigma^2 = \text{Var}(\overline{M})$ $= \dfrac{1}{1600}(20\text{Var}(X) + 20\text{Var}(Y))$ $= \dfrac{1}{80}(\text{Var}(X) + \text{Var}(Y)) = 0.5625$ $\overline{M} \sim N(44.5, 0.5625)$ Since $P(\lvert\overline{M} - 44.5\rvert < k) = 0.9545$ $\Rightarrow P(\overline{M} < 44.5 - k) = \dfrac{1 - 0.9545}{2} = 0.02275$ $\therefore 44.5 - k = 43.000$ $\Rightarrow k = 1.50$ (3 s.f)  <u>Alternative</u> $\overline{M} \sim N(44.5, \sigma^2)$ Since $P(\lvert\overline{M} - 44.5\rvert < 2\sigma) = 0.9545$ $\therefore k = 2\sigma = 2\sqrt{0.5625} = 1.50$ (3sf) *<u>Marks of students</u>* are independent of one another.
7	(i) Let μ be the mean of X. $H_0: \mu = 7$ $H_1: \mu \ne 7$

	$s^2 = \frac{30}{29}(\text{sample variance}) = \frac{30}{29}(4) = \frac{120}{29}$ Under H_0, since the sample size is large, the test statistic is $\overline{T} \sim N\left(7, \frac{4}{29}\right)$ approximately by Central Limit Theorem. (ii) Since the claim is rejected i.e. to reject H_0 at 1% significance level.  From GC, $c_1 = 6.04$ and $c_2 = 7.96$. $\bar{t} \leq 6.04$ or $\bar{t} \geq 7.96$ (iii) From the two tail test, we know that $\text{p-value (two tail)} \leq 0.01$. For a one-tail test, $\text{p-value(one tail)} = \frac{\text{p-value (two tail)}}{2} \leq 0.005 < 0.01$, therefore we reject H_0 and conclude that there is sufficient evidence at 1% significance level to say that mean waiting time is more than 7 minutes. Alternatively, From the two tail test and $\bar{t} > 7$, $P(\overline{T} > \bar{t}) < 0.005$. Thus, $P(\overline{T} > \bar{t}) < 0.005 < 0.01$. p-value for one-tail test = $P(\overline{T} > \bar{t}) < 0.01$. Therefore we reject H_0 and conclude that there is sufficient evidence at 1% significance level to say that mean waiting time is more than 7 minutes.
8	**(i)** No, because correlation does not imply causation / The increase in the monthly sales of refrigerators could be due to other factors such as a rise in the income level.

(ii)



There appears to be a curvilinear/non-linear relationship between x and y, thus a linear model is not likely to be appropriate.

(iii)

(A) r = 0.9684 (to 4dp)

(B) r = 0.9495 (to 4dp)

Since the r value between $e^{\sqrt{x}}$ and y has an absolute value closer to 1, $y = a + b e^{\sqrt{x}}$ is the better model.

(iv) New regression line (8 data points) for y on $e^{(\sqrt{x})}$ is $y = 10.876 + 0.09906e^{(\sqrt{x})}$

$\bar{y} = \dfrac{11.3+12.5+12.9+13.6+14.8+17+19.3+25.1}{8} = 15.813$

Since $\overline{e^{(\sqrt{x})}}$ and $\bar{y}$ lie on the new regression line y on $e^{(\sqrt{x})}$,

and letting $x = m$ when $y = 11.3$,

$$15.813 = 10.876 + 0.09906\left(\frac{\sum_{i=1}^{7} e^{(\sqrt{x_i})} + e^{(\sqrt{m})}}{8}\right)$$

Using GC (1-var stats), $\sum_{i=1}^{7} e^{(\sqrt{x_i})} = 390.96$

$$\therefore 15.813 = 10.876 + 0.09906\left(\frac{390.96 + e^{(\sqrt{m})}}{8}\right)$$

$\therefore e^{(\sqrt{m})} = 7.7479 \quad \Rightarrow \quad m = 4.19 \approx 4$

Monthly advertising expenditure = $400 (nearest hundred)

9

(i) Assume that the:

- weights of the 5 people chosen are independent of each other
- sample is chosen randomly.

(ii) $\mathrm{P}(X=1) < \mathrm{P}(X=2)$

and $\mathrm{P}(X=2) > \mathrm{P}(X=3)$

${}^5C_1 p(1-p)^4 < {}^5C_2 p^2(1-p)^3$

and ${}^5C_2 p^2(1-p)^3 > {}^5C_3 p^3(1-p)^2$

Since $(1-p) > 0$ and $p > 0$,

$1-p < 2p$ and $1-p > p$

$p > \frac{1}{3}$ and $p < \frac{1}{2}$

$\therefore \frac{1}{3} < p < \frac{1}{2}$ (shown)

(iii) $\bar{x} = 1.965$ (from GC)

Since $n = 5$, $np \approx 1.965 \Rightarrow p \approx 0.393$

(iv) $X \sim \mathrm{B}(5, 0.393)$

$\mathrm{P}((X_1 \geq 4) \cap (X_1 > X_2))$

$= \mathrm{P}(X_1 = 4)\mathrm{P}(X_2 \leq 3) + \mathrm{P}(X_1 = 5)\mathrm{P}(X_2 \leq 4)$

$= 0.0724(0.91823) + (0.00937)(0.99063)$

$= 0.0758$ (3 sf)

10

(i) Number of ways $= \dfrac{12!}{3!2!} = 39916800$

(ii) Method 1

I S D R I B U I O N

I S D R I B U I O N

I S D R I B U I O N

There are 3 ways to slot in the 2 T's

Total number of ways

= total number of ways to arrange the remaining ten letters $\times 3 = \dfrac{10!}{3!} \times 3 = 1814400$

Method 2

| T _ _ _ _ _ _ _ _ _ T | _ | _ |

Case 1: One I included between the two T's

Number of ways = ${}^{7}C_{6}\times 8!\times\frac{3!}{2!}=120960$

Case 2: Two I's included between the two T's

Number of ways = ${}^{7}C_{6}\times\frac{8!}{2!}\times 3!=846720$

Case 3: Three I's included between the two T's

Number of ways = ${}^{7}C_{5}\times\frac{8!}{3!}\times 3!=846720$

Total number of ways $=120960+2(846720)=1814400$

(iii) Method 1

Case 1: Both I together but both T separated

II D S R B U O N

7 single letters (excluding Is and Ts) and 1 block of 2Is.

Number of ways = $8!\times {}^{9}C_{2}=1451520$

Case 2: Both T together but I separated

Number of ways = $8!\times {}^{9}C_{2}=1451520$ (same approach as case 1)

Case 3: Both I together and both T together

II TT D S R B U O N

7 single letters (excluding Is and Ts),1 block of Is and 1 block of Ts.

Number of ways = $9!=362880$

Total number of ways in complement

= $(1451520\times 2)+362880$ = 3265920

Method 2

Number of ways in which both T are together = $\frac{10!}{2!}$

Number of ways in which both I are together = $\frac{10!}{2!}$

Number of ways in which both pairs of identical letters are together = 9!

Total number of ways in complement $=2\times\frac{10!}{2!}-9!=3265920$

Required probability = $1-\dfrac{3265920}{\dfrac{11!}{2!2!}}=0.673$

(b)(i) P(holder is not involved in any accident | the holder is classified as 'average risk')
$=100\%-15\%=85\%=0.85$

(ii) Probability of a randomly chosen policy holder not involved
in any car accident
= (0.1)(0.99) + (0.6)(0.85) + (0.3)(0.75)
= 0.834 or $\dfrac{417}{500}$

(iii) P(policy holder is 'low risk' | has met at least one car accident)

$=\dfrac{\text{P(holder is classified as 'low risk' and met with at least 1 accident)}}{\text{P(holder meets with at least 1 accident)}}$

$=\dfrac{0.1(0.01)}{1-0.834}$

$=0.00602$ (to 3sf) or $\dfrac{1}{166}$

(iv) Let C be the cost of insuring a randomly chosen 'low risk' policy holder (in thousands).

c	0	5	10	50	100
P(C = c)	0.99	(0.01)(0.75) = 0.0075	(0.01)(0.15) = 0.0015	(0.01)(0.08) = 0.0008	(0.01)(0.02) = 0.0002

(v) E(C) = 100(0.0002)+50(0.0008)+10(0.0015)+5(0.0075)= 0.1125

Note: "Profit = Premium Charged – Cost Incurred for Repair"

1000(0.1125) + 200 = 312.5
The company should charge $312.50 for a car insurance plan for 'low risk'.

Alternative (Using "Profit = Premium – Cost Incurred")

Let P be the premium charged by Adiva for a 'low risk' holder.

$$P(0.99)+(P-5000)(0.0075)+(P-10000)(0.0015)$$
$$+(P-50000)(0.0008)+(P-100000)(0.0002)=200$$

Solving, $P=312.5$
The company should charge $312.50 for a car insurance plan for 'low risk'.

HCI Paper 1

1 The *floor function*, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x. For example, $\lfloor -2.1 \rfloor = -3$ and $\lfloor 3.5 \rfloor = 3$.

The function f is defined by

$$\mathrm{f}(x) = \begin{cases} \lfloor x \rfloor & \text{for } x \in \mathbb{R}, \quad -1 \le x < 2, \\ 0 & \text{for } x \in \mathbb{R}, \quad 2 \le x < 3, \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

It is given that $\mathrm{f}(x) = \mathrm{f}(x+4)$.

(i) Find the values of $\mathrm{f}(-1.2)$ and $\mathrm{f}(3.6)$. [2]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-2 \le x < 4$. [2]

(iii) Hence evaluate $\int_{-2}^{4} \mathrm{f}(x)\,\mathrm{d}x$. [1]

2 By writing $\sec^3 x = \sec x \sec^2 x$, find $\int \sec^3 x \,\mathrm{d}x$.

Hence find the exact value of $\int_0^{\tan^{-1} 2} \sec^3 x \,\mathrm{d}x$. [6]

3 **(i)** By first expressing $3x - x^2 - 4$ in completed square form, show that $3x - x^2 - 4$ is always negative for all real values of x. [2]

(ii) Hence, or otherwise, without the use of a calculator, solve the inequality

$$\frac{\left(3x - x^2 - 4\right)\left(x-1\right)^2}{x^2 - 2x - 5} \le 0,$$

leaving your answer in exact form. [4]

4 The complex number z is given by $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta \le \pi$. It is given that the complex number $w = \left(-\sqrt{3} - \mathrm{i}\right)z$.

(i) Find $|w|$ in terms of r, and $\arg w$ in terms of θ. [2]

(ii) Given that $\dfrac{z^8}{w^*}$ is purely imaginary, find the three smallest values of θ in terms of π. [5]

5 **(a)** It is given that three non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy the equation $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{c})=\mathbf{b}\times\mathbf{c}$, where $\mathbf{b}\neq\mathbf{c}$. Find a linear relationship between $\mathbf{a}$, $\mathbf{b}$ an $\mathbf{c}$. [3]

(b) A point A with position vector $\overrightarrow{OA}=\alpha\,\mathbf{i}+\beta\,\mathbf{j}+\gamma\,\mathbf{k}$, where α, β and γ are real constants, has direction cosines $\cos\theta$, $\cos\phi$ and $\cos\omega$, where θ, ϕ and ω are the angles $\overrightarrow{OA}$ make with the positive x, y and z-axes respectively.

(i) Express the direction cosines $\cos\theta$, $\cos\phi$ and $\cos\omega$ in terms of α, β and γ.

Hence find the value of $\cos^2\theta+\cos^2\phi+\cos^2\omega$. [3]

(ii) Hence show that $\cos 2\theta+\cos 2\phi+\cos 2\omega=-1$. [2]

6 A particle moving along a path at time t, where $0<t<\frac{\pi}{3}$, is defined parametrically by

$$x=\cot 3t \quad\text{and}\quad y=2\operatorname{cosec}3t+1 .$$

(a) The tangent to the path at the point $P\left(\cot 3p,\ 2\operatorname{cosec}3p+1\right)$ meets the y-axis at the point Q. Show that the coordinates of Q is $\left(0,\ 2\sin 3p+1\right)$. [4]

(b) The distance of the particle from the point $R(0,1)$ is denoted by s, where $s^2=x^2+(y-1)^2$. Find the exact rate of change of the particle's distance from R at time $t=\frac{\pi}{4}$. [4]

7

(i) It is given that $\ln y=2\sin x$. Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-y\ln y+\dfrac{1}{y}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$. [2]

(ii) Find the first four terms of the Maclaurin series for y in ascending powers of x. [4]

(iii) Using appropriate expansions from the List of Formulae (MF26), verify the expansion found in part **(ii)**. [2]

(iv) Given that x is sufficiently small for x^4 and higher powers of x to be neglected, deduce an approximation for $\mathrm{e}^{(2\sin x)-\ln(\sec x)}$ in ascending powers of x. [2]

8 **(a)** A curve is defined parametrically by the equations

$$x = \sin t \quad \text{and} \quad y = \cos^3 t, \qquad -\pi \le t \le \pi.$$

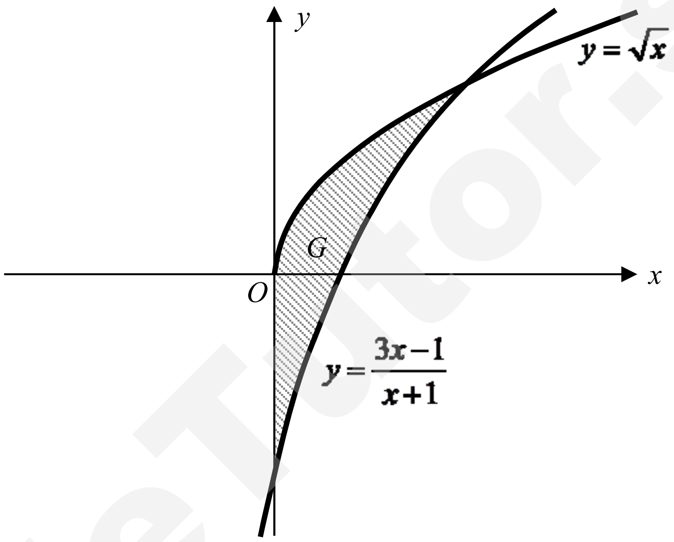
(i) Show that the area enclosed by the curve is given by

$$k\int_0^{\frac{\pi}{2}} \cos^4 t \, \mathrm{d}t,$$

where k is a constant to be determined. [3]

(ii) Hence find the exact area enclosed by the curve. [3]

(b) In the diagram, the region G is bounded by the curves $y = \dfrac{3x-1}{x+1}$, $y = \sqrt{x}$ and the y-axis.



Find the exact volume of the solid generated when G is rotated about the y-axis through 2π radians. [6]

9

A curve C_1 has equation $y = \dfrac{ax^2 - bx}{x^2 - c}$, where a, b and c are constants. It is given that C_1 passes through the point $\left(3, \frac{9}{5}\right)$ and two of its asymptotes are $y = 2$ and $x = -2$.

(i) Find the values of a, b and c. [3]

In the rest of the question, take the values of a, b and c as found in part **(i)**.

(ii) Using an algebraic method, find the exact set of values of y that C_1 cannot take. [3]

(iii) Sketch C_1, showing clearly the equations of asymptotes and the coordinates of the turning points. [3]

(iv) It is given that the equation $\mathrm{e}^y = x - r$, where $r \in \mathbb{Z}^+$, has exactly one real root. State the range of values of r. [1]

(v) The curve C_2 has equation $y = 2 + \dfrac{3x+5}{x^2 - 2x - 3}$. State a sequence of transformations which transforms C_1 to C_2. [3]

10 Food energy taken in by a man goes partly to maintain the healthy functioning of his body and partly to increase body mass. The total food energy intake of the man per day is assumed to be a constant denoted by I (in joules). The food energy required to maintain the healthy functioning of his body is proportional to his body mass M (in kg). The increase of M with respect to time t (in days) is proportional to the energy not used by his body. If the man does not eat for one day, his body mass will be reduced by 1%.

(i) Show that I, M and t are related by the following differential equation:

$$\frac{dM}{dt} = \frac{I - aM}{100a}, \text{ where } a \text{ is a constant.}$$

State an assumption for this model to be valid. [3]

(ii) Find the total food energy intake per day, I, of the man in terms of a and M if he wants to maintain a constant body mass. [1]

It is given that the man's initial mass is 100kg.

(iii) Solve the differential equation in part **(i)**, giving M in terms of I, a and t. [3]

(iv) Sketch the graph of M against t for the case where $I > 100a$. Interpret the shape of the graph with regard to the man's food energy intake. [3]

(v) If the man's total food energy intake per day is $50a$, find the time taken in days for the man to reduce his body mass from 100kg to 90kg. [2]

11 A manual hoist is a mechanical device used primarily for raising and lowering heavy loads, with the motive power supplied manually by hand. Three hoists, A, B and C are used to lift a load vertically.

(i) For hoist A, the first pull will raise the load by a vertical distance of 45 cm. On each subsequent pull, the load will raise 1.6 cm lesser than the vertical distance covered by the previous pull. Determine the number of pulls needed for the load to achieve maximum total height. Hence find this maximum total height. [4]

(ii) For hoist B, the first pull will raise the load by a vertical distance of 45 cm. On each subsequent pull, the vertical distance raised will be 95% of the distance covered by the previous pull. Find the theoretical maximum total height that the load can reach. [2]

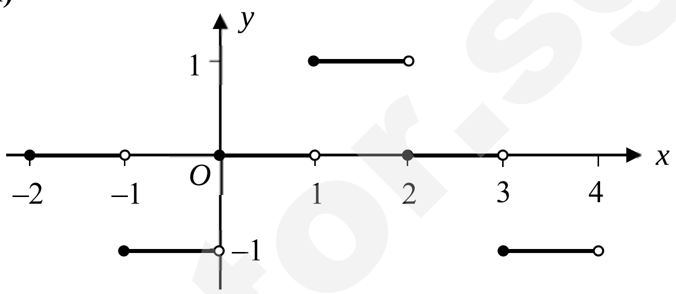
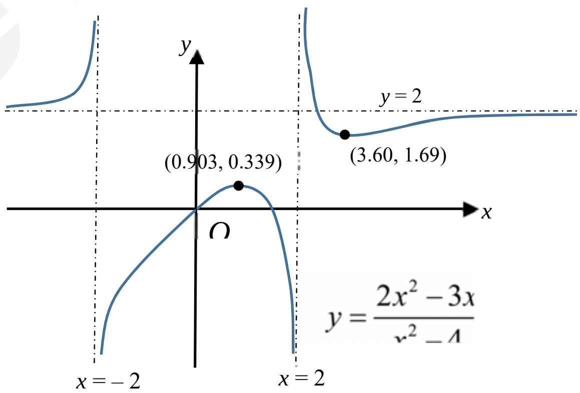
(iii) For hoist C, every pull will raise the load by a constant vertical distance of 45 cm. However, after each pull, the load will slip and drop by 2% of the total vertical height the load has reached. Show that just before the 4^{th} pull, the load would have reached a total vertical height of 130 cm, correct to 3 significant figures.

Hence show that before the $(n+1)^{\text{th}}$ pull, the load would have reached a total vertical height of $X + Y(0.98)^{n+1}$, where X and Y are integers to be determined. [5]

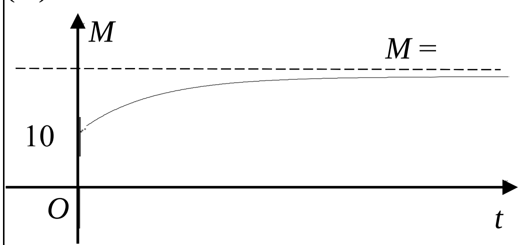
(iv) Explain clearly if hoist C can lift the load up a building of height 25 metres. [2]

ANNEX B

<u>HCI H2 Math JC2 Preliminary Examination Paper 1</u>

QN	Topic Set	Answers
1	Functions	**(i)** $\mathrm{f}(-1.2)=\mathrm{f}(2.8)=0$ $\mathrm{f}(3.6)=\mathrm{f}(-0.4)=-1$ **(ii)** 
2	Graphs and Transformation	**(i)** Since $y=2$ is a horizontal asymptote, $a=2$. Since $x=-2$ is a vertical asymptote, $c=4$. $\left(3,\frac{9}{5}\right)$ lies on $y=\dfrac{2x^2-bx}{x^2-4} \Rightarrow b=3$ **(ii)** required set is $\left\{y\in\mathbb{R}:1-\frac{\sqrt{7}}{4}<y<1+\frac{\sqrt{7}}{4}\right\}$ **(iii)**  **(iv)** $r\geq 2$ **(v)** 1. Translation of C_1 1 unit in the negative x-direction to get $y=2+\dfrac{8-3(x+1)}{(x+1)^2-4}=2+\dfrac{-3x+5}{x^2+2x-3}$ followed by 2. Reflection of $y=2+\dfrac{-3x+5}{x^2+2x-3}$ in the y-axis to get C_2.

3	Sigma Notation and Method of Difference	NA
4	Binomial Expansion	NA
5	AP and GP	**(i)** $n \le 29.125$ Hence number of pulls needed to achieve maximum total height is 29. Maximum total height $= \frac{29}{2}\left[2(45)+(29-1)(-1.6)\right]$ $= 655.4$ cm **(ii)** Maximum total height $= \dfrac{45}{1-0.95} = 900$ cm **(iii)** Before 4th pull, total height reached $= \dfrac{0.98(45)(1-0.98^3)}{1-0.98}$ $= 129.67164$ $= 130$ cm (3 s.f.) Before $(n+1)^{\text{th}}$ pull, total height reached $= \dfrac{0.98(45)(1-0.98^n)}{1-0.98}$ $= 2205 - 2250(0.98)^{n+1}$, where $X = 2205$, $Y = -2250$ **(iv)** Maximum total height reached by load using hoist C will approach 2205 cm. Therefore the hoist C cannot be used to lift the load up the building of 2500 cm
6	Equations and Inequalities	**(i)** $3x - x^2 - 4 = -\left(x-\dfrac{3}{2}\right)^2 - \dfrac{7}{4} \le -\dfrac{7}{4} < 0$ **(ii)** $x < 1-\sqrt{6}$ or $x > 1+\sqrt{6}$ or $x = 1$
7	Differentiation & Applications	**(a)** At point P, $\dfrac{dy}{dx}\Big\|_{t=p} = 2\cos 3p$ Equation of tangent at P: $y-(2\operatorname{cosec} 3p+1) = 2\cos 3p(x-\cot 3p)$ Hence the coordinates of Q is $(0, 2\sin 3p+1)$ **(b)** $\dfrac{ds}{dt} = -5\operatorname{cosec}^2 3\left(\frac{\pi}{4}\right)\cot 3\left(\frac{\pi}{4}\right)$ $= -5(2)(-1)$ $= 10$ unit/s
8	Integration techniques	$\int \sec^3 x \, dx$ $= \int \sec x \sec^2 x \, dx$

		$2\int \sec^3 x \, dx = \sec x \tan x + \ln\lvert \sec x + \tan x \rvert$ $\int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \ln\lvert \sec x + \tan x \rvert) + C$ $\int_0^{\tan^{-1} 2} \sec^3 x \, dx = \sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2)$
9	Application of Integration	**(a)(i)** Area $= 4\int_0^1 y \, dx$ $= 4\int_0^{\frac{\pi}{2}} (\cos^3 t)\cos t \, dt$ $= 4\int_0^{\frac{\pi}{2}} \cos^4 t \, dt$ (shown) $\therefore \ k = 4$ **(a)(ii)** $\frac{3\pi}{4}$ unit^2 **(b)** $\frac{29\pi}{5} - 8\pi \ln 2$ unit^3
10	Maclaurin series	**(ii)** $y = 1 + 2x + 2x^2 + x^3 + \ldots$ **(iv)** $e^{(2\sin x)}\cos x \approx 1 + 2x + \frac{3}{2}x^2 + \ldots$
11	Differential Equations	**(i)** Assumption (any 1 below): • The man does not exercise so that no food energy is used up through exercising. • The man does not fall sick so that no food energy is used up to help him recover from his illness. • The man does not consume weight enhancing/loss supplements that affect his food energy gain/loss other than maintaining the healthy functioning of his body and increasing body mass. **(ii)** For $\frac{dM}{dt}$ to be zero, $I = aM$ **(iii)** $M = \frac{I}{a} - \left(\frac{I}{a} - 100\right)e^{\frac{-t}{100}}$ **(iv)** 

		Explanation (any 1 below): • The man consumes more food than is necessary for maintaining a healthy functioning body. Therefore the graph shows that his body mass will increase. • Since $I > 100a$, hence $\frac{I}{a} > 100$. The man's body mass is always less than $\frac{I}{a}$. • In the long run, the man's body mass will approach $\frac{I}{a}$. **(v)** $t = -100\ln\frac{4}{5} = 22.3$ days
12	Complex numbers	**(i)** $\lvert w\rvert = 2r$, $\arg w = -\frac{5\pi}{6} + \theta$ **(ii)** $9\theta - \frac{5\pi}{6} = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ The three smallest values of θ are $\frac{\pi}{27}$, $\frac{4\pi}{27}$ and $\frac{7\pi}{27}$.
13	Vectors	**(a)** Since $\underset{\sim}{a}$ is non-zero and $\underset{\sim}{b} \neq \underset{\sim}{c}$, $\therefore$ $\underset{\sim}{a}$ is parallel to $(\underset{\sim}{c} - \underset{\sim}{b})$. $\therefore$ $\underset{\sim}{a} = k(\underset{\sim}{c} - \underset{\sim}{b})$, $k \in \mathbb{R}$. **(b)(i)** $\cos\theta = \frac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$; $\cos\phi = \frac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$; $\cos\omega = \frac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$; $\cos^2\theta + \cos^2\phi + \cos^2\omega = 1$
14	P&C, Probability	NA
15	DRV	NA
16	Binomial Distribution	NA
17	Normal Distribution	NA
18	Sampling	NA
19	Hypothesis Testing	NA
20	Correlation & Linear Regression	NA

H2 Mathematics 2017 Prelim Exam Paper 1 Solution

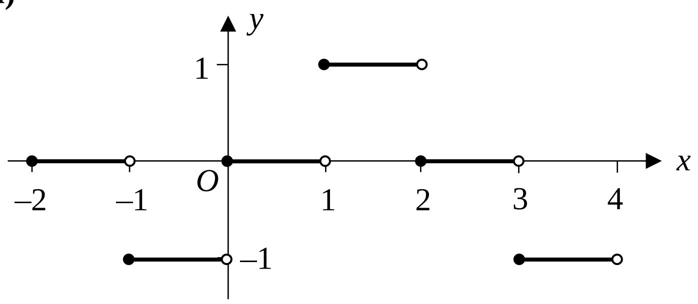
Section A: Pure Mathematics

1

(i) $\mathrm{f}(-1.2)=\mathrm{f}(2.8)=0$

$\mathrm{f}(3.6)=\mathrm{f}(-0.4)=-1$

(ii)



(iii) $\int_{-2}^{4}\mathrm{f}(x)\mathrm{d}x=-1+1-1=-1$

2

$u=\sec x\Rightarrow u'=\sec x\tan x$

$v'=\sec^2 x\Rightarrow v=\tan x$

$\int \sec^3 x\ \mathrm{d}x$

$=\int \sec x\ \sec^2 x\ \mathrm{d}x$

$=\sec x\tan x-\int \sec x\tan^2 x\ \mathrm{d}x$

$=\sec x\tan x-\int \sec x(\sec^2 x-1)\ \mathrm{d}x$

$=\sec x\tan x-\int \sec^3 x-\sec x\ \mathrm{d}x$

$=\sec x\tan x-\int \sec^3 x\ \mathrm{d}x+\ln\left|\sec x+\tan x\right|$

$2\int \sec^3 x\ \mathrm{d}x=\sec x\tan x+\ln\left|\sec x+\tan x\right|$

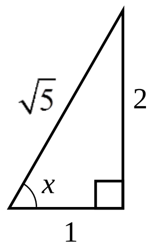
$\int \sec^3 x\ \mathrm{d}x=\frac{1}{2}(\sec x\tan x+\ln\left|\sec x+\tan x\right|)+C$

$\int_0^{\tan^{-1}2}\sec^3 x\ \mathrm{d}x$

$=\frac{1}{2}\Big[\sec x\tan x+\ln\left|\sec x+\tan x\right|\Big]_0^{\tan^{-1}2}$

$=\frac{1}{2}\Big[\sqrt{5}\times 2+\ln(\sqrt{5}+2)\Big]$

$=\sqrt{5}+\frac{1}{2}\ln(\sqrt{5}+2)$



3

(i)

$$3x-x^2-4=-\left(x^2-3x+4\right)$$
$$=-\left(\left(x-\frac{3}{2}\right)^2+\frac{7}{4}\right)$$
$$=-\left(x-\frac{3}{2}\right)^2-\frac{7}{4}$$

Since $\left(x-\frac{3}{2}\right)^2\geq 0$ for all $x\in\mathbb{R}$, $-\left(x-\frac{3}{2}\right)^2\leq 0$

Hence $3x-x^2-4=-\left(x-\frac{3}{2}\right)^2-\frac{7}{4}\leq-\frac{7}{4}<0$

$\therefore$ $3x-x^2-4$ is always negative for all values of x.

(ii)

$$\frac{\left(3x-x^2-4\right)(x-1)^2}{x^2-2x-5}\leq 0$$

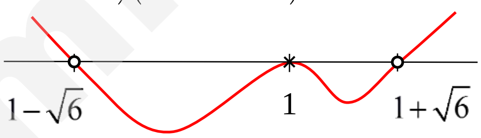
Since $3x-x^2-4$ is always negative, $\dfrac{(x-1)^2}{x^2-2x-5}\geq 0$

<u>Method 1</u> (Quadratic formula)

Let $x^2-2x-5=0$

$$\therefore\ x=\frac{2\pm\sqrt{(-2)^2-4(1)(-5)}}{2(1)}=\frac{2\pm\sqrt{24}}{2}=1\pm\sqrt{6}$$

Hence $\dfrac{(x-1)^2}{\left(x-(1-\sqrt{6})\right)\left(x-(1+\sqrt{6})\right)}\geq 0$

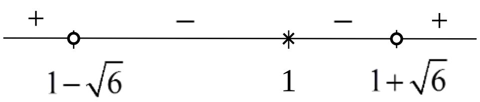


$\therefore$ $x<1-\sqrt{6}$ or $x>1+\sqrt{6}$ or $x=1$

<u>Method 2</u> (Complete the square)

$$\frac{(x-1)^2}{(x-1)^2-6}\geq 0$$

$$\frac{(x-1)^2}{\left(x-(1-\sqrt{6})\right)\left(x-(1+\sqrt{6})\right)}\geq 0$$



$\therefore$ $x<1-\sqrt{6}$ or $x>1+\sqrt{6}$ or $x=1$

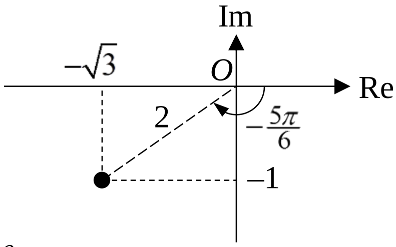
4 **(i)**

<u>Method 1</u>

$$w=\left(-\sqrt{3}-\mathrm{i}\right)z$$

$$=[2\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}]r\,\mathrm{e}^{\mathrm{i}\theta}$$

$$=2r\,\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}+\theta\right)}$$



$$\therefore |w|=2r,\ \arg w=-\frac{5\pi}{6}+\theta$$

<u>Method 2</u>

$$|w|=\left|\left(-\sqrt{3}-\mathrm{i}\right)z\right|$$

$$=\left|\left(-\sqrt{3}-\mathrm{i}\right)\right||z|$$

$$=2r$$

$$\arg w=\arg\left(\left[-\sqrt{3}-\mathrm{i}\right]z\right)$$

$$=\arg\left(-\sqrt{3}-\mathrm{i}\right)+\arg z$$

$$=-\frac{5\pi}{6}+\theta$$

(ii)

<u>Method 1</u>

$$\arg\left(\frac{z^8}{w^*}\right)=\arg(z^8)-\arg(w^*)$$ $\boxed{\arg(w^*)=-\arg(w)}$

$$=8\theta+\arg w$$

$$=8\theta+\left(-\frac{5\pi}{6}+\theta\right)$$ From **(ii)**

$$=9\theta-\frac{5\pi}{6}$$

For $\frac{z^8}{w^*}$ to be purely imaginary,

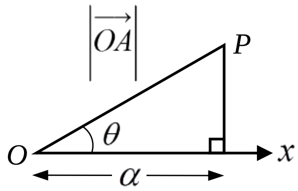
$$\arg\left(\frac{z^8}{w^*}\right)=\ \dots,-\frac{3\pi}{2},\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \dots$$

$$\therefore\ 9\theta-\frac{5\pi}{6}=\ \dots,-\frac{3\pi}{2},\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \dots$$

$$9\theta=\dots,-\frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{4\pi}{3},\ \frac{7\pi}{3},\ \dots$$

$$\theta=\dots,-\frac{2\pi}{27},\ \frac{\pi}{27},\ \frac{4\pi}{27},\ \frac{7\pi}{27},\ \dots$$

$\therefore$ the three smallest values of θ are $\frac{\pi}{27}$, $\frac{4\pi}{27}$ and $\frac{7\pi}{27}$.

	<u>Method 2</u> $\dfrac{z^8}{w^*}=\dfrac{(r\,\mathrm{e}^{\mathrm{i}\theta})^8}{2r\,\mathrm{e}^{\mathrm{i}\left[-\left(-\frac{5\pi}{6}+\theta\right)\right]}}=\dfrac{r^8\,\mathrm{e}^{\mathrm{i}(8\theta)}}{2r\,\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{6}-\theta\right)}}$ $=\dfrac{r^7}{2}\mathrm{e}^{\mathrm{i}\left[8\theta-\left(\frac{5\pi}{6}-\theta\right)\right]}$ $=\dfrac{r^7}{2}\mathrm{e}^{\mathrm{i}\left(9\theta-\frac{5\pi}{6}\right)}$ For $\dfrac{z^8}{w^*}$ to be purely imaginary, $\arg\left(\dfrac{z^8}{w^*}\right)=\dfrac{\pi}{2}+k\pi$, where $k\in\mathbb{Z}$ $\therefore\ 9\theta-\dfrac{5\pi}{6}=\dfrac{\pi}{2}+k\pi$ $9\theta=\dfrac{4\pi}{3}+k\pi$ $\theta=\dfrac{4\pi}{27}+\dfrac{k\pi}{9}$ When $k=-2$, $\theta=-\frac{2\pi}{27}$ When $k=-1$, $\theta=\frac{\pi}{27}$ When $k=0$, $\theta=\frac{4\pi}{27}$ When $k=1$, $\theta=\frac{7\pi}{27}$ $\therefore$ the three smallest values of θ are $\dfrac{\pi}{27}$, $\dfrac{4\pi}{27}$ and $\dfrac{7\pi}{27}$.
5	**(a)** $(\underset{\sim}{a}+\underset{\sim}{b})\times(\underset{\sim}{a}+\underset{\sim}{c})=\underset{\sim}{b}\times\underset{\sim}{c}$ $(\underset{\sim}{a}\times\underset{\sim}{a})+(\underset{\sim}{a}\times\underset{\sim}{c})+(\underset{\sim}{b}\times\underset{\sim}{a})+(\underset{\sim}{b}\times\underset{\sim}{c})=\underset{\sim}{b}\times\underset{\sim}{c}$ $(\underset{\sim}{a}\times\underset{\sim}{c})+(\underset{\sim}{b}\times\underset{\sim}{a})=\underset{\sim}{0}$ $(\underset{\sim}{a}\times\underset{\sim}{c})-(\underset{\sim}{a}\times\underset{\sim}{b})=\underset{\sim}{0}$ $\underset{\sim}{a}\times(\underset{\sim}{c}-\underset{\sim}{b})=\underset{\sim}{0}$ Since $\underset{\sim}{a}$ is non-zero and $\underset{\sim}{b}\neq\underset{\sim}{c}$, $\therefore$ $\underset{\sim}{a}$ is parallel to $(\underset{\sim}{c}-\underset{\sim}{b})$. $\therefore$ $\underset{\sim}{a}=k(\underset{\sim}{c}-\underset{\sim}{b})$, $k\in\mathbb{R}$. **(b)(i)** $\left\|\overrightarrow{OA}\right\|=\sqrt{\alpha^2+\beta^2+\gamma^2}$ $\therefore\ \cos\theta=\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$ 

	$\cos\phi = \dfrac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$ $\cos\omega = \dfrac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$ $\cos^2\theta+\cos^2\phi+\cos^2\omega$ $=\left(\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\right)^2+\left(\dfrac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\right)^2+\left(\dfrac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\right)^2$ $=\dfrac{\alpha^2+\beta^2+\gamma^2}{\alpha^2+\beta^2+\gamma^2}$ $=1$ **(b)(ii)** $\cos 2\theta+\cos 2\phi+\cos 2\omega$ $=2\cos^2\theta-1+2\cos^2\phi-1+2\cos^2\omega-1$ $=2\left(\cos^2\theta+\cos^2\phi+\cos^2\omega\right)-3$ $=2(1)-3$ $=-1$ (shown)
6	**(a)** $x=\cot 3t \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t}=-3\operatorname{cosec}^2 3t$ $y=2\operatorname{cosec}3t+1 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}t}=-6\operatorname{cosec}3t\cot 3t$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\dfrac{-6\operatorname{cosec}3t\cot 3t}{-3\operatorname{cosec}^2 3t}$ $=\dfrac{2\cot 3t}{\operatorname{cosec}3t}$ $=2\cos 3t$ At point P, $\dfrac{\mathrm{d}y}{\mathrm{d}x}\Big\vert_{t=p}=2\cos 3p$ Equation of tangent at P: $y-\left(2\operatorname{cosec}3p+1\right)=2\cos 3p\left(x-\cot 3p\right)$ When tangent meets y-axis, $x=0$. Hence $y=-\left(2\cos 3p\right)\left(\cot 3p\right)+\left(2\operatorname{cosec}3p+1\right)$ $y=\dfrac{-2\left(\cos^2 3p\right)}{\sin 3p}+\dfrac{2}{\sin 3p}+1$

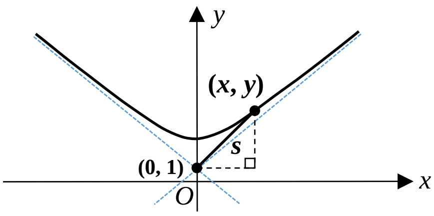
$$y = \frac{-2(\cos^2 3p - 1)}{\sin 3p} + 1$$

$$y = \frac{-2(-\sin^2 3p)}{\sin 3p} + 1$$

$$y = 2\sin 3p + 1$$

Hence the coordinates of Q is $\left(0,\ 2\sin 3p + 1\right)$. (shown)

(b)



Method 1

$$s^2 = x^2 + \left(y-1\right)^2$$
$$= \cot^2 3t + \left(2\operatorname{cosec} 3t + 1 - 1\right)^2$$
$$= (\operatorname{cosec}^2 3t - 1) + 4\operatorname{cosec}^2 3t$$
$$= 5\operatorname{cosec}^2 3t - 1$$

Differentiate w.r.t. t,

$$2s\frac{\mathrm{d}s}{\mathrm{d}t} = 10\operatorname{cosec} 3t(-\operatorname{cosec} 3t \cot 3t)(3)$$
$$= -30\operatorname{cosec}^2 3t \cot 3t$$
$$s\frac{\mathrm{d}s}{\mathrm{d}t} = -15\operatorname{cosec}^2 3t \cot 3t$$

When $t = \frac{\pi}{4}$, $s^2 = (2\sqrt{2} + 1 - 1)^2 + (-1)^2 = 9$

$\therefore\ s = 3$ (since $s > 0$)

$$\therefore\ \frac{\mathrm{d}s}{\mathrm{d}t} = -5\operatorname{cosec}^2 3\left(\tfrac{\pi}{4}\right)\cot 3\left(\tfrac{\pi}{4}\right)$$
$$= -5(2)(-1)$$
$$= 10 \text{ unit/s}$$

Method 2

$$s^2 = x^2 + \left(y-1\right)^2$$
$$= \cot^2 3t + \left(2\operatorname{cosec} 3t + 1 - 1\right)^2$$
$$= \cot^2 3t + 4\operatorname{cosec}^2 3t$$

Differentiate w.r.t. t,

$$2s\frac{\mathrm{d}s}{\mathrm{d}t} = 2\cot 3t(-\operatorname{cosec}^2 3t)(3) + 8\operatorname{cosec} 3t(-\operatorname{cosec} 3t \cot 3t)(3)$$
$$= -6\operatorname{cosec}^2 3t \cot 3t - 24\operatorname{cosec}^2 3t \cot 3t$$
$$= -30\operatorname{cosec}^2 3t \cot 3t$$
$$s\frac{\mathrm{d}s}{\mathrm{d}t} = -15\operatorname{cosec}^2 3t \cot 3t$$

	When $t=\frac{\pi}{4}$, $s^2=(2\sqrt{2}+1-1)^2+(-1)^2=9$ $\therefore\ s=3$ (since $s>0$) $\therefore\ \frac{\mathrm{d}s}{\mathrm{d}t}=-5\operatorname{cosec}^2 3\left(\frac{\pi}{4}\right)\cot 3\left(\frac{\pi}{4}\right)$ $=-5(2)(-1)$ $=10$ unit/s <u>Method 3</u> $s^2=x^2+(y-1)^2$ Differentiate w.r.t. t , $2s\frac{\mathrm{d}s}{\mathrm{d}t}=2x\frac{\mathrm{d}x}{\mathrm{d}t}+2(y-1)\frac{\mathrm{d}y}{\mathrm{d}t}$ $s\frac{\mathrm{d}s}{\mathrm{d}t}=x\frac{\mathrm{d}x}{\mathrm{d}t}+(y-1)\frac{\mathrm{d}y}{\mathrm{d}t}$ When $t=\frac{\pi}{4}$, $x=\frac{1}{\tan\left(\frac{3\pi}{4}\right)}=-1,\qquad y=\frac{2}{\sin\left(\frac{3\pi}{4}\right)}+1=2\sqrt{2}+1$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-3\operatorname{cosec}^2 3t=\frac{-3}{\sin^2\left(\frac{3\pi}{4}\right)}=-6$ $\frac{\mathrm{d}y}{\mathrm{d}t}=-6\cot 3t\operatorname{cosec} 3t=\frac{-6}{\tan\left(\frac{3\pi}{4}\right)}\times\frac{1}{\sin\left(\frac{3\pi}{4}\right)}=6\sqrt{2}$ $s^2=(2\sqrt{2}+1-1)^2+(-1)^2=9$ $\therefore\ s=3$ (since $s>0$) Hence $\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{1}{s}\left[x\frac{\mathrm{d}x}{\mathrm{d}t}+(y-1)\frac{\mathrm{d}y}{\mathrm{d}t}\right]$ $=\frac{1}{3}\left[(-1)(-6)+(2\sqrt{2})(6\sqrt{2})\right]$ $=10$ unit/s
7	**(i)** <u>Method 1</u> $\ln y=2\sin x$ $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=2\cos x$ $\frac{\mathrm{d}y}{\mathrm{d}x}=2y\cos x$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2y\sin x+2\cos x\frac{\mathrm{d}y}{\mathrm{d}x}=-y\ln y+\frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ (shown) <u>Method 2</u> $y=\mathrm{e}^{2\sin x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=(2\cos x)\mathrm{e}^{2\sin x}$

$$\frac{dy}{dx} = (2\cos x)y$$

$$\frac{d^2y}{dx^2} = -2y\sin x + 2\cos x\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -y\ln y + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 \quad \text{(shown)}$$

(ii)

$$\frac{d^3y}{dx^3} = -y\left(\frac{1}{y}\frac{dy}{dx}\right) - \ln y\frac{dy}{dx} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^3 + \frac{2}{y}\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$$

When $x = 0$, $y = 1$, $\frac{dy}{dx} = 2$, $\frac{d^2y}{dx^2} = 4$, $\frac{d^3y}{dx^3} = 6$

$$y = 1 + 2x + \frac{4x^2}{2!} + \frac{6x^3}{3!} + \ldots$$

$$y = 1 + 2x + 2x^2 + x^3 + \ldots$$

(iii)

Method 1

$$y = e^{2\sin x}$$

$$= 1 + (2\sin x) + \frac{(2\sin x)^2}{2} + \frac{(2\sin x)^3}{6} + \ldots$$

$$= 1 + 2(x - \frac{x^3}{6} + \ldots) + \frac{[2(x + \ldots)]^2}{2} + \frac{[2(x + \ldots)]^3}{6} + \ldots$$

$$= 1 + 2x - \frac{x^3}{3} + 2x^2 + \frac{4x^3}{3} + \ldots$$

$$= 1 + 2x + 2x^2 + x^3 + \ldots$$

Method 2

$$y = e^{2(x - \frac{x^3}{3!})}$$

$$= 1 + 2(x - \frac{x^3}{6}) + \frac{[2(x - \frac{x^3}{6})]^2}{2} + \frac{[2(x - \frac{x^3}{6})]^3}{6} + \ldots$$

$$= 1 + 2x - \frac{2x^3}{6} + \frac{4x^2}{2} + \frac{8x^3}{6} + \ldots$$

$$= 1 + 2x + 2x^2 + x^3 + \ldots$$

(iv)

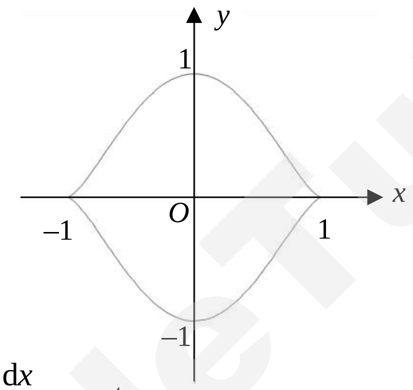
$$e^{(2\sin x) - \ln(\sec x)} = e^{(2\sin x)}e^{-\ln\sec x} = e^{(2\sin x)}e^{\ln\cos x}$$

$$= e^{(2\sin x)}\cos x$$

Method 1

$$e^{(2\sin x)}\cos x \approx (1 + 2x + 2x^2 + x^3)(1 - \frac{x^2}{2})$$

$$= 1 - \frac{x^2}{2} + 2x - \frac{2x^3}{2} + 2x^2 + x^3 + \ldots$$

	$=1+2x+\frac{3}{2}x^2+\ldots$ <u>Method 2</u> $y=e^{2\sin x}$ $\frac{dy}{dx}=(2\cos x)e^{2\sin x}$ $\therefore\ \cos x e^{2\sin x}=\frac{1}{2}\frac{dy}{dx}$ $=\frac{1}{2}\frac{d}{dx}(1+2x+2x^2+x^3+\ldots)$ ← From **(iii)** $=\frac{1}{2}(2+4x+3x^2+\ldots)$ $=1+2x+\frac{3}{2}x^2+\ldots$
8	**(a)(i)**  $x=\sin t \Rightarrow \frac{dx}{dt}=\cos t$ When $x=0$, $t=0$. When $x=1$, $t=\frac{\pi}{2}$. Area $=4\int_0^1 y\ dx$ $=4\int_0^{\frac{\pi}{2}}(\cos^3 t)\cos t\ dt$ $=4\int_0^{\frac{\pi}{2}}\cos^4 t\ dt$ (shown) $\therefore\ k=4$ **(a)(ii)** Area $=4\int_0^{\frac{\pi}{2}}\cos^4 t\ dt$ $=\int_0^{\frac{\pi}{2}}(2\cos^2 t)^2\ dt$

$$= \int_0^{\frac{\pi}{2}} (1+\cos 2t)^2 \ \mathrm{d}t$$

$$= \int_0^{\frac{\pi}{2}} 1+2\cos 2t+\cos^2 2t \ \mathrm{d}t$$

$$= \int_0^{\frac{\pi}{2}} 1+2\cos 2t+\frac{1+\cos 4t}{2} \ \mathrm{d}t$$

$$= \int_0^{\frac{\pi}{2}} \frac{3}{2}+2\cos 2t+\frac{\cos 4t}{2} \ \mathrm{d}t$$

$$= \left[\frac{3t}{2}+\sin 2t+\frac{\sin 4t}{8}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3\pi}{4} \ \text{unit}^2$$

(b)
From GC, coordinates of intersection = (1, 1)

<u>Method 1</u>

$$y=\frac{3x-1}{x+1} \Rightarrow xy+y=3x-1 \Rightarrow x=\frac{1+y}{3-y}$$

Required volume

$$= \pi\int_{-1}^{1}\left(\frac{1+y}{3-y}\right)^2 \ \mathrm{d}y-\pi\int_0^1 (y^2)^2 \ \mathrm{d}y$$

$$= \pi\int_{-1}^{1}\left(\frac{4}{3-y}-1\right)^2 \mathrm{d}y-\pi\int_0^1 y^4 \ \mathrm{d}y$$

$$= \pi\int_{-1}^{1}\left(\frac{16}{(3-y)^2}-\frac{8}{3-y}+1\right)\mathrm{d}y-\pi\left[\frac{y^5}{5}\right]_0^1$$

$$= \pi\left[\frac{16}{3-y}+8\ln|3-y|+y\right]_{-1}^{1}-\frac{\pi}{5}$$

$$= \pi\left[8+8\ln 2+1-(4+8\ln 4-1)\right]-\frac{\pi}{5}$$

$$= \pi\left[6+8\ln 2-16\ln 2\right]-\frac{\pi}{5}$$

$$= \frac{29\pi}{5}-8\pi\ln 2 \quad \text{unit}^3$$

<u>Method 2</u>

$$y=\frac{3x-1}{x+1} \Rightarrow xy+y=3x-1 \Rightarrow x=\frac{1+y}{3-y}$$

Required volume

$$= \pi \int_{-1}^{1} \left(\frac{1+y}{3-y} \right)^2 \, dy - \pi \int_{0}^{1} (y^2)^2 \, dy$$

$$= \pi \int_{-1}^{1} \frac{y^2+2y+1}{y^2-6y+9} \, dy - \pi \int_{0}^{1} y^4 \, dy$$

$$= \pi \int_{-1}^{1} 1 + \frac{8y-8}{y^2-6y+9} \, dy - \pi \left[\frac{y^5}{5} \right]_0^1$$

$$= \pi [y]_{-1}^{1} + 4\pi \int_{-1}^{1} \frac{2y-6}{y^2-6y+9} \, dy + \pi \int_{-1}^{1} \frac{16}{(y-3)^2} \, dy - \frac{\pi}{5}$$

$$= 2\pi + 4\pi \left[\ln \left| y^2-6y+9 \right| \right]_{-1}^{1} + 16\pi \left[\frac{(y-3)^{-1}}{-1} \right]_{-1}^{1} - \frac{\pi}{5}$$

$$= \frac{9\pi}{5} + 4\pi [\ln 4 - \ln 16] + 16\pi \left[\frac{1}{3-y} \right]_{-1}^{1}$$

$$= \frac{9\pi}{5} + 4\pi \ln \frac{1}{4} + 16\pi \left[\frac{1}{2} - \frac{1}{4} \right]$$

$$= \frac{9\pi}{5} - 4\pi \ln 4 + 4\pi$$

$$= \frac{29\pi}{5} - 8\pi \ln 2 \quad \text{unit}^3$$

9 **(i)**

$$y = \frac{ax^2 - bx}{x^2 - c}$$

Since $y = 2$ is a horizontal asymptote, $a = 2$.
Since $x = -2$ is a vertical asymptote, $c = 4$.

$\left(3, \frac{9}{5}\right)$ lies on $y = \dfrac{2x^2 - bx}{x^2 - 4}$

$\therefore \frac{9}{5} = \frac{2(3)^2 - b(3)}{(3)^2 - 4} \quad \Rightarrow \quad b = 3$

(ii)

$$y = \frac{2x^2 - 3x}{x^2 - 4}$$

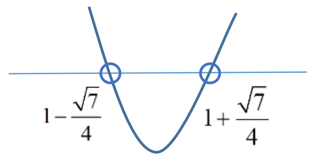
$$y\left(x^2 - 4\right) = 2x^2 - 3x$$

$$(y-2)x^2 + 3x - 4y = 0$$

For no real roots,

$$(3)^2 - 4(y-2)(-4y) < 0$$

$$16y^2 - 32y + 9 < 0$$



Method 1

$$\therefore\ y=\frac{32\pm\sqrt{(32)^2-4(16)(9)}}{2(16)}=\frac{32\pm\sqrt{448}}{32}=1\pm\frac{\sqrt{7}}{4}$$

$\therefore$ required set is $\left\{y\in\mathbb{R}:1-\frac{\sqrt{7}}{4}<y<1+\frac{\sqrt{7}}{4}\right\}$.

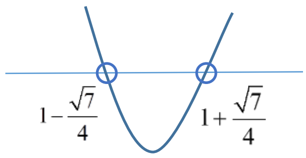
Method 2 (completing the square)

$16y^2-32y+9<0$

$y^2-2y+\frac{9}{16}<0$

$(y-1)^2-\frac{7}{16}<0$

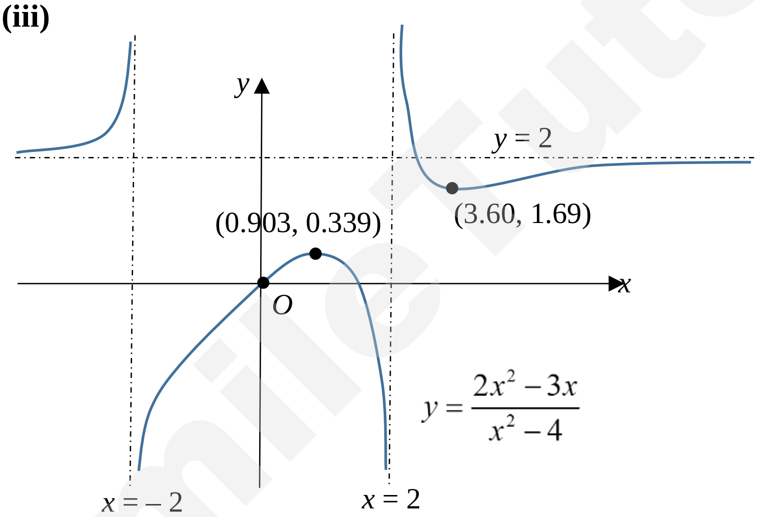
$\left(y-1+\frac{\sqrt{7}}{4}\right)\left(y-1-\frac{\sqrt{7}}{4}\right)<0$



$\therefore\ 1-\frac{\sqrt{7}}{4}<y<1+\frac{\sqrt{7}}{4}$

$\therefore$ required set is $\left\{y\in\mathbb{R}:1-\frac{\sqrt{7}}{4}<y<1+\frac{\sqrt{7}}{4}\right\}$

(iii)

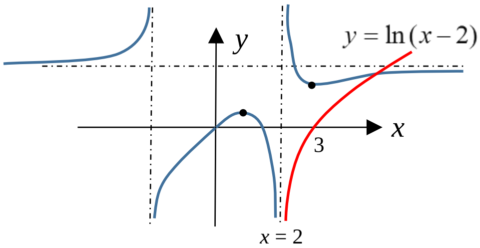


(iv)

$e^y=x-r$

$y=\ln(x-r)$

$r\geq 2$



(v)

$$C_1 : y = \frac{2x^2-3x}{x^2-4} = 2 + \frac{8-3x}{x^2-4}$$

$$\begin{array}{r} 2 \\ x^2-4 \overline{\big) 2x^2-3x} \\ \underline{2x^2-8} \\ -3x+8 \end{array}$$

$$C_2 : y = 2 + \frac{3x+5}{x^2-2x-3}$$

$$= 2 + \frac{3x+5}{(x-1)^2-4}$$

$$= 2 + \frac{8-3(1-x)}{(1-x)^2-4}$$

<u>Method 1</u>

Transformation: $x \to x+1 \to -x+1$

1. Translation of C_1 1 unit in the negative x-direction to get

$y = 2 + \dfrac{8-3(x+1)}{(x+1)^2-4} = 2 + \dfrac{-3x+5}{x^2+2x-3}$ followed by

2. Reflection of $y = 2 + \dfrac{-3x+5}{x^2+2x-3}$ in the y-axis to get C_2.

<u>Method 2</u>

Transformation: $x \to -x \to -(x-1) = -x+1$

1. Reflection of C_1 in the y-axis to get $y = 2 + \dfrac{8+3x}{x^2-4}$

followed by

2. Translation of $y = 2 + \dfrac{8+3x}{x^2-4}$ 1 unit in the positive x-direction to get C_2.

10 (i)

$\dfrac{dM}{dt} \propto I - kM$, where k is a positive constant.

$$\frac{dM}{dt} = b(I-kM)$$

If $I = 0$, $-\dfrac{1}{100}M = b(0-kM)$

$$-\frac{M}{100} = -bkM$$

$$b = \frac{1}{100k}$$

$$\frac{dM}{dt} = \frac{1}{100k}(I-kM) = \frac{I-kM}{100k}$$

$$= \frac{I-aM}{100a}\text{, where } a = k \quad \text{(shown)}$$

Assumption (any 1 below):

- The man does not exercise so that no food energy is used up through exercising.
- The man does not fall sick so that no food energy is used up to help him recover from his illness.

- The man does not consume weight enhancing/loss supplements that affect his food energy gain/loss other than maintaining the healthy functioning of his body and increasing body mass.

(ii) For $\frac{\mathrm{d}M}{\mathrm{d}t}$ to be zero, $I = aM$

(iii)

$$\int \frac{a}{I-aM}\,\mathrm{d}M = \int \frac{1}{100}\,\mathrm{d}t$$

$$-\ln\left|I-aM\right| = \frac{t}{100} + C$$

$$\ln\left|I-aM\right| = \frac{-t}{100} - C$$

$$I-aM = \pm \mathrm{e}^{\frac{-t}{100}}\mathrm{e}^{-C} = A\mathrm{e}^{\frac{-t}{100}}, \text{ where } A = \pm \mathrm{e}^{-C}$$

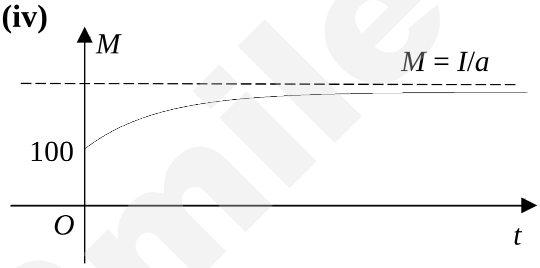
When $t = 0$, $M = 100 \Rightarrow A = I - 100a$

$$I - aM = (I-100a)\mathrm{e}^{\frac{-t}{100}}$$

$$aM = I - (I-100a)\mathrm{e}^{\frac{-t}{100}}$$

$$M = \frac{I}{a} - \left(\frac{I}{a} - 100\right)\mathrm{e}^{\frac{-t}{100}}$$

(iv)



Explanation (any 1 below):

- The man consumes more food than is necessary for maintaining a healthy functioning body. Therefore the graph shows that his body mass will increase.
- Since $I > 100a$, hence $\frac{I}{a} > 100$. The man's body mass is always less than $\frac{I}{a}$.

  In the long run, the man's body mass will approach $\frac{I}{a}$.

(v)

Given $I = 50a$,

$$90 = 50 - (50-100)\mathrm{e}^{\frac{-t}{100}}$$

Using equation found in **(iii)**

$$50\mathrm{e}^{\frac{-t}{100}} = 40$$

$$\mathrm{e}^{\frac{-t}{100}} = \frac{4}{5}$$

$$\frac{-t}{100} = \ln\frac{4}{5}$$

	$\therefore\ t = -100\ln\frac{4}{5} = 22.3$ days (3 s.f.)
11	**(i)** Method 1 Distance covered at the n^{th} pull $= 45+(n-1)(-1.6)$ $= 46.6-1.6n$ $46.6-1.6n \geq 0$ $n \leq 29.125$ Hence number of pulls needed to achieve maximum total height is 29. Maximum total height $= \frac{29}{2}\left[2(45)+(29-1)(-1.6)\right]$ $= 655.4$ cm Method 2 Distance covered at the n^{th} pull, $u_n = 45+(n-1)(-1.6)$ $= 46.6-1.6n$ Using GC, <table><tr><td>n</td><td>u_n</td></tr><tr><td>29</td><td>0.2</td></tr><tr><td>30</td><td>–1.4</td></tr></table>Hence number of pulls needed to achieve maximum total height is 29. Maximum total height $= \frac{29}{2}\left(45+0.2\right) = 655.4$ cm Method 3 Distance covered at the n^{th} pull $= 45+(n-1)(-1.6) = 0$ $\Rightarrow n = 29.125$ <table><tr><td>n</td><td>u_n</td></tr><tr><td>29</td><td>0.2</td></tr><tr><td>30</td><td>–1.4</td></tr></table>Hence number of pulls needed to achieve maximum total height is 29. Maximum total height $= \frac{29}{2}\left(45+0.2\right) = 655.4$ cm Method 4 Total height after n pulls, $S_n = \frac{n}{2}\left[2(45)+(n-1)(-1.6)\right] = 45.8n-0.8n^2$ Using GC, <table><tr><td>n</td><td>S_n</td></tr><tr><td>28</td><td>655.2</td></tr><tr><td>29</td><td>655.4</td></tr><tr><td>30</td><td>654</td></tr></table>Hence the number of pulls needed to achieve maximum total height is 29, and the maximum total height covered is 655.4 cm.

(ii)
Since $r = 0.95 < 1$, sum to infinity of G.P. exists.
$\therefore$ maximum total height $= \dfrac{45}{1-0.95} = 900$ cm

(iii)

	Total height reached
Before 2nd pull	$0.98(45)$
Before 3rd pull	$0.98\big(0.98(45)+45\big)$ $=0.98^2(45)+0.98(45)$
Before 4th pull	$0.98\big(0.98^2(45)+0.98(45)+45\big)$ $=0.98^3(45)+0.98^2(45)+0.98(45)$
$\vdots$	$\vdots$
Before $(n+1)^{\text{th}}$ pull	$0.98^n(45)+0.98^{n-1}(45)+\ldots+0.98(45)$ $=\dfrac{0.98(45)(1-0.98^n)}{1-0.98}$ [sum of G.P. with $a=45$, $r=0.98$]

$\therefore$ before 4th pull, total height reached
$= \dfrac{0.98(45)(1-0.98^3)}{1-0.98}$
$= 129.67164$
$= 130$ cm (3 s.f.)
Before $(n+1)^{\text{th}}$ pull, total height reached
$= \dfrac{0.98(45)(1-0.98^n)}{1-0.98}$
$= 2205 - 2250(0.98)^{n+1}$, where $X = 2205$, $Y = -2250$

(iv)
From **(iii)**,
Total height reached by load using hoist C $= 2205 - 2250(0.98)^{n+1}$
As $n \to \infty$, $(0.98)^{n+1} \to 0$.
Hence maximum total height $\to 2205$.
Therefore maximum total height reached by load using hoist C will approach 2205 cm. Therefore the hoist C cannot be used to lift the load up the building of 2500 cm

HCI Paper 2

1 The sum, S_n, of the first n terms of a sequence u_1, u_2, u_3, … is given by

$$S_n = b - \frac{3a}{(n+1)!},$$

where a and b are constants.

(i) It is given that $u_1 = k$ and $u_2 = \frac{2}{3}k$, where k is a constant. Find a and b in terms of k. [3]

(ii) Find a formula for u_n in terms of k, giving your answer in its simplest form. [2]

(iii) Determine, with a reason, if the series $\sum_{r=1}^{\infty} u_r$ converges. [1]

2 The complex numbers z and w satisfy the following equations

$$2z + 3w = 20,$$

$$w - zw^* = 6 + 22\mathrm{i}.$$

(i) Find z and w in the form $a + b\mathrm{i}$, where a and b are real, $a \neq 0$. [5]

(ii) Show z and w on a single Argand diagram, indicating clearly their modulus. State the relationship between z and w with reference to the origin O. [2]

3 The function f is defined by

$$\mathrm{f} : x \mapsto \sqrt{3}\sin x + \cos x, \quad x \in \mathbb{R}, \ -\pi < x < \tfrac{\pi}{6}.$$

(i) Express f in the form $R\sin(x+\alpha)$, where R and α are exact constants to be determined, $R > 0$, $0 \le \alpha \le \frac{\pi}{2}$. [2]

(ii) Sketch f, giving the exact coordinates of the turning point and the end-points. Deduce the exact range of f. [4]

(iii) The function g is defined by

$$\mathrm{g} : x \mapsto \frac{1}{2} - |x-1|, \quad x \in \mathbb{R}, \ -\frac{5}{2} \le x \le \frac{1}{2}.$$

Explain why the composite function fg exists. Find the range of fg. [3]

(iv) The domain of f is restricted such that the function f^{-1} exists. Find the largest domain of f for which f^{-1} exists. Define f^{-1} in a similar form. [4]

4 Referred to the origin O, the position vector of a point A is $-\mathbf{i}+2\mathbf{j}+4\mathbf{k}$. A plane p contains A and is parallel to the vectors $4\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+\mathbf{k}$.

(i) Find a cartesian equation of p. [2]

(ii) A plane q has equation $x-2y+z=2$. Find a vector equation of the line l where p and q meet. [1]

A point B lies on l such that AB is perpendicular to l.

(iii) Find the position vector of B. [3]

(iv) Find the length of projection of AB on q. [2]

(v) A point C lies on q such that AC is perpendicular to q. Find the position vector of C. Hence find a cartesian equation of the line of reflection of AB in q. [6]

5 The independent random variables X and Y are normally distributed with the same mean 7 but different variances $\text{Var}(X)$ and $\text{Var}(Y)$, respectively. It is given that $\text{P}(X<10)=\text{P}(Y>6)$.

(i) Show that $\text{Var}(X)=9\text{Var}(Y)$. [3]

(ii) If $\text{Var}(Y)=1$, find $\text{P}(X<9)$. [2]

6 A biased tetrahedral (4-sided) die has its faces numbered '-1', '0', '2' and '3'. It is thrown onto a table and the random variable X denotes the number on the face in contact with the table. The probability distribution of X is as shown.

x	-1	0	2	3
$\text{P}(X=x)$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$

(i) The random variable Y is defined by X_1+X_2, where X_1 and X_2 are 2 independent observations of X. Show that $\text{P}(Y=2)=\frac{3}{16}$. [2]

(ii) In a game, a player pays \$2 to throw two such biased tetrahedral dice simultaneously on a table. For each die, the number on the face in contact with the table is the score of the die. The player receives \$16 if the maximum of the two scores is -1, and receives \$3 if the sum of the two scores is prime. For all other cases, the player receives nothing. Find the player's expected gain in the game. [4]

7 Mandy has 10 beads, of which 5 are spherical and 5 are cubical, each of different colours. She wishes to decorate a card by forming a circle using 8 of the 10 beads. Find the number of ways Mandy can arrange the beads if

(i) there are no restrictions, [1]

(ii) 3 particular beads are used and not all are next to one another, [3]

(iii) spherical beads and cubic beads must alternate. [3]

8 A man wishes to buy a 4-digit number lottery. He plays by randomly choosing any number from 0000 to 9999. It is assumed that each number is equally likely to be chosen. Find the probability that a randomly chosen 4-digit number has

(i) four different digits, [1]

(ii) exactly one of the first three digits is the same as the last digit, and the last digit is even, [3]

(iii) four different digits with the first digit greater than 6, given that the 4-digit number has odd and even digits that alternate. [4]

9 In a large shipment of glass stones used for the *Go* board game, a proportion p of the glass stones is chipped. The glass stones are sold in boxes of 361 pieces each. Let X denote the number of chipped glass stones in a box.

(i) Based on this context, state two assumptions in order for X to be well modelled by a binomial distribution. [2]

In the rest of the question, assume that X follows a binomial distribution.

(ii) It is known that the probability of a box containing at most 2 chipped glass stones is 0.90409. Find p. [2]

(iii) A box is deemed to be of inferior quality if it contains more than 2 chipped glass stones. Find the probability that, in a batch of 20 boxes of glass stones, there are more than 5 boxes of inferior quality in the batch. [3]

(iv) Each week, a distributor purchases 50 batches of glass stones, each batch consisting of 20 boxes of glass stones. A batch will be rejected if it contains more than 5 boxes of inferior quality. The distributor will receive a compensation of \$10 for each rejected batch in the first 20 weeks of a year, and a compensation of \$20 for each rejected batch in the remaining weeks of the year. Assuming that there are 52 weeks in a year, find the probability that the total compensation in a year is more than \$250. [5]

10 A large cohort of students sat for a mathematics examination. Based on selected data of the examination results, the following table shows y, the proportions of students who scored x marks.

x	20	30	40	50	60	70	80	90
y	0.00029	0.00174	0.00663	0.0161	0.0252	0.0252	0.0161	0.00663

(i) Draw a scatter diagram for these values, labelling the axes. [2]

(ii) Explain why, in this context, a linear model is not appropriate. [1]

It is decided to fit a model of the form $\ln y = -a(x-m)^2 + b$, where $a > 0$ and m is a suitable constant, to the data. The product moment correlation coefficient between $(x-m)^2$ and $\ln y$ is denoted by r. The table below gives values of r for some possible values of m.

m	62.5	65	67.5
r	□□0.9899292		□□0.9938968

(iii) Calculate the value of r for $m = 65$, giving your answer correct to 7 decimal places. [1]

(iv) Use the table and your answer in part **(iii)** to suggest with a reason which of 62.5, 65 or 67.5 is the most appropriate value for m. [1]

(v) Using the value of m found in part **(iv)**, calculate the values of a and b, and use them to predict the proportion of students who scored 45 marks.
Comment on the reliability of your prediction. [5]

11 Yummy Berries Farm produces blueberries and raspberries packed in boxes.

(a) Yummy Berries Farm claims that the mass, x grams, of each box of blueberries is no less than 125 grams. After receiving ~~a~~ complaints from consumers, the Consumers Association of Singapore (CASE) took a random sample of 50 boxes of blueberries from Yummy Berries Farm and the mass of each box was recorded. The data obtained are summarised in the table.

x (grams)	120	121	122	123	124	125	126	127	128	129	130
No. of boxes	3	6	6	6	3	10	3	4	6	2	1

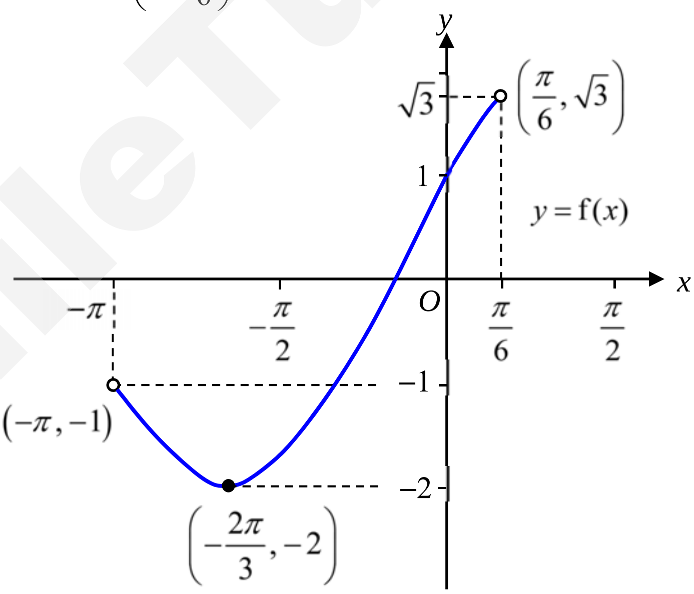
(i) Find unbiased estimates of the population mean and variance. [2]

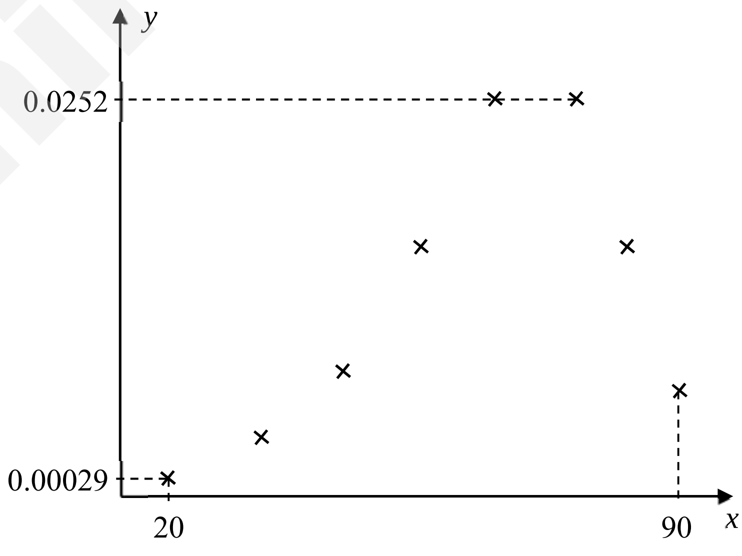
(ii) Test, at the 10% level of significance, whether Yummy Berries Farm has overstated its claim.
State, giving a reason, whether any assumptions about the masses of boxes of blueberries are needed in order for the test to be valid. [6]

(b) The masses of boxes of raspberries, each of y grams, are assumed to have a mean of 170 grams with standard deviation 15 grams. CASE took a random sample of n boxes of raspberries and the mean mass of boxes of raspberries from the sample is found to be 165 grams. A test is to be carried out at the 5% level of significance to determine if the mean mass of the boxes of raspberries is not 170 grams. Find the minimum number of boxes of raspberries to be taken for which the result of the test would be to reject the null hypothesis. [4]

ANNEX B

2017 HCI H2 Maths Preliminary Examination Paper 2

Qn /No	Topic Set	Answers
1	AP and GP	(i) $a=\frac{2}{3}k$, $b=2k$ (ii) $U_n=\frac{2k}{n!}\left(\frac{n}{n+1}\right)$ (iii) $S_n \to 2k$, $\sum_{r=1}^{\infty} u_r$ converges.
2	Complex Numbers	(i) $w=6+2\mathrm{i}$, $z=1-3\mathrm{i}$ (ii) $\angle WOZ$ is $90°$
3	Functions	(i) $\mathrm{f}(x)=2\sin\left(x+\frac{\pi}{6}\right)$ (ii)  $R_{\mathrm{f}}=[-2,\sqrt{3})$ (iii) $R_{\mathrm{fg}}=[-2,1]$ (iv) largest $D_{\mathrm{f}}=[-\frac{2\pi}{3},\frac{\pi}{6})$, $\mathrm{f}^{-1}: x \mapsto \sin^{-1}\left(\frac{x}{2}\right)-\frac{\pi}{6}$, $x\in\mathbb{R}$, $-2\le x<\sqrt{3}$

4	Vectors	(i) $x+y-2z=-7$; (ii) $\underset{\sim}{r}=\begin{pmatrix}-4\\-3\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\1\end{pmatrix}, \alpha\in\mathbb{R}$ (iii) $\overrightarrow{OB}=\begin{pmatrix}0\\1\\4\end{pmatrix}$; (iv) $\frac{\sqrt{2}}{2}$; (v) $\overrightarrow{OC}=\begin{pmatrix}-\frac{1}{2}\\1\\\frac{9}{2}\end{pmatrix}$, $x=0$, $y=5-z$.
5	Normal Distribution	(ii) $\text{P}(X<9)=0.748$
6	DRV	(ii) $-\$0.25$
7	P&C	(i) 226800; (ii) 90720; (iii) 3600
8	Probability	(i) $\frac{63}{125}$; (ii) $\frac{243}{2000}$; (iii) $\frac{3}{125}$
9	Binomial Distribution; Sampling	(i) Assumptions • The probability of a randomly chosen glass stone being chipped is constant. • Whether a glass stone is chipped or not is independent of that of any other glass stones. (ii) $p=0.00300$; (iii) 0.00923; (iv) 0.953
10	Correlation & Linear Regression	(i)  (ii) The scatter diagram displays a curvilinear relationship which suggests the presence of a maximum point. Hence a linear model is inappropriate. (iii) $r=-0.9999984$ (7 decimal places)

		(iv) $m = 65$. Of the 3 negative r values, the r value corresponding to $m = 65$ is closest to -1 (v) $a \approx 0.00222$, $b \approx -3.63$, $\ln y = -0.00222(x-65)^2 - 3.63$, 0.0109, Since $x = 45$ is within data range and $r = -0.9999984$ is very close to -1, the prediction is reliable.
11	Hypothesis Testing	(a) (i) $\overline{x} = 124.4$, $s^2 = 7.43$; (ii) p-value $= 0.0598 < 0.1$, we reject H_0 and conclude that at the 10% level of significance, there is sufficient evidence that Yummy Berries Farm has overstated its claim. No assumptions about masses of boxes of blueberries are needed. Since $n = 50$ is sufficiently large, by Central Limit Theorem, the <u>mean</u> mass of boxes of raspberries will follows a normal distribution approximately. (b) least n is 35

1

(i) $S_1 = b - \frac{3a}{2!} = b - \frac{3a}{2} = k \quad \ldots(1)$

$S_2 = b - \frac{3a}{3!} = b - \frac{a}{2} = k + \frac{2}{3}k = \frac{5}{3}k \quad \ldots(2)$

(2) – (1),

$$-\frac{a}{2} - \left(-\frac{3a}{2}\right) = \frac{5}{3}k - k$$

$$\therefore \quad a = \frac{2}{3}k$$

$$\therefore \ b = k + \frac{3a}{2} = k + \frac{3}{2}\left(\frac{2}{3}k\right) = 2k$$

(ii) $S_n = 2k - \frac{2k}{(n+1)!}$

$$u_n = S_n - S_{n-1}$$

$$= \left(2k - \frac{2k}{(n+1)!}\right) - \left(2k - \frac{2k}{n!}\right)$$

$$= \frac{2k}{n!} - \frac{2k}{(n+1)!}$$

$$= \frac{2k}{n!}\left(1 - \frac{1}{n+1}\right)$$

$$= \frac{2k}{n!}\left(\frac{n}{n+1}\right)$$

$$= \frac{2kn}{(n+1)!}$$

(iii) $\sum_{r=1}^{n} u_r = S_n = 2k - \frac{2k}{(n+1)!}$

As $n \to \infty$, $\frac{1}{(n+1)!} \to 0$.

$$\therefore \ S_n = 2k - \frac{2k}{(n+1)!} \to 2k$$

Hence the series $\sum_{r=1}^{\infty} u_r$ converges.

2 **(i)**

$$2z+3w=20 \quad ...(1)$$
$$w-zw^*=6+22\text{i} \quad ...(2)$$

From (1), $z=\dfrac{20-3w}{2}$

Substitute into (2),

$$w-\left(\frac{20-3w}{2}\right)w^*=6+22\text{i}$$
$$2w-(20-3w)w^*=12+44\text{i}$$
$$2w-20w^*+3ww^*=12+44\text{i}$$

Let $w=a+b\text{i}$

$$2(a+b\text{i})-20(a-b\text{i})+3(a+b\text{i})(a-b\text{i})=12+44\text{i}$$
$$2a+2b\text{i}-20a+20b\text{i}+3(\text{a}^2+\text{b}^2)=12+44\text{i}$$
$$\left(3a^2-18a+3b^2\right)+(22b)\text{i}=12+44\text{i}$$

Comparing real and imaginary parts,

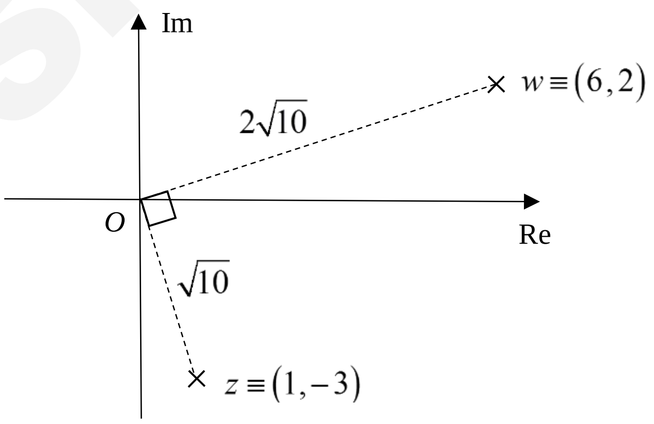
$$22b=44$$
$$\therefore\ b=2$$
$$3a^2-18a+3(2)^2=12$$
$$3a^2-18a+12=12$$
$$3a(a-6)=0$$

$a=0$ (rejected since $a\neq 0$), $a=6$

$\therefore w=6+2\text{i}$

$$z=\frac{20-3(6+2\text{i})}{2}$$
$$z=1-3\text{i}$$

(ii)



$\angle WOZ$ is $90°$

3

(i)

$\mathrm{f}(x) = \sqrt{3}\sin x + \cos x$

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\cos\alpha = \sqrt{3} \quad \ldots(1)$

$R\sin\alpha = 1 \quad \ldots(2)$

$(1)^2 + (2)^2$,

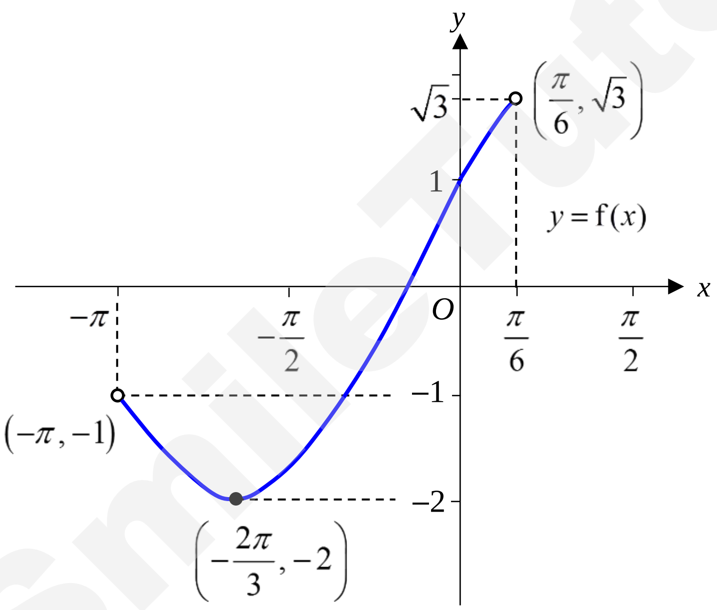
$\therefore\ R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$(1)/(2), \quad \tan\alpha = \dfrac{1}{\sqrt{3}}$,

$\therefore\ \alpha = \dfrac{\pi}{6}$

Hence $\mathrm{f}(x) = 2\sin\left(x + \dfrac{\pi}{6}\right)$

(ii)



When $y = -2$,

$2\sin(x + \dfrac{\pi}{6}) = -2$

$\sin(x + \dfrac{\pi}{6}) = -1$

$x + \dfrac{\pi}{6} = -\dfrac{\pi}{2} \Rightarrow x = -\dfrac{2\pi}{3}$

$\therefore$ turning point is $\left(-\dfrac{2\pi}{3}, -2\right)$.

$R_{\mathrm{f}} = [-2, \sqrt{3})$

(iii)

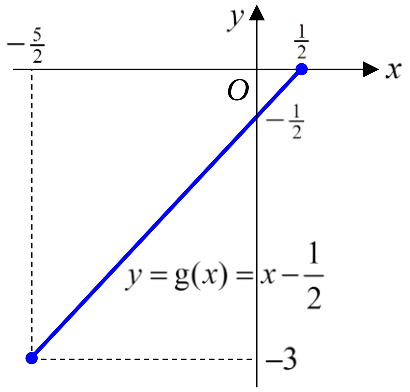
Since $-\frac{5}{2} \le x \le \frac{1}{2}$,

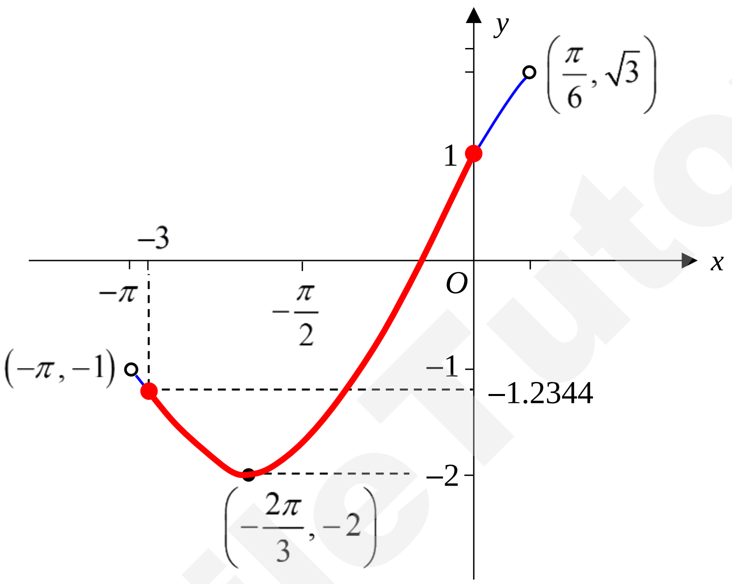
$g(x) = \frac{1}{2} + x - 1 = x - \frac{1}{2}$

$R_g = [-3, 0]$

$D_f = (-\pi, \frac{\pi}{6})$

Since $R_g \subset D_f$, fg exists





$$\underbrace{[-\tfrac{5}{2}, \tfrac{1}{2}]}_{D_g} \rightarrow \underbrace{[-3, 0]}_{R_g = \text{restricted } D_f} \rightarrow \underbrace{[-2, 1]}_{\text{restricted } R_f = R_{fg}}$$

$\therefore\ R_{fg} = [-2, 1]$

(iv)

From the graph,

largest domain for f $= [-\frac{2\pi}{3}, \frac{\pi}{6})$

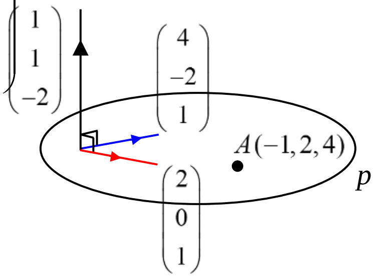
Let $y = 2\sin(x + \frac{\pi}{6})$

$x = \sin^{-1}\left(\frac{y}{2}\right) - \frac{\pi}{6}$

$f^{-1} : x \mapsto \sin^{-1}\left(\frac{x}{2}\right) - \frac{\pi}{6}\ ,\quad x \in \mathbb{R},\quad -2 \le x < \sqrt{3}.$

4

(i) $\begin{pmatrix}4\\-2\\1\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\-2\\4\end{pmatrix}=-2\begin{pmatrix}1\\1\\-2\end{pmatrix}$



$\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}-1\\2\\4\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=-7$

$\therefore$ Cartesian equation of p is $x+y-2z=-7$.

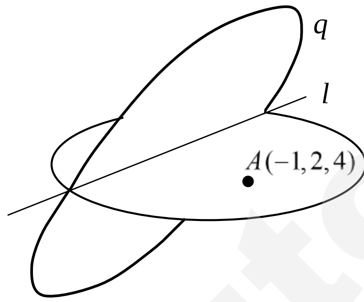
(ii) $x+y-2z=-7$

$x-2y+z=2$

Using GC,

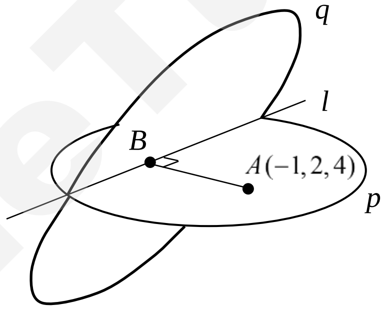
a vector equation of l is

$\underset{\sim}{r}=\begin{pmatrix}-4\\-3\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\1\end{pmatrix}, \alpha\in\mathbb{R}$.



(iii)

$\overrightarrow{OB}=\begin{pmatrix}-4+\alpha\\-3+\alpha\\\alpha\end{pmatrix}$



$\overrightarrow{AB}=\begin{pmatrix}-4+\alpha\\-3+\alpha\\\alpha\end{pmatrix}-\begin{pmatrix}-1\\2\\4\end{pmatrix}$

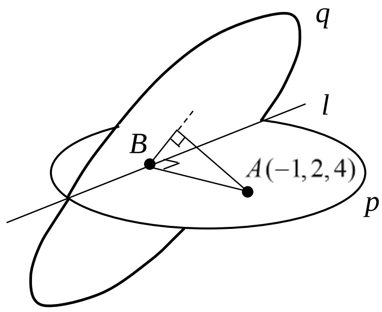
$=\begin{pmatrix}\alpha-3\\\alpha-5\\\alpha-4\end{pmatrix}$

$\begin{pmatrix}\alpha-3\\\alpha-5\\\alpha-4\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=0$

$\alpha-3+\alpha-5+\alpha-4=0 \quad\Rightarrow\quad \alpha=4$

$\therefore\ \overrightarrow{OB}=\begin{pmatrix}0\\1\\4\end{pmatrix}=\underset{\sim}{j}+4\underset{\sim}{k}$

(iv) Equation of q: $\underset{\sim}{r}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=2$



$\overrightarrow{AB}=\begin{pmatrix}4-3\\4-5\\4-4\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$

$\therefore$ length of projection of AB on q is

$\left|\overrightarrow{AB}\times\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}\right|=\frac{1}{\sqrt{6}}\left|\begin{pmatrix}1\\-1\\0\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}\right|=\frac{1}{\sqrt{6}}\left|\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right|=\frac{1}{\sqrt{6}}\left(\sqrt{3}\right)=\frac{\sqrt{2}}{2}$

5 **(i)**

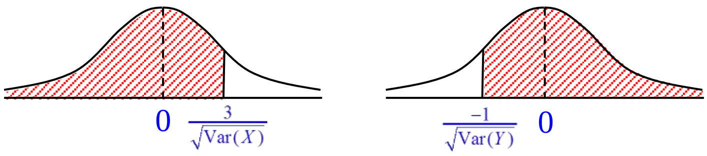
$X \sim \mathrm{N}\left(7, \mathrm{Var}(X)\right)$

$Y \sim \mathrm{N}\left(7, \mathrm{Var}(Y)\right)$

$\mathrm{P}(X < 10) = \mathrm{P}(Y > 6)$

$$\mathrm{P}\left(Z < \frac{10-7}{\sqrt{\mathrm{Var}(X)}}\right) = \mathrm{P}\left(Z > \frac{6-7}{\sqrt{\mathrm{Var}(Y)}}\right)$$

$$\mathrm{P}\left(Z < \frac{3}{\sqrt{\mathrm{Var}(X)}}\right) = \mathrm{P}\left(Z > \frac{-1}{\sqrt{\mathrm{Var}(Y)}}\right)$$



$$\therefore \quad \frac{3}{\sqrt{\mathrm{Var}(X)}} = -\left(\frac{-1}{\sqrt{\mathrm{Var}(Y)}}\right)$$

$$3\sqrt{\mathrm{Var}(Y)} = \sqrt{\mathrm{Var}(X)}$$

Hence $\mathrm{Var}(X) = 9\mathrm{Var}(Y)$ (shown)

(ii)

$\mathrm{Var}(X) = 9(1) = 9$

$X \sim \mathrm{N}\left(7, 9\right)$

$\therefore \ \mathrm{P}(X < 9) = 0.748$

6	**(i)** $\mathrm{P}(Y=2)$ $=2\mathrm{P}(X_1=2 \text{ and } X_2=0)+2\mathrm{P}(X_1=3 \text{ and } X_2=-1)$ $=2\mathrm{P}(X_1=2)\mathrm{P}(X_2=0)+2\mathrm{P}(X_1=3)\mathrm{P}(X_2=-1)$ $=2\left(\frac{1}{8}\right)\left(\frac{1}{2}\right)+2\left(\frac{1}{8}\right)\left(\frac{1}{4}\right)$ $=\frac{3}{16}$ **(ii)** $\mathrm{P}(\text{max of 2 scores}=-1)$ $=\mathrm{P}(X_1=-1)\mathrm{P}(X_2=-1)$ $=\left(\frac{1}{8}\right)^2$ $=\frac{1}{64}$ When sum of scores is prime, then $Y=2$, 3 or 5 . From (i), $\mathrm{P}(Y=2)=\frac{3}{16}$ $\mathrm{P}(Y=3)=2\mathrm{P}(X_1=0)\mathrm{P}(X_2=3)$ $=2\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$ $=\frac{1}{4}$ $\mathrm{P}(Y=5)=2\mathrm{P}(X_1=3)\mathrm{P}(X_2=2)$ $=2\left(\frac{1}{8}\right)\left(\frac{1}{4}\right)$ $=\frac{1}{16}$ $\therefore$ Expected gain $=16\left(\frac{1}{64}\right)+3\left(\frac{3}{16}+\frac{1}{4}+\frac{1}{16}\right)-2$ $=-0.25$ Hence expected gain is $-\$0.25$. [Or expected loss is $\$0.25$.] Alternatively, $\therefore$ Expected gain $=(16-2)\left(\frac{1}{64}\right)+(3-2)\left(\frac{3}{16}+\frac{1}{4}+\frac{1}{16}\right)-2\left[1-\left(\frac{1}{64}+\frac{3}{16}+\frac{1}{4}+\frac{1}{16}\right)\right]$ $=-0.25$ Hence expected gain is $-\$0.25$. [Or expected loss is $\$0.25$.]

7

(i)

No. of ways $= {}^{10}C_8\,(8-1)! = 226800$

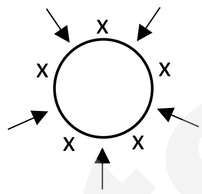
(ii)

<u>Method 1</u>: (method of complementation)

No. of ways $= \underbrace{{}^{7}C_5\,(8-1)!}_{\substack{\text{No. of ways}\\ \text{without restriction}\\ \text{[5 other beads, and}\\ \text{together with 3}\\ \text{particular beads}\\ \text{arranged in a circle]}}} - \underbrace{{}^{7}C_5\,3!(6-1)!}_{\substack{\text{No. of ways}\\ \text{with 3 particular beads}\\ \text{all together}\\ \text{[5 other beads, with 3}\\ \text{particular beads grouped}\\ \text{as 1 unit and 3! ways to}\\ \text{arrange among themselves,}\\ \text{and all 6 units arranged}\\ \text{in a circle]}}} = 90720$

<u>Method 2</u>: (method of slotting)

Case 1: (all not next to one another)



No. of ways

$= \underbrace{{}^{7}C_5\,(5-1)!}_{\substack{\text{5 other beads used}\\ \text{as seperators, and}\\ \text{arranged in a circle}}} \times \underbrace{{}^{5}C_3\,3!}_{\substack{\text{3 out of 5 slots}\\ \text{for 3 particular}\\ \text{beads, and 3! ways}\\ \text{to arrange among}\\ \text{themselves}}} = 30240$

Case 2: (2 together, 1 not)

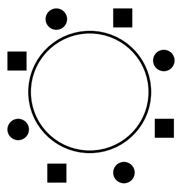
No. of ways $= \underbrace{{}^{7}C_5\,(5-1)!}_{\substack{\text{5 other beads used}\\ \text{as seperators, and}\\ \text{arranged in a circle}}} \times \underbrace{{}^{3}C_2\,2!}_{\substack{\text{2 of 3 particular}\\ \text{beads together,}\\ \text{and 2! ways to}\\ \text{arrange among}\\ \text{themselves}}} \times \underbrace{{}^{5}C_2\,2!}_{\substack{\text{2 out of 5 slots}\\ \text{for 3 particular}\\ \text{beads grouped}\\ \text{as 2 units (2}\\ \text{together, 1 not),}\\ \text{and 2! ways to}\\ \text{arrange among}\\ \text{themselves}}} = 60480$

$\therefore$ total no. of ways $= 30240 + 60480 = 90720$

(iii)

If spherical beads and cubic beads alternate, then there must be 4 spherical beads and 4 cubic beads.

No. of ways



$= \underbrace{{}^{5}C_4\,(4-1)!}_{\substack{\text{4 spherical beads and}\\ \text{arranged in a circle}}} \times \underbrace{{}^{5}C_4\,4!}_{\substack{\text{4 cubic beads,}\\ \text{and 4! ways to}\\ \text{arrange among}\\ \text{themselves}}} = 3600$

8

(i)

<u>Method 1</u>: (using permutations)

$$\text{Probability } = \frac{10\times 9\times 8\times 7}{10^4} = \frac{63}{125} \quad \text{[or 0.504]}$$

<u>Method 2</u>: (using probability)

$$\text{Probability } = \frac{10}{10}\times\frac{9}{10}\times\frac{8}{10}\times\frac{7}{10} = \frac{63}{125} \quad \text{[or 0.504]}$$

(ii)

<u>Method 1</u>: (using permutations)

$$\text{No. of ways}\ \underbrace{\overset{\text{thousands}}{\boxed{9}}\,\overset{\text{hundreds}}{\boxed{9}}\,\overset{\text{tens}}{\boxed{1}}}_{\text{3 ways to arrange digit same as last even digit}}\,\overset{\text{unit}}{\boxed{5}}$$

$$\text{Required probability } = \frac{[(9\times 9\times 1)\times 3]\times 5}{10^4}$$

$$= \frac{243}{2000} \quad \text{[or 0.1215]}$$

<u>Method 2</u>: (using permutations and combinations)

Case 1: The other 2 digits are different

$$\text{No. of ways}\ \underbrace{\overset{\text{thousands}\quad\text{hundreds}}{\boxed{\ {}^9C_2\ }}\,\overset{\text{tens}}{\boxed{1}}}_{\text{3! ways to arrange}}\,\overset{\text{unit}}{\boxed{5}}$$

$$\text{Probability } = \frac{[({}^9C_2\times 1)\times 3!]\times 5}{10^4} = \frac{27}{250} \quad \text{[or 0.108]}$$

Case 2: The other 2 digits are the same

$$\text{No. of ways}\ \underbrace{\overset{\text{thousands}\quad\text{hundreds}}{\boxed{\ {}^9C_1\ }}\,\overset{\text{tens}}{\boxed{1}}}_{\frac{3!}{2!}\text{ ways to arrange}}\,\overset{\text{unit}}{\boxed{5}}$$

$$\text{Probability } = \frac{[({}^9C_1\times 1)\times \frac{3!}{2!}]\times 5}{10^4} = \frac{27}{2000} \quad \text{[or 0.0135]}$$

$$\text{Required probability } = \frac{27}{250}+\frac{27}{2000} = \frac{243}{2000} \quad \text{[or 0.1215]}$$

<u>Method 3</u>: (using probability)

Case 1: The other 2 digits are different

Probability $=\frac{9}{10}\times\frac{8}{10}\times\frac{1}{10}\times\frac{5}{10}\times\frac{3!}{2!}=\frac{27}{250}$ [or 0.108]

Case 2: The other 2 digits are the same

Probability $=\frac{9}{10}\times\frac{1}{10}\times\frac{1}{10}\times\frac{5}{10}\times\frac{3!}{2!}=\frac{27}{2000}$ [or 0.0135]

Required probability $=\frac{27}{250}+\frac{27}{2000}=\frac{243}{2000}$ [or 0.1215]

(iii)

Let A be the event '4 different digits with 1st digit greater than 6'.
Let B be the event 'odd and even digits that alternate'.

<u>Method 1</u>: (using permutations)

Case 1: 1st digit is even, i.e. 8, and odd and even digits alternate

	thousands	hundreds	tens	unit
No. of ways	1	5	4	4

Probability $=\frac{1\times5\times4\times4}{10^4}=\frac{1}{125}$ [or 0.008]

Case 2: 1st digit is odd, i.e. 7 or 9, and odd and even digits alternate

	thousands	hundreds	tens	unit
No. of ways	2	5	4	4

Probability $=\frac{2\times5\times4\times4}{10^4}=\frac{2}{125}$ [or 0.016]

Hence $P(A\cap B)=\frac{1}{125}+\frac{2}{125}=\frac{3}{125}$ [or 0.024]

$$P(B)=P(\text{'odd,even,odd,even' or 'even,odd,even,odd'})$$
$$=\frac{2\times(5\times5\times5\times5)}{10^4}$$
$$=\frac{1}{8}$$ [or 0.125]

$$\therefore\ P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{3}{125}}{\frac{1}{8}}=\frac{24}{125}$$ [or 0.192]

<u>Method 2</u>: (using probability)

Case 1: 1st digit is even, i.e. 8, and odd and even digits alternate

Probability $=\frac{1}{10}\times\frac{5}{10}\times\frac{4}{10}\times\frac{4}{10}=0.008$

Case 2: 1st digit is odd, i.e. 7 or 9, and odd and even digits alternate

$$\text{Probability } = \frac{2}{10}\times\frac{5}{10}\times\frac{4}{10}\times\frac{4}{10} = 0.016$$

Hence $\text{P}\left(A\cap B\right) = \frac{1}{125}+\frac{2}{125} = \frac{3}{125}$ [or 0.024]

$$\text{P}\left(B\right) = \text{P}\left(\text{'odd,even,odd,even' or 'even,odd,even,odd'}\right)$$

$$= 2\times\left(\frac{5}{10}\times\frac{5}{10}\times\frac{5}{10}\times\frac{5}{10}\right)$$

$$= 0.125$$

$\therefore\ \text{P}\left(A\,|\,B\right) = \frac{\text{P}\left(A\cap B\right)}{\text{P}\left(B\right)} = \frac{\frac{3}{125}}{\frac{1}{8}} = \frac{24}{125}$ [or 0.192]

9

(i)

Assumptions

- The probability of a randomly chosen glass stone being chipped is constant.
- Whether a glass stone is chipped or not is independent of that of any other glass stones.

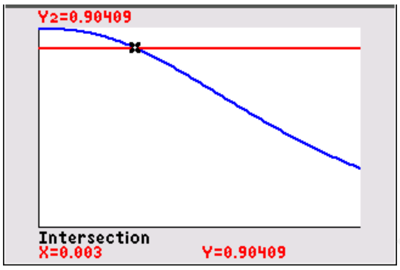
(ii)

$X \sim \mathrm{B}(361, p)$

$\mathrm{P}(X \le 2) = 0.90409$

Using GC,

$p = 0.00300$



(iii)

$P(X > 2) = 1 - P(X \le 2) = 1 - 0.90409 = 0.09591$

Let Y be number of boxes with more than 2 chipped glass stones, out of 20 boxes.

$$Y \sim \mathrm{B}(20, 0.09591)$$

$$\begin{aligned}\mathrm{P}(Y > 5) &= 1 - \mathrm{P}(Y \le 5) \\ &= 1 - 0.9907736392 \\ &= 0.0092263608 \\ &\approx 0.00923\end{aligned}$$

(iv)

Let A be the number of rejected batches, out of 50 batches.

$$A \sim \mathrm{B}(50, 0.0092263608)$$

$\mathrm{E}(A) = 50(0.0092264) = 0.46132$

$\mathrm{Var}(A) = 50(0.0092264)(1 - 0.0092264) = 0.45706$

Let $M_1 = A_1 + \ldots + A_{20}$

Since $n = 20$ is sufficiently large, by CLT,

$M_1 \sim \mathrm{N}(20 \times 0.46132,\ 20 \times 0.45706)$

$= \mathrm{N}(9.2264,\ 9.1412)$ approximately

Let $M_2 = A_{21} + \ldots + A_{52}$

Since $n = 32$ is sufficiently large, by CLT,

$M_2 \sim \mathrm{N}(32 \times 0.46132,\ 32 \times 0.45706)$

$= \mathrm{N}(14.76224,\ 14.62592)$ approximately

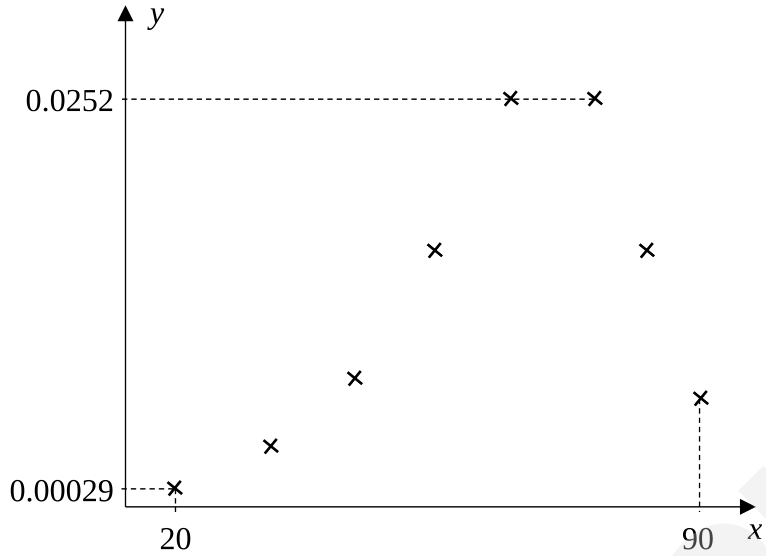
Let $T = 10M_1 + 20M_2$

Hence $T \sim \mathrm{N}\left(10(9.2264) + 20(14.76224),\ 10^2(9.1412) + 20^2(14.62592)\right)$

$= \mathrm{N}(387.5088, 6764.488)$ approximately

	$\therefore\ \mathrm{P}(T > 250) = 0.952729 \approx 0.953$

10 **(i)**



(ii)
The scatter diagram displays a curvilinear relationship which suggests the presence of a maximum point. Hence a linear model is inappropriate.

(iii)
$r = -0.9999984$ (7 decimal places)

(iv)
$m = 65$. Of the 3 negative r values, the r value corresponding to $m = 65$ is closest to -1 .

(v)
Using GC with $m = 65$,
$a \approx 0.0022230 \approx 0.00222$ (3 s.f.)
$b \approx -3.6269 \approx -3.63$ (3 s.f.)
$\therefore \ln y = -0.002223(x-65)^2 - 3.6269$
or $\ln y = -0.00222(x-65)^2 - 3.63$
When $x = 45$,
$y \approx 0.0109$ (3 s.f.)

Since $x = 45$ is within data range and $r = -0.9999984$ is very close to -1 , the prediction is reliable.

11 **(a)(i)**

Using GC,

Unbiased estimate of the population mean,

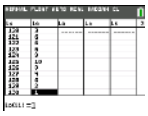
$\bar{x} = 124.4$ g

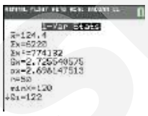
Unbiased estimate of the population variance,

$s^2 = 2.725540575^2$

$= 7.428571429$

$= 7.43 \quad (3 \text{ s.f.})$





(a)(ii)

Let μ g be the population mean mass of a box of blueberries.

$\mathrm{H}_0 : \mu = 125$

$\mathrm{H}_1 : \mu < 125$

Under H_0, test statistic

$$Z = \frac{\bar{X} - 125}{\sqrt{\frac{7.428571429}{50}}} \sim \mathrm{N}(0,1) \text{ approximately by CLT}$$

Level of significance: 10%

Critical region: Reject H_0 if p-value ≤ 0.1

Since p-value $= 0.0598 < 0.1$, we reject H_0 and conclude that at the 10% level of significance, there is sufficient evidence that Yummy Berries Farm has overstated its claim.
No assumptions about masses of boxes of blueberries are needed. Since $n = 50$ is sufficiently large, by Central Limit Theorem, the mean mass of boxes of raspberries will follows a normal distribution approximately.

(b)
Let μ_1 g be the population mean mass of a box of raspberries.

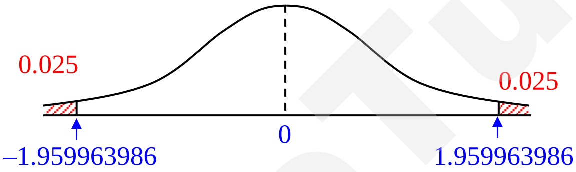
$H_0 : \mu_1 = 170$

$H_1 : \mu_1 \neq 170$

Under H_0, assuming n is large,

test statistic $Z = \dfrac{\overline{Y} - 170}{\frac{15}{\sqrt{n}}} \sim N(0,1)$ approximately by CLT

Level of significance: 5%
Critical region: Reject H_0 if p-value $\leq$ 0.05
i.e. Reject H_0 if z-value ≤ -1.959963986 or z-value ≥ 1.959963986



$$\frac{165-170}{\frac{15}{\sqrt{n}}} \leq -1.959963986 \quad \text{or} \quad \frac{165-170}{\frac{15}{\sqrt{n}}} \geq 1.959963986$$

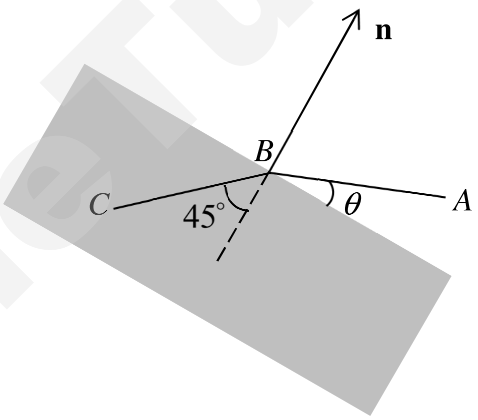
$$\sqrt{n} \geq 5.87989 \quad \text{or} \quad \sqrt{n} \leq -5.87989 \text{ (rejected)}$$

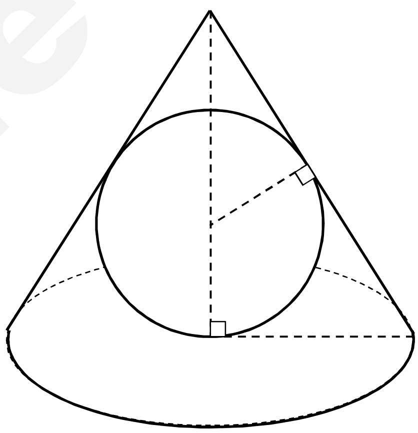
$\therefore\ n \geq 34.573$
Hence least n is 35.

1	Without using a graphic calculator, solve the inequality $\dfrac{4x^2+7x+1}{3x+1} \le x+2$. [3] Hence solve the inequality $\dfrac{4x+7\sqrt{x}+1}{3\sqrt{x}+1} \le \sqrt{x}+2$. [2]
2	**(i)** Find $\int n\cos^{-1}(nx)\,\mathrm{d}x$, where n is a positive constant. [3] **(ii)** Hence find the exact value of $\int_0^{\frac{1}{2n}} n\cos^{-1}(nx)\,\mathrm{d}x$. [2]
3	The vectors **p** and **q** are given by $\mathbf{p} = 2\mathbf{i}+\mathbf{j}+a\mathbf{k}$ and $\mathbf{q} = b\mathbf{i}+\mathbf{j}$, where a and b are non-zero constants. **(i)** Find $(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})$ in terms of a and b. [2] Given that the **i**- and **j**- components of the answer to part **(i)** are equal, find the value of b. [1] Use the value of b you have found to solve parts **(ii)** and **(iii)**. **(ii)** Given that the magnitude of $(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})$ is 80, find the possible exact values of a. [2] **(iii)** Given instead that $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular, find the exact value of $\lvert\mathbf{p}\rvert$. [3]
4	A graphic calculator is **not** to be used in answering this question. **(a)** The equation $w^3+pw^2+qw+30=0$, where p and q are real constants, has a root $w=2-\mathrm{i}$. Find the values of p and q, showing your working. [3] **(b)** The equation $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=0$ has a root $z=3+u\mathrm{i}$, where u is real constant. Find the value of u and hence find the second root of the equation in cartesian form, $a+b\mathrm{i}$, showing your working. [5]

5	A sequence $u_1,\ u_2,\ u_3,...$ is such that $$u_n = \frac{1}{2n^2(n-1)^2} \quad \text{and} \quad u_{n+1} = u_n - \frac{2}{n(n-1)^2(n+1)^2}\text{, for all } n \geq 2\ .$$ **(i)** Find $\displaystyle\sum_{n=2}^{N} \frac{2}{n(n-1)^2(n+1)^2}$. [3] **(ii)** Explain why $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n(n-1)^2(n+1)^2}$ is a convergent series, and state the value of the sum to infinity. [2] **(iii)** Using your answer in part **(i)**, find $\displaystyle\sum_{n=1}^{N} \frac{2N}{(n+1)n^2(n+2)^2}$. [2]
6	**(i)** The variables x and y are related by $$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + ky = 2 \quad \text{and} \quad y = 1 \text{ at } x = 0\text{,}$$ where k is a constant. Show that $(x+y)\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1+k)\dfrac{\mathrm{d}y}{\mathrm{d}x} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0$. [1] By further differentiation of this result, find the Maclaurin series for y, up to and including the term in x^3, giving the coefficients in terms of k. [4] **(ii)** Given that x is small, find the series expansion of $\mathrm{g}(x) = \dfrac{1}{\sin^2\left(2x + \dfrac{\pi}{2}\right)}$ in ascending powers of x, up to and including the term in x^2. If the coefficient of x^2 in the expansion of $\mathrm{g}(x)$ is equal to twice the coefficient of x^2 in the Maclaurin series for y found in part **(i)**, find the value of k. [4]

7	A population of a certain organism grows from an initial size of 5. After 5 days, the size of the population is 20, and after t days, the size of the population is M. The rate of growth of the population is modelled as being proportional to $\left(100^2 - M^2\right)$. **(i)** Write down a differential equation modelling the population growth and find M in terms of t. [6] **(ii)** Find the size of the population after 15 days, giving your answer correct to the nearest whole number. [2] **(iii)** Find the least number of days required for the population to exceed 80. [2]
8	It is given that $\mathrm{f}(x) = \begin{cases} 2x-1 & 0 \le x \le 2, \\ 2-(x-3)^3 & 2 < x \le 4, \\ 1 & \text{otherwise.} \end{cases}$ Sketch, on separate diagrams, for $0 \le x \le 8$, the graphs of **(i)** $y = \mathrm{f}(x)$ and state the range of f, [5] **(ii)** $y = \dfrac{1}{\mathrm{f}(x)}$. [4] In each graph, indicate clearly the coordinates of the end points, points of intersection with the axes and stationary point, if any. State clearly the equation of any asymptote. **(iii)** Deduce the value of $\int_{-6}^{-4} \mathrm{f}(-x)\,\mathrm{d}x$. [1]
9	Given that $\mathrm{f}(x) = \sin 2x + \cos 2x$, express $\mathrm{f}(x)$ as $R\sin(2x+\alpha)$, where $R > 0$, $0 < \alpha < \dfrac{\pi}{2}$ and R and α are constants to be found. [2] **(i)** Describe a sequence of transformations involved that transformed $y = \sin x$ to $y = \mathrm{f}(x)$. [3]

	(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $0 \le x \le \frac{3\pi}{8}$, indicating clearly the exact coordinates of the maximum point and the end points of the graph. [3]
	(iii) The region bounded by the curve $y = \mathrm{f}(x)$, the line $x = \frac{\pi}{8}$ and both axes is rotated about the y-axis through 2π radians. Find the volume of the solid of revolution correct to 4 decimal places. [4]
10	When a light ray passes from air to glass, it is deflected through an angle. The light ray ABC starts at point $A(1, 2, 2)$ and enters a glass object at point $B(0, 0, 2)$. The surface of the glass object is a plane with normal vector $\mathbf{n}$. The diagram shows a cross-section of the glass object in the plane of the light ray and $\mathbf{n}$.  **(i)** Find a vector equation of the line AB. [1] The surface of the glass object is a plane with equation $x + z = 2$. AB makes an acute angle θ with the plane. **(ii)** Calculate the value of θ, giving your answer in degrees. [2] The line BC makes an angle of 45° with the normal to the plane, and BC is parallel to the unit vector $\begin{pmatrix} -\frac{2}{3} \\ p \\ q \end{pmatrix}$.

	(iii) By considering a vector perpendicular to the plane containing the light ray and **n**, or otherwise, find the values of p and q. [4] The light ray leaves the glass object through a plane with equation $3x+3z=-4$. **(iv)** Find the exact thickness of the glass object, taking one unit as one cm. [2] **(v)** Find the exact coordinates of the point at which the light ray leaves the glass object. [3]
11	[It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$ and the volume and surface area of a sphere of radius r are $\frac{4}{3}\pi r^3$ and $4\pi r^2$ respectively.] In a distant Northern kingdom of Drivenbell, Elsanna builds a spherical snowball with radius 3 m. The snowball is inscribed in a right conical container of base radius r m and height h m. The container is specially designed to allow the snowball to remain intact with fixed radius 3 m (see diagram).  **(i)** By considering the slant height of the cone, show that $r=\dfrac{3h}{\sqrt{\left(h^2-6h\right)}}$. [3] **(ii)** Use differentiation to find the values of h and r that give a minimum volume for the container. Find the value of the minimum volume. [6]

	The snowball is being removed from the container and it starts to melt under room temperature. **(iii)** Assuming that the snowball remains spherical as it melts, find the rate of decrease of its volume at the instant when the radius of the sphere is 2.5 m, given that the surface area is decreasing at 0.75 m^2 per minute at this instant. [5]

ANNEX B

IJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Equations and Inequalities	$x \le -1$ or $-\frac{1}{3} < x \le 1$; $0 \le x \le 1$
2	Integration techniques	(i) $(nx)\cos^{-1}(nx) - \sqrt{(1-n^2x^2)} + C$ (ii) $\frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$
3	Vectors	(i) $20\begin{pmatrix} -a \\ ab \\ 2-b \end{pmatrix}$ $b = -1$ (ii) $\pm\frac{\sqrt{14}}{2}$ (iii) $\frac{5\sqrt{2}}{2}$
4	Complex numbers	(a) $p=2$, $q=-19$ (b) $u = -5$, $z = 2 + 3\mathrm{i}$
5	Sigma Notation and Method of Difference	(i) $\frac{1}{8} - \frac{1}{2N^2(N+1)^2}$ (ii) $\frac{1}{8}$ (iii) $\frac{N}{8}\left[1 - \frac{4}{(N+1)^2(N+2)^2}\right]$
6	Maclaurin series	(i) $y = 1 + (2-k)x + \left(\frac{3k-6}{2}\right)x^2 + \left(k^2 - 6k + 8\right)x^3 + \ldots$ (ii) $1 + 4x^2 + \ldots$; $k = \frac{10}{3}$
7	Differential Equations	(i) $\frac{\mathrm{d}M}{\mathrm{d}t} = k\left(100^2 - M^2\right)$, $k > 0$

		$M=\dfrac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1}$ (ii) $M\approx 47$ (iii) 35
8	Graphs and Transformation	(i) $R_f=[-1,3]$ (iii) 2
9	Application of Integration	$\mathrm{f}(x)=\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)$ (i) A: A translation of $\frac{\pi}{4}$ units in the negative *x*-direction B: A scaling/stretch with scale factor $\frac{1}{2}$ parallel to the *x*-axis. C: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the *y*-axis. (iii) 0.6506
10	Vectors	(i) $\underset{\sim}{r}=\begin{pmatrix}1\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}$ (ii) $\theta=18.4^\circ$ (iii) $p=-\frac{2}{3}$; $q=-\frac{1}{3}$ (iv) $\frac{5\sqrt{2}}{3}$ cm (v) $\left(-\frac{20}{9},-\frac{20}{9},\frac{8}{9}\right)$
11	Differentiation & Applications	(ii) $h=12$; $r=\frac{6}{\sqrt{2}}$; $V=72\pi$ m^3 (iii) 0.9375 m^3 per minute

1

$$\frac{4x^2+7x+1}{3x+1} \le x+2$$

$$\frac{4x^2+7x+1-(x+2)(3x+1)}{3x+1} \le 0$$

$$\frac{4x^2+7x+1-(3x^2+x+6x+2)}{3x+1} \le 0$$

$$\frac{x^2-1}{3x+1} \le 0$$

$$\frac{(x-1)(x+1)}{3x+1} \le 0$$

$$\begin{array}{ccccccc} - & & + & & - & & + \\ \hline & -1 & & -\frac{1}{3} & & 1 & \end{array}$$

$$\therefore \quad x \le -1 \quad \text{or} \quad -\frac{1}{3} < x \le 1$$

$$\frac{4x+7\sqrt{x}+1}{3\sqrt{x}+1} \le \sqrt{x}+2$$

Replace x with $\sqrt{x}$,

$$\therefore \sqrt{x} \le -1 \qquad \text{or} \qquad -\frac{1}{3} < \sqrt{x} \le 1$$

(rejected as $\sqrt{x} \ge 0$)

Since $\sqrt{x} \ge 0$,

$$-\frac{1}{3} < \sqrt{x} \le 1 \quad \Rightarrow \quad 0 \le \sqrt{x} \le 1$$

$$0 \le x \le 1$$

2

(i)

$$\int n\cos^{-1}(nx)\,\mathrm{d}x$$

$$=(nx)\cos^{-1}(nx)-\int (nx)\left(-\frac{n}{\sqrt{1-(nx)^2}}\right)\mathrm{d}x$$

$$=(nx)\cos^{-1}(nx)-\frac{1}{2}\int \left(-2n^2x\right)\left(1-n^2x^2\right)^{-1/2}\,\mathrm{d}x$$

$$=(nx)\cos^{-1}(nx)-\frac{1}{2}\times\frac{\left(1-n^2x^2\right)^{1/2}}{\frac{1}{2}}+C$$

$$=(nx)\cos^{-1}(nx)-\sqrt{\left(1-n^2x^2\right)}+C$$

	(ii) $\int_0^{\frac{1}{2n}} n\cos^{-1}(nx)\,\mathrm{d}x$ $=\left[(nx)\cos^{-1}(nx)-\sqrt{(1-n^2x^2)}\ \right]_0^{\frac{1}{2n}}$ $=\left[\frac{1}{2}\cos^{-1}\frac{1}{2}-\sqrt{1-\frac{1}{4}}\right]-(0-1)$ $=\frac{\pi}{6}-\frac{\sqrt{3}}{2}+1 \quad or \quad \frac{\pi}{6}+\frac{2-\sqrt{3}}{2}$
3	(i) $(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})=4\mathbf{p}\times\mathbf{p}+10\mathbf{p}\times\mathbf{q}-10\mathbf{q}\times\mathbf{p}-25\mathbf{q}\times\mathbf{q}$ $=20\mathbf{p}\times\mathbf{q}$ $=20\begin{pmatrix}2\\1\\a\end{pmatrix}\times\begin{pmatrix}b\\1\\0\end{pmatrix}$ $=20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}$ Alternative: $(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})=\left(2\begin{pmatrix}2\\1\\a\end{pmatrix}-5\begin{pmatrix}b\\1\\0\end{pmatrix}\right)\times\left(2\begin{pmatrix}2\\1\\a\end{pmatrix}+5\begin{pmatrix}b\\1\\0\end{pmatrix}\right)$ $=\begin{pmatrix}4-5b\\-3\\2a\end{pmatrix}\times\begin{pmatrix}4+5b\\7\\2a\end{pmatrix}$ $=\begin{pmatrix}-6a-14a\\-(8a-10ab-8a-10ab)\\28-35b+12+15b\end{pmatrix}$ $=\begin{pmatrix}-20a\\20ab\\40-20b\end{pmatrix}=20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}$ Given that the **i**- and **j**- components of the vector $20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}$ are equal, $-a=ab$ $ab+a=0$ $a(b+1)=0$ Since $a\neq 0$, thus $b=-1$

(ii)

$$|(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})|=80$$

$$\left|20\begin{pmatrix}-a\\ ab\\ 2-b\end{pmatrix}\right|=80$$

$$\left|\begin{pmatrix}-a\\ -a\\ 2+1\end{pmatrix}\right|=4$$

$$\sqrt{2a^2+9}=4$$

$$2a^2+9=16$$

$$a^2=\frac{7}{2}$$

$$a=\pm\sqrt{\frac{7}{2}}\quad or\quad \pm\frac{\sqrt{14}}{2}$$

(iii)

Since $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular,

$$(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q})=0$$

$$4|\mathbf{p}|^2-25|\mathbf{q}|^2=0$$

$$|\mathbf{p}|^2=\frac{25}{4}|\mathbf{q}|^2$$

$$=\frac{25}{4}\left((-1)^2+1^2\right)$$

$$=\frac{25}{2}$$

$$|\mathbf{p}|=\frac{5\sqrt{2}}{2}$$

Alternative:

$$(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q})=\begin{pmatrix}4+5\\ -3\\ 2a\end{pmatrix}\cdot\begin{pmatrix}4-5\\ 7\\ 2a\end{pmatrix}$$

$$=16-25-21+4a^2$$

$$=4a^2-30$$

Since $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular,

$$(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q})=0$$

$$4a^2-30=0$$

$$a^2=\frac{15}{2}$$

$$|\mathbf{p}|=\sqrt{2^2+1+a^2}=\sqrt{5+\frac{15}{2}}=\sqrt{\frac{25}{2}}=\frac{5\sqrt{2}}{2}$$

4	(a) **Method 1** Since the coefficients are real, $w=2+\mathrm{i}$ is another root of the equation. $(w-2+\mathrm{i})(w-2-\mathrm{i})=(w-2)^2-(\mathrm{i})^2$ $=w^2-4w+4+1$ $=w^2-4w+5$ $w^3+pw^2+qw+30=0$ $(w^2-4w+5)(w+6)=0$ (By inspection) Comparing coefficients of w^2, $p=6-4=2$ Comparing coefficients of w, $q=-24+5=-19$ **Method 2** Substitute $w=2-\mathrm{i}$ (or $w=2+\mathrm{i}$) into the given eqn, $(2-\mathrm{i})^3+p(2-\mathrm{i})^2+q(2-\mathrm{i})+30=0$ $(3-4\mathrm{i})(2-\mathrm{i})+p(3-4\mathrm{i})+q(2-\mathrm{i})+30=0$ $(6-3\mathrm{i}-8\mathrm{i}-4)+p(3-4\mathrm{i})+q(2-\mathrm{i})+30=0$ $(32+3p+2q)+(-11-4p-q)\mathrm{i}=0$ Comparing the real parts, $3p+2q=-32$ --- (1) Comparing the imaginary parts, $4p+q=-11$ ---- (2) (1) – (2) × 2: $3p-8p=-32+11\times 2$ $-5p=-10$ $p=2$ From (2): $q=-11-4\times 2=-19$ $\therefore p=2$, $q=-19$ (b) Substitute $z=3+u\mathrm{i}$ into the given eqn, $(3+u\mathrm{i})^2+(-5+2\mathrm{i})(3+u\mathrm{i})+(21-\mathrm{i})=0$ $9+6u\mathrm{i}-u^2-15-5u\mathrm{i}+6\mathrm{i}-2u+21-\mathrm{i}=0$ $(15-2u-u^2)+(u+5)\mathrm{i}=0$ Compare imaginary coefficient: $u+5=0$ $u=-5$ $\therefore z=3-5\mathrm{i}$ [Note: if using $15-2u-u^2=0$, need to reject $u=3$] **Method 1** Let the other root be *w*. $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=(z-3+5\mathrm{i})(z-w)$ Comparing coefficients of *z*, $-5+2\mathrm{i}=-w-3+5\mathrm{i}$ $w=2+3\mathrm{i}$

Method 2

Let the other solution be $a+bi$,

$z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})$

$=(z-(3-5\mathrm{i}))(z-(a+b\mathrm{i}))$

$=z^2-(a+b\mathrm{i})z-(3-5\mathrm{i})z+(3-5\mathrm{i})(a+b\mathrm{i})$

$=z^2-\left[a+3+(b-5)\mathrm{i}\right]z+(3-5\mathrm{i})(a+b\mathrm{i})$

Compare the z term: $-(a+3)=-5 \quad \Rightarrow a=2$

$-(b-5)=2 \quad \Rightarrow b=3$

$\therefore z=2+3\mathrm{i}$ is another root.

5

(i)

$$\sum_{n=2}^{N}\frac{2}{n(n-1)^2(n+1)^2}$$

$$=\sum_{n=2}^{N}\left[u_n-u_{n+1}\right]$$

$$=\left[\begin{array}{l}(u_2-\cancel{u_3})\\+(\cancel{u_3}-\cancel{u_4})\\+(\cancel{u_4}-\cancel{u_5})\\ \ldots\ldots\\ \ldots\ldots\\+(\cancel{u_{N-1}}-\cancel{u_N})\\+(\cancel{u_N}-u_{N+1})\end{array}\right]$$

$$=u_2-u_{N+1}$$

$$=\frac{1}{2(2^2)(2-1)^2}-\frac{1}{2(N+1)^2((N-1)+1)^2}$$

$$=\frac{1}{8}-\frac{1}{2N^2(N+1)^2}$$

(ii)

As $N\to\infty$, $\dfrac{1}{2N^2(N+1)^2}\to 0$

$\displaystyle\sum_{n=2}^{\infty}\frac{2}{n(n-1)^2(n+1)^2}\to\frac{1}{8}$ which is a constant, hence it is a convergent series.

$$\sum_{n=2}^{\infty}\frac{2}{n(n-1)^2(n+1)^2}=\frac{1}{8}-0$$

$$=\frac{1}{8}$$

	(iii) **Method 1** $$\sum_{n=1}^{N}\frac{2N}{(n+1)n^2(n+2)^2}=N\sum_{n=1}^{N}\frac{2}{(n+1)n^2(n+2)^2}$$ $$=N\sum_{n=2}^{N+1}\frac{2}{(n)(n-1)^2(n+1)^2}$$ $$=N\left[\frac{1}{8}-\frac{1}{2(N+1)^2(N+2)^2}\right]$$ $$=\frac{N}{8}\left[1-\frac{4}{(N+1)^2(N+2)^2}\right]$$ **Method 2** By listing the terms $$\sum_{n=2}^{N}\frac{2}{n(n-1)^2(n+1)^2}$$ $$=\frac{2}{2(1)^2(3)^2}+\frac{2}{3(2)^2(4)^2}+\cdots+\frac{2}{N(N-1)^2(N+1)^2}$$ $$\sum_{n=1}^{N}\frac{2N}{(n+1)n^2(n+2)^2}$$ $$=N\left[\frac{2}{2(1)^2(3)^2}+\frac{2}{3(2)^2(4)^2}+\cdots+\frac{2}{(N+1)(N)^2(N+2)^2}\right]$$ $$=N\sum_{n=2}^{N+1}\frac{2}{n(n-1)^2(n+1)^2}$$ $$=N\left[\frac{1}{8}-\frac{1}{2(N+1)^2(N+2)^2}\right]$$ $$=\frac{N}{8}\left[1-\frac{4}{(N+1)^2(N+2)^2}\right]$$
6	(i) $$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+ky=2 \qquad \cdots(1)$$ Differentiating (1) w.r.t. x: $$(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+k\frac{\mathrm{d}y}{\mathrm{d}x}=0$$ $$(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+k)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0 \qquad \cdots(2)$$

	Differentiating (2) w.r.t. x: $(x+y)\frac{d^3y}{dx^3}+\left(1+\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+(1+k)\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=0$ $(x+y)\frac{d^3y}{dx^3}+\left(2+3\frac{dy}{dx}+k\right)\frac{d^2y}{dx^2}=0$ $x=0,\quad y=1:\ \frac{dy}{dx}=2-k$ $\frac{d^2y}{dx^2}=3k-6$ $\frac{d^3y}{dx^3}=6k^2-36k+48=6\left(k^2-6k+8\right)$ $\therefore y\ =1+(2-k)x+\left(\frac{3k-6}{2!}\right)x^2+\left(\frac{6\left(k^2-6k+8\right)}{3!}\right)x^3+...$ $=1+(2-k)x+\left(\frac{3k-6}{2}\right)x^2+\left(k^2-6k+8\right)x^3+...$ (ii) $\sin\left(2x+\frac{\pi}{2}\right)=\sin 2x\cos\frac{\pi}{2}+\cos 2x\sin\frac{\pi}{2}=\cos 2x$ $\frac{1}{\sin^2\left(x+\frac{\pi}{2}\right)}=\frac{1}{\cos^2 2x}$ $\approx\left(1-\frac{(2x)^2}{2}\right)^{-2}$ $=\left(1-2x^2\right)^{-2}$ $=1+4x^2+...$ $4=2\left(\frac{3k-6}{2}\right)$ $k=\frac{10}{3}$
7	(i) $\frac{dM}{dt}=k\left(100^2-M^2\right)$, $k>0$ Since $\frac{dM}{dt}>0$ and $M>0,\Rightarrow\left(100^2-M^2\right)>0$ and $0<M<100$ $\int\frac{1}{\left(100^2-M^2\right)}\,dM=\int k\ dt$ $\frac{1}{200}\ln\left(\frac{100+M}{100-M}\right)=kt+C$

$$\ln\left(\frac{100+M}{100-M}\right)=200kt+C'$$

$$\frac{100+M}{100-M}=Ae^{200kt}\quad \text{, where } A=e^{C'}$$

When $t=0,\quad M=5\quad\Rightarrow\quad A=\frac{105}{95}=\frac{21}{19}$

When $t=5,\quad M=20\quad\Rightarrow\quad \frac{3}{2}=\frac{21}{19}e^{1000k}$

$$e^{1000k}=\frac{19}{14}\quad or\quad 200k=\frac{1}{5}\ln\left(\frac{19}{14}\right)$$

Thus $\dfrac{100+M}{100-M}=\dfrac{21}{19}\left(e^{1000k}\right)^{\frac{t}{5}}=\dfrac{21}{19}\left(\dfrac{19}{14}\right)^{\frac{t}{5}}$

$$100+M=\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}(100-M)$$

$$M\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1\right]=100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]$$

$$M=\frac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1}\quad OR\quad \frac{100\left[21\left(\frac{19}{14}\right)^{\frac{t}{5}}-19\right]}{21\left(\frac{19}{14}\right)^{\frac{t}{5}}+19}\quad OR\quad \frac{100\left[\left(\frac{19}{14}\right)^{\frac{t}{5}}-\frac{19}{21}\right]}{\left(\frac{19}{14}\right)^{\frac{t}{5}}+\frac{19}{21}}$$

(ii)

When $t=15$, $M=\dfrac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{3}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{3}+1}=46.847$

$M\approx 47$ (nearest whole number)

(iii)

<u>Method 1:</u> Graphical Method

Sketch the graphs of M=f(t) and M=80

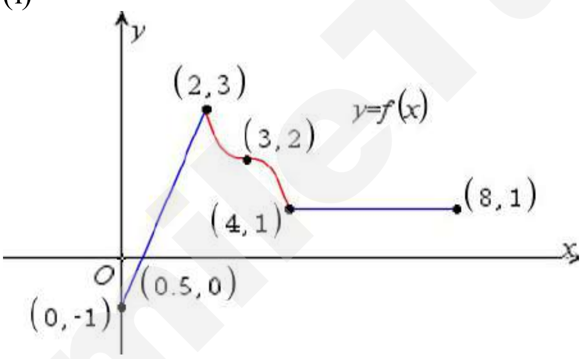
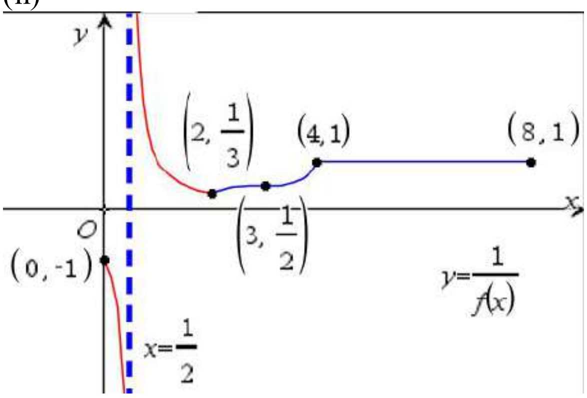
From the graph, when $t>34.336397,\quad M>80$

Least number of days required is 35.

<u>Method 2:</u> Use GC table

$$\left.\begin{array}{l}\text{When } t=34,\ M=79.627<80\\ \text{When } t=35,\ M=80.718>80\\ \text{When } t=36,\ M=81.756>80\end{array}\right\}\quad \boxed{\Rightarrow\ t\geq 35}$$

Thus least number of days required is 35.

	Method 3: $$\frac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1}>80$$ $$\frac{5}{4}\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]>\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1$$ $$\frac{1}{4}\cdot\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}>\frac{9}{4}$$ $$\left(\frac{19}{14}\right)^{\frac{t}{5}}>\frac{57}{7}$$ $$t>\frac{5\ln\left(\frac{57}{7}\right)}{\ln\left(\frac{19}{14}\right)}=34.336397$$ Least number of days required is 35.
8	(i)  Range of f is $[-1, 3]$ or $R_f=[-1,3]$ or $R_f=\{y:-1\le y\le 3\}$ (ii) 

(iii)

$$\int_{-6}^{-4} \mathrm{f}(-x)\,\mathrm{d}x = \int_{4}^{6} \mathrm{f}(x)\,\mathrm{d}x$$
$$= \text{area of rectangle}$$
$$= 2$$

9

$\mathrm{f}(x) = \sin 2x + \cos 2x$

$R = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\tan\alpha = 1 \quad \Rightarrow \quad \alpha = \dfrac{\pi}{4}$

$\mathrm{f}(x) = \sin 2x + \cos 2x = \sqrt{2}\sin\left(2x + \dfrac{\pi}{4}\right)$

(i)

Transforming $y = \sin x$ to $y = \sqrt{2}\sin\left(2x + \dfrac{\pi}{4}\right)$

Sequence of Transformation:

Either

A: A translation of $\dfrac{\pi}{4}$ units in the negative x-direction

B: A scaling/stretch with scale factor $\dfrac{1}{2}$ parallel to the x-axis.

C: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the y-axis.

Acceptable sequence: ABC, ACB, CAB.

OR $y = \sqrt{2}\sin\left[2\left(x + \dfrac{\pi}{8}\right)\right]$

D: A scaling/stretch with scale factor $\dfrac{1}{2}$ parallel to the x-axis.

E: A translation of $\dfrac{\pi}{8}$ units in the negative x-direction.

F: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the y-axis.

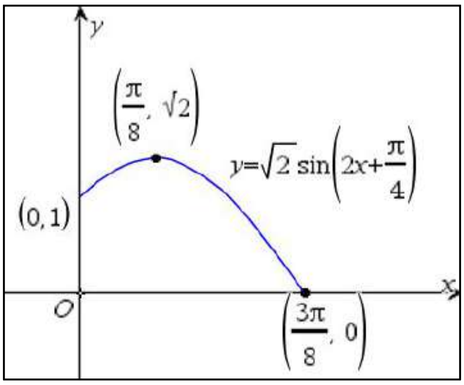
Acceptable sequence: DEF, DFE, FDE

(ii)

Max point occurs when $\sin\left(2x + \dfrac{\pi}{4}\right) = 1$

$\Rightarrow \quad \left(2x + \dfrac{\pi}{4}\right) = \dfrac{\pi}{2}$

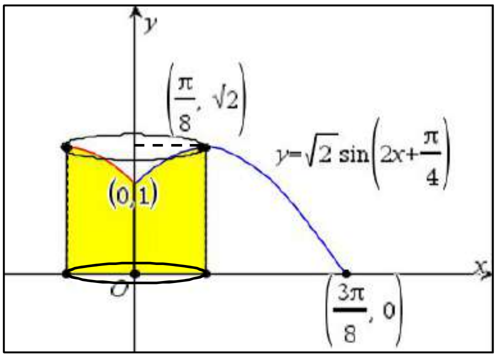
$\Rightarrow \quad x = \dfrac{\pi}{8},\ y = \sqrt{2}$



(iii)

$y = \sqrt{2}\sin\left(2x + \frac{\pi}{4}\right)$

The curve is one-one for $0 \le x \le \frac{\pi}{8}$, thus inverse function exists.



$\sin\left(2x + \frac{\pi}{4}\right) = \frac{y}{\sqrt{2}}$

$2x + \frac{\pi}{4} = \sin^{-1}\frac{y}{\sqrt{2}}$

$x = \frac{1}{2}\left[\sin^{-1}\left(\frac{y}{\sqrt{2}}\right) - \frac{\pi}{4}\right]$

Volume = Volume of cylinder - $\pi\int_1^{\sqrt{2}} x^2 \, \mathrm{d}y$

$= \pi\left(\frac{\pi}{8}\right)^2\sqrt{2} - \pi\int_1^{\sqrt{2}} \frac{1}{4}\left[\sin^{-1}\left(\frac{y}{\sqrt{2}}\right) - \frac{\pi}{4}\right]^2 \mathrm{d}y$

$= 0.6506458$

≈ 0.6506 (4 d.p.)

10

(i)

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$

$l_{AB} : \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ *or* $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$ or equivalent

(ii)

$\sin\theta = \frac{\left|\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right|}{\sqrt{5}\sqrt{2}} = \frac{1}{\sqrt{10}}$

$\theta = 18.4^\circ$

(iii)

Let $\underset{\sim}{m}$ be a vector perpendicular to the plane containing the light ray and $\underset{\sim}{n}$.

$\underset{\sim}{m} = \underset{\sim}{n} \times \overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\begin{pmatrix} -\frac{2}{3} \\ p \\ q \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{2}} \Rightarrow \frac{2}{3} - q = 1$$

$$q = -\frac{1}{3}$$

$$\begin{pmatrix} -\frac{2}{3} \\ p \\ q \end{pmatrix} \perp \underset{\sim}{m} \Rightarrow \begin{pmatrix} -\frac{2}{3} \\ p \\ -\frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0$$

$$-\frac{4}{3} - p + \frac{2}{3} = 0 \Rightarrow p = -\frac{2}{3}$$

(iv)
Glass upper surface is $x + z = 2$

Glass bottom surface is $3x + 3z = -4 \Rightarrow x + z = -\frac{4}{3}$

Distance between two planes $= \frac{\left| 2 - \left(-\frac{4}{3} \right) \right|}{\sqrt{2}} = \frac{10}{3\sqrt{2}} = \frac{5\sqrt{2}}{3}$

Thickness of the glass object is $\frac{5\sqrt{2}}{3}$ cm

(v)
Let the point at which the light ray leaves the glass object be F.

Method 1:

$$l_{BF} : \underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad \text{or} \quad \underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

At F,

$$\left[\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = -4$$

$$6 + \mu(6+3) = -4$$

$$\mu = -\frac{10}{9}$$

OR

$$\boxed{\begin{gathered} \left[\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \right] \cdot \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = -4 \\ 6 + \mu(-2-1) = -4 \\ \mu = \frac{10}{3} \end{gathered}}$$

The coordinates of F are

$$\left(-\frac{20}{9}, -\frac{20}{9}, \frac{8}{9} \right)$$

Method 2:

$$\cos 45^\circ = \frac{\frac{5\sqrt{2}}{3}}{BF} \Rightarrow \left| \overrightarrow{BF} \right| = \frac{5\sqrt{2}}{3} \times \sqrt{2} = \frac{10}{3}$$

(or using Pythagoras' theorem)

	$\vec{BF}=\frac{10}{3}\begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}=-\frac{10}{9}\begin{pmatrix}2\\2\\1\end{pmatrix}$ $\vec{OF}=-\frac{10}{9}\begin{pmatrix}2\\2\\1\end{pmatrix}+\begin{pmatrix}0\\0\\2\end{pmatrix}=\frac{1}{9}\begin{pmatrix}-20\\-20\\8\end{pmatrix}$ The coordinates of F are $\left(-\frac{20}{9},-\frac{20}{9},\frac{8}{9}\right)$
11	(i) Let l be the slant height of the cone. $l^2=h^2+r^2 \quad ----(1)$ Using similar triangles, $\frac{h-3}{l}=\frac{3}{r}$ $l=\frac{rh-3r}{3} \quad ----(2)$ Equating (1) and (2), $\left(\frac{rh-3r}{3}\right)^2=h^2+r^2 \quad ----(*)$ $r^2h^2-6r^2h+9r^2=9h^2+9r^2$ $r^2(h^2-6h)=9h^2$ $\therefore r=\frac{3h}{\sqrt{h^2-6h}} \qquad (\text{Since } r>0)$ (ii) Volume of cone, $V=\frac{1}{3}\pi r^2h$ $=\frac{1}{3}\pi\left(\frac{3h}{\sqrt{h^2-6h}}\right)^2h$ $=\frac{3\pi h^3}{h^2-6h}$ $=\frac{3\pi h^2}{h-6}$ $\frac{dV}{dh}=\frac{6\pi h(h-6)-3\pi h^2}{(h-6)^2}$ $=\frac{3\pi h^2-36\pi h}{(h-6)^2}$

	$\frac{dV}{dh}=0 \quad \Rightarrow \quad 3\pi h^2-36\pi h=0$ $h(h-12)=0$ $h=12$ or $h=0$ (reject $\because h>0$)

h	12^-	12	12^+
Sign of $\frac{dV}{dh}$	− ve	0	+ ve
Tangent	\	—	/

Thus, V is a minimum when $h=12$

When $h=12$,

$$r=\frac{3(12)}{\sqrt{(12)^2-6(12)}}=\frac{6}{\sqrt{2}} \qquad (\approx 4.2426)$$

$$V=\frac{3\pi(12)^2}{12-6}=72\pi \qquad (\approx 226.195)$$

(iii)

Let R be the radius of the snowball

$$S=4\pi R^2 \quad \Rightarrow \quad \frac{dS}{dt}=8\pi R\frac{dR}{dt}$$

$$V=\frac{4}{3}\pi R^3 \quad \Rightarrow \quad \frac{dV}{dt}=4\pi R^2\frac{dR}{dt}$$

When $R=2.5$, $\frac{dS}{dt}=-0.75 \quad \Rightarrow \quad 8\pi(2.5)\frac{dR}{dt}=-0.75$

$$\frac{dR}{dt}=-\frac{3}{80\pi} \quad or \quad -\frac{0.0375}{\pi} \quad or \quad -0.0119366$$

$$\frac{dV}{dt}=4\pi(2.5)^2\left(-\frac{3}{80\pi}\right)=-\frac{15}{16} \quad or \quad -0.9375$$

At the instant when $R=2.5$ m, the rate of decrease of volume is 0.9375 m^3 per minute.

1 The complex number z is such that $|z| = 1$ and $\arg z = \theta$, where $0 < \theta < \dfrac{\pi}{4}$.

(i) Mark a possible point A representing z on an Argand diagram. Hence, mark the points B and C representing z^2 and $z + z^2$ respectively on the same Argand diagram corresponding to point A. [2]

(ii) State the geometrical shape of $OACB$. [1]

(iii) Express $z + z^2$ in polar form, $p\cos(q\theta)\left[\cos(k\theta) + \mathrm{i}\sin(k\theta)\right]$, where p, q and k are constants to be determined. [2]

2 The function f is given by $\mathrm{f} : x \mapsto 3 + \dfrac{1}{x-2}$ for $x \in \mapsto,\ x > 2$.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Explain why the composite function f^2 exists. [1]

(iii) Find the value of x for which $\mathrm{f}^2(x) = x$. Explain why this value of x satisfies the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

3 It is given that a curve C has parametric equations

$$x = t^2 - t, \quad y = \frac{1}{t^2 + 1} \quad \text{for } -2 \le t < 2.$$

(i) Sketch C, indicating clearly the coordinates of the end points and the points where C cuts the y-axis. [4]

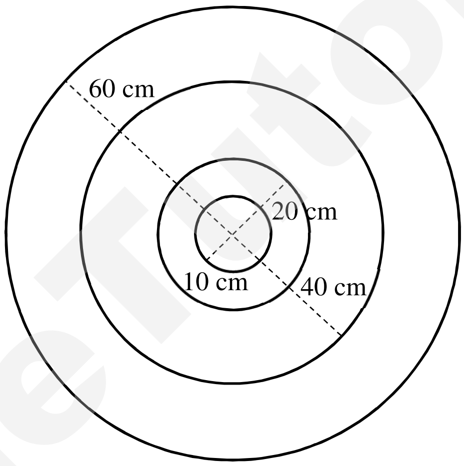
(ii) Find the equation of the tangent to C that is parallel to the y-axis. [4]

(iii) Express the area of the region bounded by C, the tangent found in part **(ii)** and both axes, in the form

$$\int_a^b \mathrm{f}(t)\, \mathrm{d}t,$$

where the function f and the constants a and b are to be determined. Hence find this area, leaving your answer in exact form. [5]

4	A farmer owns a plot of farmland. To prepare for wheat planting, the farmer has to plough the farmland before sowing wheat seeds. At the start of the first week, 300 m^2 of the farmland is ploughed. The farmer ploughs another 100 m^2 of the farmland at the beginning of each subsequent week. To sow wheat seeds, the farmer is considering two different options. **(a)** In the first option, the farmer sows wheat seeds on 60% of the **unsown** ploughed land at the end of each week. **(i)** Find the area of **unsown** ploughed land at the end of the second week. [1] **(ii)** Show that the area of **unsown** ploughed land at the end of the nth week is given by $$\left[0.4^n(300)+k\left(1-0.4^{n-1}\right)\right] \text{ m}^2,$$ where k is an exact constant to be determined. [3] **(iii)** Find the number of complete weeks required for the area of **unsown** ploughed land to first fall below 70 m^2. [3] **(b)** In the second option, the farmer sows 80 m^2 of the **unsown** ploughed land at the end of the first week. At the end of each subsequent week, he sows 20 m^2 of the **unsown** ploughed land more than in the previous week. This means that the area of sown ploughed land is 100 m^2 in the second week, 120 m^2 in the third week, and so on. **(i)** Find, in terms of n, the area of **unsown** ploughed land at the end of the nth week. [4] **(ii)** Find the number of complete weeks required for the farmer to finish sowing all the ploughed farmland in this option. Deduce the area of ploughed land to be sown in the final week. [4]
5	A group of twelve people consists of six married couples. Each couple consists of a husband and a wife. **(i)** The twelve people are to stand in a straight line. Find the number of different arrangements if each husband must stand next to his wife. [2] **(ii)** The group of twelve people finds a round table with ten chairs. Assuming only ten people are to be seated, find the probability that five married couples are seated such that each husband sits next to his wife and husbands and wives alternate. [3]

6	Seven red counters and two blue counters are placed in a bag. All the counters are indistinguishable except for their colours. Clark and Kara take turns to draw a counter from the bag at random with replacement. The first player to draw a blue counter wins the game and the game ends immediately. If Clark draws first, find the probability that **(i)** Clark wins the game at his third draw, [2] **(ii)** Kara wins the game. [3]
7	An archer shoots an arrow into a circular target board that has a radius of 60 cm. The target board further consists of three inner concentric circular sections, with radii 40 cm, 20 cm and 10 cm respectively as shown in the diagram.  The archer scores • 50 points if the arrow lands in the centre circle of radius 10 cm, • 20 points if the arrow lands in the ring with outer radius 20 cm, • 10 points if the arrow lands in the ring with outer radius 40 cm, • 0 point otherwise. Assume that the arrow will definitely hit the target board and is equally likely to hit any portion of the target board. **(i)** Let X be the number of points scored for one arrow shot. Find the expectation of X, leaving your answer in 4 significant figures. [3] **(ii)** Interpret, in this context, the value obtained in part **(i)**. [1] **(iii)** The archer shot at the target board forty times. Find the probability that the average score obtained by the archer is between 10 and 20 points (inclusive). [4]

8	At a hospital, records show that 84.5% of patients turn up for their appointments. It is known that on any day, the doctor has time to see 20 patients. On one particular day, there are 20 patients who make appointments to see the doctor. **(i)** State, in this context, one condition that must be met for the number of patients who turn up for their appointments to be well modelled by a binomial distribution. [1] For the remainder of this question, assume that the condition stated in part (**i**) is met. **(ii)** Find the probability that more than 15 patients turn up for their appointments. [2] **(iii)** Given that at least 12 patients turn up for their appointments, find the probability that more than 2 patients fail to turn up for their appointments. [3] **(iv)** To improve efficiency, the hospital decides to increase the number of appointments that can be made on each day. Given that there will still be enough time for the doctor to see 20 patients, find the greatest number of appointments that can be made so that there is a probability of at least 0.85 of the doctor having time to see all patients who turn up. [2]
9	In order to recruit the best possible employees, a large corporation has designed an entrance test that consists of three components, namely Logical Reasoning, Personality and Communication. The scores obtained by candidates in each of the three components are independent random variables L, P and C which are normally distributed with means and standard deviations as shown in the table.

	Mean	Standard deviation
Logical Reasoning, L	35.2	5.2
Personality, P	24.6	3.8
Communication, C	29.3	4.3

(i) For a particular role in the corporation, the Logical Reasoning and the Personality scores of a candidate is valued and hence a special score of $3L+2P$ is computed.

(a) Find the special score that is exceeded by only 1% of candidates taking the test. Leave your answer in 1 decimal place. [4]

(b) Five candidates are selected randomly. Find the probability that three of them obtained a special score of more than 150, and the other two obtained less than 140. [3]

(ii) For another role in the corporation, a candidate must achieve a result such that his special score of $3L+2P$ differs from $5C$ by less than 25. Find the percentage of candidates who will be able to achieve this. [4]

10	The following table shows the mass (m) of a foetus, in grams, taken at various weeks (t).

t	12	16	20	24	28	32	36
m	14	100	300	600	1005	1702	2622

(i) Draw a scatter diagram to illustrate the data, labelling the axes clearly. [1]

(ii) Calculate the product moment correlation coefficient between t and m, giving your answer correct to 5 decimal places. Explain why this value does not necessarily mean that the linear model is the best model for the relationship between t and m. [2]

It is proposed that the mass of the foetus at week t can be modelled by

$$m = at^b,$$

where a and b are positive constants.

(iii) By using logarithm to transform $m = at^b$ into a linear equation, calculate the value of the product moment correlation coefficient and give two reasons why this model may be a better model. [4]

(iv) Calculate the values of a and b. [2]

(v) Using the equation of a suitable regression line, estimate the mass of the foetus at 26 weeks, giving your answer to the nearest grams. Comment on the reliability of the estimate. [2]

11 The mass of strawberry jam in a randomly chosen jar follows a normal distribution and has a mean mass of 200 grams. A retailer suspects that the mean mass of the strawberry jam is being overstated. He takes a random sample of 30 jars of strawberry jam and weighs the content, x grams, in each jar. The results are summarized as follows.

$$\sum(x-200) = -66 \quad \text{and} \quad \sum(x-200)^2 = 958$$

(i) Test at 2% significance level, whether the retailer's suspicion is justifiable. [6]

(ii) Explain, in this context, the meaning of 'at 2% significance level'. [1]

(iii) Suppose the retailer now decides to test whether the mean mass differs from 200 grams at 2% significance level. Without carrying out the test, explain whether the conclusion would change in part **(i)**. [1]

The manufacturing process has now been improved and the population standard deviation is 3.5 grams. The retailer selects a new random sample of 20 jars of strawberry jam and the sample mean is found to be k grams. Find the range of possible values of k so that the retailer's suspicion that the mean mass differs from 200 grams is not justified at the 2% significance level. Give your answer correct to one decimal place. [4]

ANNEX B

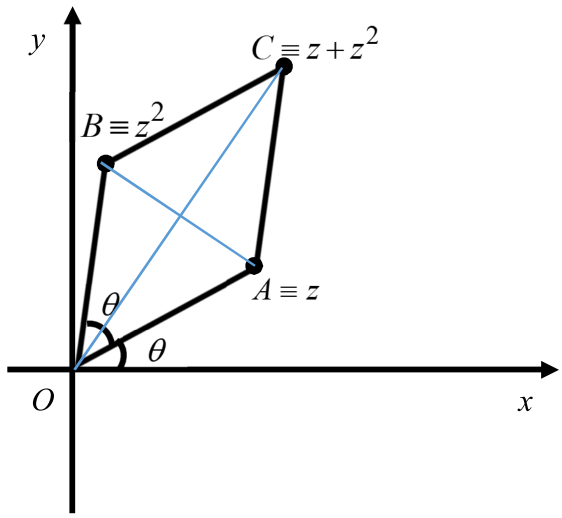
IJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Complex numbers	(ii) rhombus (iii) $2\cos\frac{\theta}{2}\left[\cos\frac{3\theta}{2}+\mathrm{i}\sin\frac{3\theta}{2}\right]$
2	Functions	(i) $\mathrm{f}^{-1}(x)=2+\frac{1}{x-3},\ x\in\mathbb{R},\ x>3$ (iii) $x=3.62$
3	Differentiation & Applications	(ii) $x=-\frac{1}{4}$ (iii) $\ln\frac{8}{5}-\frac{\pi}{4}+\tan^{-1}\frac{1}{2}$
4	AP and GP	(a)(i) $88\ \mathrm{m}^2$ (a)(ii) k is $\frac{200}{3}$ (a)(iii) 5 (b)(i) $-10n^2+30n+200$ (b)(ii) number of complete weeks is 7; $120\,\mathrm{m}^2$
5	P&C, Probability	(i) 46080 (ii) 0.0000120
6	P&C, Probability	(i) 0.0813 (ii) 0.4375
7	DRV	(i) 6.389 (iii) 0.00965
8	Binomial Distribution	(i) Whether a randomly chosen patient turns up for an appointment is independent of any other patient. (ii) 0.812 (iii) 0.618 (iv) 22
9	Normal Distribution	(i)(a) $a=195.2$ (i)(b) 0.0875 (ii) 61.3%
10	Correlation & Linear Regression	(ii) $r=0.94597$ (iii) $r=0.990$ (iv) $a=2.30\times10^{-4}$, $b=4.59$ (v) $m=728$ Since the value of 26 is within the range of values of t and the value of r is close to 1, this estimate is reliable.

11	Hypothesis Testing	(ii) At 2% significance level means that there is a probability of 0.02 that **<u>the test will indicate</u>** that the mean mass of the strawberry jam in the jar is less than 200 g when in fact it is 200 g. (iii) This will result in a different conclusion; $198.2 < k < 201.8$

1

(i)



(ii)

Since $OACB$ is a parallelogram with 4 equal sides, it is a **rhombus**.

(iii)

$$z+z^2$$
$$=\cos\theta+\mathrm{i}\sin\theta+\left(\cos\theta+\mathrm{i}\sin\theta\right)^2$$
$$=\cos\theta+\mathrm{i}\sin\theta+\cos^2\theta+2\mathrm{i}\cos\theta\sin\theta-\sin^2\theta$$
$$=\left(\cos\theta+\cos2\theta\right)+\mathrm{i}\left(\sin\theta+\sin2\theta\right)$$
$$=2\cos\frac{3\theta}{2}\cos\frac{\theta}{2}+2\mathrm{i}\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$$
$$=2\cos\frac{\theta}{2}\left[\cos\frac{3\theta}{2}+\mathrm{i}\sin\frac{3\theta}{2}\right]$$

Alternative

$$\arg\left(z+z^2\right)=\theta+\frac{\theta}{2}=\frac{3}{2}\theta$$
$$\left|z+z^2\right|=2OM=2\cos\left(\frac{\theta}{2}\right)$$
$$z+z^2=2\cos\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{3}{2}\theta\right)+\mathrm{i}\sin\left(\frac{3}{2}\theta\right)\right]$$
$$\therefore p=2,\ q=\frac{1}{2},\ k=\frac{3}{2}$$

2

(i)

$$\mathrm{f}:x\mapsto 3+\frac{1}{x-2},\ x\in\mathbb{R},\ x>2$$

Let $y=\mathrm{f}(x)$.

$$y = 3 + \frac{1}{x-2}$$

$$x - 2 = \frac{1}{y-3}$$

$$x = 2 + \frac{1}{y-3}$$

$$\therefore \ \mathrm{f}^{-1}(x) = 2 + \frac{1}{x-3}, \ x \in \square, \ x > 3$$

(ii)

$$\mathrm{D_f} = (2, \infty)$$

$$\mathrm{R_f} = (3, \infty)$$

Since $\mathrm{R_f} \subseteq \mathrm{D_f}$, the composite function f^2 exists.

(iii)

$$\mathrm{f}^2(x) = x$$

$$\mathrm{f}\left(3 + \frac{1}{x-2}\right) = x$$

$$3 + \frac{1}{3 + \frac{1}{x-2} - 2} = x$$

$$3 + \frac{1}{\left(\frac{x-1}{x-2}\right)} = x$$

$$\frac{3(x-1) + (x-2)}{x-1} = x$$

$$4x - 5 = x(x-1)$$

$$x^2 - 5x + 5 = 0$$

Using GC, $x = 1.38$ (rej $\because 1.38 \notin \mathrm{D_f}$) or $x = 3.62$

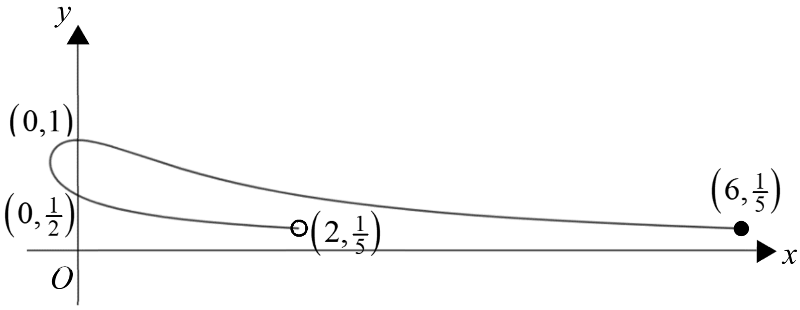
$$\mathrm{ff}(x) = x$$

$$\mathrm{f}^{-1}\mathrm{ff}(x) = \mathrm{f}^{-1}(x)$$

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$$

Therefore $x = 3.62$ satisfies $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.

3	



When $x = 0$, $t(t-1) = 0 \quad \Rightarrow t = 0$ or $t = 1$

$\Rightarrow y = 1$ or $y = \frac{1}{2}$

Coordinates are $(0,1)$ and $\left(0, \frac{1}{2}\right)$.

(ii)

$\frac{dx}{dt} = 2t - 1, \quad \frac{dy}{dt} = \frac{-2t}{\left(t^2+1\right)^2}$

$$\therefore \frac{dy}{dx} = \frac{-2t}{\left(t^2+1\right)^2} \times \frac{1}{2t-1}$$

$$= \frac{-2t}{\left(t^2+1\right)^2 (2t-1)}$$

When tangent is parallel to y-axis,

$$\left(t^2+1\right)^2 (2t-1) = 0 \Rightarrow t = \frac{1}{2} \qquad \left(\because \left(t^2+1\right)^2 > 0\right)$$

Equation of tangent: $x = -\frac{1}{4}$

(iii)

Area of the required region

$$= \int_{-1/4}^{0} y \, dx$$

$$= \int_{1/2}^{1} \frac{1}{t^2+1}(2t-1)\, dt$$

$$= \int_{1/2}^{1} \frac{2t}{t^2+1} - \frac{1}{t^2+1}\, dt$$

$$= \left[\ln\left(t^2+1\right) - \tan^{-1} t\right]_{1/2}^{1}$$

$$= \left[\left(\ln 2 - \frac{\pi}{4}\right) - \left(\ln \frac{5}{4} - \tan^{-1} \frac{1}{2}\right)\right]$$

$$= \ln \frac{8}{5} - \frac{\pi}{4} + \tan^{-1} \frac{1}{2}$$

When $x = -\frac{1}{4}$, $t = \frac{1}{2}$

When $x = 0$, $t = 1$

4	(a)(i)

Area of **unsown** ploughed land

$= 0.4\left[0.4(300)+100\right]$

$= 88 \text{ m}^2$

(a)(ii)

n	Beginning of week	End of week
1	300	$0.4(300)$
2	$0.4(300)+100$	$0.4\left[0.4(300)+100\right]$ $=0.4^2(300)+0.4(100)$
3	$0.4^2(300)+0.4(100)$ $+100$	$0.4\left[0.4^2(300)+0.4(100)+100\right]$ $=0.4^3(300)+0.4^2(100)+0.4(100)$
..	…	…
n	…	$0.4^n(300)+0.4^{n-1}(100)+...$ $+0.4^2(100)+0.4^1(100)$

Area of land **unsown** ploughed land at the end of nth week

$$= 0.4^n(300)+100\left[\frac{0.4\left(1-0.4^{n-1}\right)}{1-0.4}\right]$$

$$= \left[0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)\right] \text{ m}^2$$

$\therefore$ the value of k is $\frac{200}{3}$.

(a)(iii)

Method 1

$$0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)<70$$

$$0.4^n(300)+\frac{200}{3}-\frac{200}{3}(0.4)^{-1}0.4^n<70$$

$$\frac{400}{3}\left(0.4^n\right)<\frac{10}{3}$$

$$0.4^n<\frac{1}{40}$$

$$n>\ln\left(\tfrac{1}{40}\right)\Big/\ln 0.4$$

$$n>4.02588$$

Hence the number of complete weeks required is 5.

Method 2

$$0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)<70$$

Using GC,
when n = 4, unsown ploughed land = 70.08 (> 70)
when n = 5, unsown ploughed land = 68.032 (< 70)
when n = 6, unsown ploughed land = 67.213 (< 70)

Hence the number of complete weeks required is 5.

(b)(i)

n	Beginning of week	End of week
1	300	$300-80$
2	$300+(100)-80$	$300+(100)-80-100$
3	$300+2(100)-80-100$	$300+2(100)-80-100-120$
..	$\dots$	$\dots$
n	$\dots$	$300+(n-1)(100)-80-100$ $-\cdots-\left[80+20(n-1)\right]$

Area of **unsown** ploughed land at the end of nth week

$$=300+100(n-1)-\frac{n}{2}\left[2(80)+20(n-1)\right]$$

$$=300+100n-100-\frac{n}{2}(140+20n)$$

$$=300+100n-100-70n-10n^2$$

$$=-10n^2+30n+200$$

(b)(ii)
For the farmer to finish sowing all the ploughed farmland,

$$-10n^2+30n+200\le 0$$

Method 1:
Solving the inequality,
$n\ge 6.21699$ or $n\le -3.21699$ (rejected)
Hence the number of complete weeks is 7.

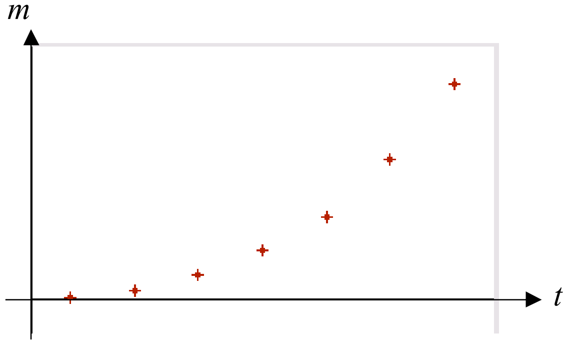
Method 2:
Using GC to set up a table,
When $n=6$, area unsown $=20$
When $n=7$, area unsown $=-80$
When $n=8$, area unsown $=-200$
Hence the number of complete weeks is 7.

	In week 6, the area of **unsown** ploughed land $=-10(6)^2+30(6)+200=20\ \text{m}^2$ $\therefore$ area of ploughed land to be **sown** in week 7 (the final week) $=20+100=120\ \text{m}^2$
5	(i) Number of arrangements = $6!\times 2^6=46080$ (ii) Required probability $=\dfrac{{}^6C_5\times(5-1)!\times 2}{{}^{12}C_{10}\times(10-1)!}$ $=\dfrac{288}{23950080}$ $=0.0000120$ (3 sig fig)
6	(i) P(Clark wins in 3rd draw) $=\dfrac{7}{9}\times\dfrac{7}{9}\times\dfrac{7}{9}\times\dfrac{7}{9}\times\dfrac{2}{9}$ $=0.081322$ $=0.0813$ (ii) P(Kara wins) $=\dfrac{7}{9}\times\dfrac{2}{9}+\left(\dfrac{7}{9}\right)^3\times\dfrac{2}{9}+\left(\dfrac{7}{9}\right)^5\times\dfrac{2}{9}+\ldots.$ $=\dfrac{2}{9}\left[\dfrac{7}{9}+\left(\dfrac{7}{9}\right)^3+\left(\dfrac{7}{9}\right)^5+\ldots\right]$ $=\dfrac{2}{9}\left(\dfrac{\frac{7}{9}}{1-\left(\frac{7}{9}\right)^2}\right)$ $=0.4375\ \textit{or}\ \dfrac{7}{16}$
7	(i) Given that X is the number of points scored for one arrow shot. $\text{P}(X=50)=\dfrac{\pi(10)^2}{\pi(60)^2}=\dfrac{1}{36}$ $\text{P}(X=20)=\dfrac{\pi(20)^2-\pi(10)^2}{\pi(60)^2}=\dfrac{1}{12}$ $\text{P}(X=10)=\dfrac{\pi(40)^2-\pi(20)^2}{\pi(60)^2}=\dfrac{1}{3}$

	$\mathrm{E}(X)=(10)\left(\frac{1}{3}\right)+(20)\left(\frac{1}{12}\right)+(50)\left(\frac{1}{36}\right)$ $=6.389$ (4 sig fig) (ii) If the archer is to shoot at the target board repeatedly, then in the long run his average score will be 6.389 points. (iii) $\mathrm{Var}(X)=(10)^2\left(\frac{1}{3}\right)+(20)^2\left(\frac{1}{12}\right)+(50)^2\left(\frac{1}{36}\right)-(6.38888)^2$ $=95.2932$ Let $\overline{X}=\frac{X_1+X_2+...+X_{40}}{40}$. Since n = 40 is large, by Central Limit Theorem, $\overline{X}\sim \mathrm{N}\left(6.38888,\frac{95.2932}{40}\right)$ approximately. Required probability $=\mathrm{P}(10<\overline{X}<20)$ $=0.00965$ (3 sig fig)
8	(i) Whether a randomly chosen patient turns up for an appointment is independent of any other patient. (ii) Let X be the number of patients who turn up for their appointments, out of 20 appointments. $X\sim \mathrm{B}(20,0.845)$ $\mathrm{P}(X>15)$ $=1-\mathrm{P}(X\le 15)$ $=0.812$ (3 sig fig) (iii) Required probability $=\mathrm{P}(X\le 17 \mid X\ge 12)$ $=\frac{\mathrm{P}(12\le X\le 17)}{\mathrm{P}(X\ge 12)}$ $=\frac{\mathrm{P}(X\le 17)-\mathrm{P}(X\le 11)}{1-\mathrm{P}(X\le 11)}$ $=0.618$ (3 sig fig)

	(iv) Let Y be the number of patients who turn up for their appointments, out of n appointments. $Y \sim \mathrm{B}(n, 0.845)$ $\mathrm{P}(Y \leq 20) \geq 0.85$ --- (*) Using GC, When $n = 21$, $\mathrm{P}(Y \leq 20) = 0.9709 \quad (> 0.85)$ When $n = 22$, $\mathrm{P}(Y \leq 20) = 0.8762 \quad (> 0.85)$ When $n = 23$, $\mathrm{P}(Y \leq 20) = 0.7146 \quad (< 0.85)$ $\therefore$ Largest n is 22.
9	(i)(a) Given: $L \sim \mathrm{N}(35.2, 5.2^2)$ $P \sim \mathrm{N}(24.6, 3.8^2)$ $C \sim \mathrm{N}(29.3, 4.3^2)$ Let $T = 3L + 2P$. $\mathrm{E}(T) = 3 \times 35.2 + 2 \times 24.6 = 154.8$ $\mathrm{Var}(T) = 3^2 \times 5.2^2 + 2^2 \times 3.8^2 = 301.12$ $\therefore T \sim \mathrm{N}(154.8, 301.12)$ Let a be the required score exceed by 1% of the candidates. $\mathrm{P}(T > a) = 0.01$ $\Rightarrow \mathrm{P}(T \leq a) = 0.99$ Using GC, $a = 195.2$ (1 dec pl) (i)(b) Required probability $= [\mathrm{P}(T > 150)]^3 [\mathrm{P}(T < 140)]^2 \times \left(\frac{5!}{2!3!}\right)$ $= 0.0875$ (3 sig fig) (ii) Consider $A = 3L + 2P - 5C$ $\mathrm{E}(A) = 154.8 - 5(29.3) = 8.3$ $\mathrm{Var}(A) = 301.12 + 5^2(4.3^2) = 763.37$ $\therefore A \sim \mathrm{N}(8.3, 763.37)$ Required probability $= \mathrm{P}(\|A\| < 25)$ $= \mathrm{P}(-25 < A < 25)$ $= 0.613$ (3 sig fig) Required percentage = 61.3%

10	(i)



(ii)
The product moment correlation coefficient between t and m is $r = 0.94597$ (5 d.p.).

A value of 0.94597 for r suggests that there is a strong positive linear correlation between t and m. However, the points on the scatter diagram **show a curvilinear relationship**. Therefore this value of r does not necessarily mean that the linear model is best model for the relationship between t and m.

(iii)

$$m = at^b$$

$$\ln m = \ln\left(at^b\right)$$

$$\ln m = b\ln t + \ln a$$

The product moment correlation coefficient between ln t and ln m is $r = 0.98967 = 0.990$ (3 sig fig)

Reason 1: From the scatter diagram, as t increases, the **weight of the foetus increases at an increasing rate**.

Reason 2: The value of r between ln t and ln m is 0.98967, which is closer to 1 as compared to that between t and m, hence indicating a **stronger positive linear correlation** between ln t and ln m.
Hence $m = at^b$ is a better model.

(iv)
From GC,
$\ln m = -8.3764 + 4.5938 \ln t$ (5 sig fig)

$\ln a = -8.3764$
$a = 2.30 \times 10^{-4}$ and $b = 4.59$

(v)
When $t = 26$, $\ln m = -8.3764 + 4.5938 \ln 26$
$m = 728$ (nearest grams)
Since the value of 26 is within the range of values of t and the value of r is close to 1, this estimate is reliable.

11	(i) Let X be the random variable denoting the mass of strawberry jam, in grams, in a randomly chosen jar. Unbiased estimate of population mean $\bar{x} = \frac{-66}{30} + 200 = 197.8$ Unbiased estimate of population variance $s^2 = \frac{1}{29}\left[958 - \frac{(-66)^2}{30}\right] = 28.02759$ $H_0 : \mu = 200$ $H_1 : \mu < 200$ Test at 2% significance level Assume H_0 is true. $\bar{X} \sim N\left(200, \frac{28.02759}{30}\right)$ Test statistic: $Z = \frac{\bar{X} - 200}{\sqrt{28.02759/30}} \sim N(0,1)$ Using GC, p-value = 0.011420121 < 0.02 Reject H_0 and conclude that there is sufficient evidence at 2% level of significance that the mean mass of strawberry jam in each jar is overstated. Therefore the retailer's suspicion is justifiable. (ii) At 2% significance level means that there is a probability of 0.02 that **<u>the test will indicate</u>** that the mean mass of the strawberry jam in the jar is less than 200 g when in fact it is 200 g. (iii) $H_0 : \mu = 200$ $H_1 : \mu \neq 200$ For a two tailed test, the p-value will be twice of 0.0114 which is 0.0228. This value is now more than the 0.02 where we do not reject H_0 at 2% significance level. As such this will result in a different conclusion. (iv) $H_0 : \mu = 200$ $H_1 : \mu \neq 200$ Test at 2% significance level Assume H_0 is true. $\bar{X} \sim N\left(200, \frac{3.5^2}{20}\right)$. Test statistic: $Z = \frac{\bar{X} - 200}{\sqrt{3.5^2/20}} \sim N(0,1)$

	For the retailer's suspicion that the mean mass differs from 200 g to be not justified, **do not reject H_0.** $\Rightarrow$ z-value falls outside the critical region $-2.32635 < z\text{-value} < 2.32635$ $-2.32635 < \dfrac{k-200}{3.5/\sqrt{20}} < 2.32635$ $-1.82066 < k-200 < 1.82066$ $198.17934 < k < 201.82066$ $\Rightarrow 198.2 < k < 201.8$ (to 1 d.p)

H2 Mathematics 2017 Prelim Exam Paper 1 Question

1	Mr Subash returned to Singapore after his tour in Europe and wishes to convert his foreign currencies back to Singapore Dollars (S$). Three money changers offer the following exchange rates:

Money Changer	1 Swiss Franc	1 British Pound	1 Euro	Total amount of S$ Mr Subash would receive after currency conversion
A	S$1.35	S$1.80	S$1.55	S$1151.50
B	S$1.40	S$1.85	S$1.65	S$1208.25
C	S$1.45	S$1.75	S$1.60	S$1189.25

How much of each currency has Mr Subash left after his tour? [4]

2

(a) Find $\int \sin(2\theta)\cos(3\theta)\,\mathrm{d}\theta$. [2]

(b) Use the substitution $\theta = \sqrt{x}$ to find the exact value of $\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \theta^3 \cos\left(\theta^2\right) \mathrm{d}\theta$. [5]

3

(i) Using the formula for $\sin P - \sin Q$, show that

$$\sin\left[(2r+1)\theta\right] - \sin\left[(2r-1)\theta\right] \equiv 2\cos(2r\theta)\sin\theta.$$ [1]

(ii) Given that $\sin\theta \neq 0$, using the method of differences, show that

$$\sum_{r=1}^{n} \cos(2r\theta) = \frac{\sin\left[(2n+1)\theta\right] - \sin\theta}{2\sin\theta}.$$ [2]

(iii) Hence find $\sum_{r=1}^{n} \cos^2\left(\frac{r\pi}{5}\right)$ in terms of *n*.

Explain why the infinite series

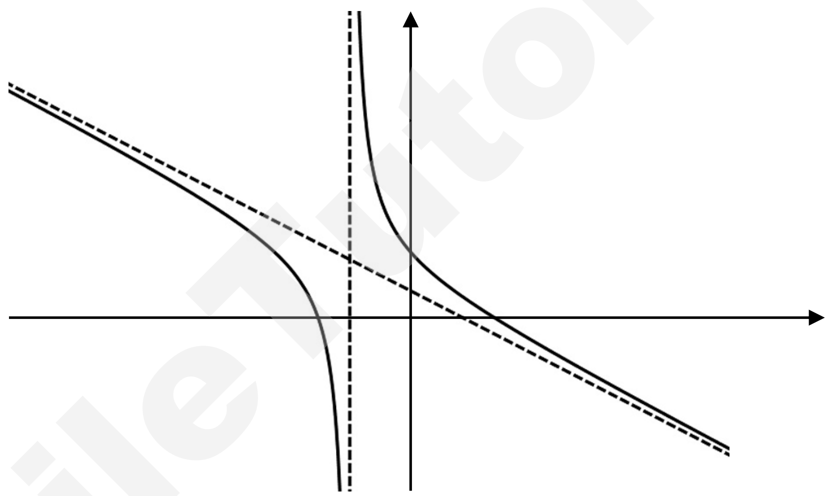
$$\cos^2\left(\frac{\pi}{5}\right) + \cos^2\left(\frac{2\pi}{5}\right) + \cos^2\left(\frac{3\pi}{5}\right) + \ldots$$

is divergent. [3]

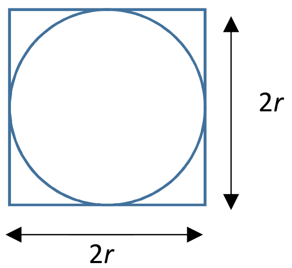
4

A fund is started at \$6000 and compound interest of 3% is added to the fund at the end of each year. If withdrawals of \$*k* are made at the beginning of each of the subsequent years, show that the amount in the fund at the beginning of the $(n+1)$th year is

$$\$\frac{100}{3}\left[(180-k)(1.03)^n + k\right].$$ [5]

	(i) It is given that $k = 400$. At the beginning of which year, for the first time, will the amount in the fund be less than \$1000? [2] **(ii)** If the fund is fully withdrawn at the beginning of sixteenth year, find the least value of k to the nearest integer. [2]
5	**(a)** The curve C has the equation $$(x-2)^2 = a^2\left(1-y^2\right), \qquad 1<a<2.$$ Sketch C, showing clearly any intercepts and key features. [2] **(b)** The diagram shows the graph of $y = \mathrm{f}(x)$, which has an oblique asymptote $y = 1-x$, a vertical asymptote $x=-1$, x-intercepts at $\left(\sqrt{2},0\right)$ and $\left(-\sqrt{2},0\right)$, and y-intercept at $(0,2)$.  Sketch, on separate diagrams, the graphs of **(i)** $y=\dfrac{1}{\mathrm{f}(x)}$, [3] **(ii)** $y=\mathrm{f}'(x)$, [3] showing clearly all relevant asymptotes and intercepts, where possible.
6	With respect to the origin O, the position vectors of the points U, V and W are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively. The mid-points of the sides VW, WU and UV of the triangle UVW are M, N and P respectively. **(i)** Show that $\overrightarrow{UM} = \frac{1}{2}\left(\mathbf{v}+\mathbf{w}-2\mathbf{u}\right)$. [2] **(ii)** Find the vector equations of the lines UM and VN. Hence show that the position vector of the point of intersection, G, of UM and VN is $\frac{1}{3}\left(\mathbf{u}+\mathbf{v}+\mathbf{w}\right)$. [5]

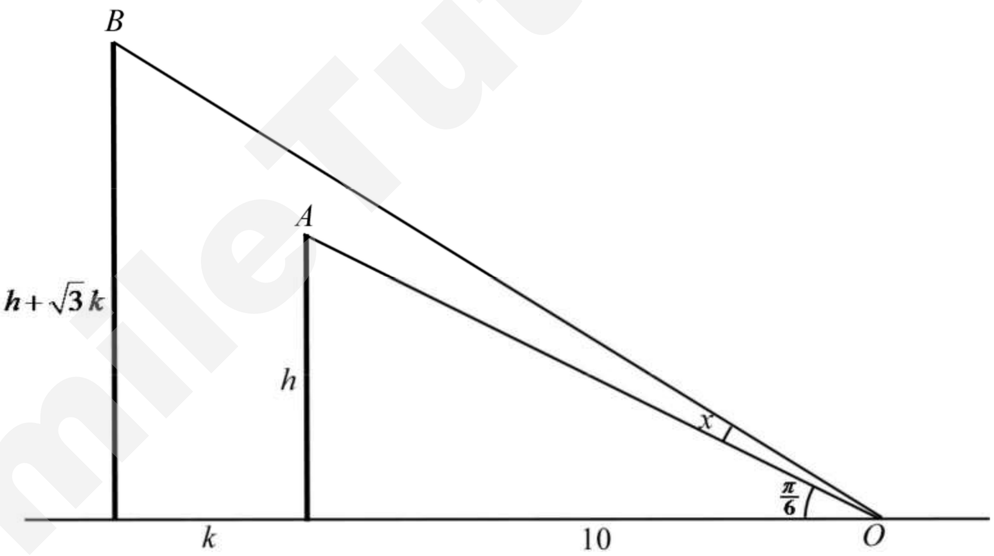
	(iii) It is now given that $\mathbf{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathbf{w} = \begin{pmatrix}0\\0\\1\end{pmatrix}$. Find the direction cosines of $\overrightarrow{OG}$. [2]
7	**(a)** If $u = 2 - \mathrm{i}\sin^2\theta$ and $v = 2\cos^2\theta + \mathrm{i}\sin^2\theta$ where $-\pi < \theta \le \pi$, find $u - v$ in terms of $\sin^2\theta$, and hence determine the exact expression for $\lvert u - v \rvert$ and the exact value of $\arg(u - v)$. [6] **(b)** The roots of the equation $x^2 + (\mathrm{i} - 3)x + 2(1 - \mathrm{i}) = 0$ are α and β, where α is a real number and β is not a real number. Find α and β. [4]
8	**(a)** When a liquid is poured onto a flat surface, a circular patch is formed. The area of the circular patch is expanding at a constant rate of $6\pi\ \mathrm{cm}^2/\mathrm{s}$. **(i)** Find the rate of change of the radius 24 seconds after the liquid is being poured. [3] **(ii)** Explain whether the rate of change of the radius will increase or decrease as time passes. [1] **(b)** A cylindrical can of volume $355\ \mathrm{cm}^3$ with height h cm and base radius r cm is made from 3 pieces of metal. The curved surface of the can is formed by bending a rectangular sheet of metal, assuming that no metal is wasted in creating this surface. The top and bottom surfaces of the can are cut from square sheets of metal with length $2r$ cm, as shown below. The cost of the metal sheets is \$$K$ per cm^2.



(i) Show that the total cost of metal used, denoted by \$$C$, is given by

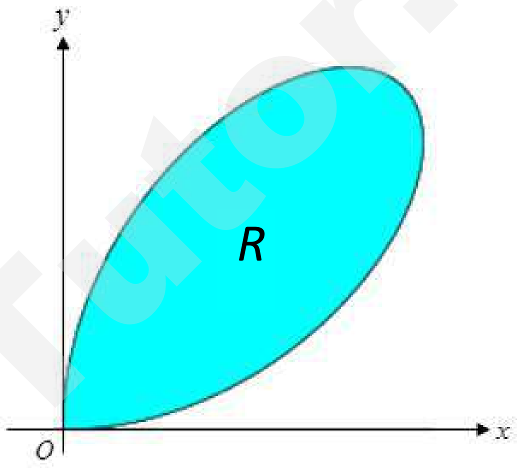
$$C = K\left(\frac{710}{r} + 8r^2\right).$$ [3]

(ii) Use differentiation to show that, when the cost of metal used is a minimum, then $\frac{h}{r} = \frac{8}{\pi}$. [5]

9	**(i)** Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$ where R and α are exact positive constants to be found. [2] **(ii)** State a sequence of transformations which transform the graph of $y = \cos x$ to the graph of $y = \sqrt{3}\cos x - \sin x$. [2] The function f is defined by $\mathrm{f} : x \mapsto \sqrt{3}\cos x - \sin x,\ 0 \le x \le 2\pi$. **(iii)** Sketch the graph of $y = \mathrm{f}(x)$ and state the range of f. [3] The function g is defined by $\mathrm{g} : x \mapsto \mathrm{f}(x),\ 0 \le x \le k$. **(iv)** Given that g^{-1} exists, state the largest exact value of k and find $\mathrm{g}^{-1}(x)$. [3] The function h is defined by $\mathrm{h} : x \mapsto x - 2,\ x \ge 0$. **(v)** Explain why the composite function fh does not exist. [1]
10	 A laser from a fixed point O on a flat ground projects light beams to the top of two vertical structures A and B as shown above. To project the beam to the top of A, the laser makes an angle of elevation of $\frac{\pi}{6}$ radians. To project the beam to the top of B, the laser makes an angle of elevation of $\left(\frac{\pi}{6} + x\right)$ radians. The two structures A and B are of heights h m and $\left(h + \sqrt{3}\,k\right)$ m respectively and are 10 m and (10 + k) m away from O respectively. **(i)** Show that the length of the straight beam from O to A is $\frac{20}{\sqrt{3}}$ m. [1] **(ii)** Show that the length of AB is $2k$ m and that the angle of elevation of B from A is $\frac{\pi}{3}$ radians. [3]

	(iii) Hence, using the sine rule, show that $k=\dfrac{10\sin x}{\sqrt{3}\sin\left(\dfrac{\pi}{6}-x\right)}$.	[2]
	(iv) If x is sufficiently small, show that $k\approx\dfrac{20}{\sqrt{3}}\left(x+ax^2\right)$, where a is a constant to be determined.	[6]
11	**(a)** The diagram below shows a section of *Folium of Descartes* curve which is defined parametrically by	

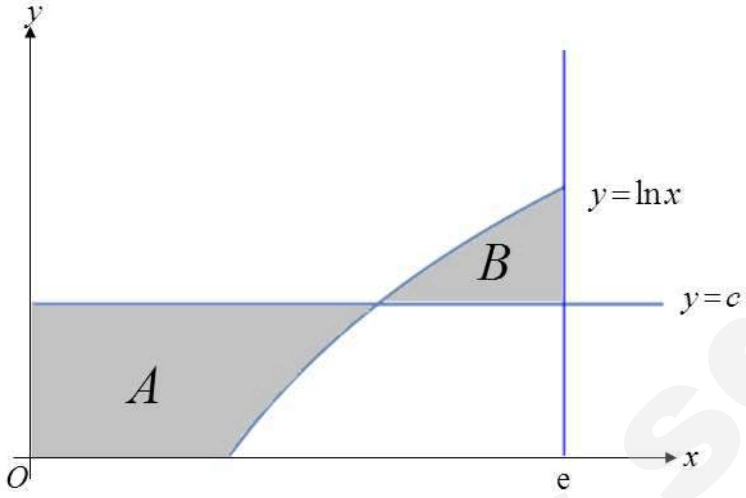
$$x=\frac{3m}{1+m^3},\quad y=\frac{3m^2}{1+m^3},\quad m\geq 0.$$



(i) It is known that the curve is symmetrical about the line $y=x$. Find the values of m where the curve meets the line $y=x$. [1]

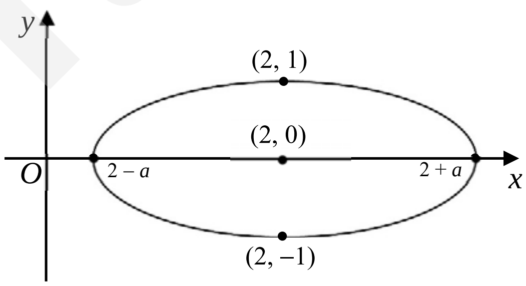
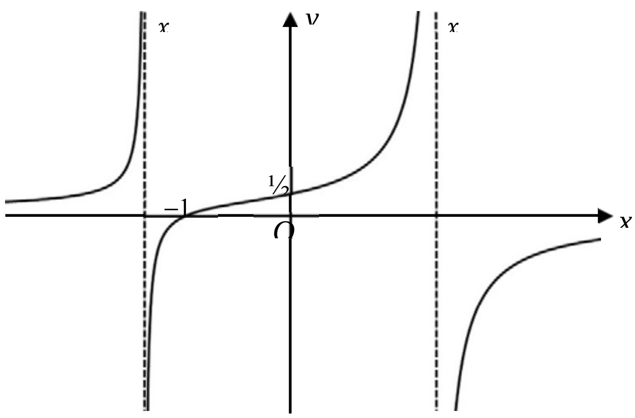
(ii) Region R is the region enclosed by the curve in the first quadrant. Show that the area of R is given by $2\left(\int_0^{\frac{3}{2}} x\,\mathrm{d}y-\dfrac{9}{8}\right)$, and evaluate this integral. [5]

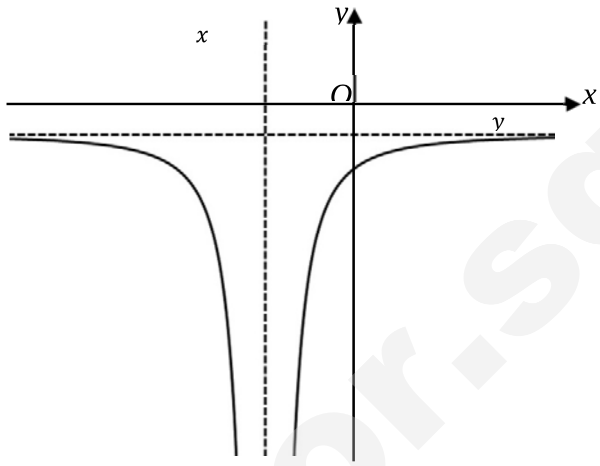
(b) The diagram below shows a horizontal line $y=c$ intersecting the curve $y=\ln x$ at a point where the x-coordinate is such that $1<x<e$.

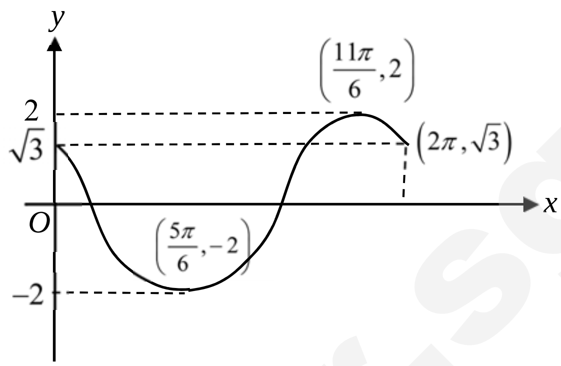
	 The region A is bounded by the curve, the line $y=c$, the x-axis and the y-axis while the region B is bounded by the curve and the lines $x=e$ and $y=c$. Given that the volumes of revolution when A and B are rotated completely about the y-axis are equal, show that $c=\dfrac{e^2+1}{2e^2}$. [6]

ANNEX B

XXJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Equations and Inequalities	He has <u>250 francs</u>, <u>125 pounds</u> and <u>380 euros</u> left.
2	Integration techniques	2(a) $\frac{1}{2}\cos\theta-\frac{1}{10}\cos(5\theta)+c$ 2(b) $-\frac{1}{2}-\frac{\pi}{4}$
3	Sigma Notation and Method of Difference	3(iii) $\dfrac{\sin\frac{(2n+1)\pi}{5}}{4\sin\frac{\pi}{5}}-\frac{1}{4}+\frac{1}{2}n$
4	AP and GP	4(i) at the beginning of <u>19th</u> year 4(ii) Least $k=$ <u>503</u>
5	Graphs and Transformation	5(a)  5(b) (i) 

		5(b) (ii) 
6	Vectors	6(ii) Line *UM*: $\underline{\underline{\mathbf{r}=\mathbf{u}+\lambda\left(\mathbf{w}+\mathbf{v}-2\mathbf{u}\right)}},\ \lambda\in\mathbb{R}$ Line *VN*: $\underline{\underline{\mathbf{r}=\mathbf{v}+\mu\left(\mathbf{w}+\mathbf{u}-2\mathbf{v}\right)}},\ \mu\in\mathbb{R}$ 6(iii) Direction cosines of $\overrightarrow{OG}$ are $\underline{\underline{\sqrt{\frac{1}{3}},\sqrt{\frac{1}{3}},\sqrt{\frac{1}{3}}}}$
7	Complex numbers	7(a) $u-v=2\sin^2\theta-2\mathrm{i}\sin^2\theta$ $\lvert u-v\rvert=2\sqrt{2}\sin^2\theta,\ \arg\left(u-v\right)=-\frac{\pi}{4}$ 7(b) $\alpha = 2,\ \beta = 1-\mathrm{i}$
8	Differentiation & Applications	8(a) (i) $\frac{1}{4}$ cm/ s , (ii) $\frac{\mathrm{d}r}{\mathrm{d}t}$ will <u>decrease</u> as time passes
9	Functions	9(i) $R=2,\ \alpha=\frac{\pi}{6}$ 9(ii) $y=\cos x\xrightarrow{A}y=\cos(x+\alpha)\xrightarrow{B}y=R\cos(x+\alpha)$ *A*: Translation by α radians in the negative *x*-direction, followed by *B*: Scaling parallel to the *y*-axis by a scale factor *R*.

		9(iii)  Range of f, $R_f = [-2, 2]$ 9(iv) Largest $k = \frac{5\pi}{6}$ $g^{-1}(x) = \cos^{-1}\frac{x}{2} - \frac{\pi}{6}$ 9(v) Since $R_h = [-2, +\infty)$ and $D_f = [0, 2\pi]$, $R_h \not\subset D_f$, fh does not exist.
10	Binomial Expansion	10(iv) $a = \sqrt{3}$
11	Application of Integration	11(a) (i) $m = 0$ or 1 11(a) (ii) $\frac{3}{2}$ units^2

H2 Mathematics 2017 Prelim Exam Paper 1 Solution

1

Let x, y and z be the amount of Francs, Pounds & Euro Mr Subash has left respectively.

$$1.35x+1.80y+1.55z=1151.50$$
$$1.40x+1.85y+1.65z=1208.25$$
$$1.45x+1.75y+1.60z=1189.25$$

Using GC, $x = 250$, $y = 125$, $z = 380$.
He has <u>250 francs</u>, <u>125 pounds</u> and <u>380 euros</u> left.

2

(a) By Factor Formula,

$$\begin{aligned}\sin(2\theta)\cos(3\theta) &= \frac{1}{2}\left[\sin(5\theta)+\sin(-\theta)\right]\\ &= \frac{1}{2}\left[\sin(5\theta)-\sin(\theta)\right]\\ \int \sin(2\theta)\cos(3\theta)\,\mathrm{d}\theta &= \int\frac{1}{2}\left[\sin(5\theta)-\sin(\theta)\right]\mathrm{d}\theta\\ &= \underline{\underline{\frac{1}{2}\cos\theta-\frac{1}{10}\cos(5\theta)+c}}\end{aligned}$$

(b) $\theta=\sqrt{\pi}\Rightarrow\sqrt{x}=\sqrt{\pi}\Rightarrow x=\pi$

$$\theta=\sqrt{\frac{\pi}{2}}\Rightarrow\sqrt{x}=\sqrt{\frac{\pi}{2}}\Rightarrow x=\frac{\pi}{2}$$

$$\theta=\sqrt{x}\ \Rightarrow\ \frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}.$$

$$\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}}\theta^3\cos\left(\theta^2\right)\mathrm{d}\theta$$

$$=\int_{\frac{\pi}{2}}^{\pi}x\sqrt{x}(\cos x)\left(\frac{1}{2\sqrt{x}}\right)\mathrm{d}x$$

$$=\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}x\cos x\,\mathrm{d}x$$

$$\begin{array}{c|c} u = x & \dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos x \\ \dfrac{\mathrm{d}u}{\mathrm{d}x} = 1 & v = \sin x \end{array}$$

$$=\frac{1}{2}\left[\left[x\sin x\right]_{\frac{\pi}{2}}^{\pi}-\int_{\frac{\pi}{2}}^{\pi}1(\sin x)\,\mathrm{d}x\right]$$

$$=\frac{1}{2}\left(0-\frac{\pi}{2}+\left[\cos x\right]_{\frac{\pi}{2}}^{\pi}\right)$$

$$=\frac{1}{2}\left[-\frac{\pi}{2}+(-1-0)\right]$$

$$=\underline{\underline{-\frac{1}{2}-\frac{\pi}{4}}}$$

3 **(i)**

$$\sin\left[(2r+1)\theta\right]-\sin\left[(2r-1)\theta\right]$$
$$\equiv 2\cos\frac{(2r+1)\theta+(2r-1)\theta}{2}\sin\frac{(2r+1)\theta-(2r-1)\theta}{2}$$
$$\equiv 2\cos(2r\theta)\sin\theta \quad \text{[Shown]}$$

(ii) From (i), $\sin\left[(2r+1)\theta\right]-\sin\left[(2r-1)\theta\right]\equiv 2\cos(2r\theta)\sin\theta$

$$\Rightarrow \cos(2r\theta)=\frac{\sin\left[(2r+1)\theta\right]-\sin\left[(2r-1)\theta\right]}{2\sin\theta}$$

$$\therefore \sum_{r=1}^{n}\cos(2r\theta) = \sum_{r=1}^{n}\frac{\sin\left[(2r+1)\theta\right]-\sin\left[(2r-1)\theta\right]}{2\sin\theta}$$

$$=\frac{1}{2\sin\theta}\left[\begin{array}{l} \sin 3\theta-\sin\theta \\ +\ \sin 5\theta-\sin 3\theta \\ +\ \sin 7\theta-\sin 5\theta \\ +\ \cdots \\ +\ \sin(2n-1)\theta-\sin(2n-3)\theta \\ +\ \sin(2n+1)\theta-\sin(2n-1)\theta \end{array}\right]$$

$$=\frac{\sin\left[(2n+1)\theta\right]-\sin\theta}{2\sin\theta} \quad \text{[Shown]}$$

(iii) $$\sum_{r=1}^{n}\cos^2\left(\frac{r\pi}{5}\right) = \sum_{r=1}^{n}\frac{\cos\left(\frac{2r\pi}{5}\right)+1}{2}$$

$$=\frac{1}{2}\sum_{r=1}^{n}\cos\left(\frac{2r\pi}{5}\right)+\sum_{r=1}^{n}\frac{1}{2} \qquad \left(\text{Let } \theta=\frac{\pi}{5}\right)$$

$$=\frac{1}{2}\left[\frac{\sin\frac{(2n+1)\pi}{5}-\sin\frac{\pi}{5}}{2\sin\frac{\pi}{5}}\right]+\frac{1}{2}n$$

$$=\frac{\sin\frac{(2n+1)\pi}{5}}{4\sin\frac{\pi}{5}}-\frac{1}{4}+\frac{1}{2}n$$

As $n\to\infty$, $-\frac{1}{4}+\frac{1}{2}n\to\infty$ and $\left|\sin\frac{(2n+1)\pi}{5}\right|\le 1$,

$$\therefore \sum_{r=1}^{n}\cos^2\left(\frac{r\pi}{5}\right)\to\infty.$$

$\therefore$ the series $\cos^2\left(\frac{\pi}{5}\right)+\cos^2\left(\frac{2\pi}{5}\right)+\cos^2\left(\frac{3\pi}{5}\right)+\ldots$ is divergent.

4

Yr	Amount at the beginning of yr	Amount at the end of yr
1	6000	6000(1.03)
2	$6000(1.03)-k$	$[6000(1.03)-k](1.03)$ $=6000(1.03)^2-k(1.03)$
3	$6000(1.03)^2-k(1.03)-k$ $=6000(1.03)^2-k(1.03)-k$	$\left[6000(1.03)^2-k(1.03)-k\right](1.03)$ $=6000(1.03)^3-k(1.03)^2-k(1.03)$

By inspection, amount in the fund at the end of nth year

$= 6000(1.03)^n-k(1.03)^{n-1}-k(1.03)^{n-2}-...-k(1.03)$

Amount in the fund at the beginning of $(n+1)$th year

$= 6000(1.03)^n-k(1.03)^{n-1}-k(1.03)^{n-2}-...-k(1.03)-k$

$=6000(1.03)^n-k\left[1+1.03+(1.03)^2+\cdots+(1.03)^{n-1}\right]$

$=6000(1.03)^n-k\left\{\dfrac{1\left[1-(1.03)^n\right]}{1-1.03}\right\}$

$=6000(1.03)^n+\dfrac{100}{3}k\left[1-(1.03)^n\right]$

$=\dfrac{100}{3}\left[180(1.03)^n+k-k(1.03)^n\right]$

$=\dfrac{100}{3}\left[(180-k)(1.03)^n+k\right]$ [Shown]

(i) Given $k = 400$,

$$\frac{100}{3}\left[(180-400)(1.03)^n+400\right] < 1000$$

$$-220(1.03)^n+400 < 30$$

$$(1.03)^n > \frac{37}{22} \text{ (or 1.6818)}$$

$$n\ln 1.03 > \ln\frac{37}{22}$$

$$n > \frac{\ln\frac{37}{22}}{\ln 1.03} = 17.6 \text{ (3 sf)}$$

$$\text{Least } n = 18$$

Or: use GC, table of values gives

$$\begin{aligned} \text{least } n &= 18 \\ n+1 &= 19 \end{aligned}$$

Therefore, at the beginning of 19th year, the amount in the fund will be less than \$1000 for the first time

(ii) When $n+1=16 \Rightarrow n=15$,

$$\begin{aligned} \frac{100}{3}\left[(180-k)(1.03)^{15}+k\right] &\le 0 \\ (180-k)(1.03)^{15}+k &\le 0 \\ 180(1.03)^{15}+k\left[1-(1.03)^{15}\right] &\le 0 \\ k\left[1-(1.03)^{15}\right] &\le -180(1.03)^{15} \\ k\left[(1.03)^{15}-1\right] &\ge 180(1.03)^{15} \\ k &\ge \frac{180(1.03)^{15}}{(1.03)^{15}-1} \\ k &\ge 502.6 \\ \text{Least } k &= 503 \text{ (nearest integer)} \end{aligned}$$

Or: from GC (plot graph or table of values),

$$\text{least } k = 503 \text{ (nearest integer)}$$

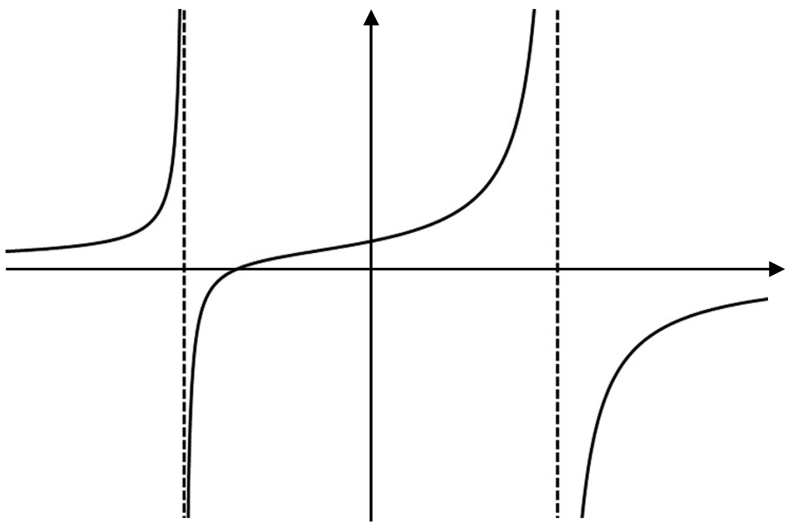
5 **(a)** $(x-2)^2 = a^2(1-y^2)$

$$\Rightarrow \frac{(x-2)^2}{a^2}+y^2=1$$

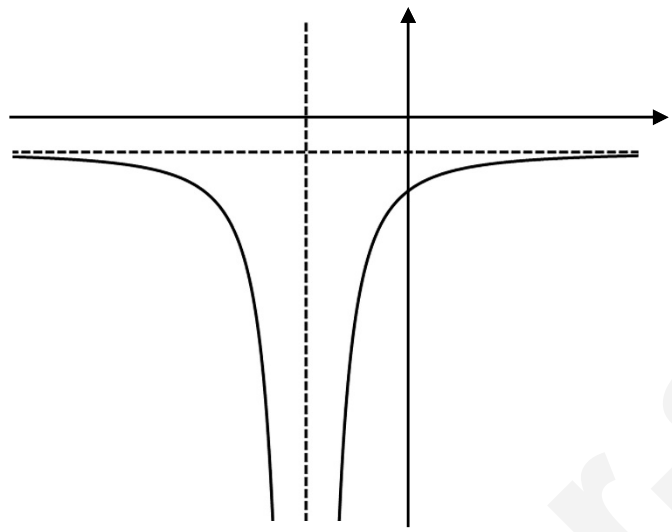
$$\Rightarrow \frac{(x-2)^2}{a^2}+\frac{(y-0)^2}{1^2}=1,$$

$1<a<2$

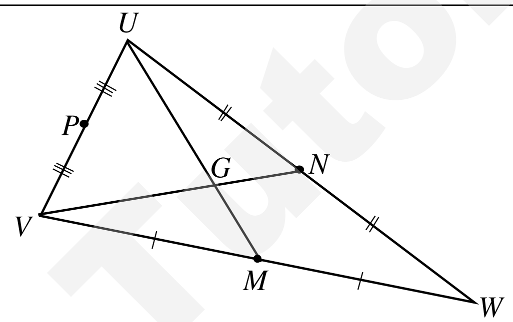
(b)(i) $y=\dfrac{1}{\mathrm{f}(x)}$



(b)(ii) $y = \mathrm{f}'(x)$



6



(i)

By Ratio Theorem, $$\overrightarrow{UM} = \frac{\overrightarrow{UW}+\overrightarrow{UV}}{2} = \frac{\mathbf{w}-\mathbf{u}+\mathbf{v}-\mathbf{u}}{2} = \frac{1}{2}(\mathbf{v}+\mathbf{w}-2\mathbf{u}) \quad \text{(Shown)}$$

(ii) Vector equation of line *UM* is $\mathbf{r} = \mathbf{u} + \lambda(\mathbf{w}+\mathbf{v}-2\mathbf{u}),\ \lambda \in \because$

$$\overrightarrow{VN} = \frac{\overrightarrow{VW}+\overrightarrow{VU}}{2} = \frac{\mathbf{w}-\mathbf{v}+\mathbf{u}-\mathbf{v}}{2} = \frac{1}{2}(\mathbf{w}+\mathbf{u}-2\mathbf{v})$$

Vector equation of line *VN* is $\mathbf{r} = \mathbf{v} + \mu(\mathbf{w}+\mathbf{u}-2\mathbf{v}),\ \mu \in \because$

At point of intersection *G*,

$$\mathbf{u} + \lambda(\mathbf{w}+\mathbf{v}-2\mathbf{u}) = \mathbf{v} + \mu(\mathbf{w}+\mathbf{u}-2\mathbf{v})$$

For **u**: $1 - 2\lambda = \mu$

For **w**: $\lambda = \mu$

Solving, $\lambda = \frac{1}{3} = \mu$

	$\overrightarrow{OG} = \mathbf{u}+\frac{1}{3}(\mathbf{w}+\mathbf{v}-2\mathbf{u})$ $= \frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$ (Shown) **(iii)** $\mathbf{u}=\begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{v}=\begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{w}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ $\overrightarrow{OG} = \frac{1}{3}\left[\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}\right]=\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{pmatrix}$ $\left\|\overrightarrow{OG}\right\| = \sqrt{3\left(\frac{1}{3^2}\right)}=\sqrt{\frac{1}{3}}$ Direction cosines of $\overrightarrow{OG}$ are $\frac{\frac{1}{3}}{\sqrt{\frac{1}{3}}}, \frac{\frac{1}{3}}{\sqrt{\frac{1}{3}}}, \frac{\frac{1}{3}}{\sqrt{\frac{1}{3}}}$, i.e., $\underline{\underline{\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}}}$
7	**(a)** $u=2-\mathrm{i}\sin^2\theta,\ v=2\cos^2\theta+\mathrm{i}\sin^2\theta$ $u-v = 2-\mathrm{i}\sin^2\theta-2\cos^2\theta-\mathrm{i}\sin^2\theta$ $= 2-2\cos^2\theta-2\mathrm{i}\sin^2\theta$ $= 2(1-\cos^2\theta)-2\mathrm{i}\sin^2\theta$ $= \underline{\underline{2\sin^2\theta-2\mathrm{i}\sin^2\theta}}$ or $\underline{\underline{2(\sin^2\theta)(1-\mathrm{i})}}$ $\|u-v\| = 2\left\|\sin^2\theta-\mathrm{i}\sin^2\theta\right\|$ $= 2\sqrt{\sin^4\theta+\sin^4\theta}$ $= 2\sqrt{2\sin^4\theta}$ $= \underline{\underline{2\sqrt{2}\sin^2\theta}}$ or $2\left\|\sin^2\theta\right\|\left\|1-\mathrm{i}\right\|$ $= 2(\sin^2\theta)\sqrt{1+1}$ $= \underline{\underline{2\sqrt{2}\sin^2\theta}}$ Note that $u-v$ lies in the 4th quadrant. $\arg(u-v) = -\tan^{-1}\frac{2\sin^2\theta}{2\sin^2\theta}$ $= -\tan^{-1}1 = \underline{\underline{-\frac{\pi}{4}}}$ Or: $\arg(u-v) = \arg(2\sin^2\theta-2\mathrm{i}\sin^2\theta)=\arg\left[2(\sin^2\theta)(1-\mathrm{i})\right]$ $= \arg(2\sin^2\theta)+\arg(1-\mathrm{i})$

$$= \ 0+\left(-\frac{\pi}{4}\right) = -\frac{\pi}{4}$$

(b) Method 1 Solve α first then factorise quadratic expression or use sum of roots

$$x^2+(\mathrm{i}-3)x+2(1-\mathrm{i}) = 0$$

Sub. $x=\alpha \in \because$,

$$\alpha^2+(\mathrm{i}-3)\alpha+2(1-\mathrm{i}) = 0$$

$$\left(\alpha^2-3\alpha+2\right)+\mathrm{i}(\alpha-2) = 0$$

Comparing imaginary parts,

$$\alpha-2 = 0$$

$$\alpha = 2$$

$$x^2+(\mathrm{i}-3)x+2(1-\mathrm{i}) = (x-2)(x-\beta)$$

Comparing constants,

$$2(1-\mathrm{i}) = 2\beta$$

$$\therefore \beta = 1-\mathrm{i}$$

Or: Sum of roots, $\alpha+\beta = -(\mathrm{i}-3)$

$$2+\beta = 3-\mathrm{i}$$

$$\therefore \beta = 1-\mathrm{i}$$

Method 2 Factorise the quadratic expression first

$$x^2+(\mathrm{i}-3)x+2(1-\mathrm{i}) = (x-\alpha)(x-\beta)$$

Comparing coefficients of x,

$$\mathrm{i}-3 = -(\alpha+\beta)$$

$$\alpha+\beta = 3-\mathrm{i} \qquad (1)$$

Comparing constants,

$$\alpha\beta = 2-2\mathrm{i} \qquad (2)$$

From (1), $\beta = 3-\mathrm{i}-\alpha \qquad (3)$

Sub. (3) into (2), $\alpha(3-\mathrm{i}-\alpha) = 2-2\mathrm{i}$

$$3\alpha-\alpha^2-\alpha\mathrm{i} = 2-2\mathrm{i}$$

Comparing imaginary parts, $\alpha = 2$

Sub. into (3), $\beta = 3-\mathrm{i}-2$

$$\therefore \beta = 1-\mathrm{i}$$

Or:

Let $\beta=a+b\mathrm{i}$, where $a\in \because, b\in \because$ and $b\neq 0$

$$x^2+(\mathrm{i}-3)x+2(1-\mathrm{i}) = (x-\alpha)\left[x-(a+b\mathrm{i})\right]$$

Comparing coefficients of x,

$$\mathrm{i}-3 = -a-b\mathrm{i}-\alpha$$

$$b = -1 \qquad \text{(Comparing imaginary parts)}$$

$$a+\alpha = 3 \quad (1) \qquad \text{(Comparing real parts)}$$

Comparing constants,

$$2-2\mathrm{i} = \alpha(a+b\mathrm{i})$$

$$= \alpha(a-\text{i}) = \alpha a - \alpha\text{i}$$

$$\underline{\underline{\alpha = 2}} \qquad \text{(Comparing imaginary parts)}$$

Sub. into (1), $a = 3-\alpha = 3-2 = 1$

$$\therefore \underline{\underline{\beta = 1-\text{i}}}$$

<u>Method 3 Solve *x* first using quadratic formula</u>

$$x^2 + (\text{i}-3)x + 2(1-\text{i}) = 0$$

$$x = \frac{-(\text{i}-3) \pm \sqrt{(\text{i}-3)^2 - 4(1)[2(1-\text{i})]}}{2}$$

$$= \frac{3-\text{i} \pm \sqrt{\text{i}^2 - 6\text{i} + 9 - 8 + 8\text{i}}}{2} = \frac{3-\text{i} \pm \sqrt{2\text{i}}}{2}$$

$$= \frac{3-\text{i} \pm (1+\text{i})}{2} \qquad \text{(use GC to find } \sqrt{2\text{i}}\text{)}$$

$$= 2 \text{ or } 1-\text{i}$$

$$\therefore \underline{\underline{\alpha = 2}} \text{ and } \underline{\underline{\beta = 1-\text{i}}}$$

For comparison purpose:

If GC is **not** used to find $\sqrt{2\text{i}}$, then the algebraic works will look as follows:

Let $\sqrt{2\text{i}} = a + b\text{i}$, where $a \in \because, b \in \because$

$2\text{i} = a^2 - b^2 + 2ab\text{i}$

Compring real parts, $a^2 - b^2 = 0$

$a^2 = b^2$

$a = \pm b \quad (1)$

Compring imaginary parts, $ab = 1 \quad (2)$

When $a = b$,

Sub. into (2), $a^2 = 1$

$a = \pm 1$

When $a = 1$, $b = 1$. When $a = -1$, $b = -1$

$\pm\sqrt{2\text{i}} = \pm(1+\text{i})$

When $a = -b$

Sub. into (2), $-b^2 = 1 \quad (\text{NA} \because b \in \because)$

$$\therefore x = \frac{3-\text{i} \pm (1+\text{i})}{2} = 2 \text{ or } 1-\text{i}$$

$$\therefore \underline{\underline{\alpha = 2}} \text{ and } \underline{\underline{\beta = 1-\text{i}}}$$

8 **(a)(i)** Let A cm^2 be area of the circular patch.

$$A = \pi r^2$$

$$\frac{\text{d}A}{\text{d}r} = 2\pi r$$

Given $\frac{\text{d}A}{\text{d}t} = 6\pi$ cm^2/s, a **constant**

	This means that, in 1 s, A increases by $6\pi \text{ cm}^2$ **constantly**. When $t=0$, $A = 0$ When $t=24$, $A = 24\times 6\pi = 144\pi$ $\pi r^2 = 144\pi$ $r = 12$ (reject $r=-12$ since $r>0$) $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi(12) = 24\pi$ $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$ $6\pi = 24\pi\frac{\mathrm{d}r}{\mathrm{d}t}$ $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{4}$ $\therefore$ rate of change of the radius is $\underline{\underline{\frac{1}{4}\text{ cm/ s}}}$. **(a)(ii)** $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$ $6\pi = 2\pi r\frac{\mathrm{d}r}{\mathrm{d}t}$ $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{6\pi}{2\pi r} = \frac{3}{r}$ <u>Method 1</u> As r increases, $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{r}$ decreases, $\therefore \frac{\mathrm{d}r}{\mathrm{d}t}$ will $\underline{\underline{\text{decrease}}}$ as time passes. <u>Method 2</u> $\frac{\mathrm{d}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)}{\mathrm{d}t} = \frac{\mathrm{d}\left(\frac{3}{r}\right)}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$ $= \frac{-3}{r^2}\left(\frac{3}{r}\right) = \frac{-9}{r^3} < 0$ $\therefore \frac{\mathrm{d}r}{\mathrm{d}t}$ will $\underline{\underline{\text{decrease}}}$ as time passes. **(b)(i)** $V = \pi r^2 h$ $355 = \pi r^2 h$ $\pi r h = \frac{355}{r}$ $C = K(2\pi r h) + 2K(4r^2)$ $= K\left[2\left(\frac{355}{r}\right)+8r^2\right]$

$$= K\left(\frac{710}{r}+8r^2\right) \qquad \text{(Shown)}$$

(b)(ii) $\dfrac{dC}{dr} = \left(-\dfrac{710}{r^2}+16r\right)K$

For C to be a minimum, $\dfrac{dC}{dr}=0$.

$$-\frac{710}{r^2}+16r = 0$$

$$-710+16r^3 = 0$$

$$r^3 = \frac{355}{8}$$

$$r = \sqrt[3]{\frac{355}{8}} = 3.54 \text{ (3 sf)}$$

$$\frac{d^2C}{dr^2} = \left(\frac{1420}{r^3}+16\right)K = \left(\frac{1420}{\frac{355}{8}}+16\right)K = 48K > 0$$

Or

r	3.5	$\sqrt[3]{\frac{355}{8}} \approx 3.54$	3.6
$\frac{dC}{dr}$	$-1.96K < 0$	0	$2.82K > 0$

So, $r = \sqrt[3]{\frac{355}{8}}$ does give the minimum cost.

Recall $355 = \pi r^2 h$

$$h = \frac{355}{\pi r^2}$$

$$\therefore \frac{h}{r} = \frac{355}{\pi r^3} = \frac{355}{\pi\left(\frac{355}{8}\right)}$$

$$= \frac{8}{\pi} \qquad \text{(Shown)}$$

9

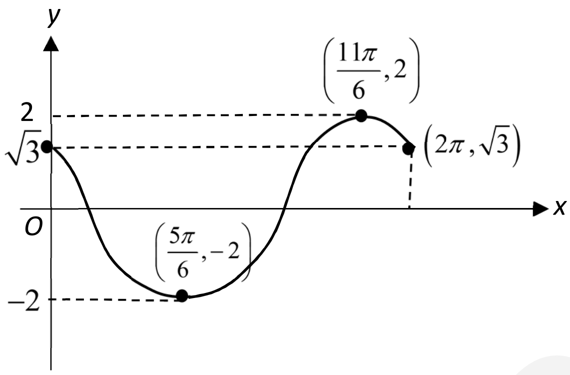
(i) $\sqrt{3}\cos x - \sin x = R\cos(x+\alpha)$

$$R = \sqrt{\left(\sqrt{3}\right)^2+1^2} = \sqrt{4} = \underline{\underline{2}}$$

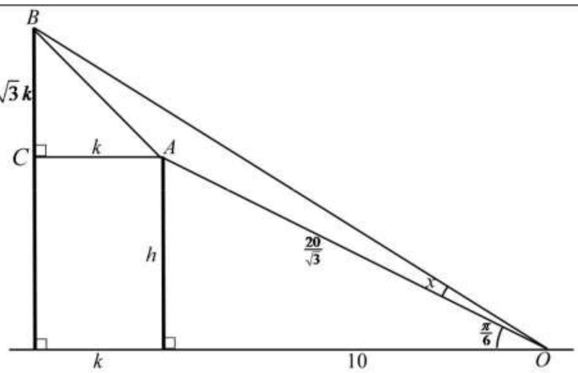
$$\alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \underline{\underline{\frac{\pi}{6}}}$$

(ii) $y = \sqrt{3}\cos x - \sin x = 2\cos\left(x+\dfrac{\pi}{6}\right)$

$$y=\cos x \xrightarrow{A} y=\cos(x+\alpha) \xrightarrow{B} y=R\cos(x+\alpha)$$

	A: Translation by α radians in the negative *x*-direction, followed by *B*: Scaling parallel to the *y*-axis by a scale factor *R*. [can be *B* followed by *A*] **(iii)** $\mathrm{f}: x \mapsto \sqrt{3}\cos x - \sin x,\ 0 \le x \le 2\pi$  Range of f, $R_f = \underline{\underline{[-2, 2]}}$. **(iv)** $\mathrm{g}: x \mapsto \mathrm{f}(x),\ 0 \le x \le k.$ Largest $k = \underline{\underline{\frac{5\pi}{6}}}$. Let $y = \mathrm{g}(x)$. $y = 2\cos\left(x + \frac{\pi}{6}\right)$ $\cos\left(x + \frac{\pi}{6}\right) = \frac{y}{2}$ $\Rightarrow x = \cos^{-1}\frac{y}{2} - \frac{\pi}{6}$ $\therefore \mathrm{g}^{-1}(x) = \underline{\underline{\cos^{-1}\frac{x}{2} - \frac{\pi}{6}}}$ **(v)** $\mathrm{h}: x \mapsto x - 2,\ x \ge 0$ Since $R_h = [-2, +\infty)$ and $D_f = [0, 2\pi]$, $R_h \not\subset D_f$, fh does not exist.
10	

(i)



$$\cos\frac{\pi}{6}=\frac{10}{OA}$$
$$\frac{\sqrt{3}}{2}=\frac{10}{OA}$$
$$OA=\frac{20}{\sqrt{3}}\text{ m}$$

(Shown)

(ii)

$$AB = \sqrt{k^2+3k^2}=\sqrt{4k^2}=2k \qquad \text{(Shown)}$$
$$\angle BAC = \tan^{-1}\frac{\sqrt{3}\,k}{k}=\tan^{-1}\sqrt{3}$$
$$= \frac{\pi}{3} \qquad \text{(Shown)}$$

(iii)

$$\angle CBO=\tfrac{\pi}{2}-\left(\tfrac{\pi}{6}+x\right)=\tfrac{\pi}{3}-x$$
$$\angle CBA=\tfrac{\pi}{2}-\tfrac{\pi}{3}=\tfrac{\pi}{6}$$
$$\angle ABO=\tfrac{\pi}{3}-x-\tfrac{\pi}{6}=\tfrac{\pi}{6}-x$$

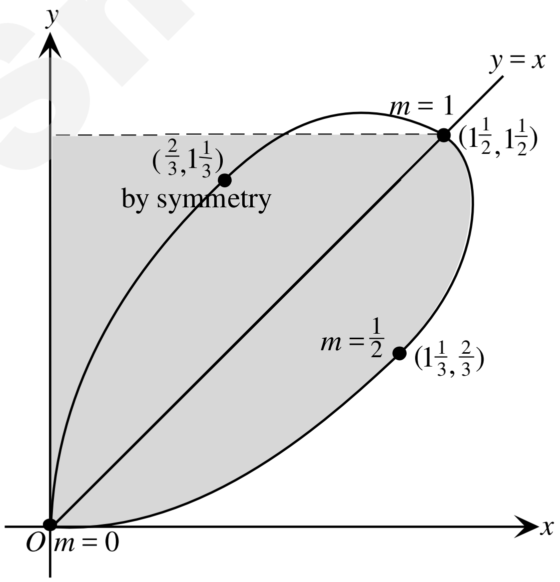
Or:

$$\angle BAO=2\pi-\tfrac{\pi}{2}-\tfrac{\pi}{3}-\tfrac{\pi}{3}\quad(\angle\text{ at a pt})$$
$$=\tfrac{5\pi}{6}$$
$$\angle ABO=\pi-x-\tfrac{5\pi}{6}=\tfrac{\pi}{6}-x$$

In ΔABO,

$$\frac{2k}{\sin x}=\frac{\frac{20}{\sqrt{3}}}{\sin\left(\frac{\pi}{6}-x\right)}$$
$$k=\frac{10\sin x}{\sqrt{3}\sin\left(\frac{\pi}{6}-x\right)}$$

(iv)

	$k=\dfrac{10\sin x}{\sqrt{3}\sin\left(\frac{\pi}{6}-x\right)}$ $=\dfrac{10\sin x}{\sqrt{3}\left(\sin\frac{\pi}{6}\cos x-\cos\frac{\pi}{6}\sin x\right)}$ $\approx\dfrac{10x}{\sqrt{3}\left[\frac{1}{2}\left(1-\frac{x^2}{2}\right)-\frac{\sqrt{3}}{2}x\right]}$ $=\dfrac{10x}{\frac{\sqrt{3}}{2}\left[\left(1-\frac{x^2}{2}\right)-\sqrt{3}x\right]}$ $=\dfrac{20x}{\sqrt{3}}\left[1-\left(\sqrt{3}x+\frac{x^2}{2}\right)\right]^{-1}$ $\approx\dfrac{20x}{\sqrt{3}}\left(1+\sqrt{3}x\right)$ $=\underline{\underline{\dfrac{20}{\sqrt{3}}\left(x+\sqrt{3}x^2\right)}}$
11	**(a)(i)** $x=\dfrac{3m}{1+m^3},\quad y=\dfrac{3m^2}{1+m^3},\quad m\geq 0$ $y = x$ $\dfrac{3m^2}{1+m^3}=\dfrac{3m}{1+m^3}$ $m(m-1)=0$ $m=\underline{\underline{0\text{ or }1}}$ **(a)(ii)** When $m=0$, $y=0$. When $m=1$, $y=\dfrac{3}{1+1}=\dfrac{3}{2}$.  **<u>Notes</u>**: Use GC to trace the path to see how m varies when the point moves along the path.

Area of (lower) half of the "leaf" is

$$\frac{1}{2}A = \int_0^{\frac{3}{2}} x\,\mathrm{d}y - \text{area of } \Delta \qquad \left(\text{Note: } \int_0^{\frac{3}{2}} x\,\mathrm{d}y = \text{shaded area}\right)$$

$$A = 2\left[\int_0^{\frac{3}{2}} x\,\mathrm{d}y - \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\right]$$

$$= 2\left(\int_0^{\frac{3}{2}} x\,\mathrm{d}y - \frac{9}{8}\right) \qquad \text{(Shown)}$$

$$2\left(\int_0^{\frac{3}{2}} x\,\mathrm{d}y - \frac{9}{8}\right) = 2\int_0^1 \frac{3m}{1+m^3}\left[\frac{6m\left(1+m^3\right)-3m^2\left(3m^2\right)}{\left(1+m^3\right)^2}\right]\mathrm{d}m - \frac{9}{4}$$

$$= 2\int_0^1 \frac{3m\left(6m-3m^4\right)}{\left(1+m^3\right)^3}\mathrm{d}m - \frac{9}{4}$$

$$= \frac{15}{4} - \frac{9}{4} \qquad \text{(by GC)}$$

$$= \underline{\underline{\frac{3}{2}}}$$

(b)

$$y = \ln x$$
$$x = \mathrm{e}^y$$

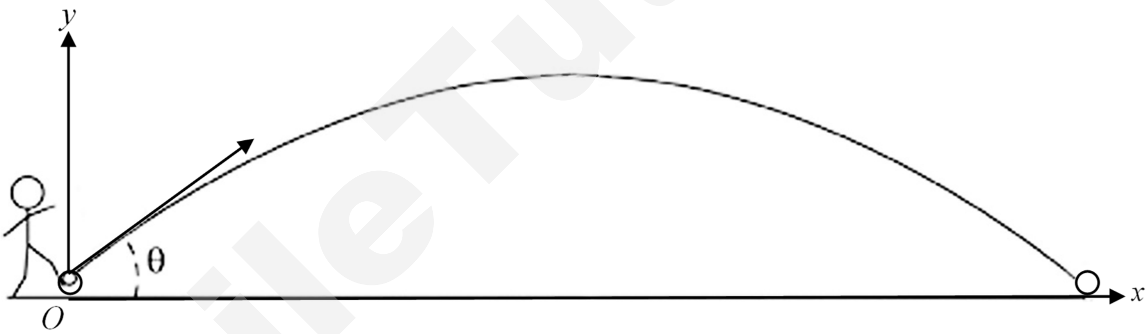
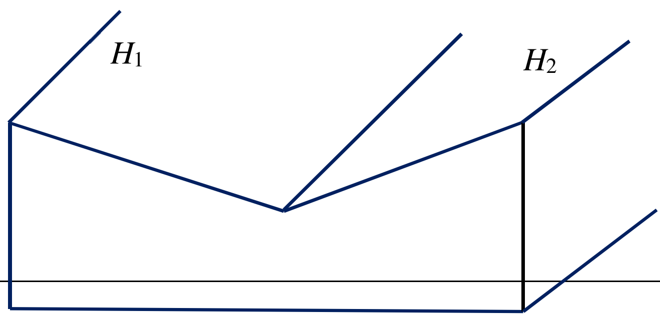
$$V_A = \pi\int_0^c \left(\mathrm{e}^y\right)^2 \mathrm{d}y$$
$$= \pi\int_0^c \mathrm{e}^{2y}\,\mathrm{d}y$$
$$= \pi\left[\frac{1}{2}\mathrm{e}^{2y}\right]_0^c$$
$$= \frac{\pi}{2}\left(\mathrm{e}^{2c}-1\right)$$

$$V_B = (1-c)\pi\mathrm{e}^2 - \pi\int_c^1 \left(\mathrm{e}^y\right)^2 \mathrm{d}y \quad \text{or} \quad \pi\int_c^1 \left[\mathrm{e}^2 - \left(\mathrm{e}^y\right)^2\right]\mathrm{d}y$$
$$= \pi(1-c)\mathrm{e}^2 - \pi\left[\frac{1}{2}\mathrm{e}^{2y}\right]_c^1$$
$$= \pi(1-c)\mathrm{e}^2 - \frac{\pi}{2}\left(\mathrm{e}^2 - \mathrm{e}^{2c}\right)$$

$$V_A = V_B$$
$$\frac{\pi}{2}\left(\mathrm{e}^{2c}-1\right) = \pi(1-c)\mathrm{e}^2 - \frac{\pi}{2}\left(\mathrm{e}^2 - \mathrm{e}^{2c}\right)$$
$$\mathrm{e}^{2c} - 1 = 2\mathrm{e}^2(1-c) - \mathrm{e}^2 + \mathrm{e}^{2c}$$
$$= 2\mathrm{e}^2 - 2c\mathrm{e}^2 - \mathrm{e}^2 + \mathrm{e}^{2c}$$
$$2c\mathrm{e}^2 = \mathrm{e}^2 + 1$$

	$c = \dfrac{e^2+1}{2e^2}$ (Shown)

H2 Mathematics 2017 Prelim Exam Paper 2 Question

1	It is given that $y=\ln(1+\sin x)$. **(i)** Find $\dfrac{dy}{dx}$. Show that $\dfrac{d^2y}{dx^2}=-e^{-y}$. [4] **(ii)** Express $\dfrac{d^4y}{dx^4}$ in terms of $\dfrac{dy}{dx}$ and e^{-y}. [3] **(iii)** Hence, find the first four non-zero terms in the Maclaurin series for $\ln(1+\sin x)$. [3]
2	John kicked a ball at an acute angle θ made with the horizontal, and it moved in a projectile motion, as shown in the diagram. The initial velocity of the ball is u m s^{-1}. Taking John's position where he kicked the ball as the origin O, the ball's displacement curve is given by the parametric equations: horizontal displacement, $x=ut\cos\theta$, vertical displacement, $y=ut\sin\theta-5t^2$, where u and θ are constants and t is the time in seconds after the ball is kicked.  **(i)** Show that $\dfrac{dy}{dx}=\tan\theta-\dfrac{10}{u}t\sec\theta$. [2] **(ii)** If the initial velocity of the ball is 30 m s^{-1}, find the equation of the tangent to the displacement curve at the point where $t=\dfrac{1}{2}$, giving your answer in the form $y=(a\tan\theta+b\sec\theta)x+c$, where a, b and c are constants to be determined. [3]
3	Peter is using equations of planes to model two hillsides that meet along a river. The river is modelled by the line where the two planes meet. 

	One of the hillsides, H_1, contains the points *A*, *B* and *C* with coordinates (3, 0, 2), (1, 0, 3) and $(2, -3, 5)$ respectively. The point *A* is on the river. The other hillside H_2 has equation $2x - y + kz = 14,$ where *k* is a constant. **(i)** Find a vector equation of H_1 in scalar product form. [4] **(ii)** Show that $k = 4$ and deduce that point *B* is also on the river. [3] **(iii)** Write down a cartesian equation of the river. [1] **(iv)** Show that *B* is the point on the river that is nearest to *C*. Hence find the exact distance from *C* to the river. [3] **(v)** Find the acute angle between *BC* and H_2. [2]
4	To determine whether the amount of preservatives in a particular brand of bread meets the safety limit of preservatives present, the Food Regulatory Authority (FRA) conducted a test to examine the growth of fungus on a piece of bread over time after its expiry date. The piece of bread has a surface area of $100\ \text{cm}^2$. The staff from FRA estimate the amount of fungus grown and the rate at which it is growing by finding the area of the piece of bread the fungus covers over time. They believe that the area, $A\ \text{cm}^2$, of fungus present *t* days after the expiry date is such that the rate at which the area is increasing is proportional to the product of the area of the piece of bread covered by the fungus and the area of the bread not covered by the fungus. It is known that the initial area of fungus is $20\ \text{cm}^2$ and that the area of fungus is $40\ \text{cm}^2$ five days after the expiry date. **(i)** Write down a differential equation expressing the relation between *A* and t. [1] **(ii)** Find the value of *t* at which 50% of the piece of bread is covered by fungus, giving your answer correct to 2 decimal places. [6] **(iii)** Given that this particular brand of bread just meets the safety limit of the amount of preservatives present when the test is concluded 2 weeks after the expiry date, find the range of values of *A* for any piece of bread of this brand to be deemed safe for human consumption in terms of the amount of preservatives present, giving your answer correct to 2 decimal places. [2] **(iv)** Write the solution of the differential equation in the form $A = \mathrm{f}(t)$ and sketch this curve. [3]
5	The probability distribution of a discrete random variable, *X*, is shown below. <table><tr><td>x</td><td>1</td><td>2</td></tr><tr><td>$\mathrm{P}(X = x)$</td><td>a</td><td>b</td></tr></table> Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ in terms of *a*. [5]
6	**(i)** Find the number of 3-digit numbers that can be formed using the digits 1, 2 and 3 when **(a)** no repetitions are allowed, [1] **(b)** any repetitions are allowed, [1]

(c) each digit may be used at most twice. [2]

(ii) Find the number of 4-digit numbers that can be formed using the digits 1, 2 and 3 when each digit may be used at most twice. [5]

7 At a canning factory, cans are filled with potato puree. The machine which fills the cans is set so that the volume of potato puree in a can has mean 420 millilitres. After the machine is recalibrated, a quality control officer wishes to check whether the mean volume has changed. A random sample of 30 cans of potato puree is selected and the volume of the puree in each can is recorded. The sample mean volume is $\bar{x}$ millilitres and the sample variance is 12 millilitres^2.

(i) Given that $\bar{x} = 418.55$, carry out a test at the 1% level of significance to investigate whether the mean volume has changed. State, giving a reason, whether it is necessary for the volumes to have a normal distribution for the test to be valid. [6]

(ii) Use an algebraic method to calculate the range of values of $\bar{x}$, giving your answer correct to 2 decimal places, for which the result of the test at the 1% level of significance would be to reject the null hypothesis. [3]

8 In this question you should state clearly the values of the parameters of any normal distribution you use.

The mass of a tomato of variety *A* has normal distribution with mean 80 g and standard deviation 11 g.

(i) Two tomatoes of variety *A* are randomly chosen. Find the probability that one of the tomatoes has mass more than 90 g and the other has mass less than 90 g. [3]

The mass of a tomato of variety *B* has normal distribution with mean 70 g. These tomatoes are packed in sixes using packaging that weighs 15 g.

(ii) The probability that a randomly chosen pack of 6 tomatoes of variety *B* including packaging, weighs less than 450 g is 0.8463. Show that the standard deviation of the mass of a tomato of variety *B* is 6 g, correct to the nearest gram. [4]

(iii) Tomatoes of variety *A* are packed in fives using packaging that weighs 25 g. Find the probability that the total mass of a randomly chosen pack of variety *A* is greater than the total mass of a randomly chosen pack of variety *B*, using 6 g as the standard deviation of the mass of a tomato of variety *B*. [5]

9 A jar contains 5 identical balls numbered 1 to 5. A fixed number, *n*, of balls are selected and the number of balls with an even score is denoted by *X*.

(i) Explain how the balls should be selected in order for *X* to be well modelled by a binomial distribution. [2]

Assume now that *X* has the distribution $\mathrm{B}\left(n, \frac{2}{5}\right)$.

(ii) Given that $n = 10$, find $\mathrm{P}(X \geq 4)$. [2]

(iii) Given that the mean of *X* is 4.8, find *n*. [2]

(iv) Given that P(*X* = 0 or 1) < 0.01, write down an inequality for *n* and find the least value of *n*. [3]

Shawn and Arvind take turns to draw one ball from the jar at random. The first person who draws a ball with an even score wins the game. Shawn draws first.

(v) Show that the probability that Shawn wins the game is $\frac{3}{5}$ if the selection of balls is done without replacement. [2]

(vi) Find the probability that Shawn wins the game if the selection of balls is done with replacement. [2]

10 **(a)** Traffic engineers are studying the correlation between traffic flow on a busy main road and air pollution at a nearby air quality monitoring station. Traffic flow, x, is recorded automatically by sensors and reported each hour as the average flow in vehicles per hour for the preceding hour. The air quality monitoring station provides, each hour, an overall pollution reading, y, in a suitable unit (higher readings indicate more pollution). Data for a random sample of 8 hours are as follows.

Traffic flow, x	1796	1918	2120	2315	2368	2420	2588
Pollution reading, y	1.0	2.2	3.5	4.2	4.3	4.5	4.9

(i) Draw the scatter diagram for these values, labelling the axes. [2]

It is thought that the pollution y can be modelled by one of the formulae

$$y = a + bx \qquad\qquad y^2 = c + dx$$

where a, b, c and d are constants.

(ii) Find the value of the product moment correlation coefficient between
(a) x and y,
(b) x and y^2. [2]

(iii) Use your answers to parts (i) and (ii) to explain which of $y = a + bx$ or $y^2 = c + dx$ is the better model. [2]

(iv) It is required to estimate the value of y for which $x = 2000$. Find the equation of a suitable regression line, and use it to find the required estimate. [2]

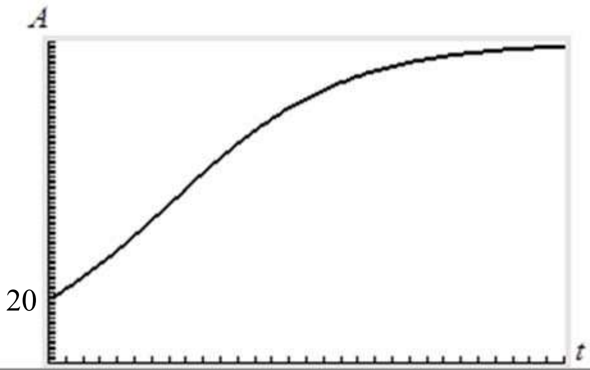
(v) The local newspaper carries a headline "Heavy traffic causes air pollution". Comment briefly on the validity of this headline in the light of your results. [1]

(b) The diagram below shows an old research paper that has been partially destroyed. The surviving part of the paper contains incomplete information about some bivariate data from an experiment. Calculate the missing constant at the end of the equation of the second regression line. [3]

The mean of x is 4.4. The
The equation of the regression line of y on x is $y = 2.5x + 3.8$
The equation of the regression line of x on y is $x = 1.5y$

ANNEX B

JJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Maclaurin series	(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos x}{1+\sin x}$ (ii) $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -\left(\mathrm{e}^{-y}\right)^2 - \mathrm{e}^{-y}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ or $-\mathrm{e}^{-y}\left[\mathrm{e}^{-y} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]$ (iii) $\ln\left(1+\sin x\right) = x - \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 - \dfrac{1}{12}x^4 + \ldots$
2	Differentiation & Applications	(ii) $y = \left(\tan\theta - \dfrac{1}{6}\sec\theta\right)x + \dfrac{5}{4}$
3	Vectors	(i) $\mathbf{r}\bullet\begin{pmatrix}3\\5\\6\end{pmatrix} = 21$ (iii) $\dfrac{x-3}{-2} = z-2,\ y=0$ or $\dfrac{x-1}{-2} = z-3,\ y=0$ (iv) $\sqrt{14}$ (v) $49.3°$ or 0.861 rad
4	Differential Equations	(i) $\dfrac{\mathrm{d}A}{\mathrm{d}t} = kA\left(100-A\right)$ (ii) 7.07 days (iii) $79.58 \le A \le 100$ (iv) $A = \dfrac{100\mathrm{e}^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}{4+\mathrm{e}^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}$ or $\dfrac{100\mathrm{e}^{0.196t}}{4+\mathrm{e}^{0.196t}}$ 
5	DRV	$\mathrm{E}\left(X\right) = 2-a$ and $\mathrm{Var}\left(X\right) = a-a^2$
6	P&C, Probability	(i) (a) 6, (b) 27, (c) 24 (ii) 54

7	Hypothesis Testing	(i) Since p-value = 0.0242 $> \alpha = 0.01$, we do not reject $\mathrm{H_0}$ at 1% level of significance and conclude that there is insufficient evidence that the population mean volume has changed. It is not necessary for the volumes to have a normal distribution for the test to be valid as $n = 30$ is large. (ii) $\bar{x} \le 418.34$ or $\bar{x} \ge 421.66$
8	Normal Distribution	(i) 0.297 (iii) 0.364
9	Binomial Distribution	(i) (1) Selection of balls is done with replacement. (2) The balls are thoroughly mixed before each selection. (ii) 0.618 (iii) 12 (iv) $\left(\frac{3}{5}\right)^n + n\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^{n-1} < 0.01$, least n = 14 (vi) $\frac{5}{8}$ or 0.625
10	Correlation & Linear Regression	(a) (ii) (a) 0.959, (b) 0.995 (iii) $y^2 = c + dx$ is the better model since • From **(i)**, the points on the scatter diagram seem to lie on a concave downward curve. • From **(ii)**, the product moment correlation coefficient between x and y^2 is closer to 1, as compared to that between x and y. (iv) $y^2 = 0.0279x - 48.0$, y = 2.79 when x 2000. (v) May not be valid as coorelation does not necessarily imply causation. (b) 17.8

H2 Mathematics 2017 Prelim Exam Paper 2 Solution

1

(i)

$$y = \ln(1+\sin x) \quad \Rightarrow e^y = 1+\sin x$$

$$\frac{dy}{dx} = \frac{\cos x}{1+\sin x} \qquad \text{[B1]}$$

$$\frac{d^2y}{dx^2} = \frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x-1}{(1+\sin x)^2} \qquad \text{[A1]}$$

$$= \frac{-(\sin x+1)}{(1+\sin x)^2}$$

$$= \frac{-1}{1+\sin x}$$

$$= \frac{-1}{e^y}$$

$$= -e^{-y} \quad \text{(Shown)}$$

(ii)

$$\frac{d^3y}{dx^3} = -e^{-y}\left(-\frac{dy}{dx}\right)$$

$$= e^{-y}\frac{dy}{dx}$$

$$\frac{d^4y}{dx^4} = e^{-y}\frac{d^2y}{dx^2}+e^{-y}\left(-\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)$$

$$= e^{-y}\left(-e^{-y}\right)-e^{-y}\left(\frac{dy}{dx}\right)^2 \qquad \text{(from (i))}$$

$$= -\left(e^{-y}\right)^2-e^{-y}\left(\frac{dy}{dx}\right)^2 \text{ or } -e^{-y}\left[e^{-y}+\left(\frac{dy}{dx}\right)^2\right]$$

(iii) When $x=0$,

$$y = \ln 1 = 0$$

$$\frac{dy}{dx} = \frac{\cos 0}{1+\sin 0} = 1$$

$$\frac{d^2y}{dx^2} = -e^0 = -1$$

$$\frac{d^3y}{dx^3} = 1$$

$$\frac{d^4y}{dx^4} = -1-1 = -2$$

	$\therefore \ln(1+\sin x) = \quad 0 + x + \frac{(-1)}{2}x^2 + \frac{1}{3!}x^3 + \frac{(-2)}{4!}x^4 + \ldots$ $= \quad x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \ldots$
2	**(i)** $\frac{dx}{dt} = u\cos\theta, \quad \frac{dy}{dt} = u\sin\theta - 10t,$ $\frac{dy}{dx} = \frac{u\sin\theta - 10t}{u\cos\theta}$ $= \tan\theta - \frac{10t}{u\cos\theta}$ $= \tan\theta - \frac{10}{u}t\sec\theta$ (Shown) **(ii)** When $u = 30$ and $t = \frac{1}{2}$, $x = 15\cos\theta, \; y = 15\sin\theta - \frac{5}{4}, \; \frac{dy}{dx} = \tan\theta - \frac{1}{6}\sec\theta$ Equation of tangent is $y - 15\sin\theta + \frac{5}{4} = \left(\tan\theta - \frac{1}{6}\sec\theta\right)(x - 15\cos\theta)$ $= \left(\tan\theta - \frac{1}{6}\sec\theta\right)x - 15\sin\theta + \frac{5}{2}$ $\therefore y = \left(\tan\theta - \frac{1}{6}\sec\theta\right)x + \frac{5}{4}$
3	**(i)** $A(3, 0, 2)$, $B(1, 0, 3)$, $C(2, -3, 5)$ $\overrightarrow{AB} = \begin{pmatrix}1\\0\\3\end{pmatrix} - \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}-2\\0\\1\end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix}2\\-3\\5\end{pmatrix} - \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}-1\\-3\\3\end{pmatrix}$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}-2\\0\\1\end{pmatrix} \times \begin{pmatrix}-1\\-3\\3\end{pmatrix} = \begin{pmatrix}3\\5\\6\end{pmatrix}$ Take $\mathbf{n}_1 = \begin{pmatrix}3\\5\\6\end{pmatrix}$, $\mathbf{a} \bullet \mathbf{n}_1 = \begin{pmatrix}1\\0\\3\end{pmatrix} \bullet \begin{pmatrix}3\\5\\6\end{pmatrix} = 3 + 0 + 18 = 21$ A vector equation of H_1 is $\mathbf{r} \bullet \begin{pmatrix}3\\5\\6\end{pmatrix} = 21$

(ii) Equation of H_2 is $2x - y + kz = 14$.

Sub. $A(3, 0, 2)$ into equation of H_2,

$2(3) - 0 + k(2) = 14$

$\therefore\ k = 4$ (Shown)

Sub. $B(1, 0, 3)$ into LHS of equation of H_2,

LHS = $2x - y + 4z = 2(1) - 0 + 4(3) = 14 =$ RHS

$\therefore\ B$ is also in H_2.

Since B is in both H_1 and H_2, $\therefore\ B$ is on the river. (Deduced)

(iii) Recall $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, using $A(3, 0, 2)$ or $B(1, 0, 3)$,

a cartesian equation of the river (line AB) is

$\underline{\underline{\dfrac{x-3}{-2} = z - 2,\ y = 0}}$ or $\underline{\underline{\dfrac{x-1}{-2} = z - 3,\ y = 0}}$

(iv) Since $\overrightarrow{BC}\cdot\overrightarrow{AB} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}\cdot\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 1(-2) + (-3)(0) + 2(1) = 0$,

BC is perpendicular to AB.

$\therefore\ B$ is the point on the river that is nearest to C.

Exact distance from C to the river

$$= \left|\overrightarrow{BC}\right| = \left|\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}\right| = \sqrt{1+9+4} = \underline{\underline{\sqrt{14}}}$$

(v) Acute angle between BC and H_2

$$\theta = \sin^{-1}\frac{\left|\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}\right|}{\sqrt{14}\sqrt{21}} = \sin^{-1}\frac{13}{\sqrt{14}\sqrt{21}}$$

$= \underline{\underline{49.3°}}$ or $\underline{\underline{0.861 \text{ rad}}}$

4

(i) $\dfrac{\mathrm{d}A}{\mathrm{d}t} = kA(100 - A)$

(ii) $$\int \frac{1}{A(100-A)}\,\mathrm{d}A = \int k\ dt$$

By partial fractions,

$$\frac{1}{A(100-A)} = \frac{1}{100A} + \frac{1}{100(100-A)}$$

$$\therefore \frac{1}{100}\int\left(\frac{1}{A}+\frac{1}{100-A}\right)\,\mathrm{d}A = kt+c$$

$$\frac{1}{100}\left(\ln|A|-\ln|100-A|\right) = kt+c \quad (\because A>0 \text{ and } 100-A>0)$$

$$\frac{1}{100}\left[\ln A-\ln(100-A)\right] = kt+c$$

$$\ln\frac{A}{100-A} = 100(kt+c)$$

$$\frac{A}{100-A} = \mathrm{e}^{100(kt+c)} = \mathrm{e}^{100kt}\mathrm{e}^{100c} = D\mathrm{e}^{k_1 t}$$

where $k_1 = 100k$ and $D = \mathrm{e}^{100c}$.

When $t = 0, A = 20$,

$$\frac{20}{100-20} = D$$

$$D = \frac{1}{4}$$

When $t = 5, A = 40$,

$$\frac{40}{100-40} = \frac{1}{4}\mathrm{e}^{5k_1}$$

$$\frac{1}{4}\mathrm{e}^{5k_1} = \frac{2}{3}$$

$$\mathrm{e}^{5k_1} = \frac{8}{3}$$

$$5k_1 = \ln\frac{8}{3}$$

$$k_1 = \frac{1}{5}\ln\frac{8}{3}$$

$$\therefore \frac{A}{100-A} = \frac{1}{4}\mathrm{e}^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$

When $A = 0.5\times100 = 50$,

$$\frac{50}{100-50} = \frac{1}{4}\mathrm{e}^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$

$$1 = \frac{1}{4}\mathrm{e}^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$

$$\mathrm{e}^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t} = 4$$

$$\left(\frac{1}{5}\ln\frac{8}{3}\right)t = \ln 4$$

$$t = \frac{\ln 4}{\frac{1}{5}\ln\frac{8}{3}} = 7.07 \ \text{(2 dp)}$$

The required time is 7.07 days.

(iii) When $t = 14$ (days),

$$\frac{A}{100-A} = \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)(14)}$$

Method 1 Solve algebraically

$$\frac{A}{100-A} = 3.8963 \text{ (5 sf)}$$

$$A = (100-A)(3.8963)$$

$$= 389.63 - 3.8963A$$

$$4.8963A = 389.63$$

$$A = 79.58 \text{ (2 dp)}$$

For the bread to be deemed safe for human consumption in terms of the amount of preservatives present, $\underline{\underline{79.58 \le A \le 100}}$

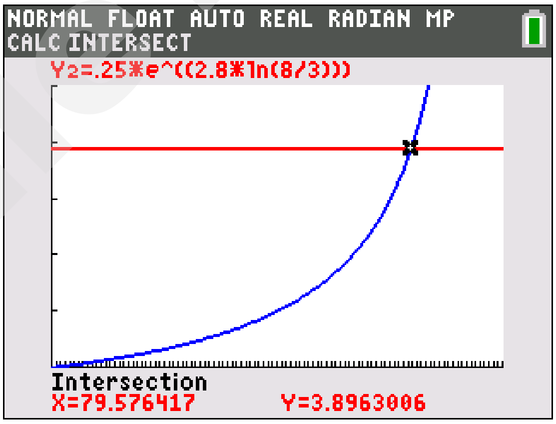
Method 2 Use GC to plot graphs

Use GC to plot $y = \dfrac{A}{100-A}$ and $y = \dfrac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)(14)}$ (≈ 3.8963)

Look for the point of intersection (adjust window).

$$A = 79.58 \text{ (2 dp)}$$

For the bread to be deemed safe for human consumption in terms of the amount of preservatives present, $\underline{\underline{79.58 \le A \le 100}}$



(iv)

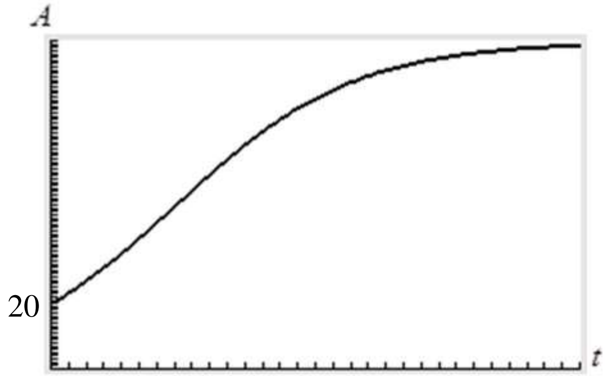
$$\frac{A}{100-A} = \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$

$$A = \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}(100-A)$$

$$4A = e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}(100-A) = 100e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t} - Ae^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$

$$\left[4 + e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}\right]A = 100e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$

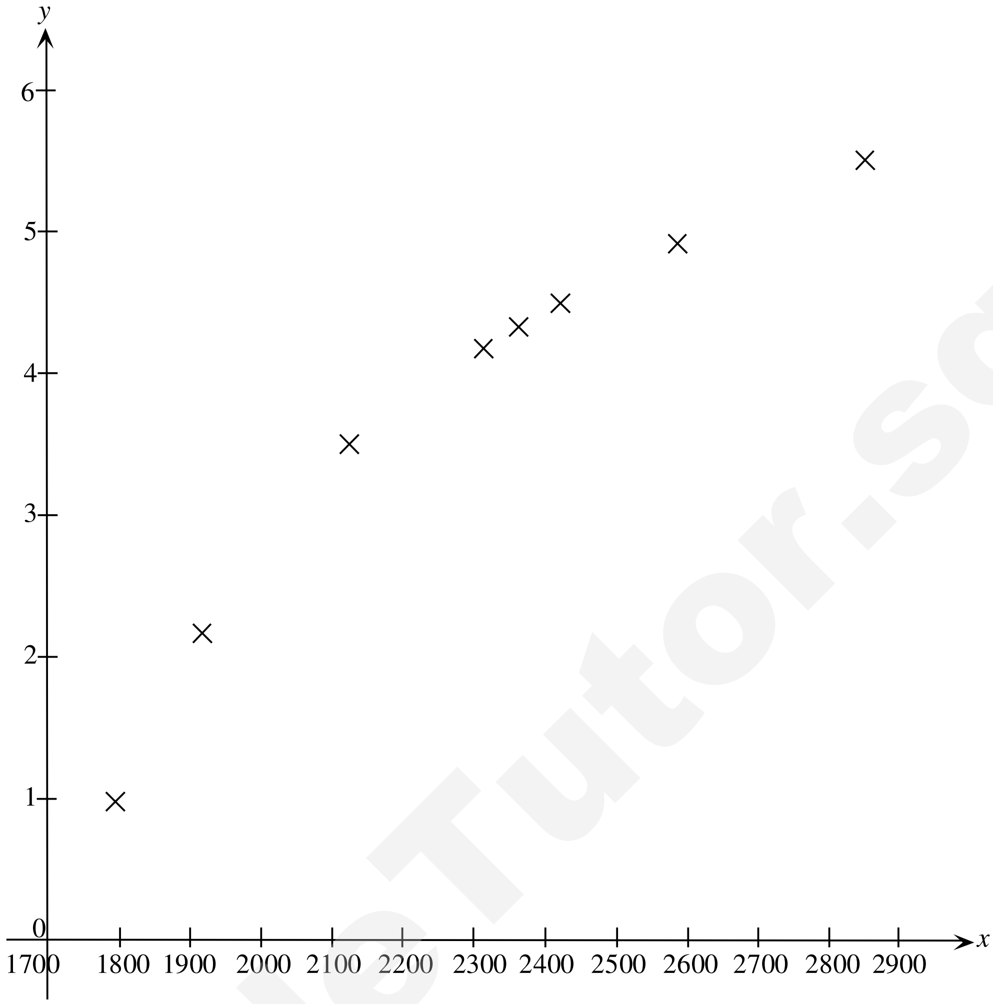
$$A = \underline{\underline{\frac{100e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}{4 + e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}}} \text{ or } \underline{\underline{\frac{100e^{0.196t}}{4 + e^{0.196t}}}}$$

	
5	$b=1-a$ | x | 1 | 2 | | $\mathrm{P}(X=x)$ | a | $1-a$ | $\mathrm{E}(X) = 1(a)+2(1-a)$ $= \underline{\underline{2-a}}$ $\mathrm{E}(X^2) = 1^2(a)+2^2(1-a)$ $= 4-3a$ $\mathrm{Var}(X) = \mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ $= 4-3a-(2-a)^2$ $= 4-3a-(4-4a+a^2)$ $= \underline{\underline{a-a^2}}$
6	**(i)** Use 1, 2 and 3 to form 3-digit numbers **(a)** $3! = \underline{\underline{6}}$ **(b)** $3\times3\times3 = \underline{\underline{27}}$ **(c)** Method 1 Consider the complement Number of 3-digit numbers with all 3 digits the same (AAA) = 3 Required number = $27-3 = \underline{\underline{24}}$ Method 2 Consider cases Case 1 Each digit is used exactly once Number of 3-digit numbers = 6 (from (i)(a)) Case 2 One digit is used twice (AAB) Number of 3-digit numbers = ${}^3\mathrm{P}_2\times\frac{3!}{2!}$ = 18 (${}^3\mathrm{P}_2 = 3\times2$: 3 ways to select a digit to be used twice; 2 ways to select another digit) Total number of 3-digit numbers = 6 + 18 = $\underline{\underline{24}}$ **(ii)** Use 1, 2 and 3 to form 4-digit numbers

		<u>Method 1 Consider the complement</u> Total number of 4-digit numbers = $3^4 = 81$ <u>Case 1 AAAB</u> Number of 4-digit numbers = ${}^3P_2 \times \frac{4!}{3!}$ = 24 (${}^3P_2 = 3\times2$: 3 ways to select a digit to be used thrice; 2 ways to select another digit) <u>Case 2 AAAA</u> Number of 4-digit numbers = 3 Total number of 4-digit numbers = $81-(24+3) = \underline{\underline{54}}$ <u>Method 2 Consider cases</u> <u>Case 1 AABC</u> Number of 4-digit numbers = $3\times\frac{4!}{2!}$ = 36 (3 ways to select the digit to be used twice) <u>Case 2 AABB</u> Number of 4-digit numbers = ${}^3C_2 \times \frac{4!}{2!\times2!}$ = 18 (${}^3C_2$ ways to select the 2 digits each to be used twice) Total number of 4-digit numbers = 36 + 18 = $\underline{\underline{54}}$
7	**(i)**	$H_0 : \mu = 420$ $H_1 : \mu \neq 420$ $s^2 = \frac{30}{29}(12)$ = 12.414 Under H_0, since $n = 30$ is large, by Central Limit Theorem, $\bar{X} \sim N\left(420, \frac{12.414}{30}\right)$ approximately. Hence it is <u>not necessary</u> for the volumes to have a normal distribution for the test to be valid. Test statistic $Z = \frac{\bar{X}-420}{\sqrt{\frac{12.414}{30}}} \sim N(0,1)$ approximately α = 0.01 From GC, $z = \frac{418.55-420}{\sqrt{\frac{12.414}{30}}} = -2.2541$ p-value = 0.0242 (3 sf) Since p-value = 0.0242 $> \alpha = 0.01$, we <u>do not reject H_0</u> at 1% level of

	significance and conclude that there is <u>insufficient</u> evidence that the population mean volume has changed. **(ii)** $\alpha = 0.01 \Rightarrow \frac{\alpha}{2} = 0.005$ Reject H_0 if $z \le -2.5758$ or $z \ge 2.5758$ $\frac{\bar{x}-420}{\sqrt{\frac{12.414}{30}}} \le -2.5758$ or $\frac{\bar{x}-420}{\sqrt{\frac{12.414}{30}}} \ge 2.5758$ $\bar{x} \le 420 - 2.5758\sqrt{\frac{12.414}{30}}$ or $\bar{x} \ge 420 + 2.5758\sqrt{\frac{12.414}{30}}$ $\underline{\underline{\bar{x} \le 418.34}}$ or $\underline{\underline{\bar{x} \ge 421.66}}$
8	Let A g be the mass of a tomato of variety A and B g be the mass of a tomato of variety B. $A \sim \mathrm{N}(80, 11^2)$ **(i)** $\mathrm{P}(A > 90) = 0.18165$ P(one greater than 90 g and one less than 90 g) $= 2 \times \mathrm{P}(A > 90) \times \mathrm{P}(A < 90)$ $= 2(0.18165)(1 - 0.18165)$ $= \underline{\underline{0.297}}$ (3 sf) Let $B \sim \mathrm{N}(70, \sigma^2)$. **(ii)** Let $S_B = B_1 + B_2 + \ldots + B_6 + 15$ $S_B \sim \mathrm{N}(6 \times 70 + 15, 6\sigma^2)$ i.e., $\mathrm{N}(435, 6\sigma^2)$ $\mathrm{P}(S_B < 450) = 0.8463$ $\mathrm{P}\left(Z < \frac{450-435}{\sqrt{6}\,\sigma}\right) = 0.8463$ $\frac{15}{\sqrt{6}\,\sigma} = 1.0207$ $\sigma = \frac{15}{1.0207\sqrt{6}} = 6$ (nearest g) (Shown) **(iii)** $S_B \sim \mathrm{N}(435, 216)$ Let $S_A = A_1 + A_2 + \ldots + A_5 + 25$ $S_A \sim \mathrm{N}(5 \times 80 + 25, 5 \times 11^2)$ i.e., $\mathrm{N}(425, 605)$ $S_A - S_B \sim \mathrm{N}(425 - 435, 605 + 216) = \mathrm{N}(-10, 821)$ $\mathrm{P}(S_A > S_B) = \mathrm{P}(S_A - S_B > 0)$ $= \underline{\underline{0.364}}$ (3 sf)
9	**(i)** (1) Selection of balls is done with replacement.

	(2) The balls are thoroughly mixed before each selection.
	(ii) Given $X \sim \mathrm{B}\left(10, \frac{2}{5}\right)$ $\mathrm{P}(X \geq 4) = 1 - \mathrm{P}(X \leq 3)$ $= \underline{\underline{0.618}}$ (3 sf)
	(iii) Given $\mathrm{E}(X) = 4.8$ $\Rightarrow \frac{2}{5}n = 4.8$ $n = \underline{\underline{12}}$
	(iv) Given $X \sim \mathrm{B}\left(n, \frac{2}{5}\right)$ $\mathrm{P}\left(X = 0 \text{ or } 1\right) < 0.01$ $\Rightarrow \mathrm{P}\left(X = 0\right) + \mathrm{P}(X = 1) < 0.01$ $\Rightarrow \left(\frac{3}{5}\right)^n + n\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^{n-1} < 0.01$ From GC, least $n = \underline{\underline{14}}$
	(v) Without replacement, P(Shawn wins the game) $= \frac{2}{5} + \frac{3}{5}\left(\frac{2}{4}\right)\left(\frac{2}{3}\right)$ $= \frac{3}{5}$ (Shown)
	(vi) With replacement, P(Shawn wins the game) $= \frac{2}{5} + \frac{3}{5}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right) + \frac{3}{5}\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right) + \ldots$ $= \frac{2}{5} + \frac{2}{5}\left(\frac{3}{5}\right)^2 + \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^4 + \ldots$ $= \dfrac{\frac{2}{5}}{1 - \left(\frac{3}{5}\right)^2}$ $= \underline{\underline{\frac{5}{8}}}$ or $\underline{\underline{0.625}}$
10	**(i)**



(ii) **(a)** Between x and y: $r = \underline{\underline{0.959}}$

(b) Between x and y^2: $r = \underline{\underline{0.995}}$

(iii) From (i), since as x increases, y increases at a decreasing rate, the points on the scatter diagram take the shape of the graph of $y^2 = c + dx$.

Or: From **(i)**, the points on the scatter diagram seem to lie on a concave downward curve.

From **(ii)**, the product moment correlation coefficient between x and y^2 is closer to 1, as compared to that between x and y,

$\therefore$ the model $\underline{\underline{y^2 = c + dx}}$ is the better model.

(iv) From GC, the regression line of y^2 on x is

$y^2 = 0.027897x - 47.985$

$\underline{\underline{y^2 = 0.0279x - 48.0}}$ (3 sf)

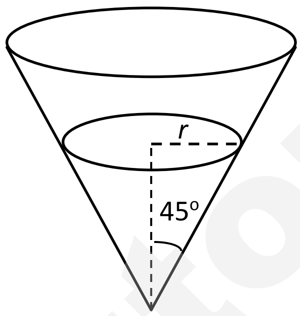
When $x = 2000$,

	$y^2 = 0.027897(2000) - 47.985$ $= 7.809$ $\therefore\ y = \underline{\underline{2.79}}$ (3 sf) or $\underline{\underline{2.8}}$ (1 dp, as shown in the table of values) **(v)** May not be valid as correlation does not necessarily imply causation. Or: May not be valid as there could be other factors relating traffic flow and air pollution. **(b)** $y = 2.5x + 3.8$ $\overline{y} = 2.5\overline{x} + 3.8$ $= 2.5(4.4) + 3.8$ $= 14.8$ Let $x = 1.5y - k$ $\overline{x} = 1.5\overline{y} - k$ $4.4 = 1.5(14.8) - k$ $k = 22.2 - 4.4$ $= \underline{\underline{17.8}}$

H2 Mathematics 2017 Prelim Paper 1 Question
Answer all questions [100 marks].

1 Water is leaking at a rate of 2 cm^3 per minute from a container in the form of a cone, with its axis vertical and vertex downwards. The semi-vertical angle of the cone is 45^{o} (see diagram). At time t minutes, the radius of the water surface is r cm. Find the rate of change of the depth of water when the depth of water in the container is 0.3cm. [4]

[The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.]



2 Without using a calculator, solve the inequality

$$\frac{x}{x-1} \leq \frac{4}{x+2}.$$ [5]

3 **Do not use a calculator in answering this question.**
Showing your working, find the complex numbers z and w which satisfy the simultaneous equations

$$4\text{i}z - 3w = 1 + 5\text{i} \quad \text{and}$$
$$2z + (1+\text{i})w = 2 + 6\text{i}.$$ [5]

4 **(a)** The points A and B relative to the origin O have position vectors $3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $-3\mathbf{i} + 2\mathbf{j}$ respectively.

(i) Find the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [2]

(ii) Hence or otherwise, find the shortest distance from B to line OA. [2]

(b) The points C, D and E relative to the origin O have non-zero and non-parallel position vectors $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$ respectively. Given that $(\mathbf{c} \times \mathbf{d}) \bullet \mathbf{e} = 0$, state with reason(s) the relationship between O, C, D and E. [2]

5 **(i)** Prove by the method of differences that

	$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{k}{2(n+1)(n+2)},$$ where k is a constant to be determined. [5]
	(ii) Explain why $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$ is a convergent series, and state its value. [2]
	(iii) Using your answer in part **(i)**, show that $\sum_{r=1}^{n} \frac{1}{(r+2)^3} < \frac{1}{4}$. [2]
6	A curve C has equation $y = \frac{ax+b}{cx+1}$, where a, b and c are positive real constants and $b > \frac{a}{c}$.
	(i) Sketch C, stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. [3]
	The curve C is transformed by a scaling parallel to y-axis by factor $\frac{1}{2}$ and followed by a translation of 2 units in the positive x-direction.
	(ii) Find the equation of the new curve in the form of $y = \mathrm{f}(x)$. [2]
	It is given that the new curve $y = \mathrm{f}(x)$ passes through the points with coordinates $\left(3, \frac{3}{2}\right)$ and $(6,1)$, and that $y = \frac{3}{4}$ is one of the asymptotes of the new curve $y = \mathrm{f}(x)$.
	(iii) Find the values of a, b and c. [5]
7	**(i)** Given that $\mathrm{f}(x) = \tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right)$, show that $\mathrm{f}'(x) = \frac{1}{2}\left[1 + (\mathrm{f}(x))^2\right]$, and find $\mathrm{f}(0)$, $\mathrm{f}'(0)$, $\mathrm{f}''(0)$ and $\mathrm{f}'''(0)$. Hence write down the first four non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. [7]
	(ii) The first three non-zero terms in the Maclaurin series for $\mathrm{f}(x)$ are equal to the first three non-zero terms in the series expansion of $\frac{\cos(ax)}{1+bx}$. By using appropriate expansions from the List of Formulae (MF26), find the possible value(s) for the constants a and b. [5]

8	10 pirates live on a pirate ship and they are ranked based on their seniority. **(a)** One day, the pirates found a treasure chest that consists of some gold coins. The rule which the pirates adhered by to divide all the gold coins are based on their seniority and is as follows: The most senior pirate will get 3 gold coins more than the 2nd most senior pirate. The 2nd most senior pirate will also get 3 gold coins more than the 3rd most senior pirate and so on. Thus, the most junior pirate will get the least number of gold coins. **(i)** If the treasure chest contains 305 gold coins, find the number of gold coins the most senior pirate will get. [3] **(ii)** Find the least number of gold coins the treasure chest must contain if all pirates get some (at least one) gold coins each. [2] **(b)** The pirates need to take turns, one at a time, to be on the lookout for their ship. Each day (24 hours) is divided into 10 shifts rotated among the 10 pirates. The 1st lookout shift starts from 10pm daily and it starts with the most junior pirate to the most senior pirate. The length of their shift is also based on their seniority. The length of shift for the most senior pirate is 10% less than that of the 2nd most senior pirate. The length of shift for the 2nd most senior pirate is 10% less than that of the 3rd most senior pirate and so on. Thus, the most junior pirate has the longest shift. **(i)** Show that the length of shift for the most junior pirate is 3.6848 hours, correct to 4 decimal places. [2] **(ii)** Calculate the length of shift for the 6th most junior pirate. Find the start time of his shift, giving your answer to the nearest minute. [4]
9	A curve C has parametric equations $$x = \sqrt{2}\cos\frac{t}{2}, \quad y = \sqrt{2}\sin t, \quad \text{for } -2\pi \le t \le 2\pi.$$ **(i)** Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and verify that curve C has a stationary point at P with parameter $\frac{\pi}{2}$. Hence find the equation of the normal to the curve at point P. [3] **(ii)** Sketch C, indicating clearly all turning points and axial intercepts in exact form. [4]

(iii) Find the exact area bounded by the curve C. (You may first consider the area bounded by the curve C and the positive x-axis in the first quadrant.) [6]

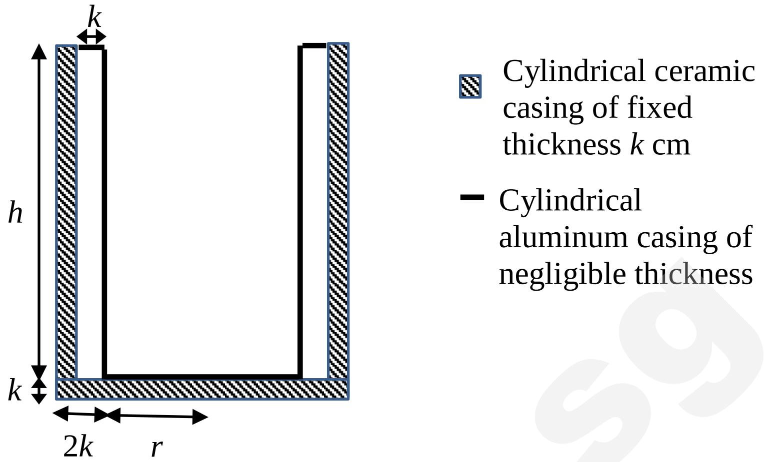
10 The plane p_1 has equation $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 16 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, where λ and μ are real parameters. The point A has position vector $5\mathbf{i} - 6\mathbf{j} + 7\mathbf{k}$.

(i) Find a cartesian equation of p_1. [3]

(ii) Find the position vector of the foot of perpendicular from A to p_1. [4]

The plane p_2 has equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = 52$. The plane p_3 is obtained by reflecting p_2 about p_1. By considering the relationship between A and p_2, or otherwise, find a cartesian equation of p_3. [6]

11 A company intends to manufacture a cylindrical double-walled ceramic vacuum flask which can hold a fixed V cm^3 of liquid when filled to the brim. The cylindrical vacuum flask is made up of an inner cylindrical aluminum casing (of negligible thickness) with height h cm and radius r cm and an outer cylindrical ceramic casing of fixed thickness k cm. There is a fixed k cm gap between the sides of the inner casing and outer casing where air has been removed to form a vacuum. The diagram below shows the view of the vacuum flask if it is dissected vertically through the centre.



Let the volume of the outer ceramic casing be C cm^3.

(i) Show that the volume of the ceramic casing can be expressed as

$$C = k\left(\frac{2V}{r} + \frac{3kV}{r^2} + \pi\left(r + 2k\right)^2\right) .$$ [4]

(ii) Let r_1 be the value of r which gives the minimum value of C. Show that r_1 satisfies the equation $\pi r^4 + 2\pi k r^3 - rV - 3kV = 0$. [3]

For the rest of the question, it is given that $k = \dfrac{1}{4}$ and $V = 250$.

(iii) Find the minimum volume of the ceramic casing, proving that it is a minimum. [3]

(iv) Sketch the graph showing the volume of the ceramic casing as the radius of the aluminum casing varies. [2]

− End Of Paper −

ANNEX B

MJC H2 Math JC2 Preliminary Examination Paper 1

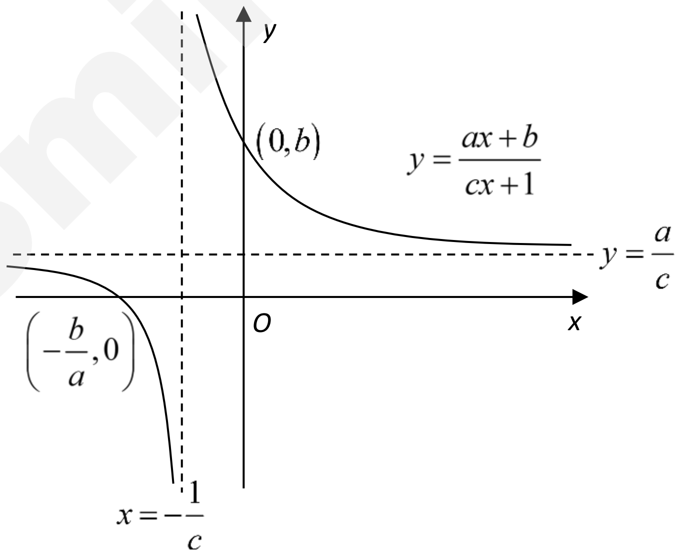
QN	Topic Set	Answers
1	Differentiation & Applications	7.07
2	Equations and Inequalities	$-2 < x < 1$
3	Complex numbers	$w = -3 + 5\text{i}$ and $z = 5 + 2\text{i}$
4	Vectors	(a)(i) 2.35 radian (a)(ii) $h = 2.58$
5	Sigma Notation and Method of Difference	(i) $k = 1$ (ii) $\frac{1}{4}$
6	Graphs and Transformation	(ii) $y = \frac{1}{2}\left[\frac{a(x-2)+b}{c(x-2)+1}\right]$ (iii) $a = 3,\ b = 6$ and $c = 2$
7	Maclaurin series	(i) $\text{f}(0) = 1;\ \text{f}'(0) = 1;\ \text{f}''(0) = 1;\ \text{f}'''(0) = 2;$ $\text{f}(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 \cdots$ (ii) $a = \pm 1$; $b = -1$
8	AP and GP	(a)(i) 44 (a)(ii) 145 (b)(ii) 1.05pm
9	Application of Integration	(i) $\frac{\text{d}y}{\text{d}x} = -\frac{2\cos t}{\sin \frac{t}{2}}$; $x = 1$ (iii) $\frac{16}{3}$ units^2
10	Vectors	(i) $-3x + y + 5z = 84$ (ii) $\overrightarrow{OF} = \begin{pmatrix} -1 \\ -4 \\ 17 \end{pmatrix}$ (iii) $-31x + 22y + 5z = 308$
11	Differentiation & Applications	(iii) 49.7 cm^3

H2 Mathematics 2017 Prelim Exam Paper 1 Solution

1	Solution: $\tan 45^\circ = \frac{r}{h} \Rightarrow r = h$ $V = \frac{1}{3}\pi r^2 h$ $V = \frac{1}{3}\pi h^3$ $\frac{\mathrm{d}V}{\mathrm{d}h} = \pi h^2$ When $h = 0.3$, $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V}\frac{\mathrm{d}V}{\mathrm{d}t}$ $= \frac{1}{\pi(0.3)^2}(-2)$ $= -\frac{200}{9\pi} = -7.07$ (3s.f) The depth of water is decreasing at 7.07 cm per minute.
2	Solution: $\frac{x}{x-1} \le \frac{4}{x+2}$ $\frac{x}{x-1} - \frac{4}{x+2} \le 0$ $\frac{x(x+2)-4(x-1)}{(x-1)(x+2)} \le 0$ $\frac{x^2+2x-4x+4}{(x-1)(x+2)} \le 0$ $\frac{x^2-2x+4}{(x-1)(x+2)} \le 0$ $\frac{(x-1)^2+3}{(x-1)(x+2)} \le 0$ Since $(x-1)^2+3>0$ for all $x \in \mathbb{R}$, $(x-1)(x+2)<0$ $-2<x<1$
3	Solution: $4\mathrm{i}z - 3w = 1 + 5\mathrm{i}$ --------(1) $2z + (1+\mathrm{i})w = 2 + 6\mathrm{i}$ --------(2)

	$(2)\times 2\mathrm{i}$ $4\mathrm{i}z+2\mathrm{i}(1+\mathrm{i})w=2\mathrm{i}(2+6\mathrm{i})$ $4\mathrm{i}z+2\mathrm{i}w-2w=4\mathrm{i}-12$ --------(3) $(3)-(1)$: $4\mathrm{i}z+2\mathrm{i}w-2w-(4\mathrm{i}z-3w)=(4\mathrm{i}-12)-(1+5\mathrm{i})$ $w+2\mathrm{i}w=-13-\mathrm{i}$ $(1+2\mathrm{i})w=-13-\mathrm{i}$ $w=\left(\frac{-13-\mathrm{i}}{1+2\mathrm{i}}\right)\left(\frac{1-2\mathrm{i}}{1-2\mathrm{i}}\right)$ $w=\frac{-13+26\mathrm{i}-\mathrm{i}-2}{(1)^2-(2\mathrm{i})^2}$ $w=\frac{-15+25\mathrm{i}}{5}$ $w=-3+5\mathrm{i}$ Substitute $w=-3+5\mathrm{i}$ into (2) $2z=2+6\mathrm{i}-(1+\mathrm{i})(-3+5\mathrm{i})$ $2z=2+6\mathrm{i}-(-3+5\mathrm{i}-3\mathrm{i}-5)$ $2z=2+6\mathrm{i}-(-8+2\mathrm{i})$ $2z=10+4\mathrm{i}$ $z=5+2\mathrm{i}$ $\therefore w=-3+5\mathrm{i}$ and $z=5+2\mathrm{i}$.
4	Solution: (a)(i) Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. $\cos\theta=\frac{\begin{pmatrix}3\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}-3\\2\\0\end{pmatrix}}{\left\|\begin{pmatrix}3\\-1\\3\end{pmatrix}\right\|\left\|\begin{pmatrix}-3\\2\\0\end{pmatrix}\right\|}$ $\theta=\cos^{-1}\left(\frac{-11}{\sqrt{19}\sqrt{13}}\right)=134.4^\circ$ (1 d.p) = 2.35 radian (3 s.f) (a)(ii) Let h be the shortest distance from B to line OA. $\sin 134.42^\circ=\frac{h}{\|\mathbf{b}\|}$ $h=\sqrt{13}\sin 134.42^\circ$ $=2.5752$ $=2.58$ units (3 s.f)

	(b) Let $\mathbf{c}\times\mathbf{d}=\mathbf{s}$. 1) $\mathbf{s}\bullet\mathbf{e}=0 \Rightarrow$ $\mathbf{s}$ is perpendicular to **e**. 2) $\mathbf{c}\times\mathbf{d}=\mathbf{s}\Rightarrow$ $\mathbf{s}$ is perpendicular to both **c** and **d**. Since $\mathbf{s}$ is <u>perpendicular to **c**, **d** and **e**</u> and <u>**c**, **d** and **e** passes through common point *O*</u> $\Rightarrow$ points *O*, *C*, *D* and *E* are coplanar.
5	Solution: (i) Let $\dfrac{1}{r(r+1)(r+2)}=\dfrac{A}{r}+\dfrac{B}{r+1}+\dfrac{C}{r+2}$ Using 'cover-up' rule, $A=\dfrac{1}{2}, \qquad B=-1, \qquad C=\dfrac{1}{2}$ $\therefore \dfrac{1}{r(r+2)}=\dfrac{1}{2r}-\dfrac{1}{r+1}+\dfrac{1}{2(r+2)}$ $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\sum_{r=1}^{n}\left(\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}\right)$ $=\Big[\dfrac{1}{2}-\dfrac{1}{2}+\dfrac{1}{6}$ $+\dfrac{1}{4}-\dfrac{1}{3}+\dfrac{1}{8}$ $+\dfrac{1}{6}-\dfrac{1}{4}+\dfrac{1}{10}$ $+\dfrac{1}{8}-\dfrac{1}{5}+\dfrac{1}{12}$ $+\ldots$ $+\dfrac{1}{2(n-2)}-\dfrac{1}{n-1}+\dfrac{1}{2n}$ $+\left(\dfrac{1}{2n-1}\right)-\dfrac{1}{n}+\dfrac{1}{2(n+1)}$ $+\dfrac{1}{2n}-\dfrac{1}{n+1}+\dfrac{1}{2(n+2)}\Big]$ $=\dfrac{1}{2}-\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{2(n+1)}-\dfrac{1}{n+1}+\dfrac{1}{2(n+2)}$ $=\dfrac{1}{4}-\dfrac{1}{2(n+1)}+\dfrac{1}{2(n+2)}$ $=\dfrac{1}{4}-\dfrac{1}{2(n+1)(n+2)}$ (proven) $\therefore k=1$

	(ii) As $n \to \infty, \dfrac{1}{2(n+1)(n+2)} \to 0, \displaystyle\sum_{r=1}^{n} \dfrac{1}{r(r+1)(r+2)} \to \dfrac{1}{4}$ $\therefore \displaystyle\sum_{r=1}^{\infty} \dfrac{1}{r(r+1)(r+2)}$ is a convergent series. $\therefore \displaystyle\sum_{r=1}^{\infty} \dfrac{1}{r(r+1)(r+2)} = \dfrac{1}{4}$ (iii) For all $r \geq 1$, $(r+2)^3 > r(r+1)(r+2)$ $\dfrac{1}{(r+2)^3} < \dfrac{1}{r(r+1)(r+2)}$ $\displaystyle\sum_{r=1}^{n} \dfrac{1}{(r+2)^3} < \sum_{r=1}^{n} \dfrac{1}{r(r+1)(r+2)}$ $\displaystyle\sum_{r=1}^{n} \dfrac{1}{(r+2)^3} < \dfrac{1}{4} - \dfrac{1}{2(n+1)(n+2)}$ $\displaystyle\sum_{r=1}^{n} \dfrac{1}{(r+2)^3} < \dfrac{1}{4}$ $\left(\because \dfrac{1}{2(n+1)(n+2)} > 0 \text{ for all } n \geq 1\right)$
6	Solution: (i)  (ii) Equation of new curve: $y = \dfrac{1}{2}\left[\dfrac{a(x-2)+b}{c(x-2)+1}\right]$

	(iii) Since the new curve $y=\mathrm{f}(x)$ passes through the points with coordinates $\left(3,\frac{3}{2}\right)$ and $(6,1)$: $\frac{3}{2}=\frac{1}{2}\left[\frac{a(3-2)+b}{c(3-2)+1}\right]$ $3=\frac{a+b}{c+1}$ $a+b=3c+3$ $a+b-3c=3$ --------(1) $1=\frac{1}{2}\left[\frac{a(6-2)+b}{c(6-2)+1}\right]$ $2=\frac{4a+b}{4c+1}$ $4a+b=8c+2$ $4a+b-8c=2$ ---------(2) Since $y=\frac{3}{4}$ is one of the asymptotes of $y=\mathrm{f}(x)$, $\frac{3}{4}=\frac{1}{2}\left(\frac{a}{c}\right)$ $\frac{a}{c}=\frac{3}{2}$ $2a-3c=0$ --------(3) Solving equations (1), (2) and (3) using GC, $a=3,\ b=6$ and $c=2$.
7	Solution: (i) $\mathrm{f}(x)=\tan\left(\frac{1}{2}x+\frac{1}{4}\pi\right)$ $\mathrm{f}'(x)=\frac{1}{2}\sec^2\left(\frac{1}{2}x+\frac{1}{4}\pi\right)$ $=\frac{1}{2}\left[1+\tan^2\left(\frac{1}{2}x+\frac{1}{4}\pi\right)\right]$ $=\frac{1}{2}\left(1+(\mathrm{f}(x))^2\right)$ (shown) $\mathrm{f}''(x)=\mathrm{f}(x)\mathrm{f}'(x)$ $\mathrm{f}'''(x)=\mathrm{f}(x)\mathrm{f}''(x)+(\mathrm{f}'(x))^2$ $\mathrm{f}(0)=1$, $\mathrm{f}'(0)=1$, $\mathrm{f}''(0)=1$,

	$f'''(0)=2$. $\therefore \quad f(x)=1+x+\frac{1}{2}x^2+\frac{1}{3}x^3\cdots$ (ii) $\frac{\cos(ax)}{1+bx}=\left(1-\frac{(ax)^2}{2!}+...\right)\left(1+(-1)(bx)+\frac{(-1)(-2)}{2!}(bx)^2+...\right)$ $=\left(1-\frac{a^2x^2}{2}+...\right)\left(1-bx+b^2x^2+...\right)$ $\approx 1-bx+b^2x^2-\frac{a^2x^2}{2}$ $=1-bx+\left(b^2-\frac{a^2}{2}\right)x^2$ Comparing coefficients, $x:b=-1$ $x^2:b^2-\frac{a^2}{2}=\frac{1}{2}\Rightarrow\frac{a^2}{2}=\frac{1}{2}\Rightarrow a=\pm 1$
8	Solution: (a)(i) Let a be the number of gold coins the most junior pirate will get. $\frac{10}{2}\left[2a+(10-1)(3)\right]=305$ $a=17$ No of gold coins for most senior pirate $=17+(10-1)(3)$ $=44$ (a)(ii) Least no of gold coins $=\frac{10}{2}\left[2(1)+(10-1)(3)\right]$ $=145$ (b)(i) Let b be the length of shift for the most junior pirate $\frac{b}{1-0.9}\left(1-0.9^{10}\right)=24$ $b=3.6848\text{ hr}\left(\text{to 4 d.p.}\right)$ $\quad$ (shown) (b)(ii) Length of shift for 6th most junior pirate $=3.6848(0.9)^5$ $=2.18\text{ hr}$

$$\text{Length of 1st 5 shifts} = \frac{3.6848}{1-0.9}\left(1-0.9^5\right)$$
$$= 15.090$$
$$= 15 \text{ hrs } 5 \text{ mins}$$
$$\text{Start time of shift } = 1.05\text{pm}$$

9

(i) $x=\sqrt{2}\cos\frac{t}{2} \Rightarrow \frac{dx}{dt} = -\frac{\sqrt{2}}{2}\sin\frac{t}{2}$

$y=\sqrt{2}\sin t \Rightarrow \frac{dy}{dt}=\sqrt{2}\cos t$

$\therefore \frac{dy}{dx} = -\frac{2\cos t}{\sin\frac{t}{2}}$

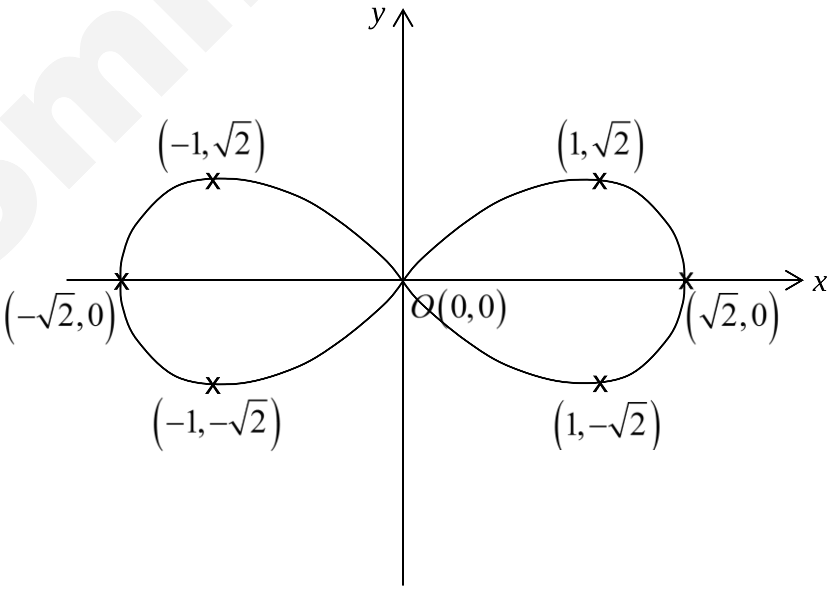
At $t=\frac{\pi}{2}$,

$\frac{dy}{dx} = -\frac{2\cos\frac{\pi}{2}}{\sin\frac{\pi}{4}} = 0$ (verified)

When $t=\frac{\pi}{2}$, $x=\sqrt{2}\cos\left(\frac{\pi}{4}\right)=1$

Equation of normal: $x=1$

(ii)



	(iii) Area $= 4\int_0^{\sqrt{2}} y \, \mathrm{d}x$ $= 4\int_\pi^0 \sqrt{2}\sin t \cdot \left(-\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right) \mathrm{d}t$ $= 4\int_0^\pi \sin t \cdot \sin\frac{t}{2} \, \mathrm{d}t$ $= 8\int_0^\pi \sin^2\frac{t}{2}\cos\frac{t}{2} \, \mathrm{d}t$ $= 8\left[\frac{2}{3}\sin^3\frac{t}{2}\right]_0^\pi$ $= \frac{16}{3}$ units^2 **Alternative Method** Area $= 4\int_0^{\sqrt{2}} y \, \mathrm{d}x$ $= 4\int_\pi^0 \sqrt{2}\sin t \cdot \left(-\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right) \mathrm{d}t$ $= 4\int_0^\pi \sin t \cdot \sin\frac{t}{2} \, \mathrm{d}t$ $= -2\int_0^\pi \cos\frac{3t}{2} - \cos\frac{t}{2} \, \mathrm{d}t$ $= -2\left[\frac{2}{3}\sin\frac{3t}{2} - 2\sin\frac{t}{2}\right]_0^\pi$ $= \frac{16}{3}$ units^2
10	Solutions: (i) $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$ $\mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 16 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} = 84$ Cartesian equation of p_1 is $-3x + y + 5z = 84$. (ii) $\overrightarrow{OA} = \begin{pmatrix} 5 \\ -6 \\ 7 \end{pmatrix}$ Let the foot of perpendicular from A to p_1 be F.

	$\overrightarrow{OF}=\begin{pmatrix}5\\-6\\7\end{pmatrix}+\beta\begin{pmatrix}-3\\1\\5\end{pmatrix}=\begin{pmatrix}5-3\beta\\-6+\beta\\7+5\beta\end{pmatrix}$ for some $\beta\in\mathbb{R}$ Since F lies on p_1, $\begin{pmatrix}5-3\beta\\-6+\beta\\7+5\beta\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\5\end{pmatrix}=84$ $35\beta+14=84$ $\beta=2$ $\therefore\overrightarrow{OF}=\begin{pmatrix}5-6\\-6+2\\7+10\end{pmatrix}=\begin{pmatrix}-1\\-4\\17\end{pmatrix}$ Note that A lies on p_2 since $\begin{pmatrix}5\\-6\\7\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix}=52$. Let A' be the point of reflection of A about p_1. Note that A' lies on p_3. $\overrightarrow{OF}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$ $\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}=2\begin{pmatrix}-1\\-4\\17\end{pmatrix}-\begin{pmatrix}5\\-6\\7\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}$ $p_1:\ -3x+y+5z=84$. $p_2:\ \mathbf{r}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix}=52\ \Rightarrow\ x-2y+5z=52$ By GC, the line of intersection between p_1 and p_2 is $\mathbf{r}=\begin{pmatrix}-44\\-48\\0\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix},\quad\alpha\in\mathbb{R}$ A vector parallel to p_3 is $\overrightarrow{OA'}-\begin{pmatrix}-44\\-48\\0\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}-\begin{pmatrix}-44\\-48\\0\end{pmatrix}=\begin{pmatrix}37\\46\\27\end{pmatrix}$ $\begin{pmatrix}37\\46\\27\end{pmatrix}\times\begin{pmatrix}3\\4\\1\end{pmatrix}=\begin{pmatrix}-62\\44\\10\end{pmatrix}=2\begin{pmatrix}-31\\22\\5\end{pmatrix}$

$$\mathbf{r}\cdot\begin{pmatrix}-31\\22\\5\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}\cdot\begin{pmatrix}-31\\22\\5\end{pmatrix}=308$$

A cartesian equation of p_3 is $-31x+22y+5z=308$

11 Solution:

(i)

$$V=\pi h r^2$$

$$h=\frac{V}{\pi r^2} \quad \text{--------} \quad (*)$$

$$C=\pi(h+k)(r+2k)^2-\pi h(r+k)^2$$

$$=\pi\left(h\left((r+2k)^2-(r+k)^2\right)+k(r+2k)^2\right)$$

$$=\pi\left(h\left((r^2+4rk+4k^2)-(r^2+2rk+k^2)\right)+k(r+2k)^2\right)$$

$$=\pi\left(\frac{V}{\pi r^2}(2rk+3k^2)+k(r+2k)^2\right) \qquad (\text{from } (*))$$

$$=k\left(V\frac{(2r+3k)}{r^2}+\pi(r+2k)^2\right)$$

$$=k\left(\frac{2V}{r}+\frac{3kV}{r^2}+\pi(r+2k)^2\right) \qquad (\text{shown})$$

(ii) $\dfrac{\mathrm{d}C}{\mathrm{d}r}=k\left(\dfrac{-2V}{r^2}-\dfrac{6kV}{r^3}+2\pi(r+2k)\right)$

When $\dfrac{\mathrm{d}C}{\mathrm{d}r}=0$,

$$k\left(\frac{-2V}{r^2}-\frac{6kV}{r^3}+2\pi(r+2k)\right)=0$$

$$-Vr-3kV+\pi r^3(r+2k)=0$$

$$\pi r^4+2k\pi r^3-Vr-3kV=0 \qquad (\text{Shown})$$

(iii) From GC,

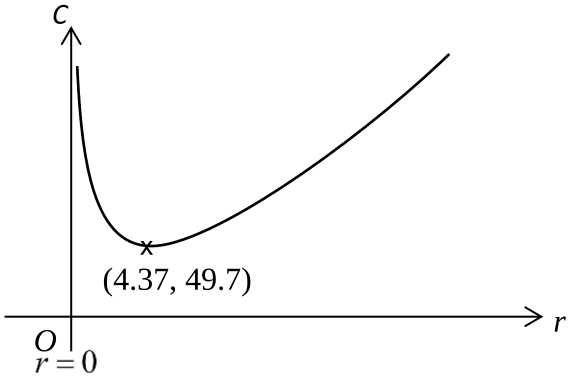
$r_1=4.3736 \quad (\text{since } r>0)$

r	r_1^-	r_1	r_1^+
$\dfrac{\mathrm{d}C}{\mathrm{d}r}$			

$\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2}=5.33>0 \Rightarrow C$ is a minimum

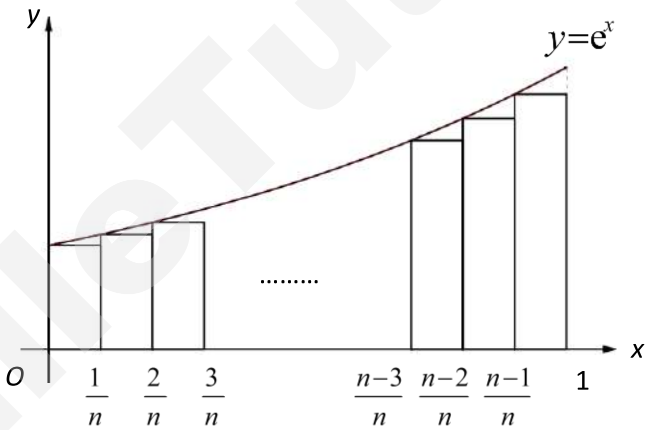
$C=49.7 \quad (3\text{ s.f.})$

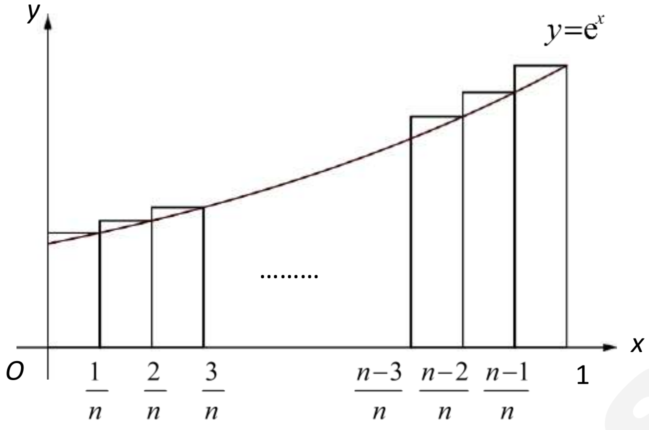
Minimum volume of ceramic casing is 49.7 cm^3.

	Note: cannot use part (iv) graph to solve this part. (iv) 

H2 Mathematics 2017 Prelim Paper 2 Question
Answer all questions [100 marks].

1	The complex number z has modulus 3 and argument $\frac{2\pi}{3}$. **(i)** Find the modulus and argument of $\frac{-2\mathrm{i}}{z^*}$, where z^* is the complex conjugate of z, leaving your answers in the exact form. [3] **(ii)** Hence express $\frac{-2\mathrm{i}}{z^*}$ in the form of $x+\mathrm{i}y$, where x and y are real constants, giving the exact values of x and y in non-trigonometrical form. [2] **(iii)** The complex number w is defined such that $w=1+\mathrm{i}k$, where k is a non-zero real constant. Given that $\frac{-2\mathrm{i}w}{z^*}$ is purely imaginary, find the exact value of k. [2]
2	Two students are investigating the rate of change of the amount of water in a reservoir, x million cubic metres, at time t hour during a rainfall. Student A suggests that x and t are related by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{2}{(t+1)^3}$. **(i)** Find the general solution of this differential equation. [3] Student B assumes that the amount of water flowing into the reservoir depends only on the rainfall and is at a constant rate of k million cubic metres per hour. The rate at which water flows out from the reservoir is proportional to the square of the amount of water in the reservoir. **(ii)** If the amount of water in the reservoir stabilizes at 0.5 million cubic metres, show that the rate of change of the amount of water in the reservoir can be modelled by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t}=k\left(1-4x^2\right)$. [2] **(iii)** Find x in terms of k and t, given that there are initially 1 million cubic metres of water in the reservoir. [5]
3	The function f is defined by $\mathrm{f}: x \mapsto \ln\left(x^2-1\right),\ x\in\mathbb{R},\ x>1.$ **(i)** Find f^{-1} in similar form. [3]

	(ii) Sketch f, f^{-1} and $\mathrm{f}^{-1}\mathrm{f}$ on the same diagram, indicating clearly all asymptotes and axial intercepts. [3]
	The functions g and h are defined by
	$\mathrm{g}: x \mapsto \begin{cases} 4(x-1)^2 & \text{for } 0 \le x < 2, \\ 8-\lvert 2x-8 \rvert & \text{for } 2 \le x < 8, \end{cases}$
	$\mathrm{h}: x \mapsto 3\sin x, \qquad 0 \le x \le \pi.$
	(iii) Sketch the graph of $y = \mathrm{g}(x)$. [3]
	(iv) Prove that the function gh exists and find the range of gh. [2]
4	The graph of $y = \mathrm{e}^x$, for $0 \le x \le 1$, is shown in the diagram below. Rectangles, each of width $\frac{1}{n}$ where *n* is an integer, are drawn under the curve.
	
	(i) Show that the total area of all the *n* rectangles, A_n, is $\dfrac{c}{n\left(\mathrm{e}^{\frac{1}{n}}-1\right)}$, where *c* is an exact constant to be found. [3]
	(ii) By considering the Maclaurin Series for $\mathrm{e}^x - 1$, or otherwise, find the value of $\lim_{x \to 0} \frac{1}{x}\left(\mathrm{e}^x - 1\right)$. [3]
	(iii) Hence, without using integration, find the exact value of $\lim_{n \to \infty} A_n$. [2]
	(iv) Give a geometrical interpretation of the value you found in part **(iii)**, and verify your answer in part **(iii)** using integration. [2]
	Another set of *n* rectangles are drawn, as shown in the diagram below.



The total area of all the n rectangles in the second diagram is denoted by B_n. By considering the concavity of the graph of $y = \mathrm{e}^x$, or otherwise, show that

$$\frac{A_n + B_n}{2} > \int_0^1 \mathrm{e}^x \, \mathrm{d}x$$

for any positive integer n. [2]

5 Andy needs two passcodes to open a treasure box. Both passcodes consist of three letters and four digits. Each of the three letters can be any of the twenty-six letters of the alphabet A-Z. Each of the four digits can be any of the ten digits 0-9.

(a) The first passcode consists of three letters followed by four digits. It is also known that no letters and digits are repeated. An example of the code is ABC1234.

(i) Find the total number of possible first passcodes. [2]

(ii) An additional hint is given to Andy to break the first passcode. The four digits of the passcode form a number which is odd and greater than 3000. Find the total number of possible first passcodes. [3]

(b) The second passcode has no fixed arrangement for the letters and digits. Given that the letters and digits can be repeated (i.e. 1AA3C34 can be a possible passcode), find the total number of possible second passcodes. [3]

6 The probability function of X is given by

$$\mathrm{P}(X = x) = \begin{cases} (2x-1)\theta & \text{if } x = 1, 2, 3 \\ k & \text{if } x = 4 \\ 0 & \text{otherwise} \end{cases}$$

where $0 < \theta < \dfrac{1}{9}$.

(i) Show that $k = 1 - 9\theta$. Find, in terms of θ, the probability distribution of *X*. [2]

(ii) Find $\mathrm{E}(X)$ in terms of θ and hence show that $\mathrm{Var}(X) = 26\theta - 196\theta^2$. [3]

(iii) The random variable Y is related to X by the formula $Y = a + bX$, where *a* and *b* are non-zero constants. Given that $\mathrm{Var}(Y) = \frac{1}{3}b^2$, find the value of θ. [3]

7 Coloured lego pieces are packed into boxes of 20 pieces by a particular manufacturer. Each box is made up of randomly chosen coloured lego pieces. The manufacturer produces a large number of lego pieces every day. On average, 15% of lego pieces are red. Explain why binomial distribution is appropriate for modelling the number of red lego pieces in a box. [2]

(i) Find the probability that a randomly chosen box of lego pieces contains at least 4 red lego pieces. [2]

(ii) A customer buys 50 randomly chosen boxes containing lego pieces. Find the probability that no more than 19 of these boxes contain at least 4 red lego pieces. [2]

It is given instead that the proportion of lego pieces that are red is now *p*. The probability that there is at least one red lego piece but fewer than four red lego pieces in a box, is 0.22198, correct to 5 significant figures. Write down an equation involving *p* and hence find the value of *p*, given that $p > 0.2$. [4]

8 In an assembly line, a machine is programmed to dispense shampoo into empty bottles and the volume of shampoo dispensed into each bottle is a normally distributed continuous random variable *X*. Under ordinary conditions, the expected value of *X* is 325 ml.

(i) After a routine servicing of the machine, the assembly manager suspects that the machine is dispensing more shampoo than expected. A random sample of 60 bottles is taken and the data is as follows:

Volume of shampoo (correct to nearest ml)	324	325	326	327	328	329	330
Number of bottles	16	20	9	8	4	1	2

Find unbiased estimates of the population mean and variance, giving your answers to 2 decimal places. [2]

Test, at the 5% significance level, whether the assembly manager's suspicion is valid. [4]

Explain what it meant by the phrase 'at 5% significance level' in the context of the question. [1]

(ii) Due to the assembly manager's suspicion, the machine is being recalibrated to dispense 325 ml of shampoo. Another random sample of 50 is taken and a two-tailed test, at the 5% significance level, concluded that the recalibration is done accurately. Given that the volume of shampoo dispensed into each bottle is normally distributed with standard deviation 1.2 ml, find the set of values the mean volume of the 50 bottles can take, giving your answers to 2 decimal places. [4]

9 The consumer price index measures the average price changes in a fixed basket of consumption goods and services commonly purchased by resident households over time. It is commonly used as a measure of consumer price inflation. In the 2013 Singapore household expenditure survey, housing and food made up about half of the average monthly expenditure of an average household.

The table below shows the housing and food price index from 2005 to 2012, where 2005 is the base period, i.e. in 2005, the price index is 100. For example, the food price index of 104.6 in 2007 means that average food prices increased by 4.6% from 2005 to 2007.

Year	2005	2006	2007	2008	2009	2010	2011	2012
Housing Price Index, x	100	100.7	102.3	116.8	123.1	124.3	[redacted]	148
Food Price Index, y	100	101.6	104.6	112.6	115.2	116.8	120.3	123.1

	(i)	Show that the value of the missing housing price index for 2011 is 136 (nearest integer), given that the regression line of y on x is $y = 54.271 + 0.48363x$, correct to 5 significant figures. [2]
	(ii)	Draw the scatter diagram for these values, labelling the axes clearly. Comment on the suitability of the linear model. [3]
	(iii)	It is required to estimate the housing price index in 2016 where the food price index in 2016 is 134.6. Find the equation of an appropriate regression line for y and $\sqrt{x}$ and use it to find the required estimate. Explain why this estimate might not be reliable. [4]
	(iv)	Find the product moment correlation coefficient between y and $\sqrt{x}$. [1]
	(v)	To simplify recordings and calculations, it would be more convenient to tabulate $\frac{x}{100}$ and $\frac{y}{100}$ instead. Without any further calculations, explain if the product moment correlation coefficient between $\sqrt{\frac{x}{100}}$ and $\frac{y}{100}$ would differ from the value obtained in part **(iv)**. [1]
10	**(i)**	Factory A produces nuts whose mass may be assumed to be normally distributed with mean μ grams and standard deviation σ grams. A random sample of 50 nuts is taken. It is given that the probability that the mean mass is less than 247 grams is 0.018079, correct to 5 significant figures. It is also given that the probability that the total mass exceeds 12600 grams is 0.78397, correct to 5 significant figures. Find the values of μ and σ, giving your answers to the nearest grams. [5]
	(ii)	(For this question, you should state clearly the values of the parameters of any normal distribution you use.) Factory B produces bolts and nuts. The masses, in grams, of bolts and nuts produced are modelled as having independent normal distributions with means and standard deviations as shown in the table:

	Mean Mass (in grams)	Standard Deviation (in grams)
Bolts	745	7.3
Nuts	250	5

	(a) Find the probability that the mass of a randomly chosen bolt differs from 3 times the mass of a randomly chosen nut by at least 40 grams. [4] **(b)** This factory introduces a new process which is able to reduce the mass of each nut by 10%. Find the probability that the total mass, after the introduction of this process, of 10 randomly chosen nuts is less than 2.24 kg. [3]

− End Of Paper −

ANNEX B

MJC H2 Math JC2 Preliminary Examination Paper 2

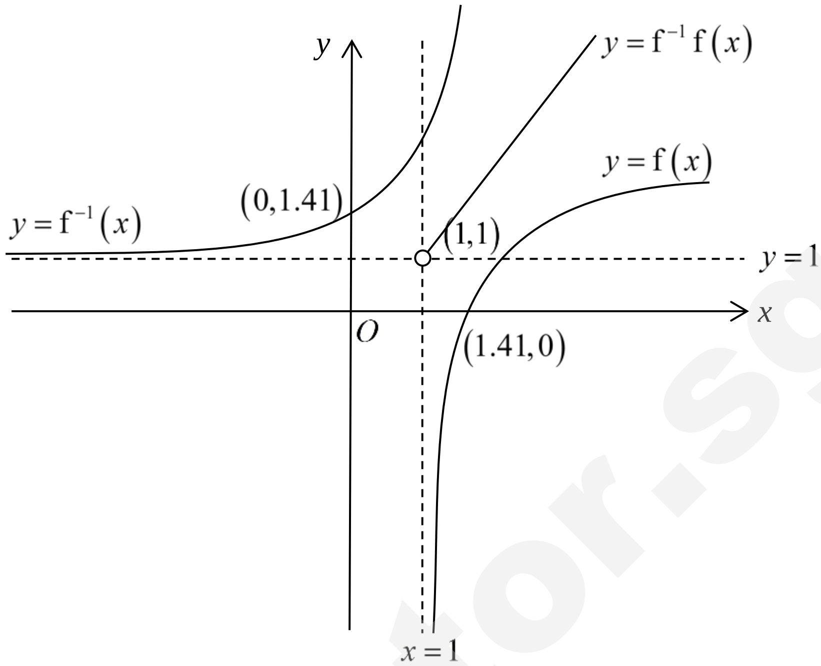
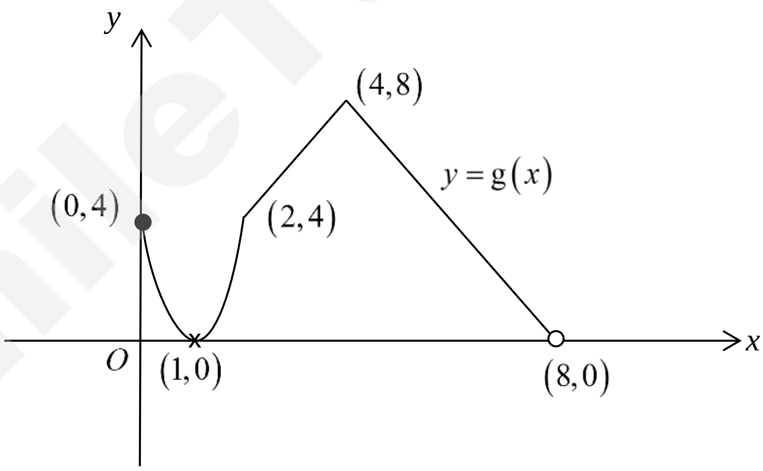
<table>
<tr><th>QN</th><th>Topic Set</th><th>Answers</th></tr>
<tr><td>1</td><td>Complex numbers</td><td>(i) $\frac{2}{3}$; $\frac{\pi}{6}$
(ii) $\frac{\sqrt{3}}{3}+\frac{1}{3}\mathrm{i}$
(iii) $k=\sqrt{3}$</td></tr>
<tr><td>2</td><td>Differential Equations</td><td>(i) $x=\frac{1}{t+1}+at+b$, where $b\in \mapsto$
(iii) $x=\frac{1+3\mathrm{e}^{4kt}}{6\mathrm{e}^{4kt}-2}$ or $\frac{1}{3\mathrm{e}^{4kt}-1}+\frac{1}{2}$</td></tr>
<tr><td>3</td><td>Functions</td><td>(i) $\mathrm{f}^{-1}: x\mapsto\sqrt{1+\mathrm{e}^{x}},\ x\in\mapsto$
(iv) $\mathrm{R}_{\mathrm{gh}}=[0,6]$</td></tr>
<tr><td>4</td><td>Integration techniques</td><td>(i) $c=\mathrm{e}-1$
(ii) 1
(iii) $\mathrm{e}-1$</td></tr>
<tr><td>5</td><td>P&C, Probability</td><td>(a)(i) 78624000
(a)(ii) 27081600
(b) 6151600000</td></tr>
<tr><td>6</td><td>DRV</td><td>(i)
<table><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>$\mathrm{P}(X=x)$</td><td>θ</td><td>3θ</td><td>5θ</td><td>$1-9\theta$</td></tr></table>
(ii) $\mathrm{E}(X)=4-14\theta$
(iii) $\theta=0.0144$</td></tr>
<tr><td>7</td><td>Binomial Distribution</td><td>(i) 0.715
(ii) 0.715
(iii) $p=0.250$</td></tr>
<tr><td>8</td><td>Hypothesis Testing</td><td>(i) $\bar{x}=325.58$; $s^2=2.35$
$p-\text{value}=0.00169<0.05$
(ii) $\{\bar{x}\in\mapsto: 324.67<\bar{x}<325.33\}$</td></tr>
<tr><td>9</td><td>Correlation & Linear Regression</td><td>(iii) $\sqrt{x}=0.0896y+0.860$; $x=167$
(iv) $r=0.979$</td></tr>
<tr><td>10</td><td>Normal Distribution</td><td>(i) $\mu=255$; $\sigma=27$
(ii)(a) 0.0214
(ii)(b) $\mathrm{P}(T<2240)=0.241$ (3s.f.)</td></tr>
</table>

H2 Mathematics 2017 Prelim Exam Paper 2 Solution

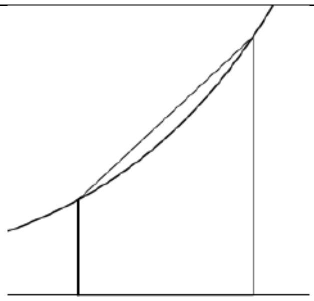
1	Solution: (i) Given $\lvert z\rvert = 3,\ \arg(z) = \dfrac{2\pi}{3}$, $\left\lvert \dfrac{-2\mathrm{i}}{z^*} \right\rvert = \dfrac{\lvert -2i\rvert}{\lvert z^*\rvert} = \dfrac{2}{3} \quad (\because \lvert z\rvert = \lvert z^*\rvert)$ $\arg\left(\dfrac{-2\mathrm{i}}{z^*}\right) = \arg(-2\mathrm{i}) - \arg(z^*)$ $= -\dfrac{\pi}{2} - \left(-\dfrac{2\pi}{3}\right) \quad (\because \arg(z^*) = -\arg(z))$ $= \dfrac{\pi}{6}$ (ii) $\dfrac{-2\mathrm{i}}{z^*} = \dfrac{2}{3}\left[\cos\left(\dfrac{\pi}{6}\right) + \mathrm{i}\sin\left(\dfrac{\pi}{6}\right)\right]$ $= \dfrac{2}{3}\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\mathrm{i}\right)$ $= \dfrac{\sqrt{3}}{3} + \dfrac{1}{3}\mathrm{i}$ (iii) $\dfrac{-2\mathrm{i}w}{z^*} = \left(\dfrac{\sqrt{3}}{3} + \dfrac{1}{3}\mathrm{i}\right)(1 + \mathrm{i}k)$ $= \dfrac{\sqrt{3}}{3} + \dfrac{\sqrt{3}}{3}k\mathrm{i} + \dfrac{1}{3}\mathrm{i} - \dfrac{1}{3}k$ Since $\dfrac{-2\mathrm{i}w}{z^*}$ is purely imaginary, $\dfrac{\sqrt{3}}{3} - \dfrac{1}{3}k = 0$ $k = \sqrt{3}$
2	Solution: (i)

	$\frac{d^2x}{dt^2} = \frac{2}{(t+1)^3}$ $\frac{dx}{dt} = \int 2(t+1)^{-3}\ dt$ $\frac{dx}{dt} = \frac{2(t+1)^{-2}}{-2} + a$, where $a \in \mathbb{R}$ $= -(t+1)^{-2} + a$ $x = \int -(t+1)^{-2} + a\ dt$ $= (t+1)^{-1} + at + b$, where $b \in \mathbb{R}$ $= \frac{1}{t+1} + at + b$, where $b \in \mathbb{R}$ (ii) $\frac{dx}{dt} = k - cx^2$, $k, c > 0$ When $x = 0.5$, $\frac{dx}{dt} = 0$ $k = c(0.5)^2$ $c = 4k$ $\frac{dx}{dt} = k - 4kx^2 = k(1-4x^2)$ (shown) (iii)

	$\frac{dx}{dt} = k - 4kx^2 = k\left(1-4x^2\right)$ $\int \frac{1}{1-4x^2}\,dx = \int k\,dt, \quad 1-4x^2 \neq 0$ $\frac{1}{2(2)}\ln\left\|\frac{1+2x}{1-2x}\right\| = kt + d \text{ , } d \in \mathbb{R}$ $\frac{1}{4}\ln\left\|\frac{1+2x}{1-2x}\right\| = kt + d$ $\ln\left\|\frac{1+2x}{1-2x}\right\| = 4kt + 4d$ $\frac{1+2x}{1-2x} = \pm e^{4kt+4d} = Ae^{4kt}$ where $A = \pm\, e^{4d}$ When $t = 0, x = 1,$ $\frac{1+2}{1-2} = A$ $A = -3$ $\frac{1+2x}{1-2x} = -3e^{4kt}$ $1+2x = -3e^{4kt} + 6xe^{4kt}$ $x\left(2-6e^{4kt}\right) = -3e^{4kt} - 1$ $x = \frac{-3e^{4kt}-1}{2-6e^{4kt}} = \frac{1+3e^{4kt}}{6e^{4kt}-2}$ or $\frac{1}{3e^{4kt}-1} + \frac{1}{2}$
3	Solution: (i) Let $y = \ln\left(x^2-1\right)$. $x = \pm\sqrt{1+e^y}$ Since $x > 1 > 0$, $\therefore x = \sqrt{1+e^y}$. $D_{f^{-1}} = R_f$ $= \mathbb{R}$ $f^{-1} : x \mapsto \sqrt{1+e^x},\ x \in \mathbb{R}$ (ii)

	 (iii)  (iv) Since $R_h = [0,3] \subseteq [0,8) = D_g$, therefore the function gh exists. Restrict D_g to be $[0,3]$ From the graph in **(iii)**, $R_{gh} = [0,6]$.
4	Solution: (i)

	$A_n = \dfrac{1}{n}\left(\mathrm{e}^0 + \mathrm{e}^{\frac{1}{n}} + \mathrm{e}^{\frac{2}{n}} + \mathrm{e}^{\frac{3}{n}} + \cdots + \mathrm{e}^{\frac{n-2}{n}} + \mathrm{e}^{\frac{n-1}{n}}\right)$ $= \dfrac{1}{n} \cdot \dfrac{\mathrm{e}^0\left(1-\left(\mathrm{e}^{\frac{1}{n}}\right)^n\right)}{1-\mathrm{e}^{\frac{1}{n}}}$ $= \dfrac{1}{n} \cdot \dfrac{1-\mathrm{e}}{1-\mathrm{e}^{\frac{1}{n}}} = \dfrac{\mathrm{e}-1}{n\left(\mathrm{e}^{\frac{1}{n}}-1\right)}$ $\therefore c = \mathrm{e}-1$ (ii) $\mathrm{e}^x - 1 = \left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\right) - 1 = x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ $\lim\limits_{x\to 0} \dfrac{1}{x}\left(\mathrm{e}^x - 1\right) = \lim\limits_{x\to 0}\left[\dfrac{1}{x}\left(x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\right)\right]$ $= \lim\limits_{x\to 0}\left(1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \ldots\right)$ $= 1$ (iii) $\lim\limits_{n\to\infty} \dfrac{\mathrm{e}-1}{n\left(\mathrm{e}^{\frac{1}{n}}-1\right)} = \lim\limits_{x\to 0} \dfrac{\mathrm{e}-1}{\frac{1}{x}\left(\mathrm{e}^x-1\right)}$ $= \mathrm{e}-1$ (iv) e – 1 is the exact area under the graph of $y = \mathrm{e}^x$ from x = 0 to x = 1. area $= \int_0^1 \mathrm{e}^x \mathrm{d}x = \mathrm{e}-1$. Since the graph of $y = \mathrm{e}^x$ is concave upwards, and $\dfrac{A_n + B_n}{2}$ is the sum of the area of n trapeziums each of width $\dfrac{1}{n}$, the area of all trapeziums will be greater than the exact area under the graph, which is $\int_0^1 \mathrm{e}^x \mathrm{d}x$.

	
5	Solution: (a)(i) Total number of possible passcodes $=26\times25\times24\times10\times9\times8\times7=78624000$ or $={}^{26}C_3\times3!\ \times\ {}^{10}C_4\times4!=78624000$ or $={}^{26}P_3\times{}^{10}P_4=78624000$ (a)(ii) <u>Case 1:</u> 1st digit 4, 6, 8 4, 6, 8 (3 choices) | 8 choices | 7 choices | 1, 3, 5, 7, 9 (5 choices) Number of possible passcodes $=26\times25\times24\times3\times8\times7\times5=13104000$ <u>Case 2:</u> 1st digit 3, 5, 7, 9 3, 5, 7, 9 (4 choices) | 8 choices | 7 choices | 4 choices Number of possible passcodes $=26\times25\times24\times4\times8\times7\times4=13977600$ Total number of possible passcodes $=13104000+13977600=27081600$ (b) Total number of possible passcodes $=26^3\times10^4\times\dfrac{7!}{4!3!}=6151600000$
6	Solution: (i) x | 1 | 2 | 3 | 4

$P(X=x)$	θ	3θ	5θ	k

Since $\sum_{\text{all } x} P(X=x) = 1$,

$\theta + 3\theta + 5\theta + k = 1$

$\therefore k = 1 - 9\theta$

Probability distribution of X is

x	1	2	3	4
$P(X=x)$	θ	3θ	5θ	$1-9\theta$

(ii)

$$
\begin{aligned}
E(X) &= 1(\theta) + 2(3\theta) + 3(5\theta) + 4(1-9\theta) \\
&= \theta + 6\theta + 15\theta + 4 - 36\theta \\
&= 4 - 14\theta
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= 1^2(\theta) + 2^2(3\theta) + 3^2(5\theta) + 4^2(1-9\theta) \\
&= \theta + 12\theta + 45\theta + 16 - 144\theta \\
&= 16 - 86\theta
\end{aligned}
$$

$$
\begin{aligned}
\text{Var}(X) &= E(X^2) - [E(X)]^2 \\
&= 16 - 86\theta - (4 - 14\theta)^2 \\
&= 16 - 86\theta - (16 - 112\theta + 196\theta^2) \\
&= 26\theta - 196\theta^2
\end{aligned}
$$

(iii)

$Y = a + bX$

$\text{Var}(Y) = \text{Var}(a + bX)$

$\text{Var}(Y) = b^2 \text{Var}(X)$

$\frac{1}{3}b^2 = b^2(26\theta - 196\theta^2)$

$196\theta^2 - 26\theta + \frac{1}{3} = 0 \quad (\because b \neq 0)$

Using GC,

$\theta = 0.0144$ or $\theta = 0.118$ (rejected $\because 0 < \theta < \frac{1}{9}$)

7	Solution:

A binomial distribution is appropriate as there is a large number of lego pieces with constant probability 0.15 of them being red suggests independence in selection. Moreover, there are only two possible outcomes (red or non red).

(i) Let X be the number of lego pieces, out of 20, that are red.

$X \sim \mathrm{B}(20, 0.15)$

$$\begin{aligned}\mathrm{P}(X \geq 4) &= 1 - \mathrm{P}(X \leq 3)\\ &= 0.35227\\ &= 0.352 \quad (3\text{s.f.})\end{aligned}$$

(ii) Let Y be the number of boxes of lego pieces, out of 50, that contain at least 4 red lego pieces.

$Y \sim \mathrm{B}(50, 0.35227)$

$$\begin{aligned}\mathrm{P}(Y \leq 19) &= 0.71498\\ &= 0.715 \quad (3\text{ s.f.})\end{aligned}$$

(iii) Let A be the number of lego pieces, out of 20, that are red.

$A \sim \mathrm{B}(20, p)$

$\mathrm{P}(1 \leq A < 4) = 0.22198$

$\mathrm{P}(A=1) + \mathrm{P}(A=2) + \mathrm{P}(A=3) = 0.22198$

$$\binom{20}{1} p(1-p)^{19} + \binom{20}{2} p^2(1-p)^{18} + \binom{20}{3} p^3(1-p)^{17} = 0.22198$$

$$20p(1-p)^{19} + 190p^2(1-p)^{18} + 1140p^3(1-p)^{17} = 0.22198$$

Since $0.2 < p < 1$, $p = 0.250$ (3 s.f)

8 Solution:

(i) Using GC,

Unbiased estimate of population mean is $\bar{x} = 325.58$ (2 d.p.)

Unbiased estimate of population variance is $s^2 = 1.5326^2 = 2.35$ (2 d.p.)

Let μ denote the population mean volume of shampoo dispensed by the machine.

Given $X \sim \mathrm{N}(\mu, \sigma^2)$ $\therefore \overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$

H_0: $\mu = 325$

H_1: $\mu > 325$

Test statistic: $Z = \dfrac{\overline{X} - \mu}{S/\sqrt{n}}$

Level of significance: 5%

Alternatively,

Reject H_0 if $z\text{-value} > 1.6449$

Reject H_0 if $p - \text{value} < 0.05$

Under H_0, using GC,

$p - \text{value} = 0.00160$ (3 s.f) or 0.00169 (3 s.f)

Conclusion:

Since $p - \text{value} = 0.00169 < 0.05$, we **reject H_0** and conclude that there is **sufficient evidence**, at the 5% significance level, that the mean volume dispensed is more than 325 ml.

Thus, the assembly manager's suspicion is valid at 5% level of significance.

There is a probability of 0.05 of concluding that the mean volume of shampoo dispensed is more than 325 ml when in fact, it is 325 ml.

(ii) H_0: $\mu = 325$

H_1: $\mu \neq 325$

Given $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$ $\therefore \overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$

Test statistic: $Z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}}$

Level of significance: 5%

Since H_0 is not rejected,

$$-1.9600 < z - \text{value} < 1.9600$$

$$-1.9600 < \frac{\bar{x} - 325}{\frac{1.2}{\sqrt{50}}} < 1.9600$$

$$324.67 < \bar{x} < 325.33 \quad (2 \text{ d.p.})$$

$\{\bar{x} \in \mathbb{R} : 324.67 < \bar{x} < 325.33\}$

9 Solution:

(i) Let k be the missing housing price index for 2011.

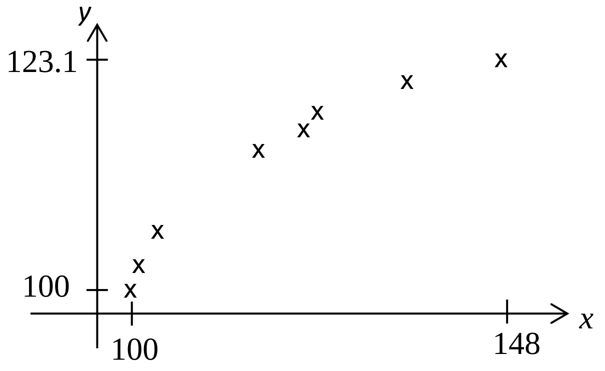
$\bar{y} = 111.775$ and $\bar{x} = \dfrac{815.2 + k}{8}$

Since $\bar{y}$ and $\bar{x}$ lies on the regression line,

$$111.775 = 54.271 + 0.48363\left(\frac{815.2 + k}{8}\right)$$

$k = 136$ (3 s.f.)(shown)

(ii)



From the scatter diagram, as x increases, y increases at a decreasing rate. Thus the linear model might not be the most appropriate model.

(iii) (Note that there is no clear independent variable.)
From GC, an appropriate regression line would be
$\sqrt{x} = 0.0896y + 0.860 \quad (3 \text{ s.f.})$

When $y = 134.6$, from GC, $x = 167 \quad (3 \text{ s.f.})$.
The estimated housing price index in 2016 is 167.

Since $y = 134.6$ <u>falls outside the data range of y</u>, the linear correlation between y and $\sqrt{x}$ might no longer hold and thus, the estimate is unreliable.

(iv) From GC, $r = 0.979 \quad (3 \text{ s.f.})$.

(v) The product moment correlation coefficient between $\sqrt{\frac{x}{100}}$ and $\frac{y}{100}$ <u>**does not differ**</u> from the value obtained in part **(iv)** as the r-value is <u>**independent of the scale of measurement.**</u>

Note that: $\sqrt{\frac{x}{100}} = \frac{\sqrt{x}}{10}$ means that the values of $\sqrt{x}$ undergo a scaling of 10 units and $\frac{y}{100}$ means that the values of y undergo a scaling of 100 units.

10 (i)
Let X be the mass of a randomly chosen nut in grams.
$X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{50}\right)$ and $X_1 + \ldots + X_{50} \sim \mathrm{N}\left(50\mu, 50\sigma^2\right)$

Given $\mathrm{P}\left(\overline{X} < 247\right) = 0.018079$ and $\mathrm{P}\left(X_1 + \ldots + X_{50} > 12600\right) = 0.78397$

Standardizing, $Z \sim \mathrm{N}(0,1)$

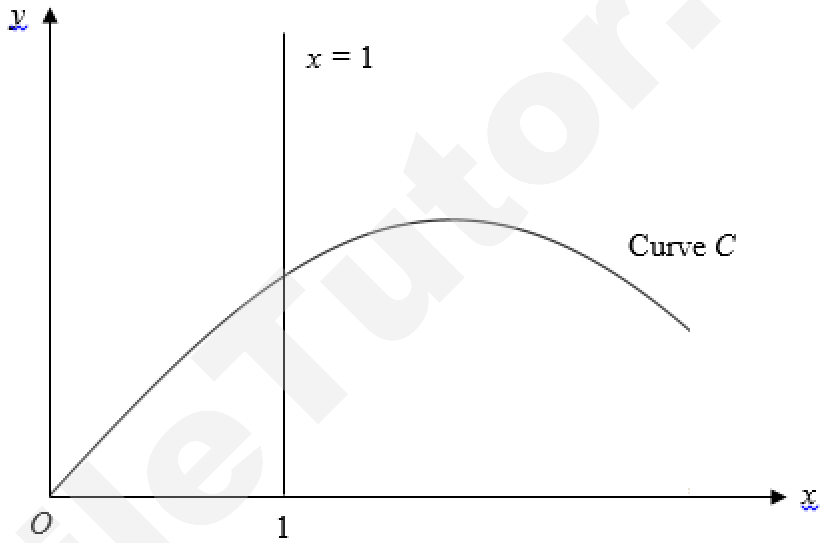
$P\left(Z < \dfrac{247-\mu}{\sigma/\sqrt{50}}\right) = 0.018079$ $\dfrac{247-\mu}{\sigma/\sqrt{50}} = -2.095146$ $\mu - 0.2962984\sigma = 247$ ...(1)	$P\left(Z > \dfrac{12600-50\mu}{\sqrt{50}\sigma}\right) = 0.78397$ $P\left(Z < \dfrac{12600-50\mu}{\sqrt{50}\sigma}\right) = 0.21603$ $\dfrac{12600-50\mu}{\sqrt{50}\sigma} = -0.7856714$ $50\mu - 0.7856714\left(\sqrt{50}\sigma\right) = 12600$...(2)

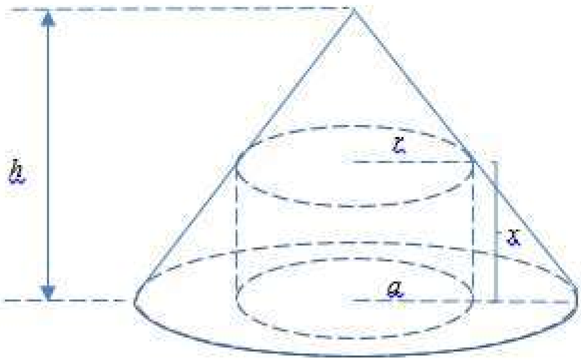
Solving equation (1) and (2), using GC,
$\mu = 255$ (nearest gram)
$\sigma = 27$ (nearest gram)

(ii)(a)
Let Y be the mass of a randomly chosen nut in grams.
$Y \sim \mathrm{N}\left(250, 5^2\right)$
Let W be the mass of a randomly chosen bolt in grams.
$W \sim \mathrm{N}\left(745, 7.3^2\right)$
$W - 3Y \sim \mathrm{N}\left(745 - 3\times 250, 7.3^2 + 3^2 \times 5^2\right)$
i.e. $W - 3Y \sim \mathrm{N}\left(-5, 278.29\right)$
$P\left(\left|W-3Y\right| \geq 40\right) = P\left(W-3Y < -40\right) + P\left(W-3Y > 40\right)$
$= 0.0214$ (3s.f.)

(ii)(b)
Let T be total mass of 10 randomly chosen nut, made using new material, in grams.
$T = 0.9Y_1 + 0.9Y_2 + \ldots + 0.9Y_{10} \sim \mathrm{N}\left(10\times 0.9 \times 250, 10 \times 0.9^2 \times 5^2\right)$
$T \sim \mathrm{N}\left(2250, 202.5\right)$
$P\left(T < 2240\right) = 0.241$ (3s.f.)

H2 Mathematics 2017 Preliminary Exam Paper 1 Question
Answer all questions [100 marks].

1	The sum of the first n terms of a sequence is denoted by S_n. The first term of the sequence is 3 and it is known that $S_3 = 21$ and $S_{10} = 210$. Given that S_n is a quadratic polynomial in n, find S_n in terms of n. [3]
2	Using the substitution $v = \sqrt{x} + 1$, find $\int \frac{1}{x + \sqrt{x}} \, \mathrm{d}x$, where $x > 0$. [3]
3	 The diagram shows the curve C with equation $y = \sin x$ and the line $x = 1$. With reference to the diagram, a student wrote down the following series $$S = \frac{1}{n}\left[\sin\left(\frac{1}{n}\right) + \sin\left(\frac{2}{n}\right) + \sin\left(\frac{3}{n}\right) + ... + \sin\left(\frac{n}{n}\right)\right].$$ **(i)** State what the series represents. [2] **(ii)** When $n \to \infty$, $S \to L$. State the geometrical meaning of L. Determine the exact value of L, leaving your answer in the form $a - \cos b$, where a and b are constants to be determined. [3] **(iii)** What can be said about the value of S in relation to the value of L? [1]
4	[It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



The diagram above shows a right circular cone with fixed radius a and fixed height h. A cylinder of radius r and height x is removed from the cone.

(i) Show that the volume of the remaining shape, V, is $\frac{\pi h}{3}\left(a^2 - 3r^2 + \frac{3r^3}{a}\right)$. [2]

(ii) As r varies, use differentiation to find the value of r that gives the minimum value of V, leaving your answer in terms of a. [4]

5 A line L passes through the points $A(3,-1,\ 0)$ and $B(11,\ 11,\ 4)$.

(i) Find the angle between L and the y-axis. [2]

(ii) State the geometrical meaning of $\left|\overrightarrow{OB}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right|$. [1]

The point $F(2a+1,\ a,\ a-1)$ is a point on L, where a is a positive constant.

The point P is such that $\overrightarrow{PF} = \begin{pmatrix}3\\0\\2\end{pmatrix}$ and the area of the triangle AFP is $\sqrt{\frac{59}{2}}$ units^2.

(iii) Determine the value of a. [3]

(iv) The point C on L is such that the ratio of the area of triangle AFP to the area of triangle FCP is $2:1$. State the ratio $AF:CF$, justifying your answer. [2]

6 **(i)** Show that $\int e^{2x}\cos x\,dx = \frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x + C$. [3]

	(ii)	Find the volume of the solid generated when the region bounded by $y = e^{x}\sqrt{(\cos x)}$ and $y = -\frac{2}{\pi}x + 1$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis, leaving your answer in exact form. [4]
7	**(i)**	Prove by the method of mathematical induction that $$\sum_{r=1}^{n} \frac{2}{r(r+2)} = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}$$ for all positive integers of n. [5]
	(ii)	Explain why $\sum_{r=1}^{n} \frac{2}{r(r+2)}$ is a convergent series, and state the value of the sum to infinity. [2]
	(iii)	Using the result in part **(i)**, find $\sum_{r=5}^{N} \frac{2}{(r-2)(r-4)}$. [2]
8		Using the substitution $y = ux$, show that the differential equation $$x\frac{dy}{dx} = 3x + y - 2$$ can be reduced to the form $$x^2\frac{du}{dx} = 3x - 2.$$ Hence, find the general solution to the differential equation $x\frac{dy}{dx} = 3x + y - 2$. [5]
	(i)	State the equation of the locus where the stationary points of the solution curves lie. [1]
	(ii)	Sketch, on a single diagram, the graph of the locus found in part **(i)** and two members of the family of solution curves, where the arbitrary constant in the general solution is equal to 1 and -1. [3]
9		It is given that

$$\mathrm{f}(x)=\begin{cases}(x-1)^2+4 & , \quad k \le x < 3,\\ 3x-1 & , \quad 3 \le x \le 4,\end{cases}$$

where $k \in \mapsto, k < 3$.

(i) Sketch, for $k=0$, the graph of $y=\mathrm{f}(x)$, stating the coordinates of the turning point. Write down the range of f. [3]

(ii) Explain why f^{-1} does not exist. State the smallest value of *k* for f^{-1} to exist. [2]

(iii) Using the value of *k* in part **(ii)**, find f^{-1} in similar form. [4]

(iv) State the geometrical relationship between f and f^{-1}. The point $P(a, b)$, where a and b are constants, lies on the graph $y=\mathrm{f}(x)$. The point Q on the graph $y=\mathrm{f}^{-1}(x)$ is the point corresponding to P. State the coordinates of Q. [2]

10 **(a)** It is given that $-1+\mathrm{i}$ is a root of the equation $2z^3+az^2+bz+(3+\mathrm{i})=0$.

(i) Find the values of the real numbers a and b. [4]

(ii) Using these values of a and b, find the other roots of this equation. [3]

(b) It is given that $w=-1+(\sqrt{3})\mathrm{i}$.

(i) Without using a calculator, find an exact expression for w^5. Give your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $0 \le \theta \le 2\pi$. [3]

(ii) Without using a calculator, find the three smallest positive whole number values of n for which $\dfrac{w^*}{w^n}$ is a real number. [4]

11 A curve C_1 is defined parametrically by the equations $x=t-\dfrac{1}{t}, \quad y=t+\dfrac{1}{t}, \ t \ne 0$.

(i) Sketch C_1, stating the equation of the asymptotes and coordinates of any points of intersection with the y-axis. [2]

(ii) Show that the equation of the normal to C_1 at the point with parameter p is given by

$$y = -\frac{p^2+1}{p^2-1}x + \frac{2(p^2+1)}{p}.$$ [4]

(iii) The normal in part **(ii)** intersects the x-axis at the point A and the y-axis at the point B. Find, in terms of p, an expression for the area of the triangle OAB. [4]

The line l is the normal to C_1 when $p = 2$.

(iv) Find the equation of l. [1]

A curve C_2 is defined parametrically by the equations $x = 3at$, $y = -t^2 + a$, $t \in \mapsto$ where a is a non-zero constant.

(v) Given that l intersects C_2, show that the parameter q of the point(s) of intersection satisfies the equation

$$q^2 - 5aq + 5 - a = 0.$$

Hence, determine the range of values of a such that l intersects C_2 at two distinct points. [3]

12 As part of a project, a group of engineering students design two robots for a game. One robot is called 'Prey' and the other robot is called 'Predator'. The two robots are designed with the following specifications.

'Prey': It is designed to leap 1 m forward for the first leap. Subsequently, it leaps 2.5 cm less than the previous leap distance. 'Prey' shuts down when the leap distance is 0 or when it is caught by 'Predator'.

'Predator': It is designed to leap 2 m forward for the first leap. Subsequently, it leaps 90% of the previous leap distance. 'Predator' shuts down when 'Prey' shuts down or when it catches 'Prey'.

Both robots take each leap at the same time and the number of leaps taken is given by n. 'Predator' starts the game from the starting line while 'Prey' starts the game 7 m in front of 'Predator'.

(i) Find the distance of 'Prey' and of 'Predator' from the starting line after n leaps, leaving your answers in terms of n. [2]

	(ii) Explain why 'Predator' has to catch 'Prey' before 'Predator's distance from the starting line reaches 20 m. [2] **(iii)** Using a graphical method, explain why 'Predator' will not catch 'Prey'. [3] **(iv)** 'Prey' now starts the game 4 m in front of 'Predator'. 'Predator' catches 'Prey' on the k-th leap. Find the value of k. Calculate the distance of 'Predator' from the starting line after completing the k-th leap. [3]

− End Of Paper −

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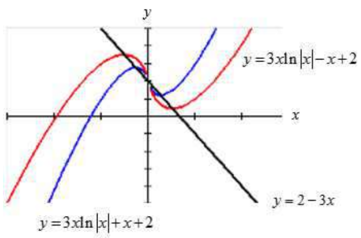
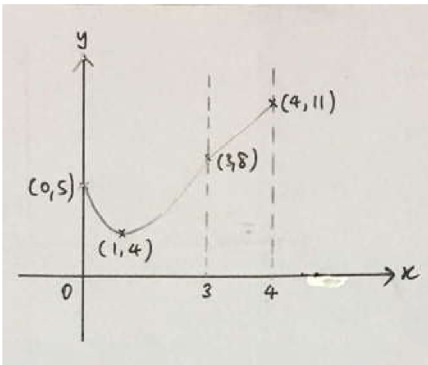
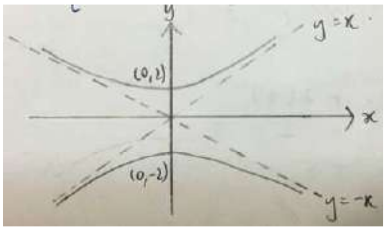
ANNEX B

MI H2 Math PU3 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Equations and Inequalities	$S_n = 2n^2 + n$
2	Integration techniques	$2\ln\left(\sqrt{x}+1\right)+c$
3	Application of Integration	(i) The sum of the areas of n rectangles with equal width from $x=0$ to $x=1$, where the top right vertex of each rectangle lies on the curve. (ii) L is the actual area under C from $x=0$ to $x=1$, $1-\cos 1$ (iii) Since the sum of the areas of the rectangles in part (i) is larger than the actual area under the curve C, $S > L$
4	Differentiation & Applications	(ii) $r = \frac{2}{3}a$
5	Vectors	(i) 36.7° (ii) The length of projection of $\overrightarrow{OB}$ onto the $z-$axis . (iii) 2 (iv) 2:1
6	Application of Integration	(ii) $\frac{1}{5}\pi e^{\pi} - \frac{2}{5}\pi - \frac{\pi^2}{6}$
7	Sigma Notation and Method of Difference	(ii) $\frac{3}{2}$ (iii) $\frac{3}{2} - \frac{2N-5}{(N-3)(N-2)}$
8	Differential Equations	$y = 3x\ln\lvert x\rvert + Cx + 2$ (i) $y = 2-3x$ (ii) 

<table>
<tr><td>9</td><td>Functions</td><td>(i)


Range of f, $R_f = [4,11]$
(ii) 1
(iii) $f^{-1}(x) = \begin{cases} 1+\sqrt{x-4}, & 4 \le x < 8, \\ \frac{1}{3}x + \frac{1}{3}, & 8 \le x \le 11, \end{cases}$
(iv) The graph $y = f^{-1}(x)$ is the reflection of the graph $y = f(x)$ in the line $y = x$. The coordinates of Q is (b, a).</td></tr>
<tr><td>10</td><td>Complex numbers</td><td>(a)(i) $a = 6, b = 7$
(a)(ii) $z = -\frac{1}{2} - \frac{1}{2}\mathrm{i}$ or $z = -\frac{3}{2} - \frac{1}{2}\mathrm{i}$
(b)(i) $32e^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$
(b)(ii) 2, 5, 8.</td></tr>
<tr><td>11</td><td>Differentiation & Applications</td><td>(i) $x = t - \frac{1}{t}, \quad y = t + \frac{1}{t}, \quad t \ne 0.$


(iii) $\frac{2\left(p^2+1\right)}{p^2}\left|p^2-1\right|$ or $\frac{2}{p^2}\left|p^4-1\right|$ units^2
(iv) $y = -\frac{5}{3}x + 5$
(v) $a < -0.978$ or $a > 0.818$ (3 s.f)</td></tr>
<tr><td>12</td><td>AP and GP</td><td>(i) $-0.0125n^2 + 1.0125n + 7$; $20\left(1-0.9^n\right)$
(iv) 8; 11.4 m</td></tr>
</table>

13	Q13 Topic	

2017 PU3 H2 Prelim II Paper 1 Suggested Solutions

Qn. No.	Question
1	Let $S_n = an^2 + bn + c$ where $a,\ b$ and c are constants $S_1 = 3 \quad \Rightarrow a+b+c=3$ $S_3 = 21 \quad \Rightarrow 9a+3b+c=21$ $S_{10} = 210 \quad \Rightarrow 100a+10b+c=210$ Using GC, $a=2,\ b=1,\ c=0 \quad \Rightarrow \quad S_n = 2n^2+n$
2	$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{2x^{\frac{1}{2}}} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}v} = 2x^{\frac{1}{2}}$ $\int \frac{1}{x+\sqrt{x}}\ \mathrm{d}x = \int \frac{1}{x+x^{\frac{1}{2}}} \frac{\mathrm{d}x}{\mathrm{d}v}\mathrm{d}v$ where $\frac{1}{x+x^{\frac{1}{2}}} \frac{\mathrm{d}x}{\mathrm{d}v} = \frac{1}{x+x^{\frac{1}{2}}}\left(2x^{\frac{1}{2}}\right) = \frac{2}{\left(x+x^{\frac{1}{2}}\right)\left(x^{-\frac{1}{2}}\right)} = \frac{2}{\left(x^{\frac{1}{2}}+1\right)} = \frac{2}{v}$ $\int \frac{1}{x+\sqrt{x}}\ \mathrm{d}x = \int \frac{2}{v}\mathrm{d}v$ $= 2\int \frac{1}{v}\ \mathrm{d}v$ $= 2\ln v + c$ $= 2\ln\left(\sqrt{x}+1\right)+c$
3	(i) The sum of the areas of n rectangles with equal width from $x=0$ to $x=1$, where the top right vertex of each rectangle lies on the curve. (ii) L is the actual area under C from $x=0$ to $x=1$. $L = \int_0^1 \sin x\ \mathrm{d}x$ $= \left[-\cos x\right]_0^1$ $= -\cos 1 + \cos 0$ $= 1-\cos 1 \qquad$ i.e. $a=1,\ b=1$ (iii) Since the sum of the areas of the rectangles in part (i) is larger than the actual area under the curve C, $S > L$

4	(i) $\frac{r}{a}=\frac{h-x}{h}$ $\therefore x=h-\frac{hr}{a}$ $V=\frac{1}{3}\pi a^2 h-\pi r^2\left(h-\frac{hr}{a}\right)$ $\quad=\frac{\pi h}{3}\left(a^2-3r^2+\frac{3r^3}{a}\right)$ (shown) (ii) $\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{\pi h}{3}\left(-6r+\frac{9r^2}{a}\right)$ For max/min volume: $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ $\frac{\pi h}{3}\left(-6r+\frac{9r^2}{a}\right)=0$ $-6r+\frac{9r^2}{a}=0$ $r\left(-6+\frac{9r}{a}\right)=0$ $r=0$ (reject as $r>0$) or $r=\frac{2}{3}a$ **Method 1 (2nd derivative test)** $\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=\frac{\pi h}{3}\left(-6+\frac{18r}{a}\right)$ At $r=\frac{2}{3}a$: $\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=\frac{\pi h}{3}\left(-6+\frac{18}{a}\left(\frac{2}{3}a\right)\right)=2\pi h>0$ Therefore the volume is a minimum when $r=\frac{2}{3}a$. **Method 2 (1st derivative test)**

r	$\frac{2}{3}a^-$	$\frac{2}{3}a$	$\frac{2}{3}a^+$
$\frac{\mathrm{d}V}{\mathrm{d}r}$	Negative	Zero	Positive

Therefore the volume is a minimum when $r=\frac{2}{3}a$.

5

(i)

$$\overrightarrow{AB}=\begin{pmatrix}8\\12\\4\end{pmatrix}=4\begin{pmatrix}2\\3\\1\end{pmatrix}$$

The required angle, $\theta=\cos^{-1}\dfrac{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}{\sqrt{14}\sqrt{1}}=36.7^{\circ}$ (1 d.p)

(ii)

The length of projection of $\overrightarrow{OB}$ onto the $z-$axis .

(iii)

$$\frac{1}{2}\left|\overrightarrow{AF}\times\overrightarrow{PF}\right|=\sqrt{\frac{59}{2}}$$

$$\frac{1}{2}\left|\begin{pmatrix}2a-2\\a+1\\a-1\end{pmatrix}\times\begin{pmatrix}3\\0\\2\end{pmatrix}\right|=\sqrt{\frac{59}{2}}$$

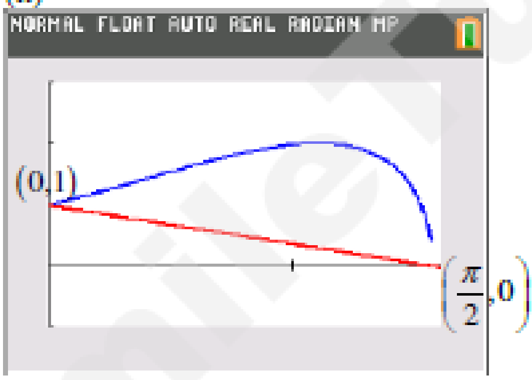
$$\left|\begin{pmatrix}2(a+1)\\1-a\\-3(a+1)\end{pmatrix}\right|=2\sqrt{\frac{59}{2}}$$

$$\sqrt{4(a+1)^2+(1-a)^2+9(a+1)^2}=2\sqrt{\frac{59}{2}}$$

$$13(a+1)^2+(1-a)^2=118$$
$$14a^2+24a-104=0$$
$$7a^2+12a-52=0$$
$$(7a+26)(a-2)=0$$
$a=-\frac{26}{7}$ (rejected as $a>0$) or $a=2$

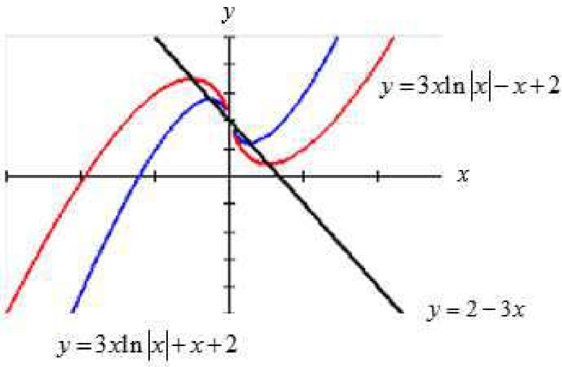
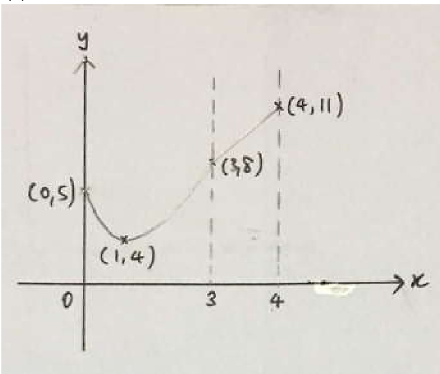
Accept: Using GC, $a=2$ or $a=-3.7143$ (rejected as $a>0$)

(iv)

	Both triangles have the same height (h). $AF:CF$ = Area of triangle AFP : Area of triangle FCP $= 2:1$
6	(i) $\int e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\cos x + \int \frac{1}{2}e^{2x}\sin x\,dx$ $= \frac{1}{2}e^{2x}\cos x + \frac{1}{2}\left[\frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx\right]$ $= \frac{1}{2}e^{2x}\cos x + \frac{1}{4}e^{2x}\sin x - \frac{1}{4}\int e^{2x}\cos x\,dx$ $\frac{5}{4}\int e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\cos x + \frac{1}{4}e^{2x}\sin x + C_1$ $\int e^{2x}\cos x\,dx = \frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x + C$ (ii)  Volume $= \pi\int y^2\,dx$ $= \pi\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx - \frac{1}{3}\pi(1)^2\left(\frac{\pi}{2}\right)$ $= \pi\left[\frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x\right]_0^{\frac{\pi}{2}} - \frac{\pi^2}{6}$ $= \pi\left[\frac{1}{5}e^{\pi}\sin\frac{\pi}{2} - \frac{2}{5}e^{0}\cos 0\right] - \frac{\pi^2}{6}$ $= \frac{1}{5}\pi e^{\pi} - \frac{2}{5}\pi - \frac{\pi^2}{6}$
7	(i) Let P_n be the statement $\sum_{r=1}^{n}\frac{2}{r(r+2)} = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}$ for $n \in Z^+$. Prove P_1 is true.

	LHS = $\dfrac{2}{(1)(1+2)}=\dfrac{2}{3}$ RHS = $\dfrac{3}{2}-\dfrac{2(1)+3}{(1+1)(1+2)}=\dfrac{4}{6}=\dfrac{2}{3}=\text{LHS}$ P_1 is true. Assume that P_k is true for some $k\in Z^+$ i.e. $\displaystyle\sum_{r=1}^{k}\frac{2}{r(r+2)}=\frac{3}{2}-\frac{2k+3}{(k+1)(k+2)}$. Prove P_{k+1} is true i.e. $\displaystyle\sum_{r=1}^{k+1}\frac{2}{r(r+2)}=\frac{3}{2}-\frac{2k+5}{(k+2)(k+3)}$. LHS $\displaystyle=\sum_{r=1}^{k}\frac{2}{r(r+2)}+T_{k+1}$ $\displaystyle=\frac{3}{2}-\frac{2k+3}{(k+1)(k+2)}+\frac{2}{(k+1)(k+3)}$ $\displaystyle=\frac{3}{2}-\frac{(2k+3)(k+3)-2(k+2)}{(k+1)(k+2)(k+3)}$ $\displaystyle=\frac{3}{2}-\frac{2k^2+7k+5}{(k+1)(k+2)(k+3)}$ $\displaystyle=\frac{3}{2}-\frac{(2k+5)(k+1)}{(k+1)(k+2)(k+3)}$ $\displaystyle=\frac{2k+5}{(k+2)(k+3)}=\text{RHS}$ P_{k+1} is true Since P_1 is true, and P_k is true implies P_{k+1} is true, by Mathematical Induction, $\displaystyle\sum_{r=1}^{n}\frac{2}{r(r+2)}=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}$ for $n\in Z^+$. (ii) $\displaystyle n\to\infty,\frac{2n+3}{(n+1)(n+2)}\to 0,\sum_{r=1}^{n}\frac{2}{r(r+2)}\to\frac{3}{2}$ The series converges to a value. $\therefore$ the series is a convergent series. $\displaystyle\sum_{r=1}^{\infty}\frac{2}{r(r+2)}=\frac{3}{2}$ (iii)

	$\sum_{r=5}^{N}\frac{2}{(r-2)(r-4)}$ $\xrightarrow[r=l+4]{\text{letting}}\sum_{l+4=5}^{l+4=N}\frac{2}{(l+4-2)(l+4-4)}$ $=\sum_{l=1}^{N-4}\frac{2}{l(l+2)}$ $=\frac{3}{2}-\frac{2(N-4)+3}{(N-4+1)(N-4+2)}$ $=\frac{3}{2}-\frac{2N-5}{(N-3)(N-2)}$
8	$y=ux$ $\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$ $x\left(u+x\frac{\mathrm{d}u}{\mathrm{d}x}\right)=3x+ux-2$ $ux+x^2\frac{\mathrm{d}u}{\mathrm{d}x}=3x+ux-2$ $x^2\frac{\mathrm{d}u}{\mathrm{d}x}=3x-2$ (shown) $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{3x-2}{x^2}$ $\int\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x=\int\frac{3x-2}{x^2}\ \mathrm{d}x$ $\int\ \mathrm{d}u=\int\frac{3}{x}-\frac{2}{x^2}\ \mathrm{d}x$ $u=3\ln\|x\|+\frac{2}{x}+C$ $\frac{y}{x}=3\ln\|x\|+\frac{2}{x}+C$ $y=3x\ln\|x\|+Cx+2$ (i) For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\Rightarrow x(0)=3x+y-2$ $\Rightarrow y=2-3x$ The equation of the locus is $y=2-3x$. (ii)

	
9	(i)  Range of f, $R_f = [4, 11]$ (ii) A horizontal line, $y = k,\ 4 < k \le 5$ intersects the graph of $y = f(x)$ at 2 points. f is not a one-one function. Hence, f^{-1} does not exist. For f^{-1} to exist, the minimum value of k is 1. (iii) Let $y = f(x)$ For $1 \le x < 3$: Let $y = (x-1)^2 + 4$ $x = 1 \pm \sqrt{y-4}$ $x = 1 + \sqrt{y-4}$ since $1 \le x < 3$ $f^{-1}(x) = 1 + \sqrt{x-4},\ 4 \le x < 8$ For $3 \le x \le 4$: Let $y = 3x - 1$ $x = \frac{1}{3}y + \frac{1}{3}$ $f^{-1}(x) = \frac{1}{3}x + \frac{1}{3},\ 8 \le x \le 11$

	$\mathrm{f}^{-1}(x)=\begin{cases}1+\sqrt{x-4} & , \quad 4\le x<8,\\ \frac{1}{3}x+\frac{1}{3} & , \quad 8\le x\le 11,\end{cases}$ (iv) The graph $y=\mathrm{f}^{-1}(x)$ is the reflection of the graph $y=\mathrm{f}(x)$ in the line $y=x$. The coordinates of Q is (b,a).
10	(a)(i) Since $-1+\mathrm{i}$ is a root of $2z^3+az^2+bz+(3+\mathrm{i})=0$, $2(-1+\mathrm{i})^3+a(-1+\mathrm{i})^2+b(-1+\mathrm{i})+(3+\mathrm{i})=0$ $4+4\mathrm{i}+a(-2\mathrm{i})-b+b\mathrm{i}+3+\mathrm{i}=0$ Comparing real parts: $4-b+3=0\Rightarrow b=7$ Comparing imaginary parts: $4-2a+b+1=0\Rightarrow a=6$ (a)(ii) $2z^3+6z^2+7z+(3+\mathrm{i})=0$ $\left[z-(-1+\mathrm{i})\right]\left[2z^2+(4+2\mathrm{i})z+(1+2\mathrm{i})\right]=0$ $z=-1+\mathrm{i}\ (\text{given})$ or $z=\dfrac{-(4+2\mathrm{i})\pm\sqrt{\left[(4+2\mathrm{i})^2-4(2)(1+2\mathrm{i})\right]}}{2(2)}$ $=\dfrac{(-4-2\mathrm{i})\pm 2}{4}$ $z=-\frac{1}{2}-\frac{1}{2}\mathrm{i}$ or $z=-\frac{3}{2}-\frac{1}{2}\mathrm{i}$ (b)(i) $\left\|-1+\mathrm{i}\sqrt{3}\right\|=2$ $\arg\left(-1+\mathrm{i}\sqrt{3}\right)=\pi-\tan^{-1}\left(\sqrt{3}\right)=\frac{2\pi}{3}$ $w^5=\left(2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}\right)^5=32\mathrm{e}^{\mathrm{i}\left(\frac{10\pi}{3}\right)}=32\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$ (b)(ii) $\dfrac{w^*}{w^n}=\dfrac{2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}}{\left[2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}\right]^n}=2^{1-n}\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)}$

Method 1

$$2^{1-n}\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)} = 2^{1-n}\left[\cos\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)\right]$$

$$= 2^{1-n}\left[\cos\left(\frac{2\pi}{3}+\frac{2n\pi}{3}\right)-\mathrm{i}\sin\left(\frac{2\pi}{3}+\frac{2n\pi}{3}\right)\right]$$

Since $\frac{w^*}{w^n}$ is a real number,

$$\sin\left(\frac{2\pi}{3}+\frac{2n\pi}{3}\right)=0$$

$$\frac{2\pi}{3}+\frac{2n\pi}{3}=\pi,2\pi,3\pi,4\pi,5\pi,6\pi...$$

$$2\pi+2n\pi=3\pi,6\pi,9\pi,12\pi,15\pi,18\pi...$$

$$2n\pi=\pi,4\pi,7\pi,10\pi,13\pi,16\pi...$$

$$n=\frac{1}{2},2,\frac{7}{2},5,\frac{13}{2},8,...$$

The 3 smallest positive whole number values of n are 2, 5 and 8.

Method 2

Since $\frac{w^*}{w^n}$ is a real number, $\arg\left(\frac{w^*}{w^n}\right)=k\pi,\ k\in\mapsto$

$$-\frac{2n\pi}{3}-\frac{2\pi}{3}=k\pi$$

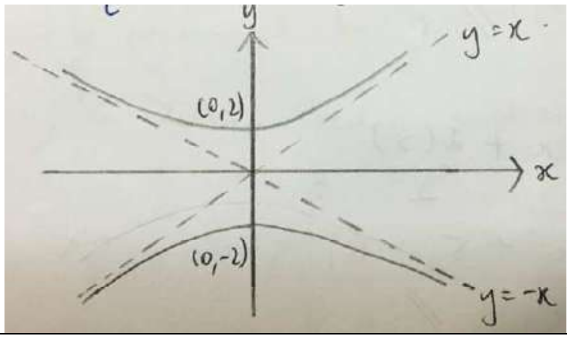
$$n=-1-\frac{3k}{2}$$

At $k=-2$: $n=2$
At $k=-4$: $n=5$
At $k=-6$: $n=8$

The 3 smallest positive whole number values of n are 2, 5 and 8.

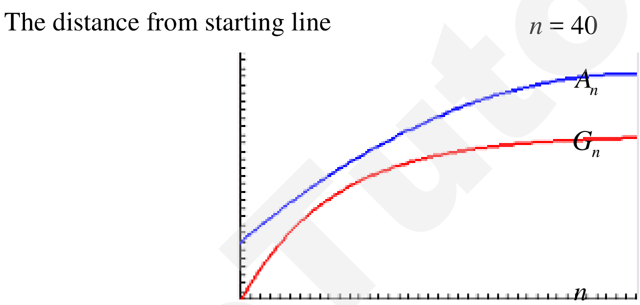
11 (i)

$x=t-\frac{1}{t},\quad y=t+\frac{1}{t},\ \ t\neq 0.$



	(ii) $x = t - \frac{1}{t}, \quad y = t + \frac{1}{t}$ $\frac{dx}{dt} = 1 + \frac{1}{t^2} = \frac{t^2+1}{t^2}, \quad \frac{dy}{dt} = 1 - \frac{1}{t^2} = \frac{t^2-1}{t^2}$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2-1}{t^2+1}$ When $t = p$, the gradient of normal $= -\frac{p^2+1}{p^2-1}$ The required equation of normal: $y - \left(p + \frac{1}{p}\right) = -\frac{p^2+1}{p^2-1}\left[x - \left(p - \frac{1}{p}\right)\right]$ $y = -\frac{p^2+1}{p^2-1}x + \frac{p^2+1}{p^2-1}\left(p - \frac{1}{p}\right) + p + \frac{1}{p}$ $y = -\frac{p^2+1}{p^2-1}x + \frac{p^2+1}{p^2-1}\left(\frac{p^2-1}{p}\right) + \frac{p^2+1}{p}$ $y = -\frac{p^2+1}{p^2-1}x + \frac{2\left(p^2+1\right)}{p}$ (shown) (iii) When $x = 0$, $y = \frac{2\left(p^2+1\right)}{p} \Rightarrow B\left(0, \ \frac{2\left(p^2+1\right)}{p}\right)$ When $y = 0$, $-\frac{p^2+1}{p^2-1}x + \frac{2\left(p^2+1\right)}{p} = 0$ $x = \frac{2\left(p^2-1\right)}{p} \Rightarrow A\left(\frac{2\left(p^2-1\right)}{p}, \ 0\right)$

	Area of triangle OAB $=\left\|\frac{1}{2}\left[\frac{2\left(p^2+1\right)}{p}\right]\left[\frac{2\left(p^2-1\right)}{p}\right]\right\|$ $=2\left\|\frac{\left(p^2+1\right)\left(p^2-1\right)}{p^2}\right\|$ $=\frac{2}{p^2}\left\|\left(p^2+1\right)\left(p^2-1\right)\right\|$ $=\frac{2\left(p^2+1\right)}{p^2}\left\|p^2-1\right\|$ or $\frac{2}{p^2}\left\|p^4-1\right\|$ units^2 (iv) When $p=2$, the equation of the normal is $y=-\frac{2^2+1}{2^2-1}x+\frac{2\left(2^2+1\right)}{2}$ $y=-\frac{5}{3}x+5$ The equation of l is $y=-\frac{5}{3}x+5$. (v) By substitution, $-q^2+a=-\frac{5}{3}(3aq)+5$ $q^2-5aq+5-a=0$ (shown) For l to intersect C_2 at 2 distinct points, $b^2-4ac>0$ $(-5a)^2-4(1)(5-a)>0$ $25a^2+4a-20>0$ $a<-0.978$ or $a>$ 0.818 (3 s.f)
12	(i) Distance of 'Prey' from starting line, A_n $=\frac{n}{2}\left[2(1)+(n-1)(-0.025)\right]+7=-0.0125n^2+1.0125n+7$ Distance of 'Predator' from starting line, G_n $=\frac{2\left(1-0.9^n\right)}{1-0.9}=20\left(1-0.9^n\right)$ (ii)

	The sum to infinity $=\dfrac{2}{1-0.9}=20$ Hence, 'Predator' has to catch 'Prey' before its distance from the starting line reaches 20 m. (iii) To determine when the leap distance of 'Prey' becomes 0 (if it is not caught): $1+(n-1)(-0.025)>0$ $n<41$ If not caught, 'Prey' will leap 40 times before the leap distance becomes 0. Plot, for $0\le n\le 40$, the graphs of $A_n=-0.0125n^2+1.0125n+7$ and $G_n=20\left(1-0.9^n\right)$ as follows:  For $0\le n\le 40$, since the two curves do not intersect, 'Predator' will not catch 'Prey'. (iv) When 'Predator' catches 'Prey', $-0.0125n^2+1.0125n+4=20\left(1-0.9^n\right)$ Using GC, $n=7.2557$ or 12.012 (rejected) 'Predator' catches 'Prey' on the 8th leap $\Rightarrow k=8.$ The required distance $=20\left(1-0.9^8\right)=11.4$ m (3 s.f)

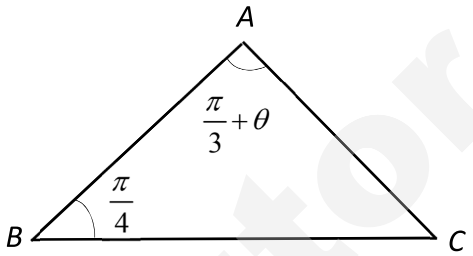
H2 Mathematics 2017 Preliminary Exam Paper 2 Question
Answer all questions [100 marks].

1	The curve C has the equation $4(x-1)^2+9y^2=36$. **(i)** Sketch, for $y \geq 0$, the curve C, stating the coordinates of the end points and the turning point. [3] **(ii)** By adding a suitable graph to your sketch in part **(i)**, solve the inequality $$2\sqrt{\left[1-\frac{(x-1)^2}{9}\right]}+2-(x-1)^2 \geq 0.$$ [2] **(iii)** Hence, solve the inequality $2\sqrt{\left[1-\frac{(e^x-1)^2}{9}\right]} \geq \left(e^x-1\right)^2-2$. [2]
2	Two loci in the Argand diagram are given by the equations $$\lvert z-2+2i \rvert = 1 \quad \text{and} \quad \arg z = -\frac{\pi}{6}.$$ The complex numbers z_1 and z_2, where $\lvert z_1 \rvert < \lvert z_2 \rvert$, correspond to the points of intersection of these loci. **(i)** Draw an Argand diagram to show both loci, and mark the points represented by z_1 and z_2. [3] **(ii)** Find the two values of z which represent points on $\lvert z-2+2i \rvert = 1$ such that $\lvert z-z_1 \rvert = \lvert z-z_2 \rvert$. [4] **(iii)** Given that the complex number w satisfies $\lvert w-2+2i \rvert \leq 1$ and $\arg w \leq -\frac{\pi}{6}$, find the range of values of $\arg(w+3i)$. [3]
3	**(a)** It is given that $\tan^{-1} y = \ln(1+x)$. **(i)** Show that $(1+x)\frac{dy}{dx} = 1+y^2$. [1]

(ii) By successively differentiating this result, find the Maclaurin series for $\tan\left[\ln\left(1+x\right)\right]$, up to and including the term in x^3. [3]

(iii) It is given that $\mathrm{f}(x) = \mathrm{e}^x \tan\left[\ln\left(1+x\right)\right]$. Using your answer to part **(a)(ii)**, estimate the value of $\mathrm{f}'\left(\frac{1}{2}\right)$. [3]

(b) The diagram shows triangle *ABC*, where $AC = k$ cm, $BC = h$ cm, $\angle BAC = \frac{\pi}{3} + \theta$ and $\angle ABC = \frac{\pi}{4}$.



Given that θ is a sufficiently small angle, show that

$$\frac{h}{k} \approx \frac{\sqrt{2}}{4}\left[2\sqrt{3} + 2\theta - \left(\sqrt{3}\right)\theta^2\right].$$ [3]

4 The plane π_1 contains the line $l_1 : \mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$, where $\lambda \in \mathbb{R}$, and is parallel to the line

$l_2 : \mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \mu\begin{pmatrix}2\\0\\1\end{pmatrix}$, where $\mu \in \mathbb{R}$.

(i) Find the vector equation of π_1 in scalar product form. [2]

(ii) Find the position vector of the foot of the perpendicular from the point $A(1, 0, 1)$ to the plane π_1. [3]

(iii) Find the position vector of the point A', which is the reflection of A about π_1. [2]

(iv) Given that the angle between $l_3 : \mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\0\end{pmatrix}$, where $\alpha \in \mathbb{R}$, and the plane

$\pi_2 : ax + 2y - z = 3$, where $a \in \mathbb{R}$, is $\dfrac{\pi}{4}$, find the value of a. [2]

(v) Find the line of intersection between the planes π_1 and π_2. [1]

(vi) π_3 has equation $bx + y + z = c$, where $b, c \in \mathbb{R}$. Given that π_1, π_2 and π_3 have no points in common, describe the geometrical relationship between the three planes. What can be said about the values of b and c? [3]

5 Resilience Primary School has 500 students who are either Chinese, Indian or Malay, as seen in the table below.

	Chinese	Indian	Malay
Boys	114	8	93
Girls	122	77	86

The National Eye Centre wishes to conduct a survey at Resilience Primary School to find out the number of hours students spend on electronic devices each week, using a sample of 50 students.

(i) Explain how stratified sampling can be carried out in this context. [2]

(ii) Give two reasons why systematic sampling may not be appropriate. [2]

6 In another survey conducted by the National Eye Centre, it was found that p% are boys and the remaining are girls. The probability that a randomly chosen boy wears spectacles is 0.3 and the probability that a randomly chosen girl wears spectacles is 0.24.

(i) Find the value of p, given that the probability that a randomly chosen child wears spectacles is 0.267. [2]

(ii) For a general value of p, the probability that a randomly chosen child that wears spectacles is a girl is denoted by $\mathrm{f}(p)$. Show that $\mathrm{f}(p) = \dfrac{4(100-p)}{(400+p)}$. Prove by differentiation that f is a decreasing function for $0 \le p \le 100$, and explain what this statement means in the context of the question. [5]

7 In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters.

The mass, in grams, of broccoli and carrots are normally distributed with means and standard deviations as shown in the table below.

	Mean (g)	Standard deviation (g)
Broccoli	μ	σ
Carrot	180	15

(i) Given that the probability that the mass of a randomly chosen broccoli does not exceed 250g is 0.788 and the probability that the mass of a randomly chosen broccoli exceeds 236g is 0.625, find the values of μ and σ. [3]

(ii) Find the probability that the mass of a randomly chosen broccoli lies within 5 grams of a randomly chosen carrot. [2]

(iii) 120 broccoli are randomly chosen. Using a suitable approximation, find the probability that there are fewer than 90 broccoli with a mass not exceeding 250g.[3]

(iv) Determine, with explanation, whether the mass of a vegetable chosen randomly from a basket containing an equal number of broccoli and carrots follows a normal distribution. [1]

8 The table gives the values of eight observations of bivariate data, *x* and *y*.

x	1	2	3	4	5	6	7	8
y	5	1	18	23	28	31	33	34

(i) Draw a scatter diagram for these values, labelling the axes clearly. Determine the outlier by labelling it as P in your scatter diagram. [2]

(ii) By omitting *P*, explain if $y = ax^2 + b$ or $y = a\ln x + b$ is the better model for the data. [2]

(iii) Using the more appropriate model found in part **(ii)**, calculate the equation of the least-squares regression line. [1]

(iv) Interpret, in the context of the question, the least squares estimates of *a* and *b*. [2]

(v) Use the regression line found in part **(iii)** to predict the value of *y* when *x* = 4.5. Comment on the reliability of your answer. [2]

9 Based on past records, the mean number of rainy days per year in Singapore was reported as 178. The authorities suspect that due to global warming, the number of rainy days has changed. A random sample of 12 years is taken and the number of rainy days per year, *X*, is summarised by

$$\sum(x-8)=2017.7, \qquad \sum x^2=372\,500.$$

(i) Calculate the unbiased estimates of the mean and variance of X. [2]

(ii) Test, at the 5% level of significance, whether the mean number of rainy days per year has changed. State any assumptions used in your calculations. [4]

(iii) Explain, in the context of the question, the meaning of the p-value. [1]

(iv) The population variance is found to be 9 and the assumption used in part **(ii)** holds true. A test at the 5% level of significance whether the mean number of rainy days per year has changed was conducted. Find the range of values of $\overline{x}$ such that the null hypothesis is not rejected. [3]

10 **(a)** Find the number of ways in which the letters of the word MILLENNIUM can be arranged if

(i) there are no restrictions, [1]

(ii) the first and last letters are the same, and the letters E and U must be separated. [2]

Four letters are randomly selected from the letters of the word MILLENNIUM to form a code word. Find the number of possible code words that can be formed. [2]

(b) Mr See (together with his wife and daughter) and Mrs Saw (together with her husband and two sons) came to visit their former teacher Mdm Rain during Millennium Institute's Homecoming Day. Find the number of ways Mr See and his family, Mrs Saw and her family, and Mdm Rain can be arranged if

(i) they are around a table with ten indistinguishable chairs, such that the children are seated together. [2]

(ii) the two empty chairs are removed and Mr See's daughter is seated between her parents and the See family are to be seated directly opposite Mdm Rain. [3]

11 In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters.

The number of people queuing to buy coffee at CoffeeVille in a period of 1 minute during lunch hour (12pm to 2pm) is a random variable with an average number of 2.9.

State, in context, a condition under which a Poisson distribution would be a suitable probability model. [1]

Assume that the number of people queuing to buy coffee at CoffeeVille in a period of 1 minute during the lunch hour follows the distribution $\text{Po}(2.9)$.

(i) State the most probable number of people queuing in 1 minute. [1]

(ii) Find the probability that in a period of 3 minutes, there are at most 5 people queuing to buy coffee. [2]

(iii) *N* periods of 3 minutes are taken. Given that the probability that at least 7 periods of 3 minutes have at most 5 people queuing to buy coffee is more than 0.99, find the least value of *N*. [3]

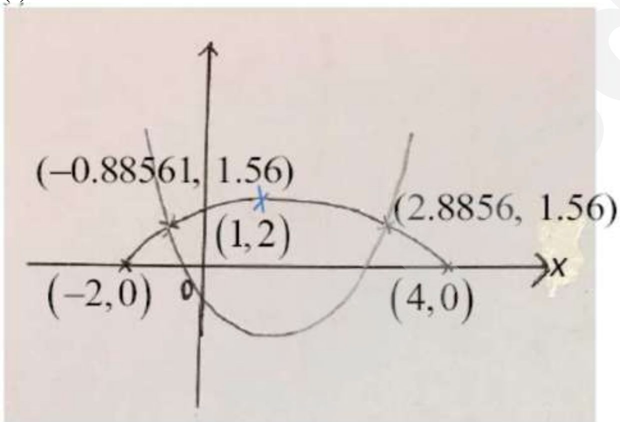
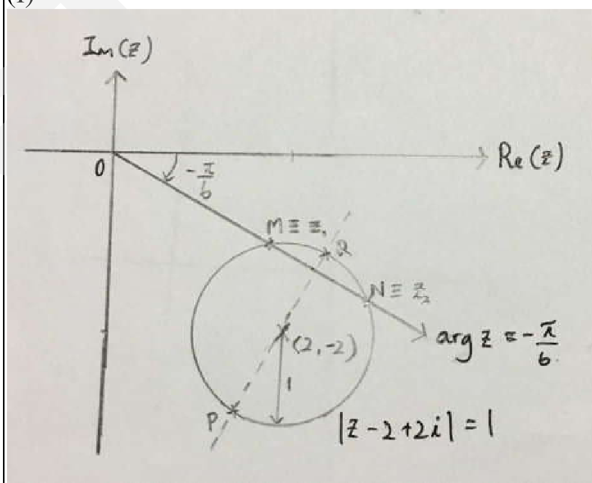
(iv) A random sample of 120 periods of 3 minutes is taken. Using a suitable approximation, find the probability that more than 12 periods of 3 minutes have exactly 4 people queuing. [3]

(v) Explain why the Poisson model would probably not be valid if applied to the operating hours of CoffeeVille from 11am to 10pm. [1]

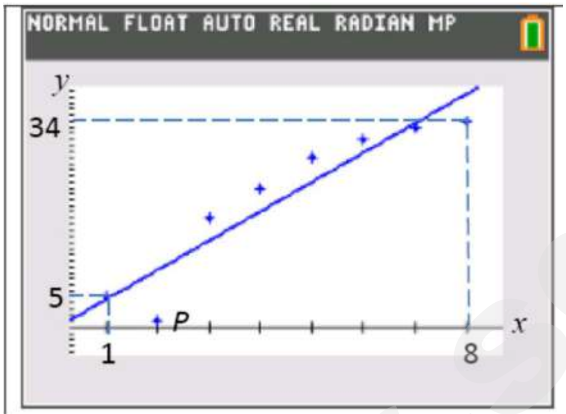
− End Of Paper −

ANNEX B

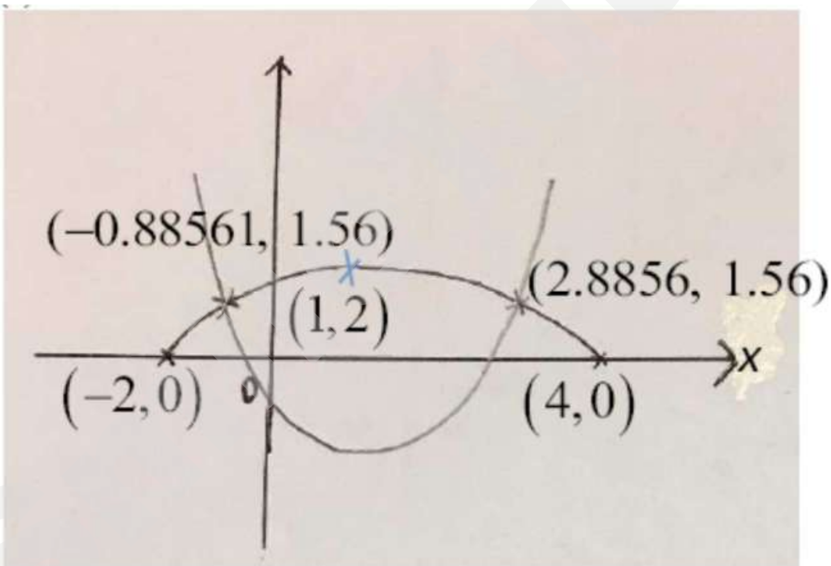
MI H2 Math PU3 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Equations and Inequalities	(i)  (ii) $-0.886 \le x \le 2.89$ (3 s.f) (iii) $x \le 1.06$ (3 s.f)
2	Complex numbers	(i)  (ii) $z=\frac{5}{2}-\left(2-\frac{\sqrt{3}}{2}\right)\text{i}$, $z=\frac{3}{2}-\left(2+\frac{\sqrt{3}}{2}\right)\text{i}$, (iii) $0 \le \arg\left[z+3\text{i}\right] \le 0.927$ (3 s.f)
3	Maclaurin series	(a)(ii) $y = x-\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots$ (a)(iii) 2
4	Vectors	(i) $\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=-3$

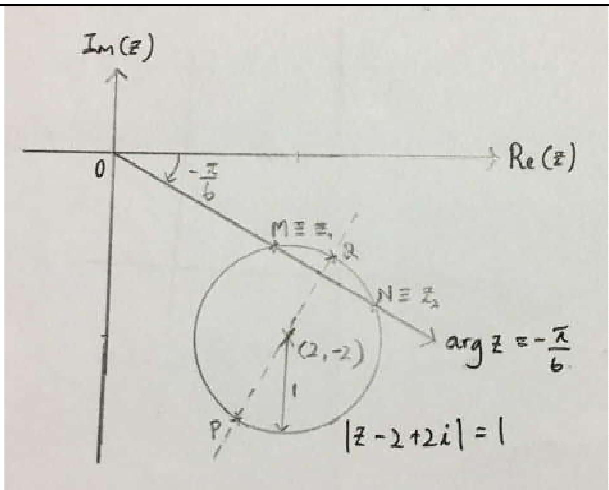
		(ii) $\overrightarrow{ON} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{5}{3} \end{pmatrix}$ (iii) $\overrightarrow{OA'} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix}$ (iv) $a = -\frac{1}{4}$ (v) $\underset{\sim}{r} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}, \beta \in \mapsto$ (vi) Either: the three planes are the sides of a triangular prism. OR: π_3 is parallel to the line of intersection of π_1 and π_2, but does not contain it, $b = -\frac{5}{4}$, $c \neq 6$
5	Sampling	(ii) $k = \frac{500}{50} = 10$ Since $k = 10 > 8 =$ number of Indian boys available, there is a possibility the Indian boys may not be represented. Systematic sampling does not ensure equal proportions of students being taken from each strata.
6	P&C, Probability	(i) $p = 45$ (ii) As the percentage of boys in the survey increases, the percentage that a girl wears spectacles decreases.
7	Normal Distribution	(i) $\mu \approx 240$, $\sigma \approx 12.5$ (ii) 0.00200 (iii) 0.129 (iv) It will not follow normal distribution as the mass of a randomly chosen vegetable from a basket containing an equal number of broccoli and carrot follows a bimodal distribution.

8	Q8 Topic	(i)  (ii) $y = a\ln x + b$ (iii) $y \approx 4.01 + 14.5\ln x\ (3\text{ s.f.})$ (iv) The expected value of y when $\ln x$ is 0 is 4.01. For every increase in $\ln x$ by 1 unit, expected value of y increases by 14.5 units. (v) $y = 25.9$, Reliable because $x = 4.5$ lies within the data range and $\|r\|$ is close to 1
9	Hypothesis Testing	(i) $\overline{x} \approx 176$, $s^2 \approx 17.9$ (accept 17.2) (ii) p-value = 0.156 (accept 0.149) (iii) p-value is the smallest level of significance for which the null hypothesis of the mean number of rainy days per year is 178 will be rejected. (iv) $176 < \overline{x} < 180$
10	P&C, Probability	(a)(i) $226\,800$ (a)(ii) $15\,120$ (a)(last part) 876 (b)(i) 15120 (b)(ii) 48
11	DRV	Average number of people queuing to buy coffee is a constant (i) 2 (ii) 0.135 (iii) 104 (iv) 0.00135 (v) Mean number of people queuing varies throughout the day.

H2 Further Mathematics 2017 Midyear Exam Paper 1 Solution

1	The curve C has the equation $4(x-1)^2+9y^2=36$. **(i)** Sketch, for $y \geq 0$, the curve C, stating the coordinates of the end points and the turning point. [3] **(ii)** By adding a suitable graph to your sketch in part **(i)**, solve the inequality $$2\sqrt{\left[1-\frac{(x-1)^2}{9}\right]}+2-(x-1)^2 \geq 0.$$ [2] **(iii)** Hence, solve the inequality $2\sqrt{\left[1-\frac{(e^x-1)^2}{9}\right]} \geq \left(e^x-1\right)^2-2$. [2]
	Solution: (i)  $4(x-1)^2+9y^2=36$ $y^2=\frac{36-4(x-1)^2}{9}$ $y^2=4\left[1-\frac{(x-1)^2}{9}\right]$ $y=2\sqrt{1-\frac{(x-1)^2}{9}}$ (for $y \geq 0$) (ii)

	$2\sqrt{1-\frac{(x-1)^2}{9}}+2-(x-1)^2 \ge 0$ $2\sqrt{1-\frac{(x-1)^2}{9}} \ge (x-1)^2-2$ The suitable graph to be added is $y=(x-1)^2-2$. From the graph, $-0.88561 \le x \le 2.8856$ $-0.886 \le x \le 2.89$ (3 s.f) (iii) By comparison, $x \to e^x$ $0 \le e^x \le 2.8856$ $\ln e^x \le \ln 2.8856$ $x \le 1.06$ (3 s.f)
2	Two loci in the Argand diagram are given by the equations $\lvert z-2+2i\rvert=1$ and $\arg z=-\frac{\pi}{6}$. The complex numbers z_1 and z_2, where $\lvert z_1\rvert<\lvert z_2\rvert$, correspond to the points of intersection of these loci. **(i)** Draw an Argand diagram to show both loci, and mark the points represented by z_1 and z_2. [3] **(ii)** Find the two values of z which represent points on $\lvert z-2+2i\rvert=1$ such that $\lvert z-z_1\rvert=\lvert z-z_2\rvert$. [4] **(iii)** Given that the complex number w satisfies $\lvert w-2+2i\rvert \le 1$ and $\arg w \le -\frac{\pi}{6}$, find the range of values of $\arg(w+3i)$. [3]
	Solution: (i) $\lvert z-2+2i\rvert=1 \Rightarrow \lvert z-(2-2i)\rvert=1$ $\arg z=-\frac{\pi}{6}$



(ii)
The 2 values of z are as indicated as P and Q on the diagram.

$b = (1)\cos\dfrac{\pi}{6}$; $a = (1)\sin\dfrac{\pi}{6}$

$b = \dfrac{\sqrt{3}}{2}$ $\qquad$ $b = \dfrac{1}{2}$

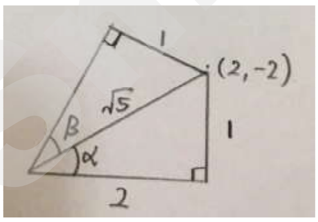
At Q: $z = \left(2 + \dfrac{1}{2}\right) - \left(2 - \dfrac{\sqrt{3}}{2}\right)\mathrm{i}$

At P: $z = \left(2 - \dfrac{1}{2}\right) - \left(2 + \dfrac{\sqrt{3}}{2}\right)\mathrm{i}$

The 2 values of z are

$\dfrac{5}{2} - \left(2 - \dfrac{\sqrt{3}}{2}\right)\mathrm{i}$ and $z = \dfrac{3}{2} - \left(2 + \dfrac{\sqrt{3}}{2}\right)\mathrm{i}$.

(iii)



Smallest value of $\arg\left[z - (-3\mathrm{i})\right] = 0$

Since $\alpha = \beta$,

Largest value of $\arg\left[z - (-3\mathrm{i})\right] = 2\tan^{-1}\dfrac{1}{2} = 0.927$ (3 s.f)

$\therefore 0 \le \arg\left[z + 3\mathrm{i}\right] \le 0.927$ (3 s.f)

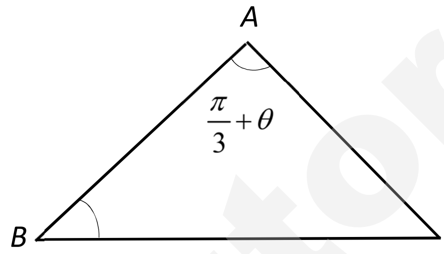
3 **(a)** It is given that $\tan^{-1} y = \ln(1 + x)$.

(i) Show that $(1 + x)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$. [1]

(ii) By successively differentiating this result, find the Maclaurin series for $\tan\left[\ln(1+x)\right]$, up to and including the term in x^3. [3]

(iii) It is given that $\mathrm{f}(x) = \mathrm{e}^x \tan\left[\ln(1+x)\right]$. Using your answer to part **(a)(ii)**, estimate the value of $\mathrm{f}'\left(\frac{1}{2}\right)$. [3]

(b) The diagram shows triangle ABC, where $AC = k$ cm, $BC = h$ cm, $\angle BAC = \frac{\pi}{3} + \theta$ and $\angle ABC = \frac{\pi}{4}$.



Given that θ is a sufficiently small angle, show that

$$\frac{h}{k} \approx \frac{\sqrt{2}}{4}\left[2\sqrt{3} + 2\theta - \left(\sqrt{3}\right)\theta^2\right].$$ [3]

Solution:

(i)

$\tan^{-1} y = \ln(1+x)$

Differentiating both sides with respect to x:

$$\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x}$$

$$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2 \text{ (shown)}$$

(ii)

$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y^2$

Differentiating both sides with respect to x:

$(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}=2y\frac{\mathrm{d}y}{\mathrm{d}x}\Rightarrow(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1-2y)\frac{\mathrm{d}y}{\mathrm{d}x}=0$

Differentiating both sides with respect to x:

$(1+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1-2y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(-2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$

$(1+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2(1-y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$

When $x=0,\ y=0,\frac{\mathrm{d}y}{\mathrm{d}x}=1,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1,\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=4$

$y=0+(1)x+(-1)\frac{x^2}{2!}+(4)\frac{x^3}{3!}+\ldots$

$y=x-\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots$

(iii)

$$
\begin{aligned}
\mathrm{f}(x)&=\mathrm{e}^x\tan\left[\ln(1+x)\right]\\
&=\mathrm{e}^x\left(x-\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots\right)\\
&=\left(1+x+\frac{1}{2}x^2+\ldots\right)\left(x-\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots\right)\\
&=x-\frac{1}{2}x^2+\frac{2}{3}x^3+x^2-\frac{1}{2}x^3+-\frac{1}{2}x^3+\ldots\\
&=x+\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots
\end{aligned}
$$

$\mathrm{f}(x)=x+\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots$

$\mathrm{f}'(x)=1+x+2x^2+\ldots$

$\mathrm{f}'\left(\frac{1}{2}\right)=1+\frac{1}{2}+2\left(\frac{1}{2}\right)^2+\ldots\approx 2$

(b)

	$\dfrac{\sin\left(\dfrac{\pi}{3}+\theta\right)}{h}=\dfrac{\sin\left(\dfrac{\pi}{4}\right)}{k}$ $\sin\left(\dfrac{\pi}{3}+\theta\right)=\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)$ $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)=\sin\left(\dfrac{\pi}{3}\right)\cos(\theta)+\cos\left(\dfrac{\pi}{3}\right)\sin(\theta)$ from MF15 $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)\approx\left(\dfrac{\sqrt{3}}{2}\right)\left(1-\dfrac{\theta^2}{2}\right)+\dfrac{\theta}{2}$ $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)\approx\dfrac{\sqrt{3}}{2}-\dfrac{\sqrt{3}\theta^2}{4}+\dfrac{\theta}{2}$ $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)\approx\dfrac{1}{4}\left(2\sqrt{3}+2\theta-\sqrt{3}\theta^2\right)$ $\dfrac{h}{k}\approx\dfrac{\sqrt{2}}{2}\left(\sqrt{3}+\theta-\dfrac{\sqrt{3}}{2}\theta^2\right)$ (shown)

4

The plane π_1 contains the line $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, where $\lambda \in \cdots$, and is parallel to the line $l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, where $\mu \in \cdots$.

(i) Find the vector equation of π_1 in scalar product form. [2]

(ii) Find the position vector of the foot of the perpendicular from the point $A(1, 0, 1)$ to the plane π_1. [3]

(iii) Find the position vector of the point A', which is the reflection of A about π_1.[2]

(iv) Given that the angle between $l_3 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, where $\alpha \in \cdots$, and the plane $\pi_2 : ax + 2y - z = 3$, where $a \in \cdots$, is $\frac{\pi}{4}$, find the value of *a*. [2]

(v) Find the line of intersection between the planes π_1 and π_2. [1]

(vi) π_3 has equation $bx + y + z = c$, where $b, c \in \cdots$. Given that π_1, π_2 and π_3 have no points in common, describe the geometrical relationship between the three planes. What can be said about the values of *b* and *c*? [3]

Solution:

(i)

$$\underset{\sim}{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -3$$

(ii)

<u>Method 1:</u>

	$l_{AN}: \underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \alpha \in \cdots$ $\overrightarrow{ON} = \begin{pmatrix} 1+\alpha \\ \alpha \\ 1-2\alpha \end{pmatrix}$, for some $\alpha \in \cdots$ Since N is the intersection point of line AN and plane, $\begin{pmatrix} 1+\alpha \\ \alpha \\ 1-2\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -3$ $1+\alpha+\alpha-2+4\alpha = -3$ $\alpha = -\frac{1}{3}$ $\overrightarrow{ON} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{5}{3} \end{pmatrix}$ <u>Method 2:</u> $\overrightarrow{AN} = \left(\overrightarrow{AB} \cdot \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}} \right) \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}}$, where $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \left(\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \cdot \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}} \right) \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}}$ $\overrightarrow{ON} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{5}{3} \end{pmatrix}$ (iii) $\overrightarrow{ON} = \frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$

	$\overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix}$ (iv) $\pi_2 : \underset{\sim}{r} \cdot \begin{pmatrix} a \\ 2 \\ -1 \end{pmatrix} = 3$ $\sin\theta = \dfrac{\left\| \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ 2 \\ -1 \end{pmatrix} \right\|}{\left(\sqrt{2}\right)\left(\sqrt{\left(a^2+4+1\right)}\right)}$ Since $\theta = \dfrac{\pi}{4}$, $\dfrac{\sqrt{2}}{2} = \dfrac{\|a-2\|}{\left(\sqrt{2}\right)\sqrt{\left(a^2+5\right)}}$ $\sqrt{\left(a^2+5\right)} = \|a-2\|$ $\left(a^2+5\right) = a^2 - 4a + 4$ $a = -\dfrac{1}{4}$ (v) Using GC: Equation of line of intersection: $\underset{\sim}{r} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}, \beta \in \cdots$ (vi) Geometrical interpretation: Either: the three planes are the sides of a triangular prism OR: π_3 is parallel to the line of intersection of π_1 and π_2, but does not contain it. $\pi_3 : \underset{\sim}{r} \cdot \begin{pmatrix} b \\ 1 \\ 1 \end{pmatrix} = c$, $\begin{pmatrix} b \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = 0 \Rightarrow b = -\dfrac{5}{4}$ $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} b \\ 1 \\ 1 \end{pmatrix} \neq c \Rightarrow c \neq 6$
5	Resilience Primary School has 500 students who are either Chinese, Indian or Malay, as

	seen in the table below. (table below) The National Eye Centre wishes to conduct a survey at Resilience Primary School to find out the number of hours students spend on electronic devices each week, using a sample of 50 students. **(i)** Explain how stratified sampling can be carried out in this context. [2] **(ii)** Give two reasons why systematic sampling may not be appropriate. [2]

	Chinese	Indian	Malay
Boys	114	8	93
Girls	122	77	86

Solution:

(i)

	Chinese	Indian	Malay
Boys	$\frac{114}{500}\times 50 \approx 11$	$\frac{8}{500}\times 50 \approx 1$	$\frac{93}{500}\times 50 \approx 9$
Girls	$\frac{122}{500}\times 50 \approx 12$	$\frac{77}{500}\times 50 \approx 8$	$\frac{86}{500}\times 50 \approx 9$

Split the students into the stratas for Chinese, Indian, Malay boys or girls as shown in the table above. Arrange the students within each strata in alphabetical order (for example). Using simple random sampling, obtain the required number in each strata.

(ii)

$$k = \frac{500}{50} = 10$$

Since $k = 10 > 8 =$ number of Indian boys available, there is a possibility the Indian boys may not be represented.

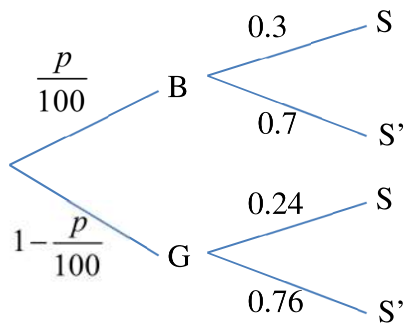
Systematic sampling does not ensure equal proportions of students being taken from each strata.

6 In another survey conducted by the National Eye Centre, it was found that p% are boys and the remaining are girls. The probability that a randomly chosen boy wears spectacles is 0.3 and the probability that a randomly chosen girl wears spectacles is 0.24.

(i) Find the value of p, given that the probability that a randomly chosen child wears spectacles is 0.267. [2]

(ii) For a general value of p, the probability that a randomly chosen child that wears spectacles is a girl is denoted by $\mathrm{f}(p)$. Show that $\mathrm{f}(p)=\dfrac{4(100-p)}{(400+p)}$. Prove by differentiation that f is a decreasing function for $0 \le p \le 100$, and explain what this statement means in the context of the question. [5]

Solution:



B = Boys

G = Girls

S = wear spectacles

S' = don't wear spectacles

(i)

$$\frac{p}{100}(0.3)+\left(1-\frac{p}{100}\right)(0.24)=0.267$$

$$0.0006p=0.027$$

$$p=45$$

(ii)

$$\mathrm{P}(\text{Girl} \mid \text{spectacles})=\frac{0.24\left(1-\frac{p}{100}\right)}{0.3\left(\frac{p}{100}\right)+0.24\left(1-\frac{p}{100}\right)}$$

$$=\frac{0.24-0.0024p}{0.003p+0.24-0.0024p}$$

$$=\frac{0.0024(100-p)}{0.0006(400+p)}$$

$$\mathrm{f}(p)=\frac{4(100-p)}{(400+p)} \quad \text{(shown)}$$

$$\mathrm{f}'(p)=\frac{(400+p)(-4)-(400-4p)}{(400+p)^2}$$

$$=\frac{-2000}{(400+p)^2}$$

Since $(400+p)^2>0$,

$$\mathrm{f}'(p)=\frac{-2000}{(400+p)^2}<0, \ \forall p \in \cdots$$

$\therefore$ f is a decreasing function.

Context: As the percentage of boys in the survey increases, the percentage that a girl wears spectacles decreases.

7	In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters.

The mass, in grams, of broccoli and carrots are normally distributed with means and standard deviations as shown in the table below.

	Mean (g)	Standard deviation (g)
Broccoli	μ	σ
Carrot	180	15

(i) Given that the probability that the mass of a randomly chosen broccoli does not exceed 250g is 0.788 and the probability that the mass of a randomly chosen broccoli exceeds 236g is 0.625, find the values of μ and σ. [3]

(ii) Find the probability that the mass of a randomly chosen broccoli lies within 5 grams of a randomly chosen carrot. [2]

(iii) 120 broccoli are randomly chosen. Using a suitable approximation, find the probability that there are fewer than 90 broccoli with a mass not exceeding 250g. [3]

(iv) Determine, with explanation, whether the mass of a vegetable chosen randomly from a basket containing an equal number of broccoli and carrots follows a normal distribution. [1]

Solution:

Let X and Y be the random variable, the mass of a broccoli and the mass of a carrot respectively

$X \sim \mathrm{N}\left(\mu, \sigma^2\right)$, $Y \sim \mathrm{N}\left(180, 15^2\right)$

(i) $\mathrm{P}(X \le 250) = 0.788$

$\mathrm{P}\left(Z \le \frac{250-\mu}{\sigma}\right) = 0.788$

$\frac{250-\mu}{\sigma} = 0.79950$

$\mu + 0.79950\sigma = 250 \quad ---(1)$

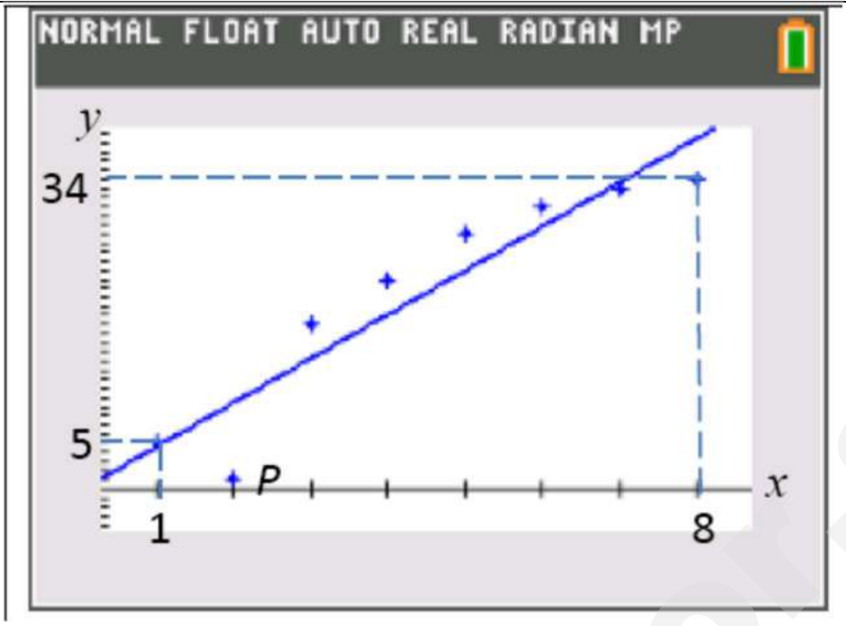
$\mathrm{P}(X > 236) = 0.625$

$\mathrm{P}\left(Z \le \frac{236-\mu}{\sigma}\right) = 0.375$

$\frac{236-\mu}{\sigma} = -0.31864$

$\mu - 0.31864\sigma = 236 \quad ---(2)$

	Using GC: $\mu \approx 239.99 \approx 240$ (3 s.f.) and $\sigma \approx 12.521 \approx 12.5$ (3 s.f.) (ii) $X - Y \sim \mathrm{N}(59.99, 381.78)$ $\mathrm{P}(\lvert X-Y \rvert \le 5) = \mathrm{P}(-5 \le X - Y \le 5)$ $= 0.00200\ (3 \text{ s.f.})$ (iii) Let W be the random variable, the number of broccoli with mass not exceeding 250g $W \sim \mathrm{B}(120, 0.788)$ Since $n = 120 > 50$, $np = 94.56 > 5$, $nq = 25.44 > 5$ $W \sim \mathrm{N}(94.56, 20.047)$ approx. $\mathrm{P}(W < 90) = \mathrm{P}(W \le 89)$ $= \mathrm{P}(W < 89.5)$ (using Continuity Correction) $= 0.129$ (3 s.f.) (iv) It will not follow normal distribution as the mass of a randomly chosen vegetable from a basket containing an equal number of broccoli and carrot follows a bimodal distribution.
8	The table gives the values of eight observations of bivariate data, x and y. <table><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td></tr><tr><td>y</td><td>5</td><td>1</td><td>18</td><td>23</td><td>28</td><td>31</td><td>33</td><td>34</td></tr></table> **(i)** Draw a scatter diagram for these values, labelling the axes clearly. Determine the outlier by labelling it as P in your scatter diagram. [2] **(ii)** By omitting P, explain if $y = ax^2 + b$ or $y = a\ln x + b$ is the better model for the data. [2] **(iii)** Using the more appropriate model found in part **(ii)**, calculate the equation of the least-squares regression line. [1] **(iv)** Interpret, in the context of the question, the least squares estimates of a and b. [2] **(v)** Use the regression line found in part **(iii)** to predict the value of y when x = 4.5. Comment on the reliability of your answer. [2]
	Solution: (i)



(ii) $y = ax^2 + b$: $r = 0.880$ (3 s.f.)

$y = a\ln x + b$: $r = 0.994$ (3 s.f.)

Since $y = a\ln x + b$ has $|r|$ closer to 1, $y = a\ln x + b$ is the better model.

(iii)

$$y = 4.0144 + 14.518\ln x$$
$$\approx 4.01 + 14.5\ln x \text{ (3 s.f.)}$$

(iv)

The expected value of y when $\ln x$ is 0 is 4.01.

For every increase in $\ln x$ by 1 unit, expected value of y increases by 14.5 units.

(v)

At $x = 4.5$, $y = 4.0144 + 14.518\ln(4.5) = 25.9$ (3 s.f.)

Reliable because $x = 4.5$ lies within the data range and $|r|$ is close to 1

9 Based on past records, the mean number of rainy days per year in Singapore was reported as 178. The authorities suspect that due to global warming, the number of rainy days has changed. A random sample of 12 years is taken and the number of rainy days per year, X, is summarised by

$$\sum(x-8) = 2017.7, \qquad \sum x^2 = 372\,500.$$

(i) Calculate the unbiased estimates of the mean and variance of X. [2]

(ii) Test, at the 5% level of significance, whether the mean number of rainy days per year has changed. State any assumptions used in your calculations. [4]

(iii) Explain, in the context of the question, the meaning of the p-value. [1]

	(iv) The population variance is found to be 9 and the assumption used in part **(ii)** holds true. A test at the 5% level of significance whether the mean number of rainy days per year has changed was conducted. Find the range of values of $\overline{x}$ such that the null hypothesis is not rejected. [3]

Solution:

(i)

$$\overline{x} = \frac{2017.7}{12} + 8 \approx 176.14 \approx 176 \text{ (3 s.f.)}$$

Method 1

$$s^2 = \frac{1}{n-1}\left(\sum x^2 - n\left(\overline{x}\right)^2\right)$$

$$= \frac{1}{11}\left(372500 - 12(176.14)^2\right)$$

$$\approx 17.855 \approx 17.9 \text{ (3 s.f.)}$$

Method 2

$$\sum x = 2017.7 + 8(12) = 2113.7$$

$$s^2 = \frac{1}{n-1}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)$$

$$= \frac{1}{11}\left(372\,500 - \frac{(2113.7)^2}{12}\right)$$

$$= 17.214 \approx 17.2 \text{ (3 sf)}$$

(ii)

Let X be the random variable, the number of rainy days per year in Singapore

$H_0 : \mu = 178$

$H_1 : \mu \neq 178$

Assume H_0 is true. $\alpha = 0.05$. Assume X follows normal distribution.

Since $n = 12 < 50$, population variance unknown,

$T \sim t(11)$ approx.

2 tail t-test used.

Method 1:

Using GC, p-value = 0.156 (3 s.f.) > 0.05 if $s^2 = 17.855$ used

[Alt: p-value = 0.149 (3 s.f.) > 0.05 if $s^2 = 17.214$ used]

	Do not reject H_0 <u>Method 2:</u> Test-statistic value: $t=\dfrac{176.14-178}{\sqrt{\dfrac{17.855}{12}}}\approx -1.52$ (3 s.f.) if $s^2=17.855$ used [Alt: $t=\dfrac{176.14-178}{\sqrt{\dfrac{17.214}{12}}}\approx -1.55$ (3 s.f.) if $s^2=17.214$ used] Critical region: $t\le -2.20$ (3 s.f.) or $t\ge 2.20$ (3 s.f.) Since test-statistic does not lie in the critical region, H_0 is not rejected. There is insufficient evidence at 5% level of significance to conclude that the mean number of rainy days per year has changed. (iii) <u>Either</u> *p*-value is the smallest level of significance for which the null hypothesis of the mean number of rainy days per year is 178 will be rejected. <u>Or</u> *p*-value is twice the probability of obtaining a test statistic less than or equal to -1.52, assuming the null hypothesis of the mean number of rainy days per year is 178 is true. (iv) $H_0: \mu = 178$ $H_1: \mu \neq 178$ Assume H_0 is true. Since *X* is normal, $\overline{X} \sim N\left(178, \dfrac{9}{12}\right)$ 2 tail *z*-test used. Since H_0 is not rejected at the 5% level of significance, $-1.9600<\dfrac{\overline{x}-178}{\left(\sqrt{\left(\dfrac{3}{4}\right)}\right)}<1.9600$ $-1.9600\sqrt{\left(\dfrac{3}{4}\right)}<\overline{x}-178<1.9600\sqrt{\left(\dfrac{3}{4}\right)}$ $176<\overline{x}<180$ (3 s.f.)
10	**(a)** Find the number of ways in which the letters of the word MILLENNIUM can be arranged if **(i)** there are no restrictions, [1]

(ii) the first and last letters are the same, and the letters E and U must be separated. [2]

Four letters are randomly selected from the letters of the word MILLENNIUM to form a code word. Find the number of possible code words that can be formed. [2]

(b) Mr See (together with his wife and daughter) and Mrs Saw (together with her husband and two sons) came to visit their former teacher Mdm Rain during Millennium Institute's Homecoming Day. Find the number of ways Mr See and his family, Mrs Saw and her family, and Mdm Rain can be arranged if

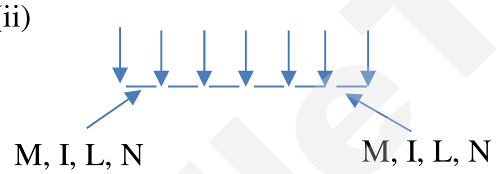
(i) they are around a table with ten indistinguishable chairs, such that the children are seated together. [2]

(ii) the two empty chairs are removed and Mr See's daughter is seated between her parents and the See family are to be seated directly opposite Mdm Rain. [3]

Solution:

(a)(i) No. of ways = $\dfrac{10!}{2!2!2!2!} = 226\,800$

(ii)



$$\text{No. of ways } = {}^4C_1 \times \frac{6!}{2!2!2!} \times {}^7C_2 \times 2!$$
$$= 15\,120$$

(a)(last part)

<u>Case 1: 2 Repeats</u>

$$\text{No. of ways } = {}^4C_2 \times \frac{4!}{2!2!} = 36$$

<u>Case 2: 1 Repeat</u>

$$\text{No. of ways } = {}^4C_1 \times {}^5C_2 \times \frac{4!}{2!} = 480$$

<u>Case 3: No Repeat</u>

$$\text{No. of ways } = {}^6C_4 \times 4! = 360$$

Total ways = 876

(b)(i)

$$\text{No. of ways } = \frac{8!}{8(2!)} \times 3!$$

	$= 15\ 120$ (b)(ii) No. of ways $= \dfrac{2!}{2} \times 2! \times 4!$ $= 48$
11	In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters. The number of people queuing to buy coffee at CoffeeVille in a period of 1 minute during lunch hour (12pm to 2pm) is a random variable with an average number of 2.9. State, in context, a condition under which a Poisson distribution would be a suitable probability model. [1] Assume that the number of people queuing to buy coffee at CoffeeVille in a period of 1 minute during the lunch hour follows the distribution $\text{Po}(2.9)$. **(i)** State the most probable number of people queuing in 1 minute. [1] **(ii)** Find the probability that in a period of 3 minutes, there are at most 5 people queuing to buy coffee. [2] **(iii)** *N* periods of 3 minutes are taken. Given that the probability that at least 7 periods of 3 minutes have at most 5 people queuing to buy coffee is more than 0.99, find the least value of *N*. [3] **(iv)** A random sample of 120 periods of 3 minutes is taken. Using a suitable approximation, find the probability that more than 12 periods of 3 minutes have exactly 4 people queuing. [3] **(v)** Explain why the Poisson model would probably not be valid if applied to the operating hours of CoffeeVille from 11am to 10pm. [1]
	Solution: Average number of people queuing to buy coffee is a constant (i) Let *X* be the random variable, for the number of people queuing to buy coffee in 1 min. $X \sim \text{Po}(2.9)$ Using GC: Mode = 2 (ii) Let *Y* be the random variable, for the number of people queuing to buy coffee in 3 min.

	$Y \sim \text{Po}(8.7)$ $\text{P}(Y \le 5) = 0.13516 \approx 0.135$ (3 s.f.) (iii) Let W be the random variable, for the number of periods of 3 min with $Y \le 5$ $W \sim \text{B}(n, 0.13516)$ $\text{P}(W \ge 7) > 0.99$ $1 - \text{P}(W \le 6) > 0.99$ $\text{P}(W \le 6) < 0.01$ Using GC:

N	$\text{P}(W \le 6)$
103	0.0104 > 0.01
104	0.00947 < 0.01
105	0.00864 < 0.01

Least value of N is 104

(iv)
Let V be the random variable, for the number of periods of 3 min with $Y = 4$

$V \sim \text{B}(120, 0.039765)$

Since $n = 120 > 50$, $np = 4.7718 < 5$
$V \sim \text{Po}(4.7718)$ approx.

$\text{P}(V > 12) = 1 - \text{P}(V \le 12) \approx 0.00135$ (3 s.f.)
(v)
Mean number of people queuing varies throughout the day.

2017 NYJC Prelim Paper 1

1 A board is such that the n^{th} row from the top has n tiles, and each row is labelled from left to right in ascending order such that the i^{th} tile is labelled i, where n and i are positive integers.

1			
1	2		
1	2	3	
$\vdots$			$\ddots$

Given that $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, by finding the sum of the numbers in the r^{th} row, show that the sum of all the numbers in n rows of tiles is $\frac{1}{6}(n)(n+1)(n+2)$. [4]

2 The curve C has equation $2x - y^2 = (x+y)^2$.

(i) Find the equations of the tangents to C which are parallel to the x-axis. [4]

(ii) The line l is tangent to C at $A(2,-2)$. If the normal to C at the origin O meets l at the point B, find the area of triangle OAB. [4]

3 **Do not use a calculator in answering this question.**

(i) Explain why the equation $z^3 + az^2 + az + 7 = 0$ cannot have more than two non-real roots, where a is a real constant. [1]

(ii) Given that $z = -7$ is a root of the equation in **(i)**, find the other roots, leaving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

(iii) Hence, solve the equation $iz^3 + 8z^2 - 8iz - 7 = 0$, leaving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

4 (i) By using the substitution $x-1=3\tan\theta$, find $\int \frac{1}{\sqrt{x^2-2x+10}}\,dx$. [5]

(ii) By expressing $x+3=A(2x-2)+B$, find $\int \frac{x+3}{\sqrt{x^2-2x+10}}\,dx$. [3]

5 (i) By considering $f(r)-f(r+1)$, where $f(r)=\frac{\sqrt{r}}{2\sqrt{r}+1}$, find

$$\sum_{r=1}^{n} \frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$$

in terms of n. [3]

(ii) Hence, find $\sum_{r=1}^{\infty} \frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$. [2]

(iii) Find the smallest integer n such that

$$\sum_{r=1}^{n} \frac{\sqrt{r+1}-\sqrt{r+2}}{\left(2\sqrt{r+1}+1\right)\left(2\sqrt{r+2}+1\right)} < -0.1.$$ [3]

6 The curve C has equation

$$y=1+\frac{2x+p}{(x-2)(x+3)},$$

where p is a constant.

(i) Find the range of values of p for which C has more than one stationary point. [4]

(ii) Sketch C for $p=7$, stating the coordinates of the turning point(s) and the points of intersection with the axes and the equations of any asymptotes. [3]

(iii) By sketching a suitable graph on the same diagram, solve the inequality

$$1+\sqrt{12-x^2} \geq \frac{2x+7}{(2-x)(x+3)}.$$ [3]

7 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \mathrm{e}^{-x^2}, \quad x \in \mathbb{R},\ x < 0,$$

$$\mathrm{g}: x \mapsto \frac{1}{x+3}, \quad x \in \mathbb{R},\ x \neq -3.$$

(i) Show that g^{-1} exists, and define g^{-1} in a similar form. [2]

(ii) State the solution set for $\mathrm{g}\mathrm{g}^{-1}(x) = x$. [1]

(iii) Explain why fg^{-1} does not exist. [1]

Let the function h be defined by

$$\mathrm{h}: x \mapsto \mathrm{g}(x), \quad x \in \mathbb{R},\ x < k,$$

where k is a real constant.

(iv) Given that $\mathrm{f}\,\mathrm{h}^{-1}$ exists, state the maximum value of k. [1]

(v) For the value of k found in **(iv)**,

(a) find the exact range of $\mathrm{f}\,\mathrm{h}^{-1}$, [2]

(b) solve $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$. [2]

8 A curve C has parametric equations

$$x = 1 + \mathrm{e}^{t} + \mathrm{e}^{-t}, \quad 2y = \mathrm{e}^{t} - \mathrm{e}^{-t}, \quad t \in \mathbb{R}.$$

(i) Show that the Cartesian equation of C is $\dfrac{(x-1)^2}{2^2} - y^2 = 1$. [2]

(ii) Sketch C, showing clearly the equations of any asymptotes and coordinates of the centre and the point(s) where the curve cuts the x-axis. [3]

(iii) Find the exact area of the region bounded by C and the line $x = 1 + \mathrm{e} + \mathrm{e}^{-1}$. [4]

(iv) Find the volume of the solid of revolution when the region bounded by C and the lines $x = 3$ and $y = 4$ is rotated completely about the y-axis. [2]

9 With reference to an origin O, a particle P moves in space with position vector $(\lambda-\mu)\mathbf{i}+(1+2\mu)\mathbf{j}+(2-3\lambda)\mathbf{k}$. Another particle Q moves along the line l with equation $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$, $t\in\mathbb{R}$.

(i) State the locus of P. [1]

(ii) Determine if the particles P and Q can meet. [3]

(iii) Find the shortest possible distance between P and Q. [2]

Another particle R moves along the line m with equation $\mathbf{r}=\begin{pmatrix}1\\k\\0\end{pmatrix}+s\begin{pmatrix}2\\-2\\-3k\end{pmatrix}$, $s\in\mathbb{R}$, where k is a constant.

(iv) Find condition(s) satisfied by k if lines l and m are skew lines. [3]

(v) A particle is shot from $X(0,-1,-5)$ perpendicularly toward the path of Q. Find the coordinates of the point where it crosses the path of Q. [2]

10 A car is travelling at a speed of 30 m/s on a road heading towards a perpendicular train track, which is elevated 30 m above the ground. The front of the car is 40 m away from the track when the front of the train first crossed the road.

If the train is travelling at 20 m/s, show that the distance between the front of the train and the car is $\sqrt{1300t^2-2400t+2500}$ m . [2]

(i) How fast is the front of both the train and the car separating 1 second later? [2]

(ii) Find the distance when the front of the train and the front of the car are closest. [4]

(iii) Find the rate of change of the angle of elevation of the front of the train from the car 1 second later. [4]

11 Suppose a point P on the rim of a wheel of radius r is initially at the point O. As the wheel roll along the x-axis without slippage, the locus of P, known as a *cycloid*, has parametric equations given by

$$x=r(\theta-\sin\theta),\ y=r(1-\cos\theta),\ \theta\geq 0.$$

(i) Sketch the locus of P for $0\leq\theta\leq 4\pi$. [2]

(ii) Show that $\frac{dy}{dx}=\cot\frac{\theta}{2}$. [3]

(iii) Show that the curve is a solution to the differential equation $\left(\frac{dy}{dx}\right)^2=\frac{2r}{y}-1$. [3]

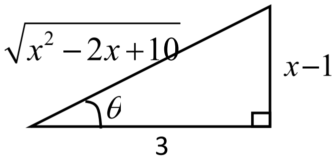
(iv) Find the exact area bounded by the locus of P and the x-axis for $0\leq x\leq 2\pi r$. [4]

——END OF PAPER——

2017 NYJC JC2 Prelim 9758/1 Solution

Qn		Remarks
1	Sum of numbers in kth row $= \sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$ Required sum $= \sum_{k=1}^{n} \frac{k\left(k+1\right)}{2}$ $= \frac{1}{2}\sum_{k=1}^{n}\left(k^2+k\right)$ $= \frac{1}{12}n\left(n+1\right)\left(2n+1\right)+\frac{1}{4}n\left(n+1\right)$ $= \frac{1}{12}n\left(n+1\right)\left(2n+1+3\right)$ $= \frac{1}{6}n\left(n+1\right)\left(n+2\right)$	A handful of students wrote $\sum_{r=1}^{k} k = \frac{1}{2}k(k+1)$, without realising that k is in fact a constant, and the expression is incorrect.
2(i)	Differentiating $2x-y^2=\left(x+y\right)^2$ ______(1) implicitly with respect to x, $2-2y\frac{dy}{dx}=2\left(x+y\right)\left(1+\frac{dy}{dx}\right)$ Where tangent is parallel to the x-axis, $\frac{dy}{dx}=0$. $2=2\left(x+y\right)$ $y=1-x$ ______(2) Sub (2) in (1), $2x-\left(1-x\right)^2=\left(x+1-x\right)^2$ $x^2-4x+2=0$ $x=\frac{-\left(-4\right)\pm\sqrt{\left(-4\right)^2-4\left(2\right)}}{2}=2\pm\sqrt{2}$ When $x=2-\sqrt{2}$, $y=1-\left(2-\sqrt{2}\right)=-1+\sqrt{2}$ When $x=2+\sqrt{2}$, $y=1-\left(2+\sqrt{2}\right)=-1-\sqrt{2}$	Many students stopped at equation (2) and incorrectly concluded that it was the required equation of tangent.

Qn		Remarks
2(ii)	$2-2y\frac{\mathrm{d}y}{\mathrm{d}x}=2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $2=2(x+y)+2(x+2y)\frac{\mathrm{d}y}{\mathrm{d}x}$ $2=2(x+y)+2(x+2y)\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-(x+y)}{x+2y}$ When $x=0,\ y=0$, $-\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}=0$. Hence normal to C at the origin is $y=0$. When $x=2,\ y=-2$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{-2}$ Tangent to C at $A(2,-2)$, $y-(-2)=-\frac{1}{2}(x-2)$ Where the normal and the tangent intersect, $2=-\frac{1}{2}(x-2)$ $x=-2$ Area of triangle $OAB=\frac{1}{2}(2)(2)=2$ units^2	This part was badly done as many wrote the equation of normal to C as $x=0$, instead of $y=0$. Due to this error, they were unable to obtain the required area.
3(i)	Since a is real, the polynomial equation has real coefficients, and thus all non-real roots must be in conjugate pairs. Since the degree of the polynomial is three, there will be 3 roots. The highest even number below 3 is 2.	Candidates showed vague understanding of conjugate root theorem. Note that it does not imply that there WILL be complex conjugate roots.
3(ii)	$z^3+az^2+az+7=0$ $(-7)^3+a(-7)^2+a(-7)+7=0$ $a=8$ $z^3+8z^2+8z+7=0$ $(z+7)(z^2+z+1)=0$ $z=-7$ or $z=\frac{-1\pm i\sqrt{3}}{2}$ $z=7e^{i\pi},\ e^{\frac{i2\pi}{3}},e^{-\frac{i2\pi}{3}}$	If candidates sub in $x=a+bi$ and $x=a-bi$ and proceed to solve for a and b, they will be penalised as the complex conjugate roots may not even exist in the first place.

Qn		Remarks
3(iii)	$-iz^3-8z^2+8iz+7=0$ $(iz)^3+8(iz)^2+8(iz)+7=0$ From (ii), replace z with iz $iz=7e^{i\pi},\ e^{\frac{i2\pi}{3}},e^{-\frac{i2\pi}{3}}$ $\Rightarrow z=-i7e^{i\pi},\ -ie^{\frac{i2\pi}{3}},-ie^{-\frac{i2\pi}{3}}$ $\Rightarrow z=e^{-\frac{i\pi}{2}}7e^{i\pi},\ e^{-\frac{i\pi}{2}}e^{\frac{i2\pi}{3}},e^{-\frac{i\pi}{2}}e^{-\frac{i2\pi}{3}}$ $\Rightarrow z=7e^{\frac{i\pi}{2}},\ e^{\frac{i\pi}{6}},e^{\frac{i5\pi}{6}}$	
4(i)	$x-1=3\tan\theta$ $\frac{dx}{d\theta}=3\sec^2\theta$ $\int\frac{1}{\sqrt{x^2-2x+10}}\,dx=\int\frac{1}{\sqrt{(x-1)^2+3^2}}\,dx$ $=\int\frac{1}{\sqrt{(3\tan\theta)^2+3^2}}\cdot 3\sec^2\theta\ d\theta$ $=\int\frac{1}{3\sec\theta}\cdot 3\sec^2\theta\ d\theta$ $=\int\sec\theta\ d\theta$ $=\ln\lvert\sec\theta+\tan\theta\rvert+C$ $=\ln\left\lvert\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right\rvert+C$	Many candidates omitted the square root in the denominator when doing the substitution or applying the trigo identity. Apply **$\tan^2\theta+1=\sec^2\theta$** Remember the modulus sign, arbitrary constant and rewrite expression in terms of x. Use the right angle triangle to give answer in terms of x. $x-1=3\tan\theta\Rightarrow\tan\theta=\frac{x-1}{3}$ 

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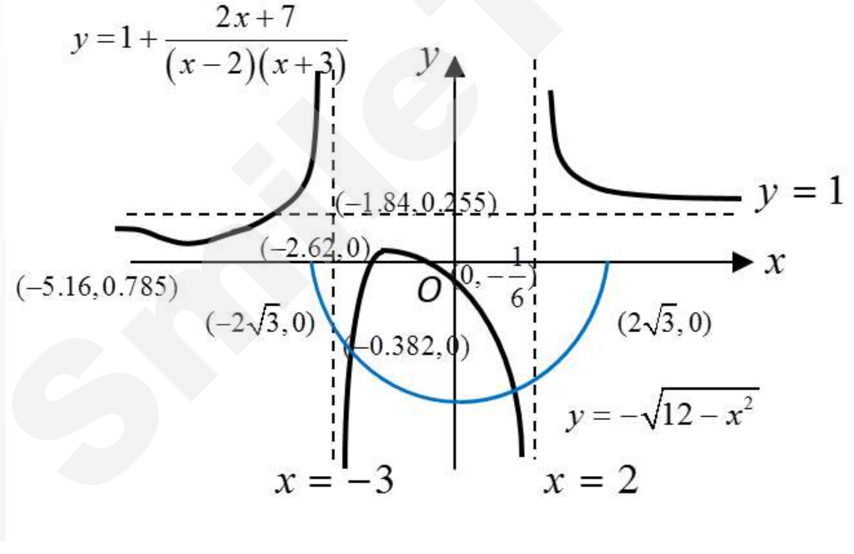
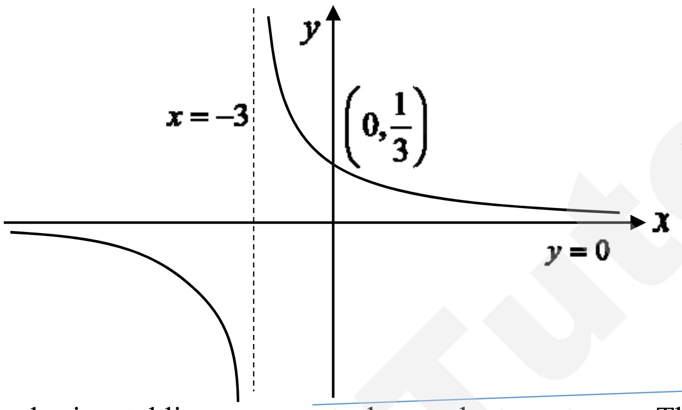
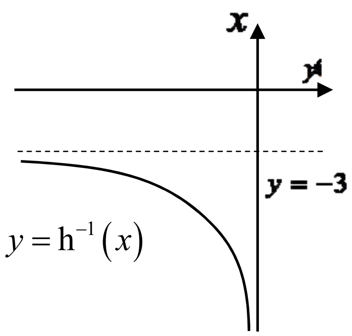
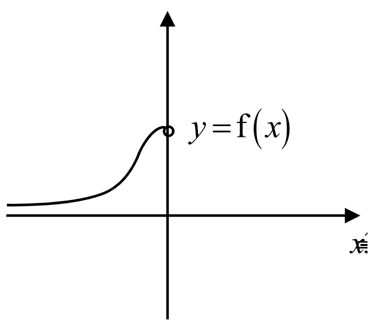
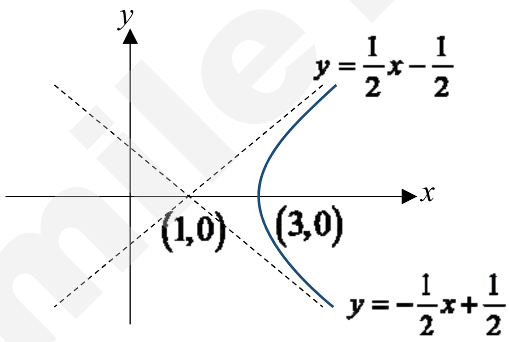
Qn		Remarks
4(ii)	$x+3=\frac{1}{2}(2x-2)+4$ $\int \frac{x+3}{\sqrt{x^2-2x+10}}\,dx$ $=\int \frac{\frac{1}{2}(2x-2)+4}{\sqrt{x^2-2x+10}}\,dx$ $=\frac{1}{2}\int \frac{2x-2}{\sqrt{x^2-2x+10}}\,dx+\int \frac{4}{\sqrt{(x-1)^2+3^2}}\,dx$ $=\frac{1}{2}\frac{\sqrt{x^2-2x+10}}{\frac{1}{2}}+4\int \frac{1}{\sqrt{(x-1)^2+3^2}}\,dx$ $=\sqrt{x^2-2x+10}+4\ln\left\lvert\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right\rvert+C$ Standard form integral $\int \underbrace{(2x-2)}_{f'(x)}\underbrace{(x^2-2x+10)}_{f(x)}^{-\frac{1}{2}}dx=\frac{(x^2-2x+10)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C$ Many students erroneously applied the formula in MF26 when the form of the integral is not the same. Ans in (i)	
5(i)	$f(r)-f(r+1)=\frac{\sqrt{r}}{2\sqrt{r}+1}-\frac{\sqrt{r+1}}{2\sqrt{r+1}+1}$ $=\frac{(2\sqrt{r}\sqrt{r+1}+\sqrt{r})-(2\sqrt{r+1}\sqrt{r}+\sqrt{r+1})}{(2\sqrt{r}+1)(2\sqrt{r+1}+1)}$ $=\frac{\sqrt{r}-\sqrt{r+1}}{(2\sqrt{r}+1)(2\sqrt{r+1}+1)}$ $\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{(2\sqrt{r}+1)(2\sqrt{r+1}+1)}=\sum_{r=1}^{n}[f(r)-f(r+1)]$ $=f(1)-f(2)$ $+f(2)-f(3)$ $+\ldots$ $+f(n)-f(n+1)$ $=f(1)-f(n+1)$ $=\frac{1}{3}-\frac{\sqrt{n+1}}{2\sqrt{n+1}+1}$	

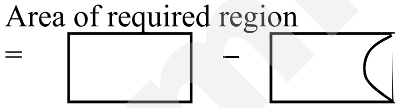
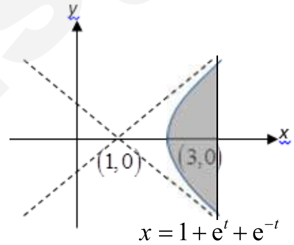
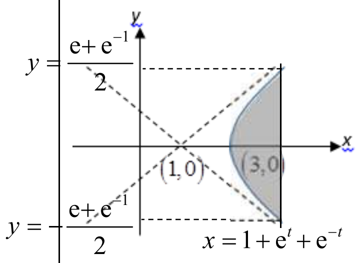
Qn		Remarks
5(ii)	$\displaystyle\sum_{r=1}^{\infty}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$ $\displaystyle=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{\sqrt{n+1}}{2\sqrt{n+1}+1}\right)$ $\displaystyle=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2+\frac{1}{\sqrt{n+1}}}\right)$ $\displaystyle=-\frac{1}{6}$	Most students have the misconception that a fraction must go to 0 when the denominator goes to infinity. However, note that the numerator goes to infinity as well, thus the expression is indeterminate until further manipulation is done to get a clearer picture.
5(iii)	$\displaystyle\sum_{r=1}^{n}\frac{\sqrt{r+1}-\sqrt{r+2}}{\left(2\sqrt{r+1}+1\right)\left(2\sqrt{r+2}+1\right)}=\sum_{r=2}^{n+1}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$ $\displaystyle=\sum_{r=1}^{n+1}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}-\frac{1-\sqrt{2}}{3(2\sqrt{2}+1)}$ $\displaystyle=\frac{1}{3}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}-\frac{1-\sqrt{2}}{3(2\sqrt{2}+1)}$ $\displaystyle=\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}$ Need $\displaystyle\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}<-0.1$ (see table below) Using GC, least $n=57$	One may alternatively input the expression as a summation into the GC, but the calculation of the sum for each n takes much longer. Some students were not careful with the number of decimal places, thus unable to compare the value with -0.1.

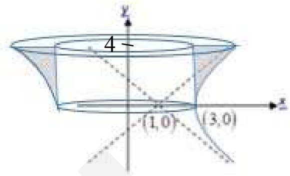
n	$\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}$
56	-0.099797
57	-0.100043

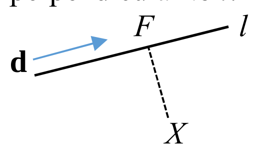
Qn		Remarks
6(i)	$y = 1 + \frac{2x+p}{x^2+x-6}$ $\frac{dy}{dx} = \frac{2(x^2+x-6)-(2x+p)(2x+1)}{(x^2+x-6)^2}$ $\frac{dy}{dx} = 0 \Rightarrow 2(x^2+x-6)-(2x+p)(2x+1)=0$ $2x^2+2px+12+p=0$ $4p^2-4(2)(12+p)>0$ $p^2-2p-24>0$ $(p+4)(p-6)>0$ $p<-4$ or $p>6$	Learn to use quotient rule.
6(ii)		Write down all the coordinates including the stationary points (in this case) as stated in the question.
6(iii)	$1+\sqrt{12-x^2} \geq \frac{2x+7}{(2-x)(x+3)}$ $1+\frac{2x+7}{(x-2)(x+3)} \geq -\sqrt{12-x^2}$ Sketch $y=-\sqrt{12-x^2}$ (as above)	Sketch $y=-\sqrt{12-x^2}$ on the same diagram with the end points touching the x-axis. Label both graphs clearly using 2 different colours e.g. blue, black or pencil.

Qn		Remarks
	$-2\sqrt{3} \le x < -3$ OR $-2.92 \le x \le 1.46$ OR $2 < x \le 2\sqrt{3}$	
7(i)	 Every horizontal line $y = k$ cuts the graph at most once. This implies g is one-one. Therefore g^{-1} exists $\mathrm{g}^{-1} : x \mapsto \frac{1}{x} - 3, x \in \mathbb{R},\ x \ne 0$	There is a need to draw the graph to show that any horizontal line will cut the graph at most once. To show that function is 1-1, there is a need to have a general equation, $y = k$. This is different from "at only one point", as the line $y = 0$ does not cut the graph.
7(ii)	$\{x \in \mathbb{R} \mid x \ne 0\}$	Question asked for "solution set" therefore answer must be written in sets.
7(iii)	$\mathrm{R}_{\mathrm{g}^{-1}} = \mathrm{D}_{\mathrm{g}} = \mathbb{R} \setminus \{-3\}$, $\mathrm{D}_{\mathrm{f}} = \mathbb{R}^{-}$. Since $\mathrm{R}_{\mathrm{g}^{-1}} \not\subset \mathrm{D}_{\mathrm{f}}$, fg^{-1} does not exist.	
7(iv)	$k = -3$	
7(v) (a)	 $\mathrm{R}_{\mathrm{fh}^{-1}} = \left(0, \mathrm{e}^{-9}\right)$	

Qn		Remarks
7(iv) (b)	$\mathrm{h}(x)=\mathrm{h}^{-1}(x)$ $\mathrm{h}(x)=x$ $\dfrac{1}{x+3}=x$ $x^2+3x-1=0$ Since $x<-3$, $x=-3.30$ (3sf) Need to reject the other root which does not lie in to range $x < -3$.	
8(i)	$(x-1)^2=\left(\mathrm{e}^t+\mathrm{e}^{-t}\right)^2=\mathrm{e}^{2t}+2+\mathrm{e}^{-2t}$ $(2y)^2=\left(\mathrm{e}^t-\mathrm{e}^{-t}\right)^2=\mathrm{e}^{2t}-2+\mathrm{e}^{-2t}$ Hence $(x-1)^2-(2y)^2=4$ $\dfrac{(x-1)^2}{2^2}-y^2=1$ <u>Alternative solution by students</u>: $(x-1)+2y=2\mathrm{e}^t$ ___(1) $(x-1)-2y=2\mathrm{e}^{-t}$ ___(2) (1)×(2): $(x-1)^2-(2y)^2=4\mathrm{e}^t\mathrm{e}^{-t}=4$	
8(ii)		Many students simply drew the whole hyperbola given by the Cartesian equation in **(i)** without taking into account the original parametric equations. Note that one needs to check the range of values the x and y coordinates can take. For all $t\in\mathbb{R}$, $x=1+\mathrm{e}^t+\mathrm{e}^{-t}>0$. Just key in parametric equations in GC.

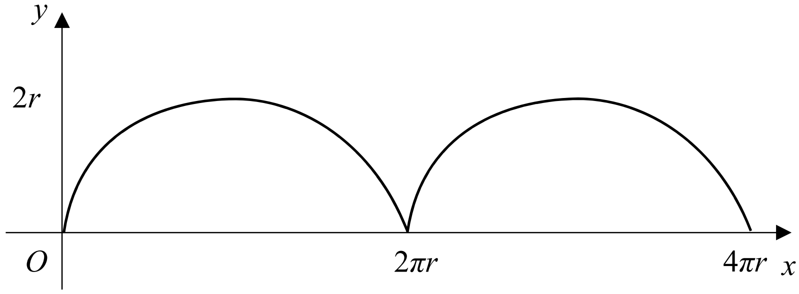
Qn		Remarks
8(iii)	When $x=3$, $3=1+\mathrm{e}^{t}+\mathrm{e}^{-t}$ $\mathrm{e}^{t}+\mathrm{e}^{-t}=2$ $t=0$ When $x=1+\mathrm{e}+\mathrm{e}^{-1}$, $t=\pm 1$ $(t=1:\ y>0,\ t=-1:\ y<0)$ $x=1+\mathrm{e}^{t}+\mathrm{e}^{-t}$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\mathrm{e}^{t}-\mathrm{e}^{-t}$ By symmetry Note that the integral $\int y\mathrm{d}x$ refers to the area <u>either</u> above the x-axis (if $y>0$) <u>or</u> below the x-axis (if $y<0$) Area of required region $=2\int_{3}^{1+\mathrm{e}+\mathrm{e}^{-1}} y\mathrm{d}x$ $=2\int_{0}^{1}\dfrac{\mathrm{e}^{t}-\mathrm{e}^{-t}}{2}\left(\mathrm{e}^{t}-\mathrm{e}^{-t}\right)\mathrm{d}t$ $=\int_{0}^{1}\left(\mathrm{e}^{t}-\mathrm{e}^{-t}\right)^{2}\mathrm{d}t$ $=\int_{0}^{1}\left(\mathrm{e}^{2t}-2+\mathrm{e}^{-2t}\right)\mathrm{d}t$ $=\left[\dfrac{1}{2}\mathrm{e}^{2t}-2t-\dfrac{1}{2}\mathrm{e}^{-2t}\right]_{0}^{1}$ $=\left[\dfrac{1}{2}\mathrm{e}^{2}-2-\dfrac{1}{2}\mathrm{e}^{-2}\right]-0$ $=\dfrac{1}{2}\left(\mathrm{e}^{2}-\mathrm{e}^{-2}\right)-2$ <u>Alternatively (more tedious)</u>: Area of required region $=$  $=\left(1+e+e^{-1}\right)\left(\dfrac{e-e^{-1}}{2}-\left(-\dfrac{e-e^{-1}}{2}\right)\right)-\int_{-\frac{\mathrm{e}-\mathrm{e}^{-1}}{2}}^{\frac{\mathrm{e}-\mathrm{e}^{-1}}{2}} x\ \mathrm{d}y$ $=\left(1+e+e^{-1}\right)\left(e-e^{-1}\right)-2\int_{0}^{\frac{\mathrm{e}-\mathrm{e}^{-1}}{2}} x\ \mathrm{d}y$ $=\left(1+e+e^{-1}\right)\left(e-e^{-1}\right)-2\int_{0}^{1}\left(1+\mathrm{e}^{t}+\mathrm{e}^{-t}\right)\dfrac{\mathrm{e}^{t}+\mathrm{e}^{-t}}{2}\mathrm{d}t$ $\vdots$ $=\dfrac{1}{2}\left(\mathrm{e}^{2}-\mathrm{e}^{-2}\right)-2$	 

Qn		Remarks
8(iv)	$\dfrac{(x-1)^2}{2^2}-y^2=1$ $(x-1)^2=2^2\left(1+y^2\right)$ $x=1+2\sqrt{1+y^2}\quad$ since $x>1$ Volume $=\pi\int_0^4 x^2\,\mathrm{d}y-\pi\left(3^2\right)(4)$ $=\pi\int_0^4\left(1+2\sqrt{1+y^2}\right)^2\mathrm{d}y-36\pi$ $=335$ units^3 (3 s.f.)	 Note that finding volume of revolution when the curve is defined parametrically is not in syllabus. Students can use the parametric equations to find volume but are not expected to do so. You should just use the Cartesian equation.
9(i)	$\overrightarrow{OP}=\begin{pmatrix}\lambda-\mu\\1+2\mu\\2-3\lambda\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-3\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\0\end{pmatrix}$ Locus of P is the plane with equation $\mathbf{r}=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-3\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\0\end{pmatrix},\quad \lambda,\mu\in\mathbb{R}$	Students are expected to identify that the locus of P is a **plane** and to give the **correct equation** (in any acceptable form)
9(ii)	Normal of the locus of P (plane), $\begin{pmatrix}1\\0\\-3\end{pmatrix}\times\begin{pmatrix}-1\\2\\0\end{pmatrix}=\begin{pmatrix}6\\3\\2\end{pmatrix}$ $\begin{pmatrix}6\\3\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\-3\end{pmatrix}=0$ Hence the line l and the plane are parallel. Equation of the plane, $\mathbf{r}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}=7$ $\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}6\\2\\3\end{pmatrix}=8\neq 7$ Hence l is parallel to the plane and does not lie in the plane. Hence points P and Q will never meet. [Note that it is not sufficient just to show that l is parallel to the plane as it may actually lie on it. One must still need to show that there is a point on l that is not on the plane] Alternatively, one can check that	Many students made calculation error in the cross product. It is strongly advised that students **check the correctness** by either using the GC or simply taking the dot product between the resulting vector and any one of the two vectors and verify that it is zero. For e.g. $\begin{pmatrix}-1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}=-6+6=0$ The marker pointed this out in the MY exam but apparently it has fallen on deaf ears

Qn		Remarks
	$\left[\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}\right]\bullet\begin{pmatrix}6\\3\\2\end{pmatrix}\neq 7$ for all $t\in\mathbb{R}$.	
9(iii)	Shortest distance between P and Q is the distance between the line and the parallel plane. $\dfrac{\left\lvert\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}-7\right\rvert}{\left\lvert\begin{pmatrix}6\\3\\2\end{pmatrix}\right\rvert}=\dfrac{2}{7}$	Many students used the wrong vector in this computation. Simply take a point on l, say $(1,1,0)$ and compute its distance from the plane
9(iv)	Lines l and m are non-parallel. Hence $k\neq 1$. If the two lines intersect, $\begin{pmatrix}1\\k\\0\end{pmatrix}+s\begin{pmatrix}2\\-2\\-3k\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$ for some $s,t\in\mathbb{R}$ $1+2s=1+2t$ ___(1) $k-2s=1-2t$ ___(2) $-3sk=-3t$ ___(3) From (1), $s=t$ Substituting $s=t$ in (2), $k=1$ $s=t$ and $k=1$ satisfies (3) Thus for the system of linear equations to be inconsistent, $k\neq 1$. Hence lines l and m are skew when $k\neq 1$.	Students must understand that if two lines are skew, then they are: ▪ Non-parallel ▪ Non-intersecting Students are expected to justify that $k\neq 1$ is the condition on k that satisfy the above two requirements
9(v)	Let F be the foot of perpendicular from X to line l. $\overrightarrow{OF}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$ for some $t\in\mathbb{R}$ $\overrightarrow{XF}=\begin{pmatrix}1\\2\\5\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$ $\left[\begin{pmatrix}1\\2\\5\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-2\\-3\end{pmatrix}=0$ $-17+17t=0$ $t=1$ $\overrightarrow{OF}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}2\\-2\\-3\end{pmatrix}=\begin{pmatrix}3\\-1\\-3\end{pmatrix}$ $F\equiv(3,-1,-3)$	Students must understand that XF and <u>not OF</u> is perpendicular to l.  Therefore $\overrightarrow{XF}\bullet\mathbf{d}=0$

2017 NYJC JC2 Prelim 9758/1 Solution

Qn		Remarks
10	Let s be the distance between the front of the car and the train at time t. $s=\sqrt{x^2+30^2}$ and $x^2=(40-30t)^2+(20t)^2$ $s=\sqrt{(40-30t)^2+(20t)^2+30^2}=\sqrt{1300t^2-2400t+2500}$	
10(i)	$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600t-2400)$ When $t=1, \frac{\mathrm{d}s}{\mathrm{d}t}=2.67$	Generally ok. There are still handful of students not differentiating the expression given.
10(ii)	At stationary point $\frac{\mathrm{d}s}{\mathrm{d}t}=0$ $\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600t-2400)=0$ $2600t-2400=0 \Rightarrow t=\frac{12}{13}$ $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600)$ $+(-\frac{1}{2})\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{3}{2}}(2600t-2400)$ When $t=\frac{12}{13}, \frac{\mathrm{d}^2s}{dt^2}>0$ $s=\sqrt{1300(\frac{12}{13})^2-2400(\frac{12}{13})+2500}=$ 19.884=19.9	
10(iii)	Let the angle of elevation be θ $\sin\theta=\frac{30}{\sqrt{1300t^2-2400t+2500}}$ $\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{1}{2}(30)(1300t^2-2400t+2500)^{-\frac{3}{2}}(2600t-2400)$ When $t=1$, $\cos\theta=\frac{\sqrt{(40-30)^2+20^2}}{\sqrt{1300-2400+2500}}=\sqrt{\frac{500}{1400}}$ $\therefore \frac{\mathrm{d}\theta}{\mathrm{d}t}= -\frac{1}{2}(30)(1300-2400+2500)^{-\frac{3}{2}}(200)\div\sqrt{\frac{500}{1400}}$ =−0.0958 rad/s (or $-5.5^\circ/\mathrm{s}$)	Most of the students are not able to form the trigo equation.

Qn		Remarks
11(i)		Candidates often forget to indicate the axial intercepts. For those whom had written down the intercepts, many simply wrote 2π and 4π without the r
11(ii)	$\frac{\mathrm{d}x}{\mathrm{d}\theta}=r(1-\cos\theta),\ \frac{\mathrm{d}y}{\mathrm{d}\theta}=r\sin\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\sin\theta}{1-\cos\theta}$ $=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ Use do formula Use trigo identity $=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=\cot\frac{\theta}{2}$	Some candidates did not make use of double angle formula and end up using a long and tedious method to arrive at the given answer
11(iii)	$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\cot^2\frac{\theta}{2}=\operatorname{cosec}^2\frac{\theta}{2}-1$ $=\frac{1}{\sin^2\frac{\theta}{2}}-1$ Use double angle formula $=\frac{2}{1-\cos\theta}-1=\frac{2r}{r(1-\cos\theta)}-1$ Thus $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{2r}{y}-1$	Badly done with poor presentation. You are required to prove that $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{2r}{y}-1$. Many candidates started by stating this equation and proceed to prove. This is not true! The key is to apply double angle formula.
11(iv)	$Area=\int_0^{2\pi r} y\,dx$ $\mathrm{d}x=\frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta=r(1-\cos\theta)\mathrm{d}\theta$ $=\int_0^{2\pi} r(1-\cos\theta)\cdot r(1-\cos\theta)\,d\theta$ $=r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$ Use double angle formula $=r^2\int_0^{2\pi}(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta)\,d\theta$ $=r^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$ $=3\pi r^2$	Many candidates are unable to write out the area expression in parametric form correctly. A handful of students did not integrate the expression correctly.

Section A: Pure Mathematics [40 marks]

1 The position vectors of points A and B with respect to the origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. Point C lies on OA produced such that $4OA = AC$ and point D lies on OB produced such that $OB = BD$. The lines BC and AD meet at the point M.

(i) Giving a necessary condition for $\mathbf{a}$ and $\mathbf{b}$, find the position vector of M in terms of $\mathbf{a}$ and $\mathbf{b}$. [5]

(ii) If $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$, find the shortest distance of M from the line OC giving your answer in the form $k|\mathbf{a} \times \mathbf{b}|$ where k is a constant to be determined. [2]

2 **(a)** Find the set of values of θ lying in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ such that the sum to infinity of the geometric series $1 + \tan\theta + \tan^2\theta + \ldots$ is greater than 2. [5]

(b) The sum of the first n terms of a positive arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m. [4]

3 It is given that $y = \ln(\cos ax - \sin ax)$, where a is a non-zero constant.

(i) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + a^2 = 0$. [3]

(ii) By further differentiation of the result in **(i)**, find, in terms of a, the Maclaurin series for y, up to and including the term in x^3. [3]

(iii) Hence show that when x is small enough for powers of x higher than 2 to be neglected and $a = 2$, then $\cos 2x - \sin 2x \approx 1 + kx + kx^2$ where k is a constant to be determined. [4]

(iv) Using appropriate expansions from the List of Formulae (**MF26**), verify the correctness of your answer in **(iii)**. [2]

4 The growth of an organism in a controlled environment is monitored and the growth rate of the organism is proportional to $(N-x)x$, where x is the population (in thousands) of the organism at time t and N is a constant such that $x<N$. The initial population of the organism is $\frac{1}{3}N$.

(i) Find x in terms of t and determine the population of the organism in the long run, giving your answer in terms of N. [6]

Another model is proposed for the growth of the organism, which assumes the growth rate is purely a function of time and is modelled by the differential equation $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}=\frac{-9t}{\left(4+9t^2\right)^2}$. It predicts that the population of the organism will also eventually stabilise.

(ii) Show that under this model, $x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+\frac{N}{3}$.

Hence state the population of the organism in the long run, giving your answer in terms of N. [6]

Section B: Probability and Statistics [60 marks]

5 From past records, the number of days of hospitalization for an individual with minor ailment can be modelled by a discrete random variable with probability density function given by

$$P(X=x)=\begin{cases}\dfrac{6-x}{15}, & \text{for } x=1,2,3,4,5,\\ 0, & \text{otherwise.}\end{cases}$$

An insurance policy pays \$100 per day for up to 3 days of hospitalization and \$25 per day of hospitalization thereafter.

(i) Calculate the expected payment for hospitalization for an individual under this policy. [4]

(ii) The insurance company will incur a loss if the total payout for 100 hospitalisation claims under this policy exceed \$24000. Using a suitable approximation, estimate the probability that the insurance company will incur a loss for 100 such claims. [4]

6 A teacher wants to randomly form two teams of 5 students from a group of 5 girls and 5 boys for a sports activity. Two of the girls, Ann and Alice, are selected as team leaders. Find the probability that one team has exactly 3 girls. [2]

The ten students are seated at a round table of 10. Find the probability that

(i) Ann and Alice are not seated together, [2]

(ii) no two of the remaining 3 girls are next to each other given that Ann and Alice are not seated together. [4]

7 In a large company, a small sample of n employees is obtained to find out their mode of transport to work. The number of employees who ride the train to work is denoted by R. Assume that R has the distribution $\mathrm{B}(n, p)$.

(i) Given that $n = 10$, find the value of p if the probability that 6 employees ride the train to work is twice the probability that 4 employees ride the train to work. [3]

(ii) Given that $p = 0.25$, find the largest value of n such that the probability that fewer than 2 employees who ride the train to work is more than 0.15. [3]

(iii) Given that $n = 11$ and $p = 0.7$, find the probability that at least 5 employees ride the train to work if at least 3 employees do not ride the train to work. [4]

8 **(a)** Comment briefly on the following statements:

(i) Flowers in a garden are watered and the product moment correlation coefficient between petal size and the amount of water given is 0.073, so it follows that there is no relation between petal size and quantity of water given to the flower. [1]

(ii) The product moment correlation coefficient between the risk of heart disease and amount of red wine intake is found to be approximately −1. Therefore we conclude that red wine intake causes the risk of heart disease to decrease. [1]

(b) The median age of residents in Singapore across the years are given in the table.

Year (x)	**1984**	**1988**	**1992**	**1996**	**2000**	**2004**	**2008**	**2012**	**2016**
Median age (y)	26.7	28.8	27.0	32.3	34.0	35.4	36.7	38.4	40.0

It is thought that the median age of residents in year x can be modelled by one of the formulae

$$y = \frac{a}{x} + b, \quad y = c \ln x + d,$$

where a, b, c and d are constants.

(i) Plot a scatter diagram on graph paper for these values, labelling the axis, using a scale of 2cm to represent 5 years on the y-axis and an appropriate scale for the x-axis. One of the values of y was quoted wrongly. Indicate this point as P on your diagram. [2]

For parts **(ii)**, **(iii)**, **(iv)** of this question, you should **exclude** the point P.

(ii) Find, correct to 5 decimal places, the value of the product moment correlation coefficient between

(A) x^{-1} and y

(B) $\ln x$ and y. [2]

(iii) Explain which model is more appropriate to predict the median age of residents in Singapore and find the equation of the least squares regression line for this model, giving your answer to 2 decimal places. [2]

(iv) Explain why neither the regression line of x^{-1} on y nor the regression line of $\ln x$ on y should be used to estimate the year when the median age is 30. [1]

(v) Give a possible reason for the rise in the median age. [1]

9 A manufacturing process produces ball bearings with diameters with known standard deviation 0.04 cm. Under normal circumstances, the manufacturing process will produce ball bearings of mean diameter 0.5 cm.

(i) During a routine quality control check, a random sample of 25 ball bearings gives a mean of 0.51 cm. Is there evidence to believe that the manufacturing process is producing ball bearings of the stated diameter? Perform an appropriate test at 5% level of significance. State a necessary assumption for the test to be valid. [4]

An enhancement on the manufacturing process will ensure that the diameters of the ball bearings produced are less variable.

(ii) Measurements of a sample of 100 ball bearings give the following summary statistics:

$$\Sigma x = 50.6,\ \Sigma(x-0.5)^2 = 0.08345.$$

Show that the unbiased estimate of the population variance is 8.07×10^{-4}.
Is there evidence at the 5% level of significance that after the enhancement, the manufacturing process is producing oversized ball bearings? [4]

(iii) Another sample of 100 ball bearings yield the same summary statistics as the previous sample in **(ii)**. Explain, with justification, whether the combined sample will give a different conclusion to **(ii)**. [4]

10 The diameters of the bolts produced by two manufacturers A and B follow a normal distribution with a standard deviation of 0.16 mm.

The mean diameter of the bolts produced by manufacturer A is 1.56 mm. Of the bolts produced by manufacturer B, 24.2% have a diameter less than 1.52 mm.

(i) Show that the mean diameter of the bolts produced by manufacturer B is 1.632 mm. [3]

(ii) Find the probability that the diameter of a randomly chosen bolt from manufacturer A differs from the diameter of a randomly chosen bolt from manufacturer B by less than 0.1 mm. [3]

(iii) Find the probability that the total diameter of 5 randomly chosen bolt from manufacturer A is more than 5 times the diameter of a randomly chosen bolt from manufacturer B. [3]

(iv) A trading company buys 44% of its stock of bolts from manufacturer A and the rest from manufacturer B. A bolt is chosen at random from the trading company's stock. Show that the probability that the diameter of the bolt is less than 1.52 mm is 0.312. [3]

——END OF PAPER——

2017 NYJC JC2 Preliminary Examination 9758/2 Solution

Qn		Remarks
1(i)	Assume that $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. $\overrightarrow{OC} = 5\mathbf{a}$, $\overrightarrow{OD} = 2\mathbf{b}$ On the line BC, $\overrightarrow{OM} = \lambda(5\mathbf{a}) + (1-\lambda)\mathbf{b}$ On the line AD, $\overrightarrow{OM} = \mu(2\mathbf{b}) + (1-\mu)\mathbf{a}$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero, non-parallel vectors, comparing coefficient $5\lambda = 1-\mu$ $2\mu = 1-\lambda$ $\Rightarrow$ $\lambda = \frac{1}{9}, \mu = \frac{4}{9}$ Thus $\overrightarrow{OM} = \frac{5}{9}\mathbf{a} + \frac{8}{9}\mathbf{b}$	Note that this is a condition on vectors, not points. You may use Ratio Theorem, or you may find direction vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$ to find the respective lines.
1(ii)	Since $\mathbf{a}$ is a unit vector in the direction of OC, shortest distance $= \left\|\overrightarrow{OM} \times \mathbf{a}\right\|$ $= \left\|\left(\frac{5}{9}\mathbf{a} + \frac{8}{9}\mathbf{b}\right) \times \mathbf{a}\right\|$ $= \frac{8}{9}\left\|\mathbf{a} \times \mathbf{b}\right\|$ $k = \frac{8}{9}$	Some students mixed up the properties for cross product and dot product. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$, $\mathbf{a} \cdot \mathbf{a} = \left\|\mathbf{a}\right\|^2$ Note that distance cannot be negative. Always check your algebraic workings when your answers appear counter-intuitive.
2(a)	For sum to infinity to exist, $\left\|\tan\theta\right\| < 1$ $-1 < \tan\theta < 1$ $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ $\frac{1}{1-\tan\theta} > 2$ $0 < 1-\tan\theta < \frac{1}{2}$ $\tan\theta > \frac{1}{2} \Rightarrow \theta > 0.464$ Since $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$, therefore $\{\theta \in \mathbb{R} \mid 0.464 < \theta < 0.786\}$ or $\theta : (0.464, 0.786)$	Most students failed to check the range of values for $\left\|r\right\| < 1$ for sum to infinity to exist. Note: Many students cross multiplied to get $1 > 2(1-\tan\theta)$. For this case it is ok as $1-\tan\theta > 0$. **In general, we should not cross multiply for inequalities** unless the term multiplied is strictly positive. Set notation.

Qn		Remarks
2(b)	$u_1 = S_1 = 2 \Rightarrow a = 8$ $u_2 = S_2 - S_1 = 10 \Rightarrow d = 8$ $u_{32} = a + (32-1)d = 2 + (32-1)8 = 250$ $\frac{u_{32}}{u_m} = \frac{u_m}{u_2} = \text{constant}$ $u_2,\ u_m$ and u_{32} are consecutive terms of GP $\Rightarrow (u_m)^2 = (10)(250) = 2500$ $u_m = 50$ (since it is a positive sequence) $50 = 2 + (m-1)8 \Rightarrow m = 7$ Alternatively, $u_n = S_n - S_{n-1}$ $= 4n^2 - 2n - \left[4(n-1)^2 - 2(n-1)\right]$ $= 8n - 6$ $\frac{u_{32}}{u_m} = \frac{u_m}{u_2}$ $\frac{8(32)-6}{8m-6} = \frac{8m-6}{8(2)-6}$ $(8m-6)^2 = (250)(10) = 2500$ $m = 7$ or $m = -5.5$ (rejected as m is a positive integer)	
3(i)	$y = \ln(\cos ax - \sin ax)$ $e^y = \cos ax - \sin ax$ $e^y \frac{dy}{dx} = -a \sin ax - a \cos ax$ $e^y \frac{d^2y}{dx^2} + e^y \left(\frac{dy}{dx}\right)^2 = -a^2 \cos ax + a^2 \sin ax$ $e^y \frac{d^2y}{dx^2} + e^y \left(\frac{dy}{dx}\right)^2 = -a^2 (\cos ax - \sin ax)$ $e^y \frac{d^2y}{dx^2} + e^y \left(\frac{dy}{dx}\right)^2 = -a^2 e^y$ $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + a^2 = 0$	A majority of students produced <u>very long, tedious and messy</u> calculations by direct differentiation when the calculation could have been much simpler by rewriting the equation into the implicit form $e^y = \cos ax - \sin ax$ and applying implicit differentiation to obtain the desired equation. Students MUST learn SMART WAYS of doing math instead of using the brute force method

Qn		Remarks
3(ii)	$\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}=0$ When $x=0,\quad y=0$ $\frac{dy}{dx}=-a,\quad \frac{d^2y}{dx^2}=-2a^2,\quad \frac{d^3y}{dx^3}=-4a^3$ $y=-ax-a^2x^2-\frac{2}{3}a^3x^3+\cdots$	Fairly straightforward application of Maclaurin's theorem to obtain series expansion.
3(iii)	$\ln\left(\cos 2x-\sin 2x\right)=-2x-4x^2-\frac{16}{3}x^3+\cdots$ $\cos 2x-\sin 2x\approx e^{-2x-4x^2}$ $\approx 1+\left(-2x-4x^2\right)+\frac{\left(-2x-4x^2\right)^2}{2!}$ (since $e^x\approx 1+x+\frac{x^2}{2!}$) $\approx 1-2x-4x^2+\frac{\left(-2x\right)^2}{2}$ $=1-2x-2x^2$ where $k=-2$	A number of students did not pay attention to the word 'Hence' which requires them to use an earlier result to deduce the next result. Many simply used the series expansions of sin x and cos x from MF26 which earn no credit
3(iv)	$\cos 2x-\sin 2x=1-\frac{(2x)^2}{2}-(2x)$ $=1-2x-2x^2$	Some students forgot the '2' and wrote $\cos 2x-\sin 2x$ $=1-\frac{x^2}{2}-x$ Some used the double angle formulae for sin 2x and cos 2x which is not necessary

<table>
<tr><th>Qn</th><th></th><th>Remarks</th></tr>
<tr><td>4(i)</td><td>
$$\frac{\mathrm{d}x}{\mathrm{d}t}=k(N-x)x$$
$$\frac{1}{(N-x)x}\frac{\mathrm{d}x}{\mathrm{d}t}=k$$
$$\int\frac{1}{(N-x)x}\,\mathrm{d}x=\int k\,\mathrm{d}t$$
$$\frac{1}{N}\int\frac{1}{N-x}+\frac{1}{x}\,\mathrm{d}x=\int k\,\mathrm{d}t$$
Remember to include the modulus sign whenever the integral involves ln.
$$\frac{1}{N}\left(-\ln|N-x|+\ln|x|\right)=kt+C$$
$$\frac{1}{N}\ln\left|\frac{x}{N-x}\right|=kt+C$$
$$\ln\left|\frac{x}{N-x}\right|=Nkt+NC$$
$$\left|\frac{x}{N-x}\right|=\mathrm{e}^{Nkt+NC}$$
$$\frac{x}{N-x}=A\mathrm{e}^{Nkt}\quad\text{where } A=\pm\mathrm{e}^{NC}$$
When $t=0$, $x=\frac{1}{3}N$,
$$A=\frac{1}{2}$$
$$\frac{x}{N-x}=\frac{1}{2}\mathrm{e}^{Nkt}$$
$$2x=(N-x)\mathrm{e}^{Nkt}$$
$$x\left(2+\mathrm{e}^{Nkt}\right)=N\mathrm{e}^{Nkt}$$
$$x=\frac{N\mathrm{e}^{Nkt}}{2+\mathrm{e}^{Nkt}}\qquad\text{equivalently, } x=\frac{N}{2\mathrm{e}^{-Nkt}+1}$$
</td><td></td></tr>
<tr><td></td><td><u>Alternative method</u> (not recommended):</td><td></td></tr>
</table>

Qn		Remarks
	$$\frac{1}{(N-x)x}\frac{\mathrm{d}x}{\mathrm{d}t}=k$$ $$-\int\frac{1}{x^2-Nx}\,\mathrm{d}x=\int k\,\mathrm{d}t$$ $$-\int\frac{1}{\left(x-\frac{N}{2}\right)^2-\left(\frac{N}{2}\right)^2}\,\mathrm{d}x=\int k\,\mathrm{d}t$$ $$-\left(\frac{1}{2\left(\frac{N}{2}\right)}\ln\left\|\frac{\left(x-\frac{N}{2}\right)-\frac{N}{2}}{\left(x-\frac{N}{2}\right)+\frac{N}{2}}\right\|\right)=kt+C$$ $$-\left(\frac{1}{N}\ln\left\|\frac{x-N}{x}\right\|\right)=kt+C$$ $$\frac{1}{N}\ln\left\|\frac{x}{N-x}\right\|=kt+C$$ $$\ln\left(\frac{x}{N-x}\right)=Nkt+NC \quad \text{since } 0<x<N$$ When $t=0$, $x=\frac{1}{3}N$, $\ln\frac{1}{2}=NC\Rightarrow C=-\frac{1}{N}\ln 2$ $$\ln\left(\frac{x}{N-x}\right)=Nkt-\ln 2$$ $$\ln\left(\frac{2x}{N-x}\right)=Nkt$$ $$\frac{2x}{N-x}=\mathrm{e}^{Nkt}$$ $$x\left(2+\mathrm{e}^{Nkt}\right)=N\,\mathrm{e}^{Nkt}$$ $x=\frac{N\,\mathrm{e}^{Nkt}}{2+\mathrm{e}^{Nkt}}$ equivalently, $x=\frac{N}{2\mathrm{e}^{-Nkt}+1}$ As $t\to\infty$, $x=\frac{N}{2\mathrm{e}^{-Nkt}+1}\to N$	We can remove the modulus sign by taking into account the range of values the variable x can take

Qn		Remarks

(ii)

$$\frac{d^2x}{dt^2}=\frac{-9t}{\left(4+9t^2\right)^2}$$

$$\int\frac{d^2x}{dt^2}\,dt=\int\frac{-9t}{\left(4+9t^2\right)^2}\,dt$$

$$=-\frac{1}{2}\int\frac{18t}{\left(4+9t^2\right)^2}\,dt$$

Integral $\int\frac{18t}{\left(4+9t^2\right)^2}\,dt$ is of the form $\int f'(t)\left[f(t)\right]^n\,dt$ where $n=-2$.

$$\frac{dx}{dt}=\frac{1}{2}\left(\frac{1}{4+9t^2}\right)+A$$

$$\int\frac{dx}{dt}\,dt=\frac{1}{2}\int\frac{1}{4+9t^2}\,dt+\int A\,dt$$

$$=\frac{1}{18}\int\frac{1}{\left(\frac{2}{3}\right)^2+t^2}\,dt+\int A\,dt$$

$$=\frac{1}{18}\cdot\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{t}{\frac{2}{3}}\right)+At+B$$

$$x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+At+B$$

Some students wrongly express the integral as $\frac{1}{2}\int\frac{1}{4+9t^2}+A\ dt$.

Note that $\int\frac{1}{2}\left(\frac{1}{4+9t^2}\right)+A\ dt=\frac{1}{2}\int\frac{1}{4+9t^2}+\underline{2}A\ dt$.

Thus it is recommended to express as two separate integrals as shown in the solution.

To find $\int\frac{1}{4+9t^2}\ dt$, it is necessary to have the coefficient of t^2 to be 1 before applying formula in MF26. Thus s

$$\int\frac{1}{4+9t^2}\ dt=\frac{1}{9}\int\frac{1}{\left(\frac{2}{3}\right)^2+t^2}\ dt=\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{t}{\frac{2}{3}}\right)=\frac{3}{2}\tan^{-1}\left(\frac{3t}{2}\right)$$

and not $\int\frac{1}{4+9t^2}\ dt=\int\frac{1}{2^2+(3t)^2}\ dt=\frac{1}{2}\tan^{-1}\left(\frac{3t}{2}\right)$.

Since the population stabilises in the long run, as $t\to\infty$, $x\to\text{finite value}$, $A=0$

When $t=0$, $x=\frac{1}{3}N$, $B=\frac{N}{3}$

Hence $x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+\frac{N}{3}$

When $t\to\infty$, $\tan^{-1}\left(\frac{3t}{2}\right)\to\frac{\pi}{2}$

Hence $x\to\frac{\pi}{24}+\frac{N}{3}$.

5(i)

Let Y be the payment for an individual. The probability table is as follows:

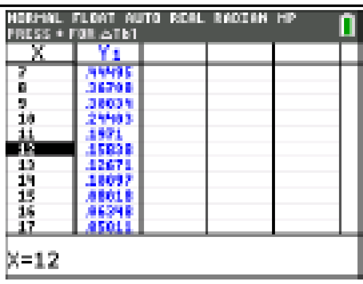
y	100	200	300	325	350
$P(Y=y)$	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{15}$

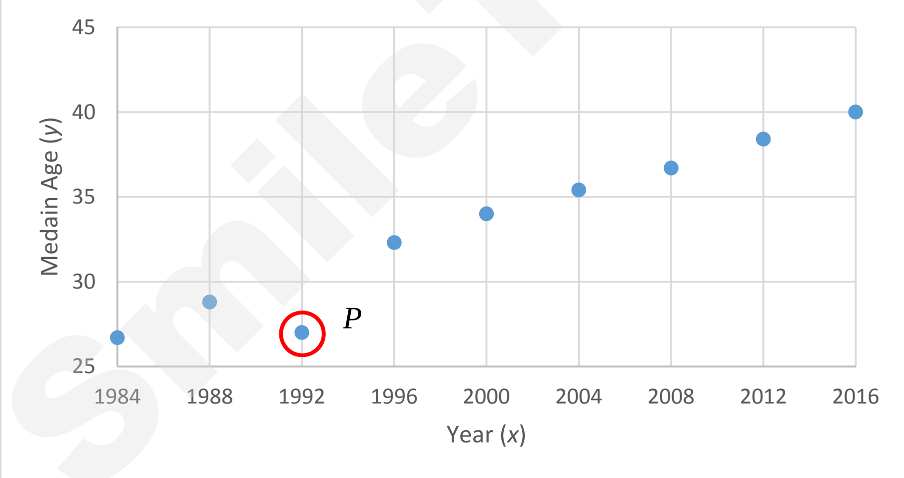
$$\mathrm{E}(Y)=100\cdot\frac{1}{3}+200\cdot\frac{4}{15}+300\cdot\frac{1}{5}+325\cdot\frac{2}{15}+350\cdot\frac{1}{15}$$

$$=213\frac{1}{3}$$

Remarks: Candidates must read the question carefully to understand the payment scale, and thereafter to write down the probability table for Y correctly. Note that there is no linear relationship between Y and X . Thus working out $E(X)$ will not obtain any credit.

Qn		Remarks
5(ii)	$\mathrm{E}(Y^2) = 100^2 \cdot \frac{1}{3} + 200^2 \cdot \frac{4}{15} + 300^2 \cdot \frac{1}{5} + 325^2 \cdot \frac{2}{15} + 350^2 \cdot \frac{1}{15}$ $= 54250$ $\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \left[\mathrm{E}(Y)\right]^2 = 8738\frac{8}{9}$ Since $n = 100$ is large, by Central Limit Theorem, $T = \sum_{i=1}^{100} Y_i \sim \mathrm{N}\left(21333.33, 873888.89\right)$ approx. $\text{Prob. Req'd} = P(T > 24000)$ ≈ 0.00217	Note that $E(Y^2)$ is not the variance of Y. Further, candidates are reminded that you should not assume that Y is normally distributed. Central Limit Theorem is necessary to obtain the approximate distribution of ΣY_i.
6	Probability required = $\frac{2 \times {}^3C_2 \times {}^5C_2}{{}^8C_4} = \frac{6}{7}$	${}^3C_2 \times {}^5C_2$ – choose 2 girls from the remaining 3 girls and 2 boys from 5 boys for the group with exactly 3 girls. Multiply by 2 because this group can be Ann or Alice's group. This is a conditional probability as Ann and Alice must be the team leaders, thus ${}^8C_4$.
6(i)	Required probability = $\frac{(8-1)!\,{}^8C_2\ 2!}{(10-1)!} = \frac{7}{9}$ OR $\quad 1 - \frac{(9-1)!\,2!}{(10-1)!} = \frac{7}{9}$	Insertion method or apply Principle of complementation

Qn		Remarks
6(ii)	Let X be the event that the remaining 3 girls are separated. Let Y be the event that Ann and Alice are not seated together. $\mathrm{P}(X \mid Y) = \dfrac{\mathrm{P}(X \cap Y)}{P(Y)}$ $= \dfrac{\mathrm{P}(X) - \mathrm{P}(X \cap Y')}{P(Y)}$ $= \dfrac{\dfrac{(7-1)!\,{}^{7}C_{3}\,3! - (6-1)!\,2!\,{}^{6}C_{3}\,3!}{(10-1)!}}{7/9}$ $= \dfrac{85/252}{7/9}$ $= \dfrac{85}{196}$	Not possible to list out all the cases.
7(i)	$\mathrm{P}(R=6) = 2\mathrm{P}(R=4)$ ${}^{10}C_{6}p^{6}(1-p)^{4} = 2\,{}^{10}C_{4}p^{4}(1-p)^{6}$ $p^{2} = 2(1-p)^{2}$ $p^{2} - 4p + 2 = 0$ $p = 0.586$	
7(ii)	$R \sim \mathrm{B}(n, 0.25)$ $\mathrm{P}(R<2) > 0.15$ $\mathrm{P}(R \le 1) > 0.15$ $n = 12$  	

Qn		Remarks
7(iii)	$R \sim \mathrm{B}(11, 0.7)$ $\mathrm{P}(R \geq 5 \mid R \leq 8) = \dfrac{\mathrm{P}(R \geq 5 \text{ and } R \leq 8)}{\mathrm{P}(R \leq 8)}$ $= \dfrac{\mathrm{P}(5 \leq R \leq 8)}{\mathrm{P}(R \leq 8)}$ $= \dfrac{\mathrm{P}(R \leq 8) - \mathrm{P}(R \leq 4)}{\mathrm{P}(R \leq 8)}$ $= 0.969$	2 major errors seen in the students' scripts. 1. Failure to recognise that this is a conditional probability. 2. Failure to count the number of cases for the numerator.
8(a) (i)	The value of 0.073 indicates that there is a weak linear correlation between petal size and the amount of water but there could be some non-linear relation.	Quite a number of students state that there is still some weak linear correlation.
8(a) (ii)	The approximate value of -1 indicates that there is a strong negative linear correlation between the risk of heart disease and amount of red wine intake. It does not mean that red wine intake decreases the risk of heart disease.	
8(b) (i)		Wrong scale is used by a handful of students.
8(b) (ii)	For Model A, $r = -0.9985438 = 0.99854$ For Model B, $r = 0.9984431 = 0.99844$	Quite a number of students chose model B instead of A simply because r is positive.
8(b) (iii)	Model A as the $\lvert r \rvert$ value is closer to 1. The suitable regression line is $y = 848.24 - \dfrac{1629165.57}{x}$	
8(b) (iv)	This is because age (x) is the controlled variable	Badly answered. Students are not able to identify controlled variable.

Qn		**Remarks**
8(b)(v)	The rise in the median age is due to the drop in the growth of the population.	
9(i)	To test $H_0: \mu = 0.5$ $H_1: \mu \neq 0.5$ Two-tailed test as the question asked for the same diameter **or not.** Level of significance: 5% Under H_0, $Z = \dfrac{\overline{X} - 0.5}{0.04/\sqrt{25}} \sim N(0,1)$ Under H_0, we have assumed that $\mu = 0.5$. Thus we have to replace μ by 0.5. Note that the population variance is given for this part! Reject H_0 if p -value ≤ 0.05 Calculation: $\overline{x} = 0.51$, p -value = 0.211 Since p -value > 0.05, we do not reject H_0. Thus there is insufficient evidence at 5% level of significance that the manufacturing process is producing ball bearings of different diameters. Distribution of the diameter of the ball bearings is normal. As sample size is small.	On the whole, this question was badly done. Many candidates still showed poor presentation. They have also illustrated poor understanding in the writing of rejection region/criteria. Common mistakes include swapping μ_0 and $\overline{x}$; identify H_1 wrongly.
9(ii)	To test $H_0: \mu = 0.5$ $H_1: \mu > 0.5$ Level of significance: 5% Under H_0, by Central Limit Theorem, $Z = \dfrac{\overline{X} - 0.5}{s/\sqrt{100}} \sim N(0,1)$ approx Reject H_0 if p -value ≤ 0.05 Calculation: $\overline{x} = 0.506$, $\Sigma(x-0.5) = 50.6 - 50 = 0.6$ We must obtain $\Sigma(x-0.5)$ first before applying formula $s^2 = \dfrac{1}{n-1}\left[\Sigma(x-0.5)^2 - \dfrac{(\Sigma(x-0.5))^2}{n}\right]$ $= \dfrac{1}{99}\left(0.08345 - \dfrac{0.6^2}{100}\right)$ $= 8.07 \times 10^{-4}$ Note that $\Sigma(x-0.5)^2$ is not sample variance as 0.5 is not the sample mean! p -value = 0.0173 Since p -value < 0.05, we reject H_0. Thus there is sufficient evidence at 5% level of significance that the manufacturing process is producing oversized ball bearings.	A handful of students identify this as a left tail test, resulting in a p -value that is more than 0.5. Note that for hypothesis testing, it does not make sense to have a p -value that is more than 0.5. As a rule of thumb, if $\overline{x} > \mu_0$, we should test $\mu > \mu_0$.

Qn		Remarks
9(iii)	For the new sample, $\bar{x} = 0.506$. However, $s^2 = \frac{1}{199}\left(2(0.08345) - \frac{(2(0.6))^2}{200}\right)$ $= 8.02513 \times 10^{-4}$ $Z_{calc} = 2.995$, p-value $= 0.00137$ Since p-value < 0.05, we will still reject H_0. The conclusion remains the same. Combined sample is **NOT** pooled sample. Please do not use this formula at all! (It's for Further Math)	For this part, candidates are supposed to either compute the p-value after the samples are combined, or give clearly supported reason why the p-value will be smaller.
10(i)	Let X denotes the diameter of bolt from manufacturer A. $X \sim \mathrm{N}(1.56, 0.16^2)$ Let Y denotes the diameter of bolt from manufacturer B. $Y \sim \mathrm{N}(\mu, 0.16^2)$ $\mathrm{P}(Y < 1.52) = 0.242$ $\mathrm{P}(Z < \frac{1.52-\mu}{0.16}) = 0.242$ $\frac{1.52-\mu}{0.16} = -0.6998836 \Rightarrow \mu = 1.63198 = 1.632$	Candidates are expected to show full working for this part as it is a 'show' question. Standardising is the preferred method. Candidates who used graphical method using did not explain the graphs used and how the final answer is attained. Tables should not be used when dealing with a non-integer value.
10(ii)	$W = X - Y \sim \mathrm{N}(1.56 - 1.632, 0.16^2 + 0.16^2)$ $\mathrm{P}(\|W\| < 0.1) = \mathrm{P}(-0.1 < W < 0.1) = 0.326$	Quite a few candidates find the distribution of $\|W\|$ instead of W, which is conceptually incorrect, and hence leading to a wrong answer.
10(iii)	$X_1 + X_2 + X_3 + X_4 + X_5 - 5Y \sim \mathrm{N}(5(1.56)\text{-}5(1.632),\ 5(0.16^2) + 5^2(0.16)^2)$ $\mathrm{P}(X_1 + X_2 + X_3 + X_4 + X_5 > 5Y)$ $= \mathrm{P}(X_1 + X_2 + X_3 + X_4 + X_5 - 5Y > 0)$ $= 0.341$	Mistakes on calculating the correct variance were not as common this time round as compared to Midyear exams. However, poor representation of the variables is still commonly seen.
10(iv)	$\mathrm{P}(X < 1.52) = 0.40129$ Prob. Req'd $= (0.44)(0.4012) + 0.56(0.242)$ $= 0.3120876$ $= 0.312$	Many candidates were unable to tackle this part. Again, as this is a 'show' question, candidates are expected to work out $\mathrm{P}(X < 1.52) = 0.40129$.

NJC Paper 1

1 Given that $\mathbf{p} = 2\mathbf{i} + \alpha\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = \alpha\mathbf{i} + \mathbf{j} + 6\mathbf{k}$, where α is a real constant and $\mathbf{w}$ is the position vector of a variable point W relative to the origin such that $\mathbf{w} \times \mathbf{p} = \mathbf{q}$.

(i) Find the value of α . [2]

(ii) Find the set of vectors $\mathbf{w}$ in the form $\{\mathbf{w} : \mathbf{w} = \mathbf{a} + \lambda\mathbf{b}, \lambda \in \mathbb{R}\}$. [3]

2 **(a)** The sum, S_n , of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by $S_n = 3 + 7^{-2n}\left(n^2\right)$.

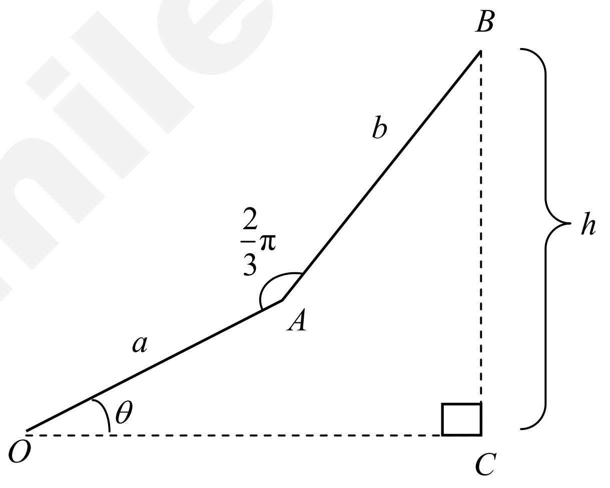
(i) Write down the value of $\sum_{r=1}^{\infty} u_r$. [1]

(ii) Find a formula for u_n for $n \geq 2$ and leave it in the form $7^{-2n}\,\mathrm{g}(n)$, where $\mathrm{g}(n)$ is an expression in terms of n. [2]

(b) Show that $\sum_{r=1}^{n}\left(\int_0^r \mathrm{e}^x - \mathrm{e}^{x-1}\ \mathrm{d}x\right) = \mathrm{e}^n + n\mathrm{e}^{-1} - (n+1)$.

Deduce the exact value of $\sum_{r=10}^{20}\left(\int_0^r \mathrm{e}^{x+2} - \mathrm{e}^{x+1}\ \mathrm{d}x\right)$. [5]

3 The diagram below shows two adjoining lines OA and AB where $OA = a$ m, $AB = b$ m and obtuse angle OAB is $\frac{2}{3}\pi$. C is a point such that OC and CB are perpendicular to each other, $BC = h$ m, and angle AOC is θ where $0 < \theta < \frac{\pi}{6}$.



(i) Show that

$h = \sqrt{a^2 + ab + b^2}\sin(\theta + \alpha)$, where α is a constant to be determined in terms of a and b. [4]

It is given that $a = 1$ and $b = 2$.

(ii) Find the rate of change of θ when $\theta = \frac{\pi}{12}$ and h is decreasing at a rate of 0.5 m per minute. [3]

(iii) When θ is a sufficiently small angle, show that $h \approx p\theta^2 + q\theta + \sqrt{3}$, where constants p and q are to be determined exactly. [3]

4 A researcher is investigating the elasticity of a new material. In the experiment, he stretched an extensible string of length 30 cm using a machine.

Each stretch is followed by a contraction. The initial stretch leads to an elongation of 10 cm and is followed by a contraction of 0.1 cm. The elongation resulting from each subsequent stretch is $\frac{10}{11}$ of the elongation caused by the previous stretch. Each subsequent contraction is 0.001 cm less than the previous contraction.

(i) Show that the length of the string after two stretches is 48.892 cm correct to 3 decimal places. [2]

(ii) Find the length of the string after it has been stretched n times, in terms of n. [3]

(iii) The string loses its elasticity completely when contraction exceeds elongation in a stretch. Find the minimum number of stretches for the string to lose its elasticity. [2]

(iv) The researcher coats a new string of the same initial length with another material. Now the string does not contract after every stretch while its elongation properties remain unchanged. Justify why it is impossible for the string to be elongated beyond 140 cm. [1]

5 **Do not use a calculator in answering this question.**

(a) Showing your working clearly, find the complex numbers z and w which satisfy the simultaneous equations

$$\mathrm{i}z + w = 2 \quad \text{and}$$

$$zw^* = 2 + 4\mathrm{i},$$

where w^* is the complex conjugate of w. [5]

(b) The complex number p is given by $a + \mathrm{i}b$, where $a > 0$, $b < 0$, $a^2 + b^2 > 1$ and $\tan^{-1}\left(\frac{b}{a}\right) = -\frac{2\pi}{9}$.

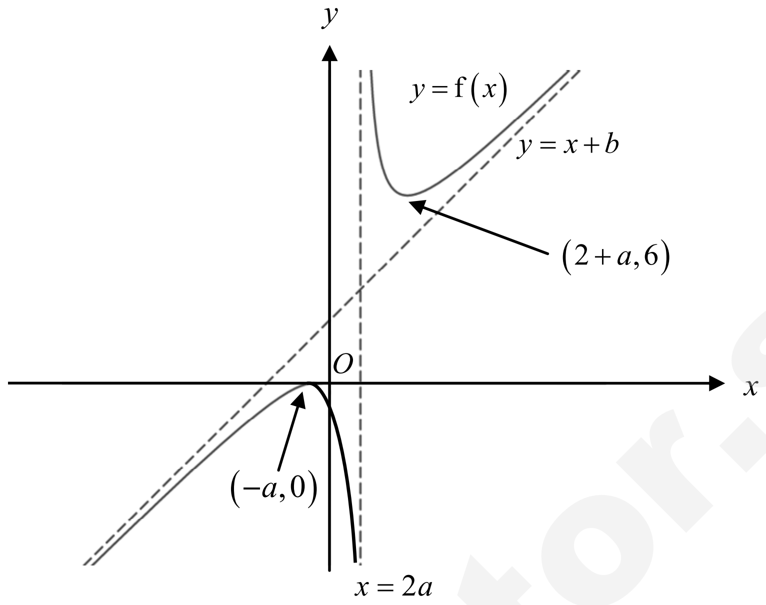
(i) Express the complex number $\frac{1}{p^2}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r is in terms of a and b, and $-\pi < \theta \le \pi$. [2]

(ii) On a single Argand diagram, illustrate the points P and Q representing the complex numbers p and $\frac{1}{p^2}$ respectively, labelling clearly their modulus and argument. [2]

(iii) It is given that $\angle OPQ = \alpha$. Using sine rule, show that $|p|^3 \approx \frac{\sqrt{3}}{2\alpha} - \frac{1}{2} - \frac{\alpha}{2\sqrt{3}}$ where α is small. [4]

6 The diagram shows the graph of the function $y = f(x)$ where, $a,\ b \in \mathbb{R},\ b \geq 2$ and $0 < a < 1$.

The coordinates of the minimum point and maximum point on the curve are $(-a,\ 0)$ and $(2+a,\ 6)$ respectively. The equations of the asymptotes are $y = x + b$ and $x = 2a$.



On separate diagrams, sketch the graphs of the following functions, labelling the coordinates of any points of intersection with the x-axis, the coordinates of any turning points and the equations of any asymptotes.

(i) $y = f(2x-1)+1$, [3]

(ii) $y = \dfrac{1}{f(x)}$. [3]

The two asymptotes of $y = f(x)$ intersect at point P. Show that P lies on the line $y = mx + (b + 2a - 2am)$ for all real values of m. Hence, state the range of values of m for which the line $y = mx + (b + 2a - 2am)$ does not cut the curve $y = f(x)$. [3]

7 **(a)** Find $\int e^x \cos(2x)\, dx$. [3]

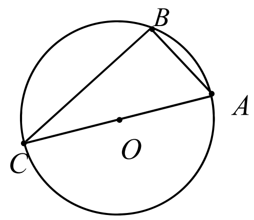
(b) The curve C has parametric equations

$$x = t - e^t, \quad y = 3\cos^2 t - 1, \quad \text{for } 0 < t < \pi.$$

(i) Use differentiation to find the exact x-coordinate of any turning point and determine the nature of the turning point. [3]

(ii) Find the exact area of the region bounded by the curve C and the line $y = 2$, expressing your answer in the form $a\pi + b + ce^{\pi}$, where a, b and c are rational numbers to be determined. [5]

8



The diagram above shows the cross-section of a sphere containing the centre O of the sphere. The points A, B and C are on the circumference of the cross-section with the line segment AC passing through O. It is given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Find $\overrightarrow{BC}$ in terms of **a** and **b**. [1]

(ii) D is a point on BC such that ΔOCD is similar to ΔACB. Find $\overrightarrow{OD}$ in terms of **a** and **b**. [2]

Point B lies on the x-z plane and has a positive z-component. It is also given that $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\angle OCB = \dfrac{\pi}{6}$.

(iii) Show that $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}$. [4]

(iv) Hence, determine whether the line passing through O and B and the line $\dfrac{x-2}{3} = \dfrac{y}{3} = z - 1$ are skew. [3]

9 The parametric equations of the curve C are

$$x = 2\sec t \text{ and } y = 3\tan t\text{, where } -\pi < t \le \pi,\ t \ne \pm\frac{\pi}{2}.$$

(i) Write down the Cartesian equation of C. [1]

(ii) Sketch the curve C, stating the equations of the asymptotes and the coordinates of the points where C crosses the axes, if any. [2]

(iii) The line $y = \sqrt{3}x + k,$ where $k < 0,$ is a tangent to C. Show that $k = -\sqrt{3}.$ [3]

The region bounded by this tangent, the curve C and the x-axis is rotated completely about the x-axis. Calculate the volume obtained. [4]

10 **(a)** By using the substitution $u=\frac{y}{x}$, show that the differential equation

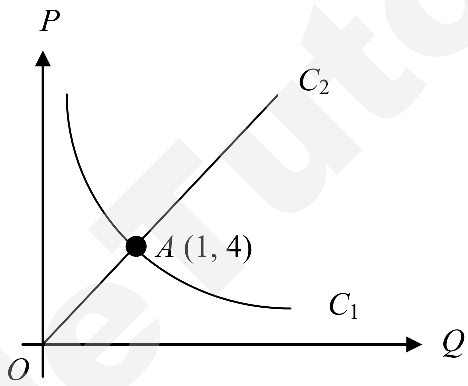
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2+xy+x^2}{x^2}, \text{ where } x>0,$$

can be reduced to $\frac{1}{u^2+1}\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$. Hence, find y in terms of x. [5]

(b) In the diagram below, the curve C_1 and the line C_2 illustrate the relationship between price (P dollars per kg) and quantity (Q tonnes) for consumers and producers respectively.

The curve C_1 shows the quantity of rice that consumers will buy at each price level while the line C_2 shows the quantity of rice that producers will produce at each price level. C_1 and C_2 intersect at point A, which has the coordinates (1, 4).

The quantity of rice that consumers will buy is inversely proportional to the price of the rice. The quantity of rice that producers will produce is directly proportional to the price.



(i) Interpret the coordinates of A in the context of the question. [1]

(ii) Solve for the equations of C_1 and C_2, expressing Q in terms of P. [2]

Shortage occurs when the quantity of rice consumers will buy exceeds the quantity of rice producers will produce. It is known that the rate of increase of P after time t months is directly proportional to the quantity of rice in shortage.

(iii) Given that the initial price is \$3 and that after 1 month, the price is \$3.65, find P in terms of t and sketch this solution curve, showing the long-term behaviour of P.

[7]

Suggest a reason why producers might use $P=aQ+b$, where $a,b\in\mathbb{R}^+$, instead of C_2 to model the relationship between price and quantity of rice produced. [1]

ANNEX B

NJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Vectors	$\alpha=-2$; $\left\{\mathbf{w}:\mathbf{w}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix},\lambda\in\mathbb{R}\right\}$
2	Sigma Notation and Method of Difference	3; $7^{-2n}(8n-7)(7-6n)$; $e^{22}-e^{11}-11e^{2}+11e$
3	Differentiation & Applications	-0.337 radians per minute; $h=-\frac{\sqrt{3}\theta^2}{2}+2\theta+\sqrt{3}$
4	AP and GP	$30+110\left(1-\left(\frac{10}{11}\right)^n\right)-\frac{n}{2000}(201-n)$; 59
5	Complex numbers	$w=3-\mathrm{i},\ z=1+\mathrm{i}$; $w=-1-\mathrm{i},\ z=1-3\mathrm{i}$; $\frac{1}{p^2}=\frac{1}{a^2+b^2}e^{\mathrm{i}\left(\frac{4\pi}{9}\right)}$; $\frac{\sqrt{3}}{2x}-\frac{1}{2}-\frac{1}{2\sqrt{3}}x$
6	Graphs and Transformation	$m\le 1$
7	Application of Integration	$\frac{e^x(\cos 2x+2\sin 2x)}{5}+c$; x-coordinate of the minimum point at is $\frac{\pi}{2}-e^{\frac{\pi}{2}}$; $-\frac{3}{2}\pi-\frac{6}{5}+\frac{6}{5}e^{\pi}$
8	Vectors	$-(\mathbf{a}+\mathbf{b})$; $\frac{\mathbf{b}-\mathbf{a}}{2}$; lines are skew
9	Application of Integration	$\frac{x^2}{4}-\frac{y^2}{9}=1$;

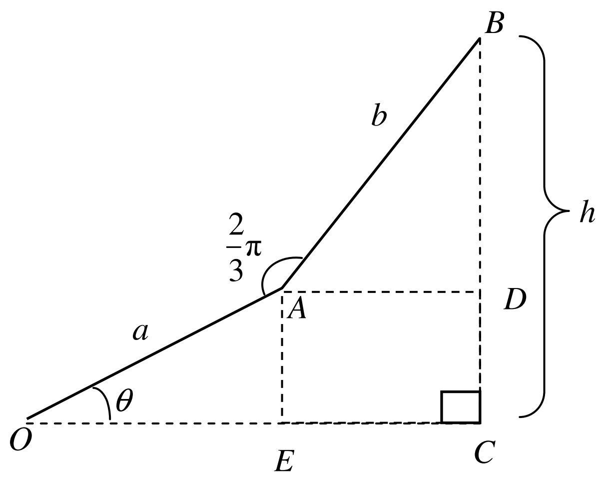
		$9.42(3\text{ sf})$
10	Differential Equations	$y = x\tan(\ln x + c)$; $C_1 : Q = \frac{4}{P}; C_2 : Q = \frac{p}{4}$; $P = \sqrt{16 - 7e^{-0.961t}}$
11	Q11 Topic	
12	Q12 Topic	
13	Q13 Topic	

H2 Mathematics 2017 Prelim Exam Paper Solution

1	**(i)** Method 1 **p.q = 0** $\mathbf{p.q} = 0$ $\begin{pmatrix}2\\\alpha\\1\end{pmatrix}.\begin{pmatrix}\alpha\\1\\6\end{pmatrix}=0$ $2\alpha+\alpha+6=0$ $\alpha=-2$ Method 2 (for marking reference) Let $\mathbf{w}=\begin{pmatrix}x\\y\\z\end{pmatrix}$. $\mathbf{w}\times\mathbf{p}=\mathbf{q}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}2\\\alpha\\1\end{pmatrix}=\begin{pmatrix}\alpha\\1\\6\end{pmatrix}$ $\begin{pmatrix}y-\alpha z\\2z-x\\\alpha x-2y\end{pmatrix}=\begin{pmatrix}\alpha\\1\\6\end{pmatrix}$ Thus, $y-\alpha z=\alpha$ ------(1) $2z-x=1$ ------(2) $\alpha x-2y=6$ ------(3) $(2)\times\alpha+(3):$ $2\alpha z-2y=\alpha+6$ $\Rightarrow 2(\alpha z-y)=\alpha+6$ $\Rightarrow 2(-\alpha)=\alpha+6$ (from (1)) $\Rightarrow \alpha=-2$ **(ii)**

	Let $\mathbf{w}=\begin{pmatrix}x\\y\\z\end{pmatrix}$. $\mathbf{w}\times\mathbf{p}=\mathbf{q}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}-2\\1\\6\end{pmatrix}$ $\begin{pmatrix}y+2z\\2z-x\\-2x-2y\end{pmatrix}=\begin{pmatrix}-2\\1\\6\end{pmatrix}$ $y+2z=-2$ ------(1) $2z-x=1$ ------(2) $-2x-2y=6$ ------(3) Let $z=\lambda, \lambda\in \cdots$: From (2): $x=-1+2\lambda$ From (1): $y+2\lambda=-2 \Rightarrow y=-2-2\lambda$ Thus, $\mathbf{w}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1+2\lambda\\-2-2\lambda\\\lambda\end{pmatrix}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}$, $\lambda\in\cdots$, which is the vector equation of the straight line. The set of vectors is $\left\{\mathbf{w}:\mathbf{w}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix},\lambda\in\cdots\right\}$
2	**(a) (i)** By GC, sum to infinity is 3. **(a) (ii)**

	$u_n = S_n - S_{n-1}$ $= 3 + 7^{-2n}\left(n^2\right) - \left[3 + 7^{-2(n-1)}\left(n-1\right)^2\right]$ $= 3 - 3 + 7^{-2n}\left(n^2\right) - 7^{-2n+2}\left(n^2 - 2n + 1\right)$ $= 7^{-2n}\left(n^2 - 49n^2 + 98n - 49\right)$ $= 7^{-2n}\left(-48n^2 + 98n - 49\right)$ $= 7^{-2n}\left(8n-7\right)\left(7-6n\right)$ where $\mathrm{g}\left(n\right) = -48n^2 + 98n - 49$ **(b)** $\sum_{r=1}^{n}\left(\int_0^r \mathrm{e}^x - \mathrm{e}^{x-1}\mathrm{d}x\right)$ $= \sum_{r=1}^{n}\left[\mathrm{e}^x - \mathrm{e}^{x-1}\right]_0^r$ $= \sum_{r=1}^{n}\left(\mathrm{e}^r - \mathrm{e}^{r-1} - \mathrm{e}^0 + \mathrm{e}^{-1}\right)$ $= \mathrm{e}^1 - \mathrm{e}^0 - \mathrm{e}^0 + \mathrm{e}^{-1}$ $+\mathrm{e}^2 - \mathrm{e}^1 - \mathrm{e}^0 + \mathrm{e}^{-1}$ $+\mathrm{e}^3 - \mathrm{e}^2 - \mathrm{e}^0 + \mathrm{e}^{-1}$ ... $+\mathrm{e}^n - \mathrm{e}^{n-1} - \mathrm{e}^0 + \mathrm{e}^{-1}$ $= \mathrm{e}^n - 1 - n(1) + n\mathrm{e}^{-1}$ $= \mathrm{e}^n + n\mathrm{e}^{-1} - \left(n+1\right)$ $\sum_{r=10}^{20}\left(\int_0^r \mathrm{e}^{x+2} - \mathrm{e}^{x+1}\ \mathrm{d}x\right)$ $= \mathrm{e}^2\sum_{r=1}^{20}\left(\int_0^r \mathrm{e}^x - \mathrm{e}^{x-1}\ \mathrm{d}x\right) - \mathrm{e}^2\sum_{r=1}^{9}\left(\int_0^r \mathrm{e}^x - \mathrm{e}^{x-1}\ \mathrm{d}x\right)$ $= \mathrm{e}^2\left[\mathrm{e}^{20} + 20\mathrm{e}^{-1} - \left(20+1\right) - \left(\mathrm{e}^9 + 9\mathrm{e}^{-1} - 10\right)\right]$ $= \mathrm{e}^{22} - \mathrm{e}^{11} - 11\mathrm{e}^2 + 11\mathrm{e}$
3	<u>Method 1</u>



$h = BD + DC,\ DC = a\sin\theta,\ BD = b\sin\angle BAD$

$\angle BAD + \angle DAE + \angle OAE = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$

$\Rightarrow \angle BAD + \frac{\pi}{2} + \left(\frac{\pi}{2} - \theta\right) = \frac{4\pi}{3}$

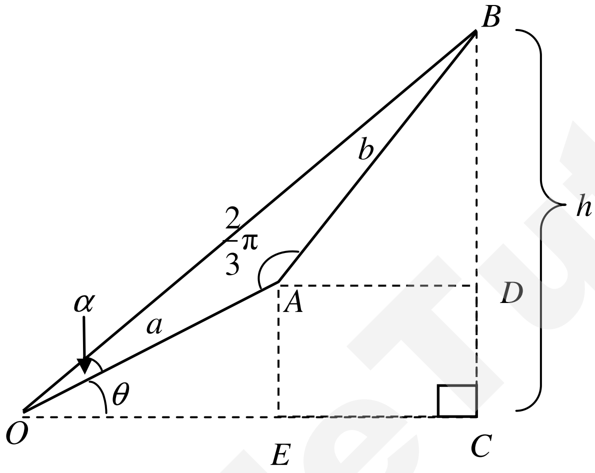
$\Rightarrow \angle BAD = \theta + \frac{\pi}{3}$

$\Rightarrow BD = b\sin\left(\theta + \frac{\pi}{3}\right)$

$\therefore h = a\sin\theta + b\sin\left(\theta + \frac{\pi}{3}\right)$

Using $\sin(A+B) = \sin A\cos B + \cos A\sin B$, we have

$$\begin{aligned} h &= a\sin\theta + b\sin\left(\theta + \frac{\pi}{3}\right) \\ &= a\sin\theta + b\sin\theta\cos\left(\frac{\pi}{3}\right) + b\cos\theta\sin\left(\frac{\pi}{3}\right) \\ &= \left(a + \frac{b}{2}\right)\sin\theta + \frac{b\sqrt{3}}{2}\cos\theta \\ &= R\sin(\theta + \alpha), \end{aligned}$$

	where $R=\sqrt{\left(a+\frac{b}{2}\right)^2+\left(\frac{\sqrt{3}}{2}b\right)^2}=\sqrt{a^2+ab+\frac{b^2}{4}+\frac{3b^2}{4}}=\sqrt{a^2+ab+b^2}$ $\tan\alpha=\dfrac{\frac{\sqrt{3}}{2}b}{a+\frac{b}{2}}=\dfrac{\sqrt{3}b}{2a+b}\Rightarrow\alpha=\tan^{-1}\left(\dfrac{\sqrt{3}b}{2a+b}\right)$ $\therefore h=\sqrt{a^2+ab+b^2}\sin\left[\theta+\tan^{-1}\left(\dfrac{\sqrt{3}b}{2a+b}\right)\right]$ Method 2  $\sin(\theta+\alpha)=\dfrac{h}{OB}$ $OB^2=a^2+b^2-2ab\cos\dfrac{2\pi}{3}$ $OB=\sqrt{a^2+b^2+ab}$ $\dfrac{\sin\alpha}{b}=\dfrac{\sin\frac{2\pi}{3}}{\sqrt{a^2+b^2+ab}}$ $\sin\alpha=\dfrac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}$ $\alpha=\sin^{-1}\dfrac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}$

$$h=\sqrt{a^2+ab+b^2}\sin\left(\theta+\sin^{-1}\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}\right)$$

(ii)

Since $a=1, b=2, \alpha=\tan^{-1}\left(\frac{2\sqrt{3}}{2+2}\right)=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$$h=\sqrt{7}\sin\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right], \frac{\mathrm{d}h}{\mathrm{d}\theta}=\sqrt{7}\cos\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

At $\theta=\frac{\pi}{12}$,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{\sqrt{7}\cos\left[\frac{\pi}{12}+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]}\times(-0.5)$$

$=-0.337$ radians per minute

(iii)

$$\alpha=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\Rightarrow\sin\alpha=\sqrt{\frac{3}{7}}, \cos\alpha=\frac{2}{\sqrt{7}}$$

$$h=\left(a+\frac{b}{2}\right)\sin\theta+\frac{b\sqrt{3}}{2}\cos\theta$$

If θ is small,

$$\begin{aligned}h&=\sqrt{7}\sin(\theta+\alpha)\\&=\sqrt{7}\sin\theta\cos\alpha+\sqrt{7}\cos\theta\sin\alpha\\&\approx\sqrt{7}\theta\left(\frac{2}{\sqrt{7}}\right)+\sqrt{7}\left(1-\frac{\theta^2}{2}\right)\left(\sqrt{\frac{3}{7}}\right)\\&=2\theta+\sqrt{3}-\frac{\sqrt{3}\theta^2}{2}\\&=-\frac{\sqrt{3}\theta^2}{2}+2\theta+\sqrt{3}\end{aligned}$$

4 **(i)**

Stretch count, n	Length of string before stretch	Elongation after stretch, u_n	Contraction after stretch, t_n	Final length of string
1	30	10	0.1	$30+10-0.1=39.9$

2	39.9	$10\left(\frac{10}{11}\right)$	$0.1-0.001$ $=0.099$	$39.9+10\left(\frac{10}{11}\right)-0.099$ $=48.8919$

$30+10-0.1+10\left(\frac{10}{11}\right)-0.099$

$=48.8919$

$=48.892\ (3\text{ dp})$

(ii)

Total length of string

$=30+u_1+u_2+...+u_n-(t_1+t_2+...+t_n)$

$=30+10+10\left(\frac{10}{11}\right)+...+10\left(\frac{10}{11}\right)^{n-1}$

$-\left(0.1+(0.1-0.001(1))+...+(0.1-0.001(n))\right)$

$$\sum_{i=1}^{n}u_i=\frac{10\left[1-\left(\frac{10}{11}\right)^n\right]}{1-\frac{10}{11}}=110\left(1-\left(\frac{10}{11}\right)^n\right)$$

$$\sum_{i=1}^{n}t_i=\frac{n}{2}\left[2(0.1)+(n-1)(-0.001)\right]=\frac{n}{2000}(201-n)$$

Length of string after n stretches

$$=30+110\left(1-\left(\frac{10}{11}\right)^n\right)-\frac{n}{2000}(201-n)$$

(iii)

$t_n>u_n$

$0.1+(n-1)(-0.001)>(10)\left(\frac{10}{11}\right)^{n-1}$

$0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}>0$

Using GC,

when $n=58$, $0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}=-7.1364\times10^{-4}$

when $n=59$, $0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}=0.00226$

Therefore, the minimum number of stretches is 59.

(iv)

$$S_\infty = 30 + \frac{10}{1-\frac{10}{11}} = 140 \quad (\text{since } 0 < r < 1)$$

Since the sum to infinity, S_∞ is 140, it is impossible for the string to be stretched beyond 140cm.

OR

The theoretical maximum is 140 cm so it is impossible for the strong to be stretched beyond 140 cm.

5 **(a)**

$= 2$ ------(1)

$2+4\text{i}$ ------(2)

From (1),

$$z = \frac{2-w}{\text{i}} = -\text{i}(2-w) \text{ ------(3)}$$

Substitute (3) into (2) and let $w = x + \text{i}y$:

$$-\text{i}(2-w)w^* = 2+4\text{i}$$
$$-\text{i}\left(2w^* - ww^*\right) = 2+4\text{i}$$
$$-\text{i}\left[2(x-\text{i}y)-\left(x^2+y^2\right)\right] = 2+4\text{i}$$
$$-2y-\text{i}\left(2x-x^2-y^2\right) = 2+4\text{i}$$

Comparing real and imaginary parts,

$$-2y = 2 \Rightarrow y = -1$$
$$-2x + x^2 + y^2 = 4$$
$$\Rightarrow -2x + x^2 + (-1)^2 = 4$$
$$\Rightarrow x^2 - 2x - 3 = 0$$
$$\Rightarrow (x-3)(x+1) = 0$$
$$\Rightarrow x = 3 \text{ or } x = -1$$

$\therefore w = 3 - \text{i}$ or $w = -1 - \text{i}$

If $w = 3 - \text{i}$, $z = -\text{i}\left(2-(3-\text{i})\right) = 1 + \text{i}$.

If $w = -1 - \text{i}$, $z = -\text{i}\left(2-(-1-\text{i})\right) = 1 - 3\text{i}$.

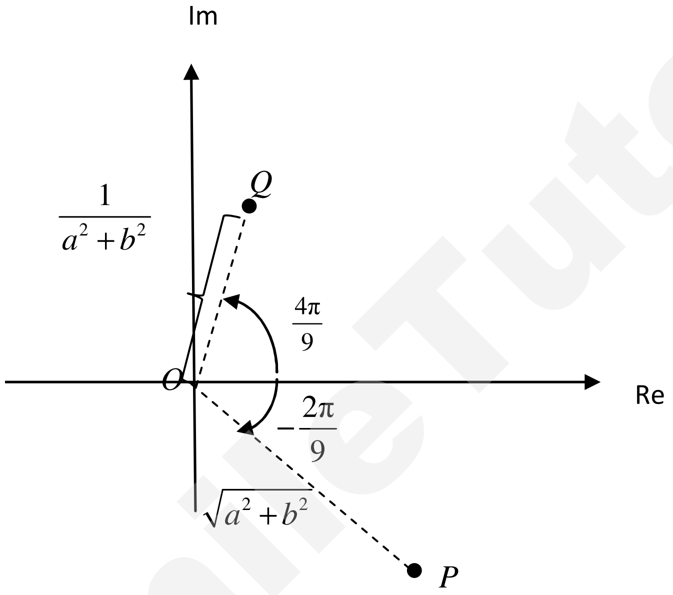
(b)(i)

$$\left|\frac{1}{p^2}\right| = \frac{1}{|p|^2} = \frac{1}{\left(\sqrt{a^2+b^2}\right)^2} = \frac{1}{a^2+b^2}$$

$$\arg\left(\frac{1}{p^2}\right) = -2\arg(p) = -2\left(\frac{-2\pi}{9}\right) = \frac{4\pi}{9}$$

$$\therefore \frac{1}{p^2} = \frac{1}{a^2+b^2}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{9}\right)}$$

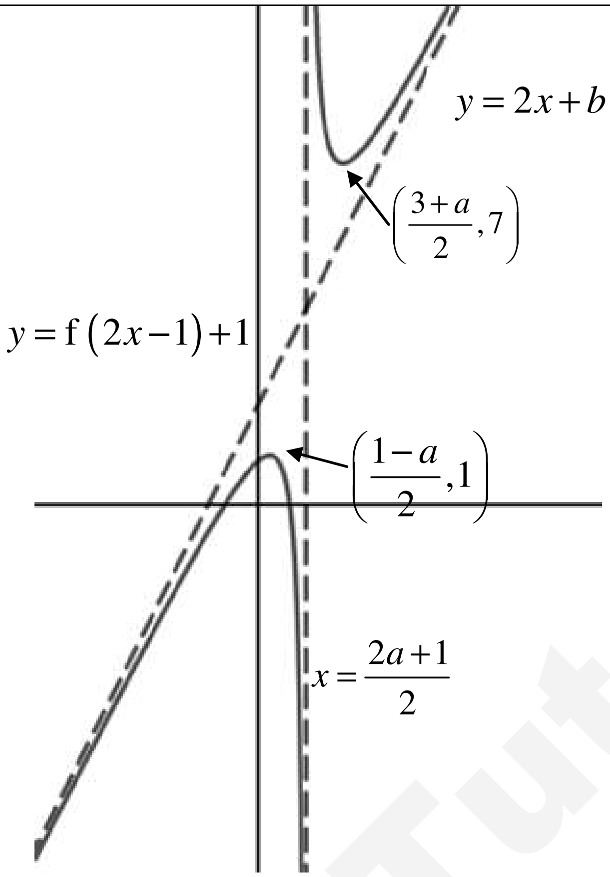
(b)(ii)



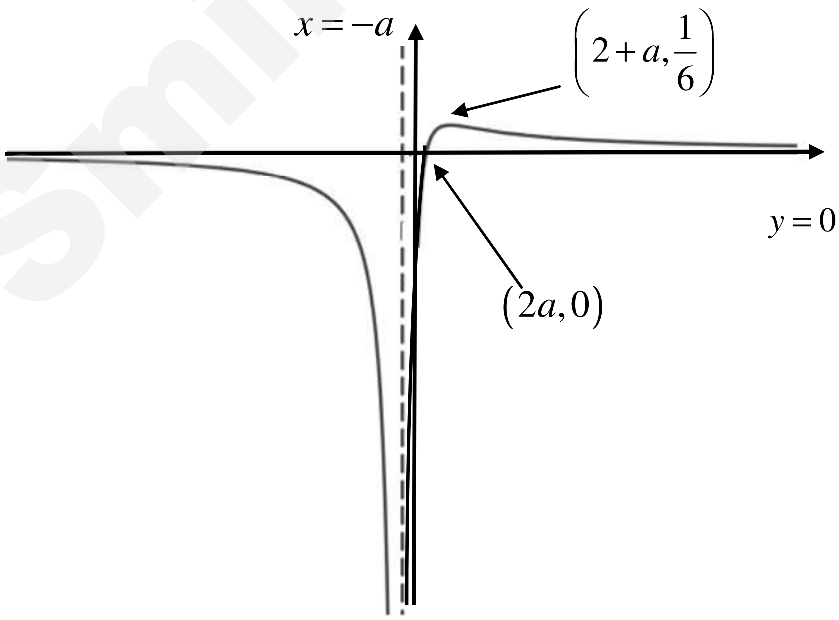
(b)(iii)

Given $\angle OPQ = \alpha$, $\dfrac{\sin\alpha}{1/\left(a^2+b^2\right)} = \dfrac{\sin\left(\frac{\pi}{3}-\alpha\right)}{\sqrt{a^2+b^2}}$

	$\left(a^2+b^2\right)^{\frac{3}{2}}=\dfrac{\sqrt{3}\cos\alpha-\sin\alpha}{2\sin\alpha}$ $\approx\dfrac{\sqrt{3}\left(1-\frac{x^2}{2}\right)-\left(x-\frac{x^3}{6}\right)}{2\left(x-\frac{x^3}{6}\right)}$ $=\frac{1}{2}\left[\frac{1}{x}\left(\sqrt{3}-x-\frac{\sqrt{3}x^2}{2}+\frac{x^3}{6}\right)\left(1-\frac{x^2}{6}\right)^{-1}\right]$ $\approx\frac{1}{2}\left[\frac{1}{x}\left(\sqrt{3}-x-\frac{\sqrt{3}x^2}{2}+\frac{x^3}{6}\right)\left(1+(-1)\left(-\frac{x^2}{6}\right)\right)\right]$ $=\frac{1}{2}\left[\left(\frac{\sqrt{3}}{x}-1-\frac{\sqrt{3}x}{2}+\frac{x^2}{6}\right)\left(1+\frac{x^2}{6}\right)\right]$ $=\frac{1}{2}\left[\frac{\sqrt{3}}{x}+\frac{\sqrt{3}}{6}x-1-\frac{x^2}{6}-\frac{\sqrt{3}x}{2}+\frac{x^2}{6}\right]$ $=\frac{1}{2}\left[\frac{\sqrt{3}}{x}-\frac{2\sqrt{3}}{6}x-1\right]$ $=\frac{\sqrt{3}}{2x}-\frac{1}{2}-\frac{1}{2\sqrt{3}}x$
6	**(i)**



(ii)



(iii)

Point P is $(2a, 2a+b)$

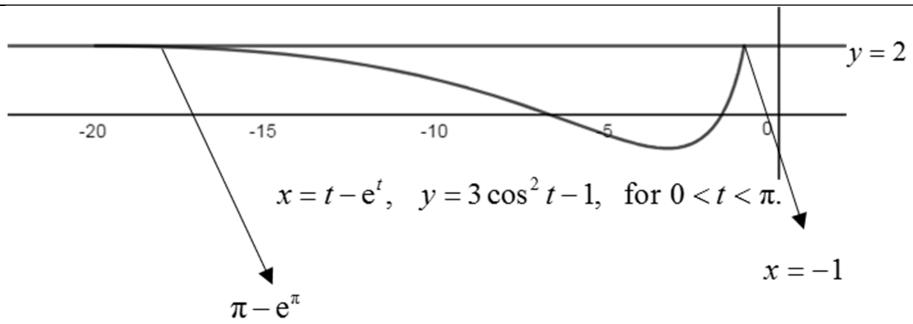
	$\dfrac{y-(2a+b)}{x-2a}=m \Rightarrow y=mx-2am+2a+b$ Hence, P lies on the line $y=mx+(b+2a-2am)$ for $m\in \cdots$. From the graph, $m\le 1$ for the line not to cut $y=\mathrm{f}(x)$.
7	**(a)** $\int e^x \cos 2x\,dx = e^x\cos 2x + 2\int e^x \sin 2x\,dx$ $= e^x\cos 2x + 2\left(e^x \sin 2x - 2\int e^x\cos 2x\,dx\right)$ $5\int e^x\cos 2x\,dx = e^x\cos 2x + 2e^x\sin 2x$ $\therefore \int e^x\cos 2x\,dx = \dfrac{e^x(\cos 2x+2\sin 2x)}{5}+c$ **(b)(i)** $x=t-e^t,\quad y=3\cos^2 t-1$ $\dfrac{dx}{dt}=1-e^t,\ \dfrac{dy}{dt}=6\cos t(-\sin t)=-3\sin(2t)$ $\dfrac{dy}{dx}=\dfrac{-3\sin(2t)}{1-e^t}$ $\dfrac{dy}{dx}=0 \Rightarrow \sin(2t)=0$ $\Rightarrow 2t=0$ (N.A.) or $2t=\pi$ or $2t=2\pi$(N.A.) $\Rightarrow t=\dfrac{\pi}{2}$

t	1.6	$\dfrac{\pi}{2}$	1.5
x	-3.35303	-3.2396811	-2.981689
$\dfrac{dy}{dx}$	-0.0443	0	0.122

NB: t increases as x decreases.

Hence x-coordinate of the minimum point at is $\dfrac{\pi}{2}-e^{\frac{\pi}{2}}$.

(b)(ii)



When $y=2$,

$2=3\cos^2 t-1$

$\Rightarrow \cos t=\pm 1$

$\Rightarrow t=0,\pi$

When $t=0, x=0-\mathrm{e}^0=-1$

When $t=\pi, x=\pi-\mathrm{e}^{\pi}=-19.9991$

Area required

$=\int_{\pi-\mathrm{e}^{\pi}}^{-1}(2-y)\,\mathrm{d}x$

$=\int_{\pi}^{0}\left(2-\left(3\cos^2 t-1\right)\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$

$=\int_{\pi}^{0}\left(3-3\cos^2 t\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$

$=3\int_{\pi}^{0}\left(1-\cos^2 t\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$

$=3\int_{\pi}^{0}\left(1-\frac{\cos 2t+1}{2}\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$

$=3\int_{\pi}^{0}\left(\frac{1}{2}-\frac{\cos 2t}{2}\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$

$=\frac{3}{2}\int_{\pi}^{0}\left(1-\cos 2t\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$

$=\frac{3}{2}\int_{\pi}^{0}\left(1-\cos 2t-\mathrm{e}^t+\mathrm{e}^t\cos 2t\right)\mathrm{d}t$

$=\frac{3}{2}\left[t-\frac{\sin 2t}{2}-\mathrm{e}^t+\frac{\mathrm{e}^t\left(\cos 2t+2\sin 2t\right)}{5}\right]_{\pi}^{0}$

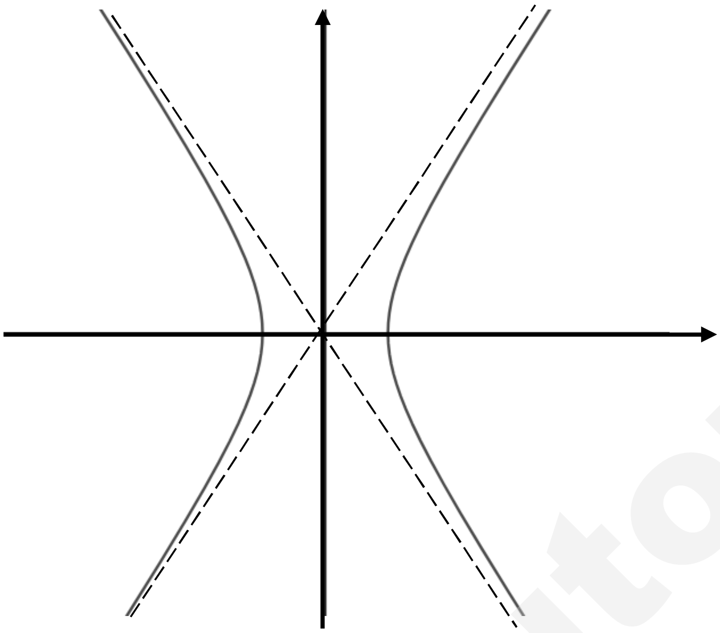
$=\frac{3}{2}\left[-\frac{4}{5}-\pi+\frac{4e^{\pi}}{5}\right]$

$=-\frac{3}{2}\pi-\frac{6}{5}+\frac{6}{5}\mathrm{e}^{\pi}$, where $a=-\frac{3}{2},\ b=-\frac{6}{5},\ c=\frac{6}{5}$

8	**(i)** $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=-\mathbf{a}-\mathbf{b}=-(\mathbf{a}+\mathbf{b})$ **(ii)** Since ΔOCD is similar to ΔACB, OD parallel to AB. $\frac{OD}{AB}=\frac{CO}{CA}=\frac{1}{2}$. $\overrightarrow{OD}=\frac{1(-\mathbf{a})+1(\mathbf{b})}{2}=\frac{\mathbf{b}-\mathbf{a}}{2}$ **(iii)** Let $\overrightarrow{OB}=\begin{pmatrix}x\\0\\z\end{pmatrix}$. $\overrightarrow{CB}=\begin{pmatrix}x\\0\\z\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}x-2\\0\\z\end{pmatrix}$ $\overrightarrow{CB}.\overrightarrow{CO}=\begin{pmatrix}x-2\\0\\z\end{pmatrix}.\begin{pmatrix}-2\\0\\0\end{pmatrix}=\left\|\overrightarrow{CB}\right\|(2)\cos\frac{\pi}{6}---(1)$ $\left\|\overrightarrow{CB}\right\|=4\cos\frac{\pi}{6}=2\sqrt{3}$ $-2x+4=2\sqrt{3}(2)\frac{\sqrt{3}}{2}$ $x=-1$ $\left\|\overrightarrow{OB}\right\|=\left\|\begin{pmatrix}-1\\0\\z\end{pmatrix}\right\|=2$ $(-1)^2+z^2=2^2$ $z^2=3$ $z=\pm\sqrt{3}$ $\overrightarrow{OB}=\begin{pmatrix}-1\\0\\\sqrt{3}\end{pmatrix}$ or $\begin{pmatrix}-1\\0\\-\sqrt{3}\end{pmatrix}$ (rejected $\because$ z-component >0).

	(iv) Equation of line passing through *OB*: $\overrightarrow{OB} = \lambda \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix},\ \lambda \in \cdots$ $\frac{x-2}{3} = \mu \Rightarrow x = 2+3\mu$ $\frac{y}{3} = \mu \Rightarrow y = 3\mu$ $z-1 = \mu \Rightarrow z = \mu+1$ Equation of line: $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}, \mu \in \cdots$ Direction vector of line is not parallel to direction vector of line passing through *O* and *B* since direction vectors of both lines are not scalar multiple of each other. Solving equations simultaneously: $\begin{pmatrix} 2+3\mu \\ 3\mu \\ \mu+1 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}$ There is no value of λ and μ that satisfy the above equation. Since the lines are not parallel and non-intersecting, the lines are skew.
9	**(i)** $x = 2\sec t$ and $y = 3\tan t$ $1+\tan^2 t = \sec^2 t$ $\Rightarrow 1+\frac{y^2}{9} = \frac{x^2}{4}$ $\Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1$

(ii)



(iii)

Method 1

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=3\sec^2 t\cdot\frac{1}{2\sec t\tan t}=1.5\operatorname{cosec} t$$

$1.5\operatorname{cosec} t=\sqrt{3}$

$t=\frac{\pi}{3}$ or $\frac{2\pi}{3}$

When $t=\frac{\pi}{3}$,

$x=2\sec\frac{\pi}{3}=4$, $y=3\tan\frac{\pi}{3}=3\sqrt{3}$.

Equation of tangent:

$y-3\sqrt{3}=\sqrt{3}(x-4)$

$\Rightarrow y=\sqrt{3}x-4\sqrt{3}+3\sqrt{3}$

$\Rightarrow y=\sqrt{3}x-\sqrt{3}$

$\therefore k=-\sqrt{3}$ (Shown)

When $t = \dfrac{2\pi}{3}$,

$x = 2\sec\dfrac{2\pi}{3} = -4$, $y = 3\tan\dfrac{2\pi}{3} = -3\sqrt{3}.$

Equation of tangent:

$y + 3\sqrt{3} = \sqrt{3}(x+4)$
$\Rightarrow y = \sqrt{3}x + 4\sqrt{3} - 3\sqrt{3}$
$\Rightarrow y = \sqrt{3}x + \sqrt{3}$
$\therefore k = \sqrt{3}$ (N.A. $\because k < 0$)

Method 2

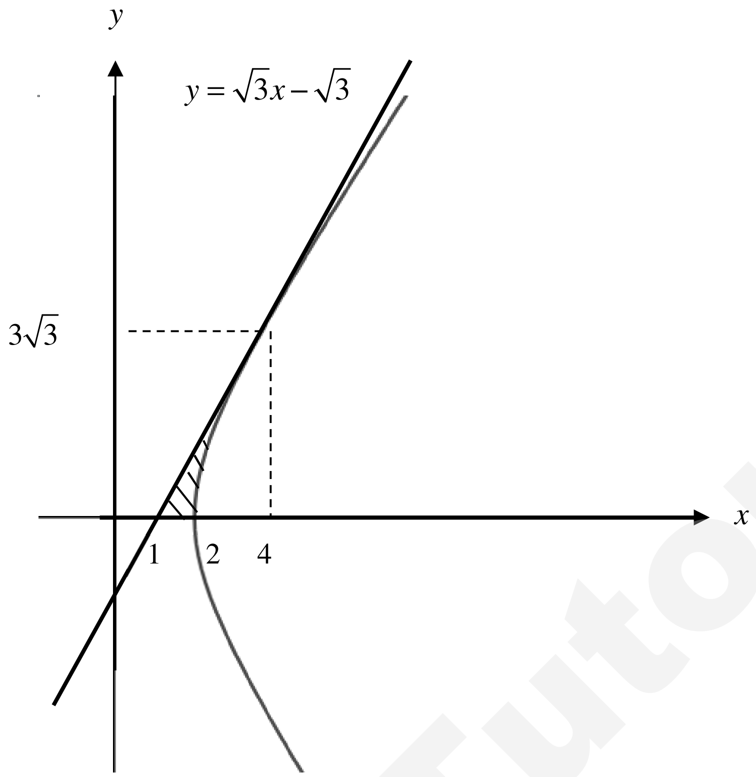
$$\frac{x^2}{4} - \frac{\left(\sqrt{3}x + k\right)^2}{9} = 1$$

$\Rightarrow -3x^2 - \left(8\sqrt{3}k\right)x - \left(36 + 4k^2\right) = 0$

Since the line $y = \sqrt{3}x + k$, where $k < 0$, is a tangent to C, there should be repeated roots.

Thus,

$\left(8\sqrt{3}k\right)^2 - 4(-3)\left(-36 - 4k^2\right) = 0$
$\Rightarrow 192k^2 - 432 - 48k^2 = 0$
$\Rightarrow 144k^2 = 432$
$\Rightarrow k^2 = 3$
$\Rightarrow k = \sqrt{3}$ (N.A. $\because k < 0$) or $k = -\sqrt{3}$ (Shown)



For $k=-\sqrt{3}$,

$-3x^2-\left(8\sqrt{3}\left(-\sqrt{3}\right)\right)x-\left(36+4(3)\right)=0$

$3x^2-24x+48=0$

$x^2-8x+16=0$

$(x-4)^2=0$

$\Rightarrow x=4$

$y^2=9\left(\frac{x^2}{4}-1\right)$

When $x=4$, $y=\sqrt{27}=3\sqrt{3}\ (y>0)$

$=\frac{1}{3}\pi\left(3\sqrt{3}\right)^2(3)-\pi\int_2^4 y^2\ \mathrm{d}x$

$=27\pi-9\pi\int_2^4\left(\frac{x^2}{4}-1\right)\ \mathrm{d}x$

$=27\pi-9\pi\left(\frac{8}{3}\right)$

$=3\pi$

$=9.42\ (3\text{ sf})$

10	**(a)**

$u=\dfrac{y}{x}\Rightarrow y=ux$, $\dfrac{dy}{dx}=\dfrac{y^2+xy+x^2}{x^2}=\left(\dfrac{y}{x}\right)^2+\dfrac{y}{x}+1$ ---(1)

$\dfrac{dy}{dx}=\dfrac{du}{dx}x+u$ -- (2)

Sub (2) into (1):

$\dfrac{du}{dx}x+u=u^2+u+1\Rightarrow\dfrac{du}{dx}x=u^2+1$

$\dfrac{1}{u^2+1}\dfrac{du}{dx}=\dfrac{1}{x}$

$\displaystyle\int\frac{1}{u^2+1}\,du=\int\frac{1}{x}\,dx$

$\tan^{-1}u=\ln|x|+c$, where c is an arbitrary constant.

$\tan^{-1}u=\ln x+c$ (since $x>0$)

$u=\tan(\ln x+c)$

$y=x\tan(\ln x+c)$

(b)(i)

Point A shows that at 4 dollars per kg, 1 tonne of rice is produced and all of it is bought by the consumers.

This is the equilibrium point where the price is 4 dollars per kg and the quantity produced/consumed is 1 tonne.

(b)(ii)

$C_1 : Q=\dfrac{k_1}{P}$

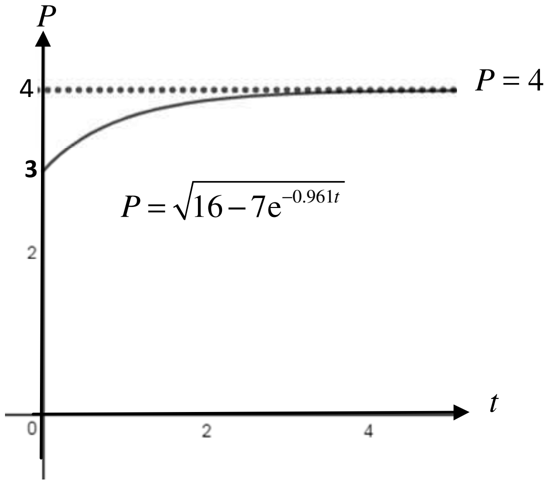
$C_2 : Q=k_2P$

When $Q=1, P=4$,

$k_1=4, k_2=\dfrac{1}{4}$.

$C_1 : Q=\dfrac{4}{P}; C_2 : Q=\dfrac{P}{4}$

	Hence, $C_1 : Q = \dfrac{4}{P}; C_2 : Q = \dfrac{p}{4}$. **(b)(iii)** $\dfrac{dP}{dt} = k_3\left(\dfrac{4}{P} - \dfrac{P}{4}\right)$ $\dfrac{dP}{dt} = k_3\left(\dfrac{16-P^2}{4P}\right)$ $\int \dfrac{4P}{16-P^2}\,dP = \int k_3\,dt$ $-2\int \dfrac{-2P}{16-P^2}\,dP = \int k_3\,dt$ $-2\ln\left\|16-P^2\right\| = k_3t + c$ $\ln\left\|16-P^2\right\| = \dfrac{-k_3}{2}t + \dfrac{-c}{2}$ $\left\|16-P^2\right\| = e^{\frac{-k_3}{2}t + \frac{-c}{2}}$ $16-P^2 = \left(\pm e^{\frac{-c}{2}}\right)e^{\frac{-k_3}{2}t}$ $16-P^2 = Ae^{Bt}, A = \pm e^{\frac{-c}{2}}, B = \dfrac{-k_3}{2}$ $\sqrt{16-Ae^{Bt}} = P \ \left(P>0\right)$ When $t=0, P=3$: $\sqrt{16-Ae^{B(0)}} = 3$ $16 - A = 3^2$ $A = 7$ When $t=1, P=3.65$: $\sqrt{16-7e^{B}} = 3.65$ $B = \ln\dfrac{16-3.65^2}{7} = -0.96102663 = -0.961$ $\therefore P = \sqrt{16-7e^{-0.961t}}$



Rice production will only occur if the price is able to at least cover the initial cost of investment.

NJC Paper 2

1 There are 3 bike-sharing companies in the current market. For each ride, α- bike charges a certain amount per 5 min block or part thereof, β- bike charges a certain amount per 10 min block or part thereof and μ- bike charges a certain amount per 15 min block or part thereof. Rebecca rode each of the bike-sharing companies' bikes once in each month. The table below shows the amount of time Rebecca clocked for each ride and her total spending for each month. In celebration of the company's first anniversary, the pricings in February and March 2017 of μ- bikes are a 5% discount off the immediate previous month's pricing.

	January 2017	**February 2017**	**March 2017**
α- bike	25 min	17 min	36 min
β- bike	30 min	10 min	39 min
μ- bike	15 min	44 min	33 min
Total spending	\$5.70	\$5.72	\$9.71

Determine which bike-sharing company offers the cheapest rate (without any discount) for a 40-min ride. Justify your answer clearly. [4]

2 A function f is said to self-inverse if $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all x in the domain of f.

The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{7-3x}{3-x}, \qquad x \in \mathbb{R}, x \neq 3,$$
$$\mathrm{g}: x \mapsto |(2-x)(1+x)|, \qquad x \in \mathbb{R}, x \in (-\infty, -1].$$

(i) Explain why f^{-1} exists and show that f is self-inverse. Hence, or otherwise, evaluate $\mathrm{f}^{2003}(5)$. [4]

(ii) Find an expression for $\mathrm{g}^{-1}(x)$. [3]

(iii) Sketch, on the same diagram, the graphs of $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$, illustrating clearly the relationship between the two graphs, and labelling the axial intercept(s), if any. Write down the set of values of x that satisfies the equation $\mathrm{g}\mathrm{g}^{-1}(x) = x$. [3]

(iv) Show that $\mathrm{f}\mathrm{g}^{-1}$ exists. Find the exact range of $\mathrm{f}\mathrm{g}^{-1}$. [3]

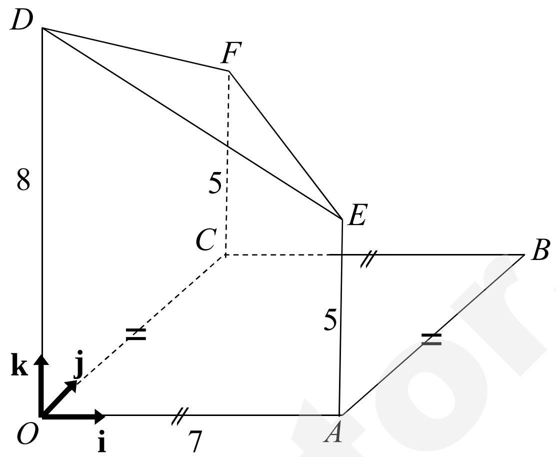
3 Using differentiation, find the Maclaurin's series of $\dfrac{\mathrm{e}^{2x}}{1+x^2}$, in ascending powers of x up to and including x^3. [6]

Let $\mathrm{h}(x) = \dfrac{\mathrm{e}^{2x}}{1+x^2}$ and the cubic polynomial obtained above be $\mathrm{f}(x)$.

Find, for $-2 \leq x \leq 2$, the set of values of x for which the value of $\mathrm{f}(x)$ is within ± 0.5 of the value of $\mathrm{h}(x)$. [3]

4 The diagram (not drawn to scale) shows the structure of a partially constructed building that is built on a horizontal ground. The building has a square base foundation of 7 m in length. Points O, A, B and C are the corners of the foundation of the building. The building currently consists of three vertical pillars OD, AE and CF of heights 8 m, 5 m and 5 m respectively.

A canvas is currently attached at D, E and F, forming a temporary shelter for the building. O is taken as the origin and vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, each of length 1 m, are taken along OA, OC and OD respectively.



(i) Find a Cartesian equation of the plane that represents the canvas DEF. [3]

(ii) Find the acute angle which the canvas DEF makes with the horizontal ground. [2]

(iii) Given that the canvas is to be extended along the plane DEF till it touches the horizontal ground, explain why point B will lie beneath the canvas. [2]

A cement roof is to be built to replace the extended canvas. A vertical partition wall is also to be built such that it is d m away from and parallel to the plane $ODFC$, where $0 < d < 7$.

(iv) Find the exact vector equation of the line where the roof meets the partition wall. Show your working clearly, leaving your answer in terms of d. [4]

(v) A lighting point, P, is to be placed on the roof such that it is closest to B. Find the position vector of P. [3]

5 A delegation of four students is to be selected from five badminton players, m floorball players, where $m > 3$, and six swimmers to attend the opening ceremony of the 2017 National Games. A pair of twins is among the floorball players. The delegation is to consist of at least one player from each sport.

(i) Show that the number of ways to select the delegation in which neither of the twins is selected is $k(m-2)(m+6)$, where k is an integer to be determined. [3]

(ii) Given that the number of ways to select a delegation in which neither of the twins is selected is more than twice the number of ways to select a delegation which includes exactly one of the twins, find the least value of m. [2]

The pair of twins, one badminton player, one swimmer and two teachers, have been selected to attend a welcome lunch at the opening ceremony. Find the number of ways in which the group can be seated at a round table with distinguishable seats if the pair of twins is to be seated together and the teachers are separated. [3]

6 In the fishery sciences, researchers often need to determine the length of a fish as a function of its age. The table below shows the average length, L inches, at age, t years, of a kind of fish called the North Sea Sole.

t	1	2	3	4	5	6	7	8
L	3.6	7.5	10.1	11.7	12.7	13.4	14.0	14.4

(i) Draw a scatter diagram of these data, and explain how you know from your diagram that the relationship between L and t should not be modelled by an equation of the form $L = at + b$. [3]

(ii) Which of the formulae $L = a\sqrt{t} + b$ and $L = c\ln t + d$, where a, b, c and d are constants, is the better model for the relationship between L and t? Explain fully how you decided, and find the constants for the better formula. [3]

(iii) Use the formula you chose from part **(ii)** to estimate the average length of a six-month old Sole. Explain whether your estimate is reliable. [2]

A popular approach to determine the average length of a fish as a function of its age is the von Bertalanffy model. The model shows the relationship between the average length that is yet to be grown, G inches, at age, t years. The maximum average length attained by the Sole is 14.8 inches.

(iv) The product moment correlation between L and t is given as r_1 while that between G and t is given as r_2 . State the relationship between r_1 and r_2 . [1]

7 There are three identically shaped balls, numbered from 1 to 3, in a bag. Balls are drawn one by one at random and with replacement. The random variable X is the number of draws needed for any ball to be drawn a second time. The two draws of the same ball do not need to be consecutive.

(i) Show that $\mathrm{P}(X = 4) = \dfrac{2}{9}$ and find the probability distribution of X. [3]

(ii) Show that $\mathrm{E}(X) = \dfrac{26}{9}$ and find the exact value of $\mathrm{Var}(X)$. [3]

(iii) The mean for forty-four independent observations of X is denoted by $\overline{X}$. Using a suitable approximation, find the probability that $\overline{X}$ exceeds 3. [3]

8 Heart rate, also known as pulse, is the number of times a person's heart beats per minute. The normal heart rate of teenagers has a mean of 75 at the resting state.

Obesity is a leading preventable cause of death worldwide. It is most commonly caused by a combination of excessive food intake, lack of physical activity and genetic susceptibility. To examine the effect of obesity on heart rate, 70 obese teenagers are randomly selected and their heart rates h are measured in a resting state. The results are summarised as follows.

$$n = 70 \qquad \sum h = 5411 \qquad \sum h^2 = 426433$$

The Health Promotion Board (HPB) wishes to test whether the mean heart rate for obese teenagers differs from the normal heart rate by carrying out a hypothesis test.

(i) Explain whether HPB should use a 1-tail test or a 2-tail test. [1]

(ii) Explain why HPB is able to carry out a hypothesis test without knowing anything about the distribution and variance of the heart rates. [2]

(iii) Find the unbiased estimates of the population mean and variance, and carry out the test at the 10% level of significance for the HPB. [6]

A researcher wishes to test whether obese teenagers have a **higher** mean heart rate. He finds that the mean heart rate for 80 randomly obese teenagers is 79.4, then carries out a hypothesis test at the 10% level of significance.

(iv) Explain, with justification, how the population variance of the heart rates will affect the conclusion made by the researcher. [3]

(v) Show that the probability of any normal variable lying within one standard deviation from its mean is approximately 0.683. [1]

By considering **(iv)** and **(v)**, explain why it is likely for the researcher to reject the null hypothesis in this test if it is assumed that heart rates follow a normal distribution at the resting state. [1]

9 The number of days of gestation for a Dutch Belted cow is normally distributed, with a mean of μ days and a standard deviation of σ days. 8.08% of this cattle breed has a gestation period shorter than 278 days whereas 21.2% has a gestation period longer than 289 days. Find the values of μ and σ, giving your answers correct to 3 significant figures. [3]

(i) Find the probability that the mean gestation period for thirty-two randomly chosen Dutch Belted cows is more than 287 days. State a necessary assumption for your calculation to be valid. [3]

For another cattle breed, the Jersey cow, the number of days of gestation is normally distributed with a mean of 278 days and a standard deviation of 2.5 days.

During gestation, a randomly chosen pregnant Dutch Belted cow eats 29 kg of feed daily while a randomly chosen pregnant Jersey cow eats 26 kg of feed daily.

(ii) Find the value of a such that during their respective gestation periods, there is a probability of 0.35 that the amount of feed consumed by a randomly chosen pregnant Jersey cow exceeds half of the amount consumed by a randomly chosen pregnant Dutch Belted cow by less than a kg. Express your answer to the nearest kg. [2]

(iii) Calculate the probability that during their respective gestation periods, the difference between the amount of feed consumed by three randomly chosen pregnant Dutch Belted cows and four randomly chosen pregnant Jersey cows is more than 4000 kg. State clearly the parameters of the distribution used in the calculation. [3]

10 Factory A manufactures a large batch of light bulbs. It is known that on average, 1 out of 200 light bulbs manufactured by Factory A, is defective. A random sample of 180 light bulbs is inspected. The batch is accepted if the sample contains less than r defective light bulbs.

(i) Explain why the context above may not be well-modelled by a binomial distribution. [1]

Assume now that the context above is well-modelled by a binomial distribution.

(ii) Determine the value of r such that the probability of accepting the batch is 0.998. [1]

In Factory B, a random sample of 30 light bulbs is taken from a large batch. If the sample contains no defective light bulbs, the batch is accepted. The batch is rejected if the sample contains more than two defective light bulbs. If the sample contains one or two defective light bulbs, a second random sample of 30 light bulbs is chosen and the batch is accepted only if this second sample contains no defectives. It is known that Factory B produces $(100p)\%$ defective light bulbs.

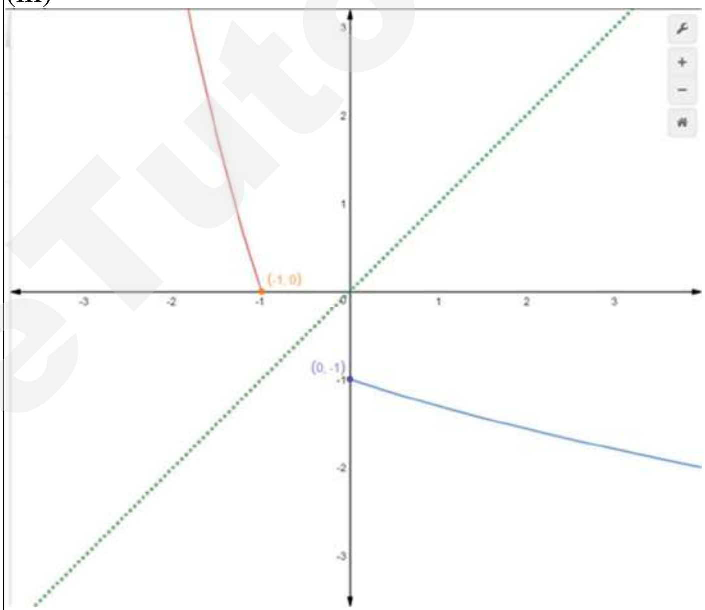
(iii) Find the probability that the batch is accepted. Leave your answer in terms of p. [3]

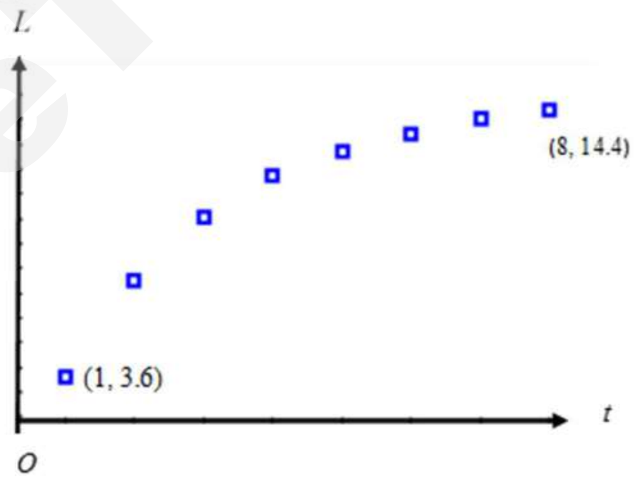
Forty random samples of 30 light bulbs are taken from each of the two factories A and B.

(iv) Given that $p = 0.007$ and there is exactly one defective bulb, find the probability that it is from Factory B. [4]

ANNEX B

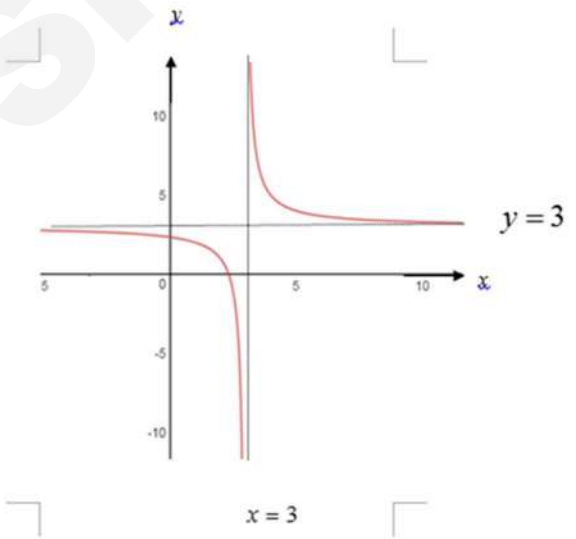
NJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Equations and Inequalities	$\alpha = \$0.41, \beta = \$0.84, \mu = \$1.14$.
2	Functions	(i) $f^{2003}(5) = 4.$ (ii) $g^{-1}(x) = \frac{1}{2} - \sqrt{x + \frac{9}{4}}$ (iii)  $x \geq 0$ (iv) $R_{\mathrm{f\,g^{-1}}} = [2.5, 3)$
3	Maclaurin series	$y \approx 1 + 2x + x^2 - \frac{2x^3}{3}$ $-0.952 \leq x \leq 1.07$
4	Vectors	(i) $3x + 3y + 7z = 56$ (ii) $\theta \approx 31.2°$ (1 dec place) (iii) (iv)

		$\mathbf{r} = \begin{pmatrix} d \\ \frac{56-3d}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}$, where $\lambda \in \cdots$. (v) $\overrightarrow{OP} = \frac{1}{67}\begin{pmatrix} 511 \\ 511 \\ 98 \end{pmatrix}$
5	P&C, Probability	(i) $k = 15$ (ii) 9 (iii) 144
6	Correlation & Linear Regression	(i)  (iii) $L = 0.261(3\text{ sf})$ (iv) $r_2 = -r_1$
7	DRV	(i) $\mathrm{P}(X = 4) = \frac{2}{9}$ (ii)

		$\mathrm{E}(X)=\frac{26}{9}$ (iii) $P\left(\overline{X}>3\right)=0.159$
8	Hypothesis Testing	(iii) p-value = 0.0768 (iv) $\sigma^2 \leq 943$
9	Normal Distribution	(i) $\mathrm{P}(\overline{D}>287)=0.0119$ (ii) $a \approx 3058$ (iii) $\mathrm{P}(\lvert C_1-C_2\rvert>4000)=0.660$
10	Binomial Distribution	(ii) $r=5$ (iii) $(1-p)^{30}+30p(1-p)^{59}+435p^2(1-p)^{58}$ (iv) 0.584
11	Q11 Topic	
12	Q12 Topic	
13	Q13 Topic	

H2 Mathematics 2017 Prelim Exam Paper Solution

1	Let α, β and μ be the original amount charged per 5 min, 10 min and 15 min block for each ride by α- bike, β- bike and μ- bike respectively. $5\alpha + 3\beta + \mu = 5.7$ ---------- (1) $4\alpha + \beta + 3(0.95\mu) = 5.72$ ---------- (2) $8\alpha + 4\beta + 3(0.95^2\mu) = 9.71$ ---------- (3) Solving the above 3 equations simultaneously by GC, $\alpha = \$0.4079329609, \beta = \$0.8402234637, \mu = \$1.139664804$ $\alpha = \$0.41, \beta = \$0.84, \mu = \$1.14$. Original pricing per 40-min block: **Using calculator values** α- bike: $\$0.4079329609 \times 8 = \$3.26$ β- bike: $\$0.84 \times 4 = \$3.36$ μ- bike: $\$1.14 \times 3 = \$3.42$ Thus, α- bike offers the cheapest rate for a 40-min ride.
2	**(i)** 

Since any horizontal line $\underline{y = a, a \in \mathbb{R}}$, intersects the graph of $y = \mathrm{f}(x)$ <u>at most once</u>, the function f is one-one. It follows that f^{-1} exists.

OR

Since any horizontal line $\underline{y = a, a \in \mathrm{R}_f}$, intersects the graph of $y = \mathrm{f}(x)$ <u>exactly once</u>, the function f is one-one. It follows that f^{-1} exists.

Let $y = \dfrac{7-3x}{3-x}$

$y(3-x) = 7-x$

$x = \dfrac{7-3y}{3-y}$

Since $\mathrm{f}^{-1}(x) = \dfrac{7-3x}{3-x}$, $x \in \mathbb{R}, x \neq 3$,

$\therefore \mathrm{f}^{-1} = \mathrm{f}$. (shown)

$D_{\mathrm{f}^{-1}} = R_{\mathrm{f}} = (-\infty, 3) \cup (3, \infty) = D_{\mathrm{f}}$

Note that $\mathrm{f}^{-1}\mathrm{f}(x) = x$. Therefore, $\mathrm{f}^{2003}(5) = \underbrace{\mathrm{fff...f}}_{2003\text{ times}}(5) = \mathrm{f}\left(\underbrace{\mathrm{f}^{-1}\mathrm{f}......\mathrm{f}^{-1}\mathrm{f}}_{1000\text{ times of }\mathrm{f}^{-1}\mathrm{f}}(5)\right) = \mathrm{f}(5) = 4.$

(ii)

$$|(2-x)(1+x)| = \begin{cases} (2-x)(1+x), & -1 \leq x \leq 2, \\ -(2-x)(1+x), & x < -1 \text{ or } x > 2. \end{cases}$$

For $x \in (-\infty, -1]$, $y = -(2-x)(1+x)$

Method 1

$x^2 - x - 2 - y = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2-y)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{9+4y}}{2}$$

Method 2

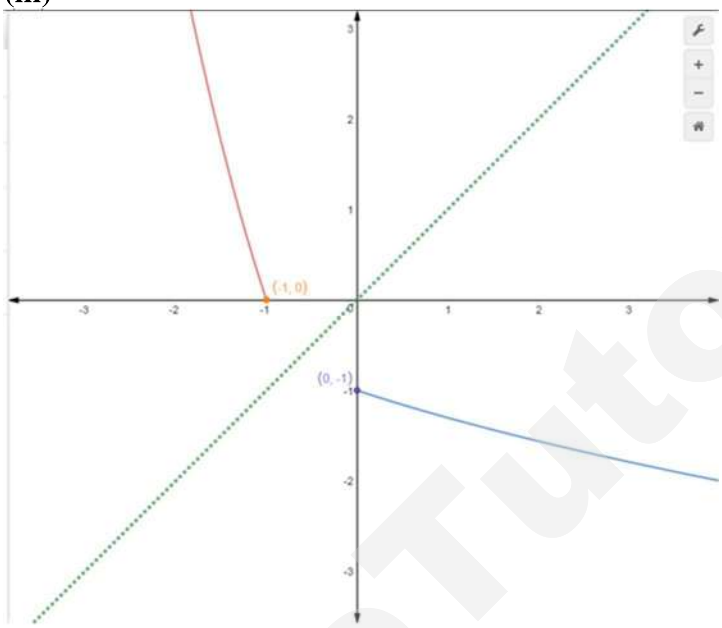
$y = x^2 - x - 2 = (x-0.5)^2 - 2.25$

$x = 0.5 \pm \sqrt{y + 2.25}$

$x = \dfrac{1+\sqrt{9+4y}}{2}$ (rejectd $\because x \le -1$) or $\dfrac{1-\sqrt{9+4y}}{2}$

$\therefore \mathrm{g}^{-1}(x) = \dfrac{1}{2} - \sqrt{x + \dfrac{9}{4}}$

(iii)



For $\mathrm{g}\,\mathrm{g}^{-1}(x) = x$,

$D_{\mathrm{g}\,\mathrm{g}^{-1}} = D_{\mathrm{g}^{-1}}$.

$\therefore x \in [0, \infty)$ or $x \ge 0$

(iv)

Since $R_{\mathrm{g}^{-1}} = (-\infty, -1]$ and $\mathrm{D_f} = (-\infty, \infty) \setminus \{3\}$

$R_{g^{-1}} \subseteq D_f$.

$\therefore \mathrm{fg}^{-1}$ exists.

Using the graph of $y = \mathrm{g}^{-1}(x)$ in part **(ii)**, $R_{\mathrm{g}^{-1}} = (-\infty, -1]$.

From graph of $y = \mathrm{f}(x)$ in **(i)** in $(-\infty, -1]$.

$\therefore R_{\mathrm{f\,g}^{-1}} = [2.5, 3)$

3

$$y=\frac{e^{2x}}{1+x^2}$$

$$\left(1+x^2\right)y=e^{2x}$$

$$\left(1+x^2\right)y'+2xy=2e^{2x}$$

$$\left(1+x^2\right)y''+2xy'+2xy'+2y=4e^{2x}$$

$$\Rightarrow\left(1+x^2\right)y''+4xy'+2y=4e^{2x}$$

$$\left(1+x^2\right)y'''+2xy''+4y'+4xy''+2y'=8e^{2x}$$

$$\Rightarrow\left(1+x^2\right)y'''+6xy''+6y'=8e^{2x}$$

When $x=0,\ y=1,\ y'=2,\ y''=2,\ y'''=-4$

$$y=\frac{e^{2x}}{1+x^2}$$

$$=1+2x+2\left(\frac{x^2}{2!}\right)-4\left(\frac{x^3}{3!}\right)+\cdots$$

$$\approx 1+2x+x^2-\frac{2x^3}{3}$$

$$a=2,\quad b=-\frac{2}{3}$$

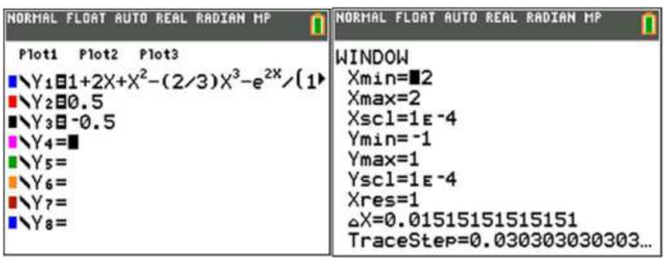
(a)

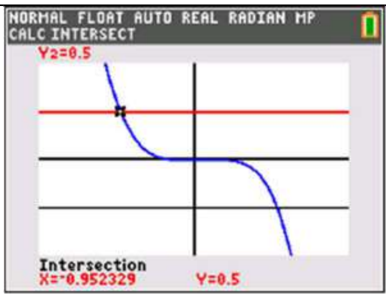
For $-2\le x\le 2$,

$$\left|f(x)-h(x)\right|\le 0.5$$

$$-0.5\le 1+2x+x^2-\frac{2x^3}{3}-\frac{e^{2x}}{1+x^2}\le 0.5$$

By GC,





From the diagram above,

$-0.95233 \le x \le 1.072619$

$\therefore -0.952 \le x \le 1.07$

4 **(i)**

$\overrightarrow{OD} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}$, $\overrightarrow{OE} = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}$, $\overrightarrow{OF} = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix}$. Hence,

$\overrightarrow{DE} = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$, $\overrightarrow{DF} = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}$ and $\overrightarrow{EF} = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \\ 0 \end{pmatrix}$

A vector perpendicular to the plane is

$= \overrightarrow{DE} \times \overrightarrow{DF}$

$= \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 21 \\ 21 \\ 49 \end{pmatrix} = 7\begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}$

Cartesian equation of the plane is

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} \Rightarrow \mathbf{r} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = 56$

$\therefore 3x + 3y + 7z = 56$

(ii)

Let the required angle be θ

$$\cos\theta = \frac{\left|\begin{pmatrix}3\\3\\7\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{3^2+3^2+7^2}\sqrt{1}} = \frac{7}{\sqrt{67}}$$

$\theta \approx 31.2°$ (1 dec place)

(or 0.545 rad)

(iii)

Method 1

$\left|\overrightarrow{OB}\right| = \sqrt{7^2+7^2} = \sqrt{98}$

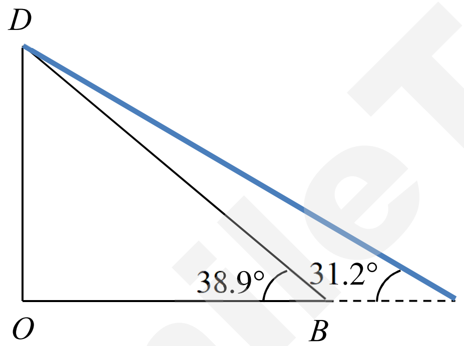
$\left|\overrightarrow{OD}\right| = 9$

Angle between DB and the ground

$= \angle OBD$

$= \tan^{-1}\left(\frac{8}{\sqrt{7^2+7^2}}\right)$

$\approx 38.9°$



From the diagram, the canvas will cover B.

Method 2

$\overrightarrow{OB} = \begin{pmatrix}7\\7\\0\end{pmatrix}$

Equation of perpendicular line passing through B, l:

$\mathbf{r} = \begin{pmatrix}7\\7\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\0\\1\end{pmatrix}, \lambda \in \mathbb{R}$

Using normal of plane to be $\begin{pmatrix}3\\3\\7\end{pmatrix}$ i.e. all entries are positive:

solve the equation of plane DEF and l:

	$\left[\begin{pmatrix}7\\7\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\cdot\begin{pmatrix}3\\3\\7\end{pmatrix}=56$ $\lambda=2$ Since $\lambda=2>0$, l and plane *DEF* intersect above the horizontal ground. So the canvas covers the point *B*. Method 3 $\begin{pmatrix}7\\7\\0\end{pmatrix}\cdot\begin{pmatrix}3\\3\\7\end{pmatrix}=42<56$ Distance from *O* to plane parallel to DEF and passing through *B* is smaller than the distance between *O* and plane *DEF*. Hence *B* is beneath the canvas. **(iv)** Normal vector of the vertical wall is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $(d,0,0)$ lies on the vertical wall. $\begin{pmatrix}d\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=d\Rightarrow \mathbf{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=d$ Hence the equation of the vertical wall is $\mathbf{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=d$. Direction vector of the line of intersection is $\begin{pmatrix}3\\3\\7\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\7\\-3\end{pmatrix}$ Let $(x,\mathrm{y},0)$ be the common point on lying on the two planes. $\begin{pmatrix}x\\y\\0\end{pmatrix}\cdot\begin{pmatrix}3\\3\\7\end{pmatrix}=56\Rightarrow 3x+3y=56$

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = d \Rightarrow x = d$$

Solving the above equations simultaneously

$$3d + 3y = 56 \Rightarrow y = \frac{56-3d}{3}$$

$$\therefore \mathbf{r} = \begin{pmatrix} d \\ \frac{56-3d}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}, \text{ where } \lambda \in \mathbb{R}.$$

(v)
For P to shine the brightest at point B, P must be as near as possible to B. Thus P is the foot of perpendicular from B to the roof.

Equation of the line passes through B and P:

$$\mathbf{r} = \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}, \alpha \in \mathbb{R}$$

Thus $\overrightarrow{OP} = \begin{pmatrix} 7+3\alpha \\ 7+3\alpha \\ 7\alpha \end{pmatrix}$ for some α.

Since P lies on the roof,

$$\begin{pmatrix} 7+3\alpha \\ 7+3\alpha \\ 7\alpha \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = 56 \Rightarrow 42 + 67\alpha = 56$$

$$\therefore \alpha = \frac{14}{67}$$

Substitute $\alpha = \frac{14}{67}$ into $\overrightarrow{OP} = \begin{pmatrix} 7+3\alpha \\ 7+3\alpha \\ 7\alpha \end{pmatrix}$.

	$\overrightarrow{OP}=\begin{pmatrix}7+3\left(\frac{14}{67}\right)\\7+3\left(\frac{14}{67}\right)\\7\left(\frac{14}{67}\right)\end{pmatrix}=\begin{pmatrix}\frac{511}{67}\\\frac{511}{67}\\\frac{98}{67}\end{pmatrix}$ **Alternatively, use projection vector:** $\overrightarrow{BD}=\begin{pmatrix}0\\0\\8\end{pmatrix}-\begin{pmatrix}7\\7\\0\end{pmatrix}=\begin{pmatrix}-7\\-7\\8\end{pmatrix}$ To check for the direction of normal vector of *DEF* **n=**$\begin{pmatrix}3\\3\\7\end{pmatrix}$ $\begin{pmatrix}3\\3\\7\end{pmatrix}.\overrightarrow{BD}=\begin{pmatrix}3\\3\\7\end{pmatrix}.\begin{pmatrix}-7\\-7\\8\end{pmatrix}=-21-21+56>0$ Hence, angle between $\overrightarrow{BD}$ and **n** is acute. $\overrightarrow{BP}=\left(\overrightarrow{BD}.\hat{\mathbf{n}}\right)\hat{\mathbf{n}}$ $=\left(\begin{pmatrix}-7\\-7\\8\end{pmatrix}.\frac{\begin{pmatrix}3\\3\\7\end{pmatrix}}{\sqrt{9+9+49}}\right)\frac{\begin{pmatrix}3\\3\\7\end{pmatrix}}{\sqrt{9+9+49}}$ $=\frac{14}{67}\begin{pmatrix}3\\3\\7\end{pmatrix}$ $\overrightarrow{OP}=\overrightarrow{OB}+\overrightarrow{BP}$ $=\begin{pmatrix}7\\7\\0\end{pmatrix}+\frac{14}{67}\begin{pmatrix}3\\3\\7\end{pmatrix}$ $=\frac{1}{67}\begin{pmatrix}511\\511\\98\end{pmatrix}$
5	**(i)**

	5	$(m-2)$	6

	Badminton Players	Floorball Players	Swimmers Players
Case 1	1	1	2
Case 2	1	2	1
Case 3	2	1	1

Case 1: Number of selections is $\binom{5}{1}\binom{m-2}{1}\binom{6}{2}$

Case 2: Number of selections is $\binom{5}{1}\binom{m-2}{2}\binom{6}{1}$

Case 3: Number of selections is $\binom{5}{2}\binom{m-2}{1}\binom{6}{1}$

Total number of selections

$$=\binom{5}{1}\binom{m-2}{1}\binom{6}{2}+\binom{5}{1}\binom{m-2}{2}\binom{6}{1}+\binom{5}{2}\binom{m-2}{1}\binom{6}{1}$$

$$=75(m-2)+30\frac{(m-2)(m-3)}{2!}+60(m-2)$$
$$=135(m-2)+15(m-2)(m-3)$$
$$=15(m-2)(9+m-3)$$
$$=15(m-2)(m+6)$$

$\therefore k=15$

Alternative method:

$$\binom{5}{1}\binom{m-2}{1}\binom{6}{1}\binom{m-2+5+6-3}{1}/2!$$

(ii)
Number of ways to select exactly one of the twins

$$=\binom{5}{1}\binom{2}{1}\binom{6}{2}+\binom{5}{1}\binom{m-2}{1}\binom{2}{1}\binom{6}{1}+\binom{5}{2}\binom{2}{1}\binom{6}{1}$$

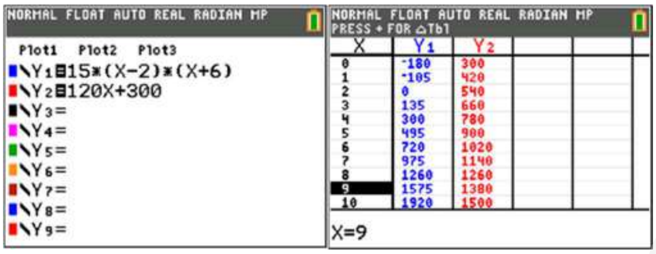
$=150+60(m-2)+120$

$=60m+150$

Number of ways that the twins are not selected > 2 times the number of ways that exactly one of the twins is selected.

$15(m-2)(m+6)>2(60m+150)$

By GC,



least value of m is 9.

Last part

Step 1: Arrange 3 units at the round table = 3!/3

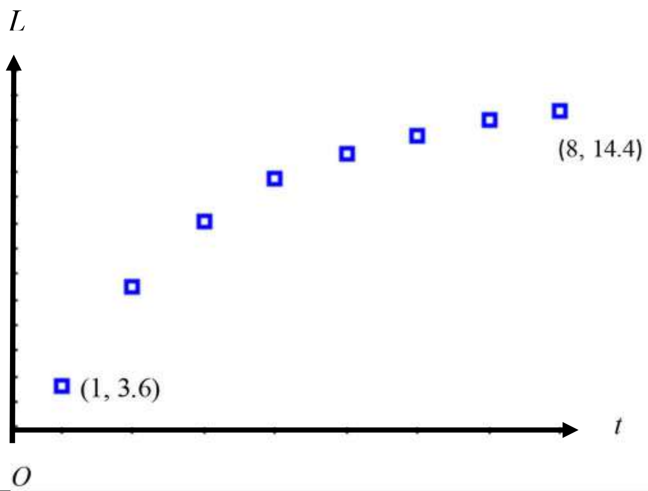
Step 2: Arrange the twins among themselves = 2!

Step 3: Slot in the teachers = $\binom{3}{2}\times 2!$

Number of ways for the twins to be seated together and teachers are separated

$=\frac{3!}{3}\times 2!\times\binom{3}{2}\times 2!\times 6=144$

6



	As L <u>increases at a decreasing rate/concave downwards</u> with respect to t, the linear model $L = at + b$ should not be used. **(ii)** The r value for $L = a\sqrt{t} + b$ is 0.972. The r value for $L = c\ln t + d$ is 0.996. Since the value of $\|r\|$ for $L = c\ln t + d$, is closer to 1, $L = c\ln t + d$ is a better model. $\therefore c = 5.28248 \approx 5.28$ $\therefore d = 3.92267 \approx 3.92$ **(iii)** $L = 5.28248\ln(0.5) + 3.92267$ $= 0.2611$ $= 0.261(3\text{ sf})$ This estimate is not reliable as as the age of the Sole is <u>out of the range of the data.</u> **(iv)** Since $G = 14.8 - L$ r_1 is positive but r_2 is negative. $\therefore r_2 = -r_1$
7	**(i)** $\mathrm{P}(X = 4)$ $= \frac{3}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{3}{3}$ $= \frac{2}{9}$ <table><tr><td>x</td><td>2</td><td>3</td><td>4</td></tr><tr><td>P($X = x$)</td><td>$\frac{1}{3}$</td><td>$\frac{4}{9}$</td><td>$\frac{2}{9}$</td></tr></table> **(ii)**

	$\mathrm{E}(X)$ $=\frac{1}{3}\times 2+\frac{4}{9}\times 3+\frac{2}{9}\times 4$ $=\frac{26}{9}$ $\mathrm{E}(X^2)$ $=\frac{1}{3}\times 2^2+\frac{4}{9}\times 3^2+\frac{2}{9}\times 4^2$ $=\frac{80}{9}$ Var(X) $=\frac{80}{9}-\left(\frac{26}{9}\right)^2$ $=\frac{44}{81}$ **(iii)** Since n =44 <u>is large</u>, by Central Limit Theorem, $\overline{X} \sim N\left(\frac{26}{9},\frac{44}{81}\div 44\right)$ approx. $P\left(\overline{X}>3\right)$ $=0.159$ (By GC)

8	**(i)** 2-tail as HPB is looking for a change in either way. **(ii)** Central Limit Theorem states that the sample <u>mean</u> heart rate will follow a normal distribution approximately when the sample is large (in this case, 70 > 20). An unbiased estimate for the unknown population variance can be found obtained from the sample. **(iii)** Unbiased estimate of population mean $\bar{h} = \frac{5411}{70} = 77.3$. Unbiased estimate of population variance, $s^2 = \frac{1}{69}\left(426433 - \frac{5411^2}{70}\right) = 118.3$. Let μ denote the mean heart rate of the teenagers in the obesity group. To test at 10% significance level: H_0: $\mu = 75$ H_1: $\mu \neq 75$ Under H_0, since n is large, by CLT, $\overline{H} \sim N\left(75, \frac{118.3}{70}\right)$ approximately, (AND/OR $\frac{\overline{H} - 75}{\sqrt{\frac{118.3}{70}}} \sim N(0,1)$) By GC, p-value = 0.0768 < 0.10. (Alternatively, CR: $\lvert z \rvert > 1.645$, $z = 1.769$ is in CR) Hence we <u>reject H_0</u> at the <u>10% level of significance</u> and conclude there is <u>sufficient evidence</u> that obesity will cause change in the mean heart rate. **(iv)** An one-tail test is used instead: H_0: $\mu = 75$ H_1: $\mu > 75$ CR: $z > z_{0.9} = 1.28155$ To reject H_0,

	$$\frac{79.4-75}{\sqrt{\frac{\sigma^2}{80}}} \geq 1.28155$$ $$\sigma^2 \leq \left(\frac{4.4\sqrt{80}}{1.28155}\right)^2 = 943 \text{ (3 sf)}$$ The researcher should conclude that obese teenagers evidentially has a higher mean heart rate if and only if the variance is not more (less) than 943. **(v)** $\mathrm{P}(\mu-\sigma < X < \mu+\sigma) = \mathrm{P}(-1 < Z < 1) = 0.68268 \approx 0.683$. Since heart rates follow a normal distribution, $\mathrm{P}(\mu-\sigma < H < \mu+\sigma) \approx 0.683$ We know that from (iv), null hypothesis will be rejected whenever $\sigma \leq 30.7$. Taking $\sigma = 30.7$, under H_0, $\mathrm{P}(75-30.7 < H < 75+30.7) \approx 0.683$ $\Rightarrow \mathrm{P}(44.3 < H < 105.7) \approx 0.683$ and null hypothesis will be rejected. We can say that for $\sigma \leq 30.7$ and when null hypothesis is rejected , $\mathrm{P}(44.3 < H < 105.7) \geq 0.683$ or $\mathrm{P}(H < 44.3) + \mathrm{P}(H > 105.7) < 0.317$ We know that the teenager's heart rate is rarely below 44.3 or above 105.7 in a resting state, so it is likely for the researcher to reject the null hypothesis. In reality, it is unlikely for sigma to be as large as 30.7 such that the probability for H to be within one standard deviation from mean to be 0.683.
9	Let D be the random variable denoting the number of days of gestation for a Dutch Belted cow. $\mathrm{P}(D < 278) = 0.0808$ $\mathrm{P}(Z < \frac{278-\mu}{\sigma}) = 0.0808$ $\frac{278-\mu}{\sigma} = -1.39971 --- (1)$ $\mathrm{P}(D > 289) = 0.212$ $\mathrm{P}(Z < \frac{289-\mu}{\sigma}) = 0.788$ $\frac{289-\mu}{\sigma} = 0.799501 --- (2)$ Solving (1)&(2), $\mu = 285.001064$ and $\sigma = 5.0017961$

$\mu = 285$ (3 s.f.) and $\sigma = 5.00$ (3 s.f.) .

(i)

$\overline{D} \sim \text{N}(285.001064, \frac{5.0017961^2}{32}).$

$\text{P}(\overline{D} > 287) = 0.0118629 = 0.0119$ (3 s.f.)

The <u>number of days of gestation for a Dutch Belted cow</u> is <u>independent</u> of <u>the number of days of gestation of another Dutch Belted cow.</u>

(ii)

$J \sim \text{N}(278, 2.50^2)$

$D \sim \text{N}(285.001064, 5.0017961^2)$

Let $X = 26J - \frac{1}{2}29D$

$X \sim \text{N}(3095.48457 \ , \ 9485.02698)$

$\text{P}(0 < X < a) = 0.35$

$\therefore a = 3057.95778 \approx 3058$

(iii)

$D \sim \text{N}(285.001064, 5.0017961^2)$

$J \sim \text{N}(278, 2.50^2)$

Let C_1 denote the random variable of the amount of feed cosumed by 3 pregnant Dutch belted cows.

Let C_2 denote the random variable of the amount of feed cosumed by 4 pregnant Jersey cows.

$C_1 = 29(D_1 + D_2 + D_3) \sim \text{N}(24795.09257, 63120.32374)$

$C_2 = 26(\text{J}_1 + J_2 + J_3 + J_4) \sim \text{N}(28912, 16900)$

$C_1 - C_2 \sim \text{N}(-4117, 80020.3237)$

$$\begin{aligned}\text{P}(|C_1 - C_2| > 4000) &= \text{P}(C_1 - C_2 < -4000) + \text{P}(C_1 - C_2 > 4000) \\ &= 0.6604182314 \\ &= 0.660 \text{ (3 s.f.)}\end{aligned}$$

Or

$D \sim \mathrm{N}(285.00, 5.0018^2)$

$J \sim \mathrm{N}(278, 2.50^2)$

Let C_1 denote the random variable of the amount of feed cosumed by 3 pregnant Dutch belted cows.

Let C_2 denote the random variable of the amount of feed cosumed by 4 pregnant Jersey cows.

$C_1 = 29(D_1 + D_2 + D_3) \sim \mathrm{N}(24795, 63120.42217)$

$C_2 = 26(J_1 + J_2 + J_3 + J_4) \sim \mathrm{N}(28912, 16900)$

$C_1 - C_2 \sim \mathrm{N}(-4117, 80020.422)$

$$\begin{aligned}\mathrm{P}(|C_1 - C_2| > 4000) &= \mathrm{P}(C_1 - C_2 < -4000) + \mathrm{P}(C_1 - C_2 > 4000)\\ &= 0.66041814\\ &= 0.660 \text{ (3 s.f.)}\end{aligned}$$

10

(i)

The event of a bulb being defective may not be independent of another bulb being defective.

(ii)

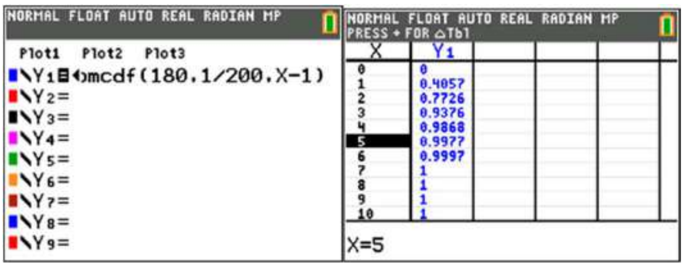
Let X be the random variable for the number of defective light bulbs produced by Factory A.

$$X \sim \mathrm{B}\left(180, \frac{1}{200}\right)$$

Given

$$\mathrm{P}(X < r) = 0.998$$
$$\Rightarrow \mathrm{P}(X \le r - 1) = 0.998$$

By GC,



$\therefore r = 5$

(iii)

Let Y be the random variable for the number of defective light bulbs produced by Factory B.

$Y \sim \mathrm{B}(30, p)$

P(the batch is accepted)

$$= \mathrm{P}(Y_1 = 0) + \mathrm{P}(Y_1 = 1 \text{ or } 2)\mathrm{P}(Y_2 = 0)$$

$$= \binom{30}{0} p^0 (1-p)^{30}$$

$$+ \left[\binom{30}{1} p(1-p)^{29} + \binom{30}{2} p^2 (1-p)^{28}\right]\binom{30}{0} p^0 (1-p)^{30}$$

$$= (1-p)^{30} + \left[30p(1-p)^{29} + 435p^2(1-p)^{28}\right](1-p)^{30}$$

$$= (1-p)^{30} + 30p(1-p)^{59} + 435p^2(1-p)^{58}$$

(iv)

Let U be the random variable for the number of defective light bulbs produced by Factory A.

Let V be the random variable for the number of defective light bulbs produced by Factory B.

$U \sim \mathrm{B}\left(1200, \frac{1}{200}\right)$

$V \sim \mathrm{B}(1200, 0.007)$

P(1 bulb from B is defective| there is exactly one defective bulb)

	$=\dfrac{\mathrm{P}(U=0,V=1)}{\mathrm{P}(U=0,V=1)+\mathrm{P}(U=1,V=0)}$ ≈ 0.5838223 $=0.584$ (3 s.f.) Reference for $\dfrac{\mathrm{P}(U=0,V=1)}{\mathrm{P}(U=0,V=1)+\mathrm{P}(U=1,V=0)}$: $\dfrac{\left[\binom{1200}{1}0.007^{1}(1-0.007)^{1199}\times\binom{1200}{0}\left(\frac{1}{200}\right)^{0}\left(1-\frac{1}{200}\right)^{1200}\right]}{\left[\begin{array}{l}\binom{1200}{1}0.007^{1}(1-0.007)^{1199}\times\binom{1200}{0}\left(\frac{1}{200}\right)^{0}\left(1-\frac{1}{200}\right)^{1200}\\+\binom{1200}{0}0.007^{0}(1-0.007)^{1200}\times\binom{1200}{1}\left(\frac{1}{200}\right)^{1}\left(1-\frac{1}{200}\right)^{1199}\end{array}\right]}$

Candidate Name: ______________________ Class: ____________

PIONEER JUNIOR COLLEGE
While I Live, I Learn

JC2 PRELIMINARY EXAM
Higher 2

MATHEMATICS **9758/01**
Paper 1
13 Sept 2017
3 hours

Additional Materials: Cover page
Answer papers
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your full name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 The first three terms of a sequence are given by $u_1 = 70$, $u_2 = 136$, $u_3 = 198$. Given that u_n is a quadratic polynomial in n, find u_n in terms of n. [4]

2 A sequence $u_0, u_1, u_2, \ldots$ is given by $u_0 = \dfrac{3}{2}$ and $u_n = u_{n-1} + 2^n - n$ for $n \geq 1$.

(i) Find u_1, u_2 and u_3. [3]

(ii) By considering $\sum_{r=1}^{n}(u_r - u_{r-1})$, find a formula for u_n in terms of n. [5]

3 By sketching the graphs of $y = e^{2x}$ and $y = 2e^{-x} - 1$, solve the inequality

$$e^{2x} \geq 2e^{-x} - 1. \quad [3]$$

Hence, without using a calculator, find

$$\int_{-1}^{2} \left| e^{2x} - 2e^{-x} + 1 \right| \mathrm{d}x,$$

giving your answer in terms of e. [4]

4 The function f is defined by

$$\mathrm{f} : x \mapsto \left| \frac{2x+6}{4-x} \right|, \qquad x \in \mathbb{R}, \quad x \neq 4.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$, giving the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. Hence state the range of f. [3]

(ii) Determine whether the function f^2 exists, justifying your answer. [1]

(iii) The function f^{-1} exists if the domain of f is further restricted to $x \leq k$. State the greatest value of k. [1]

(iv) Using the domain in **(iii)**, find $y = \mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

5 A curve is given parametrically by the equations

$$x = 2t - 1, \quad y = \frac{1}{2t+1},$$

where $t \in \mathbb{R}$, $t \neq -\frac{1}{2}$.

(i) Sketch the curve, labelling the axial intercepts and asymptotes. [2]

(ii) Find the equation of the tangent to the curve at the point $P(-1, 1)$. [3]

(iii) State the range of values of m for which the line $y = mx$ does not intersect the curve. [1]

(iv) The normal to the curve at P meets the curve again at Q. Find the coordinates of Q. [4]

6 Two expedition teams are to climb a vertical distance of 8500 m from the foot to the peak of a mountain over a period of time.

(i) Team A plans to cover a vertical distance of 400 m on the first day. On each subsequent day, the vertical distance covered is 5 m less than the vertical distance covered in the previous day. Find the number of days required for Team A to reach the peak. [2]

(ii) Team B plans to cover a vertical distance of 800 m on the first day. On each subsequent day, the vertical distance covered is 90% of the vertical distance covered in the previous day. On which day will Team A overtake Team B? [3]

(iii) Explain why Team B will never be able to reach the peak. [2]

(iv) At the end of the 15th day, Team B decided to modify their plan, such that on each subsequent day, the vertical distance covered is 95% of the vertical distance covered in the previous day. Which team will be the first to reach the peak of the mountain? Justify your answer. [5]

7 The curve C has equation $y=2+\dfrac{x-3}{(x-2)(x+1)}$.

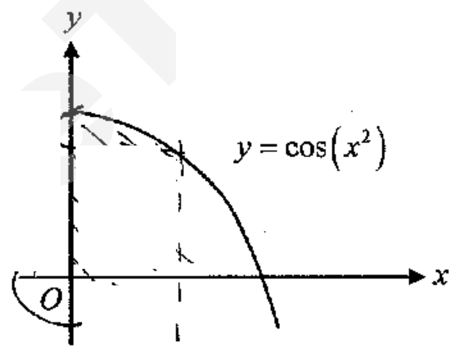
(i) Find algebraically the set of values that y can take. [5]

(ii) Sketch C, giving the coordinates of the axial intercepts, turning points and equations of any asymptotes. [3]

(iii) By adding an appropriate graph to the sketch of C, determine the range of values of k such that the equation $(x-2)^2+\dfrac{(x-3)^2}{(x-2)^2(x+1)^2}=k^2$ has at least one negative real root. [4]

8 **(a)** Find $\displaystyle\int\sqrt{\frac{1-x}{x}}\,\mathrm{d}x$ by using the substitution $x=\sin^2\theta$, where $0<\theta<\dfrac{\pi}{2}$. [6]

(b) The diagram below shows a sketch of part of the curve $y=\cos\left(x^2\right)$.



Find the exact volume of the solid generated when the region bounded by the curve $y=\cos\left(x^2\right)$, the axes and the line $x=\dfrac{\sqrt{\pi}}{2}$ is rotated through 2π radians about the y-axis. [7]

9

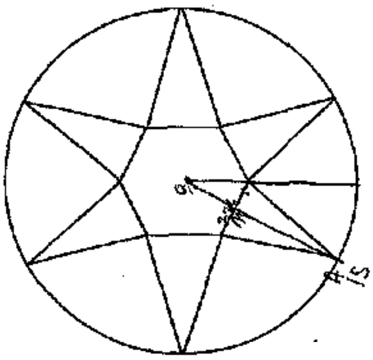


Fig. 1

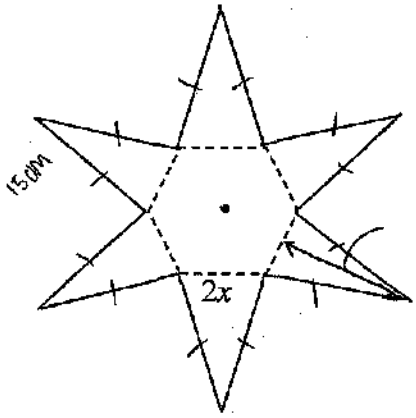


Fig. 2

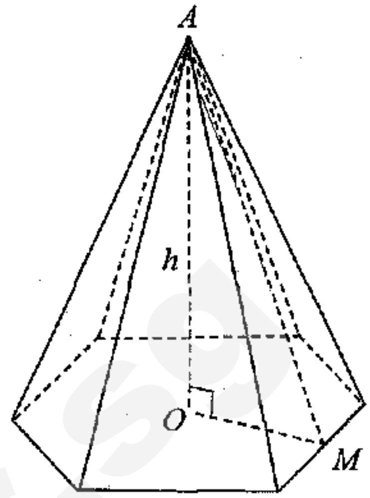


Fig. 3

Fig. 1 shows a piece of circular card of radius 15 cm. A star shape, which consists of a regular hexagon of side $2x$ cm and 6 isosceles triangles, is cut out from the card to give the shape shown in Fig. 2. The remaining card shown in Fig. 2 is folded along the dotted lines to form a pyramid of height h cm as shown in Fig. 3.
(The diagrams are not drawn to scale).

(i) By considering triangle AOM as shown in Fig. 3, where O is the centre of the hexagon and M is the midpoint of a side of the hexagon, show that

$$h^2 = 225 - 30\sqrt{3}x. \qquad [3]$$

(ii) Hence show that the volume V of the pyramid is given by

$$V^2 = 180x^4(15 - 2\sqrt{3}x). \qquad [3]$$

(iii) Use differentiation to find the maximum value of V, proving that it is a maximum. [5]

(iv) Determine the value of h for which V is maximum. [1]

10 The plane p contains the point A with coordinates $(-3,4,-2)$ and the line l with equation $x+2=\frac{4-y}{3}, z=0$.

(i) Find a cartesian equation of p. [3]

(ii) Find a vector equation of the line which is a reflection of l in the y-axis. [4]

The line m passes through A and the point $(-9,9,-6)$.

(iii) Find the acute angle between l and m. [2]

(iv) Find the coordinates of the points on m that are equidistant from p and the x-y plane. [4]

Pioneer Junior College
H2 Mathematics **JC2 2017**
JC2 H2 Preliminary Examination Paper 1 (Solution)

Q1

$u_n = an^2 + bn + c$

$u_1 = a(1)^2 + b(1) + c = 70 \quad \Rightarrow \quad a + b + c = 70 \quad (1)$

$u_2 = a(2)^2 + b(2) + c = 136 \quad \Rightarrow \quad 4a + 2b + c = 136 \quad (2)$

$u_3 = a(3)^2 + b(3) + c = 198 \quad \Rightarrow \quad 9a + 3b + c = 198 \quad (3)$

Using GC
$a = -2, \quad b = 72, \quad c = 0$
$u_n = -2n^2 + 72n$

Q2

(i)

$u_1 = u_0 + 2 - 1 = \frac{3}{2} + 2 - 1 = \frac{5}{2}$

$u_2 = u_1 + 2^2 - 2 = \frac{5}{2} + 4 - 2 = \frac{9}{2}$

$u_3 = u_2 + 2^3 - 3 = \frac{9}{2} + 8 - 3 = \frac{19}{2}$

(ii)

Note :

RHS (by sum of first n terms of GP and sum of first n terms of AP)

LHS (by method of difference)

$u_n - u_{n-1} = 2^n - n$

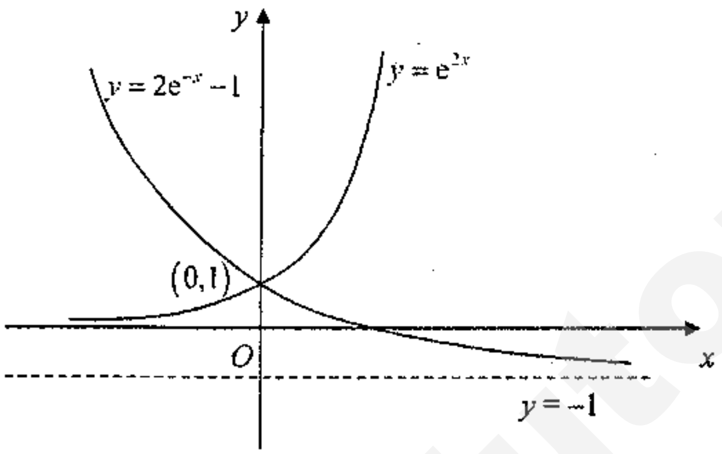
$$\sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n} 2^r - r$$
$$= \sum_{r=1}^{n} 2^r - \sum_{r=1}^{n} r$$

$$\begin{aligned}
&u_1 - u_0 \\
&+u_2 - u_1 \\
&+u_3 - u_2 \\
&+ \\
&\vdots \\
&+u_{n-2} - u_{n-3} = \frac{2(1-2^n)}{1-2} - \frac{n(n+1)}{2} \\
&+u_{n-1} - u_{n-2} \\
&+u_n - u_{n-1}
\end{aligned}$$

$$u_n - u_0 = \frac{2(1-2^n)}{1-2} - \frac{n(n+1)}{2}$$

$$u_n = -2(1-2^n) - \frac{n(n+1)}{2} + \frac{3}{2}$$
$$= 2^{n+1} - \frac{1}{2} - \frac{n(n+1)}{2}$$

Q3



$e^{2x} \geq 2e^{-x} - 1$
$x \geq 0$

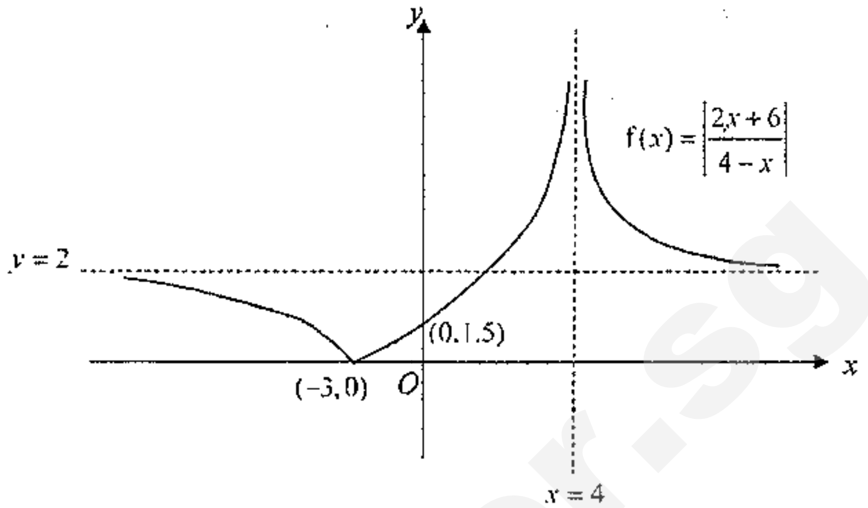
For $x \geq 0$, $e^{2x} \geq 2e^{-x} - 1 \Rightarrow e^{2x} - 2e^{-x} + 1 \geq 0$
For $x < 0$, $e^{2x} - 2e^{-x} + 1 < 0$.

$$\int_{-1}^{2} \left| e^{2x} - 2e^{-x} + 1 \right| dx = \int_{-1}^{0} -(e^{2x} - 2e^{-x} + 1)\, dx + \int_{0}^{2} (e^{2x} - 2e^{-x} + 1)\, dx$$
$$= -\left[\frac{1}{2}e^{2x} + 2e^{-x} + x\right]_{-1}^{0} + \left[\frac{1}{2}e^{2x} + 2e^{-x} + x\right]_{0}^{2}$$
$$= -\left[\left(\frac{1}{2}+2\right) - \left(\frac{1}{2}e^{-2} + 2e - 1\right)\right] + \left[\left(\frac{1}{2}e^{4} + 2e^{-2} + 2\right) - \left(\frac{1}{2}+2\right)\right]$$
$$= \frac{1}{2}e^{4} + 2e + \frac{5}{2}e^{-2} - 4$$

Q4

(i)

$R_f = [0, \infty)$



(ii)

$R_f = [0, \infty)$
$D_f = (-\infty, 4) \cup (4, \infty)$ or $D_f = \mathbb{R} \setminus \{4\}$
$R_f \not\subset D_f$
f^2 does not exists.

(iii)

$k = -3$

(iv)

For $D_f = (-\infty, -3]$

$y = -\left(\frac{2x+6}{4-x}\right)$

$y = \frac{2x+6}{x-4}$

$yx - 2x = 6 + 4y$

$x = \frac{6+4y}{y-2}$

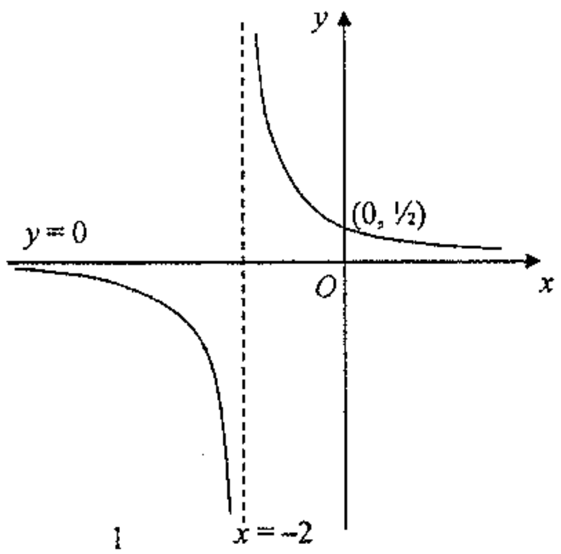
$f^{-1} : x \mapsto \frac{6+4x}{x-2}, \qquad x \in \mathbb{R},\ 0 \le x < 2$

Note :

Consider the graph without modulus.

Q5

(i)



(ii)

$x = 2t - 1 \qquad y = \dfrac{1}{2t+1}$

$\dfrac{dx}{dt} = 2 \qquad \dfrac{dy}{dt} = -\dfrac{2}{(2t+1)^2}$

$\dfrac{dy}{dx} = -\dfrac{1}{(2t+1)^2}$

At the point $P(-1, 1)$, $t = 0$

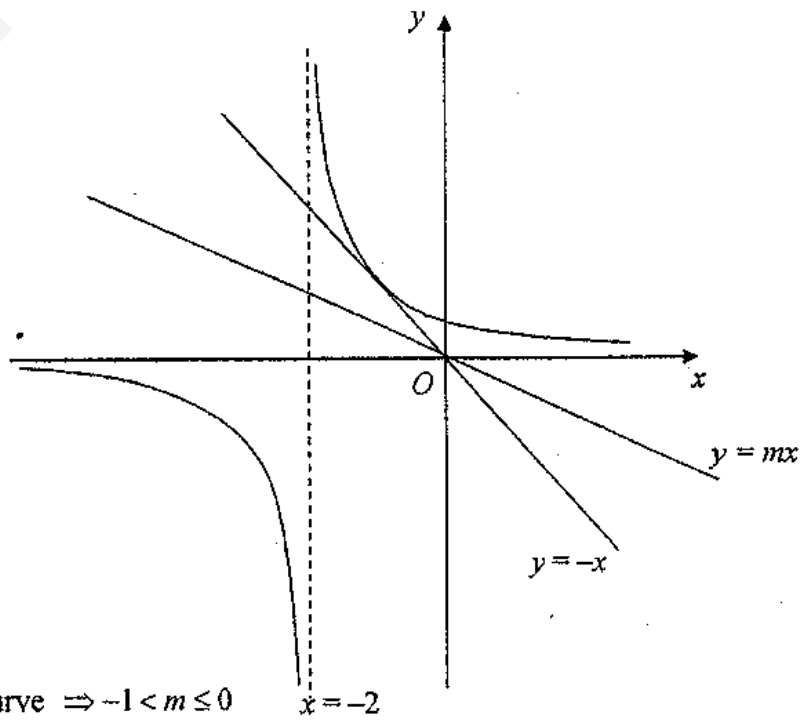
$\dfrac{dy}{dx} = -1$

Equation of tangent at P is

$y - 1 = -1(x+1)$

$y = -x$

(iii)



The line $y = mx$ does not cut the curve $\Rightarrow -1 < m \leq 0$

(iv)
Gradient of normal at $P = 1$
Equation of normal at P is
$y - 1 = x - (-1)$
$y = x + 2$

Subst $x = 2t - 1$, $y = \dfrac{1}{2t+1}$ into $y = x + 2$

$\dfrac{1}{2t+1} = 2t - 1 + 2 = 2t + 1$
$(2t+1)^2 = 1$
$2t + 1 = \pm 1$

$t = 0$ or $t = -1$
At the point Q, $t = -1$
$x = 2(-1) - 1 = -3$, $y = \dfrac{1}{2(-1)+1} = -1$

Coordinates of Q are $(-3, -1)$

Q6

(i)

AP with $a = 400$, $d = -5$
$S_n = 8500$

$\dfrac{n}{2}\left[2(400) + (n-1)(-5)\right] = 8500$

$5n^2 - 805n + 17000 = 0$
$n = 25$ or $n = 136$ (rejected as already reached peak when $n = 25$)

(ii)
GP with $a = 800$, $r = 90$
$S_{n(AP)} > S_{n(GP)}$

$\dfrac{n}{2}\left[2(400) + (n-1)(-5)\right] > \dfrac{800(1-0.9^n)}{1-0.9}$

$805n - 5n^2 > 16000\left(1 - 0.9^n\right)$

Using GC,
$n \geq 20$
A will overtake B on the 20th day.

NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl

X	Y1	Y2
18	6435	6799.2
19	6745	6919.3
20	7050	7027.4
21	7350	7124.6
22	7645	7212.2
23	7935	7291
24	8220	7361.9
25	8500	7425.7
26	8775	7483.1
27	9045	7534.8
28	9310	7581.3

X=18

(iii)

$S_\infty = \dfrac{800}{1-0.9} = 8000 (< 8500)$
Hence, Team B will never be able to reach the peak.

(iv)

$T_{15} = 800\left(0.9^{15-1}\right) = 183.014$

$S_{15} = \dfrac{800\left(1-0.9^{15}\right)}{1-0.9} = 6352.871$

Remaining distance $= 8500 - 6352.871 = 2147.129$

First term of new GP $= 183.014 \times 0.95 = 173.864$

$S_{n(New\,GP)} = 2147.129$

$\dfrac{173.864\left(1-0.95^n\right)}{1-0.95} = 2147.129$

$0.95^n = 0.38253$

$n = 18.7$

Team B will take $15 + 19 = 34$ days

Hence, Team A will reach the peak first.

Q7

(i)

Consider the graph of $y = 2 + \dfrac{x-3}{(x-2)(x+1)}$ and $y = p$ intersecting.

$p = 2 + \dfrac{x-3}{(x-2)(x+1)}$

$p - 2 = \dfrac{x-3}{x^2-x-2}$

Note :

Finding y values by $\dfrac{dy}{dx} = 0$ is not encouraged.

$px^2 - px - 2p - 2x^2 + 2x + 4 = x - 3$

$(p-2)x^2 + (1-p)x + (7-2p) = 0$

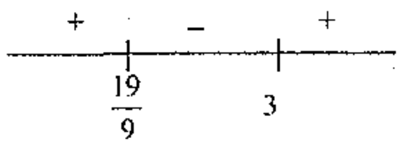
Discriminant ≥ 0

$(1-p)^2 - 4(p-2)(7-2p) \geq 0$

$1 - 2p + p^2 - 28p + 8p^2 + 56 - 16p \geq 0$

$9p^2 - 46p + 57 \geq 0$

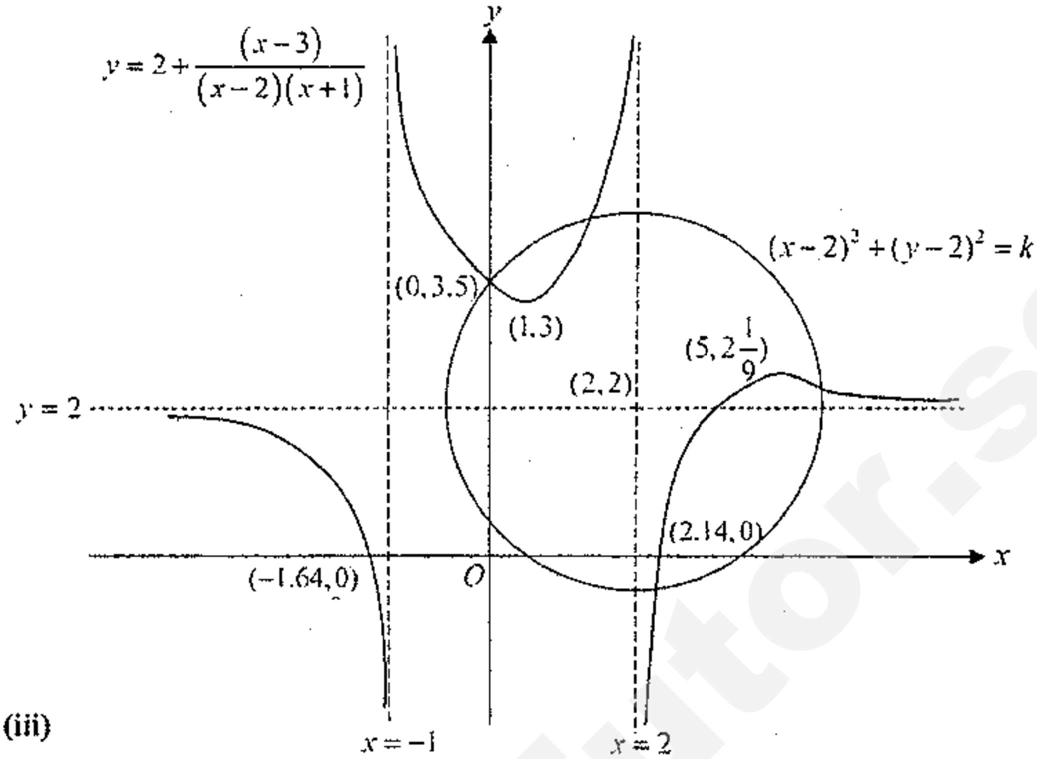
$(9p-19)(p-3) \geq 0$



$p \leq \dfrac{19}{9}$ or $p \geq 3$

$y \leq 2\dfrac{1}{9}$ or $y \geq 3$

(ii)



(iii)

$$(x-2)^2+\frac{(x-3)^2}{(x-2)^2(x+1)^2}=k^2$$

$$(x-2)^2+(y-2)^2=k^2$$

Distance from centre of circle to the $y-$intercept of $y=2+\frac{(x-3)}{(x-2)(x+1)}$

$$=\sqrt{2^2+\left(\frac{7}{2}-2\right)^2}=\frac{5}{2}$$

$k<-2.5$ or $k>2.5$

Note :

1) Be mindful of the link between (i) and (ii). The values 3 and $\frac{19}{9}$ must be some special points on the graph. Look out for those points.

Q8

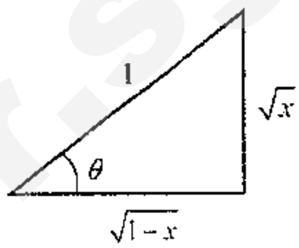
(a)

$$\int\sqrt{\frac{1-x}{x}}\,\mathrm{d}x$$
$$=\int\sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}}\,2\sin\theta\cos\theta\,\mathrm{d}\theta$$
$$=\int 2\cos^2\theta\,\mathrm{d}\theta$$
$$=\int(1+\cos 2\theta)\,\mathrm{d}\theta$$
$$=\theta+\frac{1}{2}\sin 2\theta+C$$
$$=\theta+\sin\theta\cos\theta+C$$
$$=\sin^{-1}\left(\sqrt{x}\right)+\sqrt{x\left(1-x\right)}+C$$

$x=\sin^2\theta$

$\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=2\sin\theta\cos\theta$

Since $\sqrt{x}=\sin\theta$
Consider a right angle triangle
or use trigo identity
$\cos^2\theta+\sin^2\theta=1$



(b)

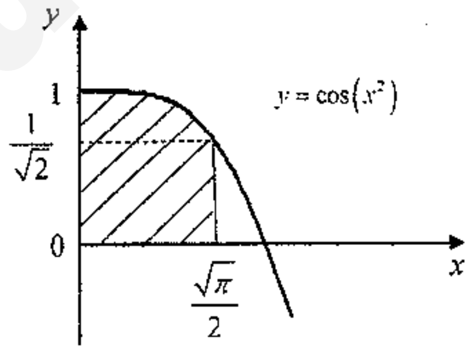
$y=\cos\left(x^2\right)$

$x=0\Rightarrow y=1$

$x=\dfrac{\sqrt{\pi}}{2}\Rightarrow y=\cos\dfrac{\pi}{4}=\dfrac{1}{\sqrt{2}}$

Required volume

$$=\pi\left(\frac{\sqrt{\pi}}{2}\right)^2\left(\frac{1}{\sqrt{2}}\right)+\pi\int_{\frac{1}{\sqrt{2}}}^{1}\cos^{-1}y\,\mathrm{d}y$$
$$=\frac{\pi^2}{4\sqrt{2}}+\pi\int_{\frac{1}{\sqrt{2}}}^{1}\cos^{-1}y\,\mathrm{d}y$$



$$\int\cos^{-1}y\,\mathrm{d}y$$
$$=y\cos^{-1}y-\int-\frac{y}{\sqrt{1-y^2}}\,\mathrm{d}y$$

Let $u=\cos^{-1}y$ $\qquad \dfrac{\mathrm{d}v}{\mathrm{d}y}=1$

$\dfrac{\mathrm{d}u}{\mathrm{d}y}=-\dfrac{1}{\sqrt{1-y^2}}$ $\qquad v=y$

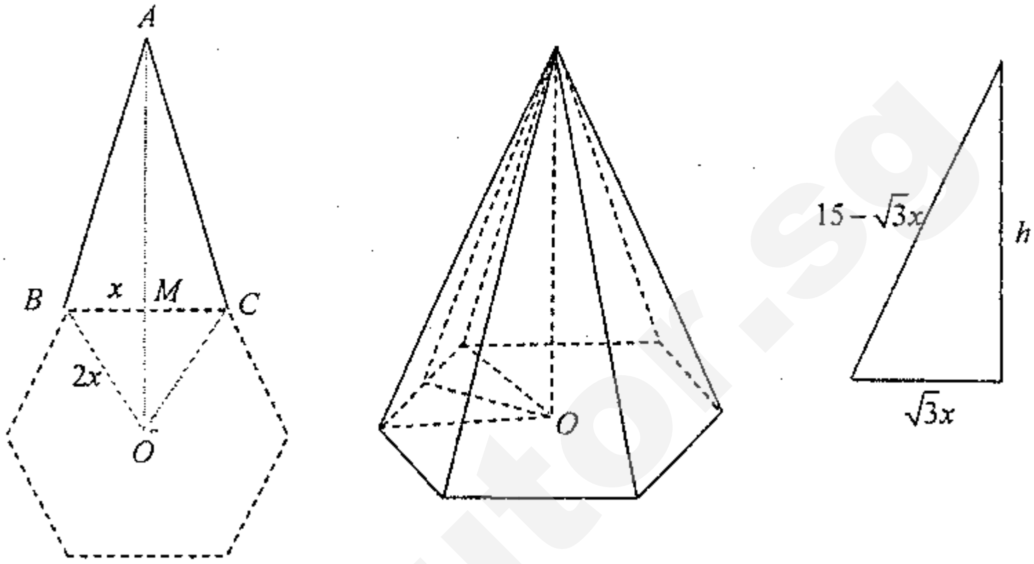
$$=y\cos^{-1}y-\frac{1}{2}\int\frac{-2y}{\sqrt{1-y^2}}\,\mathrm{d}y$$
$$=y\cos^{-1}y-\frac{1}{2}[2(1-y^2)^{\frac{1}{2}}]+c$$
$$=y\cos^{-1}y-\sqrt{1-y^2}+c$$

Required volume $=\dfrac{\pi^2}{4\sqrt{2}}+\pi\left[y\cos^{-1}y-\sqrt{1-y^2}\right]_{\frac{1}{\sqrt{2}}}^{1}$

$$= \frac{\pi^2}{4\sqrt{2}} + \pi\left[0 - \left(\frac{1}{\sqrt{2}}\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi}{\sqrt{2}}$$

Q9



$OM = \sqrt{(2x)^2 - x^2} = \sqrt{3}x$

$\therefore \quad AM = 15 - \sqrt{3}x$

Let h cm be the height of the pyramid.

$h^2 = (15 - \sqrt{3}x)^2 - (\sqrt{3}x)^2$

$\quad = 225 - 30\sqrt{3}x + 3x^2 - 3x^2$

$\quad = 225 - 30\sqrt{3}x \qquad$ (shown)

Area of hexagon $= 6\times$ area of triangle OBC

$\qquad = 6(\frac{1}{2})(2x)(\sqrt{3}x)$

$\qquad = 6\sqrt{3}x^2$

$\therefore \quad V = \frac{1}{3}(6\sqrt{3}x^2)\sqrt{225 - 30\sqrt{3}x}$

$V^2 = 180x^4(15 - 2\sqrt{3}x)$ (shown)

$V^2 = 180(15x^4 - 2\sqrt{3}x^5)$

Differentiating wrt x,

$2V\frac{dV}{dx} = 180\left(60x^3 - 10\sqrt{3}x^4\right)$

$\qquad = 1800x^3\left(6 - \sqrt{3}x\right)$

$\frac{dV}{dx} = 0 \Rightarrow x = 0$ or $x = \frac{6}{\sqrt{3}} = 2\sqrt{3}$

(NA as $x > 0$)

Alternatively,

$V = 6\sqrt{5}x^2)\sqrt{15 - 2\sqrt{3}x}$

$\frac{dV}{dx} = (6\sqrt{5}x^2)\frac{1}{2}(15 - 2\sqrt{3}x)^{-\frac{1}{2}}(-2\sqrt{3})$

$\qquad + (15 - 2\sqrt{3}x)^{\frac{1}{2}}(12\sqrt{5}x)$

$= 12\sqrt{5}x(15 - 2\sqrt{3}x)^{\frac{1}{2}} - 6\sqrt{15}x^2(15 - 2\sqrt{3}x)^{-\frac{1}{2}}$

$= 6\sqrt{5}x(15 - 2\sqrt{3}x)^{-\frac{1}{2}}[2(15 - 2\sqrt{3}x) - \sqrt{3}x]$

$= \frac{30\sqrt{5}x(6 - \sqrt{3}x)}{\sqrt{(15 - 2\sqrt{3}x)}}$

$\frac{dV}{dx} = 0 \Rightarrow x = 0$ or $x = \frac{6}{\sqrt{3}} = 2\sqrt{3}$

(NA as $x > 0$)

To Prove Maximum

Note :

You need to be clear with the values for the test and conclude < or > 0.

Method 1

$$2V\frac{d^2V}{dx^2}+2\left(\frac{dV}{dx}\right)^2=180\left[180x^2-40\sqrt{3}x^3\right]$$

$$x=2\sqrt{3},\ \frac{d^2V}{dx^2}=\frac{180}{2V}\left[180\left(2\sqrt{3}\right)^2-40\sqrt{3}\left(2\sqrt{3}\right)^3\right]=-\frac{64800}{V}<0 \text{ since } V>0$$

Method 2

x	3.4	$2\sqrt{3}=3.46$	3.5
$\frac{dV}{dx}$	$\approx\frac{7855}{2V}>0$	0	$\approx-\frac{4799}{2V}<0$

V is maximum when $x=2\sqrt{3}$ cm.
Max $V=72\sqrt{15}$ cm^3.

(iv)
When $x=2\sqrt{3},\ h^2=225-30\sqrt{3}(2\sqrt{3})=45$
$h=3\sqrt{5}$ cm (reject $h=-3\sqrt{5}$ as $h>0$)

Q10
(i)
$l: x+2=\frac{4-y}{3}, z=0$

$$l:\mathbf{r}=\begin{pmatrix}-2\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\0\end{pmatrix} \qquad \lambda\in\mathbb{R}$$

$$\begin{pmatrix}-2\\4\\0\end{pmatrix}-\begin{pmatrix}-3\\4\\-2\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$$\begin{pmatrix}1\\-3\\0\end{pmatrix}\times\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}-6\\-2\\3\end{pmatrix}$$

$$r\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}=\begin{pmatrix}-3\\4\\-2\end{pmatrix}\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}=-4$$

$p: 6x+2y-3z=-4$

(ii)

To find intersection between y-axis and l, sub $x=0$ into l

$$0+2=\frac{4-y}{3}\Rightarrow y=-2$$

Thus, point of intersection is $(0,-2,0)$.

Point of reflection of $(-2,4,0)$ about y-axis is $(2,4,0)$

$$\begin{pmatrix}2\\4\\0\end{pmatrix}-\begin{pmatrix}0\\-2\\0\end{pmatrix}=\begin{pmatrix}2\\6\\0\end{pmatrix}=2\begin{pmatrix}1\\3\\0\end{pmatrix}$$

Line of reflection, $l':\mathbf{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+s\begin{pmatrix}1\\3\\0\end{pmatrix}\qquad s\in\mathbb{R}$

(iii)

$$\begin{pmatrix}-9\\9\\-6\end{pmatrix}-\begin{pmatrix}-3\\4\\-2\end{pmatrix}=\begin{pmatrix}-6\\5\\-4\end{pmatrix}$$

$$\left|\begin{pmatrix}-6\\5\\-4\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\0\end{pmatrix}\right|=\left|\begin{pmatrix}-6\\5\\-4\end{pmatrix}\right|\left|\begin{pmatrix}1\\-3\\0\end{pmatrix}\right|\cos\theta$$

$$\cos\theta=\frac{21}{\sqrt{(-6)^2+5^2+(-4)^2}\sqrt{1^2+3^2}}=\frac{21}{\sqrt{770}}$$

$$\theta=40.8^\circ$$

(iv)

Let the point that is equidistant from both planes be C.

$\overrightarrow{OC}=\begin{pmatrix}-3\\4\\-2\end{pmatrix}+t\begin{pmatrix}6\\-5\\4\end{pmatrix}$ for some $t\in\mathbb{R}$

Distance of C from p = Distance of C from x-y plane

$$\frac{\left|\left[\begin{pmatrix}-3+6t\\4-5t\\-2+4t\end{pmatrix}-\begin{pmatrix}-3\\4\\-2\end{pmatrix}\right]\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}\right|}{\sqrt{6^2+2^2+3^2}}=\frac{\left|\left[\begin{pmatrix}-3+6t\\4-5t\\-2+4t\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{0^2+0^2+1^2}}$$

$$\frac{|36t-10t-12t|}{7}=|-2+4t|$$

$$|t|=|2t-1|$$

$t^2=4t^2-4t+1$

$3t^2-4t+1=0$

| or

$t=2t-1$ or $t=-2t+1$

$t=1$ or $t=\frac{1}{3}$

$$\overrightarrow{OC}=\begin{pmatrix}-3\\4\\-2\end{pmatrix}+(1)\begin{pmatrix}6\\-5\\4\end{pmatrix}=\begin{pmatrix}3\\-1\\2\end{pmatrix} \text{ or } \quad \overrightarrow{OC}=\begin{pmatrix}-3\\4\\-2\end{pmatrix}+\left(\frac{1}{3}\right)\begin{pmatrix}6\\-5\\4\end{pmatrix}=\left(\frac{1}{3}\right)\begin{pmatrix}-3\\7\\-2\end{pmatrix}$$

The 2 points are $(3,-1,2)$ and $\left(-1,\frac{7}{3},-\frac{2}{3}\right)$.

Candidate Name: ____________________ Class: ____________

PIONEER JUNIOR COLLEGE
While I Live, I Learn

JC2 PRELIMINARY EXAM
Higher 2

MATHEMATICS **9758/02**
Paper 2
18 Sept 2017
3 hours

Additional Materials: Cover page
Answer papers
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your full name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

Section A: Pure Mathematics [40 marks]

1 Referred to origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point P lies on OA produced such that $OA : AP = 1 : \lambda$. Point Q lies on OB, between O and B, such that $OQ : QB = 3 : 1$. The mid-point of PB is M. Show that the ratio of the area of triangle OPM to the area of triangle OQM is independent of λ. [5]

2 By differentiating $\cos x \dfrac{\mathrm{d}y}{\mathrm{d}x}$ with respect to x, solve the differential equation

$$\cos x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \sin x \frac{\mathrm{d}y}{\mathrm{d}x} = \sec^2 x + \cos 2x,$$

giving y in terms of x. [6]

3 **(a)** State a sequence of transformations which transform the graph of $y = \ln(2x+1)$ to the graph of $y = \ln\left(\dfrac{3}{2x-1}\right)$. [3]

(b) It is given that

$$\mathrm{f}(x) = \begin{cases} ax & 0 \le x < 1, \\ a & 1 \le x \le 2, \\ 3a - ax & 2 < x \le 3, \end{cases}$$

and that $\mathrm{f}(x+3) = \frac{1}{2}\mathrm{f}(x)$, for all real values of x, where a is a positive constant.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-2 \le x \le 8$. [3]

(ii) Find, in terms of a, $\int_0^2 \mathrm{f}(-x)\,\mathrm{d}x$. [1]

(iii) Find the value of the constant a for which $\int_0^\infty \mathrm{f}(x)\,\mathrm{d}x = 16$. [2]

4 **Do not use a graphic calculator in answering this question.**

The complex number z is given by $z=-1+\mathrm{i}c$, where c is a non-zero real number. Given that $\frac{z^n}{z^*}$ is purely real, find

(i) the possible values of c when $n=2$, [4]

(ii) the three smallest positive integer values of n when $c=\sqrt{3}$. [5]

5 It is given that $y=\sec 2x$.

(i) Show that $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y^2(y^2-1)$. [3]

(ii) By further differentiation, find the Maclaurin's series for y up to and including the term in x^4. [5]

(iii) By considering $\sec 2x=\frac{1}{\cos 2x}$, check on the correctness of your answer in part **(ii)**. [3]

Section B: Statistics [60 marks]

6 An unbiased disc has a single dot marked on one side and two dots marked on the other side. A tetrahedral die has faces marked with score of 1, 2, 3, and 4. The probability of getting a score of 1, 2, 3, and 4 is $\frac{1}{5}, p, \frac{1}{5}$ and q respectively, where $p, q \in [0,1]$.

A game is played by throwing the disc and the die together. The random variable S is the sum of the score showing on the die and twice the number of dots showing on the disc.

(i) Find $\mathrm{P}(S=6)$. [2]

Given that $\mathrm{P}(S=4)=\frac{1}{6}$,

(ii) calculate the values of p and q, [2]

(iii) and find the probability distribution of S. [2]

7 The masses, in kilograms, of black sea bass fish and red tilapia fish sold in a supermarket are normally distributed. The means and standard deviations of these distributions, and the selling prices, in \$ per kilogram, are given in the following table.

	Mean Mass (kg)	Standard Deviation (kg)	Selling Price (\$ per kg)
Black sea bass fish	1.10	0.20	12
Red tilapia fish	0.55	0.05	9

(i) Ayden bought 2 black sea bass fish and 3 red tilapia fish. Find the probability that he pays more than \$40. State an assumption needed in your calculation. [4]

(ii) Five red tilapia fish are randomly chosen. Find the probability that the fifth red tilapia fish is the third red tilapia fish weighing less than half a kilogram. [3]

8 The average amount of cholesterol in one standard fillet of raw red snapper from a fish farm is w mg. To lower the cost of operations, the farmer decides to use a cheaper mixture of fish feed. The farmer conducts a test to check if the average amount of cholesterol in one standard fillet of raw red snapper is affected by the change of fish feed. 50 standard fillets of raw red snapper from 50 different fish were taken and the average amount of cholesterol in these fillets is found to be 78.5 mg, with a standard deviation of 2 mg.

(i) Given that at 5% level of significance, there is insufficient evidence to conclude that the mean amount of cholesterol in one standard fillet of raw red snapper is affected, find the range of possible values of w. [5]

(ii) State, giving a reason, whether any assumptions about the population are needed in order for the test to be valid. [2]

9 A student working on a coding project studies 11-digit quaternary sequences. A quaternary sequence is a sequence formed using the digits 0, 1, 2 or 3. Examples of such sequences are 12030201131, 01122211100, 12321232123 and 00000000000. Find the number of ways that 11-digit quaternary sequences can be formed with

(i) no restriction, [1]

(ii) exactly four 0s and four 2s, [3]

(iii) at least two consecutive digits that are the same. [3]

10 For events A and B, it is given that $\mathrm{P}(A)=\frac{11}{20}$ and $\mathrm{P}(B)=\frac{1}{2}$.

(i) Find the greatest and least possible values of $\mathrm{P}(A\cap B)$. [2]

It is given in addition that $\mathrm{P}(B\mid A')=\frac{7}{9}$.

(ii) Find $\mathrm{P}(A\cup B)$. [2]

(iii) Determine if A and B are independent events. Justify your answer. [2]

(iv) Given another event C such that $\mathrm{P}(C)=\frac{2}{5}$, $\mathrm{P}(A\cup B\cup C)=\frac{19}{20}$, $\mathrm{P}(A\cap B\cap C)=\frac{1}{10}$ and $\mathrm{P}(A\cap C)=2\mathrm{P}(B\cap C)$, find $\mathrm{P}(A\cap C)$. [3]

11 Based on past statistical data, there is a 7% chance that a passenger with reservation for a flight will not show up. In order to maximise revenue, airline companies accept more reservations than the passenger capacity of its planes. State 2 assumptions needed such that the number of passengers who do not show up for a flight may be well modelled by a Binomial distribution. [2]

An airline company operates a flight from Singapore to Maluku on Boeing 737-200 planes, which has capacity of 232 passengers each.

(i) Find the probability that when 245 reservations are accepted, the flight is overbooked, i.e. there is not enough seats available for the passengers who show up. [2]

(ii) Find the maximum number of reservations that should be accepted in order to ensure that the probability of overbooking is less than 1%. [3]

This flight operates once daily throughout the year and 245 reservations are accepted for each flight.

(iii) Find the probability that no flight is overbooked in a week. [2]

(iv) Taking a year as 52 weeks, estimate the probability that the mean number of flights that is overbooked in each week for the year is not more than 1. [3]

12 **(i)** Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A), (B) and (C) below. In each case your diagram should include 5 points, approximately equally spaced with respect to x, and with all x- and y- values positive. The letters p, q, r, s, t and u represent constants.

(A) $y = p + qx^2$, where p is positive and q is negative,

(B) $y = r + se^x$, where r is negative and s is positive,

(C) $y = t + \frac{u}{x}$, where t is positive and u is positive. [3]

Daisy enrolled in a weight management programme to reduce her weight. Her weight, y kg at the end of week x of the programme are given in the table.

x	1	2	3	4	5	6	7
y	74.9	72.9	71.6	70.8	70.4	70.2	70.1

(ii) Draw a scatter diagram to illustrate the data. [2]

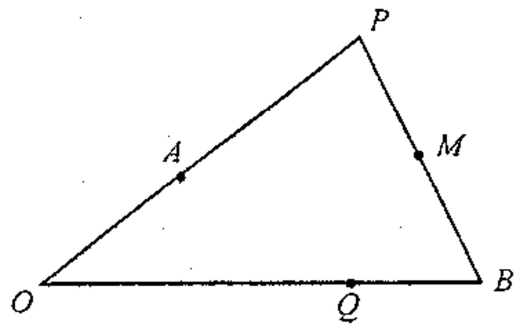
(iii) Explain which of the three cases in part **(i)** is the most appropriate for modelling these values, and calculate the product moment correlation coefficient for this case. [2]

(iv) Use the case that you identified in part **(iii)** to find the equation of a suitable regression line, and use your equation to estimate Daisy's weight at the end of week 10. [3]

(v) Given that 1 week = 7 days, re-write your equation from part **(iv)** so that it can be used to estimate the weight when the time period of the programme is given in days. [2]

Pioneer Junior College
H2 Mathematics **JC2 2017**
JC2 H2 Preliminary Examination Paper 2 (Solution)

Q1



$\overrightarrow{OP} = (\lambda+1)\mathbf{a}$

$$\overrightarrow{OM} = \frac{\overrightarrow{OP}+\overrightarrow{OB}}{2} = \frac{(\lambda+1)\mathbf{a}+\mathbf{b}}{2}$$

area of triangle OPM $= \frac{1}{2}\left|\overrightarrow{OP}\times\overrightarrow{OM}\right|$

$$= \frac{1}{2}\left|(\lambda+1)\mathbf{a}\times\frac{(\lambda+1)\mathbf{a}+\mathbf{b}}{2}\right|$$

$$= \frac{(\lambda+1)}{4}\left|\mathbf{a}\times\mathbf{b}\right|$$

area of triangle OQM $= \frac{1}{2}\left|\overrightarrow{OQ}\times\overrightarrow{OM}\right|$

$$= \frac{1}{2}\left|\frac{3}{4}\mathbf{b}\times\frac{(\lambda+1)\mathbf{a}+\mathbf{b}}{2}\right|$$

$$= \frac{3(\lambda+1)}{16}\left|\mathbf{a}\times\mathbf{b}\right|$$

Ratio of the area of triangle OPM to the area of triangle OQM is

$\frac{(\lambda+1)}{4}:\frac{3(\lambda+1)}{16} = 4:3$ (Shown)

Q2

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos x\frac{\mathrm{d}y}{\mathrm{d}x}\right]=\cos x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\sin x\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos x\frac{\mathrm{d}y}{\mathrm{d}x}\right]=\sec^2 x+\cos 2x$$
$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x}=\int \sec^2 x+\cos 2x\,\mathrm{d}x$$
$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x}=\tan x+\frac{1}{2}\sin 2x+C$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\sec x\tan x+\sin x+C\sec x$$
$$y=\int \sec x\tan x+\sin x+C\sec x\,\mathrm{d}x$$
$$y=\sec x-\cos x+C\ln\left|\sec x+\tan x\right|+D$$

Q3

(a)

Method 1

Step 1 : Translate by 1 unit in the direction of the x-axis.

Step 2 : reflection about the x-axis.

Step 3 : Translate by ln3 units in the direction of the y-axis.

$$y=\ln(2x+1)\to y=\ln\left[2(x-1)+1\right]\to y=-\ln\left(2x-1\right)\to y=\ln 3-\ln\left(2x-1\right)=\ln\left(\frac{3}{2x-1}\right)$$

Method 2

Step 1 : reflection about the x-axis.

Step 2 : Translate by 1 unit in the direction of the x-axis.

Step 3 : Translate by ln3 units in the direction of the y-axis.

$$y=\ln\left(2x+1\right)\to y=-\ln\left(2x+1\right)\to y=-\ln\left[2\left(x-1\right)+1\right]\to y=\ln 3-\ln\left(2x-1\right)=\ln\left(\frac{3}{2x-1}\right)$$

Method 3

Step 1 : reflection about the x-axis.

Step 2 : Translate by ln3 units in the direction of the y-axis.

Step 3 : Translate by 1 unit in the direction of the x-axis.

$$y=\ln\left(2x+1\right)\to y=-\ln\left(2x+1\right)\to y=\ln 3-\ln(2x+1)\to y=\ln 3-\ln\left[2\left(x-1\right)+1\right]=\ln\left(\frac{3}{2x-1}\right)$$

Method 4

Step 1 : Translate by 1 unit in the direction of the x-axis.

Step 2 : Translate by $-\ln 3$ units in the direction of the y-axis.

Step 3 : reflection about the x-axis.

$$y=\ln(2x+1)\to y=\ln[2(x-1)+1]\to y=-\ln 3+\ln(2x-1)\to y=\ln 3-\ln(2x-1)=\ln\left(\frac{3}{2x-1}\right)$$

Method 5

Step 1 : Translate by $-\ln 3$ units in the direction of the y-axis.

Step 2 : Translate by 1 unit in the direction of the x-axis.

Step 3 : reflection about the x-axis.

$$y=\ln(2x+1)\to y=-\ln 3+\ln(2x+1)\to y=-\ln 3+\ln[2(x-1)+1]\to y=\ln 3-\ln(2x-1)=\ln\left(\frac{3}{2x-1}\right)$$

Method 6

Step 1 : Translate by $-\ln 3$ units in the direction of the y-axis.

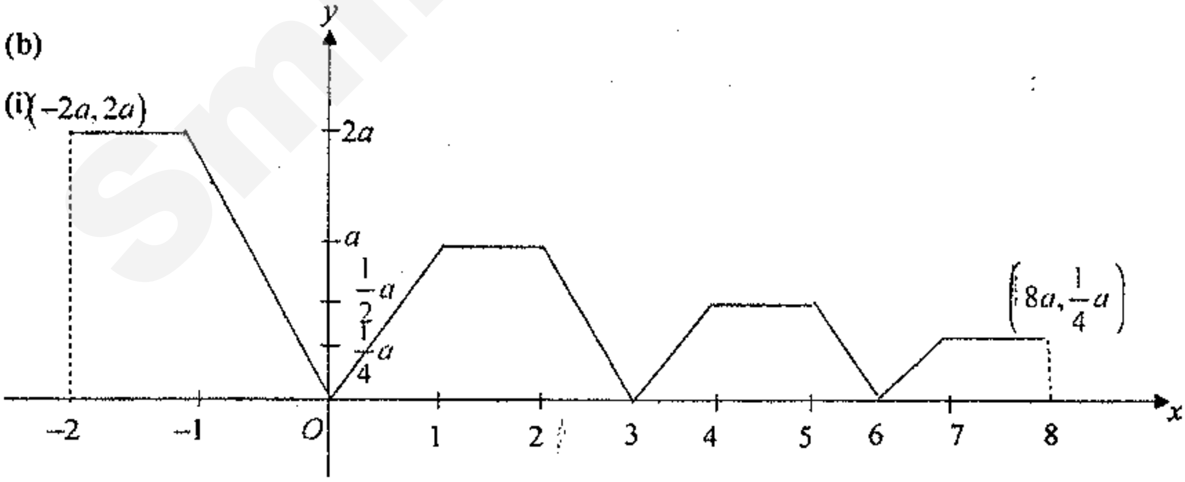
Step 2 : reflection about the x-axis.

Step 3 : Translate by 1 unit in the direction of the x-axis.

$$y=\ln(2x+1)\to y=-\ln 3+\ln(2x+1)\to y=\ln 3-\ln(2x+1)\to y=\ln 3-\ln[2(x-1)+1]=\ln\left(\frac{3}{2x-1}\right)$$

(b)

(i)



(ii)

$$\int_0^2 f(-x)\,dx = \frac{1}{2}(2+1)(2a) = 3a$$

(iii)

$$\int_0^\infty f(x)\,dx = 16$$

$$2\left(a+\frac{1}{2}a+\frac{1}{4}a+......\right)=16$$

$$2a\left(\frac{1}{1-\frac{1}{2}}\right)=16$$

$$4a=16$$

$$a=4$$

Note

(ii) :

$$\int_0^2 f(-x)\,dx = \int_{-2}^0 f(x)\,dx$$

which is the area of the trapezium from $x=-2$ to $x=0$

(iii)

Consider the graph for $x > 0$, which is equivalent to infinite number of rectangles, each of length 2, and breadth $a, \frac{1}{2}a, \frac{1}{4}a....$ which follows a GP.

Q4

(i)

$$\frac{z^2}{z^*} = \frac{(-1+ic)^2}{(-1-ic)}$$

$$= \frac{1-i2c-c^2}{(-1-ic)}$$

$$= \frac{1-i2c-c^2}{(-1-ic)} \times \frac{(-1+ic)}{(-1+ic)}$$

$$= \frac{-1+ic+i2c+2c^2+c^2-ic^3}{1+c^2}$$

Since $\frac{z^2}{z^*}$ is purely real,

$$\frac{3c-c^3}{1+c^2}=0$$

$$c(3-c^2)=0$$

$c=0$ (rej since c is non-zero) $c=\pm\sqrt{3}$

(ii)

$$z=-1+i\sqrt{3}$$

$|z|=2$, $\qquad \arg(z)=\frac{2\pi}{3}$

$$\frac{z^n}{z^*}=\frac{\left(2e^{i\frac{2\pi}{3}}\right)^n}{2e^{i\left(-\frac{2\pi}{3}\right)}}=2^{n-1}e^{i\left(\frac{2n\pi}{3}+\frac{2\pi}{3}\right)}$$

Since $\frac{z^n}{z^*}$ is purely real,

$\arg\left(\frac{z^n}{z^*}\right)=0,\pm\pi,\pm2\pi,\pm3\pi,\pm4\pi,\pm5\pi,\pm6\pi...$ or $\sin\left[(n+1)\frac{2\pi}{3}\right]=0$

$(n+1)\frac{2\pi}{3}=0,\pm\pi,\pm2\pi,\pm3\pi,\pm4\pi,\pm5\pi,\pm6\pi,..$

$n+1=0,\pm\frac{3}{2},\pm3,\pm\frac{9}{2},\pm6,...$

Considering positive integer values only,

$n+1=3,6,9...$

Three smallest positive integer values of n are $2,5,8$

Q5

(i)

$y=\sec 2x$

$\frac{dy}{dx}=2\sec 2x\tan 2x=2y\tan 2x$

$$\begin{aligned}\left(\frac{dy}{dx}\right)^2&=4y^2\tan^2 2x\\&=4y^2(\sec^2 2x-1)\\&=4y^2(y^2-1)\end{aligned}$$

(ii)

$2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=16y^3\frac{dy}{dx}-8y\frac{dy}{dx}$

$\frac{d^2y}{dx^2}=8y^3-4y$

$\frac{d^3y}{dx^3}=24y^2\frac{dy}{dx}-4\frac{dy}{dx}$

$\frac{d^4y}{dx^4}=24y^2\frac{d^2y}{dx^2}+48y\left(\frac{dy}{dx}\right)^2-4\frac{d^2y}{dx^2}$

When $x=0$, $y=1$, $\frac{dy}{dx}=0$, $\frac{d^2y}{dx^2}=4$, $\frac{d^3y}{dx^3}=0$, $\frac{d^4y}{dx^4}=80$

$y = 1 + \frac{x^2}{2!}(4) + \frac{x^4}{4!}(80) + \ldots$

$y = 1 + 2x^2 + \frac{10}{3}x^4 + \ldots$

(iii)

$\sec 2x = \frac{1}{\cos 2x}$

$\approx \frac{1}{1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}}$

$= (1 - 2x^2 + \frac{2}{3}x^4)^{-1}$

$= 1 + (-1)(-2x^2 + \frac{2}{3}x^4) + \frac{(-1)(-2)}{2!}\left(-2x^2 + \frac{2}{3}x^4\right)^2 + \ldots$

$= 1 + 2x^2 - \frac{2}{3}x^4 + 4x^4 + \ldots$

$\approx 1 + 2x^2 + \frac{10}{3}x^4$

Q6

(i)

		Score on the die			
Dots on the disc	S	$1\ (\frac{1}{5})$	$2\ (p)$	$3\ (\frac{1}{5})$	$4\ (q)$
	$1 (\frac{1}{2})$	$3\ (\frac{1}{10})$	$4\ (\frac{1}{2}p)$	$5\ (\frac{1}{10})$	$6\ (\frac{1}{2}q)$
	$2\ (\frac{1}{2})$	$5\ (\frac{1}{10})$	$6\ (\frac{1}{2}p)$	$7\ (\frac{1}{10})$	$8\ (\frac{1}{2}q)$

Since total probability = 1, $\frac{1}{5} + p + \frac{1}{5} + q = 1 \Rightarrow p + q = \frac{3}{5}$

OR: $\left(\frac{1}{10}\right) + \left(\frac{1}{2}p\right) + \left(\frac{1}{5}\right) + \left(\frac{1}{2}p + \frac{1}{2}q\right) + \left(\frac{1}{10}\right) + \left(\frac{1}{2}q\right) = 1 \Rightarrow p + q = \frac{3}{5}$

Hence $P(S = 6) = \frac{1}{2}p + \frac{1}{2}q = \frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$.

(ii)

$P(S = 4) = \frac{1}{6} \Rightarrow \frac{1}{2}p = \frac{1}{6} \Rightarrow p = \frac{1}{3}$

Since $p + q = \frac{3}{5}$, then $q = \frac{3}{5} - \frac{1}{3} = \frac{4}{15}$.

(iii)

s	3	4	5	6	7	8
$P(S = s)$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{2}{15}$

Q7

(i)

$X\sim$ mass of a black sea bass fish. $\quad X\sim \mathrm{N}\left(1.1, 0.2^2\right)$

$Y\sim$ mass of a red tilapia fish. $\quad Y\sim \mathrm{N}\left(0.55, 0.05^2\right)$

Let T be the total cost of 2 black sea bass and 3 red tilapia. Then

$T = 12\left(X_1+X_2\right)+9\left(Y_1+Y_2+Y_3\right)$

$$\begin{aligned}\mathrm{E}(T) &= (12)(2)\mathrm{E}(X)+(9)(3)\mathrm{E}(Y)\\ &= 26.4+14.85\\ &= 41.25\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(T) &= (12)^2(2)\mathrm{Var}(X)+(9)^2(3)\mathrm{Var}(Y)\\ &= 11.52+0.6075\\ &= 12.1275\end{aligned}$$

Thus $T\sim \mathrm{N}\left(41.25\ ,\ 12.1275\right)$.

$\mathrm{P}(T>40) = 0.64018 \approx 0.640$ (3 s.f.)

An assumption needed is the price / mass of all fish are independent of one another.

(ii)

Probability required $= \dfrac{4!}{2!2!}\,[\mathrm{P}\,(Y>0.5)]^2\ [\mathrm{P}\,(Y<0.5)]^3 \approx 0.0170$

Q8

(i)

$X\sim$ amount of cholesterol in one standard fillet of raw red snapper.

unbiased estimate of population variance σ^2 is $s^2 = \dfrac{n}{n-1}{\sigma_x}^2 = \dfrac{50}{49}(2)^2 = \dfrac{200}{49}$

Test $\quad \mathrm{H}_0 : \mu = w \quad$ vs $\quad \mathrm{H}_1 : \mu \neq w$

Since $n=50$ is large, by Central Limit Theorem, $\overline{X}\sim \mathrm{N}\left(w, \dfrac{\frac{200}{49}}{50}\right)$ approx.

$\overline{X}\sim N\left(w, \dfrac{4}{49}\right)$ approx.

Level of significance: 5 %

Critical region is $z<-1.9600$ or $z>1.9600$

Standardised test statistic: $z = \dfrac{78.5-w}{\frac{2}{7}}$

Since H_0 is not rejected, z lies outside the critical region.

$-1.9600 < \dfrac{78.5-w}{\frac{2}{7}} < 1.9600$

$-1.96\left(\frac{2}{7}\right) < 78.5 - w < 1.96\left(\frac{2}{7}\right)$

$-1.96\left(\frac{2}{7}\right) - 78.5 < -w < 1.96\left(\frac{2}{7}\right) - 78.5$

$-79.06 < -w < -77.94$

$77.94 < w < 79.06$

(ii)

No. It is not necessary to assume that amount of cholesterol in a standard fillet follows a normal distribution since sample size is large, by Central Limit Theorem, sample mean is normally distributed approximately.

Q9

(i)

No. of ways = $4^{11} = 4194304$

(ii)

Case 1: four 0s, four 2s, with one 1, two 3s or one 3, two 1s

No. of ways = $\frac{11!}{4!4!2!} \times 2 = 69300$

Case 2: four 0s, four 2s, with three 1s or three 3s

No. of ways = $\frac{11!}{4!4!3!} \times 2 = 23100$

Hence the total number of ways is

$69300 + 23100 = 92400$

Alternative

No. of ways = ${}^{11}C_4 \times {}^{7}C_4 \times 2^3 = 92400$

Choose 4 slots from 11 slots to place four 0s. Choose 4 slots from remaining 7 slots to place four 2s. 2 choices (digit 1 or 3) for each of the remaining 3 slots.

(iii)

No. of ways = no. of ways without restriction – no. of ways with no consecutive digits that are the same

$= 4194304 - 4 \times 3^{10} = 3958108$

Q10

(i)

Least value of $P(A \cap B) = P(A) + P(B) - 1 = \frac{11}{20} + \frac{1}{2} - 1 = \frac{1}{20}$

Greatest value of $P(A \cap B) = P(B) = \frac{1}{2}$

(ii)

$P(B \cap A') = P(B | A')P(A') = \frac{7}{9} \times \frac{9}{20} = \frac{7}{20}$

$P(A \cup B) = P(B \cap A') + P(A) = \frac{7}{20} + \frac{11}{20} = \frac{9}{10}$

(iii)

$\because \mathrm{P}(B \mid A') = \frac{7}{9} \neq \frac{1}{2} = \mathrm{P}(B)$, then B and A' are not independent events.

Hence A and B are not independent events.

OR:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

$$\frac{9}{10} = \frac{11}{20} + \frac{1}{2} - \mathrm{P}(A \cap B)$$

$$\mathrm{P}(A \cap B) = \frac{3}{20}$$

$$\mathrm{P}(A)\mathrm{P}(B) = \frac{11}{20} \times \frac{1}{2} = \frac{11}{40}$$

$$\mathrm{P}(A \cap B) \neq \mathrm{P}(A)\mathrm{P}(B)$$

Hence, A and B are not independent events.

(iv)

$$\mathrm{P}\left[C \cap (A \cup B)'\right] = \mathrm{P}(A \cup B \cup C) - \mathrm{P}(A \cup B) = \frac{19}{20} - \frac{9}{10} = \frac{1}{20}$$

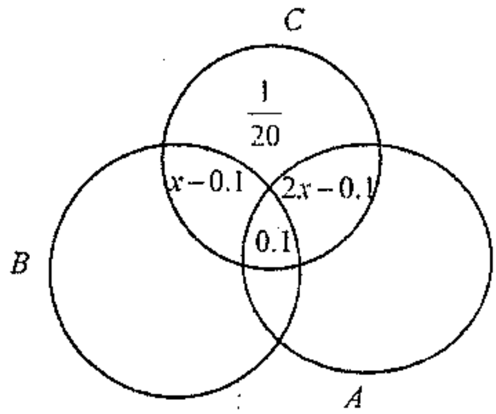
Let $\mathrm{P}(B \cap C) = x$, then $\mathrm{P}(A \cap C) = 2x$

$$\mathrm{P}(C) = \frac{1}{20} + \frac{1}{10} + \left(x - \frac{1}{10}\right) + \left(2x - \frac{1}{10}\right)$$

$$\frac{2}{5} = 3x - \frac{1}{20}$$

$$x = 0.15$$

$$\mathrm{P}(A \cap C) = 2x = 0.3$$



Or

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C)$$

$$\frac{19}{20} = \frac{11}{20} + \frac{1}{2} + \frac{2}{5} - \frac{3}{20} - 2\mathrm{P}(B \cap C) - \mathrm{P}(B \cap C) + \frac{1}{10}$$

$$3\mathrm{P}(B \cap C) = \frac{9}{20}$$

$$\mathrm{P}(B \cap C) = \frac{3}{20}$$

$$\mathrm{P}(A \cap C) = \frac{6}{20} = \frac{3}{10}$$

Q11

2 assumptions:
- Occurrence of show / no show is independent among passengers.
- Probability that a passenger does not show up is constant.

(i)
X ~ number of passengers with reservation, who show up, out of 245.
$X \sim \mathrm{B}(245\ ,\ 0.93)$
$\mathrm{P}(X > 232) = 1 - \mathrm{P}(X \le 232) = 0.118761 \approx 0.119$ (3 s.f.)

(ii)
W ~ number of passengers with reservation, who show up, out of n.
$W \sim \mathrm{B}(n\ ,\ 0.93)$
$\mathrm{P}(W > 232) < 0.01$
$1 - \mathrm{P}(W \le 232) < 0.01$
$\mathrm{P}(W \le 232) > 0.99$
Using GC,
When $n = 239$, $\mathrm{P}(W \le 232) = 0.998 > 0.99$
When $n = 240$, $\mathrm{P}(W \le 232) = 0.995 > 0.99$
When $n = 241$, $\mathrm{P}(W \le 232) = 0.989 < 0.99$
When $n = 242$, $\mathrm{P}(W \le 232) = 0.977 < 0.99$
Hence the maximum reservations that should be accepted is 240.

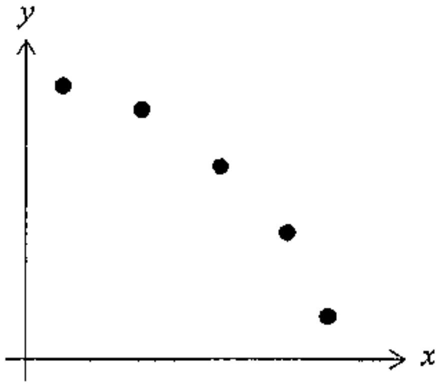
(iii)
Y ~ number of flights which is overbooked, out of 7.
$Y \sim \mathrm{B}(7\ ,\ 0.118761)$
$\mathrm{P}(Y = 0) = 0.41272 \approx 0.413$ (3 s.f.)

(iv)
Since n = 52 is large, by CLT,
$\overline{Y} \sim N(\mathrm{E}(Y), \frac{\mathrm{Var}(Y)}{52})$ approximately
$\overline{Y} \sim N\left((7)(0.118761), \frac{(7)(0.118761)(0.881239)}{52}\right)$ approximately
$\overline{Y} \sim N\left(0.831327, \frac{0.732598}{52}\right)$ approximately
$\overline{Y} \sim N(0.831327, 0.014088)$ approximately
$\mathrm{P}(\overline{Y} \le 1) = 0.9223521 \approx 0.922$ (3 s.f.)

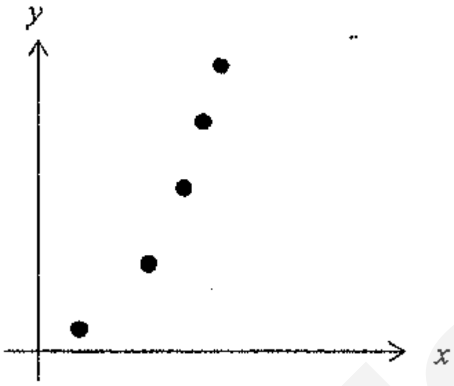
Q12

(i)

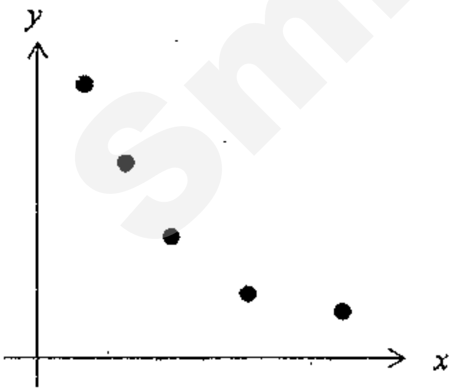
$y = p + qx^2$



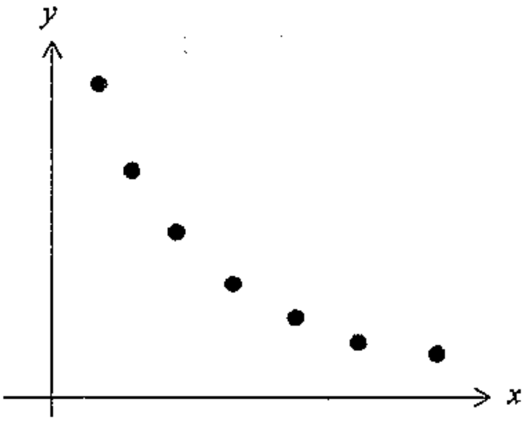
$y = r + se^x$



$y = t + \frac{u}{x}$



(ii)



(iii)

As x increases, y decreases at a decreasing rate. Hence, model (C) is the most appropriate.
Using GC, $r = 0.984$

(iv)

Equation of regression line: $y = 69.425 + \dfrac{5.75555}{x} \approx 69.4 + \dfrac{5.76}{x}$

When $x = 10$,

$y = 69.425 + \dfrac{5.75555}{10} = 70.0$

(iv)

Replace x with $\dfrac{x}{7}$,

New equation:

$y = 69.425 + \dfrac{5.75555}{\frac{x}{7}} \approx 69.4 + \dfrac{40.3}{x}$

RI H2 Mathematics 2017 Prelim Exam Paper 1 Question

1 A local wholesaler sells Pikachi plushies in two sizes, small and large. The number of Pikachi plushies bought by three particular retailers and the total amount they paid are shown in the following table.

Retailer	Small	Large	Total Amount paid
A	30	50	\$1375
B	k	$2k$	\$2704
C	$2k$	k	\$2522

Find the price of each small and each large Pikachi plushy and determine the value of k. [4]

2 A right circular cone has base radius r cm and height h cm. As r and h vary, its curved surface area, $\pi r\sqrt{\left(r^2+h^2\right)}$ cm^2, remains constant.

It is given that when $r=\sqrt{2}$ cm, the magnitude of the rate of change of h is 10 times the magnitude of the rate of change of r. Given also that $h>r$, find the height of the cone at this instant. [4]

3

(a) Find $\displaystyle\int \frac{x+2}{\sqrt{\left(1-8x-4x^2\right)}}\,\mathrm{d}x$. [4]

(b) Use the substitution $x=2\sec\theta$ to find the exact value of $\displaystyle\int_2^4 \frac{1}{x}\sqrt{\left(x^2-4\right)}\,\mathrm{d}x$. [4]

4 A curve C has equation $y=\mathrm{f}\left(x\right)$, where

$$\mathrm{f}\left(x\right)=\frac{a}{\left(x+b\right)^2}+cx,$$

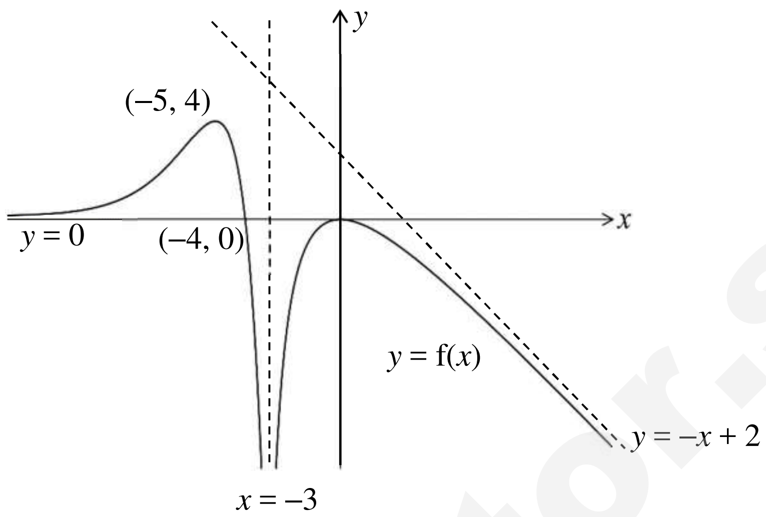
and a, b and c are constants. It is given that C has a vertical asymptote $x=-1$ and a minimum point at $(0,\ 1)$.

(i) Find the values of a, b and c. [4]

(ii) Sketch the graph of $y=\mathrm{f}\left(|x|\right)$, stating the coordinates of any point(s) of intersection with the axes and the equation(s) of any asymptote(s). [3]

(iii) Hence, solve the inequality $\mathrm{f}\left(|x|\right)-4>0$. [2]

5



The diagram shows the curve $y=\mathrm{f}(x)$. The curve has maximum points at $(-5,\ 4)$ and the origin, and crosses the x-axis at $(-4,\ 0)$. The lines $y = 0$, $x = -3$ and $y = -x + 2$ are the horizontal, vertical and oblique asymptotes to the curve respectively.

On separate diagrams, draw sketches of the graphs of

(a) $y=\dfrac{1}{\mathrm{f}\left(x\right)}$, [3]

(b) $y=\mathrm{f}'\left(x\right)$, [3]

(c) $y=\mathrm{f}\left(\dfrac{x+1}{2}\right)$, [3]

labelling clearly the equation(s) of any asymptote(s), coordinates of any axial intercept(s) and turning point(s) where applicable.

6

(i) Given that $y=\ln\left(1+\sin 2x\right)$, show that $\mathrm{e}^{y}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}^{y}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-4\sin 2x$.

Find the first three non-zero terms in the Maclaurin's series for y. [5]

(ii) It is given that the three terms found in part **(i)** are equal to the first three terms in the series expansion of $ax(1+bx)^n$ for small x. Find the exact values of the constants a, b and n and use these values to find the coefficient of x^4 in the expansion of $ax(1+bx)^n$, giving your answer as a simplified rational number. [5]

7

Mr Tan is planning to set up a home theatre in his spacious rectangular living room. A projector screen with height *a* metres is to be positioned against one of the walls *b* metres above the eye level (see Fig. 1). He is trying to decide on the horizontal distance between the sofa and the screen so that the viewing angle α of the projection screen is as large as possible.

(i) Show that $\alpha = \tan^{-1}\frac{a+b}{x} - \tan^{-1}\frac{b}{x}$, where *x* is the horizontal distance between the sofa and the screen in metres. [1]

(ii) Use differentiation to show that the value of *x* which gives the maximum value of α satisfies the equation

$$\frac{a+b}{x^2+(a+b)^2} = \frac{b}{x^2+b^2}.$$

Solve for *x* and leave your answer in terms of *a* and *b*. [4]
[It is not necessary to verify the nature of the maximum point in this part.]

Mrs Tan proposed an alternative way of arrangement. She proposed to place the sofa against the wall opposite the screen, which is *c* metres away, and to vary the vertical position of the screen placed *y* metres above the eye level in order to maximise the angle α (see Fig. 2).

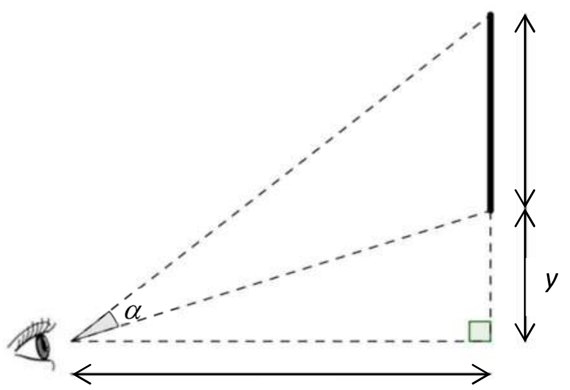


Fig. 2

(iii) Use differentiation to find the value of *y* which gives the maximum value of α, leaving your answer in terms of a. Interpret the answer in this context. [5]

8 A curve C has parametric equations

$$x = \sin^2 t, \quad y = 2\cos t, \quad \text{for } 0 \le t \le \frac{\pi}{2}.$$

(i) Find a cartesian equation of C. [2]

The tangent to the curve at the point P where $t = \frac{\pi}{3}$ is denoted by l.

(ii) Find an equation of l. [3]

(iii) On the same diagram, sketch C and l, stating the coordinates of the axial intercepts and the point of intersection. [3]

The region R is bounded by the curve C, the line l and the y-axis.

(iv) Find the exact value of the volume of revolution formed when R is rotated completely about the x-axis. [3]

9 **Do not use a calculator in answering this question.**

(a) One root of the equation $z^4 + 2z^3 + az^2 + bz + 50 = 0$, where a and b are real, is $z = 1 +$

(i) Show that $a = 7$ and $b = 30$ and find the other roots of the equation. [5]

(ii) Deduce the roots of the equation $w^4 - 2\mathrm{i}w^3 - 7w^2 + 30\mathrm{i}w + 50 = 0$. [2]

(b) Given that $p^* = \dfrac{\left(-\frac{1}{\sqrt{3}} + \mathrm{i}\right)^5}{(1-\mathrm{i})^4}$, by considering the modulus and argument of p^*, find the exact expression for p, in cartesian form $x + \mathrm{i}y$. [4]

10 In a model of forest fire investigation, the proportion of the total area of the forest which has been destroyed is denoted by x. The destruction rate of the fire is defined to be the rate of change of x with respect to the time t, in hours, measured from the instant the fire is first noticed. A particular forest fire is initially noticed when 20% of the total area of the forest is destroyed.

(a) One model of forest fire investigation shows that the destruction rate is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{10}x(1-x).$$

(i) Express the solution of the differential equation in the form $x = \mathrm{f}(t)$ and sketch the part of the curve for $t \geq 0$. [6]

(ii) Find the exact time when the destruction rate is at its maximum. [2]

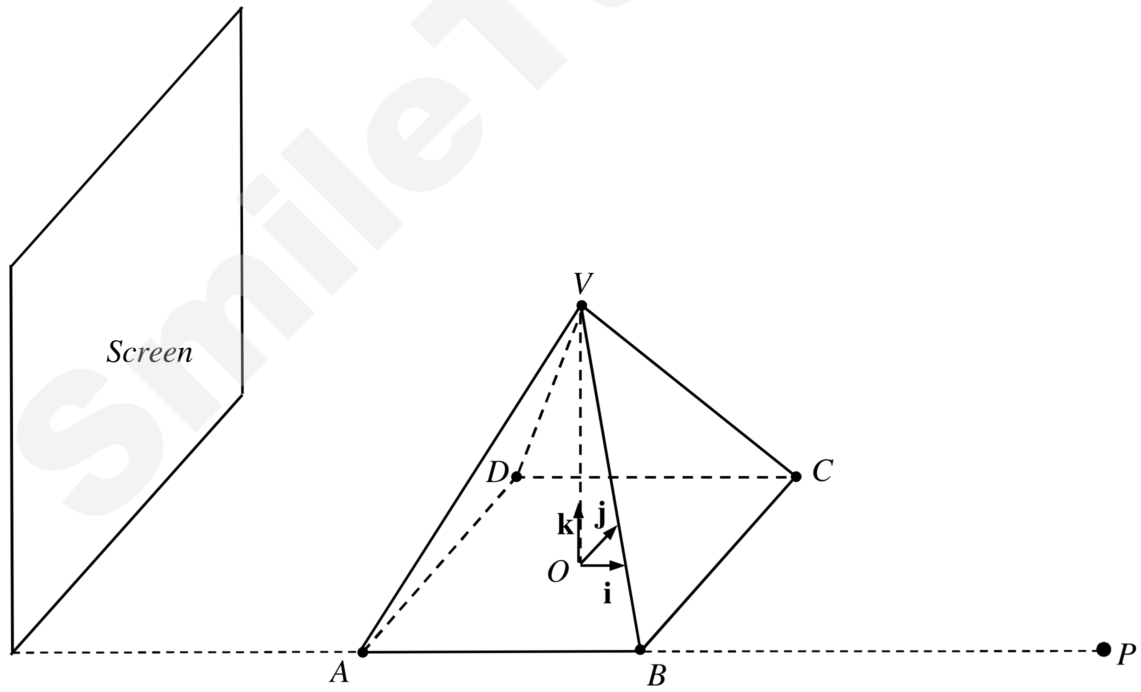
(iii) Explain briefly why this model cannot be used to estimate how long the forest has been burning when it is first noticed. [1]

(b) A second model for the investigation of forest fire is suggested and given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{5\pi\left[1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2\right]}.$$

Determine how long the forest has been burning when the fire is first noticed. [3]

11



A right opaque pyramid with square base $ABCD$ and vertex V is placed at ground level for a shadow display, as shown in the diagram. O is the centre of the square base $ABCD$, and perpendicular unit vectors **i**, **j**, **k** are in the directions of $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{OV}$ respectively. The length of AB is 8 units and the length of OV is $2h$ units.

A point light source for this shadow display is placed at the point $P(20, -4, 0)$ and a screen of height 35 units is placed with its base on the ground such that the screen lies on a plane with vector equation $\mathbf{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=\alpha$ where $\alpha<-4$ (see diagram).

(i) Find a vector equation of the line depicting the path of the light ray from P to V in terms of h. [2]

(ii) Find an inequality between α and h so that the shadow of the pyramid cast on the screen will not exceed the height of the screen. [3]

The point light source is now replaced by a parallel light source whose light rays are perpendicular to the screen and it is also given that $h = 10$.

(iii) Find the exact length of the shadow cast by the edge VB on the screen. [3]

A mirror is placed on the plane VBC to create a special effect during the display.

(iv) Find a vector equation of the plane VBC and hence find the angle of inclination made by the mirror with the ground. [4]

RAFFLES INSTITUTION
H2 Mathematics (9758)
2017 Year 6

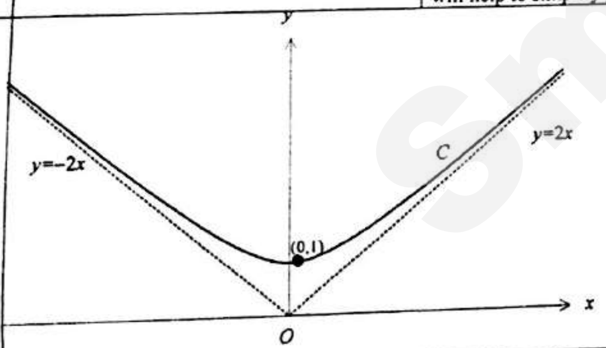
2017 H2 Math 9758 Preliminary Examination Paper 1 : Suggested Solutions

Qn 1		Comments
[4]	Let \$$x$ and \$$y$ be the price of each small and each large Pikachi plushy respectively. Retailer A : $30x + 50y = 1375 \quad \ldots (1)$ Retailer B : $kx + 2ky = 2704$ ÷ by k $x + 2y - 2704\left(\frac{1}{k}\right) = 0 \quad \ldots (2)$ Retailer C : $2kx + ky = 2522$ ÷ by k $2x + y - 2522\left(\frac{1}{k}\right) = 0 \quad \ldots (3)$ From GC : $x = 15$, $y = 18.5$, $\frac{1}{k} = \frac{1}{52}$ Hence, $k = 52$, each small Pikachi plushy costs \$15, and each large Pikachi plushy costs \$18.50.	Most students were able to form the three equations. Only a handful realized the need to linearise the equations before using GC to solve for the unknowns.

Qn 2		Comments
[4]	Let $A = \pi r\sqrt{r^2 + h^2}$. square both sides $\Rightarrow \quad A^2 = \pi^2 r^2 \left(r^2 + h^2\right)$ Differentiate w.r.t. t : $r^2\left(2r\frac{dr}{dt} + 2h\frac{dh}{dt}\right) + \left(r^2 + h^2\right)\left(2r\frac{dr}{dt}\right) = 0$ (Note: $\frac{dA}{dt} = 0$ since A is a constant) Since $r \neq 0$, $\quad \left(2r^2 + h^2\right)\frac{dr}{dt} + hr\frac{dh}{dt} = 0$ $\Rightarrow \quad \left(\frac{dh}{dt}\right) \div \left(\frac{dr}{dt}\right) = \frac{2r^2 + h^2}{-hr}$ When $r = \sqrt{2}$, $\frac{dh}{dt} = -10\frac{dr}{dt} \Rightarrow \frac{4 + h^2}{-\sqrt{2}h} = -10$ $\Rightarrow h^2 - 10\sqrt{2}h + 4 = 0$ Solving: $h = 13.9$ (3sf) or $h = 0.289$ (3sf) Since $h > r$, the height of the cone required is 13.9 cm (to 3 sf).	Students must remember that r and h are variables and not constants. When performing implicit differentiation on the variables with respect to r , h or t , you must have your $\frac{dh}{dr}$ or $\frac{dr}{dt}$ etc. Do remember to substitute the given conditions **after** differentiation and not ***before*** differentiation!

Qn 3		Comments
(a) [4]	$\int \frac{x+2}{\sqrt{1-8x-4x^2}}\,dx$ $= -\frac{1}{8}\int \frac{-8x-8}{\sqrt{1-8x-4x^2}}\,dx + \int \frac{1}{\sqrt{1-8x-4x^2}}\,dx$ complete the square $= -\frac{1}{8}\int \frac{-8x-8}{\sqrt{1-8x-4x^2}}\,dx + \int \frac{1}{\sqrt{5-4(x+1)^2}}\,dx$ $= -\frac{1}{4}\sqrt{1-8x-4x^2} + \frac{1}{2}\sin^{-1}\frac{2\sqrt{5}(x+1)}{5} + C$	This question is not well done. Most split the integrand to $\int \frac{x}{\sqrt{1-8x-4x^2}}\,dx + \int \frac{2}{\sqrt{1-8x-4x^2}}\,dx$ and have difficulty integrating the first term. There are quite a few who wrote $-\frac{1}{8}\int \frac{-8x-8}{\sqrt{1-8x-4x^2}}\,dx = -\frac{1}{8}\frac{\left(1-8x-4x^2\right)^{\frac{3}{2}}}{\frac{3}{2}} + C$
(b) [4]	$x = 2\sec\theta \Rightarrow \frac{dx}{d\theta} = 2\sec\theta\tan\theta$ When $x = 2$, $\sec\theta = 1 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$ When $x = 4$, $\sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ $\int_2^4 \frac{1}{x}\sqrt{\left(x^2-4\right)}\,dx$	There is a significant number of students who do not know how to integrate $\tan^2\theta$ and $\sec^3\theta$. Make sure you know how to integrate all the trigonometric function ($\sin x$, $\cos x$, $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$) and $(\text{trigonometric})^2$ function, such as $\sin^2 x, \cos^2 x, \tan^2 x, \sec^2 x, \operatorname{cosec}^2 x, \cot^2 x$, and be familiar with the formulae/identities given in MF26.

		Comments
	$=\int_0^{\frac{\pi}{3}}\frac{\sqrt{4\sec^2\theta-4}}{2\sec\theta}(2\sec\theta\tan\theta)\,d\theta$ $=\int_0^{\frac{\pi}{3}}\frac{2\tan\theta}{2\sec\theta}(2\sec\theta\tan\theta)\,d\theta$ since $\sqrt{4\sec^2\theta-4}=2\sqrt{\tan^2\theta}=2\tan\theta$ for $0\le\theta\le\frac{\pi}{3}$ $=\int_0^{\frac{\pi}{3}}2\tan^2\theta\,d\theta$ $=2\int_0^{\frac{\pi}{3}}\sec^2\theta-1\,d\theta$ $=2[\tan\theta-\theta]_0^{\pi/3}$ $=2\left[\sqrt{3}-\frac{\pi}{3}\right]$	The other common mistake is not changing the limits of the integration when substituting x by $2\sec\theta$.

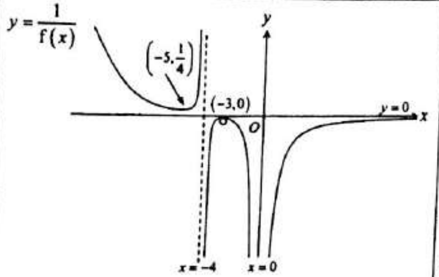
Qn 4		Comments
(i) [4]	$y=\frac{a}{(x+b)^2}+cx$ C has a vertical asymptote $x=-1$ $\Rightarrow$ $b=1$ C passes through $(0,1)$ $\Rightarrow$ $a=1$ $\frac{dy}{dx}=-\frac{2}{(x+1)^3}+c$ At $(0,1)$, $\frac{dy}{dx}=0\Rightarrow c=\frac{2}{1^3}=2$	When finding the derivative of $\frac{a}{(x+b)^2}$ with respect to x, there is no need to use the quotient rule. If the unknown constants bother you, ask yourself how you would proceed to find its derivative if you assume some real values for the unknown constants. Eg $\frac{1}{(x+1)^2}$ If you can identify the values of some of the unknowns immediately, substituting these values into the original expression will help to simplify your calculations.
(ii) [3]		Whenever you are sketching a graph, you should always remember SIA (shape, intercepts, asymptotes). Question also states that (0, 1) is a minimum point, so if it is not featured in the graph,

		something is wrong! Your graph should also be symmetrical about the y-axis.
(iii) [2]	$f(\lvert x\rvert)-4>0 \Leftrightarrow f(\lvert x\rvert)>4$ The line $y=4$ cuts the graph of $y=f(\lvert x\rvert)$ at $x=\pm1.94$ (3sf). $\therefore f(\lvert x\rvert)-4>0 \Leftrightarrow x<-1.94$ or $x>1.94$	Question did not ask for exact answers, so it is not necessary to solve for the intersection points algebraically. You just need to plot a graph and find its intersection with the x-axis using a GC. Your final answer should be symmetrical about the y axis, and remember to give the final non exact answers to 3 sf.

Qn 5

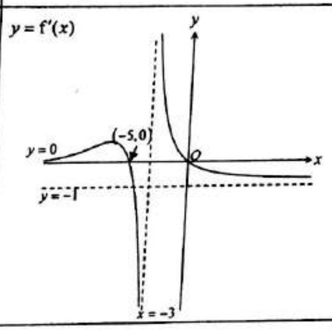
(a) [3]

$y = \dfrac{1}{f(x)}$

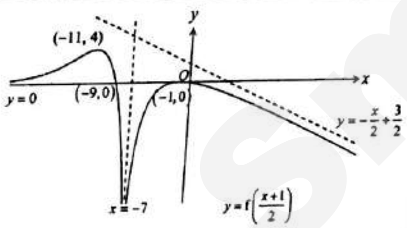


(b) [3]

$y = f'(x)$



(c) [3]



Comments

General

Co-ordinates are required for axial intercept(s) and turning point(s). We want to see $(-3,0)$ marked in part (a), and so on.

Equations of asymptotes must be labelled. Many students left out " $y=0$ " and/or " $x=0$ ".

Curve sketched near asymptote must exhibit the characteristic of approaching nearer and nearer to it.

Use ruler to draw (dotted) straight lines to represent asymptotes!

For (a):
An empty circle ought to be marked at the point $(-3,0)$ to indicate the exclusion of the point.

For (c):
We can't leave the equation of the oblique asymptote as $y=-\dfrac{x+1}{2}+2$. Simplify it.

Many students had the wrong translation unit and/or scaling factor.
There are 2 options for your consideration :

(1) $f(x) \to f\left(\dfrac{x}{2}\right) \to f\left(\dfrac{x+1}{2}\right)$

(2) $f(x) \to f\left(x+\dfrac{1}{2}\right) \to f\left(\dfrac{x}{2}+\dfrac{1}{2}\right)$

Qn 6

(i) [5]

careless

$y = \ln(1+\sin 2x)$

so $e^y = 1+\sin 2x$

Differentiating with respect to x:

$$e^y \frac{dy}{dx} = 2\cos 2x$$
$$e^y \frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 = -4\sin 2x \quad \text{(shown)}$$
$$e^y \frac{d^3y}{dx^3} + e^y\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2e^y\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + e^y\left(\frac{dy}{dx}\right)^3 = -8\cos 2x$$
$$e^y \frac{d^3y}{dx^3} + 3e^y\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + e^y\left(\frac{dy}{dx}\right)^3 = -8\cos 2x$$

When $x=0$, $e^y = 1 \Rightarrow y=0$, $\dfrac{dy}{dx}=2$, $\dfrac{d^2y}{dx^2}=-4$, $\dfrac{d^3y}{dx^3}=8$.

By Maclaurin's Theorem,

$$y = 2x - 4\left(\frac{x^2}{2!}\right) + 8\left(\frac{x^3}{3!}\right) + \cdots$$
$$= 2x - 2x^2 + \frac{4}{3}x^3 + \cdots$$

Comments

Since we need to derive the 2[nd] order DE which involves e^y, we should express $y=\ln(1+\sin 2x)$ as $e^y = 1+\sin 2x$ and apply implicit differentiation. Direct differentiation can be complicated at times.

Note that
$$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$$

An alternative solution involves applying standard Maclaurin expansion although this is not the intended method.

(ii) [5]

$ax(1+bx)^n$

$$= ax\left[1 + n(bx) + \frac{n(n-1)}{2!}(bx)^2 + \frac{n(n-1)(n-2)}{3!}(bx)^3 + \ldots\right]$$
$$= ax + nabx^2 + \frac{n(n-1)}{2}ab^2x^3 + \ldots$$

By comparing coefficients, $a=2$

$nab = -2 \Rightarrow nb = -1$

Comments

A common mistake involves omitting the term, ax, in the expansion of $ax(1+bx)^n$.
Also, since we do not know whether n is a positive integer or not, we should not use $\dbinom{n}{r}$ or $n!$.

$$\frac{n(n-1)}{2}ab^2=\frac{4}{3}$$
$$\Rightarrow n^2b^2-nb^2=\frac{4}{3}$$
$$\Rightarrow (-1)^2-(-1)b=\frac{4}{3}$$
$$\Rightarrow b=\frac{1}{3} \quad \text{and} \quad n=-3$$

x^4 term in the expansion of $2x\left(1+\frac{1}{3}x\right)^{-3}$

$$=2x\left[\frac{-3(-4)(-5)}{3!}\left(\frac{1}{3}x\right)^3\right]=-\frac{20}{27}x^4$$

$\therefore$ coefficient of $x^4=-\frac{20}{27}$

Qn 7 | **Comments**

(i) [1]

Let β be the angle of elevation of the bottom of the screen from eye-level.

$$\tan(\alpha+\beta)=\frac{a+b}{x}\Rightarrow \alpha+\beta=\tan^{-1}\frac{a+b}{x}$$
$$\tan(\beta)=\frac{b}{x}\Rightarrow \beta=\tan^{-1}\frac{b}{x}$$
$$\alpha=(\alpha+\beta)-\beta=\tan^{-1}\frac{a+b}{x}-\tan^{-1}\frac{b}{x}$$

Comments: Students should define the angles properly. They can also draw a diagram to indicate the angles α and β.

(ii) [4]

$$\frac{d\alpha}{dx}=\frac{1}{1+\left(\frac{a+b}{x}\right)^2}\left(-\frac{a+b}{x^2}\right)-\frac{1}{1+\left(\frac{b}{x}\right)^2}\left(-\frac{b}{x^2}\right)$$
$$=-\frac{a+b}{x^2+(a+b)^2}+\frac{b}{x^2+b^2}$$

For maximum α: $\frac{d\alpha}{dx}=-\frac{(a+b)}{x^2+(a+b)^2}+\frac{b}{x^2+b^2}=0$

$$\frac{(a+b)}{x^2+(a+b)^2}=\frac{b}{x^2+b^2} \quad \text{(Shown)}$$
$$(a+b)(x^2+b^2)=b[x^2+(a+b)^2]$$
$$(a+b)x^2+(a+b)b^2=bx^2+b(a+b)^2$$
$$ax^2=b(a+b)[a+b-b]=ab(a+b)$$
$$x=\sqrt{b(a+b)} \quad \text{or} \quad -\sqrt{b(a+b)} \quad \text{(NA since } x>0)$$

Comments: Students should note the application of chain rule in the showing of the differentiation in the first line.

(iii) [5]

Let the screen to be positioned y metres above the eye level.

$$\alpha=\tan^{-1}\frac{a+y}{c}-\tan^{-1}\frac{y}{c}$$
$$\frac{d\alpha}{dy}=\frac{1}{1+\left(\frac{a+y}{c}\right)^2}\left(\frac{1}{c}\right)-\frac{1}{1+\left(\frac{y}{c}\right)^2}\left(\frac{1}{c}\right)$$
$$=\frac{c}{c^2+(a+y)^2}-\frac{c}{c^2+y^2}$$
$$=\frac{c\left[c^2+y^2-c^2-(a+y)^2\right]}{\left[c^2+(a+y)^2\right](c^2+y^2)}$$
$$=\frac{c\left[y+(a+y)\right]\left[y-(a+y)\right]}{\left[c^2+(a+y)^2\right](c^2+y^2)}$$
$$=\frac{-ac[a+2y]}{\left[c^2+(a+y)^2\right](c^2+y^2)} \qquad =\frac{-2ac\left[y+\frac{a}{2}\right]}{\left[c^2+(a+y)^2\right](c^2+y^2)}$$

For maximum α: $\frac{d\alpha}{dy}=\frac{-ac[a+2y]}{\left[c^2+(a+y)^2\right](c^2+y^2)}=0$

$$\Rightarrow y=-\frac{a}{2} \text{ (since } a\neq 0 \text{ and } c\neq 0)$$

y	$\left(-\frac{a}{2}\right)^-$	$\left(-\frac{a}{2}\right)$	$\left(-\frac{a}{2}\right)^+$
$\left[y+\frac{a}{2}\right]$	<0	0	>0
$-2ac\left[y+\frac{a}{2}\right]$, where $-2ac<0$	>0	0	<0
$\frac{d\alpha}{dy}$	>0	0	<0

Therefore $y=-\frac{a}{2}$ gives the maximum viewing angle α.

Interpretation of the answer:
In order to maximise the viewing angle α, the centre of the screen need to be placed at eye level regardless of the position of the sofa.

Comments: Students should note that it is eye level and not ground level given in the question, for those who concluded that it is impossible to get y being negative.

Students should show how they deduced the signs of the derivative for $y=\left(-\frac{a}{2}\right)^-$ and $y=\left(-\frac{a}{2}\right)^+$

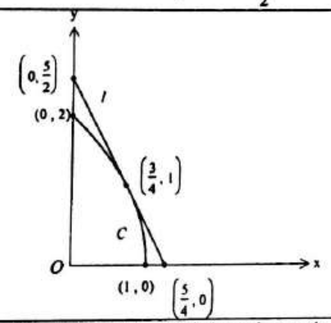
Some were also successful in using the second derivative test, although it is more demanding to obtain.

Qn 8 | **Comments**

(i) [2]

Using $\sin^2 t+\cos^2 t=1$, a cartesian equation of C is

Comments: Important to remember the Trigo identities. Answers such as $y=2\cos\left(\sin^{-1}\sqrt{x}\right)$

	$x+\left(\frac{y}{2}\right)^2=1$ $\Rightarrow y^2=4-4x,\ 0\le x\le 1,\ 0\le y\le 2$ or $\Rightarrow y=2\sqrt{1-x},\ 0\le x\le 1$	are not accepted as they are not simplified. Essential to state $0\le x\le 1$ and/or $0\le y\le 2$ as C is defined for $0\le t\le \frac{\pi}{2}$.
(ii) [3]	Differentiate with respect to x: $1+\frac{y}{2}\frac{dy}{dx}=0 \quad\Rightarrow\quad \frac{dy}{dx}=-\frac{2}{y}$ When $t=\frac{\pi}{3}$, $x=\sin^2\left(\frac{\pi}{3}\right)=\frac{3}{4}$, $y=2\cos\left(\frac{\pi}{3}\right)=1$, $\frac{dy}{dx}=-2$ Hence, an equation of l is $\quad y-1=-2\left(x-\frac{3}{4}\right)$ $y=-2x+\frac{5}{2}$	$\frac{dy}{dx}$ can also be found by parametric or explicit differentiation.
(iii) [3]	 	Sketch C for $0\le x\le 1$ only. State the coordinates of the point of intersection and axial intercepts. Sketch should illustrate that line l is a tangent to curve C at $(\frac{3}{4},1)$.
(iv) [3]	Volume of revolution of R rotated about the x-axis $=\pi\int_0^{\frac{3}{4}}\left(-2x+\frac{5}{2}\right)^2 dx-\pi\int_0^{\frac{3}{4}}(4-4x)\,dx$ $=\pi\left[\frac{\left(-2x+\frac{5}{2}\right)^3}{3(-2)}\right]_0^{\frac{3}{4}}-\pi\left[4x-2x^2\right]_0^{\frac{3}{4}}$ $=-\frac{1}{6}\pi\left[1^3-\left(\frac{5}{2}\right)^3\right]-\pi\left[3-2\left(\frac{3}{4}\right)^2\right]$	As "exact value" is required, you are to show clear working instead of using the GC to obtain the values of the integrals. (You may, of course, use the GC to check your answer).

	$=\frac{39}{16}\pi-\frac{15}{8}\pi$ $=\frac{9}{16}\pi$ units^3 OR Use volume of cone $=\frac{1}{3}\pi r^2h$, i.e. $\left[\frac{1}{3}\pi\left(\frac{5}{2}\right)^2\left(\frac{5}{4}\right)-\frac{1}{3}\pi(1)^2\left(\frac{1}{2}\right)\right]-\pi\int_0^{\frac{3}{4}}(4-4x)\,dx$	

Qn 9		Comments
(a) (i) [5]	Since $z=1+3i$ is a root and the polynomial has real coefficients, $z=1-3i$ is also a root to the polynomial. Hence a quadratic factor of the polynomial is $(z-(1+3i))(z-(1-3i))=$ $(z^2-z(1+3i+1-3i)+(1+3i)(1-3i))=(z^2-2z+10)$ $z^4+2z^3+az^2+bz+50$ $=(z^2-2z+10)(Az^2+Bz+C)$ for some constants A, B and C. By comparing coefficient of z^4 and z^3, $A=1$ and $B-2A=2\Rightarrow B=4$ By comparing the constant term, $C=5$ Hence $z^4+2z^3+az^2+bz+50=(z^2-2z+10)(z^2+4z+5)$ Comparing coefficient of z^2 and z, we have $a=-8+10+5=7$ and $b=40-10=30$ (shown). Solving $z^2+4z+5=0$, $z=\frac{-4\pm\sqrt{4^2-4(5)}}{2}=\frac{-4\pm\sqrt{-4}}{2}=\frac{-4\pm\sqrt{4}\sqrt{-1}}{2}=-2\pm i$. Hence the other roots are $z=1-3i$, $z=-2+i$ and $z=-2-i$. Alternative Solution (more tedious): Since $1+3i$ is a root, $(1+3i)^4+2(1+3i)^3+a(1+3i)^2+b(1+3i)+50=0$ --(1) $(1+3i)^2=1^2+2(3i)+(3i)^2=(1-9)+6i=-8+6i$ $(1+3i)^3=(1+3i)(-8+6i)=(-8-18)+i(6-24)=-26-18i$	Note the following instruction is given at the start of the question - **"Do not use a calculator in answering this question"**. Obviously you can use GC to check your answer, but you are required to show clear working.

$(1+3i)^4 = (-8+6i)^2 = 64 - 96i - 36 = 28 - 96i$

Applying above results on (1),

$(28-96i)+2(-26-18i)+a(-8+6i)+b(1+3i)+50=0$

$$(26-8a+b)+(-132+6a+3b)i=0$$

Comparing real and imaginary parts,

$26-8a+b=0$ and $-132+6a+3b=0$

equivalent to $-44+2a+b=0$

Solving, $-44-26+10a=0 \Rightarrow a=7$ and $b=8(7)-26=30$

$\therefore a=7,\ b=30$ (shown)

Since $z=1+3i$ is a root and the polynomial has real coefficients, $z=1-3i$ is also a root to the polynomial.

$z^4+2z^3+7z^2+30z+50$

$=(z-(1+3i))(z-(1-3i))(z^2+Az+B)$

$=(z^2-2z+10)(z^2+Az+B)$

By comparing coefficients, we have $A=4$, $B=5$.

Solving $z^2+4z+5=0$, $z=\dfrac{-4\pm\sqrt{-4}}{2}=\dfrac{-4\pm\sqrt{4}\sqrt{-1}}{2}=-2\pm i$.

Hence the other roots are $z=1-3i$, $z=-2+i$ and $z=-2-i$.

(ii) [2]

Let $z=iw$, then we get $(iw)^4+2(iw)^3+7(iw)^2+30(iw)+50=0$

$$\Rightarrow w^4-2iw^3-7w^2+30iw+50=0.$$

$z=iw \Rightarrow w=-iz.$

Hence the roots are $w=-i-3$, $w=-i+3$, $w=2i+1$ and $w=2i-1$.

Comments: Note that $\dfrac{1}{i}=-i$

(b) [4]

$$|p|=|p^*|=\frac{\left|\left(-\frac{1}{\sqrt{3}}+i\right)\right|^5}{|(1-i)|^4}=\frac{\left(\frac{2}{\sqrt{3}}\right)^5}{(\sqrt{2})^4}=\frac{32}{4}\left(\frac{1}{\sqrt{3}}\right)^5=\frac{8}{9\sqrt{3}} \text{ or } \frac{8\sqrt{3}}{27}$$

$$\arg(p)=-\arg(p^*)=-\left(5\arg\left(-\frac{1}{\sqrt{3}}+i\right)-4\arg(1-i)\right)+2\pi+2\pi$$

$$=-\left(5\left(\frac{2\pi}{3}\right)-4\left(-\frac{\pi}{4}\right)\right)+2\pi+2\pi$$

$$=-\frac{\pi}{3}$$

$$p=\frac{8}{9\sqrt{3}}\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)=\frac{8}{9\sqrt{3}}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=\frac{4}{9\sqrt{3}}-\frac{4}{9}i \text{ or } \frac{4\sqrt{3}}{27}-\frac{4}{9}i$$

Comments: Note that this question requires you to consider $|p^*|$ and $\arg(p^*)$.

Remember to simplify your surds.

Note that $\arg(1-i)=-\dfrac{\pi}{4}$

Qn 10 | **Comments**

(a)(i) [6]

$$\frac{dx}{dt}=\frac{1}{10}x(1-x)$$

$\Rightarrow \displaystyle\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx=\int\frac{1}{10}dt$

$\Rightarrow \ln\left|\dfrac{x}{1-x}\right|=\dfrac{1}{10}t+C$,

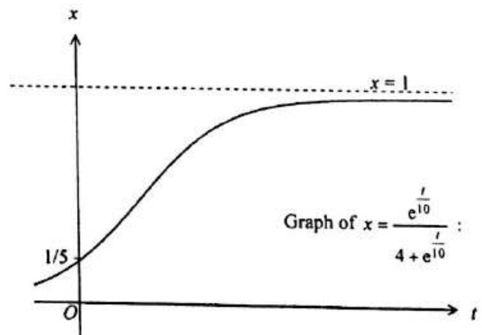
where C is an arbitrary constant

$\Rightarrow \dfrac{x}{1-x}=Ae^{\frac{t}{10}}$, where $A=\pm e^C$

$\Rightarrow \quad x=\dfrac{Ae^{\frac{t}{10}}}{1+Ae^{\frac{t}{10}}}$

When $t=0$, $x=\dfrac{1}{5}$. That is, $\dfrac{1}{5}=\dfrac{A}{1+A}\Rightarrow A=\dfrac{1}{4}$.

Hence, $x=\dfrac{\frac{1}{4}e^{\frac{t}{10}}}{1+\frac{1}{4}e^{\frac{t}{10}}}=\dfrac{e^{\frac{t}{10}}}{4+e^{\frac{t}{10}}}=1-\dfrac{4}{4+e^{\frac{t}{10}}}$.



Comments: If $\ln\dfrac{x}{1-x}$ is given instead of $\ln\left|\dfrac{x}{1-x}\right|$, then $0<x<1$ should be stated to justify the removal of the modulus.

Final answer should be in simplified form, it should not contain a fraction within a fraction like $\dfrac{\frac{1}{4}e^{\frac{t}{10}}}{1+\frac{1}{4}e^{\frac{t}{10}}}$.

(ii) [2]

Since $\dfrac{dx}{dt}=\dfrac{1}{10}x(1-x)$ is a quadratic expression, destruction rate is at its maximum when $x=\dfrac{0+1}{2}=\dfrac{1}{2}$

Or when $\dfrac{d}{dx}\left(\dfrac{dx}{dt}\right)=\dfrac{1}{10}-\dfrac{1}{5}x=0$ i.e., $x=\dfrac{1}{2}$.

Comments: The "**destruction rate** is at its maximum" refers to maximum

	Therefore, $\dfrac{\frac{1}{2}}{1-\frac{1}{2}}=\dfrac{1}{4}e^{\frac{t}{10}}$. $\Rightarrow \quad e^{\frac{t}{10}}=4 \Rightarrow \quad t=10\ln 4$	$\dfrac{dx}{dt}$, not maximum x.
(iii) [1]	From $x=\dfrac{e^{\frac{t}{10}}}{4+e^{\frac{t}{10}}}$, $\quad x=0 \Rightarrow e^{\frac{t}{10}}=0$. Since $e^{\frac{t}{10}}>0$ for all real t, there is no value of t for $x=0$. **OR** Note that $\quad x=1-\dfrac{4}{4+e^{\frac{t}{10}}}$. Since $0<\dfrac{4}{4+e^{\frac{t}{10}}}<1$, we will have $0<x<1$. Hence, $x>0$ for all real values of t, and there is no value of t for $x=0$. **OR** As $x\to-\infty$, $e^{\frac{t}{10}}\to 0^+$, $x=1-\dfrac{4}{4+e^{\frac{t}{10}}}\to 0^+$. Hence, $x=0$ is a horizontal asymptote and there are no values of t giving $x=0$.	The question asks to explain why the model **cannot** be used to estimate (i.e. why we are unable to estimate using the model), it does not ask for why the model may not give a good estimate. So answers like "extrapolation is not reliable" or "the model is not valid for $t<0$" are not accepted.
(b) [3]	$\dfrac{dx}{dt}=\dfrac{1}{5\pi\left[1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2\right]}$ $x=\dfrac{1}{5\pi}\displaystyle\int\dfrac{1}{1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2}\,dt$ $=\dfrac{10}{5\pi}\displaystyle\int\dfrac{\frac{1}{10}}{1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2}\,dt$	

typo here!

	$=\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{10}+\tan\dfrac{\pi}{10}\right)+C$ When $t=0$, $x=\dfrac{1}{5}$. Hence $\dfrac{1}{5}=\dfrac{2}{\pi}\tan^{-1}\left(\tan\dfrac{\pi}{10}\right)+C \Rightarrow C=0$ That is, $x=\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{10}+\tan\dfrac{\pi}{10}\right)$. From G.C., when $x=0$, $t=-3.25$ (3 s.f.) Hence, the forest have been burning for 3.25 hours when it is first noticed.	It is important to have "$+C$" then show that $C=0$. Without this step, no mark can be awarded for the final answer.

Qn 11		Comments
(i) [2]	$\overrightarrow{OP}=\begin{pmatrix}20\\-4\\0\end{pmatrix}$, $\overrightarrow{OV}=\begin{pmatrix}0\\0\\2h\end{pmatrix}$ $\overrightarrow{PV}=\begin{pmatrix}-20\\4\\2h\end{pmatrix}=2\begin{pmatrix}-10\\2\\h\end{pmatrix}$ Vector equation of the line depicting the path of the light ray from P to V is $\mathbf{r}=\begin{pmatrix}20\\-4\\0\end{pmatrix}+\lambda\begin{pmatrix}-10\\2\\h\end{pmatrix}$, $\lambda\in\mathbb{R}$	Note that the vector equation of a line is of the form: $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$ It is not written as $l=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$ or equation of line $=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$
(ii) [3]	$\mathbf{r}.\begin{pmatrix}1\\0\\0\end{pmatrix}=\alpha$ $\mathbf{r}=\begin{pmatrix}20\\-4\\0\end{pmatrix}+\lambda\begin{pmatrix}-10\\2\\h\end{pmatrix}$, $\lambda\in\mathbb{R}$ $\begin{pmatrix}20-10\lambda\\-4+2\lambda\\\lambda h\end{pmatrix}.\begin{pmatrix}1\\0\\0\end{pmatrix}=\alpha$ $20-10\lambda=\alpha\Rightarrow\lambda=\dfrac{20-\alpha}{10}$	"Does not exceed" means that the height is ≤ 35 units. Students who attempted this question by similar triangles must take note that α is a negative value.

	For shadow of the pyramid cast on the screen to not exceed the height of the screen, length of shadow, $\lambda h = \left(\frac{20-\alpha}{10}\right)h \le 35$ $\Rightarrow h \le \frac{350}{20-\alpha}$ since $\alpha < -4$ implies $20-\alpha > 0$	
(iii) [3]	Given that $h = 10$ $\overrightarrow{OB} = \begin{pmatrix}4\\-4\\0\end{pmatrix}$, $\overrightarrow{OV} = \begin{pmatrix}0\\0\\20\end{pmatrix} \Rightarrow \overrightarrow{BV} = \begin{pmatrix}-4\\4\\20\end{pmatrix}$ Length of the shadow cast by edge VB $= \left\|\begin{pmatrix}-4\\4\\20\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}\right\| = \left\|\begin{pmatrix}0\\20\\-4\end{pmatrix}\right\| = \sqrt{416} = 4\sqrt{26}$	Please note that the shadow cast by the edge VB on the screen is not $\|\overrightarrow{VB}\|$.
(iv) [4]	$\overrightarrow{OC} = \begin{pmatrix}4\\4\\0\end{pmatrix}$, $\overrightarrow{OV} = \begin{pmatrix}0\\0\\20\end{pmatrix}$, $\Rightarrow \overrightarrow{CV} = \begin{pmatrix}-4\\-4\\20\end{pmatrix} = -4\begin{pmatrix}1\\1\\-5\end{pmatrix}$ $\overrightarrow{BV} = \begin{pmatrix}-4\\4\\20\end{pmatrix} = 4\begin{pmatrix}-1\\1\\5\end{pmatrix}$ A vector normal to plane VBC is $\cancel{\overrightarrow{BV}\times\overrightarrow{CV}} = \begin{pmatrix}1\\1\\-5\end{pmatrix}\times\begin{pmatrix}-1\\1\\5\end{pmatrix} = \begin{pmatrix}10\\0\\2\end{pmatrix} = 2\begin{pmatrix}5\\0\\1\end{pmatrix}$ Vector equation of the plane VBC is $\mathbf{r}.\begin{pmatrix}5\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\20\end{pmatrix}.\begin{pmatrix}5\\0\\1\end{pmatrix} \Rightarrow \mathbf{r}.\begin{pmatrix}5\\0\\1\end{pmatrix} = 20$ Angle of inclination made by the mirror with the ground	Students must be more careful when computing vectors. There is a lot of computation error for $\overrightarrow{CV}$ and $\overrightarrow{BV}$. This is just a direct application of angle between 2 planes with the normal of the ground (x-y plane) taken to be $\begin{pmatrix}0\\0\\1\end{pmatrix}$. Note that this angle of inclination is **not** the same

	is $\cos^{-1}\left\|\frac{\begin{pmatrix}0\\0\\1\end{pmatrix}.\begin{pmatrix}5\\0\\1\end{pmatrix}}{\sqrt{1}\sqrt{25+1}}\right\| = \cos^{-1}\left\|\frac{1}{\sqrt{26}}\right\| = 78.7^\circ$ (correct to 1 d.p.)	as the angle between $\overrightarrow{BV}$ and $\overrightarrow{BA}$ which is equal to 78.9°

RI H2 Mathematics 2017 Prelim Exam Paper 2 Question

1 Referred to the origin O, the points A, B and C have position vectors **a**, **b** and **c** respectively such that

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}, \quad \mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{c} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

(i) Given that M is the mid-point of AC, use a vector product to find the exact area of triangle ABM. [4]

(ii) Find the position vector of the point N on the line AB such that $\overrightarrow{MN}$ is perpendicular to $\overrightarrow{AB}$. [4]

2 **(a)** **(i)** Show that $\dfrac{1}{r-1} - \dfrac{2}{r} + \dfrac{1}{r+1} = \dfrac{2}{r(r-1)(r+1)}$. [1]

(ii) Hence find $\displaystyle\sum_{r=3}^{n} \frac{4}{r(r-1)(r+1)}$.

(There is no need to express your answer as a single algebraic fraction). [4]

(b) Amy and her brother Ben are saving money together for their family trip. In the first week of 2017, Amy saves \$25 and Ben saves \$2. In each subsequent week, Amy saves \$4 more than the amount she saved in the previous week, and Ben saves 22% more than the amount he saved in the previous week.

(i) Which is the first week in which Ben saves more than Amy in that week? [2]

(ii) They need a combined total of \$2400 for the trip. How many complete weeks do Amy and Ben need to save before they can achieve their targeted amount? [2]

3 The function f is defined as follows.

$$\mathrm{f} : x \mapsto \sqrt{3}\sin x + \cos x, \quad x \in \mathbb{R}, \quad 0 < x < \pi.$$

(i) Write $\mathrm{f}(x)$ as $R\sin(x+\alpha)$, where R and α are constants with exact values to be found. [2]

(ii) Sketch the graph of $y = \mathrm{f}(x)$, stating the axial intercepts, and find the range of f. [3]

(iii) Hence, solve $\mathrm{f}(x) \le 1$ exactly. [2]

The function g is defined as follows:

$$\mathrm{g} : x \mapsto 2\cos\left(x + \frac{\pi}{6}\right), \quad x \in \mathbb{R}, \quad -\frac{\pi}{6} \le x \le b.$$

(iv) Write down the largest exact value of b, for g^{-1} to exist. [1]

(v) Taking the value of b found in part **(iv)**, show that the composite function $\mathrm{g}^{-1}\mathrm{f}$ exists and solve $\mathrm{g}^{-1}\mathrm{f}(x) = x$ exactly. [3]

4	The line l_1 has equation $\frac{x}{-3}=\frac{y}{12}=\frac{z-1}{4}$ and the line l_2 has equation $\frac{x-1}{-3}=y-4=\frac{z-1}{4}$. **(i)** Show that l_1 and l_2 are skew lines. [3] **(ii)** Find a cartesian equation of the plane *p* which is parallel to l_1 and contains l_2. [3] **(iii)** The point $A(0, a, 1)$ is equidistant from *p* and l_1. Calculate the possible values of *a* exactly. [6]
5	For events *X* and *Y*, it is given that $\mathrm{P}(X \mid Y)=\frac{1}{2}$, $\mathrm{P}(Y \mid X)=\frac{2}{3}$ and $\mathrm{P}(X \cup Y)=\frac{5}{6}$. Find **(i)** $\mathrm{P}(X)$, [3] **(ii)** $\mathrm{P}(X \cup Y')$. [2]
6	The power consumption of a randomly chosen Effixion laptop has a normal distribution. The salesman at Elf Superstore claims that the average power consumption of an Effixion laptop is 100 watts. The power consumption, *w* watts, is measured for a random sample of 50 Effixion laptops. The results are summarised as follows. $\sum(w-100)=26 \qquad \sum(w-100)^2=273$ Test whether this data provides evidence at the 3% level of significance, that the salesman has made an understatement. [6] The power consumption of another random sample of 50 Effixion laptops is measured. It is found that the sample variance is 6.25. Using this sample only, find the set of values of $\overline{w}$, correct to 2 decimal places, for which the test would result in the rejection of the null hypothesis in favour of the alternative hypothesis at the 1% level of significance. [4]
7	An unbiased cubical die has the number 1 on one face, the number 2 on two faces and the number 3 on three faces. Adrian invites Benny to play a game. In each round, Benny rolls the die twice. Adrian pays Benny \$*a* if the total score is 2 and \$3 if the total score is 3. However, if the total score is 4, Benny pays Adrian \$2. No payment is made otherwise. **(i)** Find the probability that Adrian pays Benny at least 5 times in 20 rounds. [4] The random variable *X* represents Benny's winnings in each round. **(ii)** Given that $a=6$, find the probability distribution of X. Hence, help Benny decide if he should accept Adrian's invitation to play the game. Justify your answer. [5] **(iii)** Determine the value of *a* for the game to be fair. [1]
8	**(a)** In Country S, each household's monthly income per capita is calculated by taking the gross household income divided by the total number of members in the household. It is assumed

that this amount for a randomly chosen household consisting of 3 members follows a normal distribution with mean \$2601 and standard deviation \$768.

(i) The Ministry of Education offers financial aid to students from households consisting of 3 members each and with a household monthly income per capita lower than \$1800. Find the probability that a randomly chosen household with 3 members does not qualify for financial aid. [1]

(ii) It is found that there is a 50% chance that a randomly chosen household with 3 members has a gross household income between \$5000 and \$$a$, where $a > 5000$. Find the value of a, correct to the nearest dollar. [3]

(b) Mr Tan is self-employed and his monthly income follows a normal distribution with mean \$6000 and standard deviation \$1000 whereas Mrs Tan works part-time and earns a fixed amount of \$1500 a month. Their family's monthly expenditure follows a normal distribution with mean μ dollars and standard deviation 650 dollars.

(i) It was found that 10% of the time they spend more than \$5900 in a month. Find the value of μ, correct to the nearest dollar. [2]

(ii) Mr and Mrs Tan save the remaining amount of their income after deducting their expenditure every month. Find the probability that their monthly savings in August and in September differ by more than \$1000. [4]

(iii) State an assumption needed for your calculation in part **(b)(ii)**. [1]

9 **(i)** Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A) and (B) below. In each case, your diagram should include 6 points, approximately equally spaced with respect to x, and with all x- and y-values positive. The letters a, b, c and d represent constants.

(A) $y = a + bx^2$, where a is positive and b is negative,

(B) $y = c + d\ln x$, where c is positive and d is negative. [2]

The following table shows the Gross Domestic Product (GDP) per capita, \$$x$, and infant mortality rate, y, for a sample of 9 countries.

x (\$)	1375	2502	10569	2966	11539	2036	4260	1433	7427
y	115	69	18	65	17	83	44	112	27

(ii) Draw a scatter diagram for these values, labelling the axes clearly. [2]

(iii) Calculate the product moment correlation coefficient, and explain why its value does not necessarily mean that a linear model is the best model for the relationship between x and y. [2]

	(iv)	State which of the two cases in part **(i)** is more appropriate for modelling the relationship between x and y. Calculate the product moment correlation coefficient and the equation of the appropriate regression line for this case. [3]
	(v)	Use the regression line in part **(iv)** to find an estimate of the infant mortality rate for a country with GDP per capita of $723. Comment on the reliability of your estimate. [3]

10 **(a)** It is given that the probability that 21 randomly chosen people were all born on different days of the year is 0.55631, correct to 5 decimal places.

Find the probability that in a random sample of 22 people, there are at least 2 people with the same date of birth. [3]

[You may assume there are 365 days in a year and the probability that a person is born on any of the 365 days is the same.]

(b) A soccer team consists of 1 goalkeeper, 4 defenders, 4 midfielders and 2 forwards.

Country N has a squad of 3 goalkeepers, 6 defenders, 9 midfielders and 4 forwards.

(i) How many different soccer teams can be formed by country N? [2]

One of the defenders and one of the midfielders in the squad are twin brothers.

(ii) How many different teams can be formed which include at most one of the twin brothers? [3]

The following table shows the dates of birth of the 22 players in the squad of country N:

Jersey Number	Position	Date of birth
1	Goalkeeper	29 October
2	Defender	3 May
3	Defender	17 July
4	Defender	15 May
5	Defender	14 December
6	Midfielder	12 October
7	Midfielder	15 May

Jersey Number	Position	Date of birth
12	Midfielder	15 August
13	Defender	11 July
14	Midfielder	29 March
15	Defender	22 October
16	Midfielder	13 March
17	Forward	29 November
18	Goalkeeper	20 December

8	Forward	10 May	19	Midfielder	5 February
9	Forward	1 July	20	Midfielder	1 March
10	Midfielder	1 April	21	Forward	27 March
11	Midfielder	29 October	22	Goalkeeper	31 October

(iii) Find the probability that the team formed by country N contains no players with the same date of birth. [4]

RAFFLES INSTITUTION
H2 Mathematics (9758)
2017 Year 6

2017 H2 Math 9758 Preliminary Examination Paper 2 : Suggested Solutions

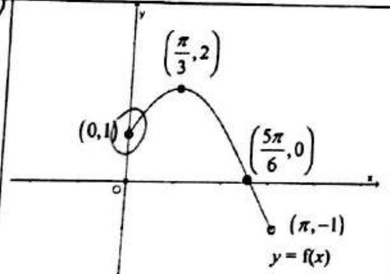
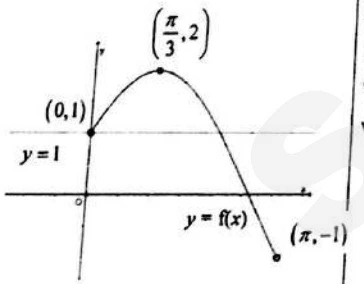
Qn 1		Comments
(i) [4]	$\overrightarrow{OA}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}5\\-2\\3\end{pmatrix}$, $\overrightarrow{OC}=\begin{pmatrix}4\\1\\-2\end{pmatrix}$ Since M is the mid-point of AC, $\overrightarrow{OM}=\dfrac{\begin{pmatrix}2\\3\\-1\end{pmatrix}+\begin{pmatrix}4\\1\\-2\end{pmatrix}}{2}=\dfrac{1}{2}\begin{pmatrix}6\\4\\-3\end{pmatrix}$ $\overrightarrow{AB}=\begin{pmatrix}5\\-2\\3\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}3\\-5\\4\end{pmatrix}$ $\overrightarrow{AM}=\dfrac{1}{2}\begin{pmatrix}6\\4\\-3\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}1\\-1\\-\frac{1}{2}\end{pmatrix}$ Area of ΔABM $=\dfrac{1}{2}\left\vert\overrightarrow{AB}\times\overrightarrow{AM}\right\vert$ $=\dfrac{1}{2}\left\vert\begin{pmatrix}3\\-5\\4\end{pmatrix}\times\begin{pmatrix}1\\-1\\-\frac{1}{2}\end{pmatrix}\right\vert$ $=\dfrac{1}{4}\left\vert\begin{pmatrix}3\\-5\\4\end{pmatrix}\times\begin{pmatrix}2\\-2\\-1\end{pmatrix}\right\vert=\dfrac{1}{4}\left\vert\begin{pmatrix}13\\11\\4\end{pmatrix}\right\vert=\dfrac{\sqrt{306}}{4}=\dfrac{3\sqrt{34}}{4}$	Many candidates did this question correctly. Some candidates wrongly expressed $\overrightarrow{OM}$ as $\frac{1}{2}\overrightarrow{AC}$. A small minority did not realize that the question asked for "exact area", and left their answer as 4.37, rounded to 3 sf, which is not acceptable. For those who attempted to express the final answer as exact, do note that values like $\dfrac{\sqrt{76.5}}{2}$ can be simplified to $\dfrac{3\sqrt{34}}{4}$ by writing 76.5 as a fraction and rationalizing the denominator.

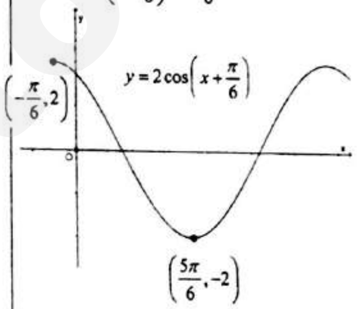
	Or: $\overrightarrow{AB}=\begin{pmatrix}5\\-2\\3\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}3\\-5\\4\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}4\\1\\-2\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}2\\-2\\-1\end{pmatrix}$ Area of ΔABC $=\dfrac{1}{2}\left\vert\overrightarrow{AB}\times\overrightarrow{AC}\right\vert$ $=\dfrac{1}{2}\left\vert\begin{pmatrix}3\\-5\\4\end{pmatrix}\times\begin{pmatrix}2\\-2\\-1\end{pmatrix}\right\vert$ $=\dfrac{1}{2}\left\vert\begin{pmatrix}13\\11\\4\end{pmatrix}\right\vert=\dfrac{\sqrt{306}}{2}$ Area of $\Delta ABM=\frac{1}{2}$Area of $\Delta ABC=\dfrac{\sqrt{306}}{4}=\dfrac{3\sqrt{34}}{4}$	
(ii) [4]	l_{AB}: $\mathbf{r}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}3\\-5\\4\end{pmatrix}$, $\lambda\in\mathbb{R}$ Since point N is on the line AB, $\overrightarrow{ON}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}3\\-5\\4\end{pmatrix}$ for some λ $\overrightarrow{MN}=\begin{pmatrix}3\lambda-1\\1-5\lambda\\\frac{1}{2}+4\lambda\end{pmatrix}$ For $\overrightarrow{MN}$ to be perpendicular to $\overrightarrow{AB}$, $\overrightarrow{MN}.\overrightarrow{AB}=0$ $\begin{pmatrix}3\lambda-1\\1-5\lambda\\\frac{1}{2}+4\lambda\end{pmatrix}.\begin{pmatrix}3\\-5\\4\end{pmatrix}=0$ $\lambda=\dfrac{3}{25}$ $\overrightarrow{ON}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+\dfrac{3}{25}\begin{pmatrix}3\\-5\\4\end{pmatrix}=\dfrac{1}{25}\begin{pmatrix}59\\60\\-13\end{pmatrix}$	Most candidates demonstrated understanding of the concept of finding $\overrightarrow{MN}$ and using it to find λ. A significant number of the candidates who got the wrong answer made careless mistakes in the following arithmetic operations i.e. $\overrightarrow{MN}=\overrightarrow{ON}-\overrightarrow{OM}$ $=\begin{pmatrix}2+3\lambda\\3-5\lambda\\-1+4\lambda\end{pmatrix}-\begin{pmatrix}3\\2\\-1.5\end{pmatrix}$ $=\begin{pmatrix}\vdots\\2.5+4\lambda\end{pmatrix}$ and $\begin{pmatrix}3\lambda-1\\1-5\lambda\\\frac{1}{2}+4\lambda\end{pmatrix}.\begin{pmatrix}3\\-5\\4\end{pmatrix}=0$ $\cdots\cdots+2+8\lambda=0$

Qn 2		Comments
(a)(i) [1]	$\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}=\frac{r(r+1)-2(r-1)(r+1)+r(r-1)}{r(r-1)(r+1)}$ $=\frac{r^2+r-2(r^2-1)+r^2-r}{r(r-1)(r+1)}$ $=\frac{2}{r(r-1)(r+1)}$	Almost all students managed to show this. Just a note that it might be easier to start from LHS and combine the fraction to arrive at RHS instead of trying to break up RHS into partial fractions.
(a)(ii) [4]	$\sum_{r=3}^{n}\frac{4}{r(r-1)(r+1)}=2\sum_{r=3}^{n}\frac{2}{r(r-1)(r+1)}$ $=2\sum_{r=3}^{n}\left(\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}\right)$ $=2\Big[\frac{1}{2}-\frac{2}{3}+\cancel{\frac{1}{4}}$ $+\frac{1}{3}-\cancel{\frac{2}{4}}+\cancel{\frac{1}{5}}$ $+\cancel{\frac{1}{4}}-\cancel{\frac{2}{5}}+\cancel{\frac{1}{6}}$ $+\cancel{\frac{1}{5}}-\cancel{\frac{2}{6}}+\cancel{\frac{1}{7}}$ $+\cancel{\dots}$ $+\cancel{\frac{1}{n-3}}-\cancel{\frac{2}{n-2}}+\cancel{\frac{1}{n-1}}$ $+\cancel{\frac{1}{n-2}}-\cancel{\frac{2}{n-1}}+\frac{1}{n}$ $+\cancel{\frac{1}{n-1}}-\frac{2}{n}+\frac{1}{n+1}\Big]$ $=2\left(\frac{1}{6}-\frac{1}{n}+\frac{1}{n+1}\right)=\frac{1}{3}-\frac{2}{n}+\frac{2}{n+1}$	Most students were able to recognize that this involved MOD. However, some common mistakes were still prevalent: 1. Forgot about the factor 2, i.e. $\sum_{r=3}^{n}\frac{4}{r(r-1)(r+1)}=\sum_{r=3}^{n}\left(\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}\right)$ 2. Did not write down the correct leftover terms after the cancellations, i.e either missed out $\frac{1}{3}$ or $\frac{1}{n}$ in the final expression. 3. Question mentioned that "There is no need to express answer as a single algebraic fraction" but that does not mean that liked terms need not be simplified. 4. Some students tried to split the sum up without realizing that r **cannot** start

		from 1, e.g. $\sum_{r=3}^{n}\left(\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}\right)$ $=\sum_{r=1}^{n}\left(\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}\right)$ $-\sum_{r=1}^{2}\left(\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}\right)$
(b)(i) [2]	Amount Amy saves in nth week $=25+(n-1)(4)=21+4n$ Amount Ben saves in nth week $=ar^{n-1}=2(1.22)^{n-1}$ When Ben saves more than Amy, $2(1.22)^{n-1}>21+4n$ From GC, in the 20th week, Amy saves \$101, Ben saves \$87.47 in the 21st week, Amy saves \$105, Ben saves \$106.72 Hence, Ben first saves more than Amy in the 21st week. Or: $2(1.22)^{n-1}>21+4n \Rightarrow 2(1.22)^{n-1}-21-4n>0$ When $n=20$, $2(1.22)^{n-1}-21-4n=-13.5<0$ When $n=21$, $2(1.22)^{n-1}-21-4n=1.72>0$	Students need to recognize the use of GC for these 2 parts based on the number of marks and also the nature of the inequality which is not easy to solve algebraically. There were a number of students who could not progress after forming the inequality. Some common mistakes made for both (i) and (ii): 1. Use the sum of AP and GP formulae for both parts. 2. Use only the nth term formulae for both parts. 3. Forgot either the sum of AP/GP formula or the nth term formula for AP/GP. 4. Take $r=0.22$.
(b)(ii) [2] use GC!	Total amount in Amy's account after nth week, $=\frac{n}{2}(2a+(n-1)d)=\frac{n}{2}(50+(n-1)(4))=\frac{n}{2}(46+4n)$ Total amount in Ben's account after nth week, $=\frac{a(r^n-1)}{r-1}=\frac{2(1.22^n-1)}{1.22-1}$ For their total saving to exceed \$2400, $\frac{n}{2}(46+4n)+\frac{2(1.22^n-1)}{1.22-1}>2400$ From GC, in the 22nd week, total savings= \$2186.89 < \$2400	

	in the 23rd week, total savings= $2458.72 > $2400 $\therefore$ least $n = 23$. Hence, their total saving will exceed $2400 after 23 complete weeks.	

Qn 3		Comments
(i) [2]	$f(x) = R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$ compare with $f(x) = \sqrt{3}\sin x + \cos x$ $\Rightarrow R\cos\alpha = \sqrt{3},\ R\sin\alpha = 1$ $\Rightarrow R = \sqrt{1^2+\sqrt{3}^2} = 2,\quad \alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ $f(x) = 2\sin\left(x+\frac{\pi}{6}\right)$	Most students are able to do this part correctly.
(ii) [3]	 The range of f is $(-1, 2]$.	Students have to learn to read the question carefully. Many did not label the intercepts. Students also need to consider the domain of f and indicate the coordinates of the end points on the sketch. Quite a handful miss out the open circle on both the end points. A few did not write down the range although they got the graph correct, which is very wasteful.
(iii) [2]		The key word here is **"exactly"**. This means that students have to show algebraic working to get the correct value of $\frac{2\pi}{3}$. Answers without working will not get any credit.

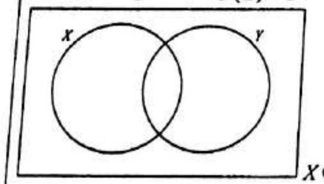
	$f(x) = 2\sin(x+\frac{\pi}{6}) = 1$ $\Rightarrow x+\frac{\pi}{6} = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6}$ $\Rightarrow x = 0$ or $\frac{2\pi}{3}$ The set of values of x is $\left[\frac{2\pi}{3}, \pi\right)$	
(iv) [1]	$g(x) = 2\cos\left(x+\frac{\pi}{6}\right),\quad -\frac{\pi}{6} \le x \le b.$  For g^{-1} to exist, g has to be a 1-1 function. The largest exact value of b, is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.	
(v) [3]	The domain of g^{-1} = the range of g = $[-2, 2]$. Range of f = $(-1,2] \subseteq$ Domain of g^{-1} = $[-2,2]$, therefore $g^{-1}f$ exists. $g^{-1}f(x) = x,\quad 0 < x < \pi$ $g(x) = f(x)$ $2\cos(x+\frac{\pi}{6}) = 2\sin(x+\frac{\pi}{6})$ $\tan(x+\frac{\pi}{6}) = 1$ $x+\frac{\pi}{6} = \frac{\pi}{4}$ $x = \frac{\pi}{12}$	Many students failed to see that the easiest way to solve this is to solve $g(x) = f(x)$. Many proceeded to find the function $g^{-1}f(x)$.

	(Note: $0 < x \le \frac{5\pi}{6}$ considering domain of f and g)	Again the key word is "**exactly**". Marks will not be awarded for answers without algebraic working.

Qn 4		Comments
(i) [3]	$l_1 : \mathbf{r} = \begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}-3\\12\\4\end{pmatrix},\ \lambda \in \mathbb{R}$ $l_2 : \mathbf{r} = \begin{pmatrix}1\\4\\1\end{pmatrix} + \mu\begin{pmatrix}-3\\1\\4\end{pmatrix},\ \mu \in \mathbb{R}$ Since $\begin{pmatrix}-3\\12\\4\end{pmatrix}$ and $\begin{pmatrix}-3\\1\\4\end{pmatrix}$ are not parallel, l_1 and l_2 are not parallel. If the two lines intersect, there will be a unique value of λ and μ for the system of equations $\begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}-3\\12\\4\end{pmatrix} = \begin{pmatrix}1\\4\\1\end{pmatrix} + \mu\begin{pmatrix}-3\\1\\4\end{pmatrix}$ $-3\lambda + 3\mu = 1 \quad (1)$ $12\lambda - \mu = 4 \quad (2)$ $4\lambda - 4\mu = 0 \quad (3)$ Using GC, no solution of λ and μ exist. Hence, the lines do not intersect. Hence, l_1 and l_2 are skew lines.	Common mistakes: - Obtained $\begin{pmatrix}-3\\0\\4\end{pmatrix}$ as the direction vector for l_2 instead of $\begin{pmatrix}-3\\1\\4\end{pmatrix}$. - Concluded that l_1 and l_2 are skew lines without showing that they are not parallel.
(ii) [3]	A normal to $p = \begin{pmatrix}-3\\12\\4\end{pmatrix} \times \begin{pmatrix}-3\\1\\4\end{pmatrix} = \begin{pmatrix}44\\0\\33\end{pmatrix} = 11\begin{pmatrix}4\\0\\3\end{pmatrix}$	Most students did well for this part, although there were a few who did not give the final equation in cartesian form.

	Since (1, 4, 1) lies on l_2 which is on p, $\begin{pmatrix}1\\4\\1\end{pmatrix}\cdot\begin{pmatrix}4\\0\\3\end{pmatrix} = 7$ Hence a cartesian equation for p is $4x + 3z = 7$.	
(iii) [6]	(0, 0, 1) is a point on l_1. $\begin{pmatrix}0\\a\\1\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\a\\0\end{pmatrix}$ Distance from A to l_1 $= \dfrac{\left\|\begin{pmatrix}0\\a\\0\end{pmatrix} \times \begin{pmatrix}-3\\12\\4\end{pmatrix}\right\|}{\left\|\begin{pmatrix}-3\\12\\4\end{pmatrix}\right\|} = \dfrac{1}{13}\left\|\begin{pmatrix}4a\\0\\3a\end{pmatrix}\right\| = \dfrac{1}{13}\lvert a\rvert\left\|\begin{pmatrix}4\\0\\3\end{pmatrix}\right\| = \dfrac{5}{13}\lvert a\rvert$ (1, 4, 1) is a point on p. $\begin{pmatrix}0\\a\\1\end{pmatrix} - \begin{pmatrix}1\\4\\1\end{pmatrix} = \begin{pmatrix}-1\\a-4\\0\end{pmatrix}$ Distance from A to p $= \dfrac{\left\lvert\begin{pmatrix}-1\\a-4\\0\end{pmatrix}\cdot\begin{pmatrix}4\\0\\3\end{pmatrix}\right\rvert}{\left\|\begin{pmatrix}4\\0\\3\end{pmatrix}\right\|} = \dfrac{1}{5}\lvert -4\rvert = \dfrac{4}{5}$	Note that this method is much faster than finding foot of perpendicular, F from A to l_1 and then taking the length of $\overline{AF}$. Instead of factorising $\lvert a\rvert$ out from $\left\|\begin{pmatrix}4a\\0\\3a\end{pmatrix}\right\|$, most students simplify the expression using definition, i.e., $\left\|\begin{pmatrix}4a\\0\\3a\end{pmatrix}\right\| = \sqrt{(4a)^2 + 0^2 + (3a)^2} = \sqrt{25a^2}$, which many mistakenly simplify as $5a$ instead of $5\lvert a\rvert$, hence obtaining only one value of a as final answer in the last step. Similarly, this method is much faster than finding foot of perpendicular, N from A to p and then taking the length of $\overline{AN}$. Many of those who attempted to find $\overline{OF}$ and $\overline{ON}$ committed careless mistakes and lost marks for not getting the correct position vectors and distances.

	Since the point A is equidistant to p and l_1, $$\frac{5}{13}\lvert a\rvert=\frac{4}{5}$$ $$a=\pm\frac{52}{25}$$	

Qn 5		Comments
[3]	$$\mathrm{P}(X\mid Y)=\frac{1}{2}$$ $$\Rightarrow \frac{\mathrm{P}(X\cap Y)}{\mathrm{P}(Y)}=\frac{1}{2}$$ $$\Rightarrow \mathrm{P}(X\cap Y)=\frac{1}{2}\mathrm{P}(Y)$$ $$\mathrm{P}(Y\mid X)=\frac{2}{3}$$ $$\Rightarrow \frac{\mathrm{P}(X\cap Y)}{\mathrm{P}(X)}=\frac{2}{3}$$ $$\Rightarrow \mathrm{P}(X\cap Y)=\frac{2}{3}\mathrm{P}(X)$$ $$\mathrm{P}(X\cap Y)=\frac{1}{2}\mathrm{P}(Y)=\frac{2}{3}\mathrm{P}(X)$$ $$\mathrm{P}(X\cup Y)=\frac{5}{6}$$ $$\Rightarrow \mathrm{P}(X)+\mathrm{P}(Y)-\mathrm{P}(X\cap Y)=\frac{5}{6}$$ $$\Rightarrow \mathrm{P}(X)+\frac{4}{3}\mathrm{P}(X)-\frac{2}{3}\mathrm{P}(X)=\frac{5}{6}$$ $$\Rightarrow \mathrm{P}(X)=\frac{1}{2}$$	Most students were able to do this part correctly. The most common method used was to apply standard results to obtain equations connecting P(X), P(Y) and P($X\cap Y$). Unfortunately, some wrong formula were seen, for example: $\mathrm{P}(X\mid Y)=\frac{\mathrm{P}(X\cap Y)}{\mathrm{P}(X)}$ instead of $\frac{\mathrm{P}(X\cap Y)}{\mathrm{P}(Y)}$, $\mathrm{P}(Y\mid X)=\frac{\mathrm{P}(Y\cap X)}{\mathrm{P}(Y)}$ instead of $\frac{\mathrm{P}(Y\cap X)}{\mathrm{P}(X)}$, and most frequently, $\mathrm{P}(X\cup Y)=\mathrm{P}(X)+\mathrm{P}(Y)+\mathrm{P}(X\cap Y)$ instead of $\mathrm{P}(X)+\mathrm{P}(Y)-\mathrm{P}(X\cap Y)$. Interestingly, there were a few students who used their GC to solve the three equations, obtaining $\mathrm{P}(X)=\frac{1}{2}$, $\mathrm{P}(Y)=\frac{2}{3}$ and $\mathrm{P}(X\cap Y)=\frac{1}{3}$ all at one go.
(ii) [2]	$$\mathrm{P}(X\cap Y)=\frac{2}{3}\mathrm{P}(X)=\frac{2}{3}\left(\frac{1}{2}\right)=\frac{1}{3}$$  $X\cup Y'$	Significant fewer number of students were able to handle this part well. The most common problem was the failure to understand $\mathrm{P}(X\cup Y')$. A large number of students tried to simplify it through the result $\mathrm{P}(X\cup Y')=\mathrm{P}(X)+\mathrm{P}(Y')-\mathrm{P}(X\cap Y')$, but often end up with a complicated sum

	$$\mathrm{P}(X\cup Y')=1-\mathrm{P}(Y)+\mathrm{P}(X\cap Y)$$ $$=1-\frac{4}{3}\left(\frac{1}{2}\right)+\frac{1}{3}$$ $$=\frac{2}{3}$$	/ difference which eventually worked out to be P(Y') instead. Those students who drew a Venn diagram were generally quick and successful in identifying the correct region and thus were able to obtain the correct answer rather effortlessly.

Qn 6		Comments
[6]	Let $X=W-100$. Then we have $\sum x=26$, $\sum x^2=273$ $$\bar{w}=\frac{1}{50}\sum x+100=\frac{26}{50}+100=100.52$$ $$s_w^{\,2}=s_x^{\,2}=\frac{1}{n-1}\left[\sum x^2-\frac{\left(\sum x\right)^2}{n}\right]$$ $$=\frac{1}{49}\left(273-\frac{26^2}{50}\right)$$ $$=5.29551$$ To test $H_0:\mu=100$ vs $H_1:\mu>100$ Perform a 1–tail test at 3% level of significance. Under $\mathrm{H_0}$, $\overline{W}\sim \mathrm{N}\left(\mu_0,\frac{s^2}{n}\right)$ **approximately** where $\mu_0=100$ and $n=50$ Using a z - test, p – value $=0.0550$ (3 s.f.) Since p – value $=0.0550>0.03$, we **do not reject** H_0 and conclude that there is **insufficient** evidence, at 3% significance level, that the salesman made an understatement on the average power consumption of the Effixion laptops.	Define your random variable if you are not using W. Do not confuse μ with $\bar{w}$. If we know the population mean, there would be no need to perform hypothesis testing. W is normally distributed, so do not quote CLT. However, there is still an approximation due to use of the unbiased **estimate** of population variance.
[4]	To test $H_0:\mu=100$ vs $H_1:\mu>100$ Perform a 1–tail test at 1% level of significance. Sample variance $=6.25$ $$s^2=\frac{n}{n-1}[\text{sample variance}]$$	Sample variance is **neither** population variance, **nor** its unbiased estimate.

	$=\dfrac{50}{49}(6.25)=6.37755$ In order to **reject** H_0 **at 1% level of significance**, we need $\mathrm{P}(\overline{W}>\overline{w})\le 0.01$ where $\overline{W}\sim \mathrm{N}\left(100,\dfrac{6.37755}{50}\right)$ approximately $\Rightarrow \mathrm{P}(\overline{W}<\overline{w})\ge 0.99$ $\Rightarrow \overline{w}\ge 100.8308$ Or $z-value \ge z_{0.99}$ $\dfrac{\overline{w}-100}{\sqrt{\dfrac{6.37755}{50}}}\ge 2.3263$ $\overline{w}\ge 100.8308$ (4 d.p.) Set of values of $\overline{w}$ is $\{\overline{w}\in\mathbb{R}:\overline{w}>100.83\}$ or $\{\overline{w}\in\mathbb{R}:\overline{w}\ge 100.84\}$	Note the alternative hypothesis is right-tailed. Standardisation is not necessary, as parameters of $\overline{W}$ are assumed to be known in the test. Due to rounding off, 100.83 is not within the range, while 100.84 is.

Qn 7		Comments
(i) [4]	Die shows: 1, 2, 3; Probability: $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$ P(Adrian pays Benny in a round) = P(total score is 2) + P(total score is 3) $=\dfrac{1}{6}\times\dfrac{1}{6}+\left(\dfrac{1}{6}\times\dfrac{1}{3}\right)2$ $=\dfrac{5}{36}$ Let Y be the number of rounds, out of 20, that Adrian pays Benny. $Y\sim \mathrm{B}\left(20,\dfrac{5}{36}\right)$ $\mathrm{P}(Y\ge 5)$ $=1-\mathrm{P}(Y\le 4)$ $=0.134$ (3 s.f.)	This is NOT the usual 6 sided die. Using a table of this form will help to determine the total score of 2 rolls easily. For P(total score is 3), there are 2 cases : 1st roll 1 & 2nd roll 2 or 1st roll 2 & 2nd roll 1. Define the random variable Y properly and state its distribution. Be careful when taking the complement as the GC can only compute $\mathrm{P}(Y\le k)$ and not $\mathrm{P}(Y<k)$ for Binomial Distributions.

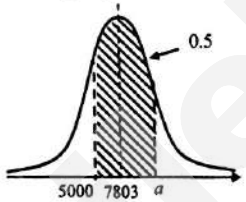
The die table in part (i):

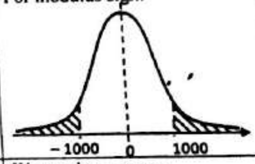
Die shows	1	2	3
Probability	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

(ii) [5]	$\mathrm{P}(\text{total score is 4})=\left(\dfrac{1}{6}\times\dfrac{1}{2}\right)2+\dfrac{1}{3}\times\dfrac{1}{3}=\dfrac{5}{18}$ Method 1 : $\mathrm{P}(\text{total score is 5 or 6})=1-\left(\dfrac{1}{36}+\dfrac{1}{9}+\dfrac{5}{18}\right)=\dfrac{7}{12}$ Method 2 : $\mathrm{P}(\text{total score is 5 or 6})=\left(\dfrac{1}{3}\times\dfrac{1}{2}\right)2+\dfrac{1}{2}\times\dfrac{1}{2}=\dfrac{7}{12}$ Given : X represents Benny's winnings in each round $\mathrm{E}(X)=-\dfrac{1}{18}$ Since $\mathrm{E}(X)<0$, Benny is expected to lose in the long run. Thus, Benny should not accept Adrian's invitation to play the game.	Method 1 is faster, but it does not "help to check" that you are on the right track. Use Method 2 (unless finding this probability is way too long/tedious) to help you check that $\sum \mathrm{P}(X=x)=1$. Many did not realize that x can be zero. Do check that $\sum \mathrm{P}(X=x)=1$. If the probabilities do not add up, check the individual probabilities and also re-evaluate if there were missing x values. $\mathrm{E}(X)$ is the long-run average winnings of Benny in one round. Thus, $\mathrm{E}(X)<0$ does not imply that Benny is expected to lose every round.
(iii) [1]	For the game to be fair, $\mathrm{E}(X)=0$. $\dfrac{a}{36}+\dfrac{1}{3}-\dfrac{10}{18}=0$ $a=8$	

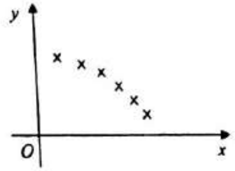
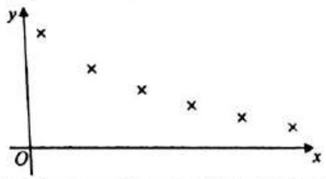
The table of X in part (ii):

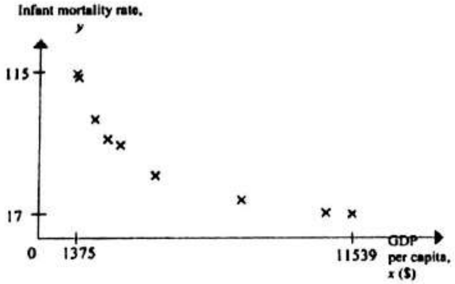
x	6	3	-2	0
$\mathrm{P}(X=x)$	$\frac{1}{36}$	$\frac{1}{9}$	$\frac{5}{18}$	$\frac{7}{12}$

Qn 8		Comments
(ai) [1]	Let X be the random variable denoting the household income per capita in dollars of a randomly chosen family in Country S. Then $X \sim \mathrm{N}(2601,\ 768^2)$. $\mathrm{P}(X>1800)=0.852$ (3s.f.)	A significant number of students misread the question and found the probability to be less than 1800 dollars instead.
(aii) [3]	Let Y be the random variable denoting the gross income in dollars of a randomly chosen family with 3 family members. $Y=3X \sim N(3\times 2601, 9\times 768^2)$ $\Rightarrow Y \sim \mathrm{N}(7803, 9\times 768^2)$ $\mathrm{P}(5000<Y<a)=0.5$ $\Rightarrow \mathrm{P}(Y<a)-\mathrm{P}(Y<5000)=0.5$ $\Rightarrow \mathrm{P}(Y<a)=0.5+\mathrm{P}(Y<5000)=0.61188$ $\Rightarrow \mathrm{P}(Y<a)=0.61188$ $\Rightarrow a=8458$ (to nearest dollars) <u>Alternative:</u> $\mathrm{P}(5000<Y<a)=0.5$ $\mathrm{P}\left(\frac{5000}{3}<X<\frac{a}{3}\right)=0.5$ since $Y=3X$ $\Rightarrow \mathrm{P}\left(X<\frac{a}{3}\right)-\mathrm{P}\left(X<\frac{5000}{3}\right)=0.5$ $\Rightarrow \mathrm{P}\left(X<\frac{a}{3}\right)=0.5+\mathrm{P}\left(X<\frac{5000}{3}\right)=0.61188$ $\Rightarrow \frac{a}{3}=2819.28$ (2dp) $\Rightarrow a=8458$ (to nearest dollar)	Many made the mistake of $Y=X_1+X_2+X_3$, which is the sum of the income per capita of three randomly chosen households. The correct relationship can be derived from the first line in the question as $X=\frac{Y}{3}$. A number of students assumed symmetry which is not true. Many did not get the correct relationship of the probabilities. A sketch of normal distribution curve may help. 
(bi) [2]	Let V be the random variable denoting the family's monthly expenditure in dollars. Then $V \sim \mathrm{N}(\mu,\ 650^2)$ $\mathrm{P}(V>5900)=0.1$ $\Rightarrow \mathrm{P}(V<5900)=0.9$ $\Rightarrow \mathrm{P}(Z<\frac{5900-\mu}{650})=0.9$ From GC: $\mathrm{P}(Z<1.28155)=0.9$ $\Rightarrow \frac{5900-\mu}{650}=1.28155$ $\Rightarrow \mu=5067$ (to the nearest dollar)	Common mistakes: - Some did not find the probability on the left tail. Note: for invNorm on TI 84 Plus C SE, the area is shaded from the left, i.e. lower tail. - Some did the standardisation wrongly as $\frac{\mu-5900}{650}$ Those who arrived at the answer 5067 with both of the above mistakes were not awarded full credits.

(bii) [4]	Let W be the random variable denoting the family's monthly saving in dollars. $W=T+1500-V$, where T denotes Mr Tan's monthly income. Then $W \sim N(7500-5067, 1000^2+0+650^2)$ i.e. $W \sim N(2433, 1422500)$ and $W_1-W_2 \sim N(0, 2845000)$ $\Rightarrow \mathrm{P}(\lvert W_1-W_2\rvert>1000)$ $=2\mathrm{P}(W_1-W_2<-1000)$ $=0.553$ (3sf)	Some did not find the distribution of the DIFFERENCE between 2 months. For modulus sign: 
(biii) [1]	It is assumed that Mr Tan's income and the family's expenditure in a particular month are independent. Alternatively, Mr Tan's income and the family's expenditure in a month are independent of what he earned and how much the family spent in another month.	We need to assume INDEPENDENCE to be able to add and subtract normal distributions to obtain new normal distributions.

Qn 9		Comments

(i) [2]	For (A): $y=a+bx^2$, with $a>0$ and $b<0$.
	
	
	For (B): $y=c+d\ln x$, with $c>0$ and $d<0$.
	
	
		A handful of students plotted points on the axes though the questions asked for positive x and y values.
		
		A number of students also plotted more than 6 data points.
		
		Students should take note that the data points should not lie close to a straight line to clearly illustrate the relationship between x and y.

(ii) [2]		Students should ensure that the scatter plot is sketched to scale. Many scatter plots were out of proportion.
(iii) [2]	From GC, the product moment correlation coefficient is -0.898 (3s.f.). Since -0.898 is close to -1, it suggests a strong negative linear correlation between X and Y. However, it can be observed from the scatter plot that the values of y are decreasing at a decreasing rate with increasing values of x, which will not be the case if the data follows a linear model (the decrease in y should be approximately constant for a linear model).	Many students fail to elaborate on the implication of the value of r in relation to the linear relationship between the variables. They also did not mention the behavior of y as x increases. Many simply mentioned that the scatter plot does not exhibit a linear relationship.
(iv) [3]	(B) is the appropriate model. Using model (B), the product moment correlation coefficient is -0.978 (3s.f.). From GC, $y = 430.30 - 45.010(\ln x)$. $\therefore y = 430 - 45.0\ln x$ (to 3s.f.)	Most students did well in this part of the question, except for those who had made errors in their data entry. Students are reminded to leave their final answers in 3 sf (unless specified otherwise). Thus, intermediate working should be at least 5 sf.
(v) [3]	At a GDP per capita of $723, the infant mortality rate is estimated to be $y = 430.30 - 45.010(\ln 723) = 133.98\ (5\text{s.f.}) = 134\ (3\text{s.f.})$ Since $x = 723$ is outside the range of the data values, the estimation is not reliable.	Many students used 3 sf answer found in (iv) to obtain the estimated value of y. This affects their accuracy of answers.

Qn 10		Comments
(a) [3]	Complement Method: Probability $= 1 - \text{P(22 people were all born on different days)}$ $= 1 - \text{P(21 people were all born on different days)} \times \dfrac{365-21}{365}$ $\approx 1 - 0.55631 \times \dfrac{344}{365}$ $= 0.476$ (3sf) Alternative method: P(at least 2 with same date of birth in 21 people) + P(21 people all born on different days and 22nd person shares same date of birth with someone else) $= (1 - 0.55631) + (0.55631) \times \dfrac{21}{365}$	It is inappropriate to describe the distribution of the number of people who shares the same date of birth as someone else in the group as following the Binomial distribution. There may be more than one common date of birth!
(bi) [2]	Number of ways $= {}^3C_1 \times {}^6C_4 \times {}^9C_4 \times {}^4C_2 = 34020$	
(ii) [3]	Complement Method: Number of ways = n(teams formed without restriction) − n(teams which include both twins) $= 34020 - {}^3C_1 \times {}^5C_3 \times {}^8C_3 \times {}^4C_2$ $= 34020 - 10080 = 23940$ Alternative method: Number of ways = n(teams which include the twin defender and not the twin midfielder) + n(teams which include the twin midfielder and not the twin defender) + n(teams which do not include both twins) $= {}^3C_1 \times {}^5C_4 \times {}^8C_4 \times {}^4C_2 + {}^3C_1 \times {}^5C_3 \times {}^8C_4 \times {}^4C_2 + {}^3C_1 \times {}^5C_4 \times {}^8C_3 \times {}^4C_2$ $= 6300 + 12600 + 5040$	1) *Read the qn carefully!* Qn asks for number of teams with **at most** one twin brother, *not both twin brothers* and *not exactly one twin brother.* 2) Complement method is more efficient since it only involves one case.
(iii) [4]	Let A denote the event that player 1 and player 11 are both in the team. Let B denote the event that player 4 and player 7 are both in the team. $n(A) = {}^6C_4 \times {}^8C_3 \times {}^4C_2 = 5040$ $n(B) = {}^3C_1 \times {}^5C_3 \times {}^8C_3 \times {}^4C_2 = 10080$ $n(A \cap B) = {}^5C_3 \times {}^7C_2 \times {}^4C_2 = 1260$ $\therefore n(A \cup B) = 5040 + 10080 - 1260 = 13860$ Hence required probability $= 1 - \dfrac{13860}{34020} = \dfrac{16}{27}$	1) Qn asks for probability, not number of ways. 2) There are many different methods available for this part, but no matter which one you apply, do always check whether your cases are mutually exclusive (ie whether there is "double" counting). There is a need to account for the duplication.

2017 RVHS Prelim Paper 1

1. (i) Describe a sequence of transformations that transform the graph of $y = \ln x$ onto the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \ln(x+a)+b$ and that a and b are constants such that $a > 1$ and $b > 1$. [2]

   (ii) By sketching the graph of $y = \mathrm{f}(x)$ or otherwise, sketch the graph of $y = \dfrac{1}{\mathrm{f}(x)}$.

   State, in terms of a and b, the coordinates of any points where $y = \dfrac{1}{\mathrm{f}(x)}$ crosses the axes and the equations of any asymptotes. [3]

2. A curve C has equation $y = \dfrac{2x^2+3}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$.

   (i) Sketch C, stating the equations of the asymptotes, axial intercepts and the coordinates of the turning points, if any. [3]

   (ii) Using part (i), solve the inequality $2x+2 \le \mathrm{e}^x - \dfrac{5}{x-1}$. [2]

   (iii) Hence, solve the inequality $2x+4 \le \mathrm{e}^{x+1} - \dfrac{5}{x}$. [2]

3. (i) By using the substitution $t = 3\sec\theta$, find $\displaystyle\int \frac{\sqrt{t^2-9}}{t}\,\mathrm{d}t$. [4]

   (ii) The curve C is defined by the parametric equations

   $$x = \ln t, \quad y = \sqrt{t^2-9}, \text{ where } t \ge 3.$$

   Find the exact value of the area of the region bounded by C, the line $x = \ln 6$ and the x-axis. [4]

4. Henry and Isaac take part in a marathon race. In their first training session, they run a distance of 2.4 km each.

(a) Henry increases the distance he runs in each subsequent training session by 400 m.

(i) Find the distance he runs in the 20^{th} session. [2]

(ii) Find the minimum number of sessions he needs to attend in order to run a total distance of 99 km. [3]

(b) (i) Isaac increases the distance he runs in each subsequent session by x %. Find x if Isaac runs a total distance of 200 km at the end of 20 sessions. [3]

(ii) Isaac feels that the training is too tough after the first session. He decides to decrease the distance he runs in each subsequent session by 5% and increase the numbers of sessions. Will he be able to run a total distance of 200 km? Justify your answer. [2]

5. With reference to the origin O, the positon vectors of three points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Given that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 3$, $\mathbf{c}$ is a unit vector and the angle AOC is $\frac{\pi}{3}$ radians.

(i) Find the value of $\mathbf{a} \cdot \mathbf{c}$ and give the geometrical interpretation of this value. [2]

(ii) Given $\mathbf{a} - \mathbf{c} = k\mathbf{b}$ where $k \in \mathbb{R},\ k \neq 0$. By considering $(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{c})$, find the exact values of k. [3]

The point M divides OC in the ratio $OM : OC = 2 : 3$.

(iii) Find the exact area of triangle AMC. [4]

6. **Do not use a calculator in answering this question.**

(a) Solve the simultaneous equations

$$z - 4w = 11 + 6\mathrm{i} \text{ and } 3z + 6\mathrm{i}w = 27$$

giving z and w in the form $x + \mathrm{i}y$ where x and y are real. [4]

(b) **(i)** The complex numbers z and w are given as $z = 4\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)$ and $w = 1 + \mathrm{i}\sqrt{3}$. w^* denotes the conjugate of w. Find the modulus r and the argument θ of $\frac{w^*}{z^2}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

(ii) Find the set of possible values of n such that $\left(\frac{w^*}{z^2}\right)^n$ is purely imaginary. [3]

7. **(a)** Show that $\int \sqrt{5 - x^2}\, \mathrm{d}x = \frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$. [4]

(b) **(i)** Let C be the curve $y^4 + x^2 = 5$. The x-coordinate of the point P on C is 1 and the y-coordinate of the point P on C is positive. Show that the gradient of the normal to C at the point P is $4\sqrt{2}$. Hence find the equation of the normal to C at the point P in exact form. [4]

(ii) The region R is bounded by the curve C. The solid S is formed by rotating the region R through π radians about the x-axis. Using part **(a)**, find the exact volume of the solid S in terms of π. [3]

8. **(a)** Using differentiation, find the exact dimensions of the rectangle of largest area that can be inscribed in the ellipse, $\frac{x^2}{9}+\frac{y^2}{36}=1$. Hence, find the area of this largest rectangle. [8]

(b) In the triangle DEF, angle $EDF=\frac{\pi}{3}$ and angle $DFE=\frac{\pi}{3}+\alpha$ and $EF=6$. Given that α is sufficiently small, show that

$$DF-DE\approx d\alpha\ ,$$

where d is an exact constant to be determined. [5]

9. The line l has equation $\frac{x-2}{4}=\frac{z+3}{1}$, $y=2$ and the plane p_1 has equation $\mathbf{r}.\begin{pmatrix}1\\1\\1\end{pmatrix}=16$.

Referred to the origin O, the position vector of the point A is $2\mathbf{i}+2\mathbf{j}-3\mathbf{k}$.

(i) Find the acute angle between the line l and the plane p_1. [2]

(ii) Find the coordinates of the foot of perpendicular, N, from point A to the plane p_1. [3]

(iii) Find the coordinates of the point B which is the reflection of A in plane p_1. [2]

(iv) Hence, determine the equation of the line which is a reflection of line l in the plane p_1. [4]

(v) Another plane, p_2, contains the point B and is parallel p_1. Determine the exact distance between p_1 and p_2. [2]

10. In a farm, the growth of the population of prawns is studied.

(a) The population of prawns of size n thousand at time t months satisfies the differential equation

$$\frac{d^2n}{dt^2} = e^{-\frac{t}{5}}.$$

(i) Find the general solution of this differential equation. [2]

(ii) It is given that initially, the size of the population of prawns is 50 000. Sketch on a single diagram, two distinct solution curves for the differential equation to illustrate the following two cases for large values of t :

I. the size of the population of prawns increases indefinitely,

II. the size of the population of prawns stabilizes at a certain positive number. [3]

(b) In order for the prawns to grow faster and be more resistance to diseases, a drug is administered to the prawns. The prawn's body metabolizes (breaks down) the drug at a rate proportional to the amount of drug, x mg, present in the body at time t hours.

(i) Given that the initial dosage is 0.1 mg, show that $x = \frac{1}{10}e^{-kt}$, where $k > 0$. [4]

(ii) The half-life of a drug is defined as the time taken for half of it to be metabolized. Given that the half-life of this drug is 4 hours, find the exact value of k. [2]

(iii) If 0.1 mg of this drug is administered to the prawn every 8 hours, show that the total amount of drug present in the prawn's body at any time t is always less than 0.15 mg. [3]

END OF PAPER

ANNEX B

RVHS H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Graphs and Transformation	(ii) y-intercept $\left(0, \dfrac{1}{(\ln a)+b}\right)$ vertical asymptote: $x = -a + e^{-b}$ horizontal asymptote: $y = 0$
2	Equations and Inequalities	(ii) $x < 1$ or $x \geq 2.34$ (iii) $x < 0$ or $x \geq 1.34$
3	Application of Integration	(i) $3\left(\dfrac{\sqrt{t^2-9}}{3} - \cos^{-1}\left(\dfrac{3}{t}\right)\right) + c$ (ii) $3\sqrt{3} - \pi$
4	AP and GP	(ai) 10 (aii) 18 (bi) 13.2% (bii) No
5	Vectors	(i) 2; $\lvert \mathbf{a} \circ \mathbf{c} \rvert$ is the length of projection of $\mathbf{a}$ onto $\mathbf{c}$ (ii) $k = \pm\dfrac{\sqrt{13}}{3}$ (iii) $\dfrac{\sqrt{3}}{3}$
6	Complex numbers	(a) $w = -1 - i$, $z = 7 + 2i$ (bi) $\dfrac{1}{8}, \dfrac{\pi}{3}$ (bii) $\left\{ n : n = \dfrac{3(2m+1)}{2},\ m \in \circ \right\}$
7	Application of Integration	(bi) $y = 4\sqrt{2}x - 3\sqrt{2}$ (bii) $\dfrac{5}{2}\pi^2$
8	Differentiation & Applications	(a) $x = \dfrac{3}{\sqrt{2}}$, $y = 3\sqrt{2}$; 36 units^2 (b) $-4\sqrt{3}\alpha$
9	Vectors	(i) $\theta = 44.4^\circ$ (ii) $(7, 7, 2)$

		(iii) $\left(12,12,7\right)$ (iv) $l_{BC}:\mathbf{r}=\begin{pmatrix}14\\2\\0\end{pmatrix}+s\begin{pmatrix}2\\-10\\-7\end{pmatrix},\ s\in \circ$ (v) $5\sqrt{3}$units
10	Differential Equations	(ai) $n=25\mathrm{e}^{-\frac{t}{5}}+Ct+D$ (bii) $k=-\frac{1}{4}\ln\frac{1}{2}=\frac{\ln 2}{4}$

1

(i)

Step 1: Translation of a units in the negative x-axis direction;

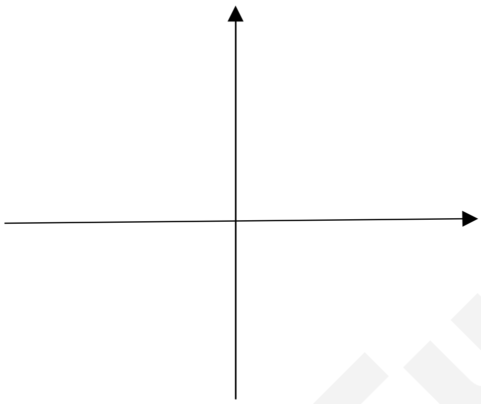
Step 2: Translation of b units in the positive y-axis direction.

OR

Step 1: Translation of b units in the positive y-axis direction;

Step 2: Translation of a units in the negative x -axis direction.

(ii)



y-intercept $\left(0, \dfrac{1}{(\ln a)+b}\right)$

vertical asymptote: $x = -a + \mathrm{e}^{-b}$

horizontal asymptote: $y = 0$

2

(i)

By long division,

$$y = \frac{2x^2+3}{x-1}$$

$$= 2x + 2 + \frac{5}{x-1}$$

y-intercept A (0, −3)

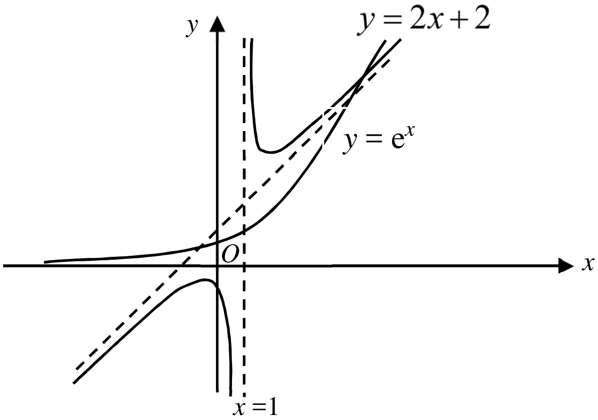
Max point B (−0.581, −2.32)

Min point C (2.58, 10.3)

(ii)

$$2x+2 \le e^x - \frac{5}{x-1}$$

$$2x+2+\frac{5}{x-1} \le e^x$$



Intersection of both curves: (2.34, 10.4)

$x<1$ or $x \ge 2.34$

(iii)

Replacing x by $x+1$

$x+1<1$ or $x+1 \ge 2.34$

$x<0$ or $x \ge 1.34$

3 (i)

Given $t = 3\sec\theta \Rightarrow \frac{dt}{d\theta} = 3\sec\theta\tan\theta$

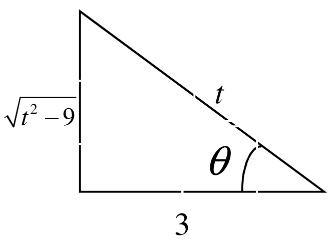
$$\int \frac{\sqrt{t^2-9}}{t}\,dt$$

$$= \int \sqrt{9\sec^2\theta - 9}\left(\frac{1}{3\sec\theta}\right)(3\sec\theta\tan\theta)\,d\theta$$

$$= 3\int \tan^2\theta\,d\theta$$

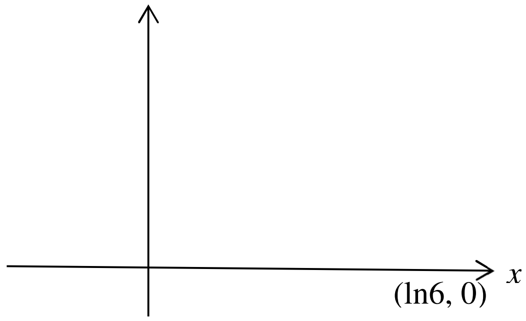
$$= 3\int \sec^2\theta - 1\,d\theta$$

$$= 3(\tan\theta - \theta) + c$$



$$= 3\left(\frac{\sqrt{t^2-9}}{3} - \cos^{-1}\left(\frac{3}{t}\right)\right) + c$$

(ii)



$$\frac{dx}{dt} = \frac{1}{t}$$

Area of $S = \int_{\ln 3}^{\ln 6} y \, dx$

$$= \int_3^6 \sqrt{t^2 - 9}\left(\frac{1}{t}\right) dt$$

$$= \int_3^6 \frac{\sqrt{t^2-9}}{t} \, dt$$

$$= 3\left[\frac{\sqrt{t^2-9}}{3} - \cos^{-1}\left(\frac{3}{t}\right)\right]_3^6$$

$$= 3\left(\frac{\sqrt{27}}{3} - \frac{\pi}{3}\right)$$

$$= 3\sqrt{3} - \pi$$

4 (ai)

AP: $a = 2.4$, $d = 0.4$

Distance he runs in the 20th session

$= 2.4 + (20-1)(0.4)$

$= 10$ km

(aii)

$S_n \geq 99$

$\Rightarrow \quad \frac{n}{2}\left[2(2.4) + (n-1)(0.4)\right] \geq 99$

$\Rightarrow \quad n\left[4.8 + 0.4n - 0.4\right] \geq 198$

$\Rightarrow \quad 0.4n^2 + 4.4n - 198 \geq 0$

$\Rightarrow \quad n \leq -28.4 \quad$ or $\quad n \geq 17.4$

(rejected as $n > 0$)

Least value of $n = 18$

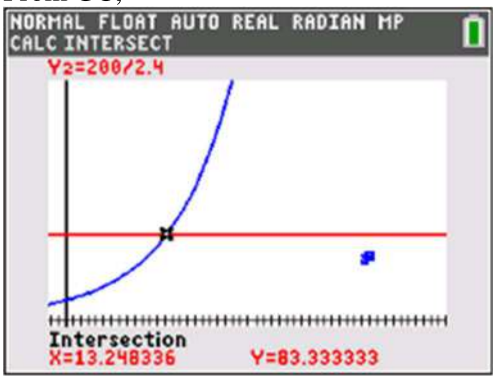
He needs a minimum of 18 sessions.

(bi)

$$S_{20}=\frac{2.4\left(\left(1+\frac{x}{100}\right)^{20}-1\right)}{\left(1+\frac{x}{100}\right)-1}=200$$

$$\frac{\left(1+\frac{x}{100}\right)^{20}-1}{\frac{x}{100}}=\frac{200}{2.4}$$

From GC,



$x = 13.2\%$

(bii)

$$\text{Sum to infinity}=\frac{2.4}{1-0.95}=48$$

Hence, total distance can never be greater than 200 km.

5 (i)

$$\mathbf{a}\cdot\mathbf{c}=4(1)\cos\frac{\pi}{3}=2$$

$|\mathbf{a}\cdot\mathbf{c}|$ is the length of projection of $\mathbf{a}$ onto $\mathbf{c}$

(ii)

$$(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{c})=k\mathbf{b}\cdot k\mathbf{b}$$

$$\mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{c}-\mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{c}=k^2\mathbf{b}\cdot\mathbf{b}$$

$$|\mathbf{a}|^2-2\mathbf{a}\cdot\mathbf{c}+|\mathbf{c}|^2=k^2|\mathbf{b}|^2$$

$$16-2(2)+1=9k^2$$

$$k^2=\frac{13}{9}$$

$$k=\pm\frac{\sqrt{13}}{3}$$

	(iii) $\overrightarrow{MC}=\frac{1}{3}\mathbf{c}$ Area of triangle AMC $=\frac{1}{2}\left\|\overrightarrow{AC}\times\overrightarrow{MC}\right\|$ $=\frac{1}{2}\left\|(\mathbf{c}-\mathbf{a})\times\frac{1}{3}\mathbf{c}\right\|$ $=\frac{1}{6}\left\|\mathbf{c}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}\right\|$ $=\frac{1}{6}\left\|\mathbf{a}\times\mathbf{c}\right\|$ $=\frac{1}{6}\left\|\mathbf{a}\right\|\left\|\mathbf{c}\right\|\sin\left(\frac{\pi}{3}\right)$ $=\frac{1}{6}(4)(1)(\frac{\sqrt{3}}{2})$ $=\frac{\sqrt{3}}{3}$
6	(a) $z-4w=11+6\mathrm{i}$ $z=4w+11+6\mathrm{i}$ Sub above equation into $3z+6iw=27$, $3(4w+11+6\mathrm{i})+6\mathrm{i}w=27$ $12w+33+18\mathrm{i}+6\mathrm{i}w=27$ $w(12+6\mathrm{i})=-6-18\mathrm{i}$ $w=\frac{-6-18\mathrm{i}}{12+6\mathrm{i}}$ $=-1-\mathrm{i}$ $z=4w+11+6\mathrm{i}$ $=4(-1-\mathrm{i})+11+6\mathrm{i}$ $=7+2\mathrm{i}$ <u>ALT</u> $z-4w=11+6\mathrm{i}$ $\times3,\quad 3z-12w=33+18\mathrm{i}\ \ldots(1)$ $3z+6\mathrm{i}w=27\ \ldots(2)$ (2) − (1), $6\mathrm{i}w+12w=-6-18\mathrm{i}$ $w=\frac{-6-18\mathrm{i}}{12+6\mathrm{i}}$ $=-1-\mathrm{i}$

$$z = 4w + 11 + 6\text{i}$$
$$= 4(-1-\text{i}) + 11 + 6\text{i}$$
$$= 7 + 2\text{i}$$

(bi)

$$|z| = 4 \qquad \& \qquad \arg z = -\frac{\pi}{3}$$

$$|w| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2 \qquad \& \qquad \arg w = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$\left|\frac{w^*}{z^2}\right| = \frac{|w^*|}{|z^2|} = \frac{|w|}{|z|^2} = \frac{2}{16} = \frac{1}{8}$$

$$\arg\left(\frac{w^*}{z^2}\right) = \arg\left(w^*\right) - \arg\left(z^2\right)$$
$$= -\arg w - 2\arg z$$
$$= -\left(\frac{\pi}{3}\right) - 2\left(-\frac{\pi}{3}\right)$$
$$= \frac{\pi}{3}$$

(bii)

$$\frac{w^*}{z^2} = \frac{1}{8}\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

$$\left(\frac{w^*}{z^2}\right)^n = \left(\frac{1}{8}\right)\left[\cos\left(\frac{n\pi}{3}\right) + i\sin\left(\frac{n\pi}{3}\right)\right]$$

For $\left(\frac{w^*}{z^2}\right)^n$ to be purely imaginary,

$$\cos\left(\frac{n\pi}{3}\right) = 0$$

$$\frac{n\pi}{3} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots \qquad \left\{n : n = \frac{3(2m+1)}{2},\ m \in \mapsto\right\}$$
$$= \frac{(2m+1)\pi}{2}, m \in \mapsto$$
$$n = \frac{3(2m+1)}{2}, m \in \mapsto$$

7 (a)

$$\int \sqrt{5-x^2}\ \text{d}x = x\sqrt{5-x^2} - \int \frac{-x^2}{\sqrt{5-x^2}}\ \text{d}x$$
$$= x\sqrt{5-x^2} - \int \frac{\left(5-x^2\right)-5}{\sqrt{5-x^2}}\ \text{d}x$$
$$= x\sqrt{5-x^2} - \int \sqrt{5-x^2}\ \text{d}x + 5\int \frac{1}{\sqrt{5-x^2}}\ \text{d}x$$
$$= x\sqrt{5-x^2} - \int \sqrt{5-x^2}\ \text{d}x + 5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)$$

$$\Rightarrow 2\int \sqrt{5-x^2}\ \mathrm{d}x = x\sqrt{5-x^2} + 5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c'$$

$$\Rightarrow \int \sqrt{5-x^2}\ \mathrm{d}x = \frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$$

(bi)

$y^4 + x^2 = 5$

Differentiating wrt x,

$4y^3 \frac{\mathrm{d}y}{\mathrm{d}x} = -2x$

When $x = 1$, $y^4 = 4$

$y = \pm\sqrt{2}$

At $\left(1, \sqrt{2}\right)$, $4\left(\sqrt{2}\right)^3 \frac{\mathrm{d}y}{\mathrm{d}x} = -2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{4\sqrt{2}}$

Gradient of normal at $\left(1, \sqrt{2}\right)$

$= -\frac{1}{-\frac{1}{4\sqrt{2}}}$

$= 4\sqrt{2}$ (shown)

Equation of normal: $y - \sqrt{2} = 4\sqrt{2}\left(x-1\right)$

$y = 4\sqrt{2}x - 3\sqrt{2}$

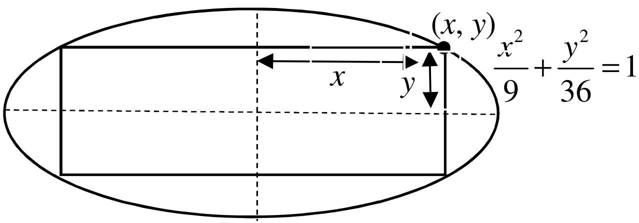
(bii)

Volume of S= $\pi\int_{-\sqrt{5}}^{\sqrt{5}} y^2\ \mathrm{d}x = 2\pi\int_0^{\sqrt{5}} \sqrt{5-x^2}\ \mathrm{d}x$

$= 2\pi\left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_0^{\sqrt{5}}$

$= 2\pi\left[\frac{5}{2}\left(\frac{\pi}{2}\right) - 0\right] = \frac{5}{2}\pi^2$

8 (a)



Let (x, y) be a point on the ellipse.

Area of rectangle, A

$= (2x)(2y)$
$= 4xy$
$= 4x\sqrt{36-4x^2}$
$= 8\sqrt{9x^2-x^4}$
$= 8\left(9x^2-x^4\right)^{\frac{1}{2}}$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 8\left(\frac{1}{2}\right)\left(9x^2-x^4\right)^{-\frac{1}{2}}\left(18x-4x^3\right)$$
$$= \frac{4\left(18x-4x^3\right)}{\sqrt{9x^2-x^4}}$$

When the area is the largest,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 0$$
$$\frac{4\left(18x-4x^3\right)}{\sqrt{9x^2-x^4}} = 0$$
$$18x-4x^3 = 0$$
$$2x\left(3-\sqrt{2}x\right)\left(3+\sqrt{2}x\right) = 0$$

$x = 0$ (rejected since $x \neq 0$)
or $x = \frac{3}{\sqrt{2}}$
or $x = -\frac{3}{\sqrt{2}}$ (rejected since $x > 0$)

When $x = \frac{3}{\sqrt{2}}$, $y = 3\sqrt{2}$

x	2.115	$\frac{3}{\sqrt{2}} \approx 2.12$	2.125
$\frac{\mathrm{d}A}{\mathrm{d}x}$	0.2013	0	-0.118
Slope			

Area of the rectangle is a maximum
Maximum area

$= 8\left(9x^2 - x^4\right)^{\frac{1}{2}}$

$= 8\sqrt{9\left(\frac{3}{\sqrt{2}}\right)^2 - \left(\frac{3}{\sqrt{2}}\right)^4}$

$= 36 \text{ units}^2$

<u>ALT</u>

Note: $A = 8\left(9x^2 - x^4\right)^{\frac{1}{2}}$

Since x, $y > 0$, value of x that maximises A also maximises A^2

$A^2 = 64\left(9x^2 - x^4\right)$

$\frac{\mathrm{d}A^2}{\mathrm{d}x} = 64\left(18x - 4x^3\right) = 0$

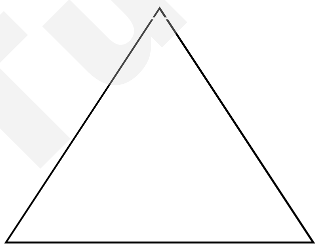
$\Rightarrow x = \frac{3}{\sqrt{2}}$

(b)

Using Sine rule,

$\frac{DF}{\sin\left(\frac{\pi}{3} - \alpha\right)} = \frac{6}{\sin\left(\frac{\pi}{3}\right)}$

$DF = 4\sqrt{3}\sin\left(\frac{\pi}{3} - \alpha\right)$



$\frac{DE}{\sin\left(\frac{\pi}{3} + \alpha\right)} = \frac{6}{\sin\left(\frac{\pi}{3}\right)}$

$DE = 4\sqrt{3}\sin\left(\frac{\pi}{3} + \alpha\right)$

$DF - DE$

$= 4\sqrt{3}\sin\left(\frac{\pi}{3} - \alpha\right) - 4\sqrt{3}\sin\left(\frac{\pi}{3} + \alpha\right)$

$= 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right) - 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha\right)$

$\approx 4\sqrt{3}\left[\frac{\sqrt{3}}{2}\left(1 - \frac{\alpha^2}{2}\right) - \frac{1}{2}\alpha - \frac{\sqrt{3}}{2}\left(1 - \frac{\alpha^2}{2}\right) - \frac{1}{2}\alpha\right]$

$= -4\sqrt{3}\alpha$

9 (i)

$$l:\mathbf{r}=\begin{pmatrix}2\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}4\\0\\1\end{pmatrix},\ \lambda\in\mapsto$$

Let θ be the angle between the line l and the plane p_1.

$$\sin\theta=\left|\frac{\begin{pmatrix}4\\0\\1\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{17}\sqrt{3}}\right|$$

$$=\left|\frac{5}{\sqrt{17}\sqrt{3}}\right|$$

$$\theta=44.4^\circ$$

(ii)

$$l_{AN}:\mathbf{r}=\begin{pmatrix}2\\2\\-3\end{pmatrix}+\mu\begin{pmatrix}1\\1\\1\end{pmatrix},\ \mu\in\mapsto$$

$$\begin{pmatrix}2+\mu\\2+\mu\\-3+\mu\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}=16$$

$$2+\mu+2+\mu-3+\mu=16$$

$$3\mu=15$$

$$\mu=5$$

Coordinates of $N=(7,7,2)$

(iii)

Since N is the midpoint of A and B, using ratio theorem,

$$\overrightarrow{ON}=\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$$

$$\overrightarrow{OB}=2\overrightarrow{ON}-\overrightarrow{OA}$$

$$=2\begin{pmatrix}7\\7\\2\end{pmatrix}-\begin{pmatrix}2\\2\\-3\end{pmatrix}=\begin{pmatrix}12\\12\\7\end{pmatrix}$$

Coordinates of $B=(12,12,7)$

(iv)

Let C be the point of intersection of the line l and the plane p_1.

$$\begin{pmatrix} 2+4\lambda \\ 2 \\ -3+\lambda \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 16$$

$$\begin{aligned} 2+4\lambda+2-3+\lambda &= 16 \\ 5\lambda &= 15 \\ \lambda &= 3 \end{aligned}$$

$$\overrightarrow{OC} = \begin{pmatrix} 14 \\ 2 \\ 0 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} 14 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 12 \\ 12 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \\ -7 \end{pmatrix}$$

$$l_{BC} : \mathbf{r} = \begin{pmatrix} 14 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ -10 \\ -7 \end{pmatrix},\ s \in \mapsto$$

(v)

Since $AN = BN$,

$$\begin{aligned} BN &= \sqrt{(2-7)^2 + (2-7)^2 + (-3-2)^2} \\ &= \sqrt{(-5)^2 + (-5)^2 + (-5)^2} \\ &= \sqrt{75}\text{units} \\ &= 5\sqrt{3}\text{units} \end{aligned}$$

10 (ai)

Let n denote the population of prawns in thousands at time t

$$\frac{\mathrm{d}^2 n}{\mathrm{d}t^2} = \mathrm{e}^{-\frac{t}{5}}$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -5\mathrm{e}^{-\frac{t}{5}} + C$$

$$n = 25\mathrm{e}^{-\frac{t}{5}} + Ct + D$$

(aii)

Given $n = 50,\ t = 0$,

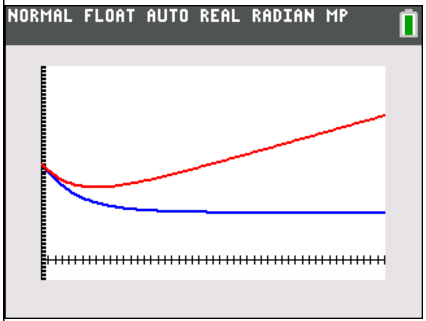
$$50 = 25 + D \Rightarrow D = 25$$

$$n = 25\mathrm{e}^{-\frac{t}{5}} + Ct + 25$$

I Requires $C > 0$ so that $n = 25\mathrm{e}^{-\frac{t}{5}} + Ct + 25 \to \infty$ as $t \to \infty$

II Requires $C = 0$ so that $n = 25e^{-\frac{t}{5}} + 25$

Then as $t \to \infty$, $n \to 25$



(bi)

$$\frac{dx}{dt} = -kx$$

$$\int \frac{1}{x}\, dx = -k\int 1\, dt$$

$$\ln|x| = -kt + C$$

$x = Ae^{-kt}$ where $A = \pm e^{C}$

At $t = 0$, $x = 0.1$,

$\therefore A = 0.1$

$x = \frac{1}{10}e^{-kt}$ (shown)

(bii)

At $t = 4$, $x = 0.05$,

$\therefore 0.05 = 0.1e^{-4k}$

$$\Rightarrow e^{-4k} = \frac{1}{2}$$

$$\Rightarrow -k = \frac{\ln\frac{1}{2}}{4}$$

$$\Rightarrow k = -\frac{1}{4}\ln\frac{1}{2} = \frac{\ln 2}{4}$$

(biii)

Total amount of drug present in the prawn's body at any time t

$$< 0.1 + 0.1e^{-\left(\frac{\ln 2}{4}\right)8} + 0.1e^{-2\left(\frac{\ln 2}{4}\right)8} + 0.1e^{-3\left(\frac{\ln 2}{4}\right)8} + \dots$$

$$= \frac{0.1}{1 - e^{-\left(\frac{\ln 2}{4}\right)8}}$$

$$= \frac{2}{15} < 0.15$$

$\therefore$ The total amount of drug present in the prawn's body at any time t is always less than 0.15 mg.

2017 RVHS Prelim Paper 2

Section A: Pure Mathematics [40 Marks]

1. The curve C is defined parametrically by equations

$$x=\cos(p), \quad y=\sin^3(p), \quad 0\le p\le 2\pi$$

The point P on C has parameter p. Given that p is increasing at a rate of 0.5 units per second, find the rate at which $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing when $p=\frac{\pi}{3}$. [4]

2. An arithmetic sequence $u_1, u_2, u_3, \ldots$ is such that the difference between the fourteenth term and the fifth term is equal to the sum of the terms between the fifth term and the fourteenth term (both inclusive). Given further that that sum of the third, fifth and fourteenth terms is 19, find the common difference of the sequence. [4]

Hence, or otherwise, find the largest of value of n such that the sum of the first n terms is positive. [2]

3. **(i)** Express $\dfrac{4r+6}{(r+1)(r+2)(r+3)}$ as partial fractions. [1]

(ii) Hence find $\displaystyle\sum_{r=1}^{n}\frac{4r+6}{(r+1)(r+2)(r+3)}$ in terms of n. [3]

(iii) Use your answer in part **(ii)** to find the sum of the infinite series

$$\frac{3}{1\times2\times3}+\frac{5}{2\times3\times4}+\frac{7}{3\times4\times5}+\frac{9}{4\times5\times6}+\cdots.$$ [3]

4. Let $y=\mathrm{f}(x)$, where $\mathrm{f}(x)=\mathrm{e}^{\sqrt{(1-x)^3}}$ for $x\le 1$.

Show that $4\sqrt{1-x}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+6(1-x)\dfrac{\mathrm{d}y}{\mathrm{d}x}-3y=0$. [4]

Hence find the Maclaurin series for f(x) up to and including the term in x^2. [3]

Using the standard series of e^x and $(1+x)^n$ given in the List of Formulae (MF26), show how you could verify the correctness of the series of $\mathrm{f}(x)$ above. [4]

5. The functions f and g are defined by

$$\mathrm{f}:x\mapsto 2x^2-x,\quad x\in\mathbb{R},\quad x\ge 0,$$

$$\mathrm{g}:x\mapsto -3+\frac{1}{\sqrt{2x+\frac{1}{2}}},\quad x\in\mathbb{R},\quad x>-\frac{1}{4}.$$

(i) Give a reason why f does not have an inverse. [1]

(ii) If the domain of f is restricted to $x\ge k$, state the least value of k for which the function f^{-1} exists, and find f^{-1} in similar form for this domain. [3]

(iii) Sketch the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ on the same diagram if the domain of f is restricted to $x\ge k$, where k is the value found in part **(ii)**. Your diagram should show clearly the relationship between the two graphs. [3]

(iv) Solve algebraically the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ for the restricted domain of f in part **(ii)**. [2]

(v) For f defined for $x\ge 0$, show that the composite function gf exists and find its range. [3]

Section B: Statistics [60 Marks]

6. A restaurant is setting up a spinning wheel for its customers to try and win vouchers. The wheel is split into 8 identical segments, comprising of \$0, \$5, \$10, \$15, \$20, \$25, \$30 and \$50.

Find the number of ways the segments can be arranged on the wheel if

(i) there are no restrictions. [1]

(ii) the \$0 segment cannot be next to the \$5 segment [2]

(iii) there must be at least two segments between the \$30 and \$50 segments. [2]

The restaurant decides to replace the \$30 and \$50 segments with another two \$0 segments.

(iv) Find the number of possible arrangements of the 8 segments. [1]

(v) Find the number of possible arrangements if the \$0 segments must be separated. [2]

7. A board game simulates players attacking each other by throwing tetrahedral (8-sided) dice. When attacking, the player throws an attack die once. An attack die has 5 of the sides printed with the number "0", 2 of the sides printed with the number "1", and 1 of the sides printed with the number "2". After the attacking player has thrown the attack die, the defending player throws a defence die once. A defence die has 2 of the sides printed with the number "0", 4 of the sides printed with the number "1" and 2 of the sides printed with the number "2". The damage dealt during a round is equal to the score shown on the attack die minus the score shown on the defence die. If the score on the defence die is more than the score on the attack die, the damage dealt will be zero.

Let A denotes the score on an attack die, and D denotes the score on a defence die.

(i) Write down the probability distributions for A and D. Hence find the expected value and variance of $A-D$. [4]

Let X denote the damage dealt during a round.

(ii) Find the probability distribution for X. Hence find the expected value and variance of X. [5]

(iii) Explain why, in the context of the question, $\mathrm{E}(X) > 0$ when $\mathrm{E}(A) < \mathrm{E}(D)$. [1]

8. A car park next to a small commercial building has a total of 12 parking lots. Land surveillance officers have been observing the usage of parking lots per day to determine if the land has been efficiently utilised. Each parking lot can be occupied by at most one vehicle per day.

(i) Denoting the number of occupied parking lots per day by X, state in context, two assumptions needed for X to be well modelled by a binomial distribution. [2]

(ii) It is further observed that for 80% of the days in the survey period, there are at least 4 occupied lots in the car park for each day. Find the probability that a parking lot is being occupied in a day. [2]

(iii) Given that at least one of the parking lots is occupied in a particular day, show that the probability that at least 2 but less than 4 lots are occupied in the particular day is given by

$$f(p)=\frac{22p^2(1-p)^9(3+7p)}{1-(1-p)^{12}}$$

where p is the probability of a parking lot being occupied in a day. What can you say about this probability if p is approximately 0.185? [5]

9. In the study of how the population of a harmful bacteria varies with temperature, scientists conducted an experiment to collect the following set of data:

Temperature (x °C)	10	12	14	16	18	20	22	24	26	28
Population (y millions)	25.4	25.1	24.4	22.9	20.8	18.3	15.4	12.2	8.8	5.3

(i) Draw a scatter diagram for the above data, labelling the axes clearly. [2]

(ii) Calculate the value of the product moment correlation coefficient. Explain why a linear model is not appropriate. [2]

It is suggested the relationship between x and y can be modelled by one of the following formulae:

$$y=a+\frac{b}{x} \quad \text{or} \quad y=a-bx^2$$

where a and b are positive constants.

(iii) Explain which of the above two models is the better model and calculate the values of a and b for the chosen model. [3]

(iv) It is required to estimate the temperature when the population of the bacteria is 10 millions. By using an appropriate regression line, find an estimate of the value of x and comment on the reliability of your answer. [2]

10. Each month the amount of electricity, X measured in kilowatt-hours (kWh), used by a household in a particular city may be assumed to follow a normal distribution with mean 950 and standard deviation σ. The charge for electricity used per month is fixed at \$0.22 per kWh.

(i) Given that 65% of the households uses less than 960 kWh of electricity in a month, find the value of σ, correct to 1 decimal place. [2]

For the rest of the question, σ is the value found in part (i).

(ii) Find the probability that the difference in the amount of electricity used among 2 randomly chosen households in a particular month is not more than 30 kWh. [3]

(iii) In the month of August, the mayor of the city decides to provide 50% and 30% subsidies for the electricity bills of households in the North and South distincts of the city respectively. Find the probability that the total electricity bill of 2 randomly chosen North district households and 1 South district household is less than \$360. [4]

(iv) In December, a random sample of n households is chosen to study the mean monthly electricity usage per household in the city. Find the least value of n if the probability of the sample mean being less than 955 kWh is at least 0.9. [3]

11. Physicists are conducting an experiment involving collisions between protons and anti-protons. The mean amount of energy, $\bar{x}$ MeV, released in n collisions is found to be 1864 MeV.

One model predicts the energy released would be 1860 MeV with standard deviation 40 MeV. This is tested at a 1% level of significance against a newer model that claims a higher value.

(i) Find the least value of n such that the hypothesis that the mean amount of energy released is 1860 MeV is rejected. [5]

Given instead that $n = 600$.

(ii) Calculate the p-value and state its meaning in context of the question. [3]

(iii) State, with a reason, whether it is necessary to assume the amount of energy released in collisions to be normally distributed for this test to be valid. [1]

Two-sigma is an indicative of how confident researchers feel their results are. For researchers to feel confident, they must be able to produce a "two-sigma" result – that is the experimental result must be at least two standard deviations away from the predicted mean under the null hypothesis.

(iv) Calculate the level of significance that corresponds to a "two-sigma" test. Hence, using your answer from part **(ii)** determine whether the experiment has met the "two-sigma" threshold. [3]

END OF PAPER

ANNEX B

RVHS H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Differentiation & Applications	0.75 units per second
2	AP and GP	-5 or $-\frac{95}{46}$; 16 or 19
3	Sigma Notation and Method of Difference	(i) $\frac{4r+6}{(r+1)(r+2)(r+3)}=\frac{1}{r+1}+\frac{2}{r+2}-\frac{3}{r+3}$ (ii) $\frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)$ (iii) $\frac{5}{4}$
4	Maclaurin series	$\mathrm{e}-\frac{3\mathrm{e}}{2}x+\frac{3\mathrm{e}}{2}x^2$
5	Functions	(ii) $k=\frac{1}{4}$; $\mathrm{f}^{-1}: x \mapsto \frac{1+\sqrt{8x+1}}{4}, x \geq -\frac{1}{8}$ (iv) $x=1$. (v) $R_{\mathrm{gf}}=(-3,-1]$
6	P&C, Probability	(i) 5040 (ii) 3600 (iii) 2160 (iv) 840 (v) 240
7	DRV	(i) $\frac{-1}{2}$, 1 (ii) $\frac{3}{16}$, $\frac{55}{256}$
8	Binomial Distribution	(ii) 0.412 (iii) $p \approx 0.185$
9	Correlation & Linear Regression	(ii) $r=-0.973$ (iii) a = 30.0 and b = 0.0308 (iv) x = 25.5 °C
10	Normal Distribution	(i) 26.0 (ii) 0.585 (iii)0.796 (iv)45

11	Hypothesis Testing	(i) 542 (ii) p-value $= 0.00715$ (iv) 2.28%

1	At point P, $x=\cos(p)$, $y=\sin^3(p)$ $\frac{\mathrm{d}y}{\mathrm{d}p}=3\cos(p)\sin^2(p)$ $\frac{\mathrm{d}x}{\mathrm{d}p}=-\sin(p)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-3\sin(p)\cos(p)=\frac{-3}{2}\sin(2p)$ Let $z=\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\mathrm{d}z}{\mathrm{d}p}\cdot\frac{\mathrm{d}p}{\mathrm{d}t}$ $=-3\cos(2p)\cdot(0.5)$ $=\frac{-3}{2}\cos(2p)$ $\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right\rvert_{p=\frac{\pi}{3}}=\frac{-3}{2}\cos\left(\frac{2\pi}{3}\right)=0.75$ Therefore, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing at 0.75 units per second when $p=\frac{\pi}{3}$.
2	Let the first term be a and the common difference be d. $\sum_{k=5}^{14}u_k=\lvert u_{14}-u_5\rvert$ $S_{14}-S_4=\lvert(a+13d)-(a+4d)\rvert$ $\frac{14}{2}(2a+13d)-\frac{4}{2}(2a+3d)=\lvert 9d\rvert$ $14a+91d-4a-6d=\lvert 9d\rvert$ $10a+85d=\lvert 9d\rvert$ $10a+85d=9d$ or $10a+85d=-9d$ $5a+38d=0$---- (1) or $5a+47d=0$---- (1) $u_3+u_5+u_{14}=19$ $(a+2d)+(a+4d)+(a+13d)=19$ $3a+19d=19$ ----- (2) Solving simultaneously, from GC, $a=38,\ d=-5$ or $a=\frac{893}{46},\ d=-\frac{95}{46}$ Hence the common difference is -5 or $-\frac{95}{46}$

$$S_n > 0 \qquad \text{or} \qquad S_n > 0$$

$$\frac{n}{2}\left(2a+(n-1)d\right)>0 \qquad \text{or} \qquad \frac{n}{2}\left(2a+(n-1)d\right)>0$$

$$n(81-5n)>0 \qquad \qquad n\left(\frac{1881}{92}-\frac{95}{92}n\right)>0$$

$$0<n<\frac{81}{5}=16.2 \qquad \text{or} \qquad 0<n<\frac{99}{5}=19.8$$

Hence, the largest value of n is 16 or 19.

3 (i)

Let $\dfrac{4r+6}{(r+1)(r+2)(r+3)}=\dfrac{A}{r+1}+\dfrac{B}{r+2}+\dfrac{C}{r+3}$

Then by cover up rule, $A=1, B=2, C=-3$

Hence, $\dfrac{4r+6}{(r+1)(r+2)(r+3)}=\dfrac{1}{r+1}+\dfrac{2}{r+2}-\dfrac{3}{r+3}$

(ii)

$$\sum_{r=1}^{n}\frac{4r+6}{(r+1)(r+2)(r+3)}=\sum_{r=1}^{n}\frac{1}{r+1}+\frac{2}{r+2}-\frac{3}{r+3}$$

$$\begin{aligned}
&= \frac{1}{2} + \frac{2}{3} - \frac{3}{4} \\
&+ \frac{1}{3} + \frac{2}{4} - \frac{3}{5} \\
&+ \frac{1}{4} + \frac{2}{5} - \frac{3}{6} \\
&+ \frac{1}{5} + \ldots\ldots \quad \ldots\ldots \\
&+ \ldots\ldots \quad \ldots\ldots - \frac{3}{n} \\
&+ \frac{1}{n-1} + \frac{2}{n} - \frac{3}{n+2} \\
&+ \frac{1}{n} + \frac{2}{n+1} - \frac{3}{n+2} \\
&+ \frac{1}{n+1} + \frac{2}{n+2} - \frac{3}{n+3} \\
&= \frac{1}{2}+\frac{2}{3}+\frac{1}{3}-\frac{3}{n+2}+\frac{2}{n+2}-\frac{3}{n+3} \\
&= \frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)
\end{aligned}$$

(iii)

$$\frac{3}{1\times2\times3}+\frac{5}{2\times3\times4}+\frac{7}{3\times4\times5}+\frac{9}{4\times5\times6}+\dots$$
$$=\frac{3}{1\times2\times3}+\frac{1}{2}\left(\frac{10}{2\times3\times4}+\frac{14}{3\times4\times5}+\frac{18}{4\times5\times6}+\dots\right)$$
$$=\frac{1}{2}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{4r+6}{(r+1)(r+2)(r+3)}$$
$$=\frac{1}{2}+\frac{1}{2}\lim_{n\to\infty}\left(\frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)\right)$$
$$=\frac{1}{2}+\frac{1}{2}\times\frac{3}{2}$$
$$=\frac{5}{4}$$

4

$$y=\mathrm{e}^{\sqrt{(1-x)^3}}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{\sqrt{(1-x)^3}}\left(\frac{3}{2}\right)(1-x)^{\frac{1}{2}}(-1)=\frac{-3}{2}y\sqrt{1-x}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-3}{2}\frac{\mathrm{d}y}{\mathrm{d}x}\sqrt{1-x}+\frac{-3}{2}y\frac{-1}{2\sqrt{1-x}}=\frac{3y}{4\sqrt{1-x}}-\frac{3\sqrt{1-x}}{2}\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$4\sqrt{1-x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3y-6(1-x)\frac{\mathrm{d}y}{\mathrm{d}x}$$

Thus,

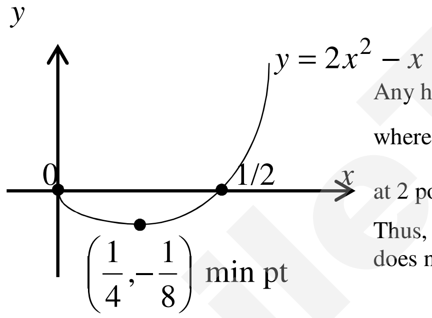
$$4\sqrt{1-x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+6(1-x)\frac{\mathrm{d}y}{\mathrm{d}x}-3y=0 \text{ (shown)}$$

When $x = 0$,

$$y=\mathrm{e}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-3\mathrm{e}}{2}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3\mathrm{e}$$

$$\mathrm{f}(x)\approx\mathrm{f}(0)+\mathrm{f}'(0)x+\frac{\mathrm{f}''(0)}{2!}x^2$$
$$\approx\mathrm{e}-\frac{3\mathrm{e}}{2}x+\frac{3\mathrm{e}}{2}x^2$$

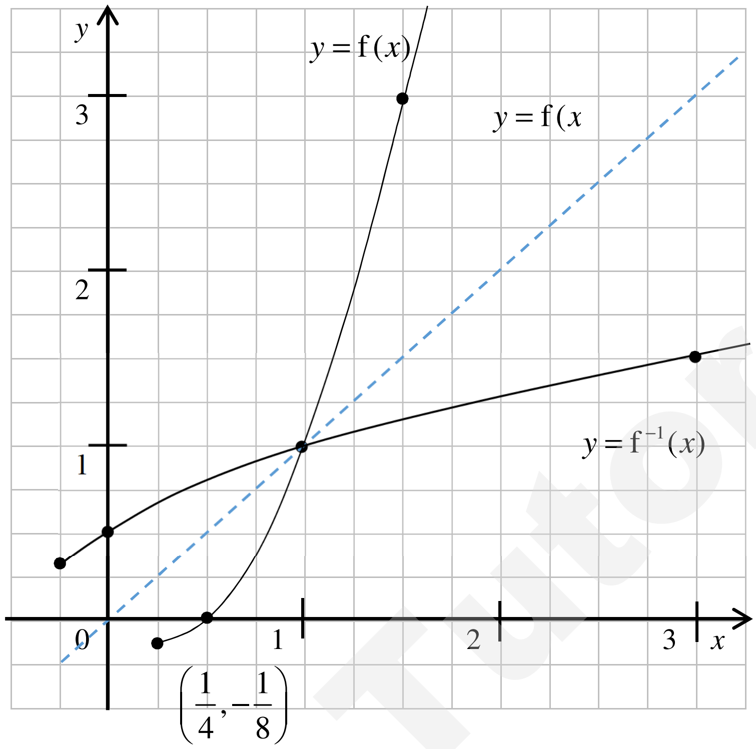
$$(1-x)^{\frac{3}{2}}\approx1-\frac{3}{2}x+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(-x)^2$$
$$=1-\frac{3}{2}x+\frac{3}{8}x^2$$

	$e^{\sqrt{(1-x)^3}} \approx e^{1-\frac{3}{2}x+\frac{3}{8}x^2}$ $= e\left(e^{-\frac{3}{2}x+\frac{3}{8}x^2}\right)$ $\approx e\left(1+\left(-\frac{3}{2}x+\frac{3}{8}x^2\right)+\frac{\left(-\frac{3}{2}x+\frac{3}{8}x^2\right)^2}{2!}\right)$ $\approx e\left(1-\frac{3}{2}x+\frac{3}{8}x^2+\frac{9}{8}x^2\right)$ $= e\left(1-\frac{3}{2}x+\frac{3}{2}x^2\right)$ $= e-\frac{3e}{2}x+\frac{3e}{2}x^2$ which is the same as the above series expansion of $f(x)$
5	(i) As shown in the following sketch:  Any horizontal line of the form $y = k$ where $-\frac{1}{4} < k \le 0$ will intersect the curve at 2 points. Thus, f is not one-one and hence f^{-1} does not exist. (ii) From the sketch of the curve, we deduce that the least value of $k = \frac{1}{4}$ for f^{-1} to exist. Next let $y = 2x^2 - x$. Then we have $y = 2\left(x^2 - \frac{1}{2}x\right)$ $= 2\left(x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right)$ $= 2\left(x - \frac{1}{4}\right)^2 - \frac{1}{8}$ $\left(x-\frac{1}{4}\right)^2 = \frac{1}{2}\left(y+\frac{1}{8}\right)$ $\Rightarrow x = \frac{1}{4} \pm \sqrt{\frac{8y+1}{16}} = \frac{1}{4} + \frac{\sqrt{8y+1}}{4}$ since $x \ge \frac{1}{4}$

Hence, $\mathrm{f}^{-1}: x \mapsto \dfrac{1+\sqrt{8x+1}}{4}, x \ge -\dfrac{1}{8}$. $D_{\mathrm{f}^{-1}} = R_{\mathrm{f}} = \left[-\dfrac{1}{8}, \infty\right)$

(iii)

Sketch of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$:



(iv)

From the sketch in part (iii) we note that to solve the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, we can also solve $\mathrm{f}(x)=x$

Thus, $2x^2 - x = x \Rightarrow 2x(x-1)=0$

Therefore, in the restricted domain of $x \ge \dfrac{1}{4}$,

the solution is $x=1$.

(v)

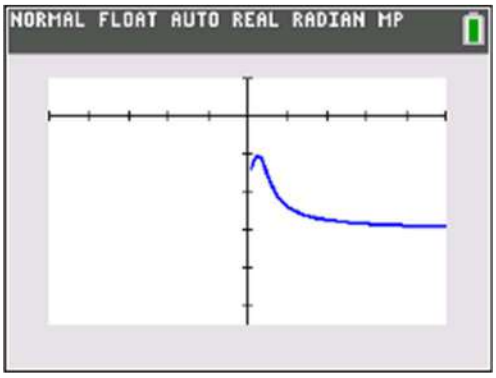
For $\mathrm{f}: x \mapsto 2x^2 - x,\ x \in \mapsto\ x \ge 0, R_{\mathrm{f}} = \left[-\dfrac{1}{8}, \infty\right)$

Also, for $\mathrm{g}: x \mapsto -3 + \dfrac{1}{\sqrt{2x+\dfrac{1}{2}}},\ x \in \mapsto\ x > -\dfrac{1}{4},\ D_{\mathrm{g}} = \left(-\dfrac{1}{4}, \infty\right)$

Since $R_{\mathrm{f}} \subseteq D_{\mathrm{g}}$, the composite function gf exists.

Then,

$$\mathrm{gf}(x) = -3 + \frac{1}{\sqrt{2(2x^2-x)+\dfrac{1}{2}}} = -3 + \frac{1}{\sqrt{4\left(x-\dfrac{1}{4}\right)^2 + \dfrac{1}{4}}}$$



Since $D_{\text{gf}} = [0, \infty)$ and $\text{gf}\left(\frac{1}{4}\right) = -3 + \frac{1}{\sqrt{\frac{1}{4}}} = -3 + 2 = -1$,

we have $R_{\text{gf}} = (-3, -1]$

<u>ALT</u>

$[0, +\infty) \to [-\frac{1}{8}, +\infty) \to (-3, -1]$

$R_{\text{gf}} = (-3, -1]$

6 (i)

$(8-1)! = 5040$

(ii)

No. of ways with \$0 and \$5 segments adjacent

$= (7-1)!2!$

$= 1440$

No. of ways without identical segments adjacent

= total no. of ways − no. of ways with identical segments adjacent

$= 5040 - 1440$

$= 3600$

(iii)

<u>Case 1: no segment separating them</u>

$(7-1)!2! = 1440$

<u>Case 2: exactly 1 segment separating them</u>

$\binom{6}{1} 2!(6-1)! = 1440$

Total number of ways = $5040 - 1440 - 1440$

$= 2160$

	<u>ALT</u> <u>Case 1</u>: exactly 2 segments separating them $\binom{6}{2}2!2!(5-1)!=1440$ <u>Case 2</u>: exactly 3 segments separating them $\dfrac{\binom{6}{3}3!2!(4-1)!}{2}=720$ Therefore, total number of ways = 2160 (iv) The segments are \$0, \$0, \$0, \$5, \$10, \$15, \$20, \$25 $\dfrac{(8-1)!}{3!}=840$ (v) Arrange the other 5 objects in (5 – 1)! = 24 ways Choose 3 spaces for the \$0 in ${}^5C_3$ = 10 ways Total = 240 ways
7	(i) Probability distribution for *A*:

a	0	1	2
P(*A* = *a*)	5/8	2/8	1/8

Probability distribution for *D*:

d	0	1	2
P(*D* = *d*)	2/8	4/8	2/8

$$\mathrm{E}(A)=\left(\frac{5}{8}\right)(0)+\left(\frac{2}{8}\right)(1)+\left(\frac{1}{8}\right)(2)=\frac{1}{2}$$

$$\mathrm{E}(D)=\left(\frac{2}{8}\right)(0)+\left(\frac{4}{8}\right)(1)+\left(\frac{2}{8}\right)(2)=1$$

$$\mathrm{E}(A-D)=\mathrm{E}(A)-\mathrm{E}(D)=\frac{-1}{2}$$

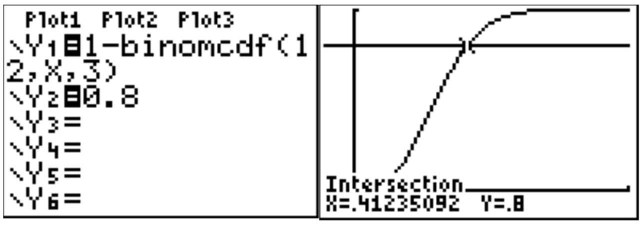
$$\mathrm{E}(A^2)=\left(\frac{5}{8}\right)(0)^2+\left(\frac{2}{8}\right)(1)^2+\left(\frac{1}{8}\right)(2)^2=\frac{3}{4}$$

$$\mathrm{E}(D^2)=\left(\frac{2}{8}\right)(0)^2+\left(\frac{4}{8}\right)(1)^2+\left(\frac{2}{8}\right)(2)^2=\frac{3}{2}$$

	$\mathrm{Var}(A)=\mathrm{E}(A^2)-\mathrm{E}(A)^2=\frac{3}{4}-\left(\frac{1}{2}\right)^2=\frac{1}{2}$ $\mathrm{Var}(D)=\mathrm{E}(D^2)-\mathrm{E}(D)^2=\frac{3}{2}-1^2=\frac{1}{2}$ $\mathrm{Var}(A-D)=\mathrm{Var}(A)+\mathrm{Var}(D)=\frac{1}{2}+\frac{1}{2}=1$ (ii) Probability distribution for *X*: (see table below) $\mathrm{E}(X)=\left(\frac{27}{32}\right)(0)+\left(\frac{1}{8}\right)(1)+\left(\frac{1}{32}\right)(2)=\frac{3}{16}$ $\mathrm{E}(X^2)=\left(\frac{27}{32}\right)(0)^2+\left(\frac{1}{8}\right)(1)^2+\left(\frac{1}{32}\right)(2)^2=\frac{1}{4}$ $\mathrm{Var}(X)=\frac{1}{4}-\left(\frac{3}{16}\right)^2=\frac{55}{256}$ (iii) If the score on the defence die is more than the score on the attack die, the damage dealt will be zero. So even though sometimes *A* – *D* will be less than zero, that is never considered when dealing damage. Hence, the expected damage must be greater than zero.
8	(i) The 2 assumptions needed for X to be well modelled by a binomial distribution are as follow: 1. The occupancy of any particular parking lot in the car park is *independent* of that of another lot. 2. The probability of a parking lot being occupied in a day is *constant* for all the car park lots in the car park. (ii) Since for 80% of the days in the survey period, there are at least 4 occupied lots for each day, we can infer that $P(X \geq 4)=1-P(X \leq 3)=0.8$ for $X \sim \mathrm{B}(12, p)$.

x	0	1	2
P(X = x)	$\mathrm{P}(A=0)$ $+\mathrm{P}(A=1)\mathrm{P}(D\geq 1)$ $+\mathrm{P}(A=2)\mathrm{P}(D=2)$ $=\frac{27}{32}$	$\mathrm{P}(A=1)\mathrm{P}(D=0)$ $+\mathrm{P}(A=2)\mathrm{P}(D=1)$ $=\frac{1}{8}$	$\mathrm{P}(A=2)\mathrm{P}(D=0)$ $=\frac{1}{32}$

We then use GC to plot the graph involving binomial cdf and determine the x coordinate of the intersection of the curve and the line $y = 0.8$ as shown below:



Hence, the value of p is 0.412 (3 s.f.)

(iii)

Let $X \sim \mathrm{B}(12, p)$

The required conditional probability, $\mathrm{f}(p)$

$$= \mathrm{P}(2 \le X < 4 \mid X \ge 1)$$

$$= \frac{\mathrm{P}(X = 2 \text{ or } X = 3)}{\mathrm{P}(X \ge 1)}$$

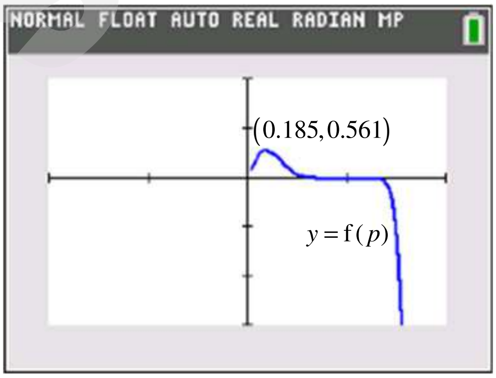
$$= \frac{\mathrm{P}(X = 2 \text{ or } X = 3)}{1 - \mathrm{P}(X = 0)}$$

$$= \frac{\binom{12}{2} p^2 (1-p)^{10} + \binom{12}{3} p^3 (1-p)^9}{1 - \left[\binom{12}{0} p^0 (1-p)^{12}\right]}$$

$$= \frac{66 p^2 (1-p)^{10} + 220 p^3 (1-p)^9}{1 - (1-p)^{12}}$$

$$= \frac{22 p^2 (1-p)^9 \left[3(1-p) + 10p\right]}{1 - (1-p)^{12}}$$

$$= \frac{22 p^2 (1-p)^9 (3 + 7p)}{1 - (1-p)^{12}}. \quad \text{(Shown)}$$

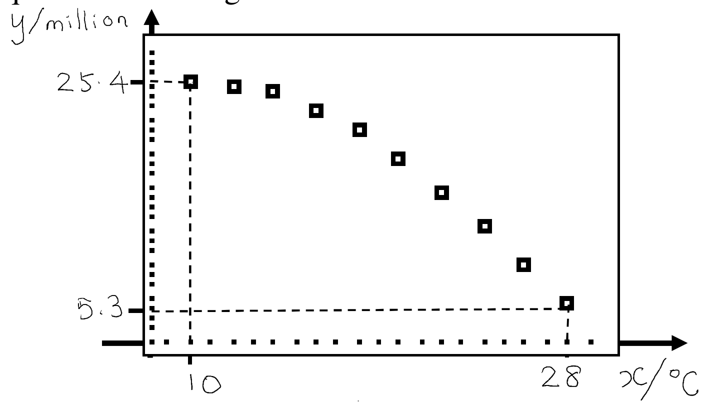


$p \approx 0.185$ give the maximum probability.

9

(i)

The required scatter diagram is as shown below:



(ii)

From GC, the correlation coefficient $r = -0.973$.
Although the value of r is close to -1 and suggests a strong negative linear relationship between x and y, the scatter diagram shows a curvilinear relationship between x and y. Thus, the a linear relationship between x and y is not appropriate.

(iii)

The scatter diagram shows that when x increases, y decreases at increasing rate. Thus, the model with $y = a - bx^2$ where a, b are positive constants is more appropriate.

Using GC, we found that $a = 29.98560169 = 30.0$ (3 s.f.)

and $b = 0.0307756388 = 0.0308$ (3 s.f.)

(For $a, b > 0$, $y = a + \dfrac{b}{x}$ decreases at a decreasing rate when x increases)

(iv)

As x is the independent variable and y is the dependent variable, we will still use the regression line $y = 30.0 - 0.0308x^2$ to estimate the value of x.
Thus, when $y = 10$, $x = 25.5$ °C (3 s.f.)
The anwer is reliable for the following reasons:

i) correlation coefficient $r = -0.995$ has absolute value close to 1
ii) the y value of 10 is within data range of the available y values.

10

(i)

$X \sim N(950,\ \sigma^2)$
Given that $P(X < 960) = 0.65$,

then $P(Z < \dfrac{960-950}{\sigma}) = 0.65$

$\Rightarrow \dfrac{960-950}{\sigma} = 0.3853204726$

$\Rightarrow \sigma = 25.95242327 = 26.0$ (1 decimal place)

(ii)

Let X_1 and X_2 be the amount of electricity used by the 2 randomly chosen household in a particular month.

Then $X_1 - X_2 \sim \mathrm{N}(0,\ 26.0^2 + 26.0^2)$

Thus, $\mathrm{P}(|X_1 - X_2| \le 30)$

$= \mathrm{P}(-30 \le X_1 - X_2 \le 30)$

$= 0.585$

(iii)

Let N_1, N_2 and S be the amount of electricity used by the 2 randomly chosen households in the North District and household in the South district respectively in August.

Then their total electricity bill = \$ T

$= \$\ 0.5\times 0.22\times(N_1 + N_2) + 0.7\times 0.22\times S$

$= \$\ 0.11N_1 + 0.11N_2 + 0.154S$

Then

$\mathrm{E}(T) = 0.11\times 950\times 2 + 0.154\times 950 = 355.3$

$\mathrm{Var}(T) = 0.11^2\times 26.0^2\times 2 + 0.154^2\times 26.0^2 = 32.391216$

So, $T \sim \mathrm{N}(355.3,\ 32.391216)$

Hence, $\mathrm{P}(T < 360) = 0.796$

(iv)

Let $\overline{X} = \dfrac{X_1 + X_2 + X_3 + ... + X_n}{n} \sim \mathrm{N}\left(950, \dfrac{26.0^2}{n}\right)$

where X_i :electricity usage for each of the randomly selected household in the month of December

Then, we have

$$\mathrm{P}\left(\overline{X} < 955\right) \ge 0.9$$

$$\Rightarrow \mathrm{P}\left(Z < \frac{955-950}{26.0/\sqrt{n}}\right) \ge 0.9$$

$$\Rightarrow \frac{955-950}{26.0/\sqrt{n}} \ge 1.281551567$$

$$\Rightarrow \frac{26.0}{\sqrt{n}} \le \frac{5}{1.281551567} = 3.901520726$$

$$\Rightarrow \sqrt{n} \ge \frac{26}{3.901520726}$$

$$\Rightarrow n \ge 44.40980429$$

Thus, the least value of n is 45.

11 (i)

Let μ denote the population mean amount of energy released in the collisions.

Test H$_0$: $\mu = 1860$

Against H$_1$: $\mu > 1860$

Using a one-tail test at 1% significance level.

Under H_0, $\overline{X} \sim N\left(1860, \frac{40^2}{n}\right)$ approx

Test statistic: $Z = \frac{\overline{X} - 1860}{40/\sqrt{n}} \sim N(0,1)$

$$z_{calc} = \frac{1864 - 1860}{40/\sqrt{n}} = \frac{\sqrt{n}}{10}$$

To reject H_0 at 1% level of significance, the critical region is:

$$z_{calc} > 2.32635$$

Hence,

$\frac{\sqrt{n}}{10} > 2.32635$

$n > 541.189$

Thus, the least value of n is 542.

(ii)

Test H_0: $\mu = 1860$

Against H_1: $\mu > 1860$

Using a one-tailed test at 1% significance level.

Under H_0, $\overline{X} \sim N\left(1860, \frac{40^2}{600}\right)$ approx

Test statistic: $Z = \frac{\overline{X} - 1860}{40/\sqrt{600}} \sim N(0,1)$

From GC,

p-value $= 0.00715$

The p-value means that the lowest level of significance at which we would reject the hypothesis that the mean amount of energy released is 1860 MeV in favour of the hypothesis that the amount is greater than 1860 MeV is 0.715 %.

(iii)

No assumption needed. This is because the sample size of 600 is large and thus by Central Limit Theorem, $\overline{X}$ follows a normal distribution.

(iv)

Let $Z \sim N(0, 1)$

$P(Z \geq 2) = 0.0228$

Hence, lowest level of significance for which the experiment meets the "two sigma" threshold is 2.28%.

Since p-value = 0.00715 < 0.0228, the result meets the "two sigma" threshold.

	<u>Alternative</u>: Under H_0, $\overline{X} \sim N\left(1860, \frac{40^2}{600}\right)$ approx $2\sigma = 2\sqrt{\frac{1600}{600}} = 3.265986$ So $\mu + 2\sigma = 1863.265986$ $P\left(\overline{X} > 1863.265986\right) \approx 0.0228$ Hence, lowest level of significance for which the experiment meets the "two sigma" threshold is 2.28%. Since $\overline{x} = 1864 > 1863.265986$ the test meets the "two sigma" threshold.

SRJC Paper 1

1 The complex numbers z and w satisfy the simultaneous equations

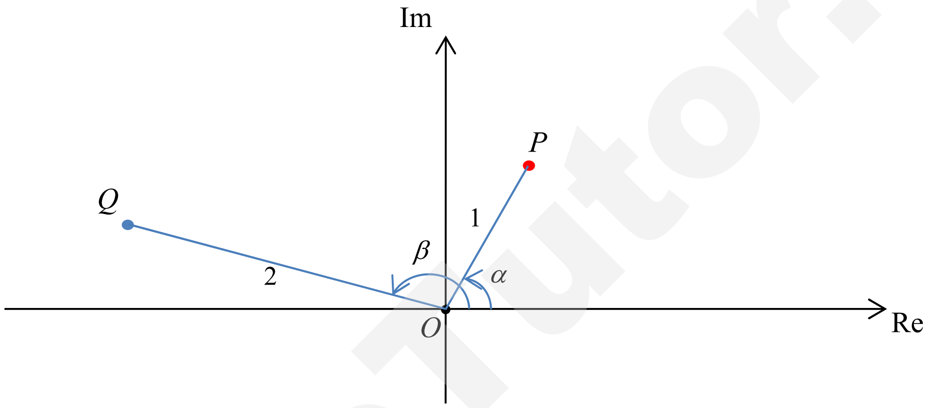
$$\mathrm{i}z + w = 2 + \mathrm{i} \quad \text{and} \quad 2w - (1+\mathrm{i})z = 8 + 4\mathrm{i} \ .$$

Find z and w in the form of $a + \mathrm{i}b$, where a and b are real. [5]

2 Solve the inequality $\dfrac{2x^2 + 2x - 1}{x^2 + 2x} \le 1$.

Hence, solve the inequality $\dfrac{2x^2 + 2|x| - 1}{x^2 + 2|x|} \le 1$. [6]

3 For $\alpha, \beta \in \mathbb{R}$ such that $2\alpha < \beta$, the complex numbers $z_1 = \mathrm{e}^{\mathrm{i}\alpha}$ and $z_2 = 2\mathrm{e}^{\mathrm{i}\beta}$ are represented by the points P and Q respectively in the Argand diagram below.



Find the modulus and argument of the complex numbers given by $\dfrac{\mathrm{i}}{2}z_2$ and $\dfrac{z_1^{\,2}}{z_2}$. [4]

Copy the given Argand diagram onto your answer script and indicate clearly the following points representing the corresponding complex numbers on your diagram.

(i) A: $\dfrac{\mathrm{i}}{2}z_2$ [1]

(ii) B: $\dfrac{z_1^{\,2}}{z_2}$ [1]

You are expected to indicate clearly the relevant moduli and arguments for parts **(i)** and **(ii)** on your Argand diagram.

If $\beta = \dfrac{11}{12}\pi$, find the smallest positive integer n such that the point representing the complex number $(z_2)^n$ lies on the negative real axis. [3]

4 The curve C has equation $4y^2 - 8y - x^2 - 4x - 4 = 0$.

(i) Using an algebraic method, find the set of values that y cannot take. [3]

(ii) Showing any necessary working, sketch C and indicate the equations of the asymptotes. [4]

5 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{\pi}{2}\tan\left(\frac{x}{2}\right), \quad x \in \mathbb{R}, -2\pi \le x \le 2\pi.$$

(i) Explain why f^{-1} does not exist. [2]

(ii) The domain of f is restricted to $(-\pi, a)$ such that a is the largest value for which the inverse function f^{-1} exists. State the exact value of a and define f^{-1} in a similar form. [3]

In the rest of the question, the domain of f is $(-\pi, a)$, where a takes the value found in part **(ii)**.

(iii) Sketch, in a single diagram, the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, labelling each graph clearly.Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$ and draw this line on your diagram. [3]

(iv) Verify that $x = \frac{\pi}{2}$ is a root of the equation $x = \mathrm{f}(x)$. Hence, explain why $x = \frac{\pi}{2}$ is also a solution to the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

6 Referred to the origin O, the two points A and B have position vectors given by **a** and **b**, where **a** and **b** are non-zero vectors. The line l has equation $\mathbf{r} = 2\mathbf{a} + \lambda(\mathbf{a} + 2\mathbf{b})$, where $\lambda \in \mathbb{R}$. The point E is a general point on l and the point D has position vector $2\mathbf{a} - \mathbf{b}$.

Given that vector **a** is a unit vector, vector **b** has a magnitude of $\sqrt{2}$ units and that $\mathbf{a} \cdot \mathbf{b} = 1$,

(i) find the angle between vectors **a** and **b**, and, [2]

(ii) by considering $\overrightarrow{DE} \bullet \overrightarrow{DE}$, find an expression for the square of the distance DE, leaving your answer in terms of λ. [3]

Hence or otherwise, find the exact shortest distance of D to l, and write down the position vector of the foot of the perpendicular from D to l, in the form $p\mathbf{a} + q\mathbf{b}$. [3]

7 **(a)** By considering the Maclaurin expansion for $\cos x$, show that the expansion of $\sec x$ up to and including the term in x^4 is given by $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$. Hence show that the expansion for $\ln(\sec x)$ up to and including the term in x^4 is given by $\left[\frac{1}{2}x^2 + Ax^4\right]$ where A is an unknown constant to be determined. [4]

(b) The variables x and y satisfy the conditions **(A)** and **(B)** as follows:

$$(1 + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y \quad \text{---(A)}$$

$$y = 0 \text{ when } x = 0 \quad \text{---(B)}$$

(i) Obtain the Maclaurin expansion of y, up to and including the term in x^3. [4]

(ii) Verify that both conditions **(A)** and **(B)** hold for the curve $\ln(1 + y) = \tan^{-1} x$.[2]

(iii) Hence, without using a graphing calculator, find an approximation for $\int_0^{\frac{1}{2}} \left(\mathrm{e}^{\tan^{-1} x} - 1\right) \mathrm{d}x$. [2]

8 **(a)** The fifth, ninth and eleventh terms of a geometric progression are also the seventh, twenty-fifth and forty-ninth terms of an arithmetic progression with a non-zero common difference respectively.

Show that $3R^6 - 7R^4 + 4 = 0$, where R is the common ratio of the geometric progression and determine if the geometric progression is convergent. [4]

(b) A semicircle with radius 12 cm is cut into 8 sectors whose areas follow a geometric progression. The first sector, which is the largest, has an area of A cm^2. The second sector has an area of Ar cm^2, the third sector has an area of Ar^2 cm^2, and so on, where r is a positive constant. Given also that the total area of the odd-numbered sectors is 10π cm^2 more than that of the even-numbered sectors, find the values of A and r. [5]

(c) The production levels of a particular coal mine in any year is 4% less than in the previous year. Show that the total production of the coal mine can never exceed 25 times the production in the first year. [2]

9

(a) Using the substitution $u = 2x + 3$, find $\int \frac{x}{(2x+3)^3}\,dx$ in the for $-\frac{Px+Q}{R(2x+3)^2}+c$

where P, Q and R are positive integers to be determined. [3]

Hence find $\int \frac{x\ln(4x+3)}{(2x+3)^3}\,dx$. [3]

(b) Find $\int \sin 4x \cos 6x \, dx$. [2]

Hence or otherwise, find $\int e^x \sin 4e^x \cos 6e^x \, dx$. [1]

10 A particle is moving along a curve, C, such that its position at time t seconds after it is set into motion is given by the parametric equations

$$x = t + e^{-2t},\ y = t - e^{-2t}.$$

(i) State the coordinates of the initial position of the particle. [1]

(ii) Explain what would happen to the path of the particle after a long time. [1]

At the time of 2 seconds after the particle was set into motion, an external force struck the particle resulting in the particle moving in a straight line along the normal to the path at the point of collision.

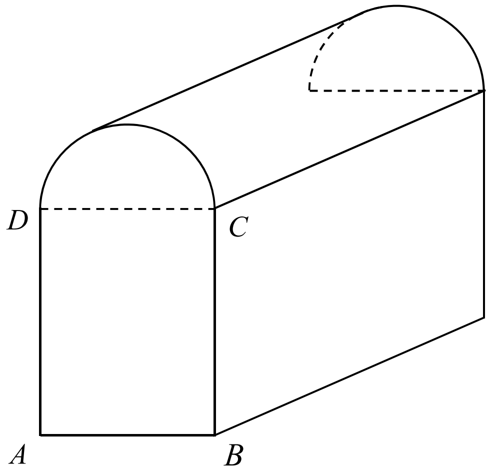
(iii) Find an equation for the normal to the curve C at the point $t = 2$, leaving your answer correct to 3 decimal places. [3]

After T seconds, where $T > 2$, the particle reaches point A, which lies on the x-axis, and stops moving.

(iv) Find the coordinates of the point A. Hence, give a sketch of the path traced by the particle, indicating the coordinates of any axial intercepts. [4]

(v) Find the total area bounded by the path of the particle in the first T seconds and the positive x-axis. [4]

11



A heavy wooden chest has a cross-sectional area made up of a rectangle and a semi-circle as shown in the diagram above. The wooden chest is constructed such that the perimeter of the cross-sectional area is 100 cm. It is given that the wooden chest is $2(a + b)$ cm long and the lengths of AB and BC are $2a$ cm and $2b$ cm respectively, where $a < 70$.

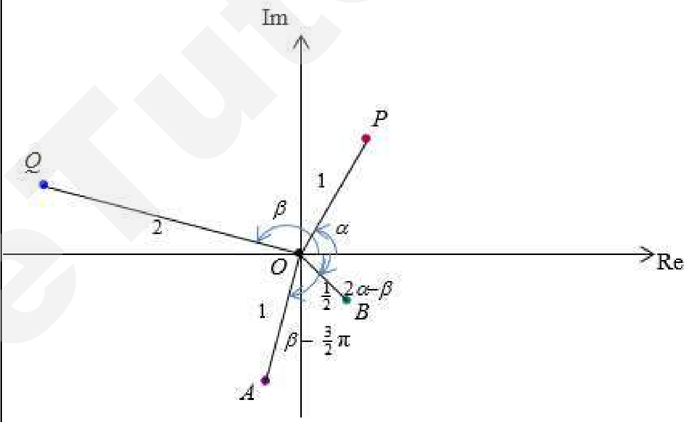
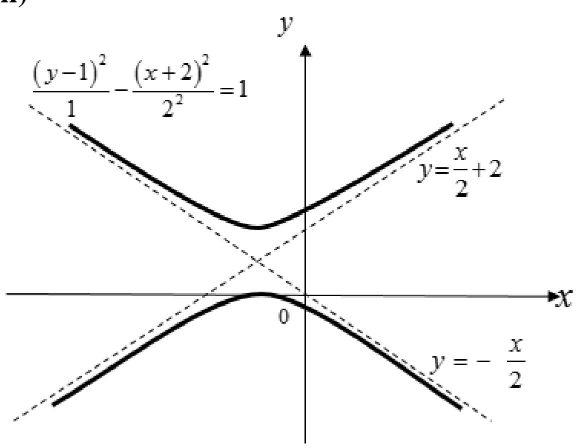
(i) Express b in terms of a. [1]

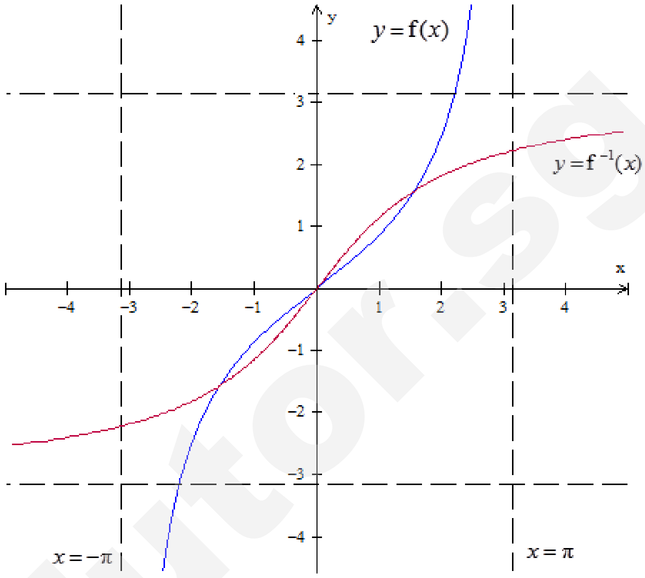
(ii) Show that the cross-sectional area of the wooden chest is given by $S = 100a - \frac{a^2}{2}(\pi + 4)$ and find the volume of the chest in terms of a and π. [4]

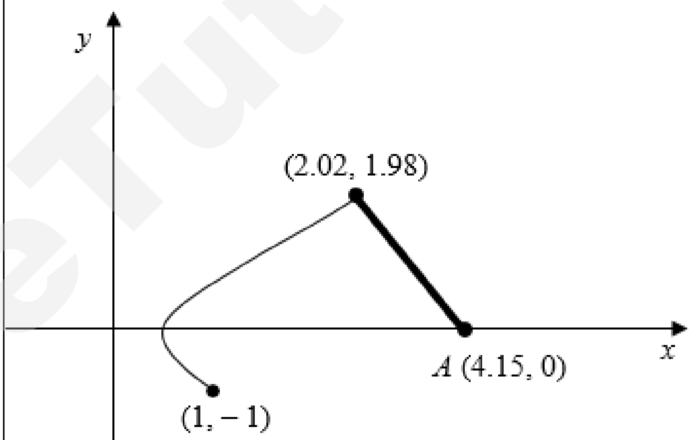
(iii) As a varies, find the value of a such that the volume of this wooden chest is greatest and find this volume correct to 2 decimal places. [5]

ANNEX B

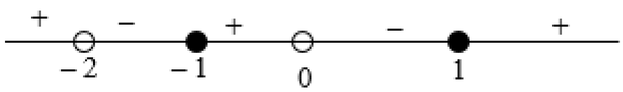
SRJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Complex numbers	$z=-1+\mathrm{i}$ and $w=3+2\mathrm{i}$
2	Equations and Inequalities	$-2<x\le -1$ or $0<x\le 1$, $-1\le x\le 1,\ x\ne 0$
3	Complex numbers	$\left\lvert \frac{\mathrm{i}}{2}z_2 \right\rvert = 1$, $\arg(\frac{\mathrm{i}}{2}z_2) = \beta - \frac{3\pi}{2}$ $\left\lvert \frac{z_1^{\,2}}{z_2} \right\rvert = \frac{1}{2}$, $\arg\left(\frac{z_1^{\,2}}{z_2}\right) = 2\alpha-\beta$ **(i) & (ii)**  Smallest n required = 12
4	Graphs and Transformation	**(i)** $0<y<2$ **(ii)** 

5	Functions	**(ii)** $a=\pi$, $f^{-1}: x \mapsto 2\tan^{-1}\left(\frac{2x}{\pi}\right)$, $x \in R$. **(iii)**  The line required is $y = x$. **(iv)** Since the graphs of $y=f(x)$ and $y=f^{-1}(x)$ intersect along the line $y = x$, and since $x=\frac{\pi}{2}$ is a root of the equation $x=f(x)$, thus, the graphs of $y=f(x)$ and $y=f^{-1}(x)$ must also intersect at the point $x=\frac{\pi}{2}$.
6	Vectors	**(i)** $\theta = 45°$ **(ii)** $13\lambda^2+10\lambda+2$ Exact shortest distance from D to l is $\frac{1}{\sqrt{13}}$ units $\overrightarrow{OF}=\frac{21}{13}\mathbf{a}-\frac{10}{13}\mathbf{b}$
7	Maclaurin series	**(a)** $\frac{1}{2}x^2+\frac{1}{12}x^4$, $A=\frac{1}{12}$ **(b) (i)** $y=x+\frac{x^2}{2}-\frac{x^3}{6}+...$ **(iii)** $\frac{55}{384}$
8	AP and GP	**(a)** $r = \pm\sqrt{2}$ so $\|r\| > 1$ Hence, the geometric progression is not convergent.

		(b) $r = 0.75610$, $A = 61.8$
9	Integration techniques	**(a)** $P = 4$, $Q = 3$ and $R = 8$ $\int \frac{x\ln(4x+3)}{(2x+3)^3}\,\mathrm{d}x = -\frac{(4x+3)\ln(4x+3)+2(2x+3)}{8(2x+3)^2}+C$ **(b)** $-\frac{1}{20}\cos 10x+\frac{1}{4}\cos 2x+C$, $-\frac{1}{20}\cos 10e^x+\frac{1}{4}\cos 2e^x+C$
10	Differentiation & Applications	**(i)** $(1,-1)$ **(ii)** The path of the particle **approaches the line $y = x$** **(iii)** $y = -0.929x + 3.857$ **(iv)** A (4.15, 0)  **(v)** 3.56 units^2
11	Differentiation & Applications	**(i)** $b = \frac{100-a(\pi+2)}{4}$ **(ii)** $5000a-75\pi a-\frac{a^3}{4}(\pi^2+2\pi-8)$ **(iii)** $a = 12.7$, greatest volume = 29671.95 cm^3

H2 Mathematics 2017 Prelim Exam Paper 1 Question
Answer all questions [100 marks].

1	$\mathrm{i}z+w=2+\mathrm{i}-----(1)$ $2w-1-\mathrm{i}z=\dfrac{20}{2-\mathrm{i}}---(2)$ Let $w=2+\mathrm{i}-\mathrm{i}z---(3)$ Substitute eq (3) into eq (2) $2(2+\mathrm{i}-\mathrm{i}z)-z-\mathrm{i}z=8+4\mathrm{i}$ $4+2\mathrm{i}-3\mathrm{i}z-z=8+4\mathrm{i}----(5)$ Let $z=a+b\mathrm{i}$ Substitute $z=a+b\mathrm{i}$ into eq(5) $4+2\mathrm{i}-3\mathrm{i}(a+b\mathrm{i})-(a+b\mathrm{i})=8+4\mathrm{i}$ $4+2\mathrm{i}-3a\mathrm{i}+3b-a-b\mathrm{i}=8+4\mathrm{i}$ Comparing real and imaginary parts: $4+3b-a=8(\text{real parts})---(6)$ $2-3a-b=4(\text{imaginary parts})---(7)$ $\text{Eq}(6)\times 3\ -\ \text{eq}(7)$ $10+10b=20$ $10b=10$ $b=1$ Since $b=1$, $4+3(1)-a=8\Rightarrow a=-1$ $\therefore z=-1+\mathrm{i}$ Substituting $z=-1+\mathrm{i}$ into eq(3) to solve for w $w=2+\mathrm{i}+\mathrm{i}+1=3+2\mathrm{i}$ Answer: $z=-1+\mathrm{i}$ and $w=3+2\mathrm{i}$
2	$\dfrac{2x^2+2x-1}{x^2+2x}\le 1$ $\dfrac{2x^2+2x-1}{x^2+2x}-1\le 0$ $\dfrac{2x^2+2x-1-x^2-2x}{x^2+2x}\le 0$ $\Rightarrow \dfrac{x^2-1}{x(x+2)}\le 0$ $\Rightarrow \dfrac{(x+1)(x-1)}{x(x+2)}\le 0$  Thus, $-2<x\le -1$ or $0<x\le 1$

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Replacing x with $|x|$,
$-2<|x|\le -1$ or $0<|x|\le 1$
$-2<|x|\le -1 \;\Rightarrow$ no solution
For $0<|x|\le 1$,
$0<|x|$ and $|x|\le 1$
$x\in \mathbb{R}, x\ne 0$ and $-1\le x\le 1$
Thus, range of values: $-1\le x\le 1,\ x\ne 0$
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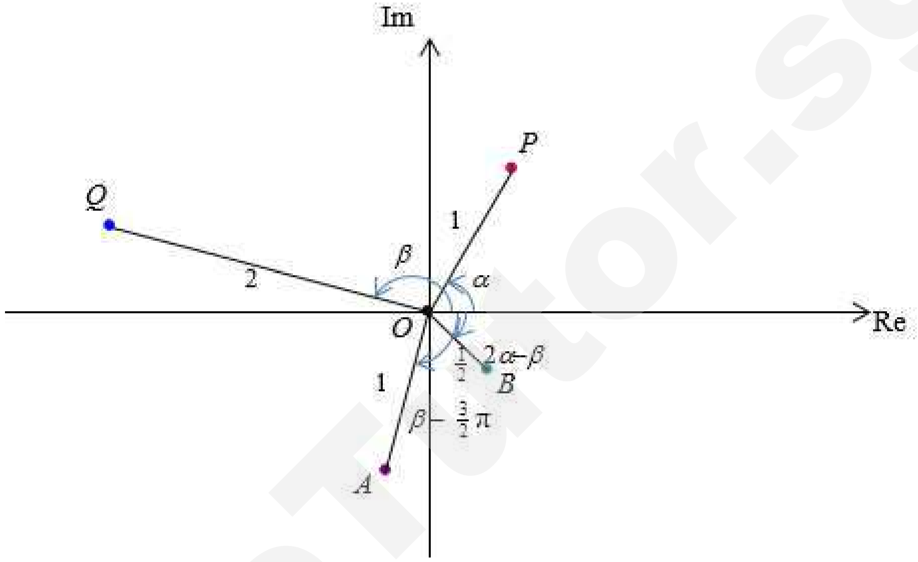

$\frac{\mathrm{i}}{2}z_2=\left(\frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)\left(2\mathrm{e}^{\mathrm{i}\beta}\right)=\mathrm{e}^{\mathrm{i}\left(\beta+\frac{\pi}{2}\right)}$
Modulus = 1
Argument = $\beta+\frac{\pi}{2}-2\pi=\beta-\frac{3\pi}{2}$
(i) Point A correctly plotted

$\frac{z_1^{\,2}}{z_2}=\frac{\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\alpha}}{2\mathrm{e}^{\mathrm{i}\beta}}=\frac{1}{2}\mathrm{e}^{\mathrm{i}(2\alpha-\beta)}$
Modulus = $\frac{1}{2}$
Argument = $2\alpha-\beta$

(ii) Point B correctly plotted

(b) $(z_2)^n=2^n\mathrm{e}^{\mathrm{i}\frac{11\pi}{12}n}$
Since the point lies on the negative real axis, $\arg(z_2)^n=(2k+1)\pi$ for $k\in\mathbb{Z}$.

$\therefore\ \frac{11}{12}n\pi=(2k+1)\pi$
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$\Rightarrow\ n = \frac{12}{11}(2k+1)$

$\Rightarrow$ Smallest n required = 12

4 **(i)** $-x^2 - 4x + \left(4y^2 - 8y - 4\right) = 0$

For values that y cannot take, there are no real solutions for x and discriminant<0.

Therefore, $(-4)^2 - 4(-1)\left(4y^2 - 8y - 4\right) < 0$

$$16 + 16y^2 - 32y - 16 < 0$$

$$y^2 - 2y < 0$$

$$y(y-2) < 0$$

$$\therefore 0 < y < 2$$

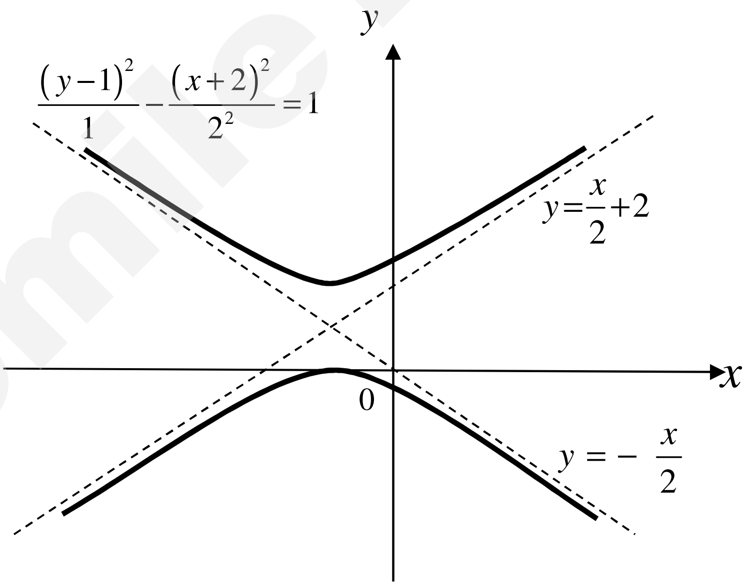
Set of values that y cannot take is $\left\{ y \in \mathbb{R} : 0 < y < 2 \right\}$.

(ii) $4y^2 - 8y - x^2 - 4x - 4 = 0$

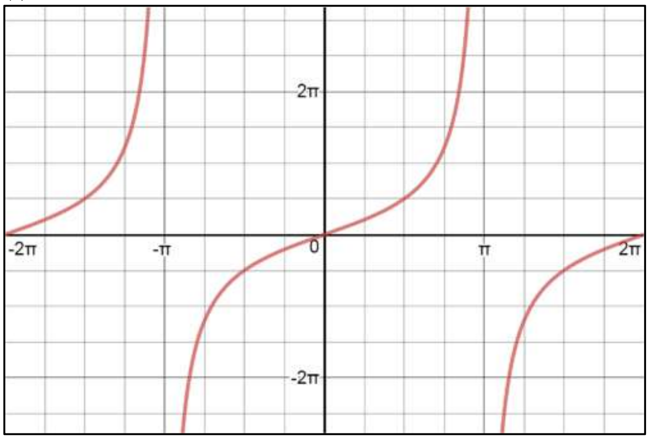
$4\left[(y-1)^2 - 1\right] - \left[(x+2)^2 - 4\right] - 4 = 0$

$4(y-1)^2 - 4 - (x+2)^2 = 0$

$\frac{(y-1)^2}{1} - \frac{(x+2)^2}{2^2} = 1$



5	**(i)**



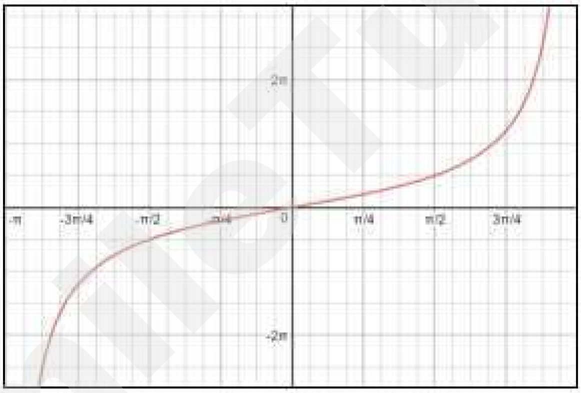
$y = \mathrm{f}(x)$

The horizontal line $y = 1$ cuts the graph of $y = \mathrm{f}(x)$ at **2 points**. Thus, $\mathrm{f}(x)$ is not a one-one function and the inverse of $\mathrm{f}(x)$ does not exist for the domain $[-2\pi, 2\pi]$.

OR

Any horizontal line $y = k$ ($k \in \mathbb{R}$) cuts the graph at more than one point. Thus, $\mathrm{f}(x)$ is not a one-one function and the inverse of $\mathrm{f}(x)$ does not exist for the domain $[-2\pi, 2\pi]$.

(ii)



$a = \pi$

To make x the subject of y

$$y = \frac{\pi}{2}\tan\left(\frac{x}{2}\right)$$

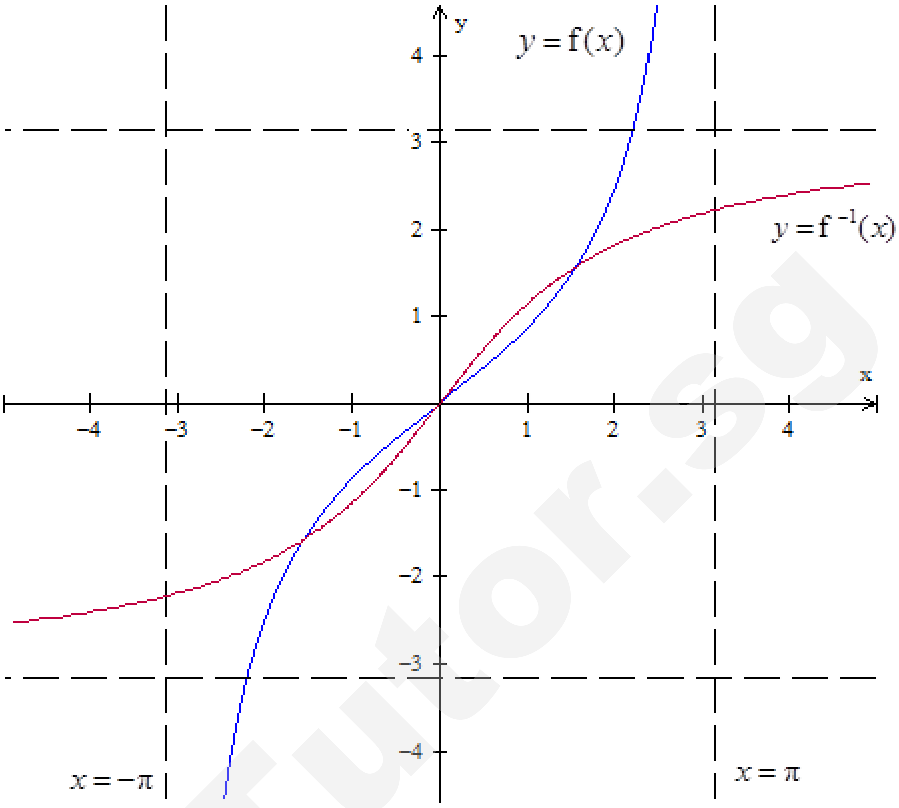
$$\frac{2y}{\pi} = \tan\left(\frac{x}{2}\right)$$

$$\tan^{-1}\left(\frac{2y}{\pi}\right) = \frac{x}{2}$$

$$\Rightarrow x = 2\tan^{-1}\left(\frac{2y}{\pi}\right)$$

$$\mathrm{f}^{-1} : x \mapsto 2\tan^{-1}\left(\frac{2x}{\pi}\right), \quad x \in R.$$

(iii)



The line required is $y = x$.

(iv) $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\tan\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

Thus, $x = \frac{\pi}{2}$ is a root of the equation $x = f(x)$.

Since the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect along the line $y = x$, and since $x = \frac{\pi}{2}$ is a root of the equation $x = f(x)$, thus, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ must also intersect at the point $x = \frac{\pi}{2}$.

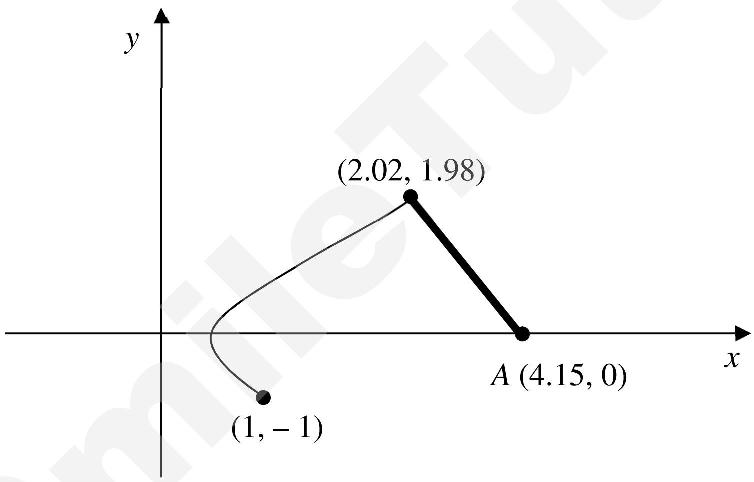
6	(i) $\mathbf{a}.\mathbf{b} = \lvert a\rvert\lvert b\rvert\cos\theta \Rightarrow \lvert 1\rvert\lvert\sqrt{2}\rvert\cos\theta$ $\mathbf{a}.\mathbf{b} = 1 \quad \therefore \cos\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ$ (by inspection) (ii) $\overline{DE} = \overline{OE} - \overline{OD} = 2\mathbf{a} + \lambda(\mathbf{a}+2\mathbf{b}) - (2\mathbf{a} - \mathbf{b}) = \mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b}),\ \lambda \in R$ To find the square of the distance DE $DE^2 = [\mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})].[\mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})]$ $= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}+2\mathbf{b}).(\mathbf{a}+2\mathbf{b}) + 2\lambda\mathbf{b}.(\mathbf{a}+2\mathbf{b})$ $= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}.\mathbf{a} + 4\mathbf{a}.\mathbf{b} + 4\mathbf{b}.\mathbf{b}) + 2\lambda(\mathbf{b}.\mathbf{a} + 2\mathbf{b}.\mathbf{b})$ $= 2 + \lambda^2(1 + 4(1) + 4(2)) + 2\lambda(1 + 2(2))$ as $\mathbf{a}.\mathbf{a} = 1$, $\mathbf{b}.\mathbf{b} = 2$ and $\mathbf{a}.\mathbf{b} = 1$ $= 2 + 13\lambda^2 + 10\lambda$ $= 13\lambda^2 + 10\lambda + 2$ (iii) **<u>Method One:</u>** $DE^2 = 13\left[\lambda^2 + \dfrac{10}{13}\lambda\right] + 2$ $= 13\left(\lambda + \dfrac{10}{26}\right)^2 + 2 - \dfrac{25}{13} = 13\left(\lambda + \dfrac{5}{13}\right)^2 + \dfrac{1}{13}$ $DE = \sqrt{13\left(\lambda + \dfrac{5}{13}\right)^2 + \dfrac{1}{13}}$ The perpendicular distance from E to l occurs when D is closest to l, that is when DE is minimum or $\lambda = -\dfrac{5}{13}$. Exact shortest distance from D to l is $\dfrac{1}{\sqrt{13}}$ units.

	Method Two: DE is minimum when DE^2 is minimum: $\frac{\mathrm{d}}{\mathrm{d}x}(DE^2) = 26\lambda + 10$ To find stationary point: When $\frac{\mathrm{d}}{\mathrm{d}x}(DE^2) = 0,\ 26\lambda + 10 = 0$ $\therefore \lambda = -\frac{5}{13}$. Since DE^2 is quadratic and coefficient of $\lambda^2 > 0$, DE^2 is minimum at $\lambda = -\frac{5}{13}$ $\therefore$ perpendicular distance from D to l occur when $\lambda = -\frac{5}{13}$. $DE^2 = 13\lambda^2 + 10\lambda + 2 = 13\left(-\frac{5}{13}\right)^2 + 10\left(-\frac{5}{13}\right) + 2 = \frac{1}{13}$ Exact shortest distance from D to l is $\frac{1}{\sqrt{13}}$ units. (iv) Let F be the foot of the perpendicular from D to l. $\overrightarrow{OF} = 2\mathbf{a} - \frac{5}{13}(\mathbf{a} + 2\mathbf{b}) = \frac{21}{13}\mathbf{a} - \frac{10}{13}\mathbf{b}$
7	**(a)** $\sec x = \frac{1}{\cos x}$ $= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots\right)^{-1}$ $= 1 + (-1)\left[-\frac{1}{2}x^2 + \frac{1}{24}x^4\right] + \frac{(-1)(-2)}{2!}\left[-\frac{1}{2}x^2 + \frac{1}{24}x^4\right]^2 + \ldots$ $= 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{4}x^4 + \ldots$ $= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$ (up to x^4) (shown) $\ln(\sec x) \approx \ln\left[1 + \frac{1}{2}x^2 + \frac{5}{24}x^4\right]$ $= \left[\frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots\right] - \frac{1}{2}\left[\frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots\right]^2$

	$=\frac{1}{2}x^2+\frac{5}{24}x^4 \; -\frac{1}{2}\left(\frac{1}{4}x^4\right)+...$ $=\frac{1}{2}x^2+\frac{1}{12}x^4$ Thus $A=\frac{1}{12}$ **(b)(i)** $\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y$ $\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}$ $\left(1+x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(1-2x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}$ At $x = 0$, $y = 0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-1$ Thus, $y=x+\frac{x^2}{2!}-\frac{x^3}{3!}+...$ i.e. $y=x+\frac{x^2}{2}-\frac{x^3}{6}+...$ **(ii)** $\ln\left(1+y\right)=\tan^{-1}x\Rightarrow\frac{1}{1+y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+x^2}$ $\therefore\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y$ so condition **(A)** is satisfied. At $x = 0$, $\ln\left(1+y\right)=\tan^{-1}0=0\Rightarrow 1+y=\mathrm{e}^0$ $\therefore y=0$ **(iii)** $\int_0^{\frac{1}{2}}\left(\mathrm{e}^{\tan^{-1}x}-1\right)\mathrm{d}x\approx\int_0^{\frac{1}{2}}\left(x+\frac{x^2}{2}-\frac{x^3}{6}\right)\mathrm{d}x=\frac{55}{384}$
8	(**a**)Let a denote the first term of the geometric progression. Likewise, let b and d denote the first term and common difference of the arithmetic progression. $\therefore$ $ar^4 = b + 6d$ …Eq(1) $ar^8 = b + 24d$ …Eq(2) $ar^{10} = b + 48d$ …Eq(3) Eq(2) – Eq(1): $ar^8 - ar^4 = 18d$ …Eq(4) Eq(3) – Eq(2): $ar^{10} - ar^8 = 24d$ …Eq(5) Eq(5)/Eq(4): $\frac{ar^8\left(r^2-1\right)}{ar^4\left(r^4-1\right)}=\frac{24d}{18d}$ $\frac{r^4}{r^2+1}=\frac{4}{3}$ $3r^4 = 4r^2 + 4$ (Shown)

	From GC, $r = \pm\sqrt{2}$ so $\lvert r\rvert > 1$ Hence, the geometric progression is not convergent. **(b)** Let a be the 1st term and r be the common ratio of the G.P. $S_8 = \dfrac{A\left(1-r^8\right)}{1-r} = 72\pi$ ----- (1) $S_{odd} - S_{even} = 10\pi$ $\Rightarrow \dfrac{A\left(1-(r^2)^4\right)}{1-r^2} - \dfrac{Ar\left(1-(r^2)^4\right)}{1-r^2} = 10\pi$ $\dfrac{A\left(1-r^8\right)}{(1-r)(1+r)}\left[1-r\right] = 10\pi$ ----- (2) (1) ÷ (2): $\dfrac{1-r}{1+r} = \dfrac{10}{72}$ $72-72r = 10+10r$ $82r = 62$ $r = 0.75610$ Substituting into equation (1), $A = 61.8$ (to 3 s.f.) Let the production level in the first year be a. Total production of the coal mine = $\dfrac{a}{1-0.96} = 25a$ Thus, the total production of the coal mine can never exceed 25 times the production in the first year.
9	(a) Given $u = 2x+3 \Rightarrow \dfrac{du}{dx} = 2$ $\displaystyle\int \frac{x}{(2x+3)^3}\,dx = \int \frac{\frac{1}{2}(u-3)}{u^3}\cdot\frac{1}{2}\,du$ $\displaystyle = \frac{1}{4}\int\left[u^{-2} - 3u^{-3}\right]du$ $\displaystyle = \frac{1}{4}\left[-u^{-1} + \frac{3}{2}u^{-2}\right] + C$ $\displaystyle = -\frac{1}{4(2x+3)} + \frac{3}{8(2x+3)^2} + C$ $\displaystyle = \frac{-2(2x+3)+3}{8(2x+3)^2} + C$

		$$=-\frac{4x+3}{8(2x+3)^2}+C$$ $P = 4$, $Q = 3$ and $R = 8$ $$\int \frac{\ln(4x+3)^x}{(2x+3)^3}\,dx$$ $$=\int \frac{x}{(2x+3)^3}\cdot\ln(4x+3)\,dx \qquad \text{Let } \frac{dv}{dx}=\frac{x}{(2x+3)^3}, u=\ln(4x+3)$$ $$=-\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2}-\int -\frac{(4x+3)}{8(2x+3)^2}\cdot\frac{4}{(4x+3)}\,dx + C$$ $$=-\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2}+\frac{1}{2}\int (2x+3)^{-2}\,dx + C$$ $$=-\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2}+\frac{1}{2}(2x+3)^{-1}\left(-\frac{1}{2}\right)+C$$ $$=-\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2}-\frac{1}{4(2x+3)}+C$$ $$=-\frac{(4x+3)\ln(4x+3)+2(2x+3)}{8(2x+3)^2}+C$$ (b) $$\int \sin 4x\cos 6x\,dx$$ $$=\frac{1}{2}\int \sin 10x+\sin(-2x)\,dx$$ $$=\frac{1}{2}\int \sin 10x-\sin 2x\,dx$$ $$=\frac{1}{2}\left[-\frac{1}{10}\cos 10x+\frac{1}{2}\cos 2x\right]+C$$ $$=-\frac{1}{20}\cos 10x+\frac{1}{4}\cos 2x+C$$ $$\int e^x\sin 4e^x\cos 6e^x\,dx$$ $$=-\frac{1}{20}\cos 10e^x+\frac{1}{4}\cos 2e^x+C$$
10	**(i)**	At the original position, $t = 0$ $x=0+e^0=1$ and $y=0-e^0=-1$ Thus the coordinates are $(1,-1)$.
	(ii)	As t tends to infinity, $e^{-2t}\to 0$ so $x\to t$ and $y\to t$ Thus, the path of the particle **approaches the line $y = x$**

	(iii) $\frac{dy}{dt} = 1 + 2e^{-2t}$ and $\frac{dx}{dt} = 1 - 2e^{-2t}$ $\frac{dy}{dx} = \frac{1+2e^{-2t}}{1-2e^{-2t}}$ At $t = 2$, $x = 2 + e^{-4} = 2.01832$, $y = 2 - e^{-4} = 1.98168$ and $\frac{dy}{dx} = \frac{1+2e^{-4}}{1-2e^{-4}}$ Gradient of normal = $\frac{2e^{-4}-1}{1+2e^{-4}} = -0.92933$ Thus, an equation for C_2 is $y - 1.98168 = -0.92933(x - 2.01832)$ i.e. $y = -0.92933x + 3.85737$ i.e. $y = -0.929x + 3.857$ (correct to 3 d.p.) **(iv)** At point A, $y = 0$ $0 = -0.929x + 3.857 \Rightarrow x = 4.15178$ Coordinates of A are (4.15, 0) Sketch of motion of particle:  (v) Consider the curve C_1 when $y = 0$, $t = e^{-2t}$ and solving by GC, $t = 0.4263$ Thus, $x = 0.85261$ Required area $= \int_{0.852}^{2.02} y \, dx + \int_{2.02}^{4.15} (-0.929x + 3.857) \, dx$ $= \int_{0.4263}^{2} (t - e^{-2t})(1 - 2e^{-2t}) dt + \int_{2.02}^{4.15} (-0.929x + 3.857) \, dx$ = 3.5576 units^2 = 3.56 units^2
11	**(i)** Perimeter of cross-sectional area $= 100 = (2a + 4b) + \frac{1}{2}(2\pi a)$ $\Rightarrow 100 = 4b + a(\pi + 2)$

$$\Rightarrow b=\frac{100-a(\pi+2)}{4}$$

(ii) $S=(2a)(2b)+\frac{1}{2}(\pi a^2)$

$$=4a\left[\frac{100-a(\pi+2)}{4}\right]+\frac{\pi}{2}a^2$$

$$=100a-a^2(\pi+2)+\frac{\pi}{2}a^2$$

$$=100a-\frac{a^2}{2}(2\pi+4-\pi)$$

$$=100a-\frac{a^2}{2}(\pi+4)\quad\text{(shown)}$$

Note that, $a+b=a+\frac{100-a(\pi+2)}{4}$

$$=\frac{4a+100-a(\pi+2)}{4}$$

$$=\frac{1}{4}\left[100+a(2-\pi)\right]$$

$$V=\left[100a-\frac{a^2}{2}(\pi+4)\right]2(a+b)$$

$$=\left[100a-\frac{a^2}{2}(\pi+4)\right]\cdot\frac{2}{4}\left[100+a(2-\pi)\right]$$

$$=\frac{a}{2}\left[100-\frac{a}{2}(\pi+4)\right]\cdot\left[100+a(2-\pi)\right]$$

$$=5000a-75\pi a-\frac{a^3}{4}(\pi^2+2\pi-8)$$

(iii) $\frac{\mathrm{d}V}{\mathrm{d}a}=5000-150\pi a-\frac{3}{4}a^2(\pi^2+2\pi-8)$

When $\frac{\mathrm{d}V}{\mathrm{d}a}=0$, using the GC, $a=12.70471$ or $a=64.36321$

For $a=12.70471$			
A	a^-	a	a^+
Sign	–	0	+
$\frac{\mathrm{d}V}{\mathrm{d}a}$	/	—	\

For $a=64.36321$			
a	a^-	a	a^+
sign	–	0	+
$\frac{\mathrm{d}V}{\mathrm{d}a}$	\	—	/

Thus when $a=12.70471=12.7$ (3 s.f.), volume is greatest.
Using the GC, greatest volume is 29671.95154=29671.95 cm^3.

− End Of Paper −

1 **(i)** Prove that $\dfrac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B$. [1]

(ii) Hence, by considering a suitable expression of A and B, find

$$\sum_{r=1}^{N}\frac{\sin x}{\cos\left[(r+1)x\right]\cos(rx)}.$$ [3]

(iii) Using your answer to part **(ii)**, find $\displaystyle\sum_{r=1}^{N}\left(\frac{\sqrt{3}}{2\cos\frac{r\pi}{3}\cos\frac{(r+1)\pi}{3}}\right)$, leaving your answer in terms of N. [2]

2 **(i)** Find $\displaystyle\int_2^n \frac{9x}{\left(x^2-1\right)^3}\,\mathrm{d}x$, where $n\geq 2$ and hence evaluate $\displaystyle\int_2^\infty \frac{9x}{\left(x^2-1\right)^3}\,\mathrm{d}x$. [3]

(ii) Sketch the curve $y=\dfrac{9x}{\left(x^2-1\right)^3}$ for $x\geq 0$. [2]

(iii) The region R is bounded by the curve, the line $y=\dfrac{2}{3}$ and the line $x=5$.

Write down the equation of the curve when it is translated by $\dfrac{2}{3}$ units in the negative y-direction. [1]

Hence or otherwise, find the volume of the solid formed when R is rotated completely about the line $y=\dfrac{2}{3}$, leaving your answer correct to 3 decimal places. [2]

3 **(a)** **(i)** Show that $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta-\dfrac{1}{3}\sin^3\theta\right)=\cos^3\theta$. [1]

(ii) Find the solution to the differential equation $\operatorname{cosec} x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\cos^2 x$ in the form $y=\mathrm{f}(x)$, given that $y=0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{3}+\dfrac{2}{\pi}$ when $x=0$. [4]

(b) Show, by means of the substitution $v=x^2y$, that the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+2y+4x^2y=0$$

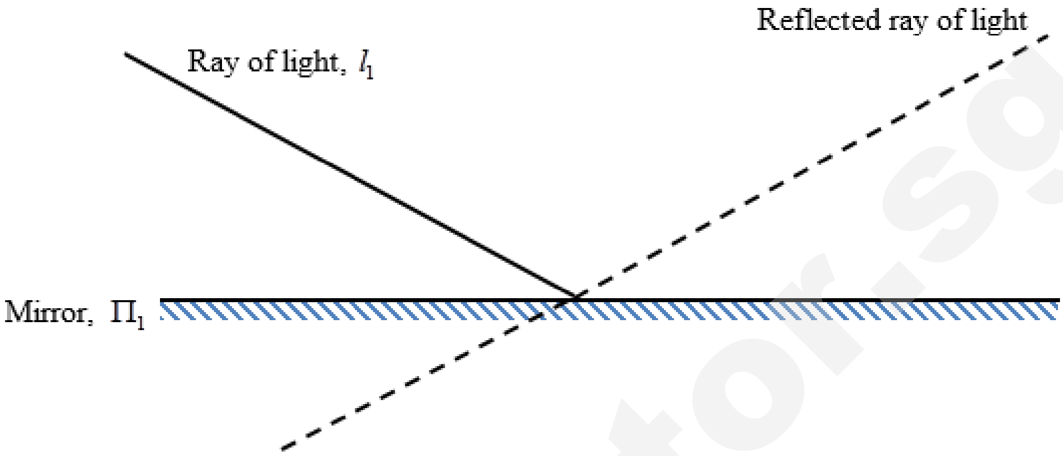
can be reduced to the form

$$\frac{\mathrm{d}v}{\mathrm{d}x}=-4vx.$$

Hence find y in terms of x, given that $y=\dfrac{1}{3}$ when $x=-3$. [6]

4 In the study of light, we may model a ray of light as a straight line.

A ray of light, l_1, is known to be parallel to the vector $2\mathbf{i}+\mathbf{k}$ and passes through the point P with coordinates $(1,1,0)$. The ray of light hits a mirror, and is reflected by the mirror which may be modelled by a plane Π_1 containing the points A, B and C with coordinates $(-1,1,0)$, $(0,0,2)$ and $(0,3,-3)$ respectively. This scenario is depicted in the diagram below:



(i) Show that an equation for plane Π_1 is given by $-x+5y+3z=6$. [3]

(ii) Find the coordinates of the point where the ray of light meets the mirror. [2]

(iii) Determine the position vector of the foot of the perpendicular from the point P to the mirror and hence, find an equation of the line that may be used to model the reflected ray of light. [6]

A second ray of light which is parallel to the mirror may be modelled by the line l_2, with Cartesian equation $\frac{x-1}{2}=\frac{z-2}{\alpha}, y=\beta$. Given that the distance between l_2 and the mirror is $\frac{14}{\sqrt{35}}$ units, find the values of the positive constants α and β. [4]

5 A random variable X has the probability distribution given in the following table.

x	2	3	4	5
$\mathrm{P}(X=x)$	0.2	a	b	0.45

Given that $E\left(\lvert X-4\rvert\right)=\frac{11}{10}$, find the values of a and b. [3]

Two independent observations of X are taken. Find the probability that one of them is 2 and the other is at most 4. [2]

6 In a large consignment of mangoes, 4.5% of the mangoes are damaged.

(i) A total of 21 mangoes are selected at random. Calculate the probability that not more than 3 mangoes are damaged. [2]

(ii) The mangoes are randomly selected and packed into boxes of 21. For shipping purposes, the boxes are packed into cartons, with each carton containing 12 boxes. A box containing more than 3 damaged mangoes is considered low standard. Calculate the probability that, in a randomly selected carton, there are at least 2 boxes which are of low standard. [3]

(iii) Find the probability that a randomly chosen box that is of low standard contains no more than five damaged mangoes. [3]

7 **(a)** Seven boys and five girls formed a group in a school orientation. During one of the game segments, they are required to arrange themselves in a row. Find the exact probability that

(i) the girls are separated from one another, [2]

(ii) there will be exactly one boy between any two girls. [2]

In another game segment, they are required to sit at a round table with twelve identical chairs. Find the exact probability that one particular boy is seated between two particular girls. [2]

(b) The events A and B are such that $\mathrm{P}\left(A\right)=\dfrac{7}{10}$, $\mathrm{P}\left(B\right)=\dfrac{2}{5}$ and $\mathrm{P}\left(A \mid B\right)=\dfrac{13}{20}$.

(i) Find $\mathrm{P}\left(A \cup B\right)$, [3]

(ii) State, with a reason, whether the events A and B are independent. [1]

(c) A man plays a game in which he draws balls, with replacement, from a bag containing 3 yellow balls, 2 red balls and 4 black balls. If he draws a black ball, he loses the game and if he draws a red ball he wins the game. If he draws a yellow ball, the ball is replaced and he draws again. He continues drawing until he either wins or loses the game. Find the probability that he wins the game. [2]

8 A company manufactures bottles of iced coffee. Machines A and B are used to fill the bottles with iced coffee.

(i) Machine A is set to fill the bottles with 500 ml of iced coffee. A random sample of 50 filled bottles was taken and the volume of iced coffee (x ml) in each bottle was measured. The following data was obtained

$$\sum x = 24965 \quad \sum (x-\bar{x})^2 = 365$$

Calculate unbiased estimates of the population mean and variance. Test at the 2% level of significance, whether the mean volume of iced coffee per bottle is 500 ml. [6]

(ii) The company claims that Machine B filled the bottles with μ_0 ml of iced coffee. A random sample of 70 filled bottles was taken and the mean is 489.1 ml with standard deviation 4 ml. Find the range of values of μ_0 for which there is sufficient evidence for the company to have overstated the mean volume at the 2% level of significance. [5]

9 An online survey revealed that 34.1% of junior college students spent between 3 to 3.8 hours on their mobile phones daily. Assuming that the amount of time a randomly chosen junior college student spends on mobile phones daily follows a normal distribution with mean 3.4 hours and standard deviation σ hours, show that $\sigma = 0.906$, correct to 3 decimal places. [3]

Find the probability that

(i) four randomly chosen students each spend between 3 to 3.8 hours daily on their mobile phones. [1]

(ii) the total time spent on their mobile phones daily by the three randomly chosen junior college students is less than twice that of another randomly chosen junior college student. [3]

(iii) State an assumption required for your calculations in **(i)** and **(ii)** to be valid. [1]

N samples, each consisting of 50 randomly selected junior college students, are selected. It is expected that 15 of these samples will have a mean daily time spent on mobile phones greater than 3.5 hours.

(iv) Estimate the value of *N*. [4]

10 In a medical study, researchers investigated the effect of varying amounts of calcium intake on the bone density of Singaporean women of age 50 years. A random sample of eighty 50-year-old Singaporean women was taken.

(i) Explain, in the context of this question, the meaning of the phrase 'random sample'. [1]

The daily calcium intake (x mg) of the women was varied and the average percentage increase in bone density (y%) was measured. The data is as shown in the table below.

x (in mg)	700	800	900	1000	1050	1100
y (%)	0.13	0.78	1.38	1.88	2.07	2.10

(ii) Calculate the product moment correlation coefficient and suggest why its value does not necessarily mean that the best model for the relationship between x and y is $y = a + bx$. [2]

(iii) Draw a scatter diagram representing the data above. [2]

The researchers suggest that the change in bone density can instead be modelled by the equation $\ln(P - y) = a + bx$.

The product moment correlation coefficient between x and $\ln(P - y)$ is denoted by r. The following table gives values of r for some possible values of P.

P	3	5	10
r		–0.993803	–0.991142

(iv) Calculate the value of r for $P = 3$, giving your answer correct to 6 decimal places. Use the table and your answer to suggest with reason, which of 3, 5 or 10 is the most appropriate value of P. [2]

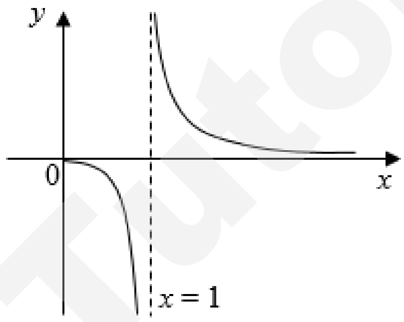
The Healthy Society wants to recommend a daily calcium intake that would ensure an average of 1.8% increase in bone density.

(v) Using the value of P found in part **(iv)**, calculate the values of a and b and use your answer to estimate the daily calcium intake that the Health Society should recommend. Comment on the reliability of the estimate obtained. [4]

(vi) Give an interpretation, in the context of the question, of the meaning of the value of P. [1]

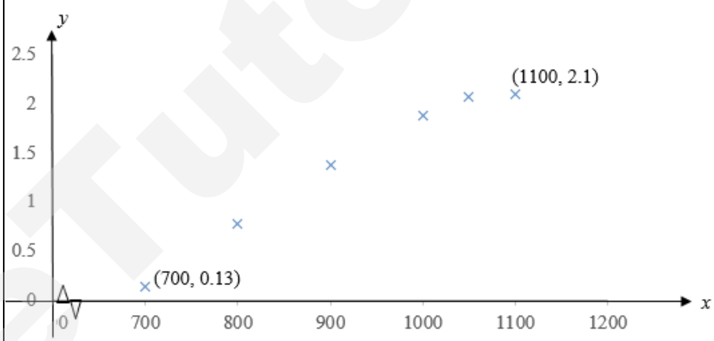
ANNEX B

SRJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Sigma Notation and Method of Difference	**(ii)** $\tan(N+1)x-\tan x$ **(iii)** $\tan\left[\dfrac{(N+1)\pi}{3}\right]-\sqrt{3}$
2	Application of Integration	**(i)** $\dfrac{1}{4}-\dfrac{9}{4\left(n^2-1\right)^2}$, $\dfrac{1}{4}$ **(ii)**  **(iii)** $y=\dfrac{9x}{\left(x^2-1\right)^3}-\dfrac{2}{3}$, 3.385 units^3
3	Differential Equations	**(a) (ii)** $y=\dfrac{1}{3}\left(\sin x-\dfrac{\sin^3 x}{3}\right)+\dfrac{2}{\pi}x$ **(b)** $y=\dfrac{3e^{18-2x^2}}{x^2}$
4	Vectors	**(ii)** (5, 1, 2) **(iii)** $\overrightarrow{OF}=\begin{pmatrix} 33/35 \\ 9/7 \\ 6/35 \end{pmatrix}$, $l_1' : \mathbf{r}=\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}+\gamma\begin{pmatrix} 72 \\ -10 \\ 29 \end{pmatrix}, \gamma \in \mapsto$ $\alpha=\dfrac{2}{3}$, $\beta=3$
5	DRV	$a=0.25$ and $b=0.1$, 0.18
6	Binomial Distribution	**(i)** 0.987 **(ii)** 0.0106 **(iii)** 0.981
7	P&C, Probability	**(a) (i)** $\dfrac{7}{99}$ **(ii)** $\dfrac{1}{198}$, $\dfrac{1}{55}$ **(b)** 0.84 **(c)** $\dfrac{1}{3}$

<table>
<tr><td>8</td><td>Hypothesis Testing</td><td>(i) $\bar{x} = 499.3$, $s^2 \approx 7.45$, p-value = 0.06974
(ii) $\mu_0 \geq 490$</td></tr>
<tr><td>9</td><td>Normal Distribution</td><td>(i) 0.0135 (ii) 0.0781
(iii) Assumption:
The amount of time spent by a randomly chosen student on mobile phones is independent of the amount of time spent by another randomly chosen student.
(iv) $N = 69$</td></tr>
<tr><td>10</td><td>Correlation & Linear Regression</td><td>(i) The phrase ‘random sample’ means that every 50-year-old Singaporean woman has an <u>equal probability of being included in the sample</u>.

(ii) $r = 0.988$

(iii)


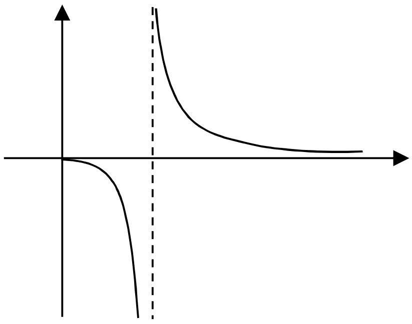
(iv) $r = -0.995337$

(v) $a = 3.24$, $b = -0.00310$
The recommended daily calcium intake is 988 mg.
Since the r value is – 0.995 is close to – 1, there is a strong negative linear correlation between $\ln(P-y)$ and x. Also since the value of $y = 1.8$ is within the data range, thus, the estimate obtained is reliable.

(vi) The value of P is the maximum percentage increase in bone density achievable as the daily calcium intake increases.</td></tr>
</table>

H2 Mathematics 2017 Prelim Exam Paper 2 Question
Answer all questions [100 marks].

1	**(i)** $\frac{\sin(A-B)}{\cos A\cos B}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B}=\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}=\tan A-\tan B$ **(ii)** $\sum_{r=1}^{N}\frac{\sin x}{\cos(r+1)x\cos rx}=\frac{\sin(2x-x)}{\cos 2x\cos x}+\frac{\sin(3x-2x)}{\cos 3x\cos 2x}+\frac{\sin(4x-3x)}{\cos 4x\cos 3x}+\ldots+\frac{\sin((N+1)x-Nx)}{\cos(N+1)x\cos Nx}$ $=(\tan 2x-\tan x)$ $+(\tan 3x-\tan 2x)$ $+(\tan 4x-\tan 3x)$ $\vdots$ $+(\tan(N-1)x-\tan(N-2)x$ $+(\tan Nx-\tan(N-1)x$ $+(\tan(N+1)x-\tan Nx$ $=\tan(N+1)x-\tan x$ **(iii)** When $x=\frac{\pi}{3}$, $\sum_{r=1}^{N}\frac{\sin x}{\cos(r+1)x\cos rx}=\sum_{r=1}^{N}\left(\frac{\sqrt{3}}{2\cos\frac{r\pi}{3}\cos\frac{(r+1)\pi}{3}}\right)$ Thus, required sum = $\tan\left[(N+1)\left(\frac{\pi}{3}\right)\right]-\tan\left(\frac{\pi}{3}\right)=\tan\left[\frac{(N+1)\pi}{3}\right]-\sqrt{3}$
2	**(i)** $\int_2^n\frac{9x}{\left(x^2-1\right)^3}\,\mathrm{d}x=\frac{9}{2}\int_2^n\frac{2x}{\left(x^2-1\right)^3}\,\mathrm{d}x$ $=\frac{9}{2}\left[-\frac{1}{2}\left(x^2-1\right)^{-2}\right]_2^n$ $=\frac{9}{2}\left[-\frac{1}{2\left(n^2-1\right)^2}+\frac{1}{18}\right]$ $=\frac{1}{4}-\frac{9}{4\left(n^2-1\right)^2}$

	$\lim_{n\to\infty}\left[\int_2^n \frac{9x}{\left(x^2-1\right)^3}\,\mathrm{d}x\right]=\lim_{n\to\infty}\left[\frac{1}{4}-\frac{9}{4\left(n^2-1\right)^2}\right]$ $=\frac{1}{4}$ **(ii)**  **(iii)** The equation of the transformed curve is $y=\frac{9x}{\left(x^2-1\right)^3}-\frac{2}{3}$. Volume of revolution $=\pi\int_2^5\left(\frac{9x}{\left((x^2-1)\right)^3}-\frac{2}{3}\right)^2\,\mathrm{d}x \;=3.385\text{ units}^3$ (to 3 d.p.)
3	**(a) (i)** $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)$ $=\cos\theta-\sin^2\theta\cos\theta$ $=\cos\theta\left(1-\sin^2\theta\right)$ $=\cos\theta\left(\cos^2\theta\right)=\cos^3\theta$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\sin x\cos^2 x$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(-\sin x\right)\left(\cos x\right)^2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(\cos x\right)^3}{3}+C$ $=\frac{1}{3}\left(\cos x\cdot\cos^2 x\right)+C$ $=\frac{1}{3}\left(\cos x\cdot\left(1-\sin^2 x\right)\right)+C$ $=\frac{1}{3}\left(\cos x-\cos x\cdot\sin^2 x\right)+C$

$$y=\frac{1}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)+Cx+D$$

When $x = 0$ and $y = 0$, $D = 0$

When $x = 0$ and $\frac{dy}{dx}=\frac{1}{3}+\frac{2}{\pi}$, $C=\frac{2}{\pi}$

$$y=\frac{1}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)+\frac{2}{\pi}x$$

(b) $v=x^2y$ --------------------(1)

$\frac{dv}{dx}=2xy+x^2\frac{dy}{dx}$ -------- (2)

$x\frac{dy}{dx}+2y+4x^2y=0$ ----- (3)

(3) × x, $\quad x^2\frac{dy}{dx}+2xy+4x^2y(x)=0$ -------- (4)

$$\frac{dv}{dx}+4x\left(x^2y\right)=0$$

$$\frac{dv}{dx}+4vx=0$$

$$\frac{dv}{dx}=-4vx \quad \text{(Shown)}$$

$$\frac{dv}{dx}=-4vx$$

$$\int\frac{1}{v}\,dv=-4\int x\,dx$$

$$\ln|v|=-2x^2+c$$

$$v=\pm e^{-2x^2+c}$$

$v=Ae^{-2x^2}$, where $A = \pm e^c$

$$x^2y=Ae^{-2x^2}$$

Given that $y=\frac{1}{3}$ when $x=-3$,

$$(-3)^2\left(\frac{1}{3}\right)=Ae^{-18}$$

$$A=3e^{18}$$

$$y = \frac{3e^{18-2x^2}}{x^2}$$

4

(i) $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \lambda \in \cdots$

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix};\ \overrightarrow{AC} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix};\ \overrightarrow{BC} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix}$$

A normal to the plane is: $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$

$$\mathbf{r} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 6$$

Thus an equation for Π_1 is $-x + 5y + 3z = 6$. (shown)

(ii) Let N be the point of intersection between the line and the plane.

$\overrightarrow{ON} = \begin{pmatrix} 1+2\lambda \\ 1 \\ \lambda \end{pmatrix}$ for some $\lambda \in \cdots$

Since N lies on the plane,

$$\begin{pmatrix} 1+2\lambda \\ 1 \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 6 \Rightarrow \lambda = 2$$

Thus, coordinates of N are (5, 1, 2).

(iii) Let the foot of the perpendicular from P to the plane be denoted by F.

$$l_{PF} : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}, \mu \in \cdots$$

Since F lies on l_{PF}, $\overrightarrow{OF} = \begin{pmatrix} 1-\mu \\ 1+5\mu \\ 3\mu \end{pmatrix}$ for some $\mu \in \cdots$

Since F lies on the plane, $\begin{pmatrix} 1-\mu \\ 1+5\mu \\ 3\mu \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 6$

Solving, $\mu = \frac{2}{35}$

	$\overrightarrow{OF}=\begin{pmatrix}33/35\\9/7\\6/35\end{pmatrix}$ Let the reflection of point *P* in the mirror be *P'*. By the midpoint theorem, $\overrightarrow{OP'}=2\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}31/35\\11/7\\12/35\end{pmatrix}$ A direction vector for the reflected line is $\begin{pmatrix}5\\1\\2\end{pmatrix}-\begin{pmatrix}31/35\\11/7\\12/35\end{pmatrix}=\begin{pmatrix}144/35\\-4/7\\58/35\end{pmatrix}=\frac{2}{35}\begin{pmatrix}72\\-10\\29\end{pmatrix}$ Thus, an equation of the reflected line is: $l_1':\mathbf{r}=\begin{pmatrix}5\\1\\2\end{pmatrix}+\gamma\begin{pmatrix}72\\-10\\29\end{pmatrix},\gamma\in\cdots$ Since l_2 is parallel to Π_1, $\begin{pmatrix}2\\0\\\alpha\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\3\end{pmatrix}=0\Rightarrow\alpha=\frac{2}{3}$ $\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}1\\\beta\\2\end{pmatrix}=\begin{pmatrix}-1\\-\beta\\0\end{pmatrix}$ Since the distance is $\frac{14}{\sqrt{35}}$, $\left\lvert\frac{\begin{pmatrix}-1\\-\beta\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\3\end{pmatrix}}{\sqrt{35}}\right\rvert=\frac{14}{\sqrt{35}}$ $\lvert 1-5\beta\rvert=14$ Solving, $\beta=-\frac{13}{5}$ (rejected) or $\beta=3$
5	$\sum_{\text{all }x}\mathrm{P}(X=x)=1=0.2+a+b+0.45\Rightarrow a+b=0.35\ \ldots\ (1)$ $E\left(\lvert X-4\rvert\right)=1\frac{1}{10}\Rightarrow\sum_{\text{all }x}\lvert x-4\rvert\mathrm{P}(X=x)=\frac{11}{10}$ $\Rightarrow 2(0.2)+a+0+0.45=\frac{11}{10}$ $\Rightarrow a=0.25$ and $b=0.1$

	P(required) $=P(X_1=2, X_2=2)+2[P(X_1=2, X_2=3)+P(X_1=2, X_2=4)]$ $=0.2\times 0.2+2[0.2\times 0.25+0.2\times 0.1]$ $=0.18$
6	**(i)** Let X be the random variable "number of damaged mangoes out of 21 mangoes". $X \sim B(21, 0.045)$ $P(X\le 3)=0.98673=0.987$ (3 s.f.) **(ii)** Let Y be the random variable "number of boxes of mangoes out of 12 boxes which are of low standard". $Y \sim B(12, 1-0.98673) \Rightarrow Y \sim B(12, 0.013268)$ $P(Y\ge 2)=1-P(Y\le 1)$ $=1-0.98936=0.01064=0.0106$ (3 s.f.) **(iii)** P(required) = $P(X\le 5 \mid$ box is of low standard) $=P(X\le 5 \mid X>3)$ $=\dfrac{P(X\le 5\cap X>3)}{P(X>3)}$ $=\dfrac{P(X=4)+P(X=5)}{1-P(Y\le 3)}$ $=\dfrac{0.011219+0.0017975}{1-0.98673}$ $=0.981$
7	**(a)(i)** Required probability = $\dfrac{7!\times {}^8C_5\times 5!}{12!}$ $=\dfrac{7}{99}$ **(a)(ii)** Required probability = $\dfrac{7!\times 4\times 5!}{12!}$ $=\dfrac{1}{198}$ Required probability = $\dfrac{(10-1)!\times 2!}{(12-1)!}$ $=\dfrac{1}{55}$

	(b)(i) $\mathrm{P}(A \mid B) = \frac{13}{20}$ $\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{13}{20}$ $\mathrm{P}(A \cap B) = \frac{13}{20}\left(\frac{2}{5}\right) = \frac{13}{50}$ (or 0.26) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ $= \frac{7}{10} + \frac{2}{5} - \frac{13}{50}$ $= \frac{21}{25}$ (or 0.84) **(b)(ii)** Since $\mathrm{P}(A \mid B) \neq \mathrm{P}(A)$, therefore events A and B are not independent. **Alternatively,** Since $\mathrm{P}(A \cap B) = \frac{13}{50}$ and $\mathrm{P}(A) \times \mathrm{P}(B) = \frac{7}{10} \times \frac{2}{5} = \frac{7}{25} \neq \mathrm{P}(A \cap B)$, therefore events A and B are not independent. **(c)** Probability of winning the game $= \frac{2}{9} + \frac{2}{9}\left(\frac{3}{9}\right) + \frac{2}{9}\left(\frac{3}{9}\right)^2 + \ldots$ $= \frac{2/9}{1 - \frac{3}{9}}$ $= \frac{1}{3}$
8	**(i)** Let X be the random variable denoting volume of the randomly chosen iced coffee bottle in ml from Machine A. $\bar{x} = \frac{24965}{50} = 499.3$ Unbiased estimate of population variance $s^2 = \frac{1}{n-1}\sum(x-\bar{x})^2 = \frac{50}{49}\left(\frac{365}{50}\right) = \frac{365}{49} = 7.4489 \approx 7.45$ $\mathrm{H}_0 : \mu = 500$ $\mathrm{H}_1 : \mu \neq 500$ Two tailed Z test at 2% level of significance Under H_0, since the sample size of 50 is large, by Central Limit Theorem

	$\overline{X} \sim \text{N}(500, \frac{7.4489}{50})$ approx. From GC, p-value = 0.06974> 0.02 Conclusion: Since the p-value is more than the level of significance, we do not reject H_0 and conclude that there is insufficient evidence at 2% that the mean volume is not 500ml. **(ii)** Let Y be the random variable denoting the volume of a randomly chosen iced coffee bottle in ml from Machine B. Unbiased estimate for population variance = $\frac{70}{69}\left(4^2\right) = 16.232$ $H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$ One tailed Z test at 2% level of significance Under H_0, since the sample size of 70 is large, by Central Limit Theorem $\overline{Y} \sim \text{N}\left(\mu_0, \frac{16.232}{70}\right)$ approx. Value of test statistic, $z_{\text{test}} = \frac{489.1-\mu_0}{\sqrt{\frac{16.232}{70}}}$ For H_0 to be rejected, p-value ≤ 0.02 $\frac{489.1-\mu_0}{\sqrt{\frac{16.232}{70}}} \leq -2.053748911$ $\mu_0 \geq 490$ (to 3 s.f.)
9	Let X denote the random variable representing the amount of time a randomly chosen junior college student spends on mobile phones each day. $\therefore X \sim \text{N}(3.4, \sigma^2)$ $\text{P}(3 < X < 3.8) = 0.341$ $\text{P}\left(\frac{3-3.4}{\sigma} < Z < \frac{3.8-3.4}{\sigma}\right) = 0.341$ $\text{P}\left(\frac{-0.4}{\sigma} < Z < \frac{0.4}{\sigma}\right) = 0.341$ $\Rightarrow \text{P}\left(Z < \frac{-0.4}{\sigma}\right) = \frac{1-0.341}{2} = 0.3295$ From GC, $\frac{-0.4}{\sigma} = -0.4412942379$ $\Rightarrow \sigma = 0.90642 = 0.906$ (3 dp) **(i)** Probability required $= (0.341)^4$ $= 0.0135$ (3 sf)

(ii) Probability required $= P(X_1 + X_2 + X_3 < 2X_4)$
$= P(X_1 + X_2 + X_3 - 2X_4 < 0)$

$X_1 + X_2 + X_3 - 2X_4 \sim N(3.4\times 3 - 2\times 3.4,\ 0.90642^2\times 3 + 2^2\times 0.90642^2)$

i.e. $X_1 + X_2 + X_3 - 2X_4 \sim N(3.4,\ 5.75118)$

$\therefore$ From GC, $(X_1 + X_2 + X_3 - 2X_4 < 0) = 0.0781$ (3 sf)

(iii) Assumption: The amount of time spent by a randomly chosen student on mobile phones is independent of the amount of time spent by another randomly chosen student.

(iv) $\overline{X} \sim N\left(3.4, \dfrac{0.90642^2}{50}\right)$

From GC, $P\left(\overline{X} > 3.5\right) = 0.217663$

Since expected number of samples with mean time exceeding 3.5 hours = 15, then $0.217663 \times N = 15$

$\Rightarrow N = 68.9 \approx 69$

10

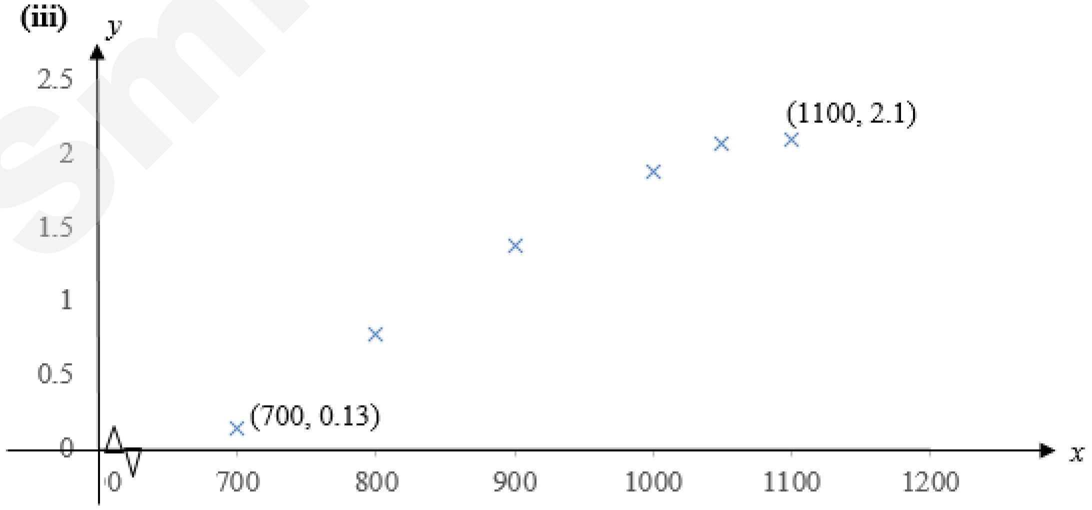
(i) The phrase 'random sample' means that every 50-year-old Singaporean woman has an **<u>equal probability of being included in the sample</u>**.

(ii) $r = 0.988$ (to 3 s.f.)

Although the r-value = 0.988 is close to 1, the value is not 1 so there may be another model with $|r|$ closer to 1.

Hence a linear model may not be the best model for the relationship between x and y.

(iii)



(iv) Using the GC, when $P = 3$, $r = -0.995337$ (to 6 d.p.)

When $P = 3$, $|r|$ is closest to 1 and thus, $P = 3$ is the most appropriate value.

(v) When $P = 3$, using the GC, $a = 3.2446 = 3.24$ (to 3 s.f.)

	$b = -0.0030988 = -0.00310$ (to 3 s.f.) When $y = 1.8$, and $P = 3$, $\ln(3-1.8) = 3.2446 - 0.0030988x$ $x = 988$ Thus, the recommended daily calcium intake is 988 mg. Since the r value is -0.995 is close to -1, there is a strong negative linear correlation between $\ln(P-y)$ and x. Also since the value of $y = 1.8$ is within the data range, thus, the estimate obtained is reliable. **(vi)** The value of P is the maximum percentage increase in bone density achievable as the daily calcium intake increases.

– End Of Paper –

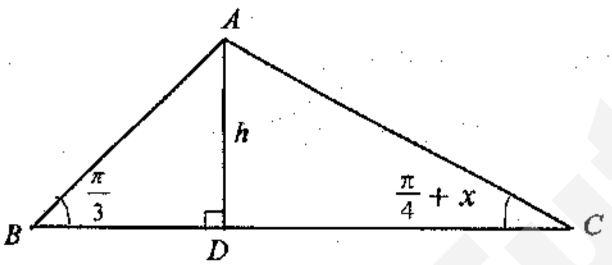
2017 SAJC Prelim Paper 1

Answer all questions [100 marks].

1 The volume of a spherical bubble is increasing at a constant rate of λ cm^3 per second. Assuming that the initial volume of the bubble is negligible, find the exact rate in terms of λ at which the surface area of the bubble is increasing when the volume of the bubble is 20 cm^3. **[5]**

[The volume of a sphere, $V = \frac{4}{3}\pi r^3$ and the surface area of a sphere, $A = 4\pi r^2$ where r is the radius of the sphere.]

2 The diagram shows the triangle ABC. It is given that the height AD is h units, $\angle ABD = \frac{\pi}{3}$ and $\angle ACD = \frac{\pi}{4} + x$.



Show that if x is sufficiently small for x^3 and higher powers of x to be neglected, then

$$BC = \frac{h}{\sqrt{3}} + \frac{h}{\tan\left(\frac{\pi}{4} + x\right)} \approx h\left(p + qx + rx^2\right)$$

for constants p, q, r to be determined in exact form. **[5]**

3 It is given that

$$f(x) = \begin{cases} b\sqrt{1 - \frac{x^2}{a^2}} & \text{for } -a < x \le a \\ -a\sqrt{1 - \frac{(x-2a)^2}{a^2}} & \text{for } a < x \le 3a \end{cases}$$

and that $f(x + 4a) = f(x)$ for all real values of x, where a and b are real constants and $0 < a < b$.

(i) Sketch the graph of $y = f(x)$ for $-a \le x \le 8a$. **[3]**

(ii) Use the substitution $x = a\cos\theta$ to find the exact value of $\int_{3a}^{4a} f(x)\, dx$ in terms of a and π. **[5]**

(i) State a sequence of transformations that would transform the curve with equation $y = e^{x^2}$ onto the curve with equation $y = f(x)$, where $f(x) = e^{ax^2} - b$, $a > 0$ and $b > 1$. **[2]**

(ii) Sketch the curve $y = f(x)$ and the curve $y = \frac{1}{f(x)}$.

You should state clearly the equations of any asymptotes, coordinates of turning points and axial intercepts. **[5]**

It is given that $\mathbf{u}+\mathbf{v}-\mathbf{w}$ is perpendicular to $\mathbf{u}-\mathbf{v}+\mathbf{w}$, where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are unit vectors.

(i) Show that the angle between $\mathbf{v}$ and $\mathbf{w}$ is 60°. **[4]**

Referred to the origin O, the points U, V and W have position vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively.

(ii) Find the exact area of triangle OVW. **[2]**

(iii) Given that $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$ are parallel, find the exact volume of the solid $OUVW$. **[2]**

[The volume of a pyramid is $\frac{1}{3}bh$, where b is the base area and h is the height of the pyramid.]

6 **(a)** **(i)** Find $\int e^x \cos nx \, dx$, where n is a positive integer. **[4]**

(ii) Hence, without the use of a calculator, find $\int_{\pi}^{2\pi} e^x \cos nx \, dx$ when n is odd. **[3]**

(b) The region bounded by the curve $y=\dfrac{\sqrt{x}}{16-x^2}$, the y-axis and the line $y=\dfrac{\sqrt{2}}{12}$ is rotated 2π radians about the x-axis. Find the exact volume of the solid obtained. **[5]**

7 **(i)** Show that for any complex number $z=re^{i\theta}$, where $r>0$, and $-\pi<\theta\le\pi$,

$$\frac{z}{z-r}=\frac{1}{2}-\frac{1}{2}\left(\cot\frac{\theta}{2}\right)i.$$

[3]

(ii) Given that $z=2e^{i\left(\frac{\pi}{3}\right)}$ is a root of the equation $z^2-2z+4=0$. State, in similar form, the other root of the equation. **[1]**

(iii) Using parts (i) and (ii), solve the equation $\dfrac{4w^2}{(w-1)^2}-\dfrac{4w}{w-1}+4=0$. **[4]**

8 In a training session, an athlete runs from a starting point S towards his coach in a straight line as shown in the diagrams below. When he reaches the coach, he runs back to S along the same straight line. A lap is completed when he returns to S. At the beginning of the training session, the coach stands at A_1 which is 30 m away from S.

After the first lap, the coach moves from A_1 to A_2 and after the second lap, he moves from A_2 to A_3 and so on. The distance between A_i to A_{i+1} is denoted by A_iA_{i+1}, $i \in \mathbb{Z}^+$.

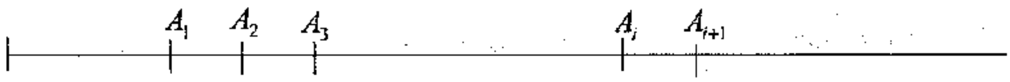


Figure 1

(i) For training regime 1 (shown in Figure 1), the coach ensures that the distance $A_iA_{i+1} = 3$ m for $i \in \mathbb{Z}^+$. Find the least number of laps that the athlete must complete so that he covers a total distance of 3000 m. **[3]**

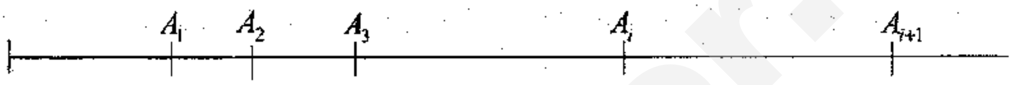


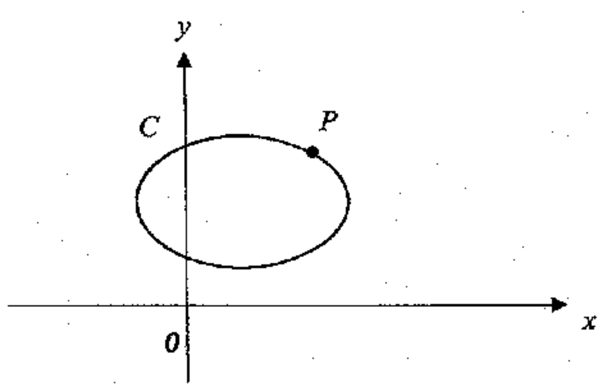
Figure 2

(ii) For training regime 2 (shown in Figure 2), after the first lap, the coach ensures that the distances $A_1A_2 = 2$ m, $A_2A_3 = 6$ m and the distance $A_{i+1}A_{i+2} = 3A_iA_{i+1}$ where $i \in \mathbb{Z}^+$. Show that the distance the coach is away from S **just before** the athlete completed r laps is $\left(3^{r-1} + 29\right)$m.

Hence find the distance run by the athlete after n complete laps. Also find how far the athlete is from the coach after he has run 8 km. **[6]**

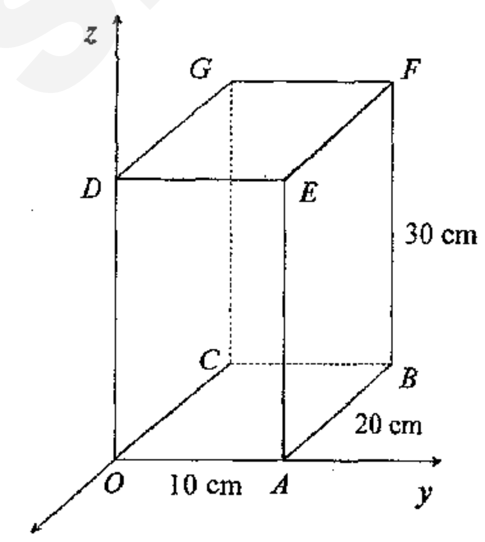
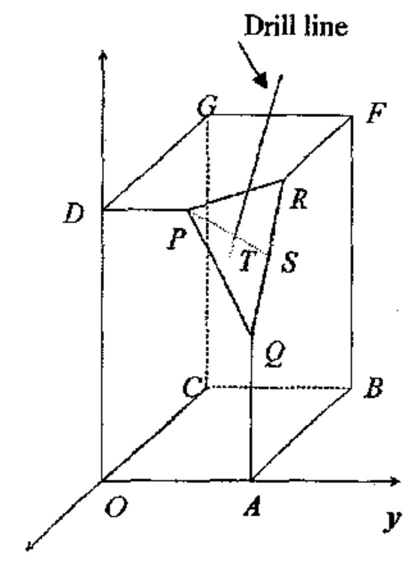
9 The diagram below shows the curve C with parametric equations

$$x = 1 + 2\sin\theta, \quad y = 4 + \sqrt{3}\cos\theta, \quad \text{for } -\pi < \theta \le \pi.$$



The point P is where $\theta = \dfrac{\pi}{6}$.

(i) Using a non-calculator method, find the equation of the normal at P. **[4]**

	(ii) The normal at the point P cuts C again at point Q, where $\theta = \alpha$. Show that $8\sin\alpha - 2\sqrt{3}\cos\alpha = 1$ and hence deduce the coordinates of Q. **[3]** **(iii)** Find the area of the region bounded by the curve C, the normal at point P and the vertical line passing through the point Q. **[4]**
10	A population of 15 foxes has been introduced into a national park. A zoologist believes that the population of foxes, x, at time t years, can be modelled by the Gompertz equation given by: $$\frac{dx}{dt} = cx\ln\left(\frac{40}{x}\right)$$ where c is a constant. **(i)** Using the substitution $u = \ln\left(\frac{40}{x}\right)$, show that the differential equation can be written as $\frac{du}{dt} = -cu$. **[2]** **(ii)** Hence find u in terms of t and show that $x = 40e^{-Be^{-ct}}$, where B is a constant. **[5]** After 3 years, the population of foxes is estimated to be 20. **(iii)** Find the values of B and c. **[3]** **(iv)** Find the population of foxes in the long run. **[1]** **(v)** Hence, sketch the graph showing the population of foxes over time. **[2]**
11	A computer-controlled machine can be programmed to make plane cuts by keying in the equation of the plane of the cut, and drill holes in a straight line through an object by keying in the equation of the drill line. A 10cm × 20 cm × 30 cm cuboid is to be cut and drilled. The cuboid is positioned relative to the x-, y-and z-axes as shown in Figure 1.  Figure 1  Figure 2

	First, a plane cut is made to remove the corner at E. The cut goes through the points P, Q and R which are the midpoints of the sides ED, EA and EF respectively. (i) Show that $\overrightarrow{PQ}=\begin{pmatrix}0\\5\\-15\end{pmatrix}$ and $\overrightarrow{PR}=\begin{pmatrix}-10\\5\\0\end{pmatrix}$. **[2]** **(ii)** Find the cartesian equation of the plane, p that contains P, Q and R. **[2]** **(iii)** Find the acute angle between p and the plane $DEFG$. **[2]** A hole is then drilled perpendicular to triangle PQR, as shown in Figure 2. The hole passes through the triangle at the point T which divides the line PS in the ratio of 4:1, where S is the midpoint of QR. **(iv)** Show that the point T has coordinates $(-4, 9, 24)$. **[3]** **(v)** State the vector equation of the drill line. **[1]** **(vi)** If the computer program continues drilling through the cuboid along the same line as in part (v), determine the side of the cuboid that the drill exits from. Justify your answer. **[4]**

– End Of Paper –

ANNEX B

SAJC H2 Math JC2 Preliminary Examination Paper 1

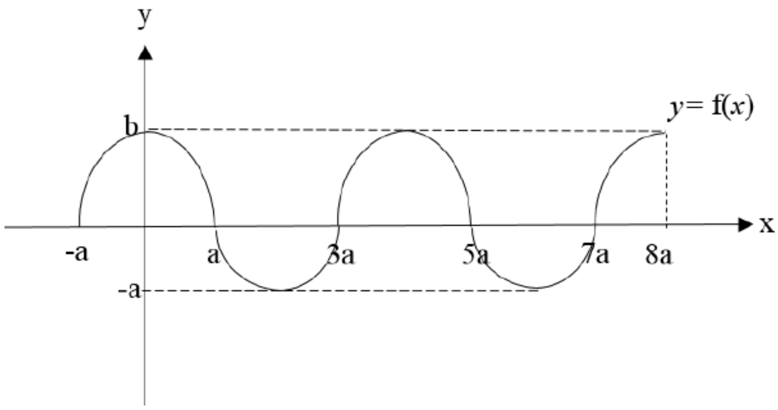
QN	Topic Set	Answers
1	Differentiation & Applications	$2\lambda\left(\frac{\pi}{15}\right)^{\frac{1}{3}}\ \text{cm}^2/\text{s}$
2	Maclaurin series	-
3	Functions	$\frac{\pi}{4}ab$
4	Graphs and Transformation	i) 1. Scale by a factor of $\frac{1}{\sqrt{a}}$ parallel to the x-axis, 2. Translate the resulting curve by b units in the negative y-direction.
5	Vectors	ii) $\frac{\sqrt{3}}{4}$ units^2 iii) $\frac{\sqrt{3}}{12}$ units^3
6	Application of Integration	a) i) $\left(\frac{n^2}{1+n^2}\right)\left[e^x\left(\frac{\sin nx}{n}\right)+\frac{1}{n}\left(\frac{e^x\cos nx}{n}\right)\right]+c$ ii) $\left(\frac{1}{1+n^2}\right)\left(e^{2\pi}+e^{\pi}\right)$ b) $\frac{5\pi}{288}$ units^3
7	Complex numbers	ii) $z=2e^{i\left(-\frac{\pi}{3}\right)}$ iii) $w=\frac{1}{2}-\frac{\sqrt{3}}{2}i$ or $w=\frac{1}{2}+\frac{\sqrt{3}}{2}i$
8	AP and GP	i) $n=24$ ii) $\left(3^n-1\right)+58n$, 5614 m away from the coach once he finishes 8 km.
9	Application of Integration	i) $y=2x+\frac{3}{2}$ ii) Q (0.421, 2.34) iii) 2.77 units^2

10	Differential Equations	iii) $x = 40e^{-0.981e^{-0.116t}}$ iv) 40
11	Vectors	ii) $3x + 6y + 2z = 90$ iii) $\theta = 73.4^0$ iv) $\mathbf{r} = \begin{pmatrix} -4 \\ 9 \\ 24 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix},\ \lambda \in \mathbb{R}$. v) The drill line will not exit from the side *GCBF*.

H2 Mathematics 2017 Prelim Exam Paper 1 Solution

1	$V=\frac{4}{3}\pi r^3$ $\frac{\mathrm{d}V}{\mathrm{d}t}=4\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}$ When V=20, $20=\frac{4}{3}\pi r^3$ $r=\left(\frac{15}{\pi}\right)^{\frac{1}{3}}$ When $r=\left(\frac{15}{\pi}\right)^{\frac{1}{3}}$, $\frac{\mathrm{d}V}{\mathrm{d}t}=\lambda$. $\lambda=4\pi\left(\frac{15}{\pi}\right)^{\frac{2}{3}}\frac{\mathrm{d}r}{\mathrm{d}t}$ $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\lambda}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}$ Surface Area, $A=4\pi r^2$ $\frac{\mathrm{d}A}{\mathrm{d}t}=8\pi r\frac{\mathrm{d}r}{\mathrm{d}t}$ When $r=\left(\frac{15}{\pi}\right)^{\frac{1}{3}}$, $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\lambda}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}$. $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$ $=8\pi\left(\frac{15}{\pi}\right)^{\frac{1}{3}}\frac{\lambda}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}$ $=2\lambda\left(\frac{\pi}{15}\right)^{\frac{1}{3}}\ \mathrm{cm}^2/\mathrm{s}$

2	(i) $BC = BD + DC$ $= \dfrac{h}{\tan\frac{\pi}{3}} + \dfrac{h}{\tan\left(\frac{\pi}{4}+x\right)}$ (ii) BC $= \dfrac{h}{\sqrt{3}} + \dfrac{h}{\dfrac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}}$ $= \dfrac{h\sqrt{3}}{3} + \dfrac{h(1-\tan x)}{1+\tan x}$ $\approx \dfrac{h\sqrt{3}}{3} + \dfrac{h(1-x)}{1+x}$ $= \dfrac{h\sqrt{3}}{3} + h(1-x)(1+x)^{-1}$ $= \dfrac{h\sqrt{3}}{3} + h(1-x)[1+(-1)x+\dfrac{(-1)(-2)}{2!})x^2+...]$ $= \dfrac{h\sqrt{3}}{3} + h(1-x)[1-x+x^2+...]$ $= \dfrac{h\sqrt{3}}{3} + h(1-2x+2x^2+...)$ $= h\left(1+\dfrac{\sqrt{3}}{3} - 2x + 2x^2\right)$

3	(i)  (ii) $\int_{3a}^{4a} \mathrm{f}(x)\, \mathrm{d}x$ $= \int_{-a}^{0} b\sqrt{1-\frac{x^2}{a^2}}\, \mathrm{d}x$ $= b\int_{\pi}^{\frac{\pi}{2}} \sqrt{1-\frac{a^2\cos^2\theta}{a^2}}\, (-a\sin\theta)\, \mathrm{d}\theta$ $= ab\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta\, \mathrm{d}\theta$ $= ab\int_{\frac{\pi}{2}}^{\pi} \frac{1-\cos 2\theta}{2}\, \mathrm{d}\theta$ $= \frac{ab}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{\pi}$ $= \frac{ab}{2}\left[\pi - \frac{\pi}{2}\right]$ $= \frac{\pi}{4}ab$

4 (i)

$y = e^{ax^2} - b = e^{(\sqrt{a}x)^2} - b$

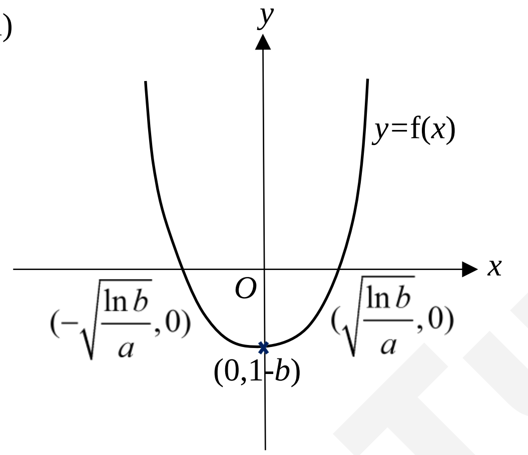
If $f(x) = e^{x^2}$, then $f(\sqrt{a}x) = e^{(\sqrt{a}x)^2}$ and so

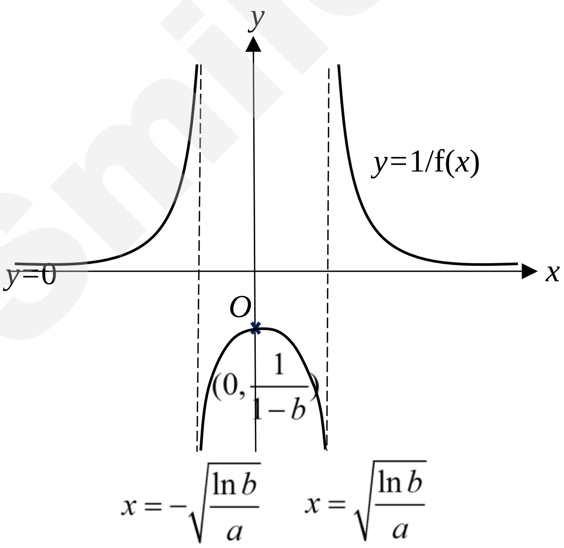
$y = f(x) \to y = f(\sqrt{a}x) \to y = f(\sqrt{a}x) + b$

Hence the sequence of transformations are:

1. Scale by a factor of $\frac{1}{\sqrt{a}}$ parallel to the x-axis,
2. Translate the resulting curve by b units in the negative y-direction.

(ii)





5

(i)

Since $\mathbf{u}+\mathbf{v}-\mathbf{w}$ is perpendicular to $\mathbf{u}-\mathbf{v}+\mathbf{w}$,

$$(\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot(\mathbf{u}-\mathbf{v}+\mathbf{w})=0$$

$$\mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$$
$$+\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}$$
$$-\mathbf{w}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{v}-\mathbf{w}\cdot\mathbf{w}=0$$

Since $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2, \mathbf{v}\cdot\mathbf{v}=|\mathbf{v}|^2, \mathbf{w}\cdot\mathbf{w}=|\mathbf{w}|^2$, and $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}, \mathbf{u}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{u}, \mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v}$,

$$|\mathbf{u}|^2-|\mathbf{v}|^2-|\mathbf{w}|^2+2\mathbf{v}\cdot\mathbf{w}=0$$

Since $\mathbf{u},\mathbf{v},\mathbf{w}$ are unit vectors, $|\mathbf{u}|=1, |\mathbf{v}|=1, |\mathbf{w}|=1$,

$$1-1-1+2\mathbf{v}\cdot\mathbf{w}=0$$

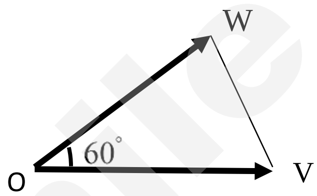
$$\mathbf{v}\cdot\mathbf{w}=\frac{1}{2}$$

$$|\mathbf{v}||\mathbf{w}|\cos\theta=\frac{1}{2}$$

$$\cos\theta=\frac{1}{2}$$

Hence, $\theta=60^\circ$

(ii)



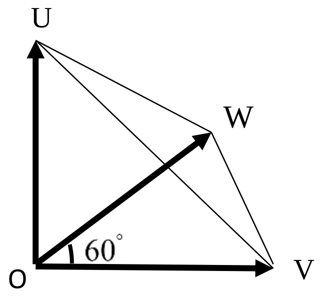
Area of $\triangle$OVW

$$=\left(\frac{1}{2}(OV)(OW)\sin 60^\circ\right)$$

$$=\left(\frac{1}{2}\right)(1)(1)\left(\frac{\sqrt{3}}{2}\right)$$

$$=\frac{\sqrt{3}}{4}\text{ units}^2$$

(iii)



Since **u** and $\mathbf{v}\times\mathbf{w}$ are parallel, we have $OU \perp OV, OU \perp OW$.

Volume of $OUVW$

$=\frac{1}{3}(\text{Area of } \triangle\text{OVW})(OU)$

$=\frac{1}{3}\left(\frac{\sqrt{3}}{4}\right)(1)$

$=\frac{\sqrt{3}}{12}$ units^3

6

(a)

(i) Using integration by parts,

$\int e^x \cos nx \, dx$

$u = e^x$	$\dfrac{dv}{dx} = \cos nx$
$\dfrac{du}{dx} = e^x$	$v = \dfrac{\sin nx}{n}$

$$= e^x\left(\frac{\sin nx}{n}\right) - \int \frac{e^x}{n}\sin nx \, dx$$

$u = e^x$	$\dfrac{dv}{dx} = \sin nx$
$\dfrac{du}{dx} = e^x$	$v = -\dfrac{\cos nx}{n}$

$$= e^x\left(\frac{\sin nx}{n}\right) - \frac{1}{n}\left[-\frac{e^x \cos nx}{n} + \int \frac{e^x \cos nx}{n} dx\right]$$

$$= e^x\left(\frac{\sin nx}{n}\right) + \frac{1}{n}\left(\frac{e^x \cos nx}{n}\right) - \frac{1}{n^2}\int e^x \cos nx \, dx$$

Rearranging,

$$\left(1+\frac{1}{n^2}\right)\int e^x \cos nx \, dx = e^x\left(\frac{\sin nx}{n}\right) + \frac{1}{n}\left(\frac{e^x \cos nx}{n}\right)$$

$$\int e^x \cos nx \, dx = \left(\frac{n^2}{1+n^2}\right)\left[e^x\left(\frac{\sin nx}{n}\right) + \frac{1}{n}\left(\frac{e^x \cos nx}{n}\right)\right] + c$$ where c is a constant

(ii) $$\int_\pi^{2\pi} e^x \cos nx \, dx = \left(\frac{n^2}{1+n^2}\right)\left[e^x\left(\frac{\sin nx}{n}\right) + \frac{1}{n}\left(\frac{e^x \cos nx}{n}\right)\right]_\pi^{2\pi}$$

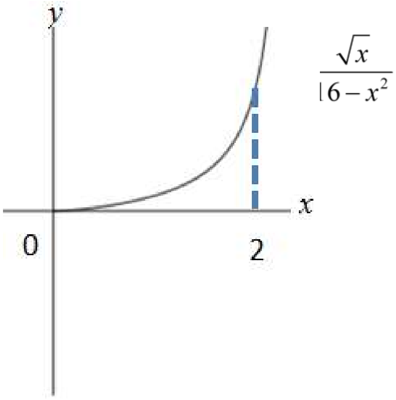
$$= \left(\frac{n^2}{1+n^2}\right)\left\{e^{2\pi}\left[\left(\frac{\sin 2n\pi}{n}\right) + \frac{1}{n}\left(\frac{\cos 2n\pi}{n}\right)\right] - e^{\pi}\left[\left(\frac{\sin n\pi}{n}\right) + \frac{1}{n}\left(\frac{\cos n\pi}{n}\right)\right]\right\}$$

For any positive integer n, $\sin 2n\pi = 0$ and $\cos 2n\pi = 1$

If n is odd, $\sin n\pi = 0$ and $\cos n\pi = -1$

$$\int_\pi^{2\pi} e^x \cos nx \, dx = \left(\frac{n^2}{1+n^2}\right)\left[e^{2\pi}\left(0+\frac{1}{n^2}\right) - e^{\pi}\left(0-\frac{1}{n^2}\right)\right] = \left(\frac{1}{1+n^2}\right)\left(e^{2\pi}+e^{\pi}\right)$$ (Ans)

(b)



$$y=\frac{\sqrt{x}}{16-x^2} \Rightarrow y^2=\frac{x}{\left(16-x^2\right)^2}$$

Hence volume required

$$=\pi r^2 h-\pi\int_0^2 y^2\,\mathrm{d}x$$

$$=\pi\left(\frac{\sqrt{2}}{12}\right)^2(2)-\pi\int_0^2\frac{x}{\left(16-x^2\right)^2}\,\mathrm{d}x$$

$$=\pi\left(\frac{\sqrt{2}}{12}\right)^2(2)-\frac{\pi}{(-2)}\int_0^2\frac{-2x}{\left(16-x^2\right)^2}\,\mathrm{d}x$$

$$=\pi\left(\frac{\sqrt{2}}{12}\right)^2(2)+\frac{\pi}{2}\left[\frac{\left(16-x^2\right)^{-1}}{-1}\right]_0^2$$

$$=\frac{4}{144}\pi+\frac{\pi}{2}\left[-\frac{1}{12}+\frac{1}{16}\right]$$

$$=\frac{5\pi}{288}\text{ units}^3$$

7	(i) $\frac{re^{i\theta}}{re^{i\theta}-r}$ $=\frac{e^{i\theta}}{e^{i\left(\frac{\theta}{2}\right)}\left(e^{i\left(\frac{\theta}{2}\right)}-e^{-i\left(\frac{\theta}{2}\right)}\right)}$ $=\frac{e^{i\left(\frac{\theta}{2}\right)}}{2i\sin\left(\frac{\theta}{2}\right)}$ $=\frac{\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)}{2i\sin\left(\frac{\theta}{2}\right)}$ $=\frac{1}{2}+\frac{1}{2i}\cot\left(\frac{\theta}{2}\right)$ $=\frac{1}{2}-\frac{1}{2}\left(\cot\left(\frac{\theta}{2}\right)\right)i$ (ii) $z=2e^{i\left(-\frac{\pi}{3}\right)}$ (iii) $\frac{4w^2}{(w-1)^2}-\frac{4w}{w-1}+4=0$ $\left(\frac{2w}{w-1}\right)^2-2\left(\frac{2w}{w-1}\right)+4=0$ Let $z=\frac{2w}{w-1}$, then $z^2-2z+4=0$ From (ii) the solutions are $z=2e^{i\left(\frac{\pi}{3}\right)}$ or $z=2e^{-i\left(\frac{\pi}{3}\right)}$ Since $z=\frac{2w}{w-1}$ $zw-z=2w$ $w(z-2)=z$ $w=\frac{z}{z-2}$ Part (i) result can be used as $z=2e^{i\left(\frac{\pi}{3}\right)}$, where $r=2$ with $\theta=\frac{\pi}{3},\ \theta=-\frac{\pi}{3}$. $w=\frac{1}{2}-\frac{1}{2}\left(\cot\frac{\pi}{6}\right)i$ or $w=\frac{1}{2}-\frac{1}{2}i\cot\left(-\frac{\pi}{6}\right)$

$w=\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$ or $w=\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$

8	**(i)** Distance travelled per lap is in AP: $a = 2(30) = 60$, $d = 2 \times 3 = 6$. Given total distance travelled > 3000 $\frac{n}{2}[\,2(60) + (n-1)6] > 3000$ $3n^2 + 57n - 3000 > 0$ $(n + 42.52)(n - 23.52) > 0$ $n < -42.52$ or $n > 23.52$ Since $n \in \mathbb{Z}^+$, least $n = 24$ **(ii)** Distance of the coach from S just before the runner completes the rth lap $=30+2(3^0)+2(3^1)+2(3^2)+...+2(3^{r-2})$ $=30+2\left(1+3+3^2+....+3^{r-2}\right)$ $=30+2\left(\frac{3^{r-1}-1}{3-1}\right)$ $=30+(3^{r-1}-1)$ $=3^{r-1}+29$ Distance covered by the athlete after n laps $=\sum_{r=1}^{n}2\left(3^{r-1}+29\right)$ $=2\sum_{r=1}^{n}3^{r-1}+\sum_{r=1}^{n}(58)$ $=2\sum_{r=1}^{n}3^{r-1}+58n$ $=2\left(\frac{3^n-1}{3-1}\right)+58n$ $=\left(3^n-1\right)+58n$ When $D = 8000$m $8000=\left(3^n-1\right)+58n$ From GC, $n=8.1254$ Hence the athlete has run 8 complete laps. The athlete has completed 7024 m Hence he still have 8000-7024=976 m On the 9th lap, the coach is $3^{9-1}+29=6590$ m from S. Hence the athlete would be 6590- 976 = 5614 m away from the coach once he finishes 8 km.

9

(i)

$$\frac{dx}{d\theta} = 2\cos\theta \ , \quad \frac{dy}{d\theta} = -\sqrt{3}\sin\theta$$

$$\frac{dy}{dx} = \frac{-\sqrt{3}\sin\theta}{2\cos\theta} = -\frac{\sqrt{3}}{2}\tan\theta$$

When $\theta = \frac{\pi}{6}$, $x = 2$, $y = \frac{11}{2}$, $\frac{dy}{dx} = -\frac{1}{2}$

Equation of normal : $y - \left(\frac{11}{2}\right) = 2(x-2)$

$$y = 2x + \frac{3}{2}$$

(ii)

$x = 1 + 2\sin\theta \cdots\cdots (1)$

$y = 4 + \sqrt{3}\cos\theta \cdots\cdots (2)$

Substitute equation (1) and (2) into $y = 2x + \frac{3}{2}$

$4 + \sqrt{3}\cos\theta = 2(1 + 2\sin\theta) + \frac{3}{2}$

$\frac{1}{2} + \sqrt{3}\cos\theta = 4\sin\theta$

$8\sin\theta - 2\sqrt{3}\cos\theta = 1$

At Point Q, $\theta = \alpha$

$8\sin\alpha - 2\sqrt{3}\cos\alpha = 1$ (shown)

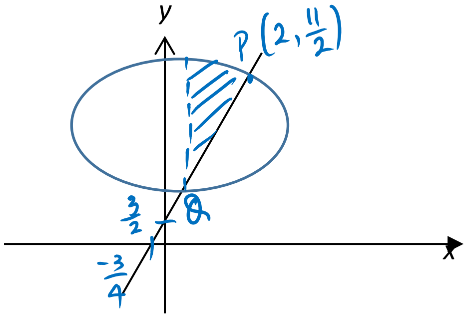
Using GC: $\alpha = -2.847916$ or $\alpha = 0.52359$ (Reject, same as $\frac{\pi}{6}$,point P)

Hence, using GC

coordinates of Q (0.42105, 2.3421)

Q (0.421, 2.34)

(iii)



	when $x = 0.42105$ $0.42105 = 1 + 2\sin\theta$ $\sin\theta = -0.289475$ $\theta = -0.29368$ or -2.8479 (at point Q) $= \int_{0.42105}^{2} y_1 \ \mathrm{d}x - \int_{0.42105}^{2} y_2 \ \mathrm{d}x$ Required Area $= \int_{-0.29368}^{\frac{\pi}{6}} \left(4 + \sqrt{3}\cos\theta\right) \ \left(2\cos\theta\right)\mathrm{d}\theta - \int_{0.42105}^{2} \left(2x + \frac{3}{2}\right) \mathrm{d}x$ $= 8.9613 - 6.1911$ $= 2.7702 \approx 2.77$ units^2 (3 s.f.)

10	(i) $\frac{dx}{dt} = cx\ln\left(\frac{40}{x}\right)$ $u = \ln\left(\frac{40}{x}\right)$ $= \ln(40) - \ln(x)$ $\frac{du}{dx} = -\frac{1}{x}$ $\frac{du}{dt} = \frac{du}{dx} \times \frac{dx}{dt}$ $= \left(-\frac{1}{x}\right)cx\ln\left(\frac{40}{x}\right)$ $= -cu$ (ii) $\frac{du}{dt} = -cu$ $\int \frac{1}{u}\,du = -\int c\ dt$ $\ln\lvert u\rvert = -ct + d$ $\lvert u\rvert = e^{-ct+d}$ $u = \pm e^{d} e^{-ct}$ $= Be^{-ct},\ B = \pm e^{d}$ Replace u with $\ln\left(\frac{40}{x}\right)$ $\ln\left(\frac{40}{x}\right) = Be^{-ct}$ $\frac{40}{x} = e^{Be^{-ct}}$ $x = \frac{40}{e^{Be^{-ct}}}$ $x = 40e^{-Be^{-ct}}$ (iii) When $t = 0,\ \ x = 15,$ $15 = 40\,e^{-B}$ $e^{-B} = \frac{3}{8}$ $B = \ln\left(\frac{8}{3}\right) = 0.98083 = 0.981$ When $t = 3,\ x = 20$

$$20 = 40e^{-Be^{-3c}}$$

$$e^{-Be^{-3c}} = \frac{1}{2}$$

$$-Be^{-3c} = \ln\frac{1}{2}$$

$$\ln\left(\frac{3}{8}\right)\left(e^{-3c}\right) = \ln\left(\frac{1}{2}\right)$$

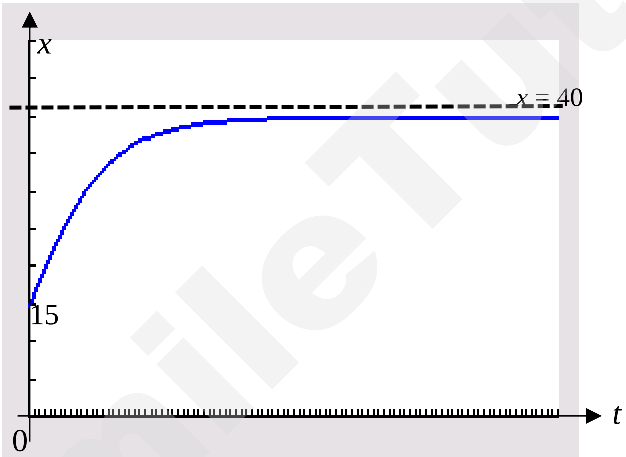
$$c = -\frac{1}{3}\ln\left(\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{3}{8}\right)}\right) = 0.11572 = 0.116$$

$$x = 40e^{-0.981e^{-0.116t}}$$

(iv)

The population of foxes in the long run is 40.

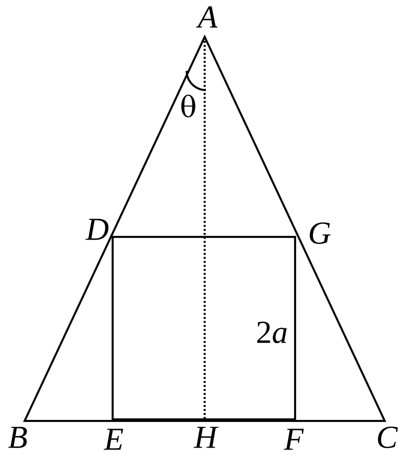
(v)



11	(i) $\overrightarrow{OP}=\begin{pmatrix}0\\5\\30\end{pmatrix}\quad \overrightarrow{OQ}=\begin{pmatrix}0\\10\\15\end{pmatrix}\quad \overrightarrow{OR}=\begin{pmatrix}-10\\10\\30\end{pmatrix}$ $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}0\\10\\15\end{pmatrix}-\begin{pmatrix}0\\5\\30\end{pmatrix}=\begin{pmatrix}0\\5\\-15\end{pmatrix}\qquad \overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}=\begin{pmatrix}-10\\10\\30\end{pmatrix}-\begin{pmatrix}0\\5\\30\end{pmatrix}=\begin{pmatrix}-10\\5\\0\end{pmatrix}$ (ii) A normal to p $\begin{pmatrix}0\\1\\-3\end{pmatrix}\times\begin{pmatrix}-2\\1\\0\end{pmatrix}=\begin{pmatrix}3\\6\\2\end{pmatrix}$ Equation of plane $\mathbf{r}\cdot\begin{pmatrix}3\\6\\2\end{pmatrix}=\begin{pmatrix}0\\10\\15\end{pmatrix}\cdot\begin{pmatrix}3\\6\\2\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}3\\6\\2\end{pmatrix}=90$ $3x+6y+2z=90$ Or any equivalent equation of plane (iii) A normal to the plane *EFGH* = $\begin{pmatrix}0\\0\\1\end{pmatrix}$ (or any equivalent vector) $\cos\theta=\dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\bullet\begin{pmatrix}3\\6\\2\end{pmatrix}\right\|}{1\times\sqrt{9+36+4}}=\dfrac{\|2\|}{\sqrt{49}}$ $\theta=73.4^0$ (iv) $\overrightarrow{OS}=\frac{1}{2}\left[\overrightarrow{OQ}+\overrightarrow{OR}\right]=\frac{1}{2}\left[\begin{pmatrix}0\\10\\15\end{pmatrix}+\begin{pmatrix}-10\\10\\30\end{pmatrix}\right]=\begin{pmatrix}-5\\10\\22\frac{1}{2}\end{pmatrix}$

H2 Mathematics 2017 Prelim Exam Paper 1 Question
Answer all questions [100 marks].

1	Without the use of a calculator, find the complex numbers z and w which satisfy the simultaneous equations $$z - w\mathrm{i} = 3$$ $$z^2 - w + 6 + 3\mathrm{i} = 0$$ **[6]**
2	The function f is defined by $\mathrm{f} : x \mapsto \dfrac{1}{x^2-1},\quad x \in \mathbb{R},\ x > 1.$ **(i)** Show that $\dfrac{2}{n-1} - \dfrac{3}{n} + \dfrac{1}{n+1} = \dfrac{An+B}{n^3-n}$, where A and B are constants to be found. [3] **(ii)** Hence find $\displaystyle\sum_{r=2}^{n} \frac{2r+6}{r^3-r}$. [4] **(iii)** Use your answer to part **(ii)** to find $\displaystyle\sum_{r=2}^{n} \frac{2r+10}{(r+1)(r+2)(r+3)}$. [1]
3	The function f is defined by $\mathrm{f} : x \mapsto \dfrac{1}{x^2-1},\quad x \in \mathbb{R},\ x > 1.$ **(i)** Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. [3] **(ii)** On the same diagram, sketch the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$ stating the equations of any asymptotes and showing the relationships between the graphs clearly. [4] **(iii)** State the set of values of x such that $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$. [1]
4	Referred to the origin O, the point A has position vector $-5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and the point B has position vector $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. The plane π has equation: $$\mathbf{r} = (1+\lambda-2\mu)\mathbf{i} + (3-2\lambda)\mathbf{j} + (\mu-2)\mathbf{k} \text{ where } \lambda, \mu \in \mathbb{R}$$ **(i)** Find the vector equation of plane π in scalar product form. [2] **(ii)** Find the position vector of the foot of perpendicular, C, from A to π. [3] The line l_1 passes through the points A and B. The line l_2 is the reflection of the line l_1 about the plane π. Find a vector equation of l_2. [3]

5	 It is given that $DEFG$ is a square with fixed side $2a$ cm and it is inscribed in the isosceles triangle ABC with height AH , where $AB = AC$ and angle $BAH = \theta$. **(i)** Taking $t = \tan\theta$, show that the area of the triangle ABC is given by $S = a^2\left(4+4t+\dfrac{1}{t}\right)$ [3] **(ii)** Find the minimum area of S in terms of a when t varies. [4] **(iii)** Hence sketch the graph showing the area of the triangle ABC as θ varies. [3]
6	**(a)** There are three yellow balls, three red balls and three blue balls. Balls of each colour are numbered 1, 2, and 3. Find the number of ways of arranging the balls in a row such that adjacent balls do not sum up to two. **[2]** **(b)** In a restaurant, there were two round tables available, a table for five and a table for six. Find the number of ways eleven friends can be seated if two particular friends are not seated next to each other. **[4]**
7	For the events A and B, it is given that $\mathrm{P}(A\cap B')=0.6,\quad \mathrm{P}(A\cup B')=0.83\quad \text{and}\quad \mathrm{P}(A\mid B')=0.83$ Find, (i) $\mathrm{P}(B)$ [2] (ii) $\mathrm{P}(A\cap B)$ [2] (iii) $\mathrm{P}(B\mid A')$ [2] Hence determine whether A and B are independent. [1]
8	A fairground game involves trying to hit a moving target with a gunshot. A round consists of a **maximum** of 3 shots. Ten points are scored if a player hits the target. The **round** ends **immediately** if the player misses a shot. The probability that Linda hits the target in a single shot is 0.6. All shots taken are independent of one another. **(i)** Find the probability that Linda scores 30 points in a round. [2] The random variable X is the number of points Linda scores in a round. **(ii)** Find the probability distribution of X. [3] **(iii)** Find the mean and variance of X. [4]

	(iv) A game consists of 2 rounds. Find the probability that Linda scores more points in round 2 than in round 1. [2]
9	Six cities in a certain country are linked by rail to city *O*. The rail company provides the information about the distance of each city to city *O* and the rail fare from that city to city *O* on its website. Charles copied the table below from the website, but he had copied one of the rail fares wrongly. City | *A* | *B* | *C* | *D* | *E* | *F* Distance, x km | 100 | 270 | 120 | 56 | 289 | 347 Rail fare, \$$y$ | 11.1 | 17.1 | 6.44 | 7.62 | 17.9 | 18.8 **(i)** Give a sketch of the scatter diagram for the data as shown on your calculator. On your diagram, circle the point that Charles has copied wrongly. [2] For parts **(ii)**, **(iii)** and **(iv)** of this question you should **exclude** the point for which Charles has copied the rail fare value wrongly. **(ii)** Find, correct to 4 decimal places, the product moment correlation coefficient between **(a)** $\ln x$ and y, **(b)** x^2 and y. [2] **(iii)** Using parts **(i)** and **(ii)**, explain which of the cases in part **(ii)** is more appropriate for modelling the data. [2] **(iv)** By using the equation of a suitable regression line, estimate the rail fare when the distance is 210 km. Explain if your estimate is reliable. [3]
10	A factory manufactures round tables in two sizes: small and large. The radius of a small table, measured in cm, has distribution $\mathrm{N}(30, 2^2)$ and the radius of a large table, measured in cm, has distribution $\mathrm{N}(50, 5^2)$. **(i)** Find the probability that the sum of the radius of 5 randomly chosen small tables is less than 160 cm. [2] **(ii)** Find the probability that the sum of the radius of 3 randomly chosen small tables is less than twice the radius of a randomly chosen large table. [2] **(iii)** State an assumption needed in your calculation in part **(ii)**. [1] A shipment of 12 large tables is to be exported. Before shipping, a check is done and the shipment will be rejected if there are at least two tables whose radius is less than 40 cm. **(iv)** Find the probability that the shipment is rejected. [3] The factory decides now to manufacture medium sized tables. The radius of a medium sized table, measured in cm, has distribution $\mathrm{N}(\mu, \sigma^2)$. It is known that 20% of the medium sized tables have radius greater than 44 cm and 30% have radius of less than 40 cm. **(v)** Find the values of μ and σ. [4]
11	The Kola Company receives a number of complaints that the volume of cola in their cans are less than the stated amount of 500 ml. A statistician decides to sample 50 cola cans to investigate the complaints. He measures the volume of cola, x ml, in each can and summarised the results as follows: $$\sum x = 24730,\ \sum x^2 = 12242631.$$

	(i) Find unbiased estimates of the population mean and variance correct to 2 decimal places and carry out the test at the 1% level of significance. [6] **(ii)** One director in the company points out that the company should test whether the volume of cola in a can is 500 ml at the 1% significance level instead. Using the result of the test conducted in **(i)**, explain how the *p*-value of this test can be obtained from *p*-value in part **(i)** and state the corresponding conclusion. [2] The head statistician agrees the company should test that the volume of cola in a can is 500 ml at the 1% level of significance. He intends to make a simple rule of reference for the production managers so that they will not need to keep coming back to him to conduct hypothesis tests. On his instruction sheet, he lists the following: 1. Collect a random sample of 40 cola cans and measure their volume. 2. Calculate the mean of your sample, $\bar{x}$ and the variance of your sample, $s_x^{\ 2}$. 3. Conclude that the volume of cola differs from 500 ml if the value of $\bar{x}$ lies..... **(iii)** Using the above information, complete the decision rule in step 3 in terms of s_x. [4] A party organiser has *n* cans of cola and 2*n* packets of grape juice. Assume now that the volume of a can of cola has mean 500 ml and variance 144 ml^2, and the volume of a packet of grape juice has mean 250 ml and variance 25 ml^2. She mixes all the cola and grape juice into a mocktail, which she pours into a 120-litre barrel. Assume that *n* is sufficiently large and that the volumes of the cans of cola and packets of grape juice are independent. **(iv)** Show that if the party organiser wants to be at least 95% sure that the barrel will not overflow, *n* must satisfy the inequality $1000n + 22.9\sqrt{n} - 120,000 \le 0$. [4]

− End Of Paper −

ANNEX B

SAJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Complex numbers	$z = 2\mathrm{i}$ or $z = -3\mathrm{i}$ $w = 2+3i$ or $w = -3+3\mathrm{i}$
2	Sigma Notation and Method of Difference	i) $\dfrac{n+3}{n^3-n}$ ii) $3-\dfrac{4}{n}+\dfrac{2}{n+1}$ iii) $\dfrac{5}{6}-\dfrac{4}{n+2}+\dfrac{2}{n+3}$
3	Functions	i) $\mathrm{f}^{-1}(x)=\sqrt{1+\dfrac{1}{x}}$; $\mathrm{D}_{\mathrm{f}^{-1}}(x)=(0,\infty)$ iii) $x>1$
4	Vectors	i) $\mathbf{r}\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}=-3$ ii) $\dfrac{1}{7}\begin{pmatrix}-37\\13\\10\end{pmatrix}$; $l_2:\ \mathbf{r}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+t\begin{pmatrix}-46\\-9\\20\end{pmatrix}$, $t\in\mathbb{R}$
5	Differentiation & Applications	ii) $8a^2$
6	P&C, Probability	a) 151200 b) 1064448
7	P&C, Probability	i) 0.277 ii) 0.107 iii) 0.580; Events A & B are not independent.
8	DRV	i) 0.216 iii) 11.76, 137.7024 iv) 0.358
9	Correlation & Linear Regression	iia) $r=0.9996$ iib) $r=0.9514$ iv) 15.73
10	Normal Distribution	i) 0.987 ii) 0.828 iv) 0.0294 v) $\sigma\approx 2.93; \mu\approx 41.5$
11	Hypothesis Testing	i) $\bar{x}=494.60; s^2=228.02$ ii) p-value $=0.00572\le 0.01$, reject H_0.

H2 Mathematics 2017 Prelim Exam Paper 2 Solution

1

Method 1

$z - w\text{i} = 3$

$\Rightarrow \quad w = \dfrac{z-3}{\text{i}} = 3\text{i} - z\text{i}$ --- (1)

Substitute (1) into $z^2 - w + 6 + 3\text{i} = 0$

$z^2 - (3\text{i} - z\text{i}) + 6 + 3\text{i} = 0$

$\Rightarrow \; z^2 + z\text{i} + 6 = 0$

$\Rightarrow \; z = \dfrac{-\text{i} \pm \sqrt{(\text{i})^2 - 4(1)(6)}}{2} = \dfrac{-\text{i} \pm \sqrt{-1-24}}{2}$

$= \dfrac{-\text{i} \pm 5\text{i}}{2}$

$\therefore \; z = 2\text{i}$ or $z = -3\text{i}$

$\Rightarrow w = 3\text{i} - (2\text{i})\text{i}$ $\qquad w = 3\text{i} - (-3\text{i})\text{i}$

$= 2 + 3i$ $\qquad = -3 + 3\text{i}$

Method 2

$z - w\text{i} = 3$

$\Rightarrow \quad z = 3 + w\text{i}$ …. (1)

Substitute (1) into $z^2 - w + 6 + 3\text{i} = 0$

$(3 + w\text{i})^2 - w + 6 + 3\text{i} = 0$

$\Rightarrow \quad 9 + 6w\text{i} - w^2 - w + 6 + 3\text{i} = 0$

$\Rightarrow \quad -w^2 - (1-6\text{i})w + 15 + 3\text{i} = 0$

$\Rightarrow \quad -w^2 - (1-6\text{i})w + 15 + 3\text{i} = 0$

$\Rightarrow \quad w^2 + (1-6\text{i})w - 15 - 3\text{i} = 0$

$\Rightarrow \quad w = \dfrac{-(1-6\text{i}) \pm \sqrt{(1-6\text{i})^2 - 4(1)(-15-3\text{i})}}{2}$

$= \dfrac{-1 + 6\text{i} \pm \sqrt{1 - 12\text{i} - 36 + 60 + 12\text{i}}}{2}$

$= \dfrac{-1 + 6\text{i} \pm \sqrt{25}}{2}$

$\therefore \; w = 2 + 3\text{i}$ or $w = -3 + 3\text{i}$

$\Rightarrow \quad z = 3 + (2+3\text{i})\text{i} = 2\text{i}$ $\qquad z = 3 + (-3+3\text{i})\text{i} = -3\text{i}$

Method 3

$w\text{i} = z - 3$

$\Rightarrow w = -\text{i}z + 3\text{i}$

$\therefore \quad z^2 - (-\text{i}z + 3\text{i}) + 6 + 3\text{i} = 0$

$\Rightarrow \quad z^2 + \text{i}z + 6 = 0$

Let $z = a + b\text{i}$ where $a, b \in \mapsto$

$(a + b\text{i})^2 + \text{i}(a + b\text{i}) + 6 = 0$

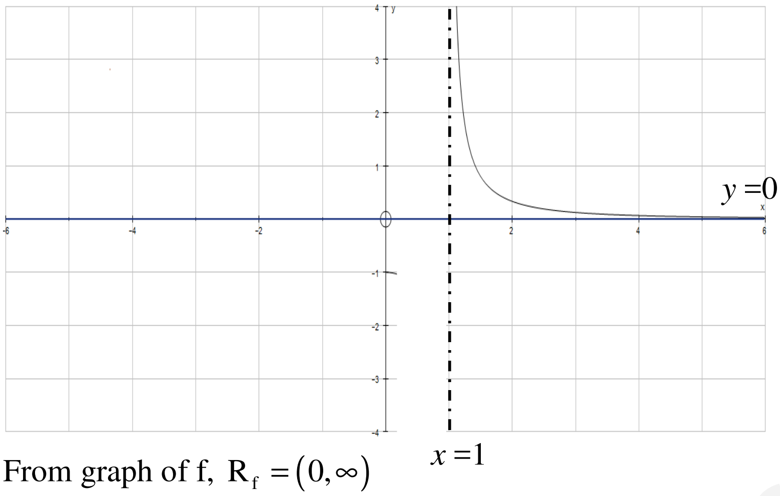
$\Rightarrow \; a^2 - b^2 + 2ab\text{i} + a\text{i} - b + 6 = 0$

$\Rightarrow \; a^2 - b^2 - b + 6 + (2ab + a)\text{i} = 0$

By comparing the real and imaginary parts,

	$a^2-b^2-b+6=0 \quad \ldots (1)$ $2ab+a=0 \quad \ldots (2)$ From (2), $a=0$ or $b=-\frac{1}{2}$ When $a=0$, $b^2+b-6=0$ $(b-2)(b+3)=0$ $b=2$ or $b=-3$ Hence $z=2\text{i}, w=-\text{i}(2\text{i})+3\text{i}=2+3\text{i}$ or $z=-3\text{i}, w=-\text{i}(-3\text{i})+3\text{i}=-3+3\text{i}$ When $b=-\frac{1}{2}$, $a^2=\frac{1}{4}-\frac{1}{2}-6=-\frac{25}{4}$ There is no real solution for a.
2	(i) $\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}$ $=\frac{2(n)(n+1)-3(n-1)(n+1)+(n-1)(n)}{(n-1)(n)(n+1)}$ $=\frac{(2n^2+2n)-(3n^2-3)+(n^2-n)}{n^3-n}$ $=\frac{n+3}{n^3-n}$ (ii) $\sum_{r=2}^{n}\frac{2r+6}{r^3-r}$ $=2\sum_{r=2}^{n}\frac{r+3}{r^3-r}$ $=2\sum_{r=2}^{n}\left(\frac{2}{r-1}-\frac{3}{r}+\frac{1}{r+1}\right)$ $=2\left[\begin{array}{l}\frac{2}{1}-\frac{3}{2}+\cancel{\frac{1}{3}}\\+\cancel{\frac{2}{2}}-\cancel{\frac{3}{3}}+\cancel{\frac{1}{4}}\\+\cancel{\frac{2}{3}}-\cancel{\frac{3}{4}}+\cancel{\frac{1}{5}}\\+\ldots\\+\cancel{\frac{2}{n-2}}-\cancel{\frac{3}{n-1}}+\frac{1}{n}\\+\cancel{\frac{2}{n-1}}-\frac{3}{n}+\frac{1}{n+1}\end{array}\right]$

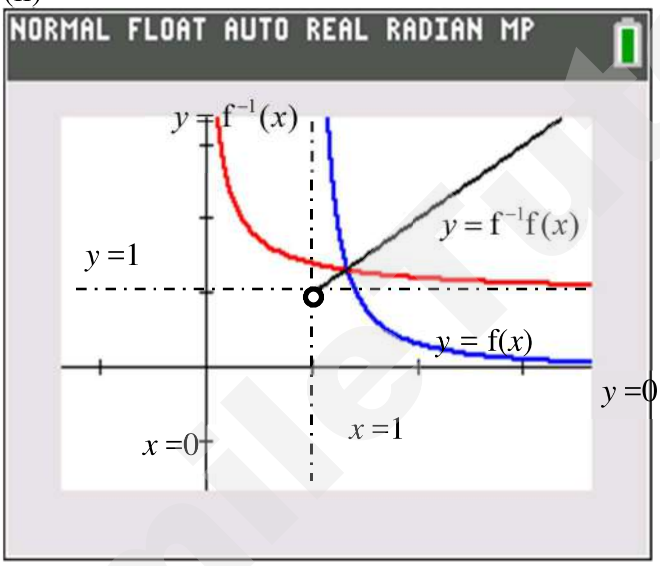
	$=2\left(\frac{2}{1}-\frac{3}{2}+\frac{2}{2}+\frac{1}{n}-\frac{3}{n}+\frac{1}{n+1}\right)$ $=2\left(\frac{3}{2}-\frac{2}{n}+\frac{1}{n+1}\right)$ $=3-\frac{4}{n}+\frac{2}{n+1}$ (iii) $\sum_{r=2}^{n}\frac{2r+10}{(r+1)(r+2)(r+3)}$ Let $r+2=p \Rightarrow r=p-2$ $=\sum_{p-2=2}^{p-2=n}\frac{2p+6}{(p-1)(p)(p+1)}$ $=\sum_{p=4}^{n+2}\frac{2p+6}{p^3-p}$ $=\sum_{p=2}^{n+2}\frac{2p+6}{p^3-p}-\sum_{p=2}^{3}\frac{2p+6}{p^3-p}$ $=\left(3-\frac{4}{n+2}+\frac{2}{n+3}\right)-\left(3-\frac{4}{3}+\frac{2}{4}\right)$ $=\frac{5}{6}-\frac{4}{n+2}+\frac{2}{n+3}$
3	(i) $\mathrm{f}: x \mapsto \frac{1}{x^2-1}$ Let $y=\frac{1}{x^2-1}$ $x^2=\frac{1}{y}+1$ $x=\pm\sqrt{1+\frac{1}{y}}$ Since $x>1$, $x=\sqrt{1+\frac{1}{y}}$ $\mathrm{f}^{-1}(x)=\sqrt{1+\frac{1}{x}}=\sqrt{\frac{1+x}{x}}$.



From graph of f, $R_f = (0, \infty)$

$\therefore D_{f^{-1}}(x) = (0, \infty)$.

(ii)



(iii)

Since $ff^{-1}(x) = f^{-1}f(x) = x$ have the same rule, we investigate the domain

$D_{f^{-1}f} = (1, \infty)$ $D_{ff^{-1}} = (0, \infty)$

Taking the intersection of these domains,

Range of values is $x > 1$.

4

(i)
Equation of plane is

$$\mathbf{r}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\0\\1\end{pmatrix},\ \lambda,\mu\in\mapsto$$

A normal vector to plane is

$$\begin{pmatrix}1\\-2\\0\end{pmatrix}\times\begin{pmatrix}-2\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\-1\\-4\end{pmatrix}$$

Hence vector equation of the plane is

$$\mathbf{r}\mapsto\begin{pmatrix}2\\1\\4\end{pmatrix}=\begin{pmatrix}1\\3\\-2\end{pmatrix}\mapsto\begin{pmatrix}2\\1\\4\end{pmatrix}$$

$$\mathbf{r}\mapsto\begin{pmatrix}2\\1\\4\end{pmatrix}=-3$$

(ii)

$$l_{AC}:\quad \mathbf{r}=\begin{pmatrix}-5\\2\\2\end{pmatrix}+s\begin{pmatrix}2\\1\\4\end{pmatrix},\ s\in\mapsto$$

Thus $\overrightarrow{OC}=\begin{pmatrix}-5\\2\\2\end{pmatrix}+s\begin{pmatrix}2\\1\\4\end{pmatrix}$ for some $s\in\mapsto$.

Since C lies on the plane:

$$\left[\begin{pmatrix}-5\\2\\2\end{pmatrix}+s\begin{pmatrix}2\\1\\4\end{pmatrix}\right]\mapsto\begin{pmatrix}2\\1\\4\end{pmatrix}=-3$$

$$2(-5+2s)+(2+s)+4(2+4s)=-3$$

$$s=-\frac{3}{21}$$

Thus $\overrightarrow{OC}=\begin{pmatrix}2\left(-\frac{3}{21}\right)-5\\ \left(-\frac{3}{21}\right)+2\\ 4\left(-\frac{3}{21}\right)+2\end{pmatrix}=\frac{1}{7}\begin{pmatrix}-37\\13\\10\end{pmatrix}$

(iii)
Using mid-point theorem

$\overrightarrow{OA'} = 2\overrightarrow{OC} - \overrightarrow{OA}$

$$= \frac{2}{7}\begin{pmatrix} -37 \\ 13 \\ 10 \end{pmatrix} - \begin{pmatrix} -5 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -39 \\ 12 \\ 6 \end{pmatrix}$$

B is the point of intersection of l_1 and π.

$\overrightarrow{BA'} = \overrightarrow{OA'} - \overrightarrow{OB}$

$$= \frac{1}{7}\begin{pmatrix} -39 \\ 12 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} -46 \\ -9 \\ 20 \end{pmatrix}$$

$$l_2: \ \mathbf{r} = \frac{1}{7}\begin{pmatrix} -39 \\ 12 \\ 6 \end{pmatrix} + t\begin{pmatrix} -46 \\ -9 \\ 20 \end{pmatrix}, \ t \in \mapsto \qquad \text{or}$$

$$l_2: \ \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} -46 \\ -9 \\ 20 \end{pmatrix}, \ t \in \mapsto$$

5

(i)

The height of triangle ADG is $\dfrac{a}{\tan\theta} = \dfrac{a}{t}$.

Hence $AH = 2a + \dfrac{a}{t} = a\left(2 + \dfrac{1}{t}\right)$.

$BH = BE + EH = 2a\tan\theta + a = a(2t+1)$

Area $S = \dfrac{1}{2}(AH)(BC)$

$$S = \frac{a}{2}\left(2 + \frac{1}{t}\right)\left(2a(2t+1)\right)$$

$$S = a^2\left(2 + \frac{1}{t}\right)(2t+1)$$

$$S = a^2\left(4 + 4t + \frac{1}{t}\right)$$

(ii)

$$\frac{dS}{dt} = a^2\left(4 - \frac{1}{t^2}\right)$$

When $\dfrac{dS}{dt} = 0$,

$$t^2 = \frac{1}{4}$$

$$\Rightarrow \ t = \pm\frac{1}{2}$$

Reject $t = \tan\theta = -\dfrac{1}{2}$ as θ is acute

$$\frac{\mathrm{d}^2 S}{\mathrm{d}t^2} = a^2\left(\frac{2}{t^3}\right)$$

When $t = \dfrac{1}{2}$, $\dfrac{\mathrm{d}^2 S}{\mathrm{d}t^2} = a^3\left(\dfrac{2}{\left(\frac{1}{2}\right)^3}\right) = 16a > 0$.

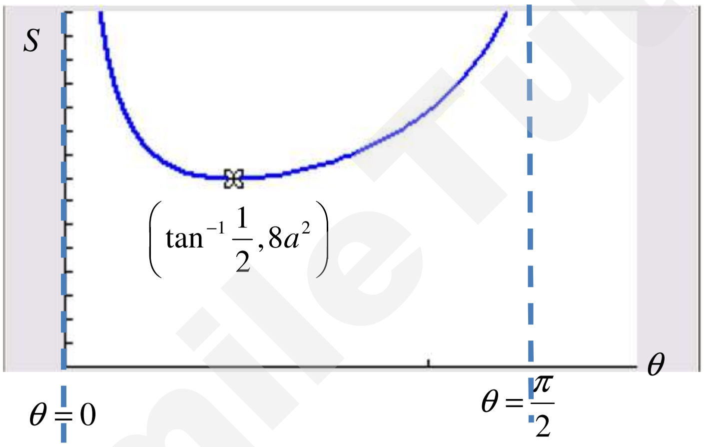
Hence the minimum value of S occurs when $t = \dfrac{1}{2}$.

Minimum $S = a^2\left(4+2+2\right) = 8a^2$.

(iii)

To sketch the graph of

$$S = a^2\left(4 + 4\tan\theta + \frac{1}{\tan\theta}\right)$$



6 (a)

Since adjacent balls do not sum up to two, balls numbered ‘1’ needs be separated.
Number of ways of arranging the other balls with no restriction = 6!
Slotting in the balls numbered ‘1’, permutation is done as balls are of different colour = ${}^{7}C_3 \times 3!$

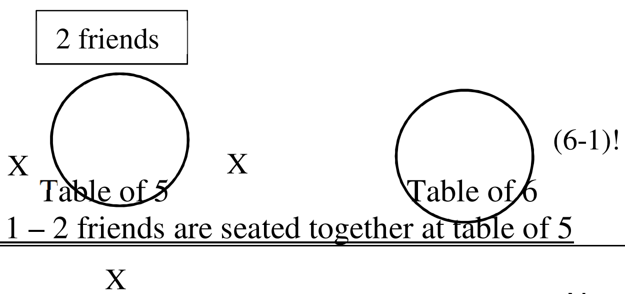
No of ways

$= 6! \times {}^{7}C_3 \times 3!$

$= 151200$

(b)

<u>Method 1</u>



<u>Case 1 – 2 friends are seated together at table of 5</u>

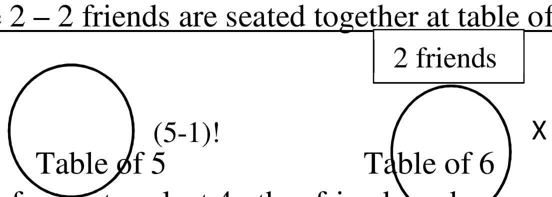
No. of ways to select 3 other friends and arrange them at the table of 5 = $^9C_3 \times (4-1)!$

No. of ways to arrange the 2 friends = 2!

No. of ways to sit the remaining friends at the table of 6
= (6-1)! = 5! = 120

Total no. of ways = $^9C_3 \times (4-1)! \times 2! \times 5! = 120960$

<u>Case 2 – 2 friends are seated together at table of 6</u>



No. of ways to select 4 other friends and arrange them at the table of 6 = $^9C_4 \times (5-1)! =$ 3024

No. of ways to sit the 2 friends at the table of 6 = 2!

No. of ways to sit the remaining friends at the table of 5
= (5-1)! = 4! = 24

Total no. of ways = $^9C_4 \times (5-1)! \times 2! \times 4! = 145152$

No of ways to arrange 11 friends without restrictions
= $^{11}C_5 \times (5-1)! \times (6-1)! = 1330560$

Total no. of ways of arranging 11 people such that 2 particular friends are not seated together
= 1330560 – 120960 – 145152 = 1064448

<u>Method 2</u>

<u>Alternative Method</u>

Case 1: Two particular friends seated at table of 5

No of ways
$= {}^9C_3 \times 2! \times 3 \times 2 \times 5!$
$= 120960$

$^9C_3$: Selection of friends to be seated at table of 5. This automatically selects friends to be seated at table of 6.

(3-1)!: Arranging the 3 other friends in table of 5.

$^3P_2$: Slotting in the 2 particular friends

5!: Arranging the 6 other friends in table of 6.

Case 2: Two particular friends seated at table of 6

No of ways
$= {}^9C_4 \times 4! \times 3! \times 4 \times 3$
$= 217728$

$^9C_4$: Selection of friends to be seated at table of 5. This automatically selects friends to be seated at table of 6.

(5-1)!: Arranging the 5 friends in table of 5.

4!: Arranging the 5 friends in table of 6.

${}^4P_2$: Slotting in the 2 particular friends

Case 3: Two particular friends seated at separate tables
No of ways
$= {}^9C_4 \times 4! \times 5! \times 2$
$= 725760$

${}^9C_4$: Selection of friends to be seated at table of 5. This automatically selects friends to be seated at table of 6.

(5-1)!: Arranging the 5 friends in table of 5.

(6-1)!: Arranging the 6 friends in table of 6.

x2: The 2 particular friends can switch tables

Total no. of ways
$= 120960 + 217728 + 725760$
$= 1064448$

7 (i)

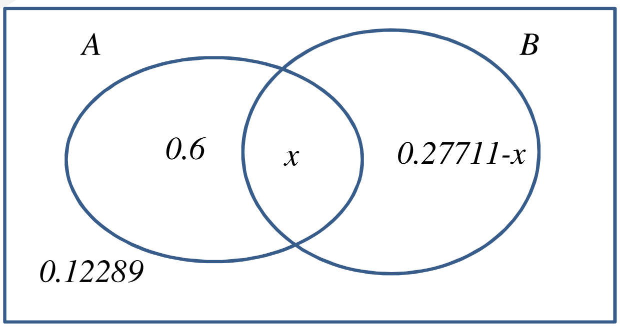
Given $P(A \mid B') = 0.83$

$$\Rightarrow \frac{P(A \cap B')}{P(B')} = 0.83$$

$$\Rightarrow \frac{0.6}{1 - P(B)} = 0.83$$

$$\Rightarrow P(B) = 1 - 0.72289 = 0.27711 = 0.277$$

(ii)
Let P($A \cap B$) = x



$$\begin{aligned} P(A \cup B) &= P(A \cap B') + P(B) \\ &= 0.6 + x + 0.27711 - x \\ &= 0.87711 \end{aligned}$$

$P(A \cup B)' = 1 - 0.87711 = 0.12289$

Since $P(A \cup B') = 0.83$

$\therefore 0.6 + x + 0.12289 = 0.83$

$\Rightarrow x = 0.10711$

$\therefore P(A \cap B) = 0.107$.

(iii)

$$P(B \mid A') = \frac{P(B \cap A')}{P(A')}$$

$$= \frac{0.27711 - 0.10711}{1 - (0.6 + 0.10711)}$$

$$= \frac{0.17}{0.29289}$$

$$= 0.58042$$

$$= 0.580$$

Since $P(B \mid A') \neq P(B) \Rightarrow B$ is not independent of A'

$\therefore$ A and B are not independent.

8

(i)

P(Linda scores 30 points) = $P(\{\text{hit, hit, hit}\})$

$= 0.6^3$

$= \frac{27}{125}$ (0.216)

(ii)

Let X be the number of points scored by Linda in a round.

X	0	10	20	30
$P(X=x)$	0.4	0.6×0.4 $=0.24$	$0.6^2\times0.4$ $=0.144$	0.216

(iii)

$E(X) = 0\times0.4 + 10\times0.24 + 20\times0.144 + 30\times0.216$

$= 11.76$

$E(X^2) = 0^2\times0.4 + 10^2\times0.24 + 20^2\times0.144 + 30^2\times0.216$

$= 276$

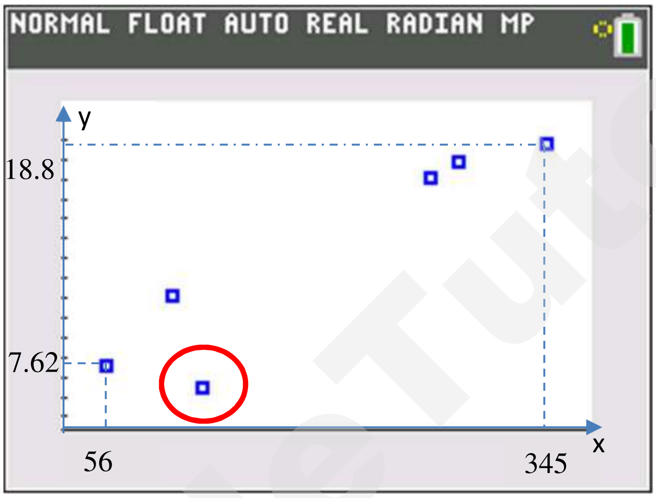
$\text{Var}(X) = E(X^2) - [E(X)]^2$

$= 276 - 11.76^2 = 137.7024$

(iv)

Let X_1 be the number of points scored by Linda in Round 1 and let X_2 be the number of points scored by Linda in Round 2.

P(Linda scores more in round 2 than in round 1)

	$= P(X_1 = 0\ \&\ X_2 \geq 10)$ $+P(X_1 = 10\ \&\ X_2 \geq 20)$ $+P(X_1 = 20\ \&\ X_2 = 30)$ $= P(X_1 = 0)P(X_2 \geq 10)$ $+P(X_1 = 10)P(X_2 \geq 20)$ $+P(X_1 = 20)P(X_2 = 30)$ $= 0.4 \times (1 - 0.4)$ $+0.24 \times (0.144 + 0.216) + 0.144 \times 0.216$ $= 0.357504 = 0.358$ (3 s.f.)
9	(i)  (ii) (a) Product moment correlation coefficient , $r = 0.99959$ (b) Product moment correlation coefficient, $r = 0.95137$ (iii) From the scatter diagram, as x increases, the value of y increases at a decreasing rate, that seems to fit model (a) better. Also, the value of $\|r\|$ for model (a) is closer to 1 as compared to model (b). (iv) We use the regression line y on $\ln x$ $y = 6.1619(\ln x) - 17.223 \approx 6.16 \ln x - 17.2$ When $x = 210$, $y = 6.1619(\ln 210) - 17.223 = 15.725 \approx 15.7$ As the value of $\|r\|$ is close to 1 and $x = 210$ is within the given data range, the estimation may be reliable.
10	(i) Let S be the random variable "radius of a small table in cm'. Let L be the random variable "radius of a large table in cm'. $S \sim \mathrm{N}(30,\ 2^2)$ $L \sim \mathrm{N}(50,\ 5^2)$

	$S_1+S_2+S_3+S_4+S_5 \sim \text{N}(5\times30,\ 5\times2^2)$ $S_1+S_2+S_3+S_4+S_5 \sim \text{N}(150,\ 20)$ $\text{P}(S_1+S_2+S_3+S_4+S_5<160)=0.98733\approx0.987$ (ii) $S_1+S_2+S_3-2L \sim \text{N}(3\times30-2\times50,\ 3\times2^2+2^2\times5^2)$ $S_1+S_2+S_3-2L \sim \text{N}(-10,\ 112)$ $P(S_1+S_2+S_3<2L)=P(S_1+S_2+S_3-2L<0)=0.82765\approx0.828$ (iii) The radii of the large and small round tables are independent of one another. (iv) Let X be the random variable "number of large tables, out of 12, with radius less than 40 cm". $X \sim \text{B}(12,\ \text{P}(L<40\ ))$ $X \sim \text{B}(12,\ 0.022750)$ $\text{P}(X\geq2\ )=1-\text{P}(X\leq1)$ $=1-0.97064$ $=0.029357$ ≈0.0294 (v) Let Y be the random variable "radius of a medium sized table in cm" $\text{P}(Y\geq44\ )=0.20$ $\text{P}(Y<44\ )=0.80$ $P\left(Z<\frac{44-\mu}{\sigma}\right)=0.80$ $\frac{44-\mu}{\sigma}=0.84162$ $\mu=44-0.84162\sigma----(1)$ $\text{P}(Y<40\ )=0.30$ $P\left(Z<\frac{40-\mu}{\sigma}\right)=0.30$ $\frac{40-\mu}{\sigma}=-0.52440$ $\mu=40+0.52440\sigma----(2)$ Solving (1) and (2), $44-0.84162\sigma=40+0.5244\sigma$ $4=1.3660\sigma$ $\sigma=2.9283\approx2.93$ $\mu=41.535\approx41.5$
11	(i) Unbiased estimate of population mean,

	$\overline{x} = \dfrac{24730}{50} = 494.60$ Unbiased estimate for population variance, $s^2 = \dfrac{1}{49}\left(12242631 - \dfrac{24730^2}{50}\right) = 228.02$ Let X be the volume of beer in one beer can in ml and μ be the population mean volume of beer of the beer cans. $H_0 : \mu = 500$ $H_1 : \mu < 500$ Under H_0 , since n = 50 is large, by the Central Limit Theorem, $\overline{X} \sim N\left(500, \dfrac{s^2}{50}\right)$ approximately. Use a left-tailed z-test at the 1% level of significance. Test statistic: $Z = \dfrac{\overline{X} - 500}{\frac{s}{\sqrt{50}}} \sim N(0,1)$. Reject H_0 if p-value ≤ 0.01. From the sample, $p - \text{value} = 0.0057248 = 0.00572$ Since p-value $= 0.00572 \leq 0.01$, we reject H_0. There is sufficient evidence at the 1% level of significance to conclude that the volume of cola in a can is less than 500 ml. (iii) Let X be the volume of cola in one can in ml and μ be the population mean volume of cola of the cans. $H_0 : \mu = 500$ $H_1 : \mu \neq 500$ Unbiased estimate of population variance, $s^2 = \dfrac{40}{39}(s_x)^2$ Under H_0 , since n = 40 is large, by the Central Limit Theorem, $\overline{X} \sim N\left(500, \dfrac{{s_x}^2}{39}\right)$ approximately. Use a two-tailed z-test at the 1% level of significance. Test statistic: $Z = \dfrac{\overline{X} - 500}{\frac{s_x}{\sqrt{39}}} \sim N(0,1)$ Critical values: $z_{crit(1)} = -2.5758$ $z_{crit(2)} = 2.5758$. Reject H_0 if $z_{cal} \leq -2.5758$ or $z_{cal} \geq 2.5758$. Since H_0 is rejected,

	$-2.5758 \le z_{cal}$ or $z_{cal} \ge 2.5758$ $-2.5758 \le \dfrac{\overline{x}-500}{\sqrt{\dfrac{s_x^{\ 2}}{39}}}$ or $\dfrac{\overline{x}-500}{\sqrt{\dfrac{s_x^{\ 2}}{39}}} \ge 2.5758$ $500-2.5758\sqrt{\dfrac{s_x^{\ 2}}{39}} \le \overline{x}$ or $\overline{x} \ge 500+2.5758\sqrt{\dfrac{s_x^{\ 2}}{39}}$ $500-0.41246s_x \le \overline{x}$ or $\overline{x} \ge 500+0.41246s_x$ $500-0.412s_x \le \overline{x}$ or $\overline{x} \ge 500+0.412s_x$ Hence the decision rule should read: Conclude that the volume of cola differs from 500 ml if the value of $\overline{x}$ lies within this range : $500-0.412s_x \le \overline{x}$ or $\overline{x} \ge 500+0.412s_x$. (iv) Let X be the volume of cola in one can in ml. since n is large, by the Central Limit Theorem, $X_1+X_2+....+X_n \sim N(500n,144n)$ approximately. Let Y be the volume of grape juice in one packet in ml. since $2n$ is large, by the Central Limit Theorem, $Y_1+Y_2+....+Y_{2n} \sim N(500n,50n)$ approximately. $X_1+X_2+....+X_n+Y_1+Y_2+....+Y_{2n} \sim N(1000n,194n)$ $P(X_1+X_2+....+X_n+Y_1+Y_2+....+Y_{2n} \le 120,000) \ge 0.95$ $P\left(Z \le \dfrac{120,000-1000n}{\sqrt{194n}}\right) \ge 0.95$ $\dfrac{120,000-1000n}{\sqrt{194n}} \ge 1.6449$ $120,000-1000n \ge 1.6449\sqrt{194n}$ $1000n+22.9\sqrt{n}-120,000 \le 0$

H2 2017 Preliminary Exam Paper 1 Question
Answer all questions [100 marks].

1 Without using a calculator, solve the inequality

$$\frac{3x^2+7x+1}{x+3} < 2x-1. \qquad [4]$$

2 The function p is defined by $\mathrm{p}: x \mapsto \dfrac{1-x^2}{1+x^2}$, $x \in \mapsto$.

(i) Find algebraically the range of p, showing your working clearly. [3]

(ii) Show that $\mathrm{p}(x) = \mathrm{p}(-x)$ for all $x \in \mapsto$ [1]

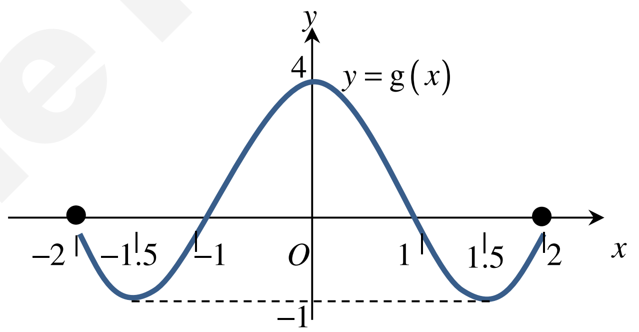
It is given that $\mathrm{q}(x) = \mathrm{p}\left(\dfrac{1}{2}x - 4\right)$, $x \in \mapsto$.

(iii) State a sequence of transformations that will transform the graph of p on to the graph of q. Hence state the line of symmetry for the graph of q. [3]

3 The function f is defined by

$$\mathrm{f}: x \mapsto (x-k)^2, \quad x < k \text{ where } k > 5.$$

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]



The diagram above shows the curve with equation $y = \mathrm{g}(x)$, where $-2 \le x \le 2$. The curve crosses the x–axis at $x=-2$, $x=-1$, $x=1$ and $x=2$, and has turning points at $(-1.5,-1)$, $(0,4)$ and $(1.5,-1)$.

(ii) Explain why the composite function fg exists. [2]

(iii) Find in terms of k,

(a)the value of fg (-1) [1]

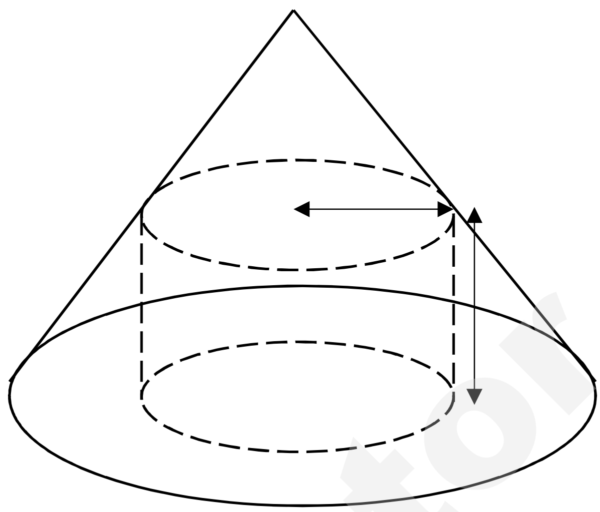
(b)the range of fg. [2]

4 It is given that $z = -1 - \mathrm{i}\sqrt{3}$.

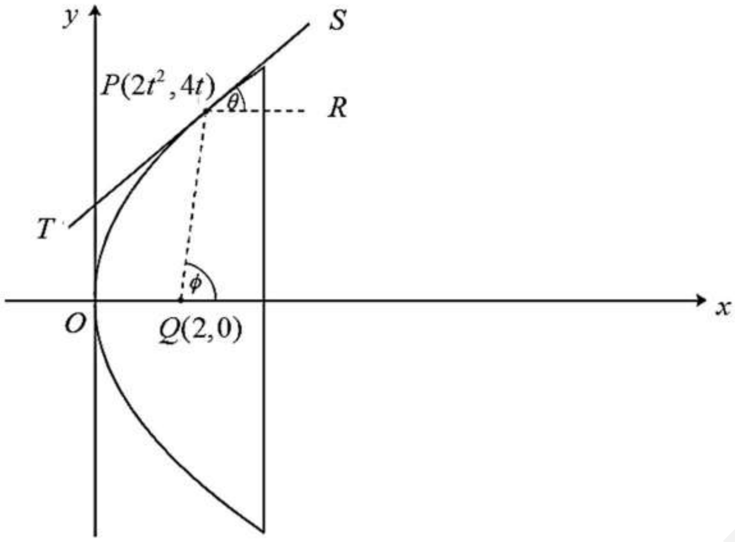
(i) Given that $\dfrac{(\mathrm{i}z)^n}{z^2}$ is purely imaginary, find the smallest positive integer n. [4]

The complex number w is such that $|wz| = 4$ and $\arg\left(\dfrac{w^*}{z^2}\right) = -\dfrac{5\pi}{6}$.

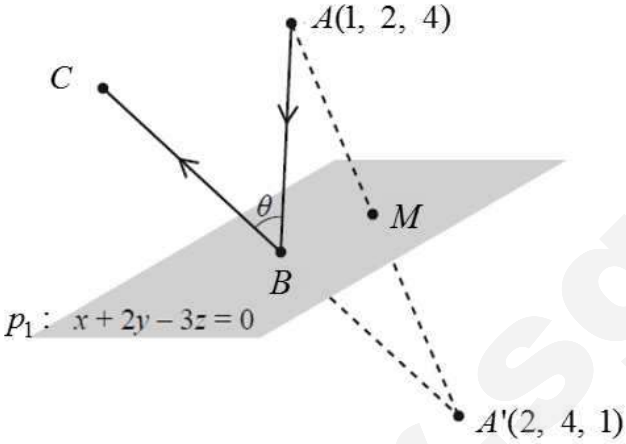
(ii) Find the value of $|w|$ and the exact value of $\arg(w)$ in terms of π. [3]

	On an Argand diagram, points A and B represent the complex numbers w and z respectively. **(iii)** Referred to the origin O, find the exact value of the angle OAB in terms of π. Hence, or otherwise, find the exact value of $\arg(z-w)$ in terms of π. [2]
5	
	A metal cylinder of radius r cm and height h cm is inscribed in a circular cone paperweight of base radius 4 cm and height 6 cm (see diagram). It is determined that the volume of the cylinder, V cm^3, should be as large as possible to provide weight to the paperweight. Show that $V=\frac{4\pi}{9}\left(36h-12h^2+h^3\right)$. [2] Hence find the exact maximum value of V. [5] The metal cylinder is known to expand under heat. An experiment shows that the height of the cylinder is increasing at a rate of 0.04 cm s^{-1} at an instant when $h=1.5$. Find the rate of change of V at this instant. [2]
6	Timber cladding is the application of timber planks over timber planks to provide the layer intended to control the infiltration of weather elements. **(a)** Using method A, 20 rectangular planks are used and the lengths of the planks form an arithmetic progression with common difference d cm. The shortest plank has length 65 cm and the longest plank has length 350 cm. **(i)** Find the value of d. [2] **(ii)** Find the total length of all the planks. [2] **(b)** Using method B, a long plank of 2000 cm is sawn off by a machine into n smaller rectangular planks. The length of the first plank is a cm and each successive plank is $\frac{8}{9}$ as long as the preceding plank. **(i)** Show that the total length of the planks sawn off can never be greater than k times the length of the first plank, where k is an integer to be determined. [2]

	(ii) Given that $a = 423$, find the greatest possible integral value of n and the corresponding length of the shortest plank. [4]
7	**(i)** Express $\frac{1}{r^2-1}$ in partial fractions, and deduce that $$\frac{1}{r(r^2-1)} = \frac{1}{2}\left[\frac{1}{r(r-1)} - \frac{1}{r(r+1)}\right].$$ [2] **(ii)** Hence, find the sum, S_n, of the first n terms of the series $$\frac{1}{2\times3} + \frac{1}{3\times8} + \frac{1}{4\times15} + \ldots .$$ [4] **(iii)** Explain why the series converges, and write down the value of the sum to infinity. [2] **(iv)** Find the smallest value of n for which S_n is smaller than the sum to infinity by less than 0.0025. [3]
8	A drug is administered by an intravenous drip. The drug concentration, x, in the blood is measured as a fraction of its maximum level. The drug concentration after t hours is modelled by the differential equation $$\frac{dx}{dt} = k\left(1+x-2x^2\right),$$ where $0 \le x < 1$, and k is a positive constant. Initially, $x = 0$. (i) Find an expression for x in terms of k and t. [5] After one hour, the drug concentration reaches 75% of its maximum level. (ii) Show that the exact value of k is $\frac{1}{3}\ln 10$, and find the time taken for the drug concentration to reach 90% of its maximum level. [3] A second model is proposed with the following differential equation $$\frac{dx}{dt} = \sin^2\left(\frac{1}{2}t\right),$$ where x is the drug concentration, measured as a fraction of its maximum level, in the blood after t hours. Initially, $x = 0$. **(iii)** Find an expression for x in terms of t. [3] **(iv)** Explain, with the aid of a sketch, why this proposed second model is inappropriate. [2]

9	
	The figure above shows a cross-section of a searchlight whose inner reflective surface is modelled, in suitable units, by the curve $$x = 2t^2, \quad y = 4t, \quad -\sqrt{2} \le t \le \sqrt{2}.$$ The inner reflective surface of the searchlight has the shape produced by rotating the curve about the x-axis. **(i)** Show that the curve has cartesian equation $y^2 = 8x$, and find the volume of revolution of the curve, giving your answer as a multiple of π. [3] $P\left(2t^2, 4t\right)$ is a point on the curve with parameter t. TS is the tangent to the curve at P, and PR is the line through P parallel to the x-axis. Q is the point $(2, 0)$. The angles that PS and QP make with the positive x-direction are θ and ϕ respectively. **(ii)** By considering the gradient of the tangent TS, show that $\cot\theta = t$. [2] **(iii)** Find the gradient of the line QP in terms of t. Hence show that $\phi = 2\theta$, and show that angle TPQ is equal to θ. [5] A lamp bulb is placed at Q. **(iv)** Use your answer to part **(iii)** to describe the direction of the reflected light from the bulb. [1] **(v)** Find a cartesian equation of the locus of the mid-point M on PQ as t varies. [2]
10	Federal Aviation Administration data shows that there were an increase in aviation incidents caused by laser illuminations reported by pilots in 2015 and 2016. A simplified laboratory model is set up to investigate the effects of a laser beam on plexiglass, a common material used to make cockpit windscreen.

The piece of plexiglass is represented by a plane p_1 with equation $x+2y-3z=0$.



Referred to the origin, a laser beam ABC is fired from the point A with coordinates $(1,\ 2,\ 4)$, and is reflected at the point B on p_1 to form a reflected ray BC as shown in the diagram above. It is given that M is the midpoint of AA', where the point A' has coordinates $(2,\ 4,\ 1)$.

(i) Show that AA' is perpendicular to p_1. [2]

(ii) By finding the coordinates of M, show that M lies in p_1. [2]

The vector equation of the line AB is $\mathbf{r}=\begin{pmatrix}1\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix},\ \lambda \in \mapsto$.

(iii) Find the coordinates of B. [2]

The acute angle between the incident ray AB and the reflected ray BC is θ (see diagram).

(iv) Given that $A'BC$ is a straight line, find the value of θ. Hence, or otherwise, write down the acute angle between the line AB and p_1. [3]

To reduce the effect of laser illumination on the pilot sitting in the cockpit at point A', a scientist proposes to include a protective film, represented by a plane p_2, such that the perpendicular distance from p_1 to p_2 is 0.5.

(v) State the possible cartesian equations of p_2. [2]

To further investigate the effects of a laser beam on plexiglass, separate laser beams are fired such that the incident ray AD is now a variable line which is also fired from the same point A and is reflected at the variable point D on p_1 to form a reflected ray DE.

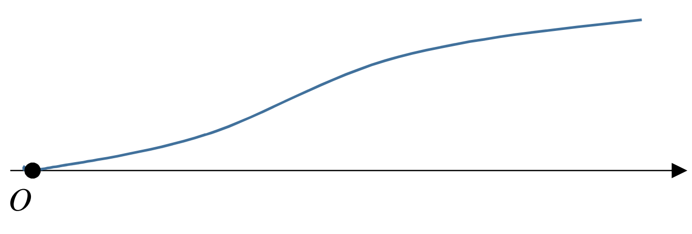
	(vi) Given that AD is perpendicular to the previous ray AB, find the minimum possible distance between B and D. [2] **(vii)** Find the acute inclination of the reflected ray DE to the z-axis when DE is inclined at $60°$ to the x-axis and $45°$ to the y-axis. [3]

− End Of Paper −

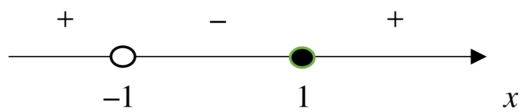
ANNEX B

TPJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Equations and Inequalities	$x < -3$
2	Graphs and Transformation	(i) $-1 < y \le 1$ (iii) Translation by 4 units in the positive x-direction, followed by -Stretch of factor 2 parallel to the x-axis. **Alternative Answers:** Stretch of factor 2 parallel to the x-axis, followed by Translation by 8 units in the positive x-direction
3	Functions	(i) $\mathrm{f}^{-1}(x) = -\sqrt{x} + k$ $\mathrm{D}_{\mathrm{f}^{-1}} = (0,\ \infty)$ (ii) $\mathrm{R_g} = [-1,\ 4]$ $\mathrm{D_f} = (-\infty,\ k)$ Since $k > 5$, $\mathrm{R_g} \subseteq \mathrm{D_f}$. Thus fg exists. (iii)(a) $\mathrm{fg}(-1) = \mathrm{f}(0) = k^2$ (b) $\mathrm{R_{fg}} = \left[(4-k)^2,\ (-1-k)^2\right]$ $= \left[(4-k)^2,\ (1+k)^2\right]$
4	Complex numbers	(i) $\therefore$ smallest positive integer $n = 5$. (ii) $\lvert w \rvert = 2$, $\arg(w) = \dfrac{13\pi}{6}$ (iii) **Hence Method:** $\arg(z-w) = -\left[\pi - \dfrac{\pi}{6} - \dfrac{\pi}{12}\right]$ $= -\left[\dfrac{5\pi}{6} - \left(\dfrac{1}{2}\left\{\pi - \dfrac{5\pi}{6}\right\}\right)\right]$ $= -\dfrac{3\pi}{4}$ (exact) **Otherwise Method:** $z - w = \left(-1-\sqrt{3}\right) + \left(-1-\sqrt{3}\right)\mathrm{i}$ $\arg(z-w) = -\left(\pi - \dfrac{\pi}{4}\right) = -\dfrac{3\pi}{4}$

5	Differentiation & Applications	$V = \frac{128\pi}{9}$ $\frac{dV}{dt} = 0.12\pi \quad cm^3s^{-1}$
6	AP and GP	(a)(i) $d = 15$ (ii) $S_{20} = 4150$ cm (b)(i) $k = 9$ (ii) $n = 6$, Length = 235 cm
7	Sigma Notation and Method of Difference	(ii) $\frac{1}{4} - \frac{1}{2(n+1)(n+2)}$ (iii) As $n \to \infty$, $\frac{1}{2(n+1)(n+2)} \to 0.$ $\frac{1}{4} - \frac{1}{2(n+1)(n+2)} \to \frac{1}{4}$ Sum to infinity $= \frac{1}{4}$ (iv)13
8	Differential Equations	(i) $x = \frac{e^{3kt} - 1}{e^{3kt} + 2}$ (ii)1.45 hours (iii) $x = \frac{1}{2}t - \frac{1}{2}\sin t$ (iv)  The graph shows that as time increases, the drug concentration still continue to increase / the curve shows a strictly increasing function beyond the maximum level of drug concentration.
9	Application of Integration	(i) 64π (iv) The reflected light from the bulb produces a horizontal beam of light/ produces a beam of line parallel to x-axis.

		(v) $y^2 = 4(x-1)$
10	Vectors	(ii) $\left(\frac{3}{2},\ 3,\ \frac{5}{2}\right)$ (iii) $(0,\ 3,\ 2)$ (iv) $\theta = 80.4°,\ 49.8°$ (v) $x+2y-3z=-\frac{\sqrt{14}}{2}$ or $x+2y-3z=\frac{\sqrt{14}}{2}$ (vi) $BD=\frac{\sqrt{6}}{\cos 49.8°}=3.79$ units (vii) $60°$

1	$\frac{3x^2+7x+1}{x+3}<2x-1$ $\frac{3x^2+7x+1}{x+3}-(2x-1)<0$ $\frac{3x^2+7x+1-(2x-1)(x+3)}{x+3}<0$ $\frac{x^2+2x+4}{x+3}<0$ $\frac{(x+1)^2+3}{x+3}<0$ Since $(x+1)^2+3>0$ for all real x, the inequality reduces to: $x+3<0$ $\Rightarrow\ x<-3$
2	Let $y=\frac{1-x^2}{1+x^2},\ x\in\mathbb{R}$: $y(1+x^2)=1-x^2$ $(y+1)x^2+(y-1)=0$ Discriminant ≥ 0: $0^2-4(y+1)(y-1)\geq 0$ $-4(y^2-1)\geq 0$ $y^2-1\leq 0$ $y^2\leq 1$ $-1\leq y\leq 1$ Since $y=-1$ is an asymptote, $-1<y\leq 1$ **<u>Alternative Method:</u>** Let $y=\frac{1-x^2}{1+x^2},\ x\in\mathbb{R}$: $y(1+x^2)=1-x^2$ $(y+1)x^2+(y-1)=0$ $x^2=\frac{1-y}{y+1},\ y\neq -1$ Since $x^2\geq 0\ \forall x\in\mathbb{R}$, $\frac{1-y}{y+1}\geq 0$  $\therefore -1<y\leq 1$

2 (ii)	$\mathrm{p}(-x)=\dfrac{1-(-x)^2}{1+(-x)^2}$ $=\dfrac{1-x^2}{1+x^2}$ $=\mathrm{p}(x)$ for all $x\in\mathbb{R}$ (shown)
2(iii)	Graph of $\mathrm{q}(x)=\mathrm{p}\left(\dfrac{1}{2}x-4\right),\ x\in\mathbb{R}$ is obtained from the graph of $\mathrm{p}(x)$ by: - Translation by 4 units in the positive *x*-direction, followed by Stretch of factor 2 parallel to the *x*-axis.
3(i)	Let $y=(x-k)^2$ $x-k=\pm\sqrt{y}$ $x=-\sqrt{y}+k \quad (\because x<k)$ $\mathrm{f}^{-1}(x)=-\sqrt{x}+k$ $\mathrm{D}_{\mathrm{f}^{-1}}=(0,\ \infty)$
3(ii)	$\mathrm{R_g}=[-1,\ 4]$ $\mathrm{D_f}=(-\infty,\ k)$ Since $k>5$, $\mathrm{R_g}\subseteq\mathrm{D_f}$. Thus fg exists.
3(iii)	$\mathrm{fg}(-1)=\mathrm{f}(0)=k^2$ Using $\mathrm{R_g}=[-1,\ 4]$, and the fact that f is a strictly decreasing function in the given domain, $\mathrm{R_{fg}}=\left[(4-k)^2,\ (-1-k)^2\right]$ $=\left[(4-k)^2,\ (1+k)^2\right]$
4(i)	$\lvert z\rvert=\sqrt{1^2+\sqrt{3}^2}=2 \qquad \arg z=-\left[\pi-\tan^{-1}\left(\dfrac{\sqrt{3}}{1}\right)\right]=-\dfrac{2\pi}{3}$ $z=2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$ $\dfrac{(\mathrm{i}z)^n}{z^2}=\dfrac{\mathrm{e}^{\mathrm{i}\left(\frac{n\pi}{2}\right)}2^n\mathrm{e}^{\mathrm{i}\left(-\frac{2n\pi}{3}\right)}}{2^2\mathrm{e}^{\mathrm{i}\left(-\frac{4\pi}{3}\right)}}$ $=2^{n-2}\mathrm{e}^{\mathrm{i}\left(\frac{n\pi}{2}-\frac{2n\pi}{3}+\frac{4\pi}{3}\right)}$ $=2^{n-2}\mathrm{e}^{\mathrm{i}\left(\frac{(8-n)\pi}{6}\right)}$ $\dfrac{(\mathrm{i}z)^n}{z^2}$ is purely imaginary: $\cos\left(\dfrac{(8-n)\pi}{6}\right)=0$ $\dfrac{(8-n)\pi}{6}=(2k+1)\dfrac{\pi}{2},\ k\in\mathbb{Z}$ $n=5-6k,\ k\in\mathbb{Z}$ **<u>Note:</u>** You may also have alternative form: $\dfrac{(8-n)\pi}{6}=(2k-1)\dfrac{\pi}{2},\ k\in\mathbb{Z}$ $n=11-6k,\ k\in\mathbb{Z}$

$\therefore$ smallest positive integer $n = 5$.

Alternative Method:

$$n\arg(\mathrm{i}z) - 2\arg(z) = n\arg(\mathrm{i}) + n\arg(z) - 2\arg(z)$$

$$= \frac{n\pi}{2} - \frac{2n\pi}{3} + \frac{4\pi}{3}$$

$$= \frac{(8-n)\pi}{6}$$

4 (ii)

$$|wz| = 4$$

$$2|w| = 4$$

$$|w| = 2$$

$$\arg\left(\frac{w^*}{z^2}\right) = -\frac{5\pi}{6}$$

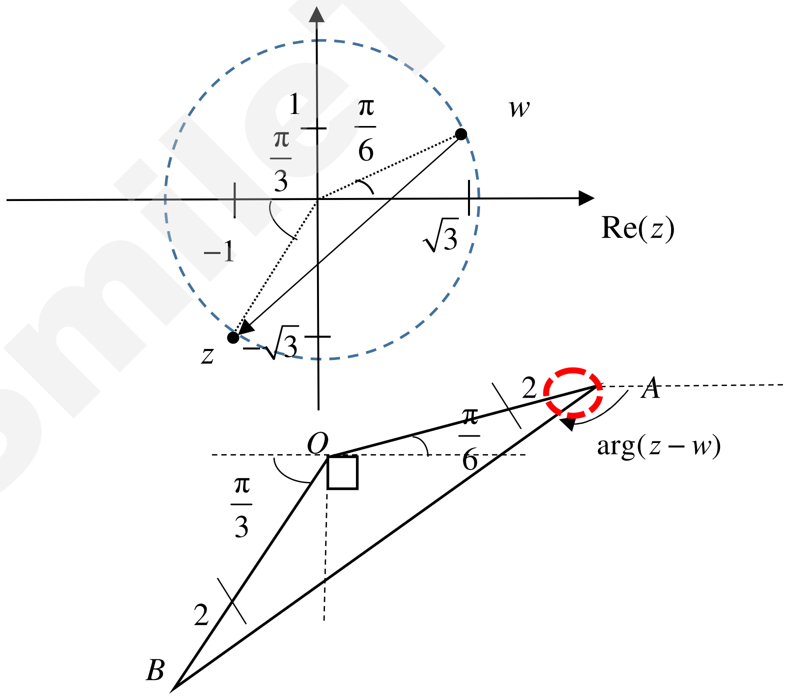
$$-\arg(w) - 2\arg(z) = -\frac{5\pi}{6}$$

$$\arg(w) = \frac{5\pi}{6} - 2\left(-\frac{2\pi}{3}\right)$$

$$= \frac{13\pi}{6}$$

Since $-\pi < \arg(w) \le \pi$, $\arg(w) = \frac{\pi}{6}$ (exact).

4(iii)



$$\angle OAB = \frac{1}{2}\left\{\pi - \left[\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + \frac{\pi}{2} + \frac{\pi}{6}\right]\right\} = \frac{\pi}{12}$$

<u>**Hence Method:**</u> $\arg(z - w) = -\left[\pi - \frac{\pi}{6} - \frac{\pi}{12}\right]$

$$= -\left[\frac{5\pi}{6} - \left(\frac{1}{2}\left\{\pi - \frac{5\pi}{6}\right\}\right)\right]$$

$$= -\frac{3\pi}{4} \quad \text{(exact)}$$

Otherwise Method:

$z-w=\left(-1-\sqrt{3}\right)+\left(-1-\sqrt{3}\right)\mathrm{i}$ $\qquad \arg(z-w)=-\left(\pi-\dfrac{\pi}{4}\right)=-\dfrac{3\pi}{4}$

5

Using similar triangles: $\dfrac{r}{4}=\dfrac{6-h}{6}$

$r=\dfrac{2}{3}\left(6-h\right)$

$$\begin{aligned}V&=\pi r^2 h\\&=\pi\left(\frac{2}{3}\left(6-h\right)\right)^2 h\\&=\frac{4\pi}{9}\left(36-12h+h^2\right)h\\&=\frac{4\pi}{9}\left(36h-12h^2+h^3\right)\quad\text{(shown)}\end{aligned}$$

For maximum V, $\dfrac{\mathrm{d}V}{\mathrm{d}h}=0$:

$\dfrac{4\pi}{9}\left(36-24h+3h^2\right)=0$

Using GC: $h=2$ or $h=6$ (Rejected as h = 6 is height of cone)

Method 1 (1st derivative sign test)

h	2^-	2	2^+
Sign of $\dfrac{\mathrm{dV}}{\mathrm{dh}}$	+	0	-
slope	/	—	\

Thus, maximum volume $V=\dfrac{128\pi}{9}$ when h = 2 cm.

Method 2 (2nd derivative test)

$\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2}=\dfrac{4\pi}{9}\left(-24+6h\right)$

When $h=2$: $\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2}=-\dfrac{16\pi}{3}<0$

Thus, maximum volume $V=\dfrac{128\pi}{9}$.

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}&=\frac{\mathrm{d}V}{\mathrm{d}h}.\frac{\mathrm{d}h}{\mathrm{d}t}\\&=\frac{4\pi}{9}\left(36-24(1.5)+3\left(1.5\right)^2\right)(0.04)\\&=0.12\pi\quad \mathrm{cm^3s^{-1}}\end{aligned}$$

(Accept: 0.377 $\mathrm{cm^3s^{-1}}$)

6(a)(i)

$u_{20}=a+(n-1)d$

$350=65+19d$

$d=15$

6(a)(ii)

$S_{20}=\dfrac{20}{2}(65+350)$

$=4150$ cm $\qquad$ (Accept: 41.5 m)

6(b)(i)	$S_\infty = \dfrac{a}{1-\frac{8}{9}}$ $= 9a$ $\therefore$ integer $k = 9$.
6 (i)	Method 1: Number of ways $= \dbinom{14}{3} \times 3! = 2184$ Method 2: Number of ways $= 14 \times 13 \times 12 = 2184$
6(b)(ii)	$S_n \le 2000$ $\dfrac{423\left[1-\left(\frac{8}{9}\right)^n\right]}{1-\frac{8}{9}} \le 2000$ $1-\left(\frac{8}{9}\right)^n \le \dfrac{2000}{3807}$ $\left(\frac{8}{9}\right)^n \ge \dfrac{1807}{3807}$ $n \le \dfrac{\ln\left(\frac{1807}{3807}\right)}{\ln\left(\frac{8}{9}\right)}$ $n \le 6.3267$ $\therefore$ Largest integer $n = 6$. Length of shortest plank is $u_6 = 423\left(\dfrac{8}{9}\right)^{6-1}$ $= 235$ cm (3 s.f.)
7(i)	$\dfrac{1}{r^2-1} = \dfrac{1}{2(r-1)} - \dfrac{1}{2(r+1)}$ $\dfrac{1}{r(r^2-1)} = \dfrac{1}{r}\left[\dfrac{1}{2(r-1)} - \dfrac{1}{2(r+1)}\right]$ $= \dfrac{1}{2}\left[\dfrac{1}{r(r-1)} - \dfrac{1}{r(r+1)}\right]$
7 (ii)	$S_n = \dfrac{1}{2\times 3} + \dfrac{1}{3\times 8} + \dfrac{1}{4\times 15} + \ldots + (n\text{th term})$

<table>
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$$=\sum_{r=2}^{n+1}\frac{1}{r(r^2-1)}$$

$$=\frac{1}{2}\sum_{r=2}^{n+1}\left[\frac{1}{r(r-1)}-\frac{1}{r(r+1)}\right]$$

$$\begin{aligned}
=\frac{1}{2}\Bigg[&\frac{1}{2\times1}-\cancel{\frac{1}{2\times3}}\\
&+\cancel{\frac{1}{3\times2}}-\cancel{\frac{1}{3\times4}}\\
&+\cancel{\frac{1}{4\times3}}-\cancel{\frac{1}{4\times5}}\\
&\vdots\\
&\vdots\\
&\vdots\\
&+\cancel{\frac{1}{(n-1)\times(n-2)}}-\cancel{\frac{1}{(n-1)\times n}}\\
&+\cancel{\frac{1}{(n)\times(n-1)}}-\cancel{\frac{1}{n\times(n+1)}}\\
&+\cancel{\frac{1}{(n+1)\times n}}-\frac{1}{(n+1)\times(n+2)}\Bigg]
\end{aligned}$$

$$=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]$$

$$=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

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<tr><td>7 (iii)</td><td>

As $n\to\infty$, $\frac{1}{2(n+1)(n+2)}\to 0.$

$$\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\to\frac{1}{4}$$

Sum to infinity $=\frac{1}{4}$

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<tr><td>7 (iv)</td><td>

$$(0<)\frac{1}{4}-S_n<0.0025$$

$$\Rightarrow(0<)\frac{1}{4}-\left[\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\right]<0.0025$$

$$\Rightarrow(0<)\frac{1}{2(n+1)(n+2)}<0.0025$$

$$\Rightarrow(n+1)(n+2)>200$$

Using G.C.
$n<-15.651$ or $n>12.651$

Since $n\in\mathbb{Z}^+$,
Smallest value of $n=13$

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8(i)	**Method 1: Using Partial Fractions**

$$\frac{1}{1+x-2x^2}\frac{dx}{dt}=k$$

$$\int \frac{1}{1+x-2x^2}\ dx=\int k\ dt$$

$$\frac{2}{3}\int \frac{1}{2x+1}\ dx-\frac{1}{3}\int \frac{1}{x-1}\ dx=\int k\ dt$$

$$\frac{1}{1+x-2x^2}=\frac{1}{(1-x)(1+2x)}$$
$$=\frac{2/3}{2x+1}-\frac{1/3}{x-1}$$

$$\frac{1}{3}\ln|2x+1|-\frac{1}{3}\ln|x-1|=kt+C$$

$$\frac{1}{3}\ln\left|\frac{2x+1}{x-1}\right|=kt+C$$

$$\frac{2x+1}{x-1}=Ae^{3kt},\ A=\pm e^{3C}$$

$$x=\frac{Ae^{3kt}+1}{Ae^{3kt}-2}$$

When $t=0,\ x=0$: $\quad 0=\frac{A+1}{A-2}\Rightarrow A=-1$

$$\therefore x=\frac{e^{3kt}-1}{e^{3kt}+2}$$

Method 2: Completing the square

$$\frac{1}{1+x-2x^2}\frac{dx}{dt}=k$$

$$\int \frac{1}{1+x-2x^2}\ dx=\int k\ dt$$

$$\int \frac{1}{-2\left(x-\frac{1}{4}\right)^2+\frac{9}{8}}\ dx=\int k\ dt$$

$$\frac{1}{2}\int \frac{1}{\left(\frac{3}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}\ dx=\int k\ dt$$

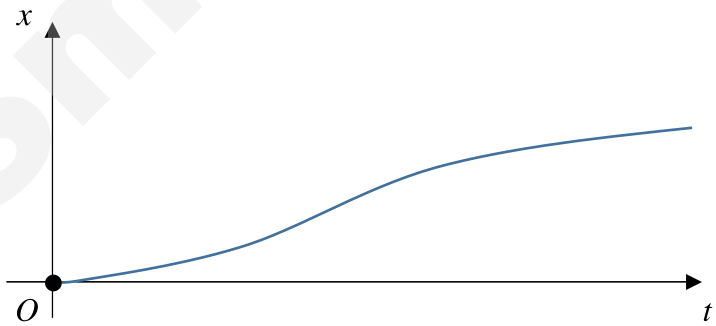
$$\frac{1}{2}\left(\frac{1}{2\left(\frac{3}{4}\right)}\right)\ln\left|\frac{\frac{3}{4}+x-\frac{1}{4}}{\frac{3}{4}-\left(x-\frac{1}{4}\right)}\right|=kt+C$$

$$\frac{1}{3}\ln\left|\frac{\frac{1}{2}+x}{1-x}\right|=kt+C$$

$$\frac{1}{3}\ln\left|\frac{2x+1}{2(1-x)}\right|=kt+C$$

$$\frac{2x+1}{2(1-x)}=Ae^{3kt},\ A=\pm e^{3C}$$

$$x=\frac{2Ae^{3kt}-1}{2\left(Ae^{3kt}+1\right)}$$

	When $t=0,\ x=0$: $\quad 0=\dfrac{2A-1}{2(A+1)} \Rightarrow A=\dfrac{1}{2}$ $\therefore x=\dfrac{e^{3kt}-1}{e^{3kt}+2}$
8 (ii)	When $t=1,\ x=\dfrac{3}{4}$: $\quad \therefore \dfrac{3}{4}=\dfrac{e^{3k}-1}{e^{3k}+2} \Rightarrow e^{3k}=10$ $\Rightarrow k=\dfrac{1}{3}\ln 10$ (shown) $\therefore x=\dfrac{10^t-1}{10^t+2}$ When $x=\dfrac{9}{10}$: $\quad \therefore \dfrac{9}{10}=\dfrac{10^t-1}{10^t+2} \Rightarrow 10^t=28$ $\Rightarrow t=\dfrac{\ln 28}{\ln 10}$ = 1.45 hours (3 s.f.) **<u>Also Accept:</u>** 86.8 mins (3 s.f.)
8 (iii)	$\dfrac{dx}{dt}=\sin^2\left(\dfrac{1}{2}t\right)$ $=\dfrac{1}{2}-\dfrac{1}{2}\cos t$ $x=\displaystyle\int \dfrac{1}{2}-\dfrac{1}{2}\cos t\ dt$ $=\dfrac{1}{2}t-\dfrac{1}{2}\sin t+C$ When $t=0,\ x=0$: $\quad C=0$ $\therefore x=\dfrac{1}{2}t-\dfrac{1}{2}\sin t$
8(iv)	 The graph shows that as time increases, the drug concentration still continue to increase / the curve shows a strictly increasing function beyond the maximum level of drug concentration.
9(i)	$y^2=(4t)^2=16t^2$ $=8(2t^2)$ $=8x$ (shown)

	$\text{Volume} = \pi\int_0^4 8x\,dx$ $= \pi\left[4x^2\right]_0^4$ $= 64\pi$
9(ii)	$\frac{dx}{dt} = 4t,\ \frac{dy}{dt} = 4$ $\frac{dy}{dx} = \frac{1}{t}$ Gradient of tangent $TS = \tan\theta$ $\therefore \tan\theta = \frac{1}{t}$ $\cot\theta = t$ (shown
9 (iii)	Gradient of line $QP = \frac{4t-0}{2t^2-2}$ $= \frac{2t}{t^2-1}$ $= \frac{2/\tan\theta}{1/\tan^2\theta - 1}$ $= \frac{2\tan\theta}{1-\tan^2\theta}$ $= \tan 2\theta$ $\tan\phi = \tan 2\theta \Rightarrow \phi = 2\theta$ (shown) $\angle QPR = 180° - \phi$ (interior angles) $= 180° - 2\theta$ (by earlier results) $\angle TPQ + (180° - 2\theta) + \theta = 180°$ $\therefore \angle TPQ = \theta$ (shown)
9 (iv)	The reflected light from the bulb <u>produces a horizontal beam</u> of light/ produces a beam of line parallel to x-axis
9 (v)	Midpoint $M = \left(\frac{2+2t^2}{2}, \frac{4t+0}{2}\right)$ $= \left(1+t^2,\ 2t\right)$ $\begin{cases} x = 1+t^2 \\ y = 2t \Rightarrow t = \frac{y}{2} \end{cases}$ Locus of midpoint M is: $x = 1 + \frac{y^2}{4}$ $y^2 = 4(x-1)$

10(i)

$$\overrightarrow{AA'} = \begin{pmatrix} 2-1 \\ 4-2 \\ 1-4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

Since $\overrightarrow{AA'} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \underset{\sim}{n}_1$,

$\overrightarrow{AA'}$ is parallel to the normal of p_1,

and thus $\overrightarrow{AA'}$ is perpendicular to p_1.

<u>Alternative Method:</u>

Since $\overrightarrow{A'A} = \begin{pmatrix} 1-2 \\ 2-4 \\ 4-1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = -\underset{\sim}{n}_1$,

$\overrightarrow{A'A}$ is parallel to the normal of p_1,

and thus $\overrightarrow{A'A}$ is perpendicular to p_1

10 (ii)

Since M is the midpoint of A and A':

$$\overrightarrow{OM} = \frac{1}{2}\left[\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right] = \begin{pmatrix} 3/2 \\ 3 \\ 5/2 \end{pmatrix}$$

Coordinates of M are $\left(\frac{3}{2},\ 3,\ \frac{5}{2}\right)$.

<u>Note:</u>
Question asks for coordinates form.

Since $\frac{3}{2} + 2(3) - 3\left(\frac{5}{2}\right) = -6 + 6 = 0$,

M lies in p_1. (shown)

10 (iii)

$\overrightarrow{OB} = \begin{pmatrix} 1+\lambda \\ 2-\lambda \\ 4+2\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

Since B lies on p_1, $(1+\lambda) + 2(2-\lambda) - 3(4+2\lambda) = 0$

$$-7 - 7\lambda = 0$$

$$\lambda = -1$$

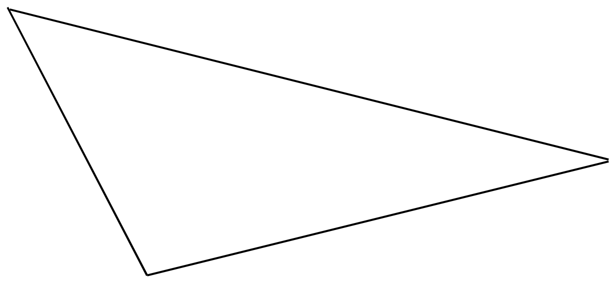
$\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$ Coordinates of B are $(0,\ 3,\ 2)$.

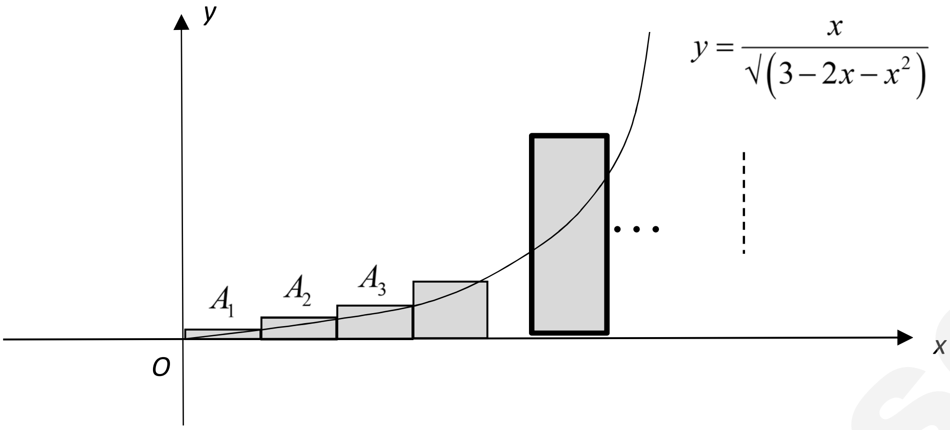
Likewise for part (**vi**).

10 (iv)	$\theta = \cos^{-1}\left\|\dfrac{\overrightarrow{BA} \cdot \overrightarrow{A'B}}{\left\|\overrightarrow{BA}\right\|\left\|\overrightarrow{A'B}\right\|}\right\|$ **Note:** You are expected to recognize that $\overrightarrow{A'B} = \overrightarrow{BC}$. $= \cos^{-1}\left\|\dfrac{\begin{pmatrix}1\\-1\\2\end{pmatrix} \cdot \begin{pmatrix}-2\\-1\\1\end{pmatrix}}{\sqrt{6}\sqrt{6}}\right\|$ $= \cos^{-1}\left\|\dfrac{1}{6}\right\|$ $= 80.4°$ (1 d.p.) Hence, acute angle between the line AB and p_1 $= \dfrac{180° - 80.4°}{2}$ $= 49.8°$ (1 d.p.)
10 (v)	Possible cartesian equations of p_2: $x + 2y - 3z = -\dfrac{\sqrt{14}}{2}$ or $x + 2y - 3z = \dfrac{\sqrt{14}}{2}$
10 (vi)	As incident ray AD varies, D is nearest to origin when OD is the shortest. Note that p_1 contains the origin. $AB = \left\|\begin{pmatrix}-1\\1\\-2\end{pmatrix}\right\| = \sqrt{6}$ $\cos 49.8° = \dfrac{\sqrt{6}}{BD}$ $\Rightarrow BD = \dfrac{\sqrt{6}}{\cos 49.8°} = 3.79$ units (3 s.f.)
10 (vii)	Let γ be the required angle of inclination: $\cos^2 60° + \cos^2 45° + \cos^2 \gamma = 1$ $\dfrac{1}{4} + \dfrac{1}{2} + \cos^2 \gamma = 1$ $\cos \gamma = \pm\dfrac{1}{2}$ $\therefore \gamma = 60°$ (since γ is acute)

End of Paper

H2 2017 Preliminary Exam Paper 2 Question
Section A: Pure Matheatics [40 marks].

1	The cubic equation $az^3 - 31z^2 + 212z + b = 0$, where a and b are real numbers, has a complex root $z = 1 - 3\mathrm{i}$. (i) Explain why the equation must have a real root. [2] (ii) Find the values of a and b and the real root, showing your working clearly. [5]
2	Relative to the origin O, the points A, B and C have position vectors $\mathbf{a}$, $\mathbf{a}+\mathbf{c}$ and $\mathbf{c}$ respectively. The point X is on AC produced such that $AC:CX$ is $2:3$ and the point Y is such that $AXYB$ is a parallelogram. **(i)**The lines AY and BX intersect at the point N. Show that $\overrightarrow{ON} = \frac{1}{4}(7\mathbf{c} - \mathbf{a})$. [3] **(ii)**Given that the area of triangle OAB is 4 square units, find the area of triangle OAN. [4] **(iii)**Give a geometrical interpretation of $\left\lvert \overrightarrow{OA} \times \frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert} \right\rvert$. Use the results from part **(ii)** to show that $\left\lvert \overrightarrow{OA} \times \frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert} \right\rvert = \frac{56}{\lvert 7\mathbf{c} - 5\mathbf{a} \rvert}$. [3]
3	**(a)** Find the series expansion of $\mathrm{e}^{2x}\ln(1+3x)$, where $-\frac{1}{3} < x \le \frac{1}{3}$, in ascending powers of x, up to and including the term in x^3. [3] **(b)** In the triangle PQR as shown in the diagram below, $PR = 1$, angle $QPR = \frac{3\pi}{4}$ radians and angle $PRQ = 2\theta$ radians.  **(i)** Show that $QR = \frac{1}{\cos 2\theta - \sin 2\theta}$. [4] **(ii)** Given that θ is sufficiently small angle, show that $QR \approx 1 + a\theta + b\theta^2$, for constants a and b to be determined. [4]
4	**(a)** Find $\int \mathrm{e}^x \sin x \, \mathrm{d}x$. [3] **(b)**



The diagram shows the curve with eq$k$uation $y=\frac{x}{\sqrt{\left(3-2x-x^2\right)}}$ for $0\le x<1$.

The region bounded by the curve, the x-axis and the line $x=k,\ 0<k<1$ is denoted by S.
It is given that n rectangles of equal width are drawn between $x=0$ and $x=k$.

(i) Show that the area of the first rectangle, $A_1=\frac{k^2}{n\sqrt{3n^2-2nk-k^2}}$. [1]

(ii) Show that the total area of all the n rectangles is

$$\sum_{r=1}^{n}\frac{rk^2}{n\sqrt{\left(3n^2-anrk-br^2k^2\right)}},$$

where a and b are constants to be determined. [2]

It is now given that $k=\left(\sqrt{3}\right)-1$.

(iii) Use integration to find the actual area of region S. Hence state the exact value of

$$\sum_{r=1}^{\infty}\frac{rk^2}{n\sqrt{\left(3n^2-anrk-br^2k^2\right)}}\ .$$ [6]

Section B: Probability and Statistics [60 marks]

5 An unbiased six-sided die is rolled twice. The random variable X represents the higher of the two values if they are different, and their common value if they are the same. The probability distribution of X is given by the formula

$$\mathrm{P}(X=r)=k(2r-1)\quad \text{for } r=1,\ 2,\ 3,\ 4,\ 5,\ 6.$$

(i) Find the exact value of k, giving your answer as a fraction in its simplest form. [2]

(ii) Find the expectation of X. [2]

A round of the game consists of rolling the unbiased six-sided die twice, and X is taken as the score for the round. A player plays three rounds of the game.

(iii) Find the probability that the total score for the three rounds is 16. [2]

6	A geologist splits rocks to look for fossils. On average 7% of the rocks selected from a particular area contain fossils. The geologist selects a random sample of 20 rocks from this area. **(i)** Find the probability that at least three of the rocks contain fossils. [2] A random sample of n rocks is selected from this area. **(ii)** The geologist wants to have a probability of 0.8 or greater of finding fossils in at least three of these rocks. Find the least possible value of n. [3] In early 2017, geologists found the fossils of *zilantophis schuberti*, a new discovered species of winged serpent. On average, the proportion of rocks that contain fossils of *zilantophis schuberti* in this area is p. It is known that the modal number of fossils of *zilantophis schuberti* in a random sample of 10 rocks is 3. **(iii)** Use this information to find exactly the range of values that p can take. [4]
7	A pilot records the take-off distance, S metres, for his private aircraft on runways at various altitudes of h metres. The data are shown in the table below.

h	0	300	600	900	1200	1500	1800
S	635	690	750	840	950	1080	1250

(i) Plot a scatter diagram on graph paper for these values, labelling the axes, using a scale of 2 cm to represent a take-off distance of 100 metres on the y-axis and an appropriate scale for the x-axis. [2]

It is thought that the take-off distance S can be modelled by one of the formulae

$$S = ah + b \qquad \text{or} \qquad S = ch^2 + d,$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) h and S,

(b) h^2 and S. [2]

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $S = ah + b$ or $S = ch^2 + d$ is the better model. [2]

(iv) Find the equation of the least-square regression line for the model you have chosen in part **(iii)**. [1]

(v) Use the equation of your regression line to estimate the take-off distance for altitude of 2200 metres. Comment on the reliability of your estimate when $h = 2200$. [2]

8	A manufacturing plant processes raw material for a supplier. An order placed with the plant is considered to be a bulk order when a worker is expected to process more than 300 kg (kilograms) of raw material. Albert uses a machine to process X kg of raw material and Bob uses a separate machine to process Y kg of raw material on a working day. X and Y are independent random variables with the distributions $\mathrm{N}(296,\ 8^2)$ and $\mathrm{N}(290,\ 12^2)$ respectively. **(i)** Find the probability that Albert processes more than 300 kg of raw material on a randomly selected working day. [2]

	(ii) Find the probability that, over a period of 15 independent working days, there are exactly four working days on which Albert processes more than 300 kg of raw material. [2]
	(iii) Find the probability that the total amount of raw material Bob processes over two working days exceeds twice the amount of raw material Albert processes on one working day. [4]
	The plant receives a bulk order and Albert wants to have a probability of at least 0.95 of meeting the order.
	(iv) This can be done by changing the value of μ, the mean amount of raw material Albert processes using the machine, but the standard deviation remains unchanged. Find the least value of μ. [3]

9 The town council is investigating the mass of rubbish in domestic dustbins. In 2016, the mean mass of rubbish in domestic dustbins was 20.0 kg per household per week. The town council starts a recycling initiative and wishes to determine whether there has been a reduction in the mass of rubbish in domestic dustbins.

The mass of rubbish in a domestic dustbin is denoted by X kg. A random sample of 50 domestic dustbins is selected and the results are summarised as follows.

$$n = 50 \qquad \sum x = 924.5 \qquad \sum x^2 = 18249.2$$

(i) Explain what is meant in this context by the term ‘a random sample’. [2]

(ii) Explain why the town council is able to carry out a hypothesis test without knowing anything about the distribution of the mass of rubbish in domestic dustbins. [2]

(iii) Find the unbiased estimates of the population mean and variance and carry out the test at the 1% level of significance for the town council. [6]

(iv) Use your results in part **(iii)** to find the range of values of n for which the result of the test would be that the null hypothesis is rejected at the 1% level of significance. [3]

10 The number of employees of a statutory board, classified by department and years of working experience, is shown below.

	5 years or less	5 to 10 years	10 years or more	Total
Human Resource Department	20	50	30	100
Legal Department	15	60	45	120
Finance Department	25	30	45	100
Total	60	140	120	320

The Managing Director of the statutory board wishes to select three employees to participate in an overseas conference. The Managing Director selects one employee from each department to participate in the conference.

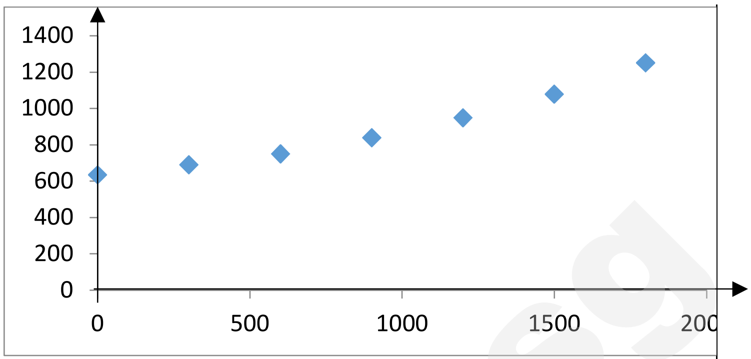
	(i)	Find the probability that two of the selected employees have years of working experience '10 years or more' and the remaining one has years of working experience '5 years or less'. [3]
	(ii)	Given that exactly one of the selected employees has years of working experience '5 years or less', find the probability that one of the selected employees is from the Legal Department and has years of working experience '5 to 10 years'. [3]

− End Of Paper −

ANNEX B

TPJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Complex numbers	(i)Since the **coefficients** of $az^3 - 31z^2 + 212z + b = 0$ are **all real**, **complex roots occur in conjugate pair**. Since a **cubic equation has three roots**, the third root must be a real root. (ii) $a = 25$, $b = 190$, $-\frac{19}{25}$
2	Vectors	(ii) 7 (iii) length of perpendicular from O to AN.
3	Maclaurin series	(a) $3x + \frac{3}{2}x^2 + 6x^3 + ...$ (b)(ii) $a = 2$, $b = 6$
4	Application of Integration	(a) $\frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + D$ (b)(iii) $\sqrt{3} - 1 - \frac{\pi}{6}$
5	DRV	(i) $\frac{1}{36}$ (ii) $\frac{161}{36}$ (iii) 0.112
6	Binomial Expansion	(i)0.161 (ii)60 (iii) $\therefore \frac{3}{11} < p < \frac{4}{11}$
7	Correlation & Linear Regression	S

		 (ii)(a)0.9809 (b)0.9960 (iii) The scatter diagram shows that S increases at an increasing rate as h increases, and for $S = ch^2 + d$, $r \approx 0.9960$ which is closer to 1, so the model $S = ch^2 + d$ is a better model. (iv) $S = 0.000182h^2 + 672$ (v)1550 Estimate for when $h = 2200$ metres is not reliable since $h = 2200$ metres is outside the range of the given data and extrapolation is not a good practice.
8	Normal Distribution	(i)0.309 (ii)0.214 (iii)0.303 (iv) 314
9	Hypothesis Testing	(i) Every dustbin has **an equal probability of being selected** and the selections of each dustbin are **made independently**. (ii) Since $n = 50$ is large, by Central Limit Theorem, the **mean** mass of rubbish in dustbins will be approximately normally distributed. (iii) 18.49, 23.6 Since p-value = 0.013937 > 0.01, we do **not** reject H_0 and conclude that there is **insufficient** evidence at 1% level of significance to claim that there has been a reduction in the mass of rubbish in dustbins. (iv) $n \geq 56,\ n \in \mathbb{Z}^+$

10	P&C, Probability	(i) $\frac{63}{800}$ (ii) $\frac{28}{61}$ (iii) 504 (iv) 3360

H2 Mathematics 2017 Preliminary Exam Paper 2 Solutions

1(i)	Since the **<u>coefficients</u>** of $az^3-31z^2+212z+b=0$ are **<u>all real</u>**, **<u>complex roots occur in conjugate pair</u>**. Since a **<u>cubic equation has three roots</u>**, the third root must be a real root.
1(ii)	Since $1-3\mathrm{i}$ is a root of $az^3-31z^2+212z+b=0$, $a(1-3\mathrm{i})^3-31(1-3\mathrm{i})^2+212(1-3\mathrm{i})+b=0$ $a(-26+18\mathrm{i})-31(-8-6\mathrm{i})+212(1-3\mathrm{i})+b=0$ $(-26a+460+b)+(18a-450)\mathrm{i}=0$ Comparing real and imaginary parts: $-26a+460+b=0$ ----------- (1) $18a-450=0$ -----------------(2) From (2), $a=25$, $b=190$ $(z-(1-3\mathrm{i}))(z-(1+3\mathrm{i}))$ $=z^2-2z+10$ $25z^3-31z^2+212z+190=(z^2-2z+10)(cz+d)$ Comparing coefficient of z^3: $c=25$ Comparing constant: $190=10d$ $d=19$ The real root is $-\dfrac{19}{25}$.
2(i)	$\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{a}+\mathbf{c}$, $\overrightarrow{OC}=\mathbf{c}$ $\overrightarrow{OX}=\overrightarrow{OA}+\overrightarrow{AX}$ $=\overrightarrow{OA}+\dfrac{5}{2}\overrightarrow{AC}$ $=\mathbf{a}+\dfrac{5}{2}(\mathbf{c}-\mathbf{a})$ $=\dfrac{1}{2}(5\mathbf{c}-3\mathbf{a})$ **<u>Alternatively:</u>** **By Ratio Theorem:** $\overrightarrow{OC}=\dfrac{2\overrightarrow{OX}+3\overrightarrow{OA}}{5}$ $\overrightarrow{OX}=\dfrac{5\overrightarrow{OC}-3\overrightarrow{OA}}{2}$ $\overrightarrow{OX}=\dfrac{1}{2}(5\mathbf{c}-3\mathbf{a})$ By midpoint theorem: $\overrightarrow{ON}=\dfrac{\overrightarrow{OB}+\overrightarrow{OX}}{2}$ $\overrightarrow{ON}=\dfrac{1}{2}\left[\mathbf{a}+\mathbf{c}+\dfrac{1}{2}(5\mathbf{c}-3\mathbf{a})\right]$ $=\dfrac{1}{4}(7\mathbf{c}-\mathbf{a})$
2(ii)	Area of triangle $OAB=\dfrac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$

	$4=\frac{1}{2}\lvert\mathbf{a}\times(\mathbf{a}+\mathbf{c})\rvert$ $=\frac{1}{2}\lvert\mathbf{a}\times\mathbf{c}\rvert \qquad (\because \mathbf{a}\times\mathbf{a}=\underset{\sim}{0})$ $\Rightarrow \lvert\mathbf{a}\times\mathbf{c}\rvert=8$ Area of triangle $OAN=\frac{1}{2}\left\lvert\overrightarrow{OA}\times\overrightarrow{ON}\right\rvert$ $=\frac{1}{2}\left\lvert\mathbf{a}\times\frac{1}{4}(7\mathbf{c}-\mathbf{a})\right\rvert$ $=\frac{7}{8}\lvert\mathbf{a}\times\mathbf{c}\rvert \qquad (\because \mathbf{a}\times\mathbf{a}=\underset{\sim}{0})$ $=\frac{7}{8}(8)$ $=7$ square units
2(iii)	$\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert$ is the length of perpendicular from O to AN. **<u>Alternative answer:</u>** $\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert$ is the shortest distance from O to AN. $\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert$ is the area of a parallelogram formed with vector $\overrightarrow{OA}$ and unit vector $\overrightarrow{AN}$ as its adjacent sides. **(Not recommended here)** Area of triangle $OAN=7$ $\frac{1}{2}\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert\left\lvert\overrightarrow{AN}\right\rvert=7$ $\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert=\frac{14}{\lvert\overrightarrow{AN}\rvert}$ $=\frac{14}{\lvert\overrightarrow{ON}-\overrightarrow{OA}\rvert}$ $=\frac{14}{\left\lvert\frac{1}{4}(7\mathbf{c}-\mathbf{a})-\mathbf{a}\right\rvert}$ $=\frac{56}{\lvert7\mathbf{c}-5\mathbf{a}\rvert}$ **(shown)**
3(a)	$\mathrm{e}^{2x}\ln(1+3x)$ $=\left(1+2x+\frac{(2x)^2}{2!}+\ldots\right)\left(3x-\frac{(3x)^2}{2}+\frac{(3x)^3}{3}-\ldots\right)$ where $-1<3x\le1$ $=\left(1+2x+2x^2+\ldots\right)\left(3x-\frac{9}{2}x^2+9x^3-\ldots\right)$

	$= 3x - \frac{9}{2}x^2 + 9x^3 + 6x^2 - 9x^3 + 6x^3 + \ldots$ $= 3x + \frac{3}{2}x^2 + 6x^3 + \ldots$ where $-\frac{1}{3} < x \le \frac{1}{3}$
3(b)(i)	$\frac{QR}{\sin\frac{3\pi}{4}} = \frac{PR}{\sin\left(\pi - \frac{3\pi}{4} - 2\theta\right)}$ $\frac{QR}{\sin\frac{3\pi}{4}} = \frac{PR}{\sin\left(\frac{\pi}{4} - 2\theta\right)}$ $QR = \frac{\sin\frac{3\pi}{4}}{\sin\frac{\pi}{4}\cos 2\theta - \cos\frac{\pi}{4}\sin 2\theta}$ $QR = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\cos 2\theta - \frac{1}{\sqrt{2}}\sin 2\theta}$ $QR = \frac{1}{\cos 2\theta - \sin 2\theta}$ (shown)
3(b)(ii)	When θ is small, $QR \approx \frac{1}{\left(1 - \frac{(2\theta)^2}{2!}\right) - 2\theta}$ $= \frac{1}{1 - 2\theta - 2\theta^2}$ $= \left(1 - \left(2\theta + 2\theta^2\right)\right)^{-1}$ $= 1 + \left(2\theta + 2\theta^2\right) + \left(2\theta + 2\theta^2\right)^2 + \ldots$ $= 1 + 2\theta + 2\theta^2 + 4\theta^2 + \ldots$ $= 1 + 2\theta + 6\theta^2 + \ldots$ $a = 2,\ b = 6$
4(a)	$\int e^x \sin x \, dx$ $= e^x \sin x - \int e^x \cos x \, dx$ $= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx\right]$ $= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$ Hence, $\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx + C$ $2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x + C$ $\int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + D$

4(b)(i)	Area of first rectangle, $x=\frac{k}{n}$: $A_1=\dfrac{k/n}{\sqrt{3-2\left(k/n\right)-\left(k/n\right)^2}}\cdot\dfrac{k}{n}=\dfrac{k^2/n^2}{\sqrt{\dfrac{3n^2-2nk-k^2}{n^2}}}=\dfrac{k^2}{n\sqrt{3n^2-2nk-k^2}}$
4(b)(ii)	Area of second rectangle, $x=\frac{2k}{n}$: $A_2=\dfrac{2k/n}{\sqrt{3-2\left(2k/n\right)-\left(2k/n\right)^2}}\cdot\dfrac{k}{n}=\dfrac{2k^2}{n\sqrt{3n^2-2n(2k)-(2k)^2}}$ Area of third rectangle, $x=\frac{3k}{n}$: $A_3=\dfrac{3k/n}{\sqrt{3-2\left(3k/n\right)-\left(3k/n\right)^2}}\cdot\dfrac{k}{n}=\dfrac{3k^2}{n\sqrt{3n^2-2n(3k)-(3k)^2}}$ By observation, combined area of *n* rectangles: $A=\sum_{r=1}^{n}\dfrac{rk^2}{n\sqrt{3n^2-2nrk-r^2k^2}}$, where $a=2$ *and* $b=1$
4(b)(iii)	$\sum_{r=1}^{\infty}\dfrac{rk^2}{n\sqrt{\left(3n^2-anrk-br^2k^2\right)}}$ $=$ Area under curve from $x=0$ to $x=\sqrt{3}-1$ $=\int_0^{\sqrt{3}-1}\dfrac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$ $=\int_0^{\sqrt{3}-1}\dfrac{-\frac{1}{2}(-2-2x)-1}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$ $=-\dfrac{1}{2}\int_0^{\sqrt{3}-1}\dfrac{-2-2x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x-\int_0^{\sqrt{3}-1}\dfrac{1}{\sqrt{4-(x+1)^2}}\,\mathrm{d}x$ $=-\dfrac{1}{2}\left[\dfrac{\sqrt{3-2x-x^2}}{1/2}\right]_0^{\sqrt{3}-1}-\left[\sin^{-1}\left(\dfrac{x+1}{2}\right)\right]_0^{\sqrt{3}-1}$ $=-\left[\sqrt{3-2x-x^2}\right]_0^{\sqrt{3}-1}-\left[\sin^{-1}\left(\dfrac{x+1}{2}\right)\right]_0^{\sqrt{3}-1}$ $=-\left[1-\sqrt{3}\right]-\left[\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)-\sin^{-1}\dfrac{1}{2}\right]$ $=\sqrt{3}-1-\dfrac{\pi}{3}+\dfrac{\pi}{6}$ $=\sqrt{3}-1-\dfrac{\pi}{6}$ (exact)

5(i)

$$\sum_{r=1}^{6} \mathrm{P}(X=r)=1$$

$$k+3k+5k+7k+9k+11k=1$$

$$k=\frac{1}{36}$$

5(ii)

$$\mathrm{E}(X)=1(k)+2(3k)+3(5k)+4(7k)+5(9k)+6(11k)$$

$$=161k$$

$$=\frac{161}{36}$$

5(iii)

Required Probability

$=\mathrm{P}(\{6,6,4\})+\mathrm{P}(\{6,5,5\})$

$$=\left(\frac{11}{36}\right)^2\left(\frac{7}{36}\right)\frac{3!}{2!}+\left(\frac{11}{36}\right)\left(\frac{9}{36}\right)^2\frac{3!}{2!}$$

$=0.112$ (3 s.f.) Accept: $\frac{1738}{15552}=\frac{869}{7776}$

6(i)

Let X be the number of rocks containing fossils out of 20 rocks.

$X \sim \mathrm{B}(20,\ 0.07)$

$\mathrm{P}(X\geq 3)=1-\mathrm{P}(X\leq 2)$

$=0.161$ (3 s.f.)

6(ii)

Let Y be the number of rocks containing fossils out of 20 rocks.

$Y \sim \mathrm{B}(n,\ 0.07)$

$\mathrm{P}(Y\geq 3)\geq 0.8$

<u>Method 1a: Using GC Table</u>

n	$\mathrm{P}(Y\geq 3)$	
59	0.79085	< 0.8
60	0.80023	> 0.8
61	0.80925	> 0.8

Hence, least $n=60$.

<u>Method 1b: Using GC Table</u>

$\mathrm{P}(Y\leq 2)\leq 0.2$

n	$\mathrm{P}(Y\leq 2)$	
59	0.20915	> 0.2
60	0.19977	< 0.2
61	0.19075	< 0.2

Hence, least $n=60$.

<u>Method 2: Using the binomial distribution function</u>

$\mathrm{P}(Y\leq 2)\leq 0.2$

$\mathrm{P}(Y=0)+\mathrm{P}(Y=1)+\mathrm{P}(Y=2)\leq 0.2$

$$0.93^n+n(0.07)(0.93)^{n-1}+\frac{n(n-1)}{2}(0.07^2)(0.93)^{n-2}\leq 0.2$$

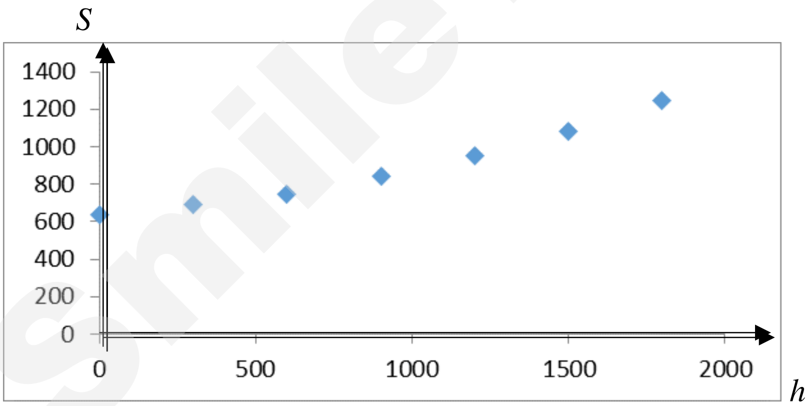
Using GC to sketch the graph:

Hence, least $n=60$.

6(iii)

Let W be the number of fossils of *zilantophis schube*

$W \sim \mathrm{B}(10,\ p)$

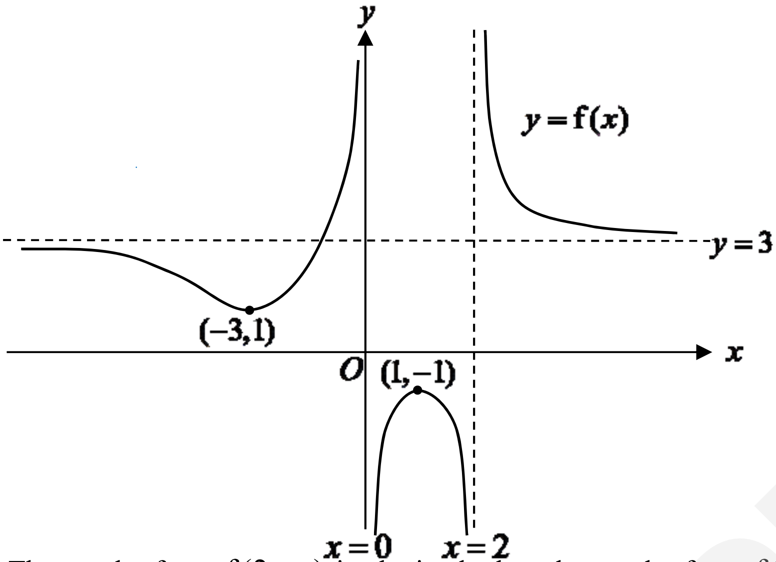
	$\mathrm{P}(W=3) > \mathrm{P}(W=2)$ $\frac{10!}{3!7!}p^3(1-p)^7 > \frac{10!}{2!8!}p^2(1-p)^8$ $120p^3(1-p)^7 > 45p^2(1-p)^8$ $8p > 3(1-p)$ (Since $0 < p < 1$) $\frac{8}{3}p > 1-p$ $p > \frac{3}{11}$ $\mathrm{P}(W=3) > \mathrm{P}(W=4)$ $\frac{10!}{3!7!}p^3(1-p)^7 > \frac{10!}{4!6!}p^4(1-p)^6$ $120p^3(1-p)^7 > 210p^4(1-p)^6$ $4(1-p) > 7p$ (Since $0 < p < 1$) $1-p > \frac{7}{4}p$ $p < \frac{4}{11}$ $\therefore \frac{3}{11} < p < \frac{4}{11}$
7(i)	
7(ii)	**(a)** $r = 0.980867 \approx 0.9809$ (4 d.p.) **(b)** $r = 0.996039 \approx 0.9960$ (4 d.p.)
7(iii)	The scatter diagram shows that <u>S increases at an increasing rate as h increases</u>, and for $S = ch^2 + d$, <u>$r \approx 0.9960$ which is closer to 1</u>, so the model $S = ch^2 + d$ is a better model.
7 (iv)	The equation of regression line is $S = 0.0001822853073h^2 + 671.7261905$ i.e. $S = 0.000182h^2 + 672$ (3 s.f.)
7 (v)	$S = 0.00018229(2200)^2 + 671.73$ $= 1554.0136$ $= 1550$ metres (3 s.f.)

	Estimate for when h = 2200 metres is <u>not</u> reliable since h = 2200 metres is outside the range of the given data and <u>extrapolation</u> is not a good practice.
8(i)	$X \sim \mathrm{N}(296,\ 8^2)$ $Y \sim \mathrm{N}(290,\ 12^2)$ Required probability = $[\mathrm{P}(X > 300)]$ $= 0.30854$ (5 s.f.) (0.3085375322) $= 0.309$ (3 s.f.)
8(ii)	Let W be the number of days in which Albert processes more than 300 kg of raw material on that day out of 15 days. $W \sim \mathrm{B}(15, 0.30854)$ $\mathrm{P}(W = 4) = 0.214$ (3 s.f.)
8(iii)	Let $S = Y_1 + Y_2 - 2X$ $\mathrm{E}(S) = \mathrm{E}(Y_1) + \mathrm{E}(Y_2) - 2\mathrm{E}(X) = 2\times290 - 2\times296 = -12$ $\mathrm{Var}(S) = 2\mathrm{Var}(Y) + 2^2\,\mathrm{Var}(X) = 2\times12^2 + 2^2\times8^2 = 544$ Hence, $S \sim \mathrm{N}(-12,\ 544)$ $\mathrm{P}(S > 0) = 0.303$ (3 s.f.) (0.3034526994)
8(iv)	$X \sim \mathrm{N}(\mu,\ 8^2)$ $\mathrm{P}(X > 300) = \mathrm{P}\left(Z > \frac{300-\mu}{8}\right) \geq 0.95$ $\mathrm{P}\left(Z \leq \frac{300-\mu}{8}\right) \leq 0.05$ $\frac{300-\mu}{8} \leq -1.6449$ $\mu \geq 313.1592$ Least value of $\mu = 314$ kg (3 s.f.)
9(i)	Every dustbin has <u>**an equal probability of being selected**</u> and the selections of each dustbin are <u>**made independently**</u>.
9(ii)	Since $n = 50$ is <u>large</u>, by <u>Central Limit Theorem</u>, the <u>**mean**</u> mass of rubbish in dustbins will be approximately normally distributed.
9 (iii)	Unbiased estimate of population mean, $\bar{x} = \frac{924.5}{50} = 18.49$ Unbiased estimate of population variance, $s^2 = \frac{1}{49}\left[18249.2 - \frac{924.5^2}{50}\right] = 23.575$ (5 s.f.) $= 23.6$ (3 s.f.) Let μ be the population mean mass of rubbish, in kg, in a domestic dustbin. To test: $\mathrm{H_0}$: $\mu = 20$ against $\mathrm{H_1}$: $\mu < 20$ at 1% level of significance Since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(20,\ \frac{23.575}{50}\right)$ approximately under $\mathrm{H_0}$. Test Statistic: $Z = \frac{\overline{X} - 20}{\sqrt{23.575/50}} \sim \mathrm{N}(0,1)$ approximately under $\mathrm{H_0}$.

	Using GC, $[\bar{x}=18.49, s^2=23.575, n=50]$ $z_{test}=-2.199$, p-value = 0.013937 (5 s.f.) Since p-value = 0.013937 > 0.01, we do **not** reject H_0 and conclude that there is **insufficient** evidence at 1% level of significance to claim that there has been a reduction in the mass of rubbish in dustbins.
9 (iv)	For H_0 to be rejected, $z_{test}=\frac{18.49-20}{\sqrt{23.575}}\times\sqrt{n}<-2.3263$ $n>55.954$ Range of values of n is $n\geq 56,\ n\in\mathbb{Z}^+$ [Also Accept: $n>55,\ n\in\mathbb{Z}$ (or equivalent form)]
10(i)	Required probability $=\frac{30}{100}\times\frac{45}{120}\times\frac{25}{100}+\frac{30}{100}\times\frac{15}{120}\times\frac{45}{100}+\frac{20}{100}\times\frac{45}{120}\times\frac{45}{100}$ $=\frac{63}{800}$
10(ii)	Required probability $=\frac{(0.2)(0.5)(0.75)+(0.8)(0.5)(0.25)}{(0.2)(0.875)(0.75)+(0.8)(0.125)(0.75)+(0.8)(0.875)(0.25)}$ $=\frac{28}{61}$
10 (iii)	Number of different possible codes $={}^9C_2\times 2!\ \times {}^7C_1$ $=504$
10 (iv)	**<u>Method 1: Complementary Method</u>** Number of possible arrangements $=\left[{}^4C_3\times{}^5C_2\times 5!\right]-\left[\left({}^4C_3\times 3!\right)\times{}^5C_2\times 3!\right]$ $=3360$ **<u>Method 2: List by Cases</u>** <u>Case 1: All the even digits are separated</u> ${}^4C_3\times{}^5C_2\times 2!\ \times 3!=480$ <u>Case 2: Exactly two even digits are next to each other (and the third even digit is separated)</u> ${}^4C_3\times\left({}^3C_2\times 2!\right)\times{}^5C_2\times 3!\ \times{}^2C_1=2880$ Number of possible arrangements $=480+2880$ $=3360$

End of Paper

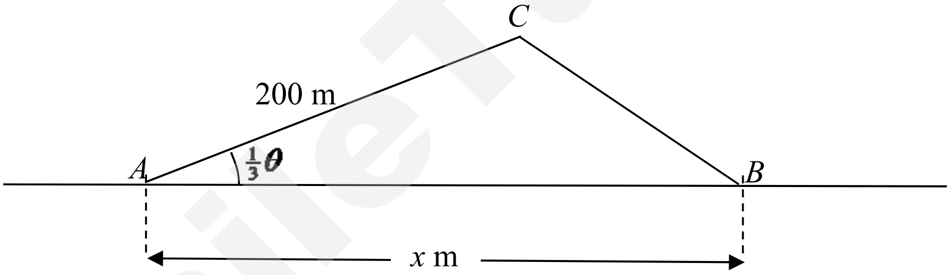
1 The graph of $y = \mathrm{f}(x)$ is shown below.



(a) The graph of $y = \mathrm{f}(2-x)$ is obtained when the graph of $y = \mathrm{f}(x)$ undergoes a sequence of transformations. Describe the sequence of transformations. [2]

(b) Sketch the graph of $y = \mathrm{f}'(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

2



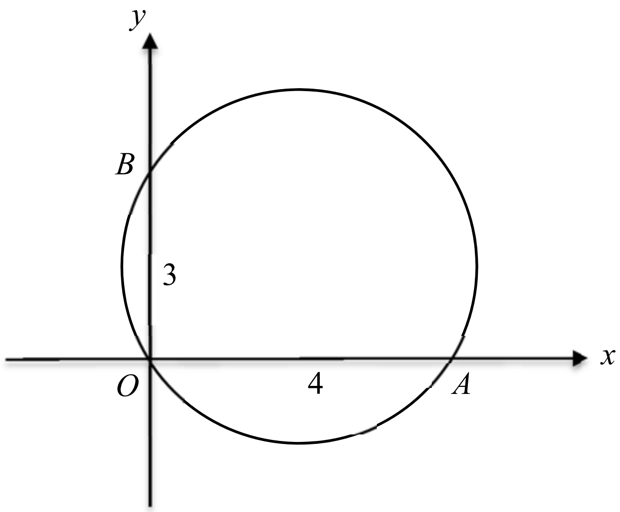
The diagram shows two points at ground level, A and B. The distance in metres between A and B is denoted by x. The angle of elevation of C from B is twice the angle of elevation of C from A. The distance AC is 200 m and $\angle BAC = \frac{1}{3}\theta$ radians. Show that

$$x = \frac{200\sin\theta}{\sin\frac{2}{3}\theta}.$$ [2]

It is given that θ is a small angle such that θ^4 and higher powers of θ are negligible. By using appropriate expansions from the List of Formulae (MF26), show that

$$x \approx \frac{2700 - 250\theta^2}{9}.$$ [4]

3



The diagram above shows a circle C which passes through the origin O and the points A and B. It is given that $OA = 4$ units and $OB = 3$ units.

(i) Show that the coordinates of the centre of C is $\left(2, \frac{3}{2}\right)$. Hence write down the equation of C in the form $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = r^2$, where r is a constant to be determined. [2]

(ii) By adding a suitable line to the diagram above, find the range of values of m for which the equation $mx - \frac{3}{2} = \sqrt{\frac{25}{4} - (x-2)^2}$ has a solution. [4]

4 The curve C has equation $y = \sin 2x + 2\cos x$, $0 \le x \le 2\pi$.

(i) Using an algebraic method, find the exact x-coordinates of the stationary points. [You do not need to determine the nature of the stationary points.] [3]

(ii) Sketch the curve C, indicating clearly the coordinates of the turning points and the intersection with the axes. [1]

(iii) Find the area bounded by the curve C and the line $y = \frac{1}{\pi}x$. [3]

5 The curve C has equation $y = kx^3$. The tangent at the point P on C meets the curve again at point Q. The tangent at point Q meets the curve again at point R. It is given that the x-coordinates of P, Q and R are p, q, and r respectively, where $p \neq 0$.

(i) Show that p and q satisfy the equation $\left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right) - 2 = 0$. [4]

(ii) Show that p, q and r are three consecutive terms of a geometric progression. Hence determine if this geometric series is convergent. [4]

[You may use the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ for $a, b \in \mathbb{R}$]

6 **(a)** The vectors **a** and **b** are the position vector of points A and B respectively. It is given that $OA = 2\sqrt{7}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{a} \cdot \mathbf{b} = -14$.

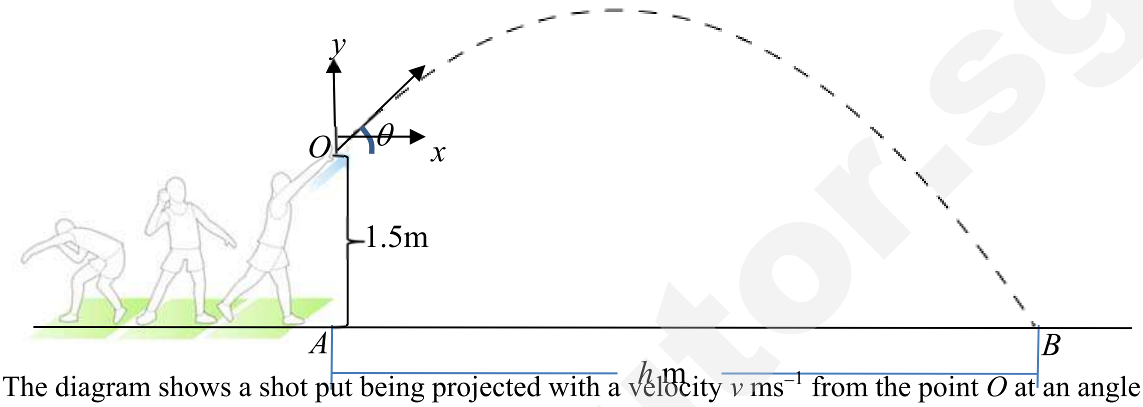
(i) Find angle AOB. [2]

(ii) State the geometrical meaning of $\left|\hat{\mathbf{a}} \cdot \mathbf{b}\right|$, where $\hat{\mathbf{a}}$ is the unit vector of **a**. [1]

(iii) Hence or otherwise, find the position vector of the foot of perpendicular from B to line OA in terms of **a**. [2]

(b) The non-zero vectors **p** and **q** are such that $\left|\mathbf{p} \times \mathbf{q}\right| = 2$. Given that **p** is a unit vector and $\mathbf{q} \cdot \mathbf{q} = 4$, show that **p** and **q** are perpendicular to each other. [3]

7



The diagram shows a shot put being projected with a velocity v ms^{-1} from the point O at an angle θ made with the horizontal. The point O is 1.5m above the point A on the ground. The x-y plane is taken to be the plane that contains the trajectory of this projectile motion with x-axis parallel to the horizontal and O being the origin. The equation of the trajectory of this projectile motion is known to be

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta},$$

where g ms^{-2} is the acceleration due to gravity.
The constant g is taken to be 10 and the distance between A and B is denoted by h m. Given that $v = 10$, show that h satisfies the equation

$$h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0$$ [3]

As θ varies, h varies. Show that stationary value of h occurs when θ satisfies the following equation

$$3\tan^2 2\theta - 20\sin 2\theta \tan 2\theta - 20\cos 2\theta - 20 = 0.$$ [5]

Hence find the stationary value of h. [2]

8 **(a)** In an Argand diagram, points P and Q represent the complex numbers $z_1 = 2 + 3\mathrm{i}$ and $z_2 = \mathrm{i}z_1$.

(i) Find the area of the triangle OPQ, where O is the origin. [2]

(ii) z_1 and z_2 are roots of the equation $(z^2 + az + b)(z^2 + cz + d) = 0$, where $a, b, c, d \in \mathbb{R}$. Find a, b, c and d. [4]

(b) Without using the graphing calculator, find in exact form, the modulus and argument of $v^* = \left(\dfrac{\sqrt{3} + \mathrm{i}}{-1 + \mathrm{i}}\right)^{14}$. Hence express v in exponential form. [5]

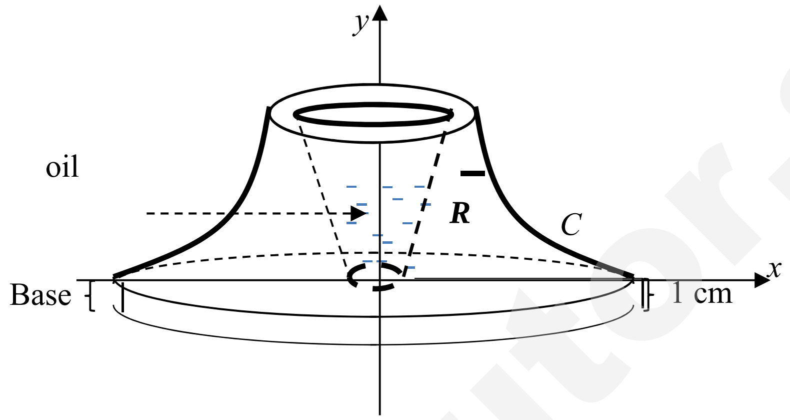
9 A curve C has parametric equation defined by

$$x = 4\sec t \text{ and } y = 8(1-\tan t), \text{ where } -\tfrac{1}{4}\pi \le t \le \tfrac{1}{4}\pi .$$

(i) Find $\frac{dy}{dx}$ in terms of t and hence show that the equation of tangent at the point $t = -\frac{1}{6}\pi$ is $y = 4x + 8(1-\sqrt{3})$. [3]

(ii) Find the Cartesian equation of C. [2]

R is the region bounded by C, the tangent in **(i)**, the normal to C at $t = 0$ and the x-axis. Part of an oil burner is formed by rotating R completely about the y-axis as shown in the diagram below (not drawn to scale). The base of the burner is a solid cylinder of thickness 1 cm. [You may assume each unit along the x and y axis to be 1 cm]



(iii) Find the volume of the material required to make the burner. [6]

10 The point A has coordinates (3, 1, 1). The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, where λ is a parameter. P is a point on l when $\lambda = t$.

(i) Find cosine of the acute angle between AP and l in terms of t. Hence or otherwise, find the position vector of the point N on l such that N is the closest point to A. [6]

(ii) Find the coordinates of the point of reflection of A in l. [2]

The line L has equation $x = -1$, $2y = z + 2$.

(iii) Determine whether L and l are skew lines. [2]

(iv) Find the shortest distance from A to L. [3]

11 A hot air balloon rises vertically upwards from the ground as the balloon operator intermittently fires and turns off the burner. At time t minutes, the balloon ascends at a rate inversely proportional to $t + \lambda$, where λ is a positive constant. At the same time, due to atmospheric factors, the balloon descends at a rate of 2 km per minute. It is also known that initially the rate of change of the height of the balloon is 1 km per minute.

(i) Find a differential equation expressing the relation between H and t, where H km is the height of the hot air balloon above ground at time t minutes. Hence solve the differential equation and find H in terms of t and λ. [7]

Using $\lambda = 15$,

(ii) Find the maximum height of the balloon above ground in exact form. [3]

(iii) Find the total vertical distance travelled by the balloon when $t = 8$. [3]

(iv) Can we claim that the rate of change of the height of the balloon above the ground is decreasing? Explain your answer. [2]

H

TEMASEK JUNIOR COLLEGE, SINGAPORE
JC 2
Preliminary Examination 2017
Higher 2

MATHEMATICS
9758/01

Paper 1
29 Aug 2017

Additional Materials: Answer paper
List of Formulae (MF26)

3 hours

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

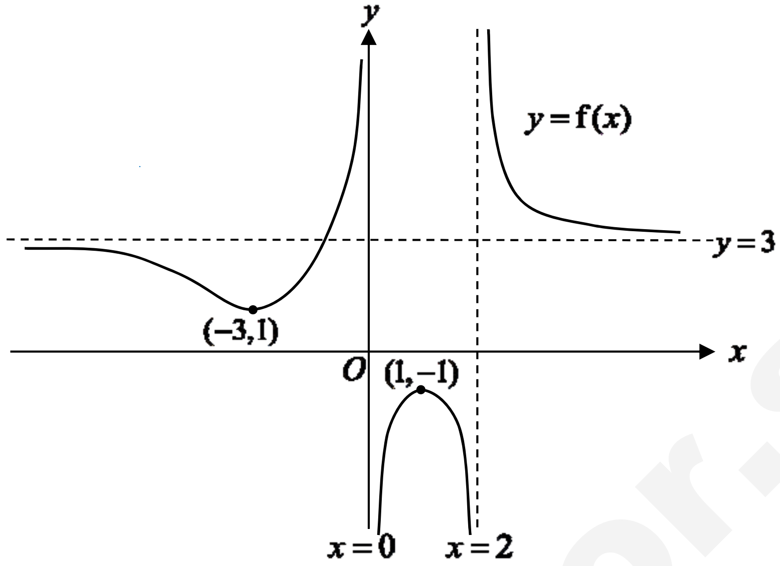
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, fasten all your work securely together.

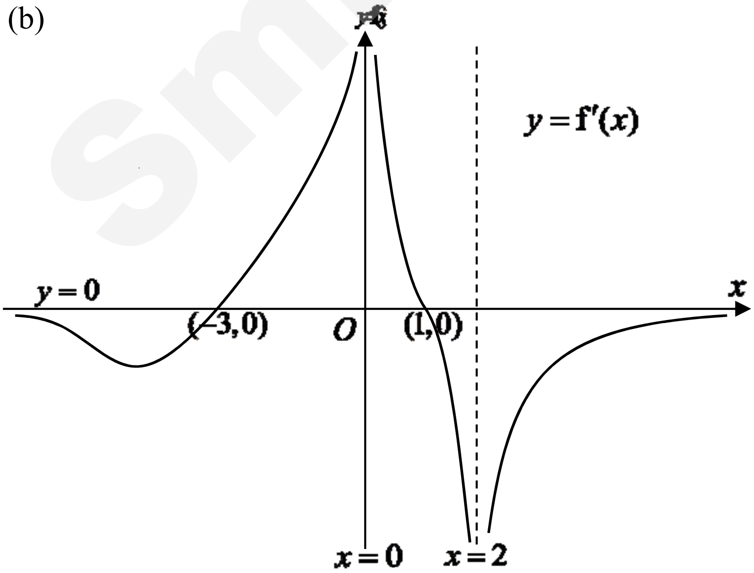
This document consists of 7 printed pages and 1 blank page.

1 The graph of $y = \mathrm{f}(x)$ is shown below.

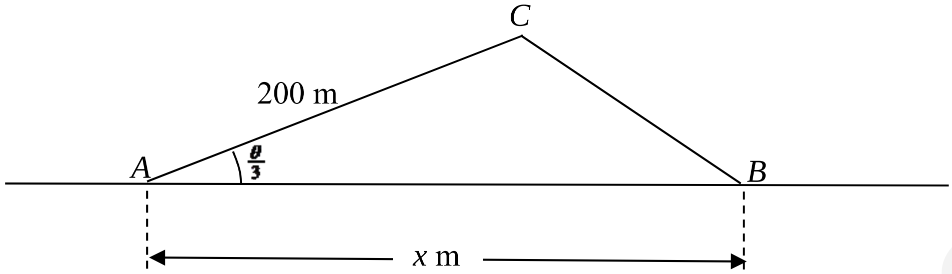


(a) The graph of $y = \mathrm{f}(2-x)$ is obtained when the graph of $y = \mathrm{f}(x)$ undergoes a sequence of transformations. Describe the sequence of transformations. [2]

(b) Sketch the graph of $y = \mathrm{f}'(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

(a) **Translation** of 2 **units** in the negative x-**direction**, followed by **reflection about** the y-axis. <u>Alternative solution</u> Reflection about the y-axis followed by translation of 2 units in the positive x-direction.	Need to use the <u>correct words</u> (**in bold**) when describing the transformations.
(b) 	All (3) asymptotes must be labelled, and intersections with axes written in coordinate form as instructed by the question.
Marker's comments This question is generally well-attempted.	

2



The diagram shows two points at ground level, A and B. The distance in metres between A and B is denoted by x. The angle of elevation of C from B is twice the angle of elevation of C from A. The distance AC is 200 m and $\angle BAC = \frac{\theta}{3}$ radians. Show that

$$x = \frac{200\sin\theta}{\sin\left(\frac{2}{3}\theta\right)}.$$ [2]

It is given that θ is a small angle such that θ^4 and higher powers of θ are negligible. By using appropriate expansions from the List of Formulae (MF26), show that

$$x \approx \frac{2700 - 250\theta^2}{9}.$$ [4]

<table>
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$\angle ABC = 2 \times \angle BAC = \frac{2\theta}{3}$

$\Rightarrow \quad \angle ACB = \pi - \theta$

Using Sine rule, $\frac{x}{\sin(\pi-\theta)} = \frac{200}{\sin\left(\frac{2\theta}{3}\right)}$

Since $\sin(\pi-\theta) = \sin\pi\cos\theta - \cos\pi\sin\theta = \sin\theta$,

$$\frac{x}{\sin\theta} = \frac{200}{\sin\left(\frac{2\theta}{3}\right)}$$

$$\Rightarrow \quad x = \frac{200\sin\theta}{\sin\left(\frac{2}{3}\theta\right)} \qquad \text{(Shown)}$$

</td>
<td>

A common mistake is

$\angle ACB = 2\theta - \frac{\theta}{3} - \frac{2\theta}{3} = \theta$.

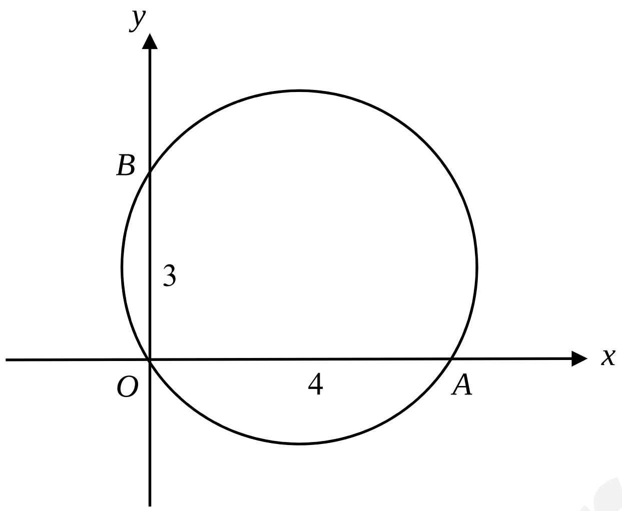
Students who made this mistake simply wanted θ to appear and do not ensure that the expression is true.

Note:
This is a "Show" question.
Thus <u>**all**</u> working/explanation <u>**should**</u> be clearly stated, i.e., need to show $\angle ACB$ and $\angle ABC$, and state the method (sine rule) used.

</td>
</tr>
</table>

$x=\dfrac{200\sin\theta}{\sin\left(\frac{2}{3}\theta\right)}\approx\dfrac{200\left(\theta-\frac{\theta^3}{3!}\right)}{\left(\frac{2}{3}\theta-\frac{\left(\frac{2}{3}\theta\right)^3}{3!}\right)}$ since θ^4 and higher powers of θ are negligible $=\dfrac{200\,\theta\left(1-\frac{\theta^2}{6}\right)}{\frac{2}{3}\theta\left(1-\frac{2\theta^2}{27}\right)}$ $=300\left(1-\frac{\theta^2}{6}\right)\left(1-\frac{2\theta^2}{27}\right)^{-1}$ $=300\left(1-\frac{\theta^2}{6}\right)\left(1+\frac{2\theta^2}{27}+...\right)$ $=300\left(1-\frac{5\theta^2}{54}+...\right)$ $\approx\dfrac{2700-250\theta^2}{9}$	Note that since "+…" is dropped, the "$\approx$" sign should be used. Take out θ and cancel for easy computation. To ensure that the final expression is a polynomial, the denominator has to be "brought up" using power −1. Then use the expansion $(1+x)^{-1}$.
Marker's comments For the 1st part, many students attempted to find *x* by considering the two triangles formed by drawing a line through *C* perpendicular to *AB*. This method is tedious. The $\sin\theta$ and $\sin\left(\frac{2}{3}\theta\right)$ in the expression to be shown would suggest using sine rule.	

3



The diagram above shows a circle C which passes through the origin O and the points A and B.

It is given that $OA = 4$ units and $OB = 3$ units.

(i) Show that the coordinates of the centre of C is $\left(2, \frac{3}{2}\right)$. Hence write down the equation of C in the form $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = r^2$, where r is a constant to be determined. [2]

(ii) By adding a suitable line to the diagram above, find the range of values of m for which the equation $mx - \frac{3}{2} = \sqrt{\frac{25}{4} - (x-2)^2}$ has a solution. [4]

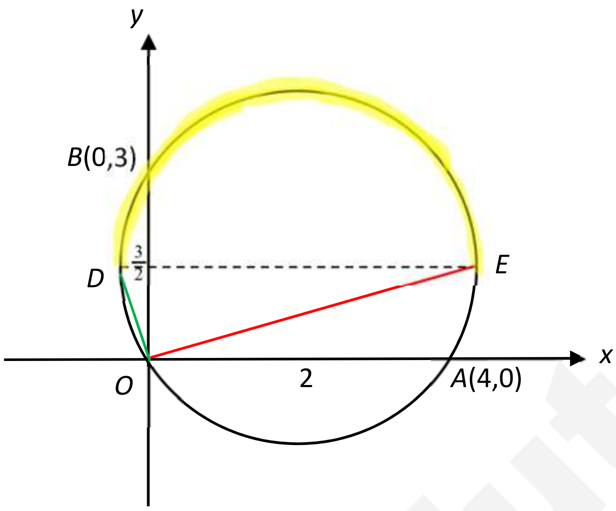
(i)	Since ΔAOB is a right-angle in a semi-circle, AB forms the diameter of the circle. Hence, centre of circle is at the mid point of AB, i.e., $\left(2,\ \frac{3}{2}\right)$. $AB = \sqrt{3^2 + 4^2} = 5 \quad \Rightarrow \quad$ radius, $r = \frac{5}{2}$ units Therefore, equation of C is $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{5}{2}\right)^2$.	Since this is a "Show" question, marks are awarded only if a clear explanation of how the centre coordinates are derived. Inefficient methods such as substituting coordinates into the circle equation and solving them simultaneously were used.

(ii) $$(x-2)^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{5}{2}\right)^2$$

$$\Rightarrow \quad y-\frac{3}{2}=\pm\sqrt{\frac{25}{4}-(x-2)^2}$$

Most students did not realise that the question only involves the top half of the circle (positive root).

Suitable line to add: $y = mx$



Realise that all possible $y = mx$ will lie between the green and red line. This motivates us to find the gradient of the green and red line.

$D=\left(-\frac{1}{2},\ \frac{3}{2}\right)$ and $E=\left(\frac{9}{2},\ \frac{3}{2}\right)$

Gradient of line $OD=-\frac{3/2}{1/2}=-3$

Gradient of line $OE=\frac{3/2}{9/2}=\frac{1}{3}$

$\therefore\ m\le -3$ or $m\ge \frac{1}{3}$

Since points D and E are the two (left/right) ends of the semicircle, their coordinates can be easily deduced using the centre coordinates and radius.

Marker's comments

Common mistakes:

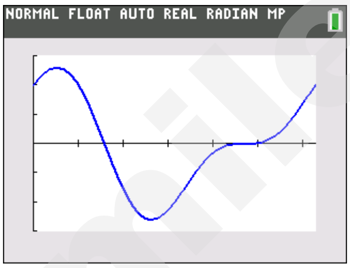
1. *Differentiating the equation of the circle to find the gradient of the tangent*:
   From the diagram, it is clear that $y = mx$ can intersect the semicircle even if it is not a tangent to the circle.
2. *Setting the discriminant to be 0*:
   The quadratic equation represents all the intersection points of $y = mx$ with the whole circle (instead of the semicircle).
3. *Stating that the range of m is* $-3\le m\le \frac{1}{3}$:
   Inaccurate deduction which can be avoided by using the diagram.

4 The curve C has equation $y = \sin 2x + 2\cos x$, $0 \le x \le 2\pi$.

(i) Using an algebraic method, find the exact x-coordinates of the stationary points. [You do not need to determine the nature of the stationary points.] [3]

(ii) Sketch the curve C, indicating clearly the coordinates of the turning points and the intersection with the axes. [1]

(iii) Find the area bounded by the curve C and the lines $y = \frac{1}{\pi}x$ and $x = \frac{5\pi}{6}$. [3]

(i) $y = \sin 2x + 2\cos x$ $\frac{dy}{dx} = 2\cos 2x - 2\sin x$ For stationary points, $\frac{dy}{dx} = 0$ $2[1 - 2\sin^2 x] - 2\sin x = 0$ $\Rightarrow 2\sin^2 x + \sin x - 1 = 0$ $\Rightarrow (2\sin x - 1)(\sin x + 1) = 0$ $\Rightarrow \sin x = 0.5$ or $\sin x = -1$ $\Rightarrow x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$ or $x = \frac{3\pi}{2}$	Differentiate and set $\frac{dy}{dx} = 0$ to find stationary points. As algebraic method is required, clear working of how the roots are arrived is expected, with usage of trigonometric identities along the way.
(ii) 	
(iii) From GC, the line $y = \frac{1}{\pi}x$ intersects the curve C at $x = 1.4544031$ Required area $= \int_{1.4544031}^{\frac{5\pi}{6}} \left[\frac{1}{\pi}x - (\sin 2x + 2\cos x)\right] dx$ $= 2.48$ (to 3 sig figs)	In order to find the area bounded by two curves, it is most important to find where the two curves intersect first, which can be done quickly using GC.

Marker's comments

Common mistakes:

1. In (i), it is unnecessary to convert $y = \sin 2x + 2\cos x = 2\sin x\cos x + 2\cos x$ because it makes the differentiation more complicated. Students should have an awareness of the approach required by the question before manipulating the given information.

   An even more serious problem was that many students were unable to solve $2\cos 2x - 2\sin x = 0$ because identities were not used to convert it into a quadratic equation. Many were also unable to solve $\sin x = 0.5$ (forgetting about the roots in other quadrants), or $\sin x = -1$ (rejecting it immediately without finding the basic angle).

2. Students were unable to identify the correct region, which resulted in them not finding the intersection between the two curves. Also, many students did not apply that the result $\int \mathrm{f}(x) - \mathrm{g}(x)\mathrm{d}x$ to find the area of the region bounded by two curves directly, and instead tried to find the area of the individual pieces which more often than not led to errors.

5 The curve C has equation $y = kx^3$. The tangent at the point P on C meets the curve again at point Q. The tangent at point Q meets the curve again at point R. If the x coordinates of P, Q and R are p, q, and r respectively where $p \neq 0$.

(i) Show that p and q satisfy the equation $\left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right) - 2 = 0$. [4]

(ii) Show that p, q and r are three consecutive terms of a geometric progression. Hence determine if this geometric series is convergent. [4]

[You may use the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ for $a, b \in \mathbb{R}$.]

(i) $y = kx^3$ $\frac{dy}{dx} = 3kx^2$ Point $P = (p,\ kp^3)$, Point $Q = (q,\ kq^3)$, Point $R = (r,\ kr^3)$	
Equation of tangent at point P: $y - kp^3 = 3kp^2(x-p)$ When tangent meets the curve again at Q: $kq^3 - kp^3 = 3kp^2(q-p)$ $q^3 - p^3 = 3p^2(q-p)$ $(q-p)(q^2 + pq + p^2) = 3p^2(q-p)$ $(q-p)(q^2 + pq - 2p^2) = 0$ $q^2 + pq - 2p^2 = 0$ since $p \neq q$ because P and Q are different points Dividing by p^2: $\left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right) - 2 = 0$ (Shown)	Note that the gradient to tangent at point P is not $3kx^2$. You need to substitute x by p in $\frac{dy}{dx} = 3kx^2$ to get the gradient of tangent at point P.

(ii) $\left(\frac{q}{p}\right)^2+\left(\frac{q}{p}\right)-2=0$ $\Rightarrow \left(\frac{q}{p}+2\right)\left(\frac{q}{p}-1\right)=0$ $\Rightarrow \frac{q}{p}=-2$ or $\frac{q}{p}=1$ (rejected since $q \neq p$) Similarly for the other case, $\frac{r}{q}=-2$ $\therefore \frac{q}{p}=\frac{r}{q}=-2$ Since the common ratio is the same, p, q and r are three consecutive terms of a geometric progression. As $\lvert\text{common ratio}\rvert=2>1$, the geometric series is not convergent.	
Marker's comments For part (i), while many students are able to find the equation of tangent, most students who had found the equation of tangent at P did not know how to continue from there. They need to observe more carefully what other information is given on the tangent to continue. In this case it is the fact that the tangent line cuts the curve again at point Q. This will lead to substituting x by q in the equation of tangent. For part (ii), students must recall the condition for a sequence to be a GP, in this case $\frac{q}{p}=\frac{r}{q}$, and work towards it. As for the second part, students must be aware that the condition for geometric series to be convergent is $\lvert\text{common ratio}\rvert<1$.	

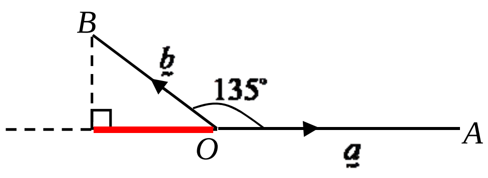
6 **(a)** The vectors **a** and **b** are the position vector of points A and B respectively. It is given that $OA = 2\sqrt{7}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{a}\cdot\mathbf{b} = -14$.

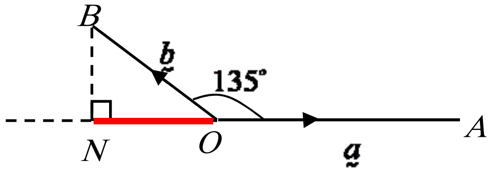
(i) Find angle AOB. [2]

(ii) State the geometrical meaning of $|\hat{\mathbf{a}}\cdot\mathbf{b}|$, where $\hat{\mathbf{a}}$ is the unit vector of **a**. [1]

(iii) Hence or otherwise, find the position vector of the foot of perpendicular from B to line OA in terms of **a**. [2]

(b) The non-zero vectors **p** and **q** are such that $|\mathbf{p}\times\mathbf{q}| = 2$. Given that **p** is a unit vector and $\mathbf{q}\cdot\mathbf{q} = 4$, show that **p** and **q** are perpendicular to each other. [3]

(a)(i) <u>Given</u>: $\lvert \underset{\sim}{a}\rvert = 2\sqrt{7}$, $\underset{\sim}{b} = \begin{pmatrix}1\\2\\-3\end{pmatrix}$ and $\underset{\sim}{a}\cdot\underset{\sim}{b} = -14$ $\Rightarrow \lvert \underset{\sim}{a}\rvert\lvert \underset{\sim}{b}\rvert \cos AOB = -14$ $\Rightarrow \left(2\sqrt{7}\right)\sqrt{1+4+9}\cos AOB = -14$ $\Rightarrow \cos AOB = -\frac{7}{\sqrt{7}\sqrt{14}} = -\frac{1}{\sqrt{2}}$ $\Rightarrow AOB = 135^\circ$	Many students are confused between angles between two vectors and acute angles between 2 lines. Note that (1) If θ is the angle between two vectors **a**, **b**, then $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ (2) If θ is the acute angle between two lines with directional vectors **a**, **b**, then $\cos\theta = \frac{\lvert\mathbf{a}\cdot\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$
(a)(ii) $\lvert\hat{\underset{\sim}{a}}\cdot\underset{\sim}{b}\rvert$ $= \lvert\hat{\underset{\sim}{a}}\rvert\lvert\underset{\sim}{b}\rvert\lvert\cos AOB\rvert$ $= \lvert\underset{\sim}{b}\rvert\lvert\cos AOB\rvert$  $\lvert\hat{\underset{\sim}{a}}\cdot\underset{\sim}{b}\rvert$ is the length of projection of $\underset{\sim}{b}$ on $\underset{\sim}{a}$.	

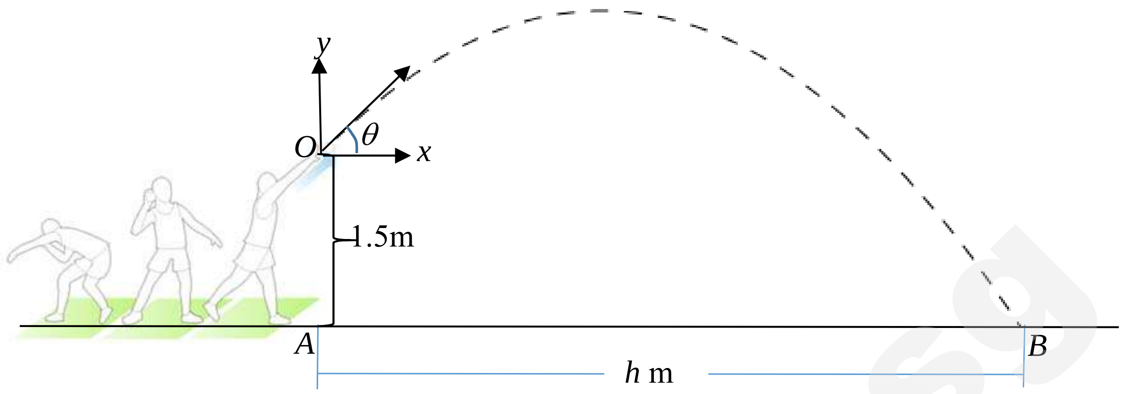
(a)(iii)  Let N be the foot of perpendicular from B to line OA. Length of projection, $ON = \left\|\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b}\right\| = \frac{\left\|\underset{\sim}{a} \cdot \underset{\sim}{b}\right\|}{\left\|\underset{\sim}{a}\right\|} = \frac{\|-14\|}{2\sqrt{7}} = \sqrt{7}$ Since $\angle AOB$ is an obtuse angle, $\overrightarrow{ON} = -\sqrt{7}\frac{\underset{\sim}{a}}{\left\|\underset{\sim}{a}\right\|} = -\frac{1}{2}\underset{\sim}{a}$	
Alternative method to (a)(iii) Let N be the foot of perpendicular from B to line OA. Since N lies on line OA, $\overrightarrow{ON} = \lambda\underset{\sim}{a}$ for some $\lambda \in \mathbb{R}$. Then, $\overrightarrow{BN} = \lambda\underset{\sim}{a} - \underset{\sim}{b}$ $\overrightarrow{BN} \perp \underset{\sim}{a}$ $\Rightarrow \overrightarrow{BN} \cdot \underset{\sim}{a} = 0$ $\Rightarrow (\lambda\underset{\sim}{a} - \underset{\sim}{b}) \cdot \underset{\sim}{a} = 0$ $\Rightarrow \lambda\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{a} \cdot \underset{\sim}{b}$ $\Rightarrow \lambda\left\|\underset{\sim}{a}\right\|^2 = -14$ $\Rightarrow \lambda\left(2\sqrt{7}\right)^2 = -14$ $\Rightarrow \lambda = -\frac{1}{2}$ Thus, $\overrightarrow{ON} = -\frac{1}{2}\underset{\sim}{a}$	
(b) <u>Given</u>: $\left\|\underset{\sim}{p}\right\| = 1$, $\underset{\sim}{q} \cdot \underset{\sim}{q} = 4$ $\Rightarrow \left\|\underset{\sim}{q}\right\|^2 = 4 \Rightarrow \left\|\underset{\sim}{q}\right\| = 2$ <u>Given</u>: $\left\|\underset{\sim}{p} \times \underset{\sim}{q}\right\| = 2$ $\Rightarrow \left\|\underset{\sim}{p}\right\|\left\|\underset{\sim}{q}\right\|\sin\theta = 2$, where θ is the angle between $\underset{\sim}{p}$ and $\underset{\sim}{q}$ $\Rightarrow (1)(2)\sin\theta = 2$ $\Rightarrow \sin\theta = 1$ $\Rightarrow \theta = 90^*$ Thus, $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are perpendicular to each other. (Shown)	Many students are not aware that $\left\|\underset{\sim}{p} \times \underset{\sim}{q}\right\| = \left\|\underset{\sim}{p}\right\|\left\|\underset{\sim}{q}\right\|\sin\theta$. Note that we do not need to mod $\sin\theta$ as θ denotes the angle between vectors which means that θ is between 0 and π. Hence, $\sin\theta$ will always be +ve.

Marker's comments

Students must know that the definition for both dot and cross product (i.e. $\left|\underset{\sim}{p} \cdot \underset{\sim}{q}\right| = \left|\underset{\sim}{p}\right|\left|\underset{\sim}{q}\right|\left|\cos\theta\right|$ and $\left|\underset{\sim}{p} \times \underset{\sim}{q}\right| = \left|\underset{\sim}{p}\right|\left|\underset{\sim}{q}\right|\sin\theta$) are very useful when solving problems that involve vectors which are not given in column vectors form.

Students who have applied using these definitions in this question had done well in this question.

7



The diagram shows a shot put being projected with a velocity v ms^{-1} from the point O at an angle θ made with the horizontal. The point O is 1.5m above the point A on the ground. The x-y plane is taken to be the plane that contains the trajectory of this projectile motion with x-axis parallel to the horizontal and O being the origin. The equation of the trajectory of this projectile motion is known to be

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta},$$

where g ms^{-2} is the acceleration due to gravity.

The constant g is taken to be 10 and the distance between A and B is denoted by h m. Given that $v = 10$, show that h satisfies the equation

$$h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0.$$ [3]

As θ varies, h varies. Show that stationary value of h occurs when θ satisfies the following equation

$$3\tan^2 2\theta - 20\sin 2\theta\tan 2\theta - 20\cos 2\theta - 20 = 0.$$ [5]

Hence find the stationary value of h. [2]

$y = x\tan\theta - \dfrac{10x^2}{2(10)^2\cos^2\theta}$ $\Rightarrow \quad y = x\tan\theta - \dfrac{x^2}{20\cos^2\theta}$ When $y = -1.5, \quad x = h$ $\therefore \quad -1.5 = h\tan\theta - \dfrac{h^2}{20\cos^2\theta}$ $\Rightarrow \quad -30\cos^2\theta = 20h\tan\theta\cos^2\theta - h^2$ $\Rightarrow \quad h^2 - 20h\sin\theta\cos\theta - 30\cos^2\theta = 0$ $\Rightarrow \quad h^2 - 10h\sin 2\theta - 15(1+\cos 2\theta) = 0$ $\Rightarrow \quad h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0$ --- (*) (Shown)	
Differentiate both sides w.r.t. θ, we have $2h\dfrac{\mathrm{d}h}{\mathrm{d}\theta} - 10\dfrac{\mathrm{d}h}{\mathrm{d}\theta}\sin 2\theta - 20h\cos 2\theta + 30\sin 2\theta = 0$ At stationary value, $\dfrac{\mathrm{d}h}{\mathrm{d}\theta} = 0$. $\therefore \quad -20h\cos 2\theta + 30\sin 2\theta = 0$ $\Rightarrow \quad h = \dfrac{30\sin 2\theta}{20\cos 2\theta} = \dfrac{3}{2}\tan 2\theta$ Sub into (*), we have $\left(\dfrac{3}{2}\tan 2\theta\right)^2 - 10\left(\dfrac{3}{2}\tan 2\theta\right)\sin 2\theta - 15\cos 2\theta - 15 = 0$ $\Rightarrow \quad \dfrac{9}{4}\tan^2 2\theta - 15\tan 2\theta\sin 2\theta - 15\cos 2\theta - 15 = 0$ $\Rightarrow \quad 3\tan^2 2\theta - 20\sin 2\theta\tan 2\theta - 20\cos 2\theta - 20 = 0$ (Shown) Using GC, $\theta = 0.71999$ (5 sig fig) Therefore, max $h = \dfrac{3}{2}\tan 2(0.71999) = 11.4$ (3 sig fig)	

Marker's comments

This question is poorly attempted in general.

(i) Students who fail to get credit for this part do not realise that $y = -1.5$ when $x = h$. There were also signs which indicate that students have difficulty applying the double-angle formula.

(ii) One common mistake made by students is to differentiate with respect to h. This is a conceptual error which indicates a poor understanding of derivatives. Many students on the other hand chose to make h the subject before differentiating, failing to realise that implicit differentiation would get the job done much easily. There was also a recurring problem of product rule when differentiating $10h\sin 2\theta$, with the erroneous result of $10\frac{\mathrm{d}h}{\mathrm{d}\theta}(-2\cos 2\theta)$.

(iii) The equation can be easily solved using the GC, though there were many attempts to solve it algebraically. Students using the GC in degree mode would fail to obtain any credit for this part.

8 **(a)** In an Argand diagram, points P and Q represent the complex numbers $z_1 = 2+3\mathrm{i}$ and $z_2 = \mathrm{i}z_1$.

(i) Find the area of the triangle OPQ, where O is the origin. [2]

(ii) z_1 and z_2 are roots of the equation $\left(z^2+az+b\right)\left(z^2+cz+d\right)=0$, where $a, b, c, d \in \mathbb{R}$. Find a, b, c and d. [4]

(b) Without using the graphing calculator, find in exact form, the modulus and argument of $v^* = \left(\dfrac{\sqrt{3}+\mathrm{i}}{-1+\mathrm{i}}\right)^{14}$. Hence express v in exponential form. [5]

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(a)(i) Since $w = \mathrm{i}z$, then $OP \perp OQ$

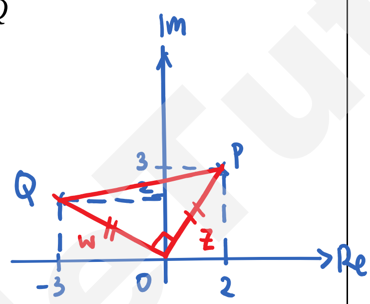
i.e. $\angle POQ = 90^o$.

Area of triangle OPQ

$= \dfrac{1}{2}|z||w|$

$= \dfrac{1}{2}|2+3\mathrm{i}|^2$

$= \dfrac{13}{2}$ units^2



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(a)(ii) Since $\left(z^2+az+b\right)\left(z^2+cz+d\right)=0$ is a polynomial with constant coefficients, complex roots occur in conjugate pairs.

Therefore, the four roots are $2+3\mathrm{i}$, $2-3\mathrm{i}$, $-3+2\mathrm{i}$ and $-3-2\mathrm{i}$.

$[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})][z-(-3+2\mathrm{i})][z-(-3-2\mathrm{i})]$

$= \left(z^2-4z+13\right)\left(z^2+6z+13\right)$

Hence, $a=-4,\ b=13,\ c=6,\ d=13$.

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Students should write out clearly the roots of the equation.

There are 2 ways to expand $[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})]$

<u>Method 1</u>

$[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})]$

$= z^2-(2+3\mathrm{i}+2-3\mathrm{i})z+(2+3\mathrm{i})(2-3\mathrm{i})$

<u>Method 2</u>

$[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})]$

$= \left[(z-2)-(3\mathrm{i})\right]\left[(z-2)+(3\mathrm{i})\right]$

$= (z-2)^2-(3\mathrm{i})^2$

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(b) $\lvert v^*\rvert = \dfrac{\lvert\sqrt{3}+\mathrm{i}\rvert^{14}}{\lvert-1+\mathrm{i}\rvert^{14}}$ $= \dfrac{2^{14}}{\left(\sqrt{2}\right)^{14}} = 2^7$ $\arg\left(v^*\right)$ $= \arg\left(\left(\dfrac{\sqrt{3}+\mathrm{i}}{-1+\mathrm{i}}\right)^{14}\right)$ $= 14\left[\arg\left(\sqrt{3}+\mathrm{i}\right) - \arg\left(-1+\mathrm{i}\right)\right]$ $= 14\left[\dfrac{\pi}{6} - \dfrac{3\pi}{4}\right]$ $= -\dfrac{49\pi}{6} \notin (-\pi, \pi]$ $\therefore\ \arg\left(v^*\right) = -\dfrac{\pi}{6}$ $\Rightarrow\ \arg\left(v\right) = \dfrac{\pi}{6}$ Since $\lvert v\rvert = \lvert v^*\rvert = 2^7$, then $v = 2^7 \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$.	Need to take note of how to present $\arg\left(v^*\right)$.

Marker's comments

(a)(i) This part was not well answered. Many students who were unclear/not aware that *OP* is perpendicular to *OQ* had problem arriving at the correct answer for the area of triangle *OPQ*. Many students used a variety of method (using vectors and cross product, shoelace method, area of trapezium/area of triangles) to find the area of triangle, some with more success than others.

(ii) Although many students were able to recognize that the complex roots occur in conjugate pairs since the coefficients of the equation are real, many students were unable to pair the factors $\left(z-(2+3\mathrm{i})\right)$ and $\left(z-(2-3\mathrm{i})\right)$, $\left(z-(-3+2\mathrm{i})\right)$ and $\left(z-(-3-2\mathrm{i})\right)$. Errors also occurred during the expansion of $\left(z-(2+3\mathrm{i})\right)\left(z-(2+3\mathrm{i})\right)$ and $\left(z-(-3+2\mathrm{i})\right)\left(z-(-3-2\mathrm{i})\right)$. A handful of students substituted $(2+3\mathrm{i})$ into the equation and managed to find a,b,c,d by comparing real and imaginary parts. Those who were unsuccessful in this method will not gain marks.

(b) Despite the statement at the start of the question, a large number of students used their GC to obtain modulus and argument to parts of the question. Many students (about 80%) of students rationalize the expression $\left(\frac{\sqrt{3}+\mathrm{i}}{-1+\mathrm{i}}\right)$ and soon realized that they did not have much success to solve the question except to use GC to obtain the modulus and argument. A number of students wrote down what they believed to be the argument of $(-1+\mathrm{i})$ without considering where the complex number was on an Argand diagram. This part clearly indicated that many students were weak in their understanding of the fundamental concepts of Complex Numbers.

9 A curve C has parametric equation defined by

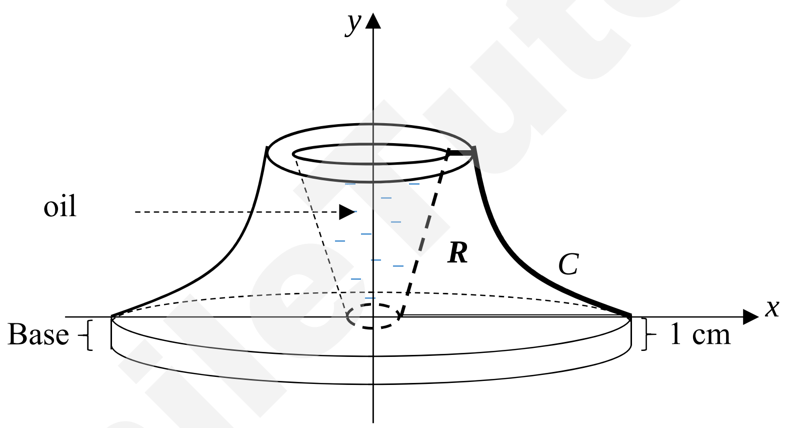
$$x = 4\sec t \text{ and } y = 8(1 - \tan t) \text{ where } -\frac{\pi}{4} \le t \le \frac{\pi}{4}.$$

(i) Find $\frac{dy}{dx}$ in terms of t and hence show that the equation of tangent at the point $t = -\frac{\pi}{6}$ is $y = 4x + 8(1 - \sqrt{3})$. [3]

(ii) Find the Cartesian equation of C. [2]

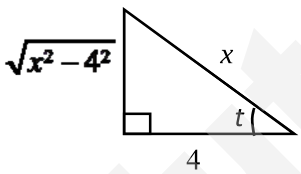
R is the region bounded by C, the tangent in **(i)**, the normal to C at $t = 0$ and the x axis. Part of an oil burner is formed by rotating R 2π radians about the y-axis as shown in the diagram below (not drawn to scale). The base of the burner is a solid cylinder of thickness 1 cm.

[You may assume each unit along the x and y axis to be 1 cm]



Find the volume of the material required to make the burner. [6]

(i) $x = 4\sec t$ and $y = 8(1 - \tan t)$ $\frac{dx}{dt} = 4\sec t \tan t$, $\frac{dy}{dt} = -8\sec^2 t$ $\frac{dy}{dx} = -\frac{2}{\sin t}$ At $t = -\frac{\pi}{6}$, gradient of tangent = 4 , $x = \frac{8}{3}\sqrt{3}$ and $y = 8\left(1 + \frac{\sqrt{3}}{3}\right)$ Equation of tangent is $y - 8\left(1 + \frac{\sqrt{3}}{3}\right) = 4\left(x - \frac{8\sqrt{3}}{3}\right)$ $y = 4x + 8(1 - \sqrt{3})$ (Shown)	Students need to know that : $\sin(-x) = -x$ $\cos(-x) = x$ $\tan(-x) = -x$

(ii) $x = 4\sec t \Rightarrow \sec^2 t = \frac{x^2}{16}$ $y = 8(1-\tan t) \Rightarrow \tan^2 t = \left(1-\frac{y}{8}\right)^2$ Since $1+\tan^2 x = \sec^2 x$, $1+\left(1-\frac{y}{8}\right)^2 = \frac{x^2}{16}$ $\frac{x^2}{16} - \frac{(y-8)^2}{64} = 1$ where $4 \le x \le 4\sqrt{2}$ and $0 \le y \le 16$ $\left(\because -\frac{\pi}{4} \le t \le \frac{\pi}{4}\right)$ Alternative method: $\sec t = \frac{x}{4} \Rightarrow \cos t = \frac{4}{x}$ $y = 8(1-\tan t)$ $y = 8\left(1 \pm \frac{\sqrt{x^2-16}}{4}\right)$  (Note that $-\frac{\pi}{4} \le t \le \frac{\pi}{4} \Rightarrow \tan t = \frac{\sqrt{x^2-16}}{4}$ or $-\frac{\sqrt{x^2-16}}{4}$)	Note that $y = 8\left(1-\tan\left(\cos^{-1}\frac{4}{x}\right)\right)$ is not in its simplest form
When C intersects x-axis, $y = 0$, $\frac{x^2}{16} - \frac{(0-8)^2}{64} = 1 \Rightarrow x^2 = 32$ $x = 4\sqrt{2}$ $(\because \text{radius} > 0)$ Volume of cylindrical base = $\pi\left(\sqrt{2}(4)\right)^2(1) = 32\pi$ Method 1 Volume of the solid that made the burner $= \frac{\pi}{4}\int_0^8 64+(y-8)^2 \, dy - \frac{\pi}{16}\int_0^8 \left(y-8\left(1-\sqrt{3}\right)\right)^2 dy + 32\pi$ $\approx 475.718 = 476$ units^3 (using GC) Method 2 Volume of solid that made the burner $= \pi\int_{\frac{\pi}{4}}^{0} (4\sec t)^2 (-8\sec^2 t) \, dt - \frac{\pi}{16}\int_0^8 \left(y-8\left(1-\sqrt{3}\right)\right)^2 dy + 32\pi \approx 476$	

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<u>Method 3</u>
Volume of solid that made the burner
$= \dfrac{\pi}{4}\int_0^8 64+(y-8)^2 \, \mathrm{d}y + 32\pi - \Bigg[\underbrace{\frac{1}{3}\pi\left(2\sqrt{3}\right)^2\left(8+8\left(\sqrt{3}-1\right)\right)}_{\text{Volume of larger cone}} - \underbrace{\frac{1}{3}\pi\left(2\sqrt{3}-2\right)^2\left(8\left(\sqrt{3}-1\right)\right)}_{\text{Vol. of smaller cone}}\Bigg]$
$\approx 476 \text{ units}^3$

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Students who use Method 3 need to realise that when finding height of the two cones, for example, the height of the larger cone, they should not be using $8+8\left(1-\sqrt{3}\right)$ since $8\left(1-\sqrt{3}\right)$ is a negative y-intercept.

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Marker's comments
(i) Generally well done.
Common errors is not knowing when to have negative sign when evaluating:
$\tan\left(-\frac{\pi}{6}\right)$ and $\sec\left(-\frac{\pi}{6}\right)$
(ii) Many students have forgotten the meaning of Cartesian equation, ended up with an equation that contains the parameter t which is wrong.
Quite a number of students leave answer as $y = 8\left(1-\tan\left(\cos^{-1}\frac{4}{x}\right)\right)$ but this is not in the simplest form.
A serious mistake made by some students is to attempt to integrate
$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{\sin t}$ but without realising that they cannot integrate $-\frac{2}{\sin t}$ with respect to x.

<u>Last part</u>
This part is very badly done.
Many either leave it blank or made a lot of careless/algebraic manipulation mistakes when trying to find x^2 in terms of y.

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10 The point A has coordinates $(3, 1, 1)$. The line l has equation $\mathbf{r} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix}$, where λ is a parameter. P is a point on l when $\lambda = t$.

(i) Find cosine of the acute angle between AP and l in terms of t. Hence or otherwise, find the position vector of the point N on l such that N is the closest point to A. [6]

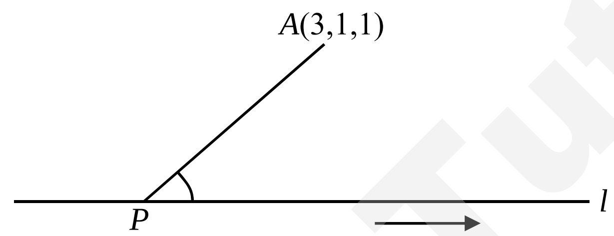
(ii) Find the coordinates of the point of reflection of A in l. [2]

The line L has equation $x = -1$, $2y = z + 2$.

(iii) Determine whether L and l are skew lines. [2]

(iv) Find the shortest distance from A to L. [3]

(i)



$\begin{pmatrix}2\\1\\1\end{pmatrix}$

P is a point on l with parameter t.

$$\Rightarrow \quad \overrightarrow{OP} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}, \quad t \in \mathbb{R}$$

$$\overrightarrow{AP} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix} - \begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\0\\-2\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}$$

Let θ be the **acute** angle between BP and l.

Then,

$$\cos\theta = \frac{\left|\overrightarrow{AP}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}{\left|\overrightarrow{AP}\right|\left|\begin{pmatrix}2\\1\\1\end{pmatrix}\right|} = \frac{\left|\left[\begin{pmatrix}-2\\0\\-2\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}{\sqrt{(-2+2t)^2 + t^2 + (-2+t)^2}\sqrt{4+1+1}}$$

$$= \frac{|(-4-2) + t(4+1+1)|}{\sqrt{4t^2 - 8t + 4 + t^2 + t^2 - 4t + 4}\sqrt{4+1+1}}$$

$$= \frac{6|t-1|}{\sqrt{6}\sqrt{6t^2 - 12t + 8}} \quad \left(\text{or} \quad \frac{\sqrt{3}|t-1|}{\sqrt{3t^2 - 6t + 4}}\right)$$

Need to read Qn carefully and do not make careless mistakes.

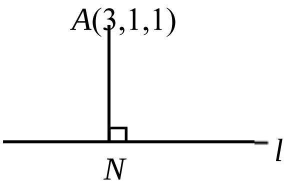
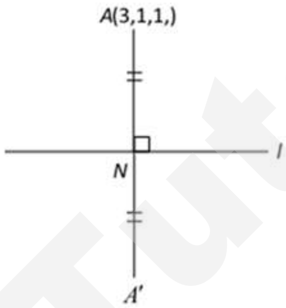
θ is acute, $\cos\theta > 0$, so numerator needs to be positive.

$$\left[\begin{pmatrix}-2\\0\\-2\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}$$

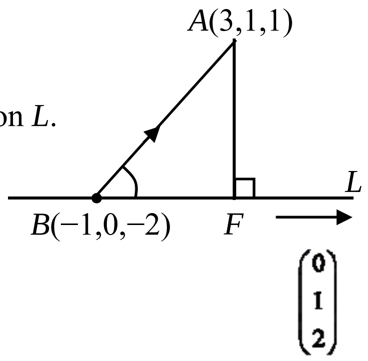
$$= \begin{pmatrix}-2\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$= (-4-2) + t(4+1+1) = -6 + 6t$$

Need to simplify the final answer, especially $\left|\overrightarrow{AP}\right|$

N is the closest point to A when $\theta = 90^\circ$.  $\Rightarrow \cos 90^\circ = 0 = \dfrac{6\lvert t-1\rvert}{\sqrt{6}\sqrt{6t^2-12t+8}}$ $\Rightarrow \quad t = 1$ Thus, $\overrightarrow{ON} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}3\\2\\0\end{pmatrix}$	<u>Alternative method</u> $\overrightarrow{ON} = \begin{pmatrix}1+2\lambda\\1+\lambda\\-1+\lambda\end{pmatrix}$, $\overrightarrow{AN} = \begin{pmatrix}2\lambda-2\\\lambda\\\lambda-2\end{pmatrix}$ $\overrightarrow{AN} \cdot \begin{pmatrix}2\\1\\1\end{pmatrix} = 0 \Rightarrow \lambda = 1$ $\therefore \overrightarrow{ON} = \begin{pmatrix}3\\2\\0\end{pmatrix}$
(ii) Let A' be the point of reflection of A in l. Using ratio theorem, $\overrightarrow{ON} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OA'}\right)$  $\Rightarrow \overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA}$ $= 2\begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\-1\end{pmatrix}$ Thus, the coordinate of A' are $(3,3,-1)$.	Do not use long way to find point of reflection. Eg. Begin with $\overrightarrow{BN} = \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{BA'}\right)$ Must answer the Qn.
Must write r (iii) $l: \quad \underset{\sim}{r} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}, \; t \in \mathbb{R}$ $L: \quad x = -1, \; 2y = z + 2 = \lambda$ i.e., $L: \; \underset{\sim}{r} = \begin{pmatrix}-1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}0\\\frac{1}{2}\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\-2\end{pmatrix} + m\begin{pmatrix}0\\1\\2\end{pmatrix}$ At point of intersection of lines l and L: $\begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\-2\end{pmatrix} + m\begin{pmatrix}0\\1\\2\end{pmatrix} \Rightarrow t = -1, \; m = 0$ Since the point $(-1,0,-2)$ lies on both l and L, the two lines **intersect** and thus cannot be skew lines. (Shown)	Cannot use the same parameter λ for both lines l and L. You may also use this equation for L: $\underset{\sim}{r} = \begin{pmatrix}-1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\2\end{pmatrix}$ Need to know how to convert Cartesian equation to vector equation of a line. Need to know how to use GC to solve the equation. 2 lines are not // and do not intersect $\Rightarrow$ they are skew lines

(iv) $L: \underset{\sim}{r} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} + m\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$



Let B be the point $(-1,0,-2)$ on L.

$$\overrightarrow{BA} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

Shortest distance from A to L

$$= \frac{\left| \overrightarrow{BA} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right|} = \frac{1}{\sqrt{1+4}} \left| \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{5}} \left| \begin{pmatrix} -1 \\ -8 \\ 4 \end{pmatrix} \right| = \frac{\sqrt{1+64+16}}{\sqrt{5}} = \frac{9\sqrt{5}}{5}$$

Alternative method

Use × product not dot product

Don't divide by $\left|\overrightarrow{BA}\right|$

Marker's comments

This is a straight forward question, but many students still did not score it well. They either made careless mistakes or cannot remember the correct formulae.

For (i), many students can get $\overrightarrow{AP} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ correctly but copied it down wrongly when they use it to find cos θ.

Many students make the following mistakes:

- $\left[\begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$
- Drop | | in the numerator part half way in the calculation or totally did not put.
- Some students used $\overrightarrow{OP}$ instead of $\overrightarrow{AP}$ to find cos θ.
- Not many students use $\theta = 90^0$ to find $\overrightarrow{ON}$.

(ii) Many students forgot to give coordinates of *A'*.

(iii) Badly done for this part.

- Quite a number of students cannot obtain the correct vector equation of line *L*.
- Of those who had the correct equation at the point of intersection, many of them gave no solution for the equation. (Do not know how to use GC to solve?)
- For those who can get the intersection point, many students conclude that: "Since there are intersection point, therefore they are skew lines."

(iv) Badly done for this part.

Careless mistake: Used line *l* instead of line *L*.

Use wrong formula: for e.g., used dot product instead of cross product

or divide by $|\overrightarrow{BA}|$

11 A hot air balloon rises vertically upwards from the ground as the balloon operator intermittently fires and turns off the burner. At time t minutes, the balloon ascends at a rate inversely proportional to $t+\lambda$, where λ is a positive constant. At the same time, due to atmospheric factors, the balloon descends at a rate of 2 km per minute. It is also known that initially the rate of change of the height of the balloon is 1 km per minute.

(i) Find a differential equation expressing the relation between H and t, where H km is the height of the hot air balloon above ground at time t minutes. Hence solve the differential equation and find H in terms of t and λ. [7]

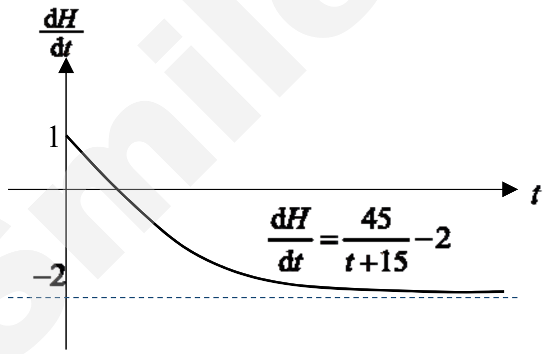
Using $\lambda = 15$,

(ii) Find the maximum height of the balloon above ground in exact form. [3]

(iii) Find the total vertical distance travelled by the balloon when $t = 8$. [3]

(iv) Can we claim that the rate of change of the height of the balloon above the ground is decreasing? Explain your answer. [2]

(i) Rate of increase in height = $\frac{k}{t+\lambda}$ where k is a positive constant Rate of decrease in height = 2 Therefore, $\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{k}{t+\lambda}-2$ Since $\frac{\mathrm{d}H}{\mathrm{d}t}=1$ when $t=0$, we have $1=\frac{k}{0+\lambda}-2$ $\Rightarrow 1=\frac{k-2\lambda}{\lambda}\ \therefore k=3\lambda$ Hence, $\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{3\lambda}{t+\lambda}-2$ (Do not combine into one single fraction!) Integrating wrt t: $H=\int\left(\frac{3\lambda}{t+\lambda}-2\right)\mathrm{d}t=3\lambda\ln\lvert t+\lambda\rvert-2t+C$ Since $t+\lambda>0$, we have $H=3\lambda\ln(t+\lambda)-2t+C$ When $t=0$, $H=0$: $0=3\lambda\ln(\lambda)+C\ \therefore C=-3\lambda\ln\lambda$ $H=3\lambda\ln(t+\lambda)-2t-3\lambda\ln\lambda$ $\therefore H=3\lambda\ln\left(\frac{t}{\lambda}+1\right)-2t$ (ii) Using $\lambda=15$, at maximum height $\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{45}{t+15}-2=0$ $\therefore t=7.5$ $\therefore H=45\ln\left(\frac{7.5}{15}+1\right)-2(7.5)=15\left(3\ln\frac{3}{2}-1\right)$	At least a few students in each class wrote rate of increase in height = $\frac{t+\lambda}{k}$, $k(t+\lambda)$ or $\frac{1}{k(t+\lambda)}$. Obviously they do not know the meaning of inversely proportional. Some leave their final answer as the general solution without finding the value of C.

(iii) When $t = 8$, $H = 45\ln\left(\frac{8}{15}+1\right)-2(8) = 45\ln\frac{23}{15}-16$ Total vertical distance travelled = Vertical distance travelled from $t=0$ to $t=7.5$ + Vertical distance travelled from $t=7.5$ to $t=8$ $= 15\left(3\ln\frac{3}{2}-1\right)+\left[15\left(3\ln\frac{3}{2}-1\right)-45\ln\frac{23}{15}+16\right]$ = 3.26 km (correct to 3 s.f.) (iv) $\frac{\mathrm{d}^2H}{\mathrm{d}t^2} = \frac{-45}{(t+15)^2} < 0$ for all real values of t, $t \geq 0$ i.e. the rate of change of the height of the balloon above ground is decreasing. Or from the graph of $\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{45}{t+15}-2$, we see that $\frac{\mathrm{d}H}{\mathrm{d}t}$ decreases as t increases.  	

Marker's comments

For part (i):

- For those who managed to get the correct DE, most are able to solve the DE using direct integration. Students lose marks if modulus is not included after integration or no reason is provided for dropping modulus.
- Students need to know that they are to find H in terms of t and λ, which means they need to find C by interpreting from the question that at $t = 0$, $H = 0$.

For part (ii):

Part (ii) was well done with only a few students not knowing how to approach the question. A few students did not read the question carefully and did not leave their answer for maximum height in exact form.

For part (iii):

Part (iii) was badly done. Many students did not realise that maximum height is reached at $t = 7.5$ (from (ii)), which means that H will decrease after 7.5 mins. Many students simply find H when $t = 8$. Some went to integrate $\frac{\mathrm{d}H}{\mathrm{d}t}$ from $t = 0$ to $t = 8$ which is incorrect.

For part (iv):

This part was also badly done. Many students conclude that as $t \to \infty$, $\frac{\mathrm{d}H}{\mathrm{d}t} \to -2$ and thus rate of change of height is decreasing, having the misconception that H decreases then rate of change of the height of the balloon is also decreasing. Some explain by drawing the graph of H instead of $\frac{\mathrm{d}H}{\mathrm{d}t}$.

Students need to know that if we want to show that H decreases with t, we need to show that $\frac{\mathrm{d}H}{\mathrm{d}t} < 0$. Similarly, if we want to show that rate of change of H, i.e. $\frac{\mathrm{d}H}{\mathrm{d}t}$ is decreasing, we need to show that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2H}{\mathrm{d}t^2} < 0$ or draw the graph of $\frac{\mathrm{d}H}{\mathrm{d}t}$ and show that it is decreasing with increasing t.

TJC Paper 2

1 Given that $\sin\left[(n+1)x\right]-\sin\left[(n-1)x\right]=2\cos nx\sin x$, show that

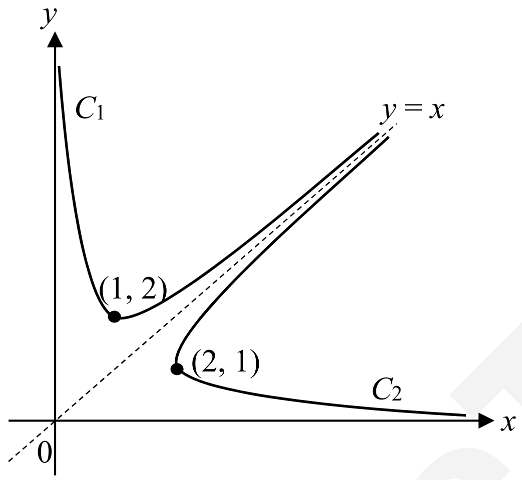
$$\sum_{r=1}^{n}\cos rx=\frac{\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x} \quad . \qquad [4]$$

Hence express

$$\cos^2\left(\frac{x}{2}\right)+\cos^2(x)+\cos^2\left(\frac{3x}{2}\right)+\ldots+\cos^2\left(\frac{11x}{2}\right)$$

in the form $a\left(\dfrac{\sin bx}{\sin cx}+d\right)$, where $a,\ b,\ c$ and d are real numbers. [3]

2



(a) The diagram above shows two curves C_1 and C_2 which are reflections of each other about the line $y=x$. State with justification, whether the following statement is true: "If C_1 is the graph of $y=\mathrm{f}(x)$, then C_2 is the graph of $y=\mathrm{f}^{-1}(x)$."

(b) The functions f and g are defined as follows

$$\mathrm{f}: x\mapsto\frac{1}{x^2-x-6},\quad x\in\mathbb{R}, x<0,\ x\neq -2$$

$$\mathrm{g}: x\mapsto\tan^{-1}\left(\frac{x}{2}\right),\quad x\in\mathbb{R}$$

(i) Sketch the graph of $y=\mathrm{f}(x)$. Determine whether f^2 exists. [3]

(ii) Find $\mathrm{f}^{-1}(x)$. [2]

(iii) Given that $\mathrm{gf}(a)=\dfrac{\pi}{4}$, find the exact value of a. [2]

3 Given that $\mathrm{e}^y=\sqrt{\mathrm{e}+x+\sin x}$. Show that

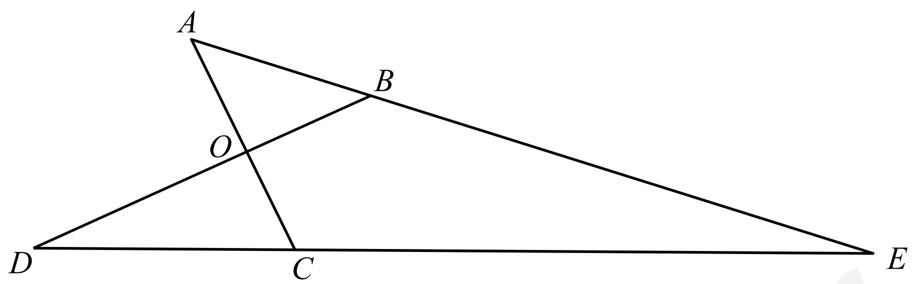
$$2\mathrm{e}^{2y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\mathrm{e}^{2y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\sin x=0\,. \qquad [2]$$

(i) Find the values of y, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x=0$. Hence, find in terms of e, the Maclaurin's series for $\ln(\mathrm{e}+x+\sin x)$, up to and including the term in x^2. [4]

(ii) By using appropriate standard series expansions from the List of Formulae (MF26), verify the correctness of the first three terms in the series expansion for $\ln(\mathrm{e}+x+\sin x)$ found in part **(i)**. [3]

(iii) Use your answer to part **(i)** to give an approximation for $\displaystyle\int_0^{\mathrm{e}^{-1}} \frac{2\mathrm{e}-4x}{\mathrm{e}^2\ln(\mathrm{e}+x+\sin x)}\,\mathrm{d}x$, giving your answer in terms of e. [3]

4



With reference to origin O, the points A, B, C and D are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=-\mathbf{a}$ and $\overrightarrow{OD}=-2\mathbf{b}$. The lines AB and DC meet at E.

Find $\overrightarrow{OE}$ in terms of **a** and **b**. [4]

Hence show that $\dfrac{BE}{AB}=3$. [1]

It is given that A and E have coordinates (1, –4, 3) and (–3, 15, –5) respectively.

(i) Show that the lines AC and BD are perpendicular. [4]

(ii) Find the equation of the plane p that contains E and is perpendicular to the line BD. [2]

(iii) Find the distance between the line AC and p. [2]

5 Four classes CG40, CG41, CG42 and CG43 are tasked to organise a College event. Each class sends 3 representatives for a meeting.

(i) In how many different ways can the 12 representatives sit in a circle so that representatives from CG40 are not seated next to each other and representatives from other classes are seated with their respective classes? [3]

The 12 representatives are to be split up into 3 groups for bonding activities. Each group must consist of a representative from each class.

(ii) In how many ways can the groups be formed? [2]

6 In a game at the carnival, a player rolls discs onto a board containing squares, each of which bears one of the numbers 1, 2, 5 or 10. If a disc does not land within a square, the player receives nothing. The probability that the disc does not land within the square is $\dfrac{3}{4}$. If a disc lands within a square, the player receives the same amount (in dollars) as the number in the square. Given that a disc falls within a square, the probabilities of landing within a square with the numbers 1, 2, 5 and 10 are 0.5, 0.3, 0.12 and 0.08 respectively. It is assumed that the rolls of the discs are independent.

(i) A player pays \$5 to play the game and is given n discs. Find n if the game is fair. [4]

(ii) If a player is allowed to roll 3 discs for \$2, find the probability that the player will have a profit of \$10. [4]

7 A factory manufactures large number of pen refills. From past records, 3% of the refills are defective. A stationery store manager wishes to purchase pen refills from the factory. To decide whether to accept or reject a batch of refills, the manager designs a sampling process. He takes a random sample of 25 refills. The batch is accepted if there is no defective refill and rejected if there are more than 2 defective refills. Otherwise, a second random sample of 25 refills is taken. The batch is then accepted if the total number of defective refills in the two samples is fewer than 4 and rejected otherwise.

(i) Find the probability of accepting a batch. [4]

(ii) If a batch is accepted, find the probability that there are 2 defective refills found in the sampling process. [3]

The stationery store manager purchases 50 boxes of 25 refills each.

(iii) Find the probability that the mean number of defective refills in a box is less than 1. [2]

8 A study is done to find out the relationship between the age of women and the steroid levels in the blood plasma. Sample data collected from 10 females with ages ranging from 8 years old to 35 years old is as shown below.

Age (years) x	8	11	14	17	20	23	26	29	32	35
Steroid Level (mmol/litre) L	4.2	11.1	16.3	19.0	25.5	26.2	24.1	33.5	20.8	17.4

(i) Give a sketch of the scatter diagram for the data. Identify the outlier and suggest a reason, in the context of the question, why this data pair is an outlier. [3]

For the remaining part of the question, the outlier is to be removed from the calculation.

(ii) Comment on the suitability of each of the following models. Hence determine the best model for predicting the steroid level of a female based on her age.

Model A: $L = a + b\ln x$

Model B: $L = c + d(x-25)^2$

Model C: $L = e + f(x-25)^4$

where a, b, c, d, e and f are constants. [3]

(iii) Using the best model in (ii), estimate the steroid level of a woman at age 40. Comment on the reliability of your estimate. [3]

(iv) It is known that body muscle mass and steroid level has a linear correlation. The muscle mass percentage m % of the 9 females were measured. An additional female, Jane, participated in the study. Jane has her muscle mass percentage and steroid level measured. The mean muscle mass percentage of the 10 females is now found to be 26.28 %. The equation of the least squares regression line of m on L for the 10 pairs of data is

$$m = 2.22 + 1.25L.$$

Calculate Jane's steroid level. [3]

9 A flange beam is a steel beam with a "H"-shaped cross section, and is used as a supporting structure in construction and civil engineering. A factory manufactures both Grade X and Grade Y flange beams. The load that can be supported by a Grade X flange beam follows a normal distribution with mean 2.43×10^5 kN and standard deviation 4.5×10^4 kN. The load that can be supported by a Grade Y flange beam is 1.5 times of the load that can be supported by a Grade X flange beam.

(i) Find the probability that the combined load that can be supported by two randomly chosen Grade Y flange beams is within 1×10^4 kN of the combined load that can be supported by three randomly chosen Grade X flange beams. [4]

(ii) A construction company wants to buy 100 sets of three Grade X flange beams. Find the probability that fewer than 95 of these sets can support more than 6×10^5 kN. [3]

The company decides to place an order with the factory for a custom-made flange beam such that the probability of being able to support a load of at least 6×10^5 kN must be at least 0.999. It is also assumed that the load that can be supported by the custom-made flange beam also follows a normal distribution.

(iii) By taking the standard deviation of a custom-made flange beam to be 3×10^4 kN, find the smallest possible mean load in kN, giving your answer correct to the nearest thousand, for the factory to meet the company's requirements for the custom-made flange beam. [5]

10 **(a)** College students intending to further their studies overseas have to sit for a mandatory Overseas Universities Test (OUT). Researcher Mr Anand wishes to find out if male college students tend to score higher for OUT compared to female college students. Mr Anand's colleague randomly selects 150 male and 150 female students from the combined student population of three particular colleges near his home to form a sample of 300 college students for the research. Explain whether this sample is a random sample. [2]

(b) The mean OUT score for all college students in 2016 is 66.
Mr Anand randomly selects 240 college students taking OUT in 2017 and their scores, x, are summarised in the following table:

Score, x	60	65	68	70	75	80
Frequency, f	40	90	63	27	18	2

(i) Write down the unbiased estimates of the population mean and variance of the OUT scores for the college students in 2017. [1]

(ii) Test, at the 10% level of significance, whether the mean OUT score for all college students in 2017 is higher than the mean score attained in 2016. [4]

(iii) Explain what is meant by the phrase "10% level of significance" in this context. [1]

(iv) Mr Anand draws a new sample of 240 male college students. Using the unbiased estimate for the population variance computed in **(i)**, find the range of values for the sample mean $\overline{x}$ that is required for this new sample to achieve a different conclusion from that in **(ii)**. [4]

(c) The 2017 OUT scores of the male and female college students are independent and assumed to be normally distributed with means and standard deviations as shown in the following table:

	Mean	Standard deviation
Male College Students	64	5.5
Female College Students	66	3.5

Mr Beng and Miss Charlene both scored 70. Explain who performed better relative to their respective gender cohort. [2]

Section A: Pure Mathematics [40 marks]

1 Given that $\sin\left[(n+1)x\right]-\sin\left[(n-1)x\right]=2\cos nx\sin x$, show that

$$\sum_{r=1}^{n}\cos rx=\frac{\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\quad .$$ [4]

Hence express

$$\cos^2\left(\frac{x}{2}\right)+\cos^2(x)+\cos^2\left(\frac{3x}{2}\right)+\ldots+\cos^2\left(\frac{11x}{2}\right)$$

in the form $a\left(\dfrac{\sin bx}{\sin cx}+d\right)$, where a,b,c and d are real numbers. [3]

Given: $2\cos nx\sin x=\sin(n+1)x-\sin(n-1)x$ Thus, $2\cos x\ \sin x=\sin 2x-\sin 0x$ $2\cos 2x\sin x=\sin 3x-\sin x$ $2\cos 3x\sin x=\sin 4x-\sin 2x$ $\ldots$ $2\cos(n-2)x\sin x=\sin(n-1)x-\sin(n-3)x$ $2\cos(n-1)x\sin x=\sin(n)x-\sin(n-2)x$ $2\cos nx\sin x=\sin(n+1)x-\sin(n-1)x$ Adding the n equations above, we have $2\sin x\sum_{r=1}^{n}\cos rx=\sin(n+1)x+\sin nx-\sin x$ $2\sin x\sum_{r=1}^{n}\cos rx=2\sin\left(\frac{2n+1}{2}\right)x\cos\frac{1}{2}x-\sin x$ $\cancel{2}\left(2\sin\frac{1}{2}x\cancel{\cos\frac{1}{2}x}\right)\sum_{r=1}^{n}\cos rx=\cancel{2}\sin\left(n+\frac{1}{2}\right)x\cancel{\cos\frac{1}{2}x}-\cancel{2}\sin\frac{1}{2}x\cancel{\cos\frac{1}{2}x}$ $2\sin\frac{1}{2}x\sum_{r=1}^{n}\cos rx=\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x$ $\therefore\ \sum_{r=1}^{n}\cos rx=\dfrac{\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$ (Shown)	

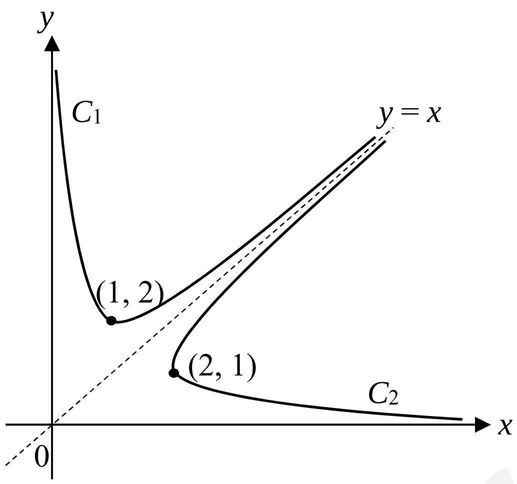
$\cos^2\left(\frac{x}{2}\right)+\cos^2(x)+\cos^2\left(\frac{3x}{2}\right)+...+\cos^2\left(\frac{11x}{2}\right)$ $=\frac{1+\cos x}{2}+\frac{1+\cos 2x}{2}+\frac{1+\cos 3x}{2}+...+\frac{1+\cos 11x}{2}$ $=\frac{1}{2}\left(11+\sum_{r=1}^{11}\cos rx\right)$ $=\frac{1}{2}\left(11+\frac{\sin\left(11+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\right)$ $=\frac{1}{2}\left(11+\frac{\sin\frac{23}{2}x}{2\sin\frac{1}{2}x}-\frac{1}{2}\right)$ $=\frac{1}{2}\left(\frac{\sin\frac{23}{2}x}{2\sin\frac{1}{2}x}+\frac{21}{2}\right)=\frac{1}{4}\left(\frac{\sin\frac{23}{2}x}{\sin\frac{1}{2}x}+21\right)$	

Marker's comments

Most students were able to make use of the given result and apply the method of differences to solve for $\sum_{r=1}^{n}\cos rx=\sum_{r=1}^{n}\frac{\sin(r+1)x-\sin(r-1)x}{2\sin x}=\frac{\sin(n+1)x+\sin nx-\sin x}{2\sin x}$. Thereafter, many students fail to apply the appropriate factor formula and double-angle formula to obtain the desired answer.

The second part of the question involves the use of double-angle formula to convert $\cos^2\left(\frac{r}{2}\right)$ into $\frac{\cos(rx)+1}{2}$, but many students chose to replace the index r by $\frac{r}{2}$, which would not allow them to achieve anything. Some students lost credit by failing to express their answer in the form as stated in the question.

2(a)



The diagram above shows two curves C_1 and C_2 which are reflections of each other about the line $y = x$. State with justification, whether the following statement is true:

"If C_1 is the graph of $y = \mathrm{f}(x)$, then C_2 is the graph of $y = \mathrm{f}^{-1}(x)$." [1]

(b) The functions f and g are defined as follows

$$\mathrm{f} : x \mapsto \frac{1}{x^2 - x - 6}, \quad x \in \mathbb{R}, x < 0,$$

$$\mathrm{g} : x \mapsto \tan^{-1}\left(\frac{x}{2}\right), \quad x \in \mathbb{R}.$$

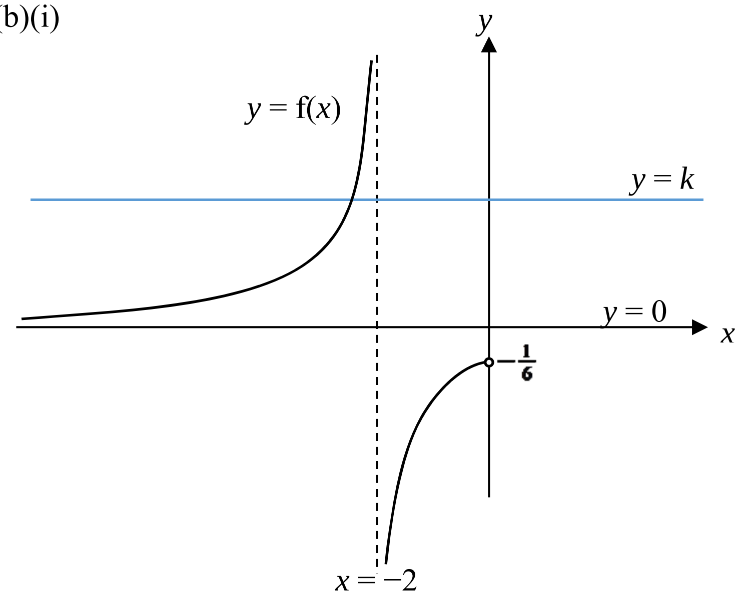
(i) Sketch the graph of $y = \mathrm{f}(x)$. Determine whether f^2 exists. [3]

(ii) Find $\mathrm{f}^{-1}(x)$. [2]

(iii) Given that $\mathrm{gf}(a) = \dfrac{\pi}{4}$, find the exact value of a. [2]

(a) From the graph of $y = \mathrm{f}(x)$ which is C_1, there exists a horizontal line $y = 3$ which cuts the graph of $y = \mathrm{f}(x)$ at 2 points. f is not one to one and thus f^{-1} does not exist. Since f^{-1} does not exist, C_2 is not the graph of $\mathrm{f}^{-1}(x)$.	

(b)(i)



Since $R_f = \left(-\infty, -\frac{1}{6}\right) \cup (0, \infty)$

and $D_f = (-\infty, 0)$

i.e. $R_f \not\subset D_f$

Therefore, f^2 does not exist.

(b)(ii) Let $y = \dfrac{1}{x^2 - x - 6}$

$$\Rightarrow \quad yx^2 - xy - 6y - 1 = 0$$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2 + 4y(6y+1)}}{2y}$$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{25y^2 + 4y}}{2y}$$

Since $x < 0$, $x = \dfrac{y - \sqrt{25y^2 + 4y}}{2y} = \dfrac{1}{2} - \dfrac{\sqrt{25y^2 + 4y}}{2y}$

Thus, $f^{-1}(x) = \dfrac{1}{2} - \dfrac{\sqrt{25x^2 + 4x}}{2x}$.

(b)(iii) $\text{gf}(a) = \dfrac{\pi}{4}$ $\tan^{-1}\left(\dfrac{\text{f}(a)}{2}\right) = \dfrac{\pi}{4}$ $\dfrac{\text{f}(a)}{2} = 1$ $\text{f}(a) = 2 \quad \Rightarrow \quad a = \text{f}^{-1}(2)$ $\therefore a = \dfrac{1}{2} - \dfrac{\sqrt{25(2)^2 + 4(2)}}{2(2)} = \dfrac{1}{2} - \dfrac{3\sqrt{3}}{2}$ **Alternative solution** $\text{gf}(a) = \dfrac{\pi}{4}$ $\tan^{-1}\left(\dfrac{\text{f}(a)}{2}\right) = \dfrac{\pi}{4}$ $\Rightarrow \quad \dfrac{\text{f}(a)}{2} = 1$ $\Rightarrow \quad \text{f}(a) = 2$ $\Rightarrow \quad \dfrac{1}{a^2 - a - 6} = 2$ $\Rightarrow \quad 2a^2 - 2a - 13 = 0$ $\Rightarrow \quad a = \dfrac{2 \pm \sqrt{4 + 4(2)(13)}}{4} = \dfrac{2 \pm 6\sqrt{3}}{4} = \dfrac{1}{2} \pm \dfrac{3}{2}\sqrt{3}$ Since $a < 0$, $a = \dfrac{1}{2} - \dfrac{3}{2}\sqrt{3}$	

Marker's comments

(a) This part is generally well-answered by students who recognised that f is not 1-1 and so the inverse cannot exist. Students who gave no or incorrect justification to why the statement is false fail to gain any credit.

(b) The graph of $y = \text{f}(x)$ is generally well-drawn and most students were able to present their sketches within the correct domain. The main issue for this part is that many students tried to justify whether f^2 exists or not in relation to whether the inverse function exists or not, showing a misconception between composite and inverse functions. Many students proceeded to obtain full credits for the desired results of parts (ii) and (iii), but students need to first be aware that no credit was deducted when the domain or range was presented wrongly.

3 Given that $e^y = \sqrt{e + x + \sin x}$. Show that

$$2e^{2y}\frac{d^2y}{dx^2} + 4e^{2y}\left(\frac{dy}{dx}\right)^2 + \sin x = 0 . \qquad [2]$$

(i) Find the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 0$. Hence, find in terms of e, the Maclaurin's series for $\ln(e + x + \sin x)$, up to and including the term in x^2. [4]

(ii) By using appropriate standard series expansions from the List of Formulae (MF26), verify the correctness of the first three terms in the series expansion for $\ln(e + x + \sin x)$ found in part **(i)**. [3]

(iii) Use your answer to part **(i)** to give an approximation for $\int_0^{e^{-1}} \frac{2e - 4x}{e^2 \ln(e + x + \sin x)}\,dx$, giving your answer in terms of e. [3]

$e^y = \sqrt{e + x + \sin x}$ $\Rightarrow \quad e^{2y} = e + x + \sin x$ Differentiate wrt x: $e^{2y}\left(2\frac{dy}{dx}\right) = 1 + \cos x$ i.e., $\quad 2e^{2y}\frac{dy}{dx} = 1 + \cos x$ Differentiate wrt x: $2\left[e^{2y}\frac{d^2y}{dx^2} + \frac{dy}{dx}e^{2y}\left(2\frac{dy}{dx}\right)\right] = -\sin x$ i.e., $\quad 2e^{2y}\frac{d^2y}{dx^2} + 4e^{2y}\left(\frac{dy}{dx}\right)^2 + \sin x = 0$ (Shown)	Square both sides first. Do not use the tedious method of working out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ directly from given equation.

(i) When $x=0$, $\mathrm{e}^{2y}=\mathrm{e}+0+0 \Rightarrow y=\dfrac{1}{2}$ $2\mathrm{e}\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+1 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\mathrm{e}}$ $2\mathrm{e}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\mathrm{e}\left(\dfrac{1}{\mathrm{e}}\right)^2+0=0 \Rightarrow \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{-2}{\mathrm{e}^2}$ $\therefore \quad y=\dfrac{1}{2}+\dfrac{1}{\mathrm{e}}x-\dfrac{2}{\mathrm{e}^2}\left(\dfrac{x^2}{2!}\right)+\cdots$ $\mathrm{e}^{2y}=\mathrm{e}+x+\sin x$ $\Rightarrow \ln(\mathrm{e}+x+\sin x)=2y$ $=2\left(\dfrac{1}{2}+\dfrac{1}{\mathrm{e}}x-\dfrac{2}{\mathrm{e}^2}\dfrac{x^2}{2!}+\cdots\right)$ i.e., $\ln(\mathrm{e}+x+\sin x)=1+\dfrac{2}{\mathrm{e}}x-\dfrac{2}{\mathrm{e}^2}x^2+\cdots$	This series expansions is for y, and **not** for $\ln(\mathrm{e}+x+\sin x)$ or e^{2y}.
(ii) $\ln(\mathrm{e}+x+\sin x)$ $=\ln(\mathrm{e}+x+x+\ldots)$ $=\ln(\mathrm{e}+2x+\ldots)$ $=\ln\left(\mathrm{e}\left(1+\dfrac{2}{\mathrm{e}}x+\ldots\right)\right)$ $=\ln\mathrm{e}+\ln\left(1+\dfrac{2}{\mathrm{e}}x+\ldots\right)$ $=1+\ln\left(1+\dfrac{2}{\mathrm{e}}x+\ldots\right)$ $=1+\left[\left(\dfrac{2}{\mathrm{e}}x\right)-\dfrac{1}{2}\left(\dfrac{2}{\mathrm{e}}x\right)^2+\cdots\right]$ $=1+\dfrac{2}{\mathrm{e}}x-\dfrac{2}{\mathrm{e}^2}x^2+\cdots$ (Verified)	Apply the following standard series expansions. $\sin x = x+\ldots$ and $\ln(1+x)=x-\dfrac{x^2}{2}+\ldots$. ($x^3$ term can be ignored as the result in part (i) is only up to x^2 term). *Note:* $\ln(\mathrm{e}+x+\sin x)\neq \ln\mathrm{e}+\ln x+\ln\sin x$ $\ln(\mathrm{e}+x+\sin x)\neq (\ln\mathrm{e})\ln(x+\sin x)$, $\ln(\mathrm{e}+x+\sin x)\neq \ln\left(1+\dfrac{x}{\mathrm{e}}+\dfrac{\sin x}{\mathrm{e}}\right)$ You can use this (ii) result to check whether you make mistakes in part (i) or (ii) if the results are different.

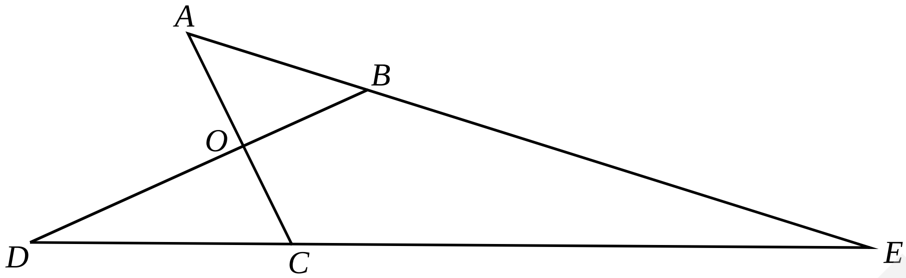
(iii) $\int_0^{e^{-1}} \frac{2e-4x}{e^2\ln(e+x+\sin x)}\,dx$ $\approx \int_0^{e^{-1}} \frac{2e-4x}{e^2\left(1+\frac{2}{e}x-\frac{2}{e^2}x^2\right)}\,dx$ $= \int_0^{e^{-1}} \frac{2e-4x}{e^2+2ex-2x^2}\,dx$ $=\left[\ln\left(e^2+2ex-2x^2\right)\right]_0^{\frac{1}{e}}$ $=\ln\left(e^2+2-2\left(\frac{1}{e}\right)^2\right)-\ln\left(e^2\right)$ $=\ln\left(e^2+2-\frac{2}{e^2}\right)-2$ $=\ln\left(e^4+2e^2-2\right)-4$	Use $\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + c$

Marker's comments

About 20% of the students use tedious method to show first part.

Badly done for part (ii). Many students leave blank for this part. For those who tried, many of them get different answers for (i) and (ii), and still wrote (verified). They should use it to check their own mistakes.

4



With reference to origin O, the points A, B, C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = -\mathbf{a}$ and $\overrightarrow{OD} = -2\mathbf{b}$. The lines AB and DC meet at E.

Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

Hence show that $\dfrac{BE}{AB} = 3$. [1]

It is given that A and E have coordinates $(1, -4, 3)$ and $(-3, 15, -5)$ respectively.

(i) Show that the lines AC and BD are perpendicular. [4]

(ii) Find the equation of the plane p that contains E and is perpendicular to the line BD. [2]

(iii) Find the distance between the line AC and p. [2]

Equation of line *AB* is $\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$. Equation of line *DC* is $\underset{\sim}{r} = -\underset{\sim}{a} + \mu(-2\underset{\sim}{b} - (-\underset{\sim}{a}))$, i.e., $\underset{\sim}{r} = -\underset{\sim}{a} + \mu(-2\underset{\sim}{b} + \underset{\sim}{a})$. To find *E*, the point of intersection of lines *AB* and *CD*, consider $\underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a}) = -\underset{\sim}{a} + \mu(-2\underset{\sim}{b} + \underset{\sim}{a})$ $\Rightarrow \quad (1-\lambda)\underset{\sim}{a} + \lambda\underset{\sim}{b} = (-1+\mu)\underset{\sim}{a} - 2\mu\underset{\sim}{b}$ $\Rightarrow \quad (2-\lambda-\mu)\underset{\sim}{a} = (-2\mu-\lambda)\underset{\sim}{b}$ Since $\underset{\sim}{a}$ is not parallel to $\underset{\sim}{b}$, $\begin{cases} 2-\mu-\lambda=0 & \cdots(1) \\ -2\mu-\lambda=0 & \cdots(2) \end{cases}$ Solving (1) and (2), we have $\mu=-2$ and $\lambda=4$ $\therefore \overrightarrow{OE} = \underset{\sim}{a} + 4(\underset{\sim}{b} - \underset{\sim}{a}) = -3\underset{\sim}{a} + 4\underset{\sim}{b}$ $\therefore \overrightarrow{BE} = \overrightarrow{OE} - \overrightarrow{OB} = -3\underset{\sim}{a} + 4\underset{\sim}{b} - \underset{\sim}{b} = 3(\underset{\sim}{b} - \underset{\sim}{a}) = 3\overrightarrow{AB}$ $\therefore \dfrac{BE}{AB} = 3$	Students must know that there is no such things as $\dfrac{vector}{vector}$. In this case, students who have written $\dfrac{3(\mathbf{b}-\mathbf{a})}{(\mathbf{b}-\mathbf{a})} = 3$ will not be given any credit.
(i) $\quad \overrightarrow{OE} = -3\underset{\sim}{a} + 4\underset{\sim}{b} = \begin{pmatrix} -3 \\ 15 \\ -5 \end{pmatrix}$ $\Rightarrow \quad -3\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + 4\underset{\sim}{b} = \begin{pmatrix} -3 \\ 15 \\ -5 \end{pmatrix}$ $\Rightarrow \quad 4\underset{\sim}{b} = \begin{pmatrix} -3 \\ 15 \\ -5 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \quad \Rightarrow \quad \underset{\sim}{b} = \begin{pmatrix} 0 \\ \frac{3}{4} \\ 1 \end{pmatrix}$ $\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \frac{3}{4} \\ 1 \end{pmatrix} = -4\left(\dfrac{3}{4}\right) + 3 = 0$ $\Rightarrow$ *OA* and *OB* are perpendicular $\Rightarrow$ *AC* and *BD* are perpendicular (as *AC* is parallel to *OA* and *BD* is parallel to *OB*)	Students must give clear explanation for every step. In this case, students must explain clearly why $OA \perp OB$ implies $AC \perp BD$.

(ii) Equation of the plane p is $\underset{\sim}{r} \cdot \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 15 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$, i.e. $\underset{\sim}{r} \cdot \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = 25$	
(iii) Distance between the line AC and the plane p = distance of O from p $= \dfrac{\begin{pmatrix} -3 \\ 15 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}}{\sqrt{3^2 + 4^2}} = 5$	
Marker's comments The first part of this question is badly done. Students must know that problems involving vectors that are not given in the column vector way are very common in this syllabus. This question is just one example which requires you to find the intersection between two lines, in which position vectors of points on the lines are as generic vectors **a** and **b**. Students are advised to do more such practices from MSM and all other vectors revision resources that are given out.	

Section B: Statistics [60 marks]

5 Four classes CG40, CG41, CG42 and CG43 are tasked to organise a College event. Each class sends 3 representatives for a meeting.

(i) In how many different ways can the 12 representatives sit in a circle so that representatives from CG40 are not seated next to each other and representatives from other classes are seated with their respective classes? [3]

The 12 representatives are to be split up into 3 groups for bonding activities. Each group must consist of a representative from each class.

(ii) In how many ways can the groups be formed? [2]

(i) Number of ways to arrange the 3 classes except CG40 $=(3!)^3(3-1)!$ Number of ways to arrange reps from CG40 for a particular arrangement of the other 3 classes = 3! Required number $=(3!)^3(3-1)!3!=2592$	
(ii) Required number $=\dfrac{(3!)^4}{3!}=216$	

Marker's comments

Students are advised to present their working for P&C questions clearly, step by step. Many students are not able to get any credit at all for this question because their answer is a one-liner answer and they got the answer wrong. Such students may be able to at least obtain one or two marks if they have presented and explained their working more clearly.

6 In a game at the carnival, a player rolls discs onto a board containing squares, each of which bears one of the numbers 1, 2, 5 or 10. If a disc does not land within a square, the player receives nothing. The probability that the disc does not land within the square is $\frac{3}{4}$. If a disc lands within a square, the player receives the same amount (in dollars) as the number in the square. Given that a disc falls within a square, the probabilities of landing within a square with the numbers 1, 2, 5 and 10 are 0.5, 0.3, 0.12 and 0.08 respectively. It is assumed that the rolls of the discs are independent.

(i) A player pays \$5 to play the game and is given n discs. Find n if the game is fair. [4]

(ii) If a player is allowed to roll 3 discs for \$2, find the probability that the player will have a profit of \$10. [4]

(i) Let Y (in dollars) be the amount received by a player for each roll.

y	0	1	2	5	10
$P(Y=y)$	$\frac{3}{4}$	$\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$	$\frac{1}{4}\times\frac{3}{10}=\frac{3}{40}$	$\frac{1}{4}\times\frac{3}{25}=\frac{3}{100}$	$\frac{1}{4}\times\frac{8}{100}=\frac{1}{50}$

For DRV questions, it is often useful to write out the probability distribution table. You should check that the probabilities add up to 1.

$$E(Y) = \left(0\times\frac{3}{4}\right)+\left(1\times\frac{1}{8}\right)+\left(2\times\frac{3}{40}\right)+\left(5\times\frac{3}{100}\right)+\left(10\times\frac{1}{50}\right) = 0.625$$

For the game to be fair,

$$E(Y_1+Y_2+\cdots+Y_n-5)=0$$
$$\Rightarrow \quad nE(Y)-5=0$$
$$\Rightarrow \quad 0.625n=5$$
$$\Rightarrow \quad n=8$$

Since there are n discs, for the game to be fair, the total expected earnings from the n discs minus the cost of 1 game must be 0.

(ii) Pay \$2 for 3 rolls with a gain of \$10 implies that the player needs to receive \$12 from 3 rolls.

Required probability

$$= 3!\times P(Y_1=0,Y_2=2,Y_3=10) + 3\times P(Y_1=1,Y_2=1,Y_3=10) + 3\times P(Y_1=2,Y_2=5,Y_3=5)$$

$$= 3!\times\left(\frac{3}{4}\right)\left(\frac{3}{40}\right)\left(\frac{1}{50}\right)+3\times\left(\frac{1}{8}\right)^2\left(\frac{1}{50}\right)+3\times\left(\frac{3}{40}\right)\left(\frac{3}{100}\right)^2$$

= 0.0789 (exact)

There are 3 cases to gain a total \$12 from 3 discs, and students should consider the order of appearance of the different scores.

Marker's comments

Common Mistakes:

Part (i):

1. Many students were unable to understand the statement "**<u>Given that</u>** a disc falls within a square…". They should realise that it is a conditional probability, which can be easily visualised using a tree diagram.
2. It is incorrect to subtract \$5 off from the score of each disc directly. This is because it implies that every disc thrown costs \$5, which is incorrect since *n* discs costs \$5 (fixed).
3. In general, we say that $\mathrm{E}(X)=0$ when a game is fair, instead of $\mathrm{E}(X)\geq 0$ which many students wrote. $\mathrm{E}(X)\geq 0$ in this question implies that the player is expected to win more than \$5, which is unfair for the game stall owner.

Part (ii):

4. Many students failed to consider the order of appearance of the scores. Many were also unable to consider all cases which could lead to \$12. Students who did not realise that the player must receive \$12 should read the question carefully.

7 A factory manufactures large number of pen refills. From past records, 3% of the refills are defective.

A stationery store manager wishes to purchase pen refills from the factory. To decide whether to accept or reject a batch of refills, the manager designs a sampling process. He takes a random sample of 25 refills. The batch is accepted if there is no defective refill and rejected if there are more than 2 defective refills. Otherwise, a second random sample of 25 refills is taken. The batch is then accepted if the total number of defective refills in the two samples is fewer than 4 and rejected otherwise.

(i) Find the probability of accepting a batch. [4]

(ii) If a batch is accepted, find the probability that there are 2 defective refills found in the sampling process. [3]

The stationery store manager purchases 50 boxes of 25 refills each.

(iii) Find the probability that the mean number of defective refills in a box is less than 1. [2]

Let X be the number of defective refills in the sample of 25 refills drawn from a batch which contains 3% defective refills. Then, $X \sim \mathrm{B}(25, 0.03)$ (i) P(accepting a batch) $= \mathrm{P}(X=0)+\mathrm{P}(X=1)\mathrm{P}(X \le 2)+\mathrm{P}(X=2)\mathrm{P}(X \le 1)$ = 0.4669747053 + 0.3473570958 + 0.1109593034 ≈ 0.9252911 = 0.925 (correct to 3 s.f.)	The cases in which the batch can be accepted should be thought through carefully.
(ii) Required probability = P(2 defective refills | batch is accepted) $= \dfrac{\mathrm{P}(X_1=1)\mathrm{P}(X_2=1)+\mathrm{P}(X_1=2)\mathrm{P}(X_2=0)}{0.9252911}$ =0.209 (correct to 3 s.f.)	The question is asking for the conditional probability of having 2 defective refills given that the batch is accepted.

(iii) $X \sim B\ (25,\ 0.03)$ Since *sample size* = 50 is large, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(25(0.03), \frac{25(0.03)(0.97)}{50}\right)$ approximately $\bar{X} \sim \mathrm{N}(0.75, 0.1455)$ Required probability $= P(\bar{X} < 1) = 0.981$ (correct to 3 s.f.) **Alternative solution** $X_1 + \ldots + X_{50} \sim \mathrm{B}(50 \times 25, 0.03)$ $X_1 + \ldots + X_{50} \sim \mathrm{B}(1250, 0.03)$ $\mathrm{P}(X_1 + \ldots + X_{50} < 50)$ $= \mathrm{P}(X_1 + \ldots + X_{50} \le 49)$ $= 0.973$	In general, when the question asks for a mean number of *X* when *X* is a discrete random variable, students should consider applying Central Limit Theorem. This is especially so if the question has keywords such as "approximate/estimate the probability".

Marker's comments

Part (i):

1. Quite a large number of students did not understand the first line and hence did not realise that the number of defective refills follow a Binomial Distribution. This leads to an attempt to list out all the cases manually. While computing the individual cases, most students using this approach did not consider the order of appearance of the "defective" refills (as per Binomial formula).
2. For students who considered the Binomial Distribution, many did not understand the selection process if a second batch is required. Many took the question at face value, i.e. $\mathrm{P}(1 \le X_1 \le 2)\cdot \mathrm{P}(X_1 + X_2 < 4)$. Students need to realise that the number of defects in the first sample affects the allowable number of defects in the second sample.

Part (ii):

3. Apart from not realising that the question is asking for the conditional probability, many students were unable to identify the cases of having P(2 defective refills $\cap$ batch is accepted). They either forgot that we can have $\mathrm{P}\left(X_1 = 2\right)\mathrm{P}\left(X_2 = 0\right)$, or thought that $\mathrm{P}\left(X_1 = 0\right)\mathrm{P}\left(X_2 = 2\right)$ was possible. The latter is not possible because if $\mathrm{P}(X_1 = 0)$, there the sample would have been accepted immediately and a second sample would not be taken.

Part (iii):

4. The computation of the parameters for $\overline{X}$ was poorly done. There was a lot of confusion about what n is. In this case, X ~ B (25, 0.03) and we have 50 samples. Hence $X_1 + \ldots + X_{50} \sim N(50\times 25\times 0.03, 50\times 25\times 0.03\times 0.97)$ approx. by CLT, and hence $\overline{X} = \dfrac{X_1 + \ldots + X_{50}}{50} \sim N(25\times 0.03, \dfrac{25\times 0.03\times 0.97}{50})$.

8 A study is done to find out the relationship between the age of women and the steroid levels in the blood plasma. Sample data collected from 10 females with ages ranging from 8 years old to 35 years old is as shown below.

Age (years) x	8	11	14	17	20	23	26	29	32	35
Steroid Level (mmol/litre) L	4.2	11.1	16.3	19.0	25.5	26.2	24.1	33.5	20.8	17.4

(i) Give a sketch of the scatter diagram for the data. Identify the outlier and suggest a reason, in the context of the question, why this data pair is an outlier. [3]

For the remaining part of the question, the outlier is to be removed from the calculation.

(ii) Comment on the suitability of each of the following models. Hence determine the best model for predicting the steroid level of a female based on her age.

Model A: $L = a + b\ln x$

Model B: $L = c + d\left(x - 25\right)^2$

Model C: $L = e + f\left(x - 25\right)^4$

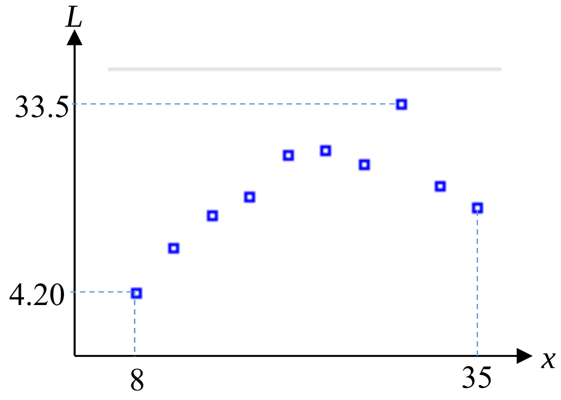
where a, b, c, d, e and f are constants. [3]

(iii) Using the best model in **(ii)**, estimate the steroid level of a woman at age 40. Comment on the reliability of your estimate. [3]

(iv) It is known that body muscle mass and steroid level has a linear correlation. The muscle mass percentage m % of the 9 females were measured. An additional female, Jane, participated in the study. Jane has her muscle mass percentage and steroid level measured. The mean muscle mass percentage of the 10 females is now found to be 26.28 %. The equation of the least squares regression line of m on L for the 10 pairs of data is

$$m = 2.22 + 1.25L.$$

Calculate Jane's steroid level. [3]

(i)  Outlier is x = 29, L = 33.5 because from age 23 onwards, there is a decreasing steroid level as age increases. However, at x = 29, the steroid level suddenly increases and this could be due to reasons such as illness/medication/pregnancy/intake of additional steroids by athlete/..etc (give any one of these reasons)	Students need to take note of what to indicate on scatter diagram: - Label of axes - Spacing and different in "height" between data points - Label min and max values The identification of outlier cannot be just circling of the point. Student must clearly states the x and L value of the outlier. Explanation of why (29, 33.5) is an outlier must be provided with a suggested possible <u>contextual</u> reason as well as an explanation of the kind of data trend that is resulted from this reason.
(ii) Model A is not suitable because as x increases, L is either only increasing (if $b > 0$) or only decreasing (if $b < 0$) which does not resemble the data trend of x and L whereby L increases but when it reaches about 23 years old, the steroid level decreases. Model B and C have similar trend as the given data set when $d < 0$ and $f < 0$ respectively and both are suitable models. But for Model C, the r-value of L and $(x-25)^4$ is -0.929 And for model B, the r-value of L and $(x-25)^2$ is -0.987 Since for model B, the r value is closer to -1, therefore model B is a better model.	Suitability of model must take into consideration the difference between the model and the data trend, using appropriate (sign) of b, d and f. As it is not possible to gauge the steepness of gradient base on the data points in the scatter diagram, thus the use of steepness to decide on whether model B or C is better is not accepted. Calculation of r must omit the outlier.

(iii) Least squares regression line equation is $L = 25.238 - 0.073667(x-25)^2$ $L = 25.2 - 0.0737(x-25)^2$ When $x = 40$, $L = 25.2 - 0.0737(40-25)^2 \approx 8.7$ The prediction is unreliable because $x = 40$ is outside the data range of 8 to 35 years old.	Calculation of equation of regression line must omit the outlier. Student must note that they cannot use the command $Y_1(40)$ directly to find L as the GC treat $(x-25)^2$ as X. Answer should be left to 1 decimal place, following that in the given table of L values.
(iv) Since $\overline{m} = 2.22 + 1.25\overline{L}$, $26.28 = 2.22 + 1.25\overline{L}$ $\Rightarrow \overline{L} = 19.248$ $\sum_{i=1}^{10} L_i = 192.48$ and since $\sum_{i=1}^{9} L_i = 164.6$, therefore Jane's steroid level is $27.88 \approx 27.9$	Students must realise that even if they are able to find the value of Jane's muscle mass t be 37.07, they cannot substitute this value into the equation to find Jane's steroid level. So even if the answer obtained is also 27.9, they are wrongly assuming that the data point lies on the regression line. Only $(\overline{L}, \overline{m})$ lies on the regression line.

Marker's comments

(i) The scatter diagram is quite well drawn but the labelling of axes, minimum and maximum values of x and L are often left out or wrongly labelled. Many students are mainly describing the data trend and the high L level of the point (29. 33.5) and did not give a contextual reason on why the data trend is as such. On the other hand, another group of students gave a very brief contextual reason but did not provide any elaborate on what this reason led to.

(ii) Many students did not explore the different possible sign of b, d and f. Most students wrongly assume that b, d and f are positive and rejected model B and C. One serious mistake that some students made is that they associate the power n in the expression $(x-25)^n$ to the number of turning points that the graph has, not realising that for all even integer n, there is only one turning point. Many students also did not read the question instruction to comment on the suitability of **each** model, they mainly compute values of r for all 3 models and conclude the one best model.

(iii) Many students left the estimated value of L to 3 s.f. instead of 1 decimal place. Some forgot to write down the equation of the regression line. Some wrongly write the equation as $L = 25.2 - 0.0737x^2$. Many forgot to omit the outlier in both part (iii) and in (ii) when finding r value.

Although most students are able to answer this part correctly, their answers are rather vague. Phrasing such as "it is an extrapolation" or "It is within data range" is not acceptable as it is unclear whether the student is referring to x or L within data range. Students must also remember to answer the question using the given term "not reliable" instead of "not accurate".

(iv) This part is generally well done but the notation of $\overline{L}$ is often not used, many just write it as L even if their subsequent workings show that they know that the value 19.248 is the mean steroid level.

9 A flange beam is a steel beam with a “H”-shaped cross section, and is used as a supporting structure in construction and civil engineering. A factory manufactures both Grade X and Grade Y flange beams. The load that can be supported by a Grade X flange beam follows a normal distribution with mean 2.43×10^5 kN and standard deviation 4.5×10^4 kN. The load that can be supported by a Grade Y flange beam is 1.5 times of the load that can be supported by a Grade X flange beam.

(i) Find the probability that the combined load that can be supported by two randomly chosen Grade Y flange beams is within 1×10^4 kN of the combined load that can be supported by three randomly chosen Grade X flange beams. [4]

(ii) A construction company wants to buy 100 sets of three Grade X flange beams. Find the probability that fewer than 95 of these sets can support more than 6×10^5 kN. [3]

The company decides to place an order with the factory for a custom-made flange beam such that the probability of being able to support a load of at least 6×10^5 kN must be at least 0.999.

It is assumed that the load that can be supported by the custom-made flange beam also follows a normal distribution.

(iii) By taking the standard deviation of a custom-made flange beam to be 3×10^4 kN, find the smallest possible mean load in kN, giving your answer correct to the nearest thousand, for the factory to meet the company’s requirements for the custom-made flange beam. [5]

Let A and B be the load that can be supported (in kN) by a Grade X and Grade Y flange beam respectively. Then, $A \sim N\left(2.43\times10^5, \left(4.5\times10^4\right)^2\right)$.	Many students did not define the random variables. Some defined it wrongly and just wrote it as "Let A be the Grade X and B be the Grade Y." There are some who took $\left(4.5\times10^4\right)$ as the variance of A.
Since $B = 1.5A$, then $B \sim N\left(1.5\left(2.43\times10^5\right), 1.5^2\left(4.5\times10^4\right)^2\right)$ i.e., $B \sim N\left(3.645\times10^5, \left(6.75\times10^4\right)^2\right)$ leave in exact decimal!	Common mistakes for Var(B): 1. Var(B) = $\left(4.5\times10^4\right)^2$ 2. Var(B) = $1.5\left(4.5\times10^4\right)^2$
(i) Want to find $\mathrm{P}\left(\left\|(B_1+B_2)-(A_1+A_2+A_3)\right\| < 1\times10^4\right)$ $\mathrm{E}\left((B_1+B_2)-(A_1+A_2+A_3)\right)$ $= 2\times3.645\times10^5 - 3\times2.43\times10^5 = 0$ $\mathrm{Var}\left((B_1+B_2)-(A_1+A_2+A_3)\right)$ $= 2\times\left(6.75\times10^4\right)^2 + 3\left(4.5\times10^4\right)^2 = 1.51875\times10^{10}$	There are still students who wrote $2B-3A$ instead of $(B_1+B_2)-(A_1+A_2+A_3)$. Students are advised not to correct their working answer to 3 s.f especially if the exact decimal answer is obtained. For e.g. in this case, if students round of their answer for E(B) to 3.65×10^5, their answer for $\mathrm{E}\left((B_1+B_2)-(A_1+A_2+A_3)\right)$ is 1000 instead of 0.
i.e. $(B_1+B_2)-(A_1+A_2+A_3) \sim N\left(0, 1.51875\times10^{10}\right)$	Must always write down the distribution after finding expectation and variance!
Required probability $= \mathrm{P}\left(\left\|(B_1+B_2)-(A_1+A_2+A_3)\right\| < 1\times10^4\right)$ $= \mathrm{P}\left(-1\times10^4 < (B_1+B_2)-(A_1+A_2+A_3) < 1\times10^4\right)$ = 0.0647 (to 3 s.f.)	Some students do not understand what is meant by A within 1×10^4 kN of B. *Note:* In general, $\mathrm{P}\left(\left\|T\right\| < 1\times10^4\right) \neq \mathrm{P}\left(T < 1\times10^4\right) + \mathrm{P}\left(T > -1\times10^4\right)$

(ii) $A_1 + A_2 + A_3 \sim N\left(3\times 2.43\times 10^5, 3\left(4.5\times 10^4\right)^2\right)$ $\mathrm{P}\left(A_1 + A_2 + A_3 > 6\times 10^5\right) \approx 0.951045$ Let T be the number of sets (out of 100 sets) of three Grade X flange beams that can support more than 6×10^5 kN. Then, $T \sim \mathrm{B}(100,\ 0.951045)$ Required probability $= \mathrm{P}(T < 95)$ $= \mathrm{P}(T \le 94)$ $= 0.365$ (to 3 s.f.)	Some students missed out on the three in the "100 sets of three Grade X flange beams" and went to find $\mathrm{P}\left(A > 6\times 10^5\right)$ Did not change $\mathrm{P}(T < 95)$ to $\mathrm{P}(T \le 94)$ when using binomcdf
(iii) Let W be the load that can be supported (in kN) by a custom-made flange beam. Given: $W \sim N\left(\mu, \left(3\times 10^4\right)^2\right)$ $\mathrm{P}(W \ge 6\times 10^5) \ge 0.999$ $\Rightarrow 1 - \mathrm{P}(W < 6\times 10^5) \ge 0.999$ $\Rightarrow \mathrm{P}(W < 6\times 10^5) \le 0.001$ $\Rightarrow \mathrm{P}\left(Z \le \frac{6\times 10^5 - \mu}{3\times 10^4}\right) \le 0.001$ By GC, $\frac{6\times 10^5 - \mu}{3\times 10^4} \le -3.0902$ $6\times 10^5 - \mu \le -92760$ $-\mu \le -92760 - 6\times 10^5$ $\mu \ge 692760$ Thus, smallest mean = 693 kN (to nearest thousand)	Some denote the smallest possible mean as $\bar{x}$ which is incorrect as $\bar{x}$ always denote sample mean. A few standardised wrongly. Wrote $\frac{\mu - 6\times 10^5}{3\times 10^4}$. *Common mistake:* Took invNorm of the right area which is 0.999 instead of left area, 0.001 Some started off with inequality but ended up with an equation after taking invNorm.

Marker's comments

- Students are advised not to use Z to denote the random variable as Z denotes the Standard Normal Variable, i.e $Z \sim \mathrm{N}(0, 1)$
- Students are advised to use exact decimal workings answer or working answers with more decimal places to avoid loss of accuracy in their final answer.
- There are at least a few in each class who do not know how to correct their answer to the nearest thousand.
- For part(iii), students are advised not to use GC Table although the unknown is to be corrected to the nearest thousand. Students who use GC table but did not show their workings clearly do not get the full marks.

10(a) College students intending to further their studies overseas have to sit for a mandatory Overseas Universities Test (OUT). Researcher Mr Anand wishes to find out if male college students tend to score higher for OUT compared to female college students. Mr Anand's colleague randomly selects 150 male and 150 female students from the combined student population of three particular colleges near his home to form a sample of 300 college students for the research. Explain whether this sample is a random sample.[2]

(b) The mean OUT score for all college students in 2016 is 66.

Mr Anand randomly selects 240 college students taking OUT in 2017 and their scores, x, are summarised in the following table:

Score, x	60	65	68	70	75	80
Frequency, f	40	90	63	27	18	2

(i) Write down the unbiased estimates of the population mean and variance of the OUT scores for the college students in 2017. [1]

(ii) Test, at the 10% level of significance, whether the mean OUT score for all college students in 2017 is higher than the mean score attained in 2016. [4]

(iii) Explain what is meant by the phrase "10% level of significance" in this context.[1]

(iv) Mr Anand draws a new sample of 240 male college students. Using the unbiased estimate for the population variance computed in **(i)**, find the range of values for the sample mean $\bar{x}$ that is required for this new sample to achieve a different conclusion from that in **(ii)**. [4]

(c) The 2017 OUT scores of the male and female college students are independent and assumed to be normally distributed with means and standard deviations as shown in the following table:

	Mean	Standard deviation
Male College Students	64	5.5
Female College Students	66	3.5

Mr Beng and Miss Charlene both scored 70. Explain who performed better relative to their respective gender cohort. [2]

(a) Sample is non-random/biased since students from other colleges do not have any chance of being selected.	Need to mention that the probability of a student being selected into the sample is not the same for every student taking OUT in 2017 since students from other colleges do not have any chance of being selected.
(b)(i) Using GC, unbiased estimate of population mean, $\bar{x} = 66.391$ $= 66.4$ (to 3 s.f.) and unbiased estimate of population variance, $s^2 = 4.1048^2 = 16.8$ (to 3 s.f.)	A number of students forgot to square the value 4.1048.
(b)(ii) Let μ be the population mean OUT score of students in 2017 . $H_0 : \mu = 66$ $H_1 : \mu > 66$ Level of significance: 10% Test Statistic: $\dfrac{\bar{X}-\mu}{s/\sqrt{n}} \sim N(0,1)$ by Central Limit Theorem since $n = 240$ is large. Under H_0, with $\bar{x} = 66.391$, $s = 4.1048$, $n = 240$, we have $p = 0.0697$ Since p-value < 0.1, we reject H_0 There is sufficient evidence at the 10% level of significance to conclude that the mean OUT score of male college students is higher than 66.	
(iii) There is a probability of 0.1 of wrongly concluding that the mean OUT score of male college students is higher than 66.	10% chance or probability of 0.1

(iv) $H_0 : \mu = 66$ $H_1 : \mu > 66$ Level of significance: 10% Do not reject H_0, $p > 0.10$ $$\frac{\bar{x} - 66}{4.1048 / \sqrt{240}} < 1.28155$$ $\Rightarrow \quad \bar{x} > 66.3396$ $\therefore \quad \bar{x} > 66.3$ (to 3 s.f.)	A number of students wrote the critical value as -1.28155 (invnorm(0.10)), without paying attention to H_1.
(c) Let M and F be the OUT scores of male and female college students respectively Given: $M \sim N(64, 5.5^2)$ and $F \sim N(66, 3.5^2)$ $P(M \leq 70) = 0.86234$ $\Rightarrow$ Mr Beng is in the 86^{th} percentile of male students (or Mr Beng scored higher than 86% of the male cohort) $P(F \leq 70) = 0.87345$ $\Rightarrow$ Miss Charlene is in the 87^{th} percentile of female students $\therefore$ Miss Charlene performed better relative to her gender cohort.	

Marker's comments

(a) Many students were able to recognise that the sample is not random. However, many of them were not able to give precise explanation. Wrong responses included mentioning the proportion of males and females, abilities of students in colleges which were not apparent in the question.

(b) (i) This part requires students to write out the unbiased estimates of the population mean and variance upon entering the data into the GC. Many students were not able to retrieve the correct unbiased estimate of population variance, they wrote down the sample variance instead. A number of students applied the formulas to find the unbiased estimates using the statistics, some with more success obtaining the values, some used the wrong statistics or wrong formula and did not obtain the correct values.

(b) (ii) Most students gained full marks here. Those who did not get (b)(i) correct would have lost some marks but not all if they have written the correct hypotheses, and conclusion given in context.

(b) (iii) This part was badly done. Many students were not able to explain precisely the phrase "10% level of significance", some students seemed to have problem remembering the definition.

(b) (iv) Students have some grasp of what was required, there were many varied errors in setting up the inequality to achieve a different conclusion from (b)(ii).

(c) Many students attempted to answer this part with lengthy paragraphs about standard deviations of the distribution of OUT scores of the male and female students. Many failed to explain using percentiles or probabilities of Beng and Charlene scoring 70 marks and above/or below. A large number of students thought they were computing the probability of Beng/Charlene scoring 70 marks using the normalpdf function. They did not understand that the probability is defined as area under the normal curve and hence the value they obtained were not able to explain who performed better in their cohort.

2017 VJC Prelim Paper 1

1. Without using a calculator, solve the inequality $\dfrac{6x-13}{x^2-4} \geqslant 1$. [4]

2. The Singapore Utility Board charges the residential users based on the usage for electricity, water and gas. Electricity and gas are charged by kilowatt hour (kWh) used while water usage is charged by cubic meters (CuM). Below are the monthly utility statements for Mr Pandy from May to August 2017.

SP Utility Bill (May 2017)

Mr Pandy
Blk 20 Marine ...

Current month charges

Electricity	514 kWh	***.**
Water	18.8 CuM	**.**
Gas	134 kWh	**.**
Total		**$155.54**

SP Utility Bill (June 2017)

Mr Pandy
Blk 20 Marine ...

Current month charges

Electricity	309 kWh	***.**
Water	11.3 CuM	**.**
Gas	89 kWh	**.**
Total		**$ 94.99**

SP Utility Bill (July 2017)

Mr Pandy
Blk 20 Marine ...

Current month charges

Electricity	639 kWh	***.**
Water	21.7 CuM	**.**
Gas	108 kWh	**.**
Total		**$ 208.40**

SP Utility Bill (August 2017)

Mr Pandy
Blk 20 Marine ...

Current month charges

Electricity	555 kWh	***.**
Water	??? CuM	**.**
Gas	128 kWh	**.**
Total		**$ 184.84**

It is known that the unit costs for electricity, water and gas remain unchanged for May and June. The unit cost for electricity was increased by 20% with effect from July 2017, while the unit cost for gas and water remain unchanged.

(i) Calculate the unit cost for electricity, water and gas for June 2017, giving your answers correct to the nearest 4 decimal places. [3]

(ii) The water usage for August 2017 was not clearly printed on the bill. Using your answers in part **(i)**, calculate the water usage for August 2017 to the nearest CuM. [2]

3. It is given that

$$f(x)=\begin{cases}(x-2)^2-1, & \text{for } 0< x\leqslant 3,\\ x-3, & \text{for } 3<x\leqslant 6,\end{cases}$$

and that $f(x)=f(x+6)$ for all real values of x.

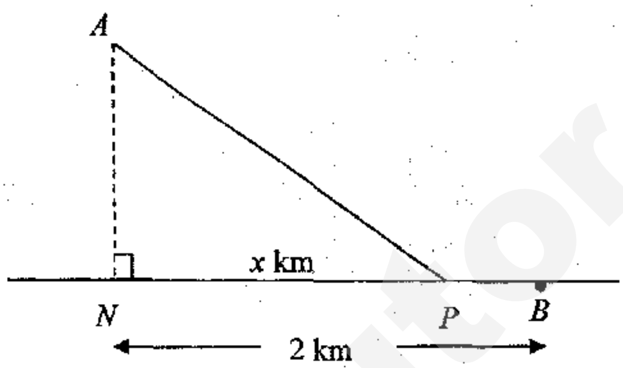
(i) Sketch the graph of $y=f(x)$ for $0<x\leqslant 10$. [3]

(ii) On a separate diagram, sketch the graph of $y = 1 + \mathrm{f}\left(\frac{1}{2}x\right)$ for $0 < x \leqslant 10$. [2]

4. The curve C has equation $(y+4)^2 - (x+3)^2 = 4$. Sketch C, giving the coordinates of any turning points and the equations of any asymptotes. [3]

Hence find the set of values of m such that the straight line with gradient m that passes through the point $(-3, -4)$ intersects C at least once. [2]

5.



Alvin is at the point A on a floating platform in the sea. He wants to reach point B located on a straight stretch of beach. N is the point on the beach nearest to A and NB = 2 km. Alvin swims at a constant speed in a straight line from A to P and then runs at a constant speed from P to B, where P is a point on the straight stretch of beach from N to B. $NP = x$ km and T minutes is the time taken for Alvin to complete the journey.

T and x satisfy the differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5.$$

(i) Solve the differential equation. [3]

(ii) Given that the minimum time taken for Alvin to complete this journey is 30 minutes, find T in terms of x. [3]

(iii) Using your answer in part **(ii)**, find the longest time taken by Alvin to complete the journey. [2]

6. The function h is defined by

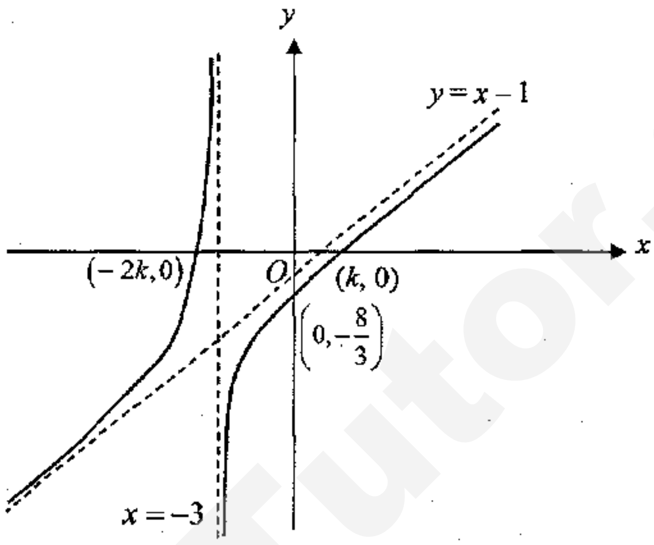
$$\mathrm{h}: x \mapsto \mathrm{e}^{x-2} - 1, \qquad \text{for } x \in \mathbb{R}.$$

(i) Find $\mathrm{h}^{-1}(x)$ and state the domain of h^{-1}. [3]

(ii) Sketch, on the same diagram, the graphs of $y = \mathrm{h}(x)$ and $y = \mathrm{h}^{-1}(x)$, giving the equations of any asymptotes and the exact coordinates of any points where the curves cross the x- and y-axes. [3]

(iii) Find the set of values of x such that $\mathrm{h}^{-1}(x) > \mathrm{h}(x)$. [2]

7. The diagram below shows the curve with equation $y = \mathrm{f}(x)$. The curve crosses the x- and y-axes at the points $(-2k, 0)$, $(k, 0)$ and $\left(0, -\frac{8}{3}\right)$ where $k > 0$. The curve has an oblique asymptote $y = x - 1$ and vertical asymptote $x = -3$.



(i) On separate diagram, sketch the graph of $y = \dfrac{1}{\mathrm{f}(x)}$, including the coordinates of the points where the graph crosses the axes and the equations of any asymptotes. [3]

(ii) It is further known that $\mathrm{f}(x) = \dfrac{x^2 + ax + b}{x + c}$ where a, b and c are constants. Find the values of a, b and c. [4]

8. It is given that $\displaystyle\sum_{r=1}^{n} \frac{r^2}{3^r} = \frac{3}{2} - \frac{n^2 + 3n + 3}{2(3^n)}$.

(i) Find $\displaystyle\sum_{r=1}^{\infty} \frac{r^2 + (-1)^r}{3^r}$. [3]

(ii) Show that $\displaystyle\sum_{r=4}^{n} \frac{(r-2)^2}{3^{r-2}} = \frac{p}{q} - \frac{an^2 - an + a}{2(3^{n-2})}$, where a, p and q are integers to be determined. [5]

9. **(a)** Given that $\int_0^a x \sin x \, \mathrm{d}x = 0.5$, where $0 < a < 2$, find an equation that is satisfied by a and use it to find the value of a. [5]

(b) Write down a definite integral that represents the area of the region bounded by the curve with equation $y=\frac{\sqrt{x}}{3-\sqrt{x}}$, the two axes and the line $x=4$.

Use the substitution $u=3-\sqrt{x}$ to find the exact value of the area. [6]

10. It is given that z_1, z_2 and z_3 are the roots of the equation

$$2z^3+pz^2+qz-4=0$$

such that $\arg z_1 < \arg z_2 < \arg z_3$ and $z_1 = 1-\mathrm{i}\sqrt{3}$. Find the values of the real numbers p and q. [3]

(i) Without using the calculator, find z_2 and z_3. [3]

In an Argand diagram, points P, Q and R represent the complex numbers z_1, $w=\sqrt{2}+\mathrm{i}\sqrt{2}$ and z_1+w respectively and O is the origin.

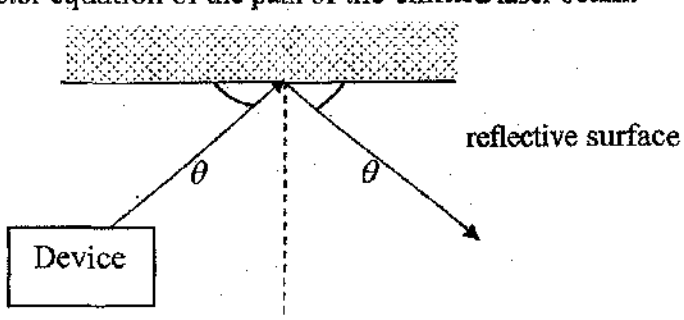
(ii) Express each of z_1 and w in the form $r\,\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. Give r and θ in exact form. [2]

(iii) Indicate P, Q and R on the Argand diagram and identify the type of the quadrilateral $OPRQ$. [3]

(iv) Find the exact value of $\arg\left(z_1^{\,4}w^*\right)$. [3]

11. Physicists are investigating the reflective property of a particular reflective surface. The diagram below shows the set-up of a particular experiment, where a laser emitting device was placed at the point with coordinates $(1, 2, 3)$. A laser beam was emitted in the direction parallel to $\mathbf{i}+\mathbf{k}$. The path of the emitted laser beam and its reflected path make the same angle θ with the reflective surface. The plane containing these two paths is perpendicular to the reflective surface.

Write down the vector equation of the path of the emitted laser beam. [1]



It is known that the reflective surface has equation $x+y+z=4$.

(i) Find θ. [3]

(ii) Show that the laser beam meets the reflective surface at the point $(0, 2, 2)$. [3]

(iii) Find the vector equation of the path of the reflected laser beam. [5]

12. A curve C has equation $y = \ln(x^2)$, $x \neq 0$.

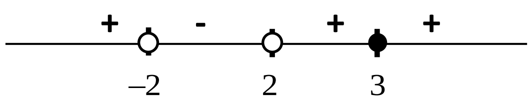
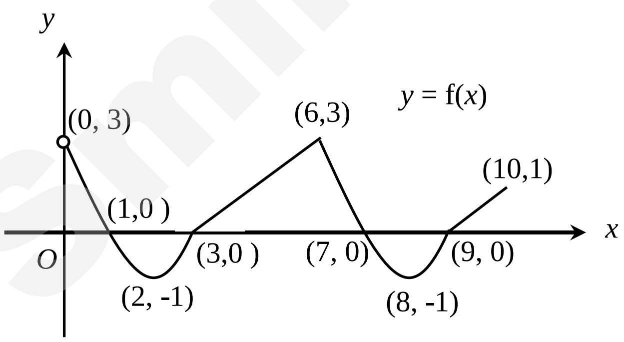
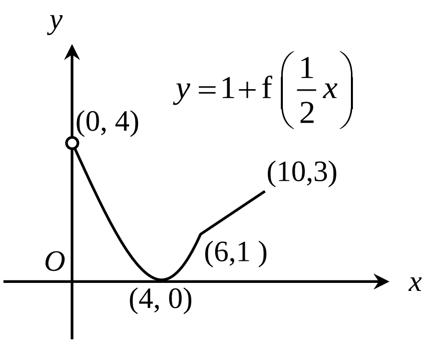
(i) Sketch C. [2]

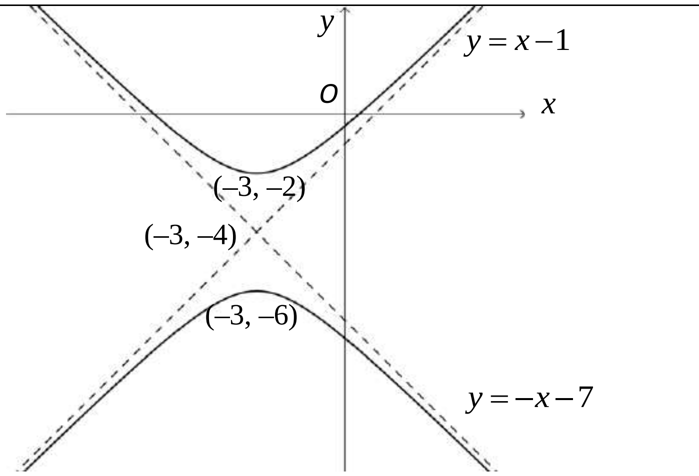
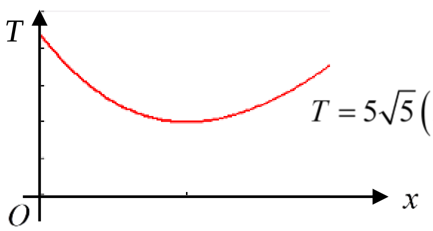
(ii) The part of C from the point $A(\mathrm{e}^{-1}, -2)$ to the point $B(\mathrm{e}^{\frac{k}{2}}, k)$, $k > 4$, and the line $y = -2$ is rotated about the y-axis to form the curved surface and the circular base of an open vase. Find the volume of the vase, giving your answer in terms of π and k, in exact form. [2]

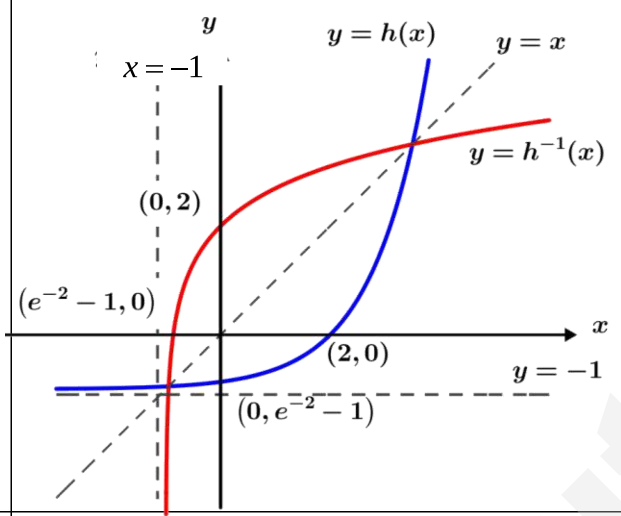
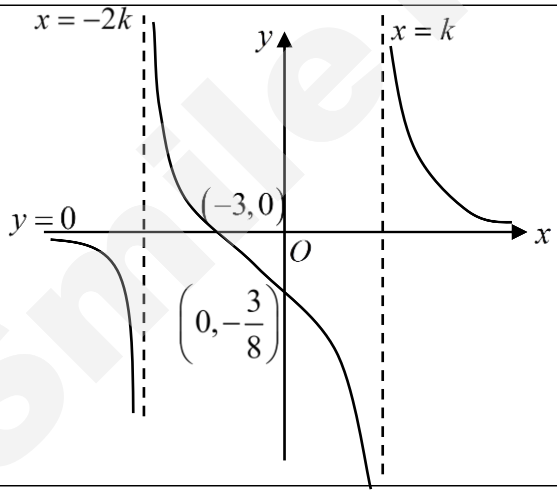
(iii) Water flows into the vase at a constant rate of 2 cm^3 per second. By first showing that the volume of water in the vase is given by $V = \pi\left(x^2 - \mathrm{e}^{-2}\right)$ when the radius of the water surface is x cm, find the rate at which x is increasing, giving your answer in terms of x. [4]

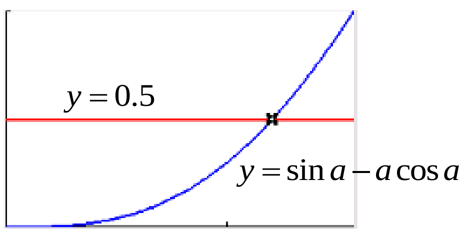
(iv) An insect lands on the inner surface of the vase at the point (e, 2) just as the incoming water reaches the depth of 2 cm. It immediately starts to crawl along C such that the x-coordinate of its location increases by a constant value of 0.03 cm per second. Find the coordinates of the point on C at which the insect will first come into contact with water. [5]

VJC H2 Maths Preliminary Examination P1 2017 Solutions

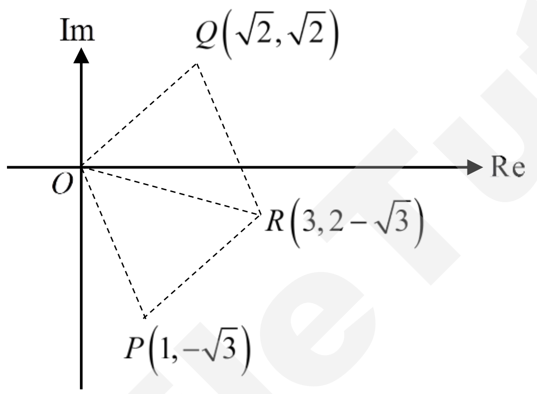
Q	Solution	Comments
1	$\dfrac{6x-13}{x^2-4} \geqslant 1$ $\dfrac{6x-13-x^2+4}{x^2-4} \geqslant 0$ $\dfrac{(x-3)^2}{(x+2)(x-2)} \leqslant 0$  $\therefore -2 < x < 2 \quad \text{or} \quad x = 3$	
2i	Let $\$E$, $\$W$ and $\$G$ be the unit cost of electricity, water and gas, respectively. $514E + 18.8W + 134G = 155.54$ $309E + 11.3W + 89G = 94.99$ $639(1.2)E + 21.7W + 108G = 208.40$ Using G.C, $E = 0.2137, \quad W = 1.1749, \quad G = 0.1761.$	
2ii	Let w be the water usage for August 2017 $(0.2137)(1.2)(555) + 1.1749w + 0.1761(128) = 184.84$ $w = 17$	
3i		
3ii		

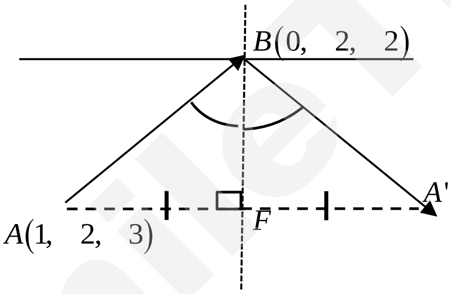
Q	Solution	Comments
4	 Intersect at least once: $\{m \in \mathbb{R}: \quad m < -1 \text{ or } \quad m > 1\}$	
5i	$\frac{dT}{dx} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5$ $T = \int \left(\frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5 \right) dx$ $= \frac{5\sqrt{5}}{2} \int 2x\left(x^2+4\right)^{-\frac{1}{2}} dx - \int 5\, dx$ $= \frac{5\sqrt{5}}{2} \frac{\left(x^2+4\right)^{\frac{1}{2}}}{\frac{1}{2}} - 5x + C$ $= 5\sqrt{5}\left(x^2+4\right)^{\frac{1}{2}} - 5x + C \qquad ---(1)$	
5ii	When t = 30, $\frac{dT}{dx} = 0: \ \frac{5\sqrt{5}x}{\sqrt{x^2+4}} = 5 \Rightarrow \sqrt{5}x = \sqrt{x^2+4}$ $5x^2 = x^2 + 4 \Rightarrow x = \pm 1$ Since $x > 0,\ x = 1$ Substitute $x = 1$ and $T = 30$ into equation (1) $30 = 5\sqrt{5}(1+4)^{\frac{1}{2}} - 5 + C \Rightarrow C = 10$ $T = 5\sqrt{5}\left(x^2+4\right)^{\frac{1}{2}} - 5x + 10$	
5iii	 $T = 5\sqrt{5}\left(x^2+4\right)^{\frac{1}{2}} - 5x + 10$ When x =0, $T = 32.361$. When x =2, $T = 31.623$ Longest time taken by Alvin is 32.4 mins.	

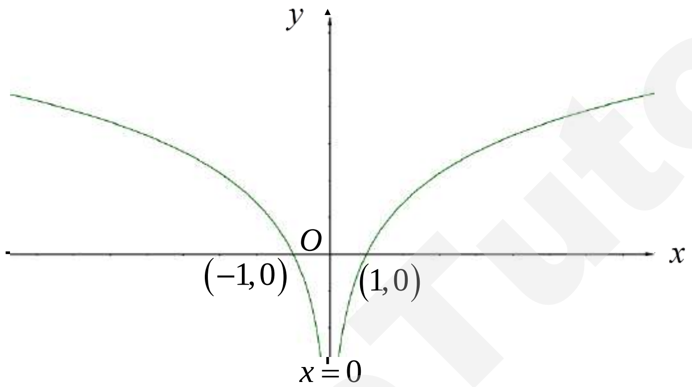
<table>
<tr><th>Q</th><th>Solution</th><th>Comments</th></tr>
<tr><td>6i</td><td>$y = e^{x-2} - 1$
$x = \ln(y+1) + 2$
$h^{-1}(x) = \ln(x+1) + 2$
Domain of h^{-1} = range of $h = (-1, \infty)$</td><td></td></tr>
<tr><td>6ii</td><td>
</td><td></td></tr>
<tr><td>6iii</td><td>Using G.C, $y = h^{-1}(x)$ and $y = h(x)$ intersects at
$x = -0.94753$ and $x = 3.50524$
Set of values of x = $\{ x \in \mathbb{R} : -0.948 < x < 3.51 \}$.</td><td></td></tr>
<tr><td>7i</td><td>
</td><td></td></tr>
<tr><td>ii</td><td>Since $x = -3$ is the vertical asymptote, $c = 3$
Given that $y = x - 1$ is an oblique asymptote,
$f(x) = x - 1 + \dfrac{A}{x+3}$
$= \dfrac{(x-1)(x+3) + A}{x+3} = \dfrac{x^2 + 2x - 3 + A}{x+3}$
By comparing coefficient of x with $\dfrac{x^2 + ax + b}{x+3}$: $a = 2$</td><td></td></tr>
</table>

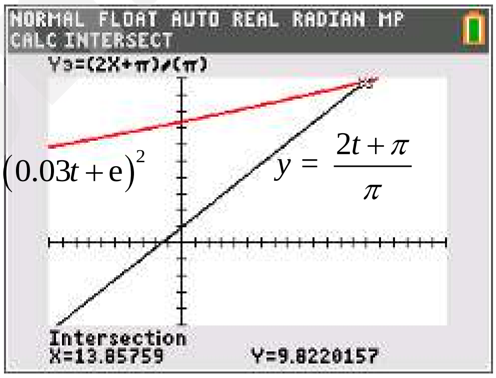
Q	Solution	Comments
	Since $\left(0,-\frac{8}{3}\right)$ is on the curve, $\frac{(0)^2+2(0)+b}{(0)+3}=-\frac{8}{3}$ $b=-8$	
8i	$\sum_{r=1}^{\infty}\frac{r^2+(-1)^r}{3^r}=\sum_{r=1}^{\infty}\frac{r^2}{3^r}+\sum_{r=1}^{\infty}\left(-\frac{1}{3}\right)^r$ $=\frac{3}{2}+\frac{\left(-\frac{1}{3}\right)}{1-\left(-\frac{1}{3}\right)}$ $=\frac{5}{4}$	
8ii	$\sum_{r=4}^{n}\frac{(r-2)^2}{3^{r-2}}=\sum_{r+2=4}^{r+2=n}\frac{r^2}{3^r}$ (replace r with $r+2$) $=\sum_{r=2}^{n-2}\frac{r^2}{3^r}$ $=\sum_{r=1}^{n-2}\frac{r^2}{3^r}-\frac{(1)^2}{3^1}$ $=\frac{3}{2}-\frac{(n-2)^2+3(n-2)+3}{2\left(3^{n-2}\right)}-\frac{1}{3}$ $=\frac{7}{6}-\frac{n^2-4n+4+3n-6+3}{2\left(3^{n-2}\right)}$ $=\frac{7}{6}-\frac{n^2-n+1}{2\left(3^{n-2}\right)}$ $\therefore p=7,\quad q=6,\quad a=1$	
9a	$\int_0^a x\sin x\,dx=0.5$ $\left[-x\cos x\right]_0^a+\int_0^a\cos x\,dx=0.5$ $\left[-a\cos a+0\right]+\left[\sin x\right]_0^a=0.5$ $-a\cos a+\sin a=0.5$ --- (1)  Using GC, a = 1.20249 =1.20 (3 s.f.)	

Q	Solution	Comments
9b	Area = $\displaystyle\int_0^4 \frac{\sqrt{x}}{3-\sqrt{x}}\ \mathrm{d}x$ Let $u = 3-\sqrt{x}$ $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\dfrac{1}{2\sqrt{x}} \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}u} = -2(3-u)$ When $x=0,\ u=3$ When $x=4,\ u=1$ $\displaystyle\int_0^4 \frac{\sqrt{x}}{3-\sqrt{x}}\ \mathrm{d}x = \int_3^1 \left(\frac{3-u}{u}\right)\left[(-2)(3-u)\right]\ \mathrm{d}u$ $\displaystyle= \int_1^3 \frac{2(3-u)^2}{u}\ \mathrm{d}u$ $\displaystyle= 2\int_1^3 \frac{9-6u+u^2}{u}\ \mathrm{d}u$ $\displaystyle= 2\int_1^3 \left(\frac{9}{u}-6+u\right)\ \mathrm{d}u$ $\displaystyle= 2\left[9\ln u - 6u + \frac{u^2}{2}\right]_1^3$ $\displaystyle= 2\left(9\ln 3 - 18 + \frac{9}{2}\right) - 2\left(-6+\frac{1}{2}\right)$ $= 18\ln 3 - 16$	
10	Since $1-\sqrt{3}\mathrm{i}$ is a root, $2\left(1-\mathrm{i}\sqrt{3}\right)^3 + p\left(1-\mathrm{i}\sqrt{3}\right)^2 + q\left(1-\mathrm{i}\sqrt{3}\right) - 4 = 0$ $2(-8) + p\left(-2-2\sqrt{3}\mathrm{i}\right) + q\left(1-\mathrm{i}\sqrt{3}\right) - 4 = 0$ $(-20-2p+q) + \left(-2\sqrt{3}p - \sqrt{3}q\right)\mathrm{i} = 0$ Compare real and imaginary parts: $-2p+q=20 \qquad ---(1)$ $-2\sqrt{3}p - \sqrt{3}q = 0 \qquad ---(2)$ $\therefore p=-5, \qquad q=10$	
10i	Since $1-\sqrt{3}\mathrm{i}$ is a root, and all coefficients are real $\Rightarrow 1+\sqrt{3}\mathrm{i}$ is also a root. $2z^3-5z^2+10z-4 = \left(z-\left(1-\sqrt{3}\mathrm{i}\right)\right)\left(z-\left(1+\sqrt{3}\mathrm{i}\right)\right)(2z+a)$ $= \left(z^2-2z+4\right)(2z+a)$	

Q	Solution	Comments
	By observation: $a=-1$ $\therefore z_2=\frac{1}{2}, \qquad z_3=1+\sqrt{3}\mathrm{i}$	
10ii	$\lvert z_1\rvert=\sqrt{1+3}=2 \qquad \lvert w\rvert=\sqrt{2+2}=2$ $\arg z_1=\arg\left(1-\sqrt{3}\mathrm{i}\right)$ $=-\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$ $=-\frac{\pi}{3}$ $\arg w=\arg\left(\sqrt{2}+\mathrm{i}\sqrt{2}\right)$ $=\frac{\pi}{4}$ $\therefore z_1=2\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}, \quad w=2\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$	
10iii	 Quadrilateral $OPRQ$ is a rhombus	
10iv	$4\arg\left(z_1\right)+\arg\left(w^*\right)=4\arg\left(z_1\right)-\arg\left(w\right)$ $=-\frac{4\pi}{3}-\frac{\pi}{4}$ $=-\frac{19\pi}{12}$ $\arg\left(z_1{}^4w^*\right)=-\frac{19\pi}{12}+2\pi$ $=\frac{5\pi}{12}$	
11	$\underset{\sim}{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad \lambda\in\mathbb{R}$	

Q	Solution	Comments
11i	$\cos\alpha = \dfrac{\begin{pmatrix}1\\0\\1\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{2}\sqrt{3}}$ $\alpha = 35.3^o$ $\theta = 90° - 35.3°$ $= 54.7°$	
11ii	Intersection of light beam with reflective surface: $\left[\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}\right].\begin{pmatrix}1\\1\\1\end{pmatrix}=4$ $6+2\lambda=4$ $\lambda=-1$ Coordinates of point of intersection = $(0, \quad 2, \quad 2)$.	
11iii	Let F be the foot of perpendicular from device to normal line and A be the point $(1, \quad 2, \quad 3)$:  $\overrightarrow{BF} = \left(\overrightarrow{BA}\cdot\hat{\underset{\sim}{n}}\right)\hat{\underset{\sim}{n}}$ $= \left[\dfrac{\begin{pmatrix}1-0\\2-2\\3-2\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{3}}\right]\dfrac{\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{3}}$ $= \dfrac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$ Using Ratio Theorem,	

Q	Solution	Comments
	$\overrightarrow{BF} = \dfrac{\overrightarrow{BA}+\overrightarrow{BA'}}{2}$ $\overrightarrow{BA'} = \dfrac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \dfrac{1}{3}\begin{pmatrix}1\\4\\1\end{pmatrix}$ Equation of reflected light path: $r = \begin{pmatrix}0\\2\\2\end{pmatrix} + \alpha\begin{pmatrix}1\\4\\1\end{pmatrix}, \quad \alpha \in \mathbb{R}$	
12i		
ii	Volume of the vase = $\pi\int_{-2}^{k} x^2\,\mathrm{d}y$ $= \pi\int_{-2}^{k} \mathrm{e}^y\,\mathrm{d}y$ $= \pi\left[\mathrm{e}^y\right]_{-2}^{k}$ $= \pi\left[\mathrm{e}^k - \mathrm{e}^{-2}\right]$	
iii	Volume of water , $V = \pi\int_{-2}^{y} \mathrm{e}^y\,\mathrm{d}y$ $= \pi\left[\mathrm{e}^y - \mathrm{e}^{-2}\right]$ $= \pi\left[\mathrm{e}^{\ln x^2} - \mathrm{e}^{-2}\right]$ $= \pi\left[x^2 - \mathrm{e}^{-2}\right]$ Given $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 2$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\mathrm{d}V}{\mathrm{d}t} \times \dfrac{\mathrm{d}x}{\mathrm{d}V}$ $= 2 \times \dfrac{1}{2\pi x}$ $= \dfrac{1}{\pi x}$	

Q	Solution	Comments
	Hence the rate at which the radius of the water surface is increasing is $\frac{1}{\pi x}$ cm per second.	
iv	For the insect, $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.03$. t seconds later, the location of the insect is at $x = 0.03t + \mathrm{e}$ For the movement of the water, $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\pi x}$ $\int \pi x \,\mathrm{d}x = \int 1 \,\mathrm{d}t$ $\frac{\pi x^2}{2} = t + C$ When $t = 0$, $x = 1$ $C = \frac{\pi}{2}$ $\frac{\pi x^2}{2} = t + \frac{\pi}{2}$ When the insect first comes into contact with water, $\frac{\pi(0.03t+\mathrm{e})^2}{2} - \frac{\pi}{2} = t$ $\pi(0.03t+\mathrm{e})^2 - \pi = 2t$ $(0.03t+\mathrm{e})^2 = \frac{2t+\pi}{\pi}$ $y = (0.03t+\mathrm{e})^2$  Using GC, t = 13.858 $x = 0.03(13.858) + \mathrm{e}$ = 3.1340 $y = \ln(3.1340)^2 = 2.28$ Hence coordinates of the point = $(3.13, 2.28)$	

2017 VJC Prelim Paper 2

Section A: Pure Mathematic [40 marks]

1. A curve C is defined by the parametric equations

$$x=\frac{t}{1+t}, \qquad y=\frac{t^2}{1+t},$$

where t takes all real values except -1.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, leaving your answer in terms of t. [3]

(i) Show that the equation of the tangent to C at the point $\left(\frac{p}{1+p}, \frac{p^2}{1+p}\right)$ is

$$y=p(p+2)x-p^2.$$ [2]

(ii) Find the acute angle between the two tangents to C which pass through the point $(2,5)$. [3]

2. Referred to the origin O, the points A, B and D are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OD}=\mathbf{d}$. The point C is such that $OACB$ is a parallelogram and angle OAC is $\frac{2\pi}{3}$ radians.

(i) Given that $\mathbf{a}$ is a unit vector and $|\mathbf{b}|=4$, find the length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OA}$. [3]

(ii) Given that $\lambda\mathbf{a}+\mu\mathbf{b}+\mathbf{d}=\mathbf{0}$ and $\lambda+\mu+1=0$, show that A, B and D are collinear. [3]

If $\mu=4$, find the area of triangle OBD, leaving your answer in the form $k|\mathbf{a}\times\mathbf{b}|$, where k is a constant to be determined. [3]

3. A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero and $a>0$. The first three terms of the geometric series are equal to the first, eighth and thirteenth terms respectively of the arithmetic series.

(i) Show that $7r^2-12r+5=0$. [2]

(ii) Deduce that the geometric series is convergent. [2]

(iii) The sum of the first n terms of the geometric series is denoted by S_n. Find the smallest value of n for S_n to be within 0.1% of the sum to infinity of the geometric series. [4]

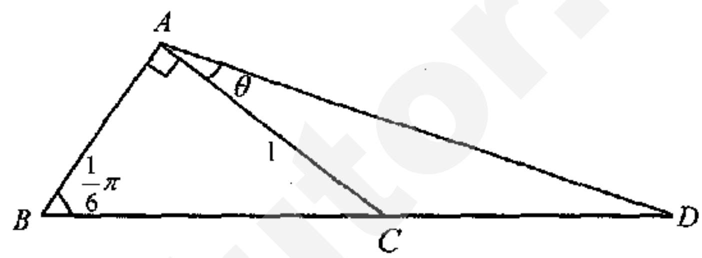
(iv) Find exactly the sum of the first 2017 terms of the arithmetic series, leaving your answer in terms of a. [3]

4. **(a)** It is given that $y=\mathrm{f}(x)$ is such that $my^2\dfrac{\mathrm{d}y}{\mathrm{d}x}-y^3=-\mathrm{e}^x\sin x$ and that the Maclaurin series for $\mathrm{f}(x)$ is given by $1+\dfrac{1}{3}x+nx^2+\ldots$, where m and n are some real constants.

(i) State the values of $\mathrm{f}(0)$ and $\mathrm{f}'(0)$. [2]

(ii) Find the values of m and n. [3]

(b) In the triangle ABC, $AC=1$, angle $BAC=\dfrac{\pi}{2}$ radians and angle $ABC=\dfrac{\pi}{6}$ radians. D is a point on BC produced such that angle $CAD=\theta$ radians (see diagram).



(i) Show that $AD=\dfrac{\sqrt{3}}{\sqrt{3}\cos\theta-\sin\theta}$. [4]

(ii) Given that θ is a sufficiently small angle, show that

$$AD\approx 1+a\theta+b\theta^2,$$

for constants a and b to be determined exactly. [3]

Section B: Statistics [60 marks]

5. John and Peter play a game of chess. It is equally likely for either player to make the first move. If John makes the first move, the probability of him winning the game is 0.3 while the probability of Peter winning the game is 0.2. If Peter makes the first move, the probability of him winning the game is 0.5 while the probability of John winning the game is 0.4. If there is no winner, then the game ends in a draw.

(i) Find the probability that Peter made the first move given that he won the game. [3]

(ii) John and Peter played a total of three games. Assuming that the results of the three games are independent, find the probability that John wins exactly one game. [3]

6. An experiment to determine the effect of a fertilizer on crop yield was carried out. A field was divided into eight plots of equal area and eight different amounts of fertilizer, one for each plot, were used. The table below shows the amount of fertilizer, x grams, and the crop yield, y grams, for each plot.

Amount of fertilizer (x)	15	22	37	55	62	69	78	90
Yield (y)	101	123	137	150	150	154	158	160

(i) Draw the scatter diagram for these values, labelling the axes. [1]

It is thought that the yield of a crop, y grams, can be modelled by one of the formulae

$$y = a + bx \quad \text{or} \quad y = c + d\ln x$$

where a, b, c and d are constants.

(ii) Find the value of the product moment correlation coefficient between

(a) x and y,

(b) $\ln x$ and y. [2]

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $y = a + bx$ or $y = c + d\ln x$ is the better model. [2]

(iv) For a plot of land, the yield of the crop was 144 grams. Using a suitable regression line estimate the amount of fertilizer used, giving your answer to the nearest gram. [2]

(v) Comment on the reliability of the model in part **(iv)** in predicting the value of y when $x = 110$. [1]

7. Four digits are randomly selected from the set $\{1,2,3,4,5,6,7,8,9\}$ to form a four-digit number. Repetitions are not allowed.

(i) Find the probability that none of the digits in the four-digit number are odd. [2]

The random variable X denotes the number of odd digits in the four-digit number formed.

(ii) Show that $P(X = 1) = \frac{10}{63}$, and find the rest of the probability distribution of X, giving each probability as a fraction in its lowest terms. [3]

(iii) Find the expectation and variance of X. [3]

(iv) Two independent observations of X are denoted by X_1 and X_2.

Find $P(|X_1 - X_2| < 3)$. [4]

8. In this question, you should state clearly the values of the parameters of any normal distribution you use.

A supermarket sells two types of durians, D25 and Musang Queen. The durians are sold by weight. The masses, in kilograms, of D25 and Musang Queen are modelled as having normal distributions. The means and standard deviations of these distributions, and the selling prices, in $ per kilogram, are shown in the following table.

	Mean (kg)	Standard deviation (kg)	Selling price ($ per kg)
D25	1.5	0.02	9
Musang Queen	1.8	0.035	18

(i) A customer buys 3 D25 durians and 2 Musang Queen durians. Find the probability that the total cost of his purchase is more than $107. [5]

(ii) State an assumption needed for your calculations in part (i). [1]

(iii) The probability that the average weight of n randomly chosen D25 durians exceeding m kg is at least 0.1. Show that n satisfies the inequality

$$(m-1.5)\sqrt{n} \leqslant 0.025631.$$

Hence find the largest possible value of n when $m = 1.51$. [4]

9. Ryde, a leading private hire car company, announced JustRyde, a new service that promises more affordable fixed fare rides and shorter waiting times. In their advertisement, Ryde claimed that the mean waiting time, in seconds, was 240. A random sample of 50 JustRyde customers is taken and their waiting times, x seconds, is recorded. The data are summarised by

$$\Sigma(x-240) = 120, \quad \Sigma(x-240)^2 = 11200.$$

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Test, at the 10% significance level, whether the population mean waiting time is more than 240 seconds. [5]

(iii) State, giving a valid reason, whether any assumptions about the population are needed in order for the test to be valid. [1]

(iv) Explain, in the context of the question, the meaning of 'at the 10% significance level'. [1]

(v) In another test, using the same data and also at the 10% significance level, the hypotheses are as follows:

Ho : the population mean waiting time is equal to k seconds.
H_1 : the population mean waiting time is not equal to k seconds.

Given that the null hypothesis is rejected in favour of the alternative hypothesis, find the set of possible values of k. [3]

10. It is a common practice for airlines to sell more plane tickets than the number of seats available. This is to maximise their profits as it is expected that some passengers will not turn up for the flight.

The plane used by Victoria Airline for her daily 10 am flight from Singapore to Hong Kong has a maximum capacity of 150 seats. For this particular flight, 154 tickets are sold every day. On average, p out of 100 customers who have purchased a plane ticket for this flight turn up. Customers who turn up after the flight is full will be turned away. The number of customers who turn up for the 10 am flight, on a randomly chosen day, is denoted by X.

(i) State, in the context of this question, two assumptions needed to model X by a binomial distribution. [2]

(ii) Explain why one of the assumptions stated in part (i) may not hold in this context. [1]

Assume now that these assumptions do in fact hold.

(iii) It is known that there is a 0.05 probability that at least 153 customers will turn up for the 10 am flight. Write down an equation for the value of p, and find this value numerically. [3]

It is given instead that $p = 94$.

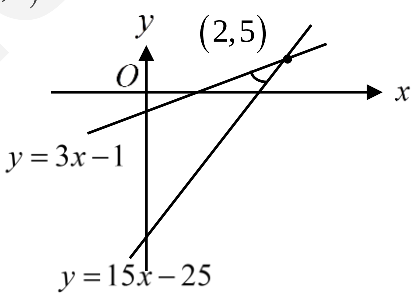
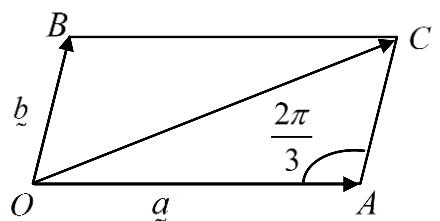
(iv) Find the probability that, on a randomly chosen day,

(a) there are at least 141 but not more than 148 customers who turn up for the 10 am flight, [2]

(b) every customer who turns up gets a seat on the 10 am flight. [1]

(v) Find the probability that every customer who turns up gets a seat on the 10 am flight on more than 5 days in a week. [3]

VJC H2 Maths Prelim P2 2017 Solutions/Mark Scheme

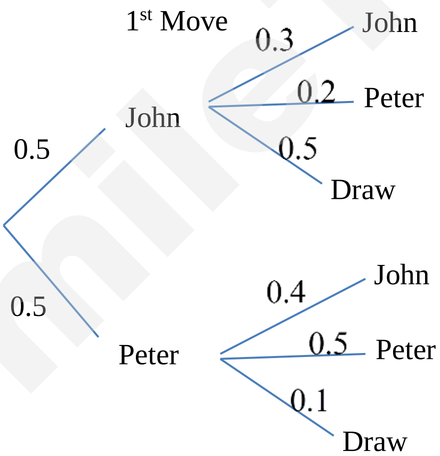
Q	Solution	
Section A: Pure Mathematics [40 marks]		
1	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$ $=\frac{(1+t)2t-t^2}{(1+t)^2}\div\frac{(1+t)(1)-t}{(1+t)^2}$ $=t^2+2t$	
1i	At point $\left(\frac{p}{1+p},\frac{p^2}{1+p}\right)$, $t=p$ Equation of tangent at point $\left(\frac{p}{1+p},\frac{p^2}{1+p}\right)$, $y-\frac{p^2}{1+p}=\left(p^2+2p\right)\left(x-\frac{p}{1+p}\right)$ $y=p(p+2)x+\frac{p^2}{1+p}-\frac{p^3}{1+p}-\frac{2p^2}{1+p}$ $y=p(p+2)x-\frac{p^2(p+1)}{1+p}$ $y=p(p+2)x-p^2$	
1ii	Tangents pass through $(2,5)$ $\Rightarrow 5=p(p+2)(2)-p^2$ $p^2+4p-5=0$ $p=-5$ or $p=1$  Equations of tangents are $y=3x-1$ and $y=15x-25$ Required acute angle between the 2 tangents $=\tan^{-1}(15)-\tan^{-1}(3)$ $=0.255$ rad or $14.6°$	
2	$\overrightarrow{OC}=\underset{\sim}{a}+\underset{\sim}{b}$  	

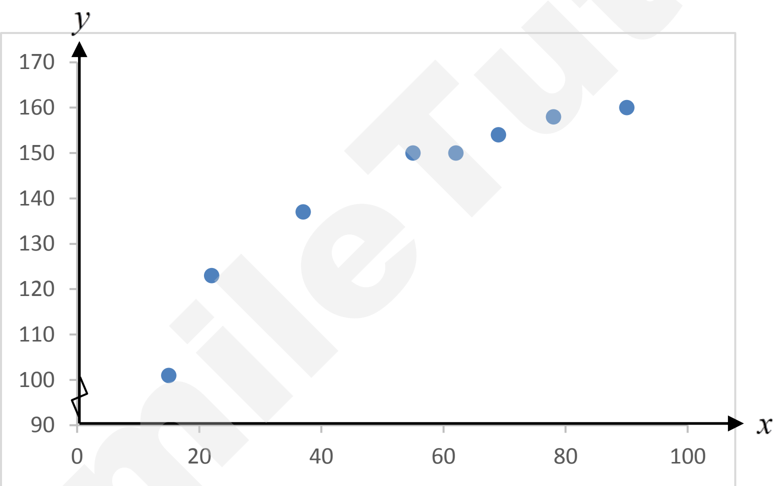
Q	Solution	
2i	Length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OA}$ $= \left\| (\underset{\sim}{a} + \underset{\sim}{b}) \cdot \hat{\underset{\sim}{a}} \right\|$ $= \left\| \underset{\sim}{a} \cdot \hat{\underset{\sim}{a}} + \underset{\sim}{b} \cdot \hat{\underset{\sim}{a}} \right\| = \left\| \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{a} \right\| \quad \because \underset{\sim}{a} = \hat{\underset{\sim}{a}}$ $= \left\| \left\|\underset{\sim}{a}\right\|^2 + \left\|\underset{\sim}{b}\right\| \left\|\underset{\sim}{a}\right\| \cos\left(\pi - \frac{2\pi}{3}\right) \right\|$ $= \left\| 1 + 4\left(\frac{1}{2}\right) \right\|$ $= 3$	
2ii	$\lambda \underset{\sim}{a} + \mu \underset{\sim}{b} + \underset{\sim}{d} = \underset{\sim}{0} \qquad ---(1)$ $\lambda + \mu + 1 = 0 \qquad ---(2)$ Sub (2) into (1): $(-1-\mu)\underset{\sim}{a} + \mu \underset{\sim}{b} + \underset{\sim}{d} = \underset{\sim}{0}$ $\mu(\underset{\sim}{b} - \underset{\sim}{a}) = \underset{\sim}{a} - \underset{\sim}{d}$ $\mu \overrightarrow{AB} = \overrightarrow{DA}$ Since $AB \mathbin{//} DA$ and A is a common point, A, B and D are collinear	
	Given $\mu = 4$, $\underset{\sim}{d} = 5\underset{\sim}{a} - 4\underset{\sim}{b}$ Area of triangle OBD $= \frac{1}{2} \left\| \underset{\sim}{b} \times \underset{\sim}{d} \right\|$ $= \frac{1}{2} \left\| \underset{\sim}{b} \times (5\underset{\sim}{a} - 4\underset{\sim}{b}) \right\|$ $= \frac{1}{2} \left\| 5\underset{\sim}{b} \times \underset{\sim}{a} - 4\underset{\sim}{b} \times \underset{\sim}{b} \right\|$ $= \frac{5}{2} \left\| \underset{\sim}{b} \times \underset{\sim}{a} \right\| \qquad (\because \underset{\sim}{b} \times \underset{\sim}{b} = \underset{\sim}{0})$ $= \frac{5}{2} \left\| \underset{\sim}{a} \times \underset{\sim}{b} \right\|$ $\therefore k = \frac{5}{2}$	
3i	$ar = a + (8-1)d \;\Rightarrow\; d = \frac{ar - a}{7}$ $ar^2 = a + (13-1)d \;\Rightarrow\; d = \frac{ar^2 - a}{12}$ $\frac{ar - a}{7} = \frac{ar^2 - a}{12}$ $12r - 12 = 7r^2 - 7$ $7r^2 - 12r + 5 = 0$	

Q	Solution	
3ii	From the GC, $r=\frac{5}{7}$ or $r=1$. Since $d\neq 0$, the terms of the geometric series are distinct we conclude that $r\neq 1$. Hence, $r=\frac{5}{7}$. As $\lvert r\rvert=\left\lvert\frac{5}{7}\right\rvert<1$, the geometric series is convergent.	
3iii	$\lvert S_\infty - S_n\rvert < 0.001S_\infty$ $\left\lvert \frac{a}{1-\frac{5}{7}} - \frac{a\left(1-\left(\frac{5}{7}\right)^n\right)}{1-\frac{5}{7}} \right\rvert < 0.001\left(\frac{a}{1-\frac{5}{7}}\right)$ $\left\lvert \frac{a}{1-\frac{5}{7}} \right\rvert \left\lvert 1-\left(1-\left(\frac{5}{7}\right)^n\right)\right\rvert < 0.001\left(\frac{a}{1-\frac{5}{7}}\right)$ $\left(\frac{5}{7}\right)^n < 0.001 \quad (\because a>0)$ $n\ln\left(\frac{5}{7}\right) < \ln 0.001$ $n > \frac{\ln 0.001}{\ln\frac{5}{7}}$ $n > 20.53$ Smallest value of n is 21.	
3iv	$d=\frac{ar-a}{7}=\frac{a\left(\frac{5}{7}\right)-a}{7}=-\frac{2}{49}a$ The sum of the first 2017 terms of the arithmetic series $=\frac{2017}{2}\left[2a+(2017-1)\left(-\frac{2}{49}a\right)\right]$ $=-\frac{566777}{7}a$	
4ai	$\mathrm{f}(x)=1+\frac{1}{3}x+nx^2+\ldots$ Comparing with $\mathrm{f}(x)=\mathrm{f}(0)+\mathrm{f}'(0)x+\frac{\mathrm{f}''(0)}{2!}x^2+\ldots$ $\Rightarrow \mathrm{f}(0)=1$	

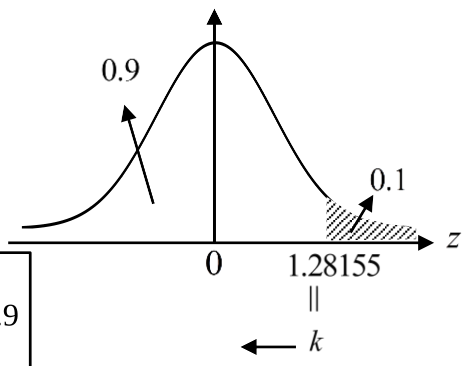
Q	Solution	
	$\Rightarrow \text{f}'(0)=\frac{1}{3}$	
4aii	Given $my^2\frac{\text{d}y}{\text{d}x}-y^3=-\text{e}^x\sin x\ ---(2)$ When $x=0$, $m(1)^2\left(\frac{1}{3}\right)-(1)^3=-\text{e}^0\sin 0$ $\frac{1}{3}m=1$ $m=3$ Differentiate (2) w.r.t. x: $3y^2\frac{\text{d}^2y}{\text{d}x^2}+6\left(\frac{\text{d}y}{\text{d}x}\right)^2-3y^2\frac{\text{d}y}{\text{d}x}=-\text{e}^x\sin x-\text{e}^x\cos x$ When $x=0$, $3(1)^2(2n)+6\left(\frac{1}{3}\right)^2-3(1)^2\left(\frac{1}{3}\right)=-1$ $6n=-\frac{2}{9}$ $n=-\frac{1}{9}$	
4bi	**<u>Method 1</u>** $\measuredangle ACD=\frac{\pi}{2}+\frac{\pi}{6}=\frac{2\pi}{3}\quad$ (ext angle of a triangle) Using Sine Rule in $\triangle ACD$ $\frac{AD}{\sin\frac{2\pi}{3}}=\frac{AC}{\sin\left(\pi-\frac{2\pi}{3}-\theta\right)}$ $AD=\frac{\sqrt{3}/2}{\sin\left(\frac{\pi}{3}-\theta\right)}$	

Q	Solution	
	$$=\frac{\sqrt{3}/2}{\sin\frac{\pi}{3}\cos\theta-\cos\frac{\pi}{3}\sin\theta}$$ $$=\frac{\sqrt{3}/2}{\left(\sqrt{3}/2\right)\cos\theta-\left(1/2\right)\sin\theta}$$ $$=\frac{\sqrt{3}}{\sqrt{3}\cos\theta-\sin\theta}$$ **<u>Method 2</u>** In right-angled $\triangle ABC$, $AB=\frac{1}{\tan\frac{\pi}{6}}=\sqrt{3}$. $\measuredangle ADB=\pi-\frac{\pi}{6}-\left(\frac{\pi}{2}+\theta\right)=\frac{\pi}{3}-\theta \quad$ (angle sum of a triangle) Using Sine Rule in $\triangle ABD$ $$\frac{AD}{\sin\frac{\pi}{6}}=\frac{AB}{\sin\left(\frac{\pi}{3}-\theta\right)}$$ $$AD=\frac{\sqrt{3}\sin\frac{\pi}{6}}{\sin\left(\frac{\pi}{3}-\theta\right)}$$ $$=\frac{\sqrt{3}/2}{\sin\frac{\pi}{3}\cos\theta-\cos\frac{\pi}{3}\sin\theta}$$ $$=\frac{\sqrt{3}/2}{\left(\sqrt{3}/2\right)\cos\theta-\left(1/2\right)\sin\theta}$$ $$=\frac{\sqrt{3}}{\sqrt{3}\cos\theta-\sin\theta}$$	
4bii	When θ is a sufficiently small angle,	

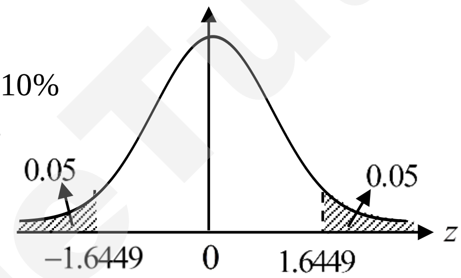
Q	Solution	
	$AD \approx \dfrac{\sqrt{3}}{\sqrt{3}\left(1-\dfrac{\theta^2}{2}\right)-\theta}$ $=\sqrt{3}\left(\sqrt{3}-\theta-\dfrac{\sqrt{3}}{2}\theta^2\right)^{-1}$ $=\left[1+\left(-\dfrac{\theta}{\sqrt{3}}-\dfrac{\theta^2}{2}\right)\right]^{-1}$ $\approx 1-\left(-\dfrac{\theta}{\sqrt{3}}-\dfrac{\theta^2}{2}\right)+\dfrac{(-1)(-2)}{2!}\left(-\dfrac{\theta}{\sqrt{3}}-\dfrac{\theta^2}{2}\right)^2$ $\approx 1+\dfrac{\theta}{\sqrt{3}}+\dfrac{\theta^2}{2}+\dfrac{\theta^2}{3}$ $=1+\dfrac{1}{\sqrt{3}}\theta+\dfrac{5}{6}\theta^2$ $\therefore a=\dfrac{1}{\sqrt{3}}, \qquad b=\dfrac{5}{6}$	
Section B: Statistics [60 marks]		
5		
5i	$\mathrm{P}(\text{Peter made first move} \mid \text{Peter won the game})$ $=\dfrac{\mathrm{P}(\text{Peter made first move and Peter won the game})}{\mathrm{P}(\text{Peter won the game})}$ $=\dfrac{0.5\times0.5}{0.5\times0.2+0.5\times0.5}$ $=\dfrac{5}{7}$	
5ii	$\mathrm{P}(\text{John wins})=0.5\times0.3+0.5\times0.4=0.35$	

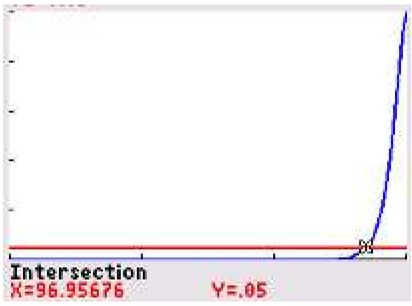
Q	Solution	
	$\text{P}(\text{John wins in exactly 1 game})$ $= (0.35)(0.65)(0.65) \times \dfrac{3!}{2!}$ $= 0.443625 \text{ or } \dfrac{3549}{8000}$ $= 0.444 (\text{to 3 s.f.})$ **<u>Alternative</u>** Let X be the number of games won by John out of 3 games. $X \sim B(3, 0.35)$ $\text{P}(\text{John wins in exactly 1 game})$ $= \text{P}(X = 1)$ $= 0.443625 \text{ or } \dfrac{3549}{8000}$ $= 0.444 (\text{to 3 s.f.})$	
6i		
6iia	From GC, r = 0.93639 = 0.936 (3 s.f)	
6iib	From GC, r = 0.98775 = 0.988 (3 s.f)	
6iii	Since 1) the points on the scatter diagram seem to lie close to an increasing curve with decreasing gradient (or close to a curve in which y increases by decreasing amounts as x increases), and 2) the product moment correlation coefficient between ln x and y of 0.988 is closer to 1 than the product moment correlation coefficient between x and y of 0.936, hence $y = c + d \ln x$ is the better model.	
6iv	From (iii), we should use the regression line of y on ln x. From GC, the equation of the regression line of y on ln x is	

<table>
<tr><th>Q</th><th>Solution</th><th></th></tr>
<tr><td></td><td>$y = 20.8496 + 31.539\ln x$
$y = 20.8 + 31.5\ln x \quad$ (3 s.f)
When y = 144, $144 = 20.8496 + 31.539\ln x$
$\therefore x = 49.635 = 50$ (nearest gram)</td><td></td></tr>
<tr><td>6v</td><td>Since x = 110 is outside the range of data values ($15 \le x \le 90$), hence the estimated value of y may not be reliable.</td><td></td></tr>
<tr><td>7i</td><td>P(no odd digits) = P(all even digits)
$= \dfrac{{}^4C_4}{{}^9C_4} \left(\text{or } \dfrac{{}^4P_4}{{}^9P_4} \right)$
$= \dfrac{1}{126}$</td><td></td></tr>
<tr><td>7ii</td><td>$\mathrm{P}(X = 1) = \dfrac{{}^5C_1\,{}^4C_3}{{}^9C_4} = \dfrac{10}{63}$
<table>
<tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>$\mathrm{P}(X = x)$</td><td>$\dfrac{1}{126}$</td><td>$\dfrac{10}{63}$</td><td>$\dfrac{10}{21}$</td><td>$\dfrac{20}{63}$</td><td>$\dfrac{5}{126}$</td></tr>
</table></td><td></td></tr>
<tr><td>7iii</td><td>$\mathrm{E}(X) = \sum_{x=0}^{4} x\mathrm{P}(X = x)$
$= 1\left(\dfrac{10}{63}\right) + 2\left(\dfrac{10}{21}\right) + 3\left(\dfrac{20}{63}\right) + 4\left(\dfrac{5}{126}\right)$
$= \dfrac{20}{9}$
$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$
$= \sum_{x=0}^{4} x^2\mathrm{P}(X = x) - \left(\dfrac{20}{9}\right)^2$
$= 1\left(\dfrac{10}{63}\right) + 4\left(\dfrac{10}{21}\right) + 9\left(\dfrac{20}{63}\right) + 16\left(\dfrac{5}{126}\right) - \left(\dfrac{20}{9}\right)^2$
$= \dfrac{50}{81}$</td><td></td></tr>
</table>

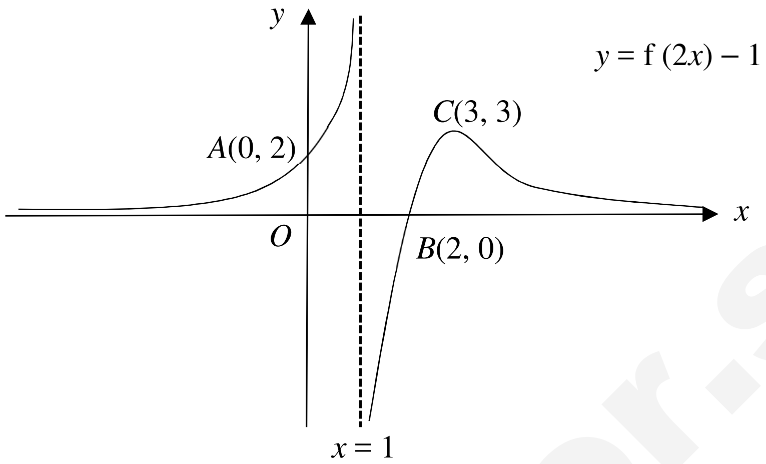
Q	Solution	
7iv	$\mathrm{P}(\lvert X_1 - X_2\rvert < 3) = \mathrm{P}(-3 < X_1 - X_2 < 3)$ $= 1 - 2\mathrm{P}(X_1 = 0 \quad \& \quad X_2 = 4)$ $-2\mathrm{P}(X_1 = 0 \quad \& \quad X_2 = 3)$ $-2\mathrm{P}(X_1 = 1 \quad \& \quad X_2 = 4)$ $= 1 - 2\left(\frac{1}{126}\right)\left(\frac{5}{126}\right) - 2\left(\frac{1}{126}\right)\left(\frac{20}{63}\right)$ $-2\left(\frac{10}{63}\right)\left(\frac{5}{126}\right)$ $= \frac{7793}{7938} \quad (\text{or } 0.982)$	
8	Let X kg and Y kg be the mass of a randomly chosen D25 durian and Musang Queen durian respectively. $X \sim \mathrm{N}(1.5, 0.02^2), \qquad Y \sim \mathrm{N}(1.8, 0.035^2)$	
8ii	Let $T = 9(X_1 + X_2 + X_3) + 18(Y_1 + Y_2)$ $\mathrm{E}(T) = \mathrm{E}[9(X_1 + X_2 + X_3) + 18(Y_1 + Y_2)]$ $= 9(3)(1.5) + 18(2)(1.8)$ $= 105.3$ $\mathrm{Var}(T) = \mathrm{Var}[9(X_1 + X_2 + X_3) + 18(Y_1 + Y_2)]$ $= (9)^2(3)(0.02)^2 + (18)^2(2)(0.035)^2$ $= 0.891$ $T \sim \mathrm{N}(105.3, 0.891)$ $\mathrm{P}(T > 107) = 0.035852$ $= 0.0359$ (3 s.f)	
8ii	The masses of all the durians are independent of each other.	
8iii	Let $\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$ $\overline{X} \sim \mathrm{N}\left(1.5, \frac{0.02^2}{n}\right)$ Given $\mathrm{P}(\overline{X} > m) \geqslant 0.1$ $\mathrm{P}\left(Z > \frac{m - 1.5}{0.02/\sqrt{n}}\right) \geqslant 0.1$ $\boxed{\Rightarrow \mathrm{P}\left(Z < \frac{m - 1.5}{0.02/\sqrt{n}}\right) \leqslant 0.9}$ 	

Q	Solution	
	From the GC, $\mathrm{P}(Z<1.28155)=0.9$ $\therefore \dfrac{m-1.5}{0.02/\sqrt{n}} \leqslant 1.28155$ $\Rightarrow (m-1.5)\sqrt{n} \leqslant 0.025631$ when $m=1.51$ $\Rightarrow (1.51-1.5)\sqrt{n} \leqslant 0.025631$ $\Rightarrow n \leqslant 6.5695$ Largest value of n is 6	
9i	Let $y=x-240$ unbiased estimate of population mean $=\bar{x}$ $=\bar{y}+240$ $=\dfrac{\sum y}{n}+240$ $=\dfrac{120}{50}+240$ $=242.4$ Unbiased estimate of population variance $=s^2$ $=\dfrac{1}{n-1}\left(\sum y^2-\dfrac{(\sum y)^2}{n}\right)$ $=\dfrac{1}{49}\left(11200-\dfrac{120^2}{50}\right)$ $=222.69=223$ (3 s.f)	
9ii	Let μ be the population mean of X. $\mathrm{H}_0 : \mu=240$ $\mathrm{H}_1 : \mu>240$ Level of significance: 10% Test Statistic : since n = 50 is sufficiently large, By Central Limit Theorem, $\bar{X}$ is approximately normal. When H_0 is true, $Z=\dfrac{\bar{X}-240}{\frac{S}{\sqrt{50}}} \sim \mathrm{N}(0,1)$ approximately	

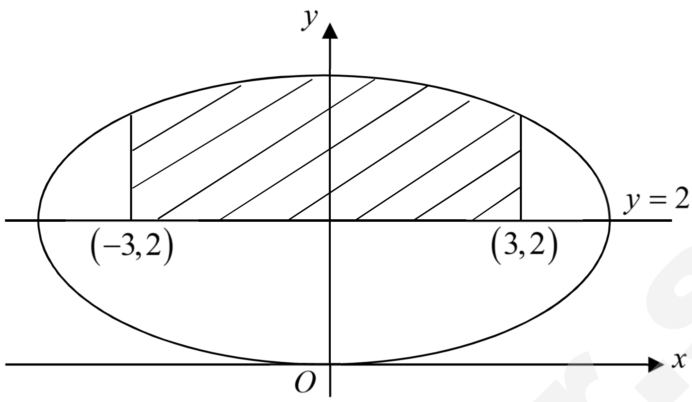
Q	Solution	
	Computation : $\bar{x} = 242.4$ $s = \sqrt{222.69} = 14.923$ $p-\text{value} = 0.128$ (3 s.f) Conclusion : Since p-value = 0.128 > 0.10, H_0 is not rejected at the 10% significance level. So there is insufficient evidence that the population mean waiting time is more than 240 seconds.	
9iii	No assumption is needed. Since the sample size is large, by Central Limit Theorem, the distribution of the sample mean ($\bar{X}$) is approximately normal.	
9iv	There is a probability of 0.10 that the test will conclude the population mean waiting time is more than 240 seconds when it is actually 240 seconds.	
9v	$H_0 : \mu = k$ $H_1 : \mu \neq k$ Level of significance: 10% For H_0 to be rejected,  $z \le -1.6449$ or $z \ge 1.6449$ $\dfrac{\bar{x}-k}{s/\sqrt{50}} \le -1.6449$ or $\dfrac{\bar{x}-k}{s/\sqrt{50}} \ge 1.6449$ $\dfrac{242.4-k}{14.923/\sqrt{50}} \le -1.6449$ or $\dfrac{242.4-k}{14.923/\sqrt{50}} \ge 1.6449$ $242.4-k \le -3.4714$ or $242.4-k \ge 3.4714$ $k \ge 245.87$ or $k \le 238.93$ $\{k \in \mathbb{R} : k \le 239 \text{ (3 s.f)} \text{ or } k \ge 246 \text{ (3 s.f)}\}$	
10i	The assumptions are (1) The probability that a customer turn up for the flight is $\dfrac{p}{100}$ <u>for all the 154 customers</u>. (2) Customers turn up independently of each other.	
10ii	Customers may be travelling in a group or as a family. Therefore, customers may not turn up independently of the others in their group.	

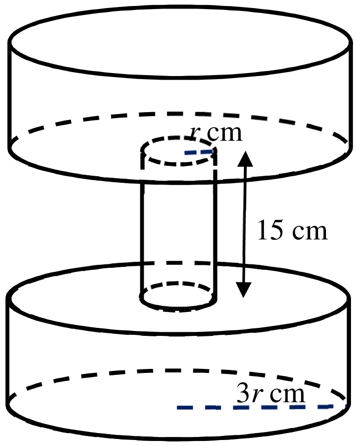
Q	Solution	
10iii	$X \sim \mathrm{B}\left(154, \frac{p}{100}\right)$ Given $\mathrm{P}(X \geqslant 153) = 0.05$ $\mathrm{P}(X = 153) + \mathrm{P}(X = 154) = 0.05$ $\binom{154}{153}\left(\frac{p}{100}\right)^{153}\left(1-\frac{p}{100}\right) + \binom{154}{154}\left(\frac{p}{100}\right)^{154} = 0.05$ $154\left(\frac{p}{100}\right)^{153}\left(1-\frac{p}{100}\right) + \left(\frac{p}{100}\right)^{154} = 0.05$  From the GC, $p = 96.9568 = 97.0$ (to 3 s.f.)	
10iv a	$X \sim \mathrm{B}(154, 0.94)$ $\mathrm{P}(141 \leqslant X \leqslant 148) = \mathrm{P}(X \leqslant 148) - \mathrm{P}(X \leqslant 140)$ $= 0.825$	
10iv b	$\mathrm{P}(X \leqslant 150) = 0.98443 = 0.984$ (to 3 s.f.)	
10v	Let Y be the number of days (out of 7) in which every customer who turns up gets a seat on the flight $Y \sim \mathrm{B}(7, 0.98443)$ $\mathrm{P}(Y > 5) = 1 - \mathrm{P}(Y \leqslant 5)$ $= 0.995$	

H2 Mathematics 2017 Preliminary Exam Paper 1 Question (9758)
Answer all questions [100 marks]

1	 The diagram shows the curve $y = \mathrm{f}(2x) - 1$ with a maximum point at $C(3, 3)$. The curve crosses the axes at the points $A(0, 2)$ and $B(2, 0)$. The line $x = 1$ and the x-axis are the asymptotes of the curve. On separate diagrams, sketch the graphs of **(i)** $y = \mathrm{f}(x)$, [2] **(ii)** $y = \mathrm{f}'(x)$, [2] stating clearly the equations of the asymptotes and the coordinates of the points corresponding to A, B and C where appropriate.
2	**(i)** Without using a calculator, solve the inequality $\dfrac{x}{x^2-5} \le 0$, giving your answer in exact form. [2] **(ii)** Hence, find the set of values of x for which $\dfrac{\sqrt{x}}{x-5} \le 0$. [2]
3	Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P on OA is such that $OP : PA = 2:3$, and the point Q on OB is such that $OQ : QB = 1:2$. Given that M is the mid-point of PQ, state the position vector of M in terms of **a** and **b**.[1] Show that the area of triangle OMP can be written as $k\lvert\mathbf{a}\times\mathbf{b}\rvert$, where k is a constant to be determined. [4]
4	Find **(a)** $\int \cos(\ln x)\ \mathrm{d}x$, [3] **(b)** $\int \dfrac{1-2x}{2x^2+1}\ \mathrm{d}x$. [3]

5	It is given that $z=\sqrt{3}+\mathrm{i}$ and $w=-1+\mathrm{i}$. **(i)** Without using a calculator, find an exact expression for $\frac{z^2}{w^*}$. Give your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [4] **(ii)** Find the exact value of the real number q such that $\arg\left(1-\frac{q}{z}\right)=\frac{\pi}{12}$. [3]
6	It is given that $y=\ln\left(3+\mathrm{e}^x\right)$. **(i)** Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{\mathrm{d}y}{\mathrm{d}x}$. [3] **(ii)** By differentiating the above result, find the first four non-zero terms of the Maclaurin series for y. Give the coefficients in exact form. [3] **(iii)** Hence find the Maclaurin series for $\frac{\mathrm{e}^{-2x}}{3+\mathrm{e}^{-2x}}$, up to and including the term in x^2. [2]
7	The curve C has equation $y=\frac{ax^2+bx-8}{x-2}$, where a and b are constants. It is given that C has asymptote $y = 3 - 2x$. **(i)** Find the value of a and show that $b = 7$. [3] **(ii)** Sketch C, stating clearly the equations of any asymptotes and the coordinates of any stationary points and any points of intersection with the axes. [3] **(iii)** By drawing another suitable curve on the same diagram, deduce the number of real roots of the equation $\left(-2x^2+7x-8\right)^2-25\left(x-2\right)^3=0$. [3]
8	Emily has 1016 toy bricks. **(i)** Emily wishes to build a brick structure with one brick in the first row, two bricks in the second row, three bricks in the third row and so on. What is the maximum number of rows that she can build and how many bricks will be left unused? [4] **(ii)** Emily keeps all her 1016 bricks in $(2k-1)$ bags of different sizes. She packs m bricks into the smallest bag. For each subsequent bag, she packs double the number of bricks she packs in the previous bag. Given that she has 64 bricks in the kth bag, find the value of m and the number of bags. [5]

9	**(a)**	By using the substitution $x = 3\sec\theta$, evaluate $\displaystyle\int_{3\sqrt{2}}^{6} \frac{3x+1}{\sqrt{x^2-9}}\,\mathrm{d}x$ exactly. [5]
	(b)	 The diagram shows an ellipse with equation $\dfrac{x^2}{16} + \dfrac{(y-2)^2}{4} = 1$. **(i)** Find the area of the shaded region, giving your answer correct to 3 decimal places. [2] **(ii)** Find the exact volume of the solid generated when the shaded region is rotated 180° about the y-axis. [4]
10	**(a)**	By using the substitution $z = x - y$, solve the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x-y-1}{x-y+1}$. Find the particular solution for which $y = 1$ when $x = 1$. [4]
	(b)	A sky diver jumped out of an aeroplane over a certain mountainous valley with zero speed and t seconds later, the speed of his descent was v metres per second. He experienced gravitational force and air resistance which affect v. Gravity would increase his speed by a constant 10 metres per second^2 and the air resistance would decrease his speed at a rate proportional to the square of his speed. It is given that when his speed reaches 50 metres per second, the rate of change of his speed is 7.5 metres per second^2. By setting up and solving a differential equation, show that $v = \dfrac{100\left(1-\mathrm{e}^{-mt}\right)}{1+\mathrm{e}^{-mt}}$, where m is a constant to be found. [7] Describe briefly what his speed would be after he had descended for a long time and just before he deployed his parachute. [1]
11		



A plastic water dumbbell consists of a cylinder as a handle and two cylinders as the weights. The handle has a radius r cm and height 15 cm. Each weight has radius $3r$ cm and height y cm. The dumbbell is made of plastic of negligible thickness and the volume of the dumbbell is a fixed value k cm^3.

(i) Given that $r = r_1$ is the value of r which gives the minimum external surface area, show that r_1 satisfies the equation $102\pi r^3 + 30\pi r^2 - k = 0$. [6]

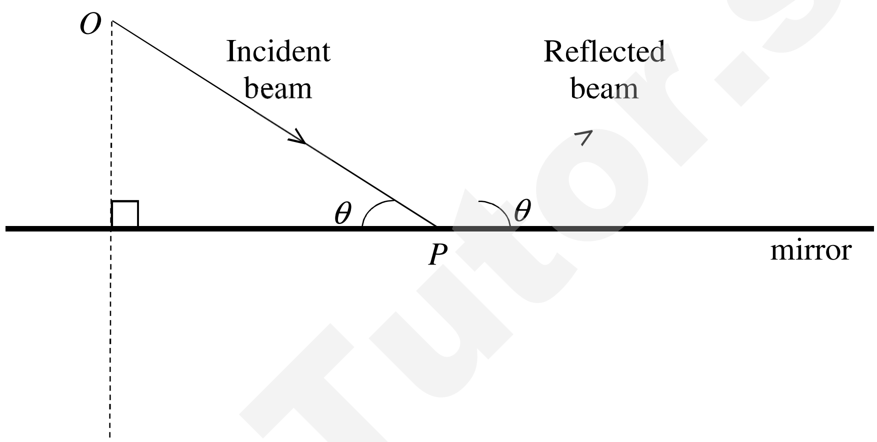
(ii) Find the value of r_1 if $k = 450$. [1]

(iii) It is given instead that $r = 2$ and $y = 7$. Water is pumped into an empty dumbbell through an opening from the top at a rate of 15 cm^{3}s^{-1}. Find the exact rate at which the depth of the water is increasing after 1 minute. [5]

[Question 12 is printed on the next page.]

12	Laser (Light Amplification by Stimulated Emission of Radiation) has many applications including medicine, data storage, military and industrial uses. It has the property of spatial coherence, which allows the laser beam to stay narrow over long distances. When a laser beam is projected onto a mirror at an angle, it reflects off the mirror at the same angle.

An engineer is designing a device that does industrial cutting using a laser beam. To make the device compact, the device has a mirror to reflect the beam before it leaves the device. The laser beam source is located at the origin O. It projects an incident beam with direction vector $\mathbf{i}+2\mathbf{j}+\mathbf{k}$. The beam hits the mirror at the point P with angle θ. The mirror has an equation $-x+2y+3z=12$.



(i) Find the acute angle θ that the beam makes with the mirror. [2]

(ii) By finding O', the image of O in the mirror, find a vector equation of the line that the reflected beam is on. [7]

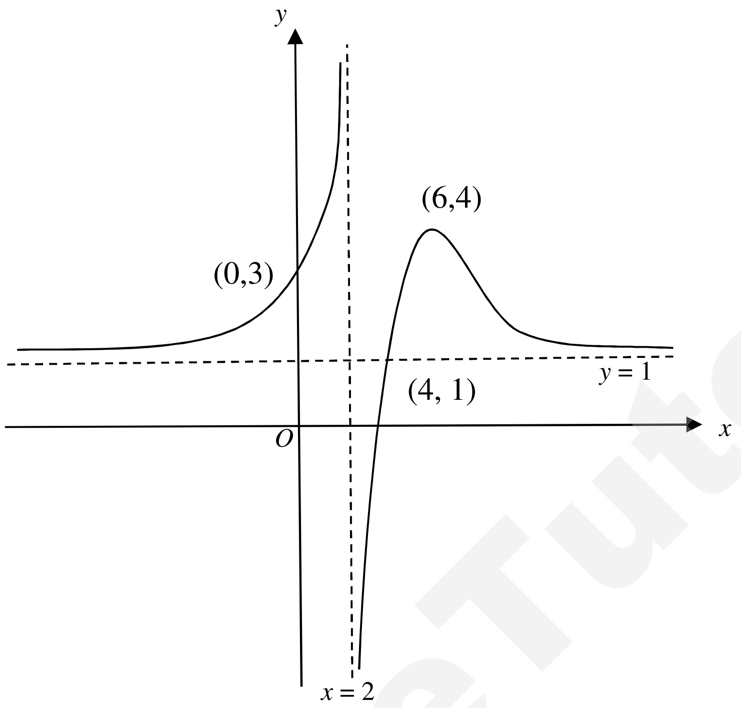
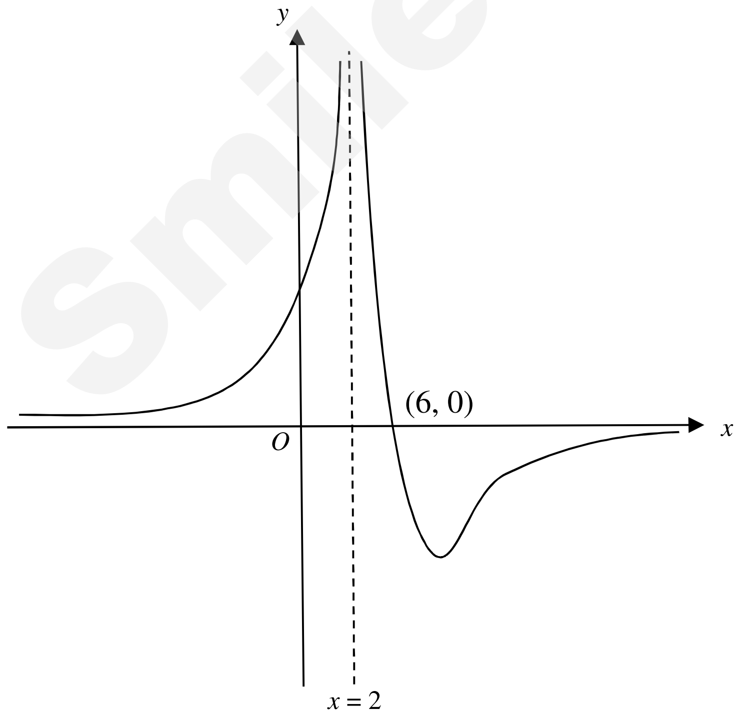
(iii) The engineer plans to install a sensor at (3, 1, 0) to monitor the heat produced by the laser. For the sensor to work properly, the sensor must be less than 2 units away from either the incident or the reflected beam. Determine if the sensor will work properly. [4]

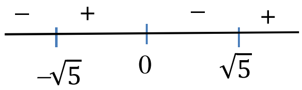
~ ~ End of paper ~ ~

YISHUN JUNIOR COLLEG5E
Mathematics Department

PRELIM Solutions

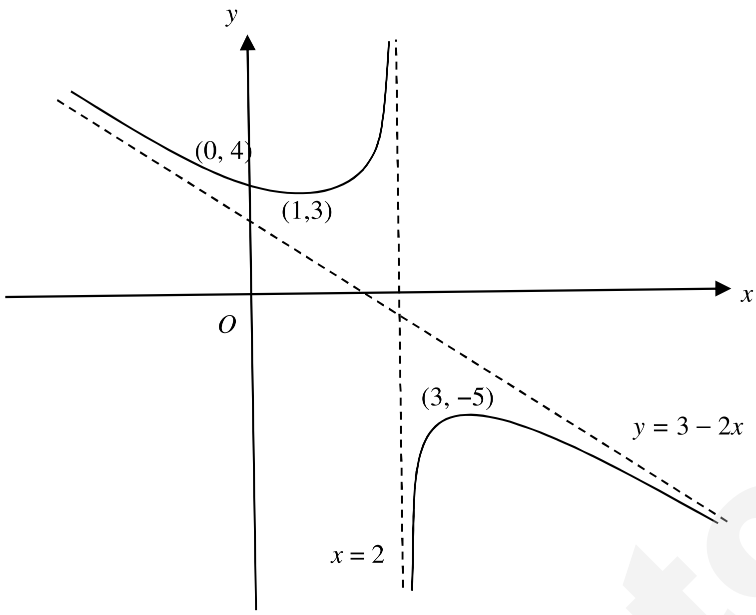
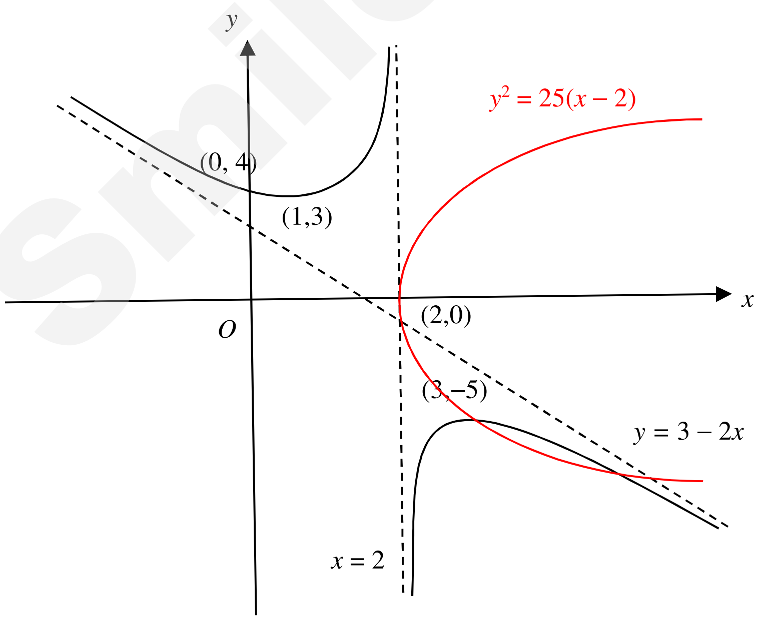
Subject : JC2 H2 MATHEMATICS 9758/9740 P1 **Date** :

Qn	**Solution**
1(i)	
(ii)	

Qn	Solution
2(i)	$\dfrac{x}{x^2-5} \le 0$ $\dfrac{x}{\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)} \le 0$  $\therefore\ x < -\sqrt{5}$ or $0 \le x < \sqrt{5}$
(ii)	$\dfrac{\sqrt{x}}{x-5} \le 0$ $\dfrac{\sqrt{x}}{\left(\sqrt{x}\right)^2-5} \le 0$ Replace x by $\sqrt{x}$ in the result from (i), $\sqrt{x} < -\sqrt{5}$ or $0 \le \sqrt{x} < \sqrt{5}$ (Reject $\because \sqrt{x} \ge 0$) or $0 \le x < 5$ Required set = $\left\{x \in \mathbb{R}:\ 0 \le x < 5\right\}$
3	$\overrightarrow{OP} = \dfrac{2}{5}\mathbf{a}$ $\quad \overrightarrow{OQ} = \dfrac{1}{3}\mathbf{b}$ $\overrightarrow{OM} = \dfrac{1}{2}\left(\dfrac{2}{5}\mathbf{a} + \dfrac{1}{3}\mathbf{b}\right)$ Area of triangle *OMP* $= \dfrac{1}{2}\left\|\left(\dfrac{1}{2}\left(\dfrac{2}{5}\mathbf{a} + \dfrac{1}{3}\mathbf{b}\right)\right) \times \dfrac{2}{5}\mathbf{a}\right\|$ $= \dfrac{1}{2}\left\|\left(\left(\dfrac{1}{5}\mathbf{a} + \dfrac{1}{6}\mathbf{b}\right)\right) \times \dfrac{2}{5}\mathbf{a}\right\|$ $= \dfrac{1}{2}\left\|\dfrac{2}{25}\mathbf{a} \times \mathbf{a} + \dfrac{1}{15}\mathbf{b} \times \mathbf{a}\right\|$ $= \dfrac{1}{2}\left\|\dfrac{1}{15}\mathbf{b} \times \mathbf{a}\right\|$ $= \dfrac{1}{30}\left\|-\mathbf{a} \times \mathbf{b}\right\|$ $= \dfrac{1}{30}\left\|\mathbf{a} \times \mathbf{b}\right\|$

Qn	Solution
4(a)	$\int \cos(\ln x)\ \mathrm{d}x = x\cos(\ln x) - \int -x\sin(\ln x).\frac{1}{x}\ \mathrm{d}x$ $= x\cos(\ln x) + \int \sin(\ln x)\ \mathrm{d}x$ $= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\ \mathrm{d}x$ $2\int \cos(\ln x)\ \mathrm{d}x = x\cos(\ln x) + x\sin(\ln x) + \text{constant}$ $\int \cos(\ln x)\ \mathrm{d}x = \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right] + C$
(b)	$\int \frac{1-2x}{2x^2+1}\ \mathrm{d}x$ $= \frac{1}{2}\int \frac{1}{x^2+\frac{1}{2}}\ \mathrm{d}x - \frac{1}{2}\int \frac{4x}{2x^2+1}\ \mathrm{d}x$ $= \frac{\sqrt{2}}{2}\tan^{-1}\sqrt{2}x - \frac{1}{2}\ln\left(2x^2+1\right) + c$
5(i)	$\lvert z\rvert = \lvert\sqrt{3}+\mathrm{i}\rvert = \sqrt{3+1} = 2$, $\lvert w\rvert = \lvert -1+\mathrm{i}\rvert = \sqrt{1+1} = \sqrt{2}$ $\arg(z) = \arg(\sqrt{3}+\mathrm{i})$ $= \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ $\arg(w) = \arg(-1+\mathrm{i})$ $= \pi - \tan^{-1}1 = \frac{3\pi}{4}$ $\frac{z^2}{w^*} = \frac{\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^2}{\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}}$ $= 2^{\frac{3}{2}}\mathrm{e}^{\mathrm{i}\frac{13\pi}{12}}$ $= 2^{\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\frac{11\pi}{12}}$
(ii)	$\arg\left(1-\frac{q}{z}\right) = \arg\left(\frac{z-q}{z}\right)$ $= \arg\left(z-q\right) - \arg\left(z\right) = \frac{\pi}{12}$ $\arg\left(z-q\right) = \frac{\pi}{12} + \frac{\pi}{6} = \frac{\pi}{4}$ $\arg\left(\left(\sqrt{3}-q\right)+\mathrm{i}\right) = \frac{\pi}{4}$ $\sqrt{3}-q = 1 \ \Rightarrow q = \sqrt{3}-1$

Qn	Solution
6(i)	$y=\ln(3+e^x)$ $\frac{dy}{dx}=\frac{e^x}{3+e^x}$ $\left(3+e^x\right)\frac{dy}{dx}=e^x$ $\frac{d^2y}{dx^2}\left(3+e^x\right)+e^x\frac{dy}{dx}=e^x$ $\frac{d^2y}{dx^2}+\frac{e^x}{\left(3+e^x\right)}\frac{dy}{dx}=\frac{e^x}{\left(3+e^x\right)}$ $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=\frac{dy}{dx}$ (proved)
(ii)	$\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}=\frac{d^2y}{dx^2}$ When $x=0,\ y=\ln 4,\ \frac{dy}{dx}=\frac{1}{4},\ \frac{d^2y}{dx^2}=\frac{3}{16},\ \frac{d^3y}{dx^3}=\frac{3}{32}$ $y=\ln 4+x\left(\frac{1}{4}\right)+\frac{x^2}{2!}\left(\frac{3}{16}\right)+\frac{x^3}{3!}\left(\frac{3}{32}\right)+\ldots$ $=\ln 4+\frac{1}{4}x+\frac{3}{32}x^2+\frac{1}{64}x^3+\ldots$
(iii)	$\frac{e^x}{3+e^x}=\frac{1}{4}+\frac{3}{16}x+\frac{3}{64}x^2+\ldots$ $\frac{e^{-2x}}{3+e^{-2x}}=\frac{1}{4}+\frac{3}{16}(-2x)+\frac{3}{64}(-2x)^2+\ldots$ $=\frac{1}{4}-\frac{3}{8}x+\frac{3}{16}x^2+\ldots$
7(i)	$a=-2$ By long division, $y=(b-4)-2x+\frac{2b-16}{x-2}$. $b-4=3\Rightarrow b=7$ (shown)

Qn	Solution
(ii)	The equation is $y = \dfrac{-2x^2+7x-8}{x-2}$ 
(iii)	$(-2x^2+7x-8)^2 - 25(x-2)^3 = 0$ $(-2x^2+7x-8)^2 = 25(x-2)^3$ $\left(\dfrac{-2x^2+7x-8}{x-2}\right)^2 = 25(x-2)$ Add graph of $y^2 = 25(x-2)$  From the graphs, the number of real roots is 2.

Qn	Solution
8(i)	This is an AP with $a = 1$, $d = 1$. For $S_n \le 1016$ $\frac{n}{2}(1+n) \le 1016$ $n^2 + n - 2032 \le 0$ From GC, $-45.58 \le n \le 44.58$ She can complete a maximum of 44 rows. $S_{44} = \frac{44}{2}(1+44) = 990$ Number of bricks left = 1016 − 990 = 26
(ii)	The sequence is a GP with common ratio 2 $S_{2k-1} = 1016$ $\frac{m\left[2^{2k-1}-1\right]}{2-1} = 1016$ $m\left[2^{2k-1}-1\right] = 1016 \quad ---(1)$ $T_k = 64 \Rightarrow m2^{k-1} = 64$ ---- (2) (1) ÷ (2): $\frac{2^{2k-1}-1}{2^{k-1}} = \frac{1016}{64}$ From GC, $k = 4$ Sub. into (2): $m2^{4-1} = 64$ $\Rightarrow m = 8$ No. of bags = 2(4) − 1 = 7
9(a)	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sec\theta\tan\theta$ $\int_{3\sqrt{2}}^{6} \frac{3x+1}{\sqrt{x^2-9}}\, \mathrm{d}x$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9\sec\theta+1}{\sqrt{9\sec^2\theta-9}}(3\sec\theta\tan\theta)\, \mathrm{d}\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9\sec\theta+1}{3\tan\theta}(3\sec\theta\tan\theta)\, \mathrm{d}\theta$

Qn	Solution
	$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 9\sec^2\theta+\sec\theta \ \mathrm{d}\theta$ $=\left[9\tan\theta+\ln\left\|\sec\theta+\tan\theta\right\|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $=9\tan\frac{\pi}{3}+\ln\left\|\sec\frac{\pi}{3}+\tan\frac{\pi}{3}\right\|-\left(9\tan\frac{\pi}{4}+\ln\left\|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right\|\right)$ $=9\sqrt{3}+\ln\left\|2+\sqrt{3}\right\|-\left(9+\ln\left\|\sqrt{2}+1\right\|\right)$ $=9\sqrt{3}-9+\ln\frac{2+\sqrt{3}}{\sqrt{2}+1}$
(b)(i)	Consider $y=2\pm 2\sqrt{1-\frac{x^2}{16}}$ Required area $=\int_{-3}^{3} 2+2\sqrt{1-\frac{x^2}{16}}\ \mathrm{d}x-2(6)$ $=10.753$ (3 dp) <u>Alternative</u> Consider $\frac{x^2}{16}+\frac{y^2}{4}=1 \Rightarrow y=\pm 2\sqrt{1-\frac{x^2}{16}}$ Required area $=\int_{-3}^{3} 2\sqrt{1-\frac{x^2}{16}}\ \mathrm{d}x$ or $4\int_{0}^{3}\sqrt{1-\frac{x^2}{16}}\ \mathrm{d}x$ $=10.753$ (3 dp)
(ii)	When $x=3$, $y=2+2\sqrt{1-\frac{9}{16}}=2+\frac{1}{2}\sqrt{7}$ When $x=0$, $y=4$ Required Volume $=\frac{\sqrt{7}}{2}\pi(3^2)+\pi\int_{2+\frac{1}{2}\sqrt{7}}^{4} 16\left(1-\frac{(y-2)^2}{4}\right)\mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\int_{2+\frac{1}{2}\sqrt{7}}^{4} 1-\frac{(y-2)^2}{4}\ \mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\left[y-\frac{(y-2)^3}{12}\right]_{2+\frac{1}{2}\sqrt{7}}^{4}$ $=18\pi+16\pi\left[4-\frac{2}{3}-2-\frac{\sqrt{7}}{2}+\frac{7\sqrt{7}}{96}\right]$ $=\frac{1}{3}\left(64-7\sqrt{7}\right)\pi$

Qn	Solution
	<u>Alternative</u> When $x=3$, $y=2\sqrt{1-\frac{9}{16}}=\frac{1}{2}\sqrt{7}$ When $x=0$, $y=2$ Required Volume $=\frac{\sqrt{7}}{2}\pi(3^2)+\pi\int_{\frac{1}{2}\sqrt{7}}^{2}16\left(1-\frac{y^2}{4}\right)\,\mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\int_{\frac{1}{2}\sqrt{7}}^{2}1-\frac{y^2}{4}\,\mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\left[y-\frac{y^3}{12}\right]_{\frac{1}{2}\sqrt{7}}^{2}$ $=\frac{9\sqrt{7}}{2}\pi+\frac{1}{6}\left[128-41\sqrt{7}\right]\pi$ $=\frac{1}{3}\left(64-7\sqrt{7}\right)\pi$
10(a)	$z=x-y\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=1-\frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{\mathrm{d}z}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x-y-1}{x-y+1}$. $\Rightarrow1-\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{z-1}{z+1}$ $\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=1-\frac{z-1}{z+1}$ $\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{2}{z+1}$ $\int(z+1)\,\mathrm{d}z=\int2\,\mathrm{d}x$ $\frac{z^2}{2}+z=2x+C$ $\frac{(x-y)^2}{2}+x-y=2x+C$ where C is a constant $\frac{(x-y)^2}{2}-x-y=C$ When $x=1$, $y=1$, $\Rightarrow C=-2$ Therefore $\frac{(x-y)^2}{2}-x-y=-2$

Qn	Solution
(b)	$\dfrac{dv}{dt}=10-kv^2$, where $k>0$ When $v=50, \dfrac{dv}{dt}=7.5$ $7.5=10-k(50)^2$ $\Rightarrow k=0.001$ $\therefore \dfrac{dv}{dt}=10-0.001v^2$ $\int \dfrac{1}{10-0.001v^2}\ dv=\int 1\ dt$ $\dfrac{1}{0.001}\int \dfrac{1}{10000-v^2}\ dv=\int 1\ dt$ $1000\int \dfrac{1}{100^2-v^2}\ dv=\int 1\ dt$ $\dfrac{1000}{200}\ln\left\|\dfrac{100+v}{100-v}\right\|=t+C$ $\ln\left\|\dfrac{100+v}{100-v}\right\|=\dfrac{1}{5}t+\dfrac{1}{5}C$ $\left\|\dfrac{100+v}{100-v}\right\|=e^{\frac{1}{5}t+\frac{1}{5}C}$ $\dfrac{100+v}{100-v}=\pm e^{\frac{1}{5}t+\frac{1}{5}C}$ $\dfrac{100+v}{100-v}=Ae^{0.2t}$ where $A=\pm e^{0.2C}$ When $t=0$, $v=0$ then $A=1$ $\dfrac{100+v}{100-v}=e^{0.2t}\Rightarrow \dfrac{100-v}{100+v}=e^{-0.2t}$ $e^{-0.2t}(100+v)=100-v$ $v\left(1+e^{-0.2t}\right)=100\left(1-e^{-0.2t}\right)$ $v=\dfrac{100\left(1-e^{-0.2t}\right)}{1+e^{-0.2t}}$ As $t\to\infty$, $e^{-0.2t}\to 0$ and $v\to 100$ The sky diver's speed would increase to a limit of 100 m/s long after he has descended and before he deployed his parachute.

Qn	Solution
11(i)	$V=2\left[\pi(3r)^2\,y\right]+\left(\pi r^2\times 15\right)$ $k=18\pi yr^2+15\pi r^2$ $y=\frac{1}{18\pi r^2}\left(k-15\pi r^2\right)$ $A=4\left[\pi(3r)^2\right]-2\pi r^2+15(2\pi r)+2y\left[2\pi(3r)\right]$ $=34\pi r^2+30\pi r+2\left[\frac{1}{18\pi r^2}\left(k-15\pi r^2\right)\right]\left[2\pi(3r)\right]$ $=34\pi r^2+30\pi r+\frac{2}{3r}\left(k-15\pi r^2\right)$ $=34\pi r^2+20\pi r+\frac{2k}{3r}$ $\frac{\mathrm{d}A}{\mathrm{d}r}=68\pi r+20\pi-\frac{2k}{3r^2}$ At minimum area, $\frac{\mathrm{d}A}{\mathrm{d}r}=0$ $68\pi r+20\pi-\frac{2k}{3r^2}=0$ $204\pi r^3+60\pi r^2-2k=0$ $102\pi r^3+30\pi r^2-k=0$ (shown)
(ii)	$102\pi r^3+30\pi r^2-450=0$ From GC, $r=1.03$ (3 s.f.)
(iii)	Volume of water pumped after 1 min = 15 (60) = 900 cm^3 Volume of a weight = $\pi(3\times 2)^2\times 7=791.68$ cm^3 Volume of the handle = $\pi(2)^2\times 15=188.50$ cm^3 Since $900<791.68+188.50=980.18$, the water level is at the handle at 1 min. Let W = volume of water in the handle and h = depth of water from the base of the handle $W=\pi(2)^2\,h=4\pi h$ $\frac{\mathrm{d}W}{\mathrm{d}h}=4\pi$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}W}{\mathrm{d}t}\times\frac{\mathrm{d}h}{\mathrm{d}W}$ $=15\times\frac{1}{4\pi}$ Thus the depth of the water is increasing at a rate of $\frac{15}{4\pi}$ cm s^{-1}.

Qn	Solution
12(i)	Let θ be the acute angle between the plane and the incident beam. $\sin\theta = \dfrac{\left\lvert \begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\3\end{pmatrix} \right\rvert}{\sqrt{1+4+1}\sqrt{1+4+9}}$ $= \dfrac{6}{\sqrt{84}}$ Therefore $\theta = 40.9°$
(ii)	Let F be the foot of the perpendicular from O to the plane. $\overrightarrow{OF} = \lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}$, for some $\lambda \in \mathbb{R}$ F is on plane $\Rightarrow \overrightarrow{OF}\bullet\begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $\Rightarrow \lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $14\lambda = 12$ $\lambda = \dfrac{6}{7}$ $\overrightarrow{OO'} = \dfrac{12}{7}\begin{pmatrix}-1\\2\\3\end{pmatrix}$ $\overrightarrow{OP} = \mu\begin{pmatrix}1\\2\\1\end{pmatrix}$, for some $\mu \in \mathbb{R}$ P is on plane $\Rightarrow \overrightarrow{OP}\bullet\begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $\Rightarrow \mu\begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $6\mu = 12$ $\mu = 2$ $\overrightarrow{OP} = \begin{pmatrix}2\\4\\2\end{pmatrix}$

Qn	Solution
	$\overrightarrow{O'P} = \frac{2}{7}\begin{pmatrix}13\\2\\-11\end{pmatrix}$ Hence $l: \mathbf{r} = \begin{pmatrix}2\\4\\2\end{pmatrix} + \gamma\begin{pmatrix}13\\2\\-11\end{pmatrix}, \gamma \in \mathbb{R}$
(iii)	Let $B \equiv (3,1,0)$. Shortest distance of *B* from incident beam $= \frac{\left\|\begin{pmatrix}3\\1\\0\end{pmatrix}\times\begin{pmatrix}1\\2\\1\end{pmatrix}\right\|}{\sqrt{1+4+1}} = \frac{\left\|\begin{pmatrix}1\\-3\\5\end{pmatrix}\right\|}{\sqrt{6}} = \sqrt{\frac{35}{6}} > 2$ $\overrightarrow{PB} = \begin{pmatrix}3\\1\\0\end{pmatrix} - \begin{pmatrix}2\\4\\2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix}$ Shortest distance of *B* from reflected beam $= \frac{\left\|\begin{pmatrix}1\\-3\\-2\end{pmatrix}\times\begin{pmatrix}13\\2\\-11\end{pmatrix}\right\|}{\sqrt{169+4+121}} = \frac{\left\|\begin{pmatrix}37\\-15\\41\end{pmatrix}\right\|}{\sqrt{294}} = \sqrt{\frac{3275}{294}} > 2$ Hence sensor will not work properly

H2 Mathematics 2017 Preliminary Exam Paper 2 Question (9758)
Answer all questions [100 marks]

Section A: Pure Mathematics [40 marks]

1	**(i)** Show that if $a_r = T_r - T_{r-1}$ for r = 1, 2, 3, …, and $T_0 = 0$, then $$\sum_{r=1}^{n} a_r = T_n .$$ [1] **(ii)** Deduce that $\sum_{r=1}^{n} \pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right] = n^2\pi^{-n}$. [3] **(iii)** Hence, find the exact value of $\sum_{r=4}^{20} \pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$. [2]
2	Functions g and h are defined by $\mathrm{g} : x \mapsto x^2 + 6x + 8,\ x \in \mapsto,\ x \le \alpha,$ $\mathrm{h} : x \mapsto -\mathrm{e}^x,\ x \in \mapsto,\ x > -2.$ **(i)** Given that the function g^{-1} exists, write down the largest value of α and define g^{-1} in similar form. State a transformation which will transform the curve $y = \mathrm{g}(x)$ onto the curve $y = \mathrm{g}^{-1}(x)$. [5] **(ii)** Given instead that $\alpha = -2$, explain why the composite function hg exists and find the exact range of hg. [2]
3	**Do not use a calculator in answering this question.** Given that $z = 1 + \mathrm{i}$ is a root of the equation $2z^4 + az^3 + 7z^2 + bz + 2 = 0$, find the values of the real numbers a and b and the other roots. [5] Deduce the roots of the equation $2z^4 + bz^3 + 7z^2 + az + 2 = 0$. [2]
4	A curve C has parametric equations $$x = t^2, \quad y = t - t^3,\ t \le 0.$$ **(i)** The point P on the curve has parameter p. Show that the equation of the tangent at P is $2py = x\left(1 - 3p^2\right) + p^2 + p^4$. [3] **(ii)** If the tangent at P passes through the point (6, 5), find the possible coordinates of P. [3] **(iii)** Find the area of the region bounded by C and the x-axis. [3]

5 The planes p_1 and p_2 have equations $x-4y+8z=4$ and $mx+ny+2z=1$ respectively, where m and n are constants.

(i) If p_1 and p_2 meet at a line that has equation $\mathbf{r}=2\mathbf{i}-0.5\mathbf{j}+\lambda(-4\mathbf{i}+\mathbf{j}+\mathbf{k})$, where $\lambda \in \mapsto$, find the values of m and n. [3]

It is given instead that $m = 1$ and $n = 2$.

(ii) Find the acute angle between p_1 and p_2. [2]

(iii) The point $(1, b, 5)$ is equidistant from p_1 and p_2. Calculate the possible value(s) of b. [6]

Section B: Statistics [60 marks]

6 **(a)** Find the number of ways to arrange the letters of the word **TOTORO** such that

(i) all the 'O's are together, [1]
(ii) all the 'O's are separated, [2]
(iii) the last letter is a consonant. [3]

(b) Tontoro soft toys are sold in four different colours, of which each varies in three sizes, small, medium and large. Each set of Tontoro soft toys consists of a small, a medium and a large sized soft toy and exactly two are of the same colour. Find the number of different possible sets of Tontoro soft toys.
[2]

7 A game is played with a set of 4 cards, each distinctly numbered 1, 2, 3 and 4. A player randomly picks a pair of cards without replacement. If the sum of the cards' numbers is an odd number, the sum is the player's score. If the sum of the two cards' number is an even number, the player randomly picks a third card from the remaining cards. The square of the third card's number is the player's score.

(i) Find the probability that a player obtains a score of 4. [2]

(ii) Find the probability distribution of a player's score, S. Hence, find the expected score of a player. [4]

(iii) Find the probability that a player obtains a score lower than 5, given that he draws three cards. [3]

8	An archaeologist examines rocks to look for fossils. On average, 10% of the rocks selected from a particular area with a large number of rocks contain fossils. The archaeologist selects a random sample of 25 rocks from this area. The number of rocks that contain fossils is denoted by X.

(i) Find the probability that more than 4 but at most 10 rocks contain fossils. [2]

(ii) Show that $\dfrac{\mathrm{P}(X=k+1)}{\mathrm{P}(X=k)}=\dfrac{25-k}{9(k+1)}$, for k = 0, 1, 2, 3, …, 24. Hence, by considering $\mathrm{P}(X=k+1)>\mathrm{P}(X=k)$, find the most probable value of X. [4]

The archaeologist explores a new area. On average, p% (p > 10) of the rocks in the new area contain fossils. A random sample of 20 rocks from the new area is selected. Given that the probability that there are two rocks that contain fossils is 0.17, find the value of p, giving your answer correct to 2 decimal places. [3]

9 A researcher investigates the relationship between the population of a particular species of bacteria in millions (b) and the surrounding temperature in °C (t). The researcher keeps records so that she can estimate the population of the bacteria at a certain temperature. Observations at different temperatures give the data as shown in the following table.

t	26.5	27.5	28.5	29.5	30.5	31.5	32.5
b	1.31	2.10	3.65	5.80	α	19.56	31.20

(i) Given that the regression line of b on t is $b=-129.368+4.75214t$, show that $\alpha=12.12$, correct to 2 decimal places. [2]

(ii) Sketch a scatter diagram for the data. [1]

(iii) Explain which of $b=ct+d$ or $b=kt^3+l$ is the more appropriate model for the relationship between b and t and find the equation of a suitable regression line for this model. [2]

(iv) Use the model you chose in part **(iii)** to estimate the population of the bacteria when the temperature is 33°C. Comment on the reliability of the estimate obtained. [2]

(v) It is given that the temperature T, in °F, is related to the temperature t, in °C, by the equation $T=1.8t+32$. Rewrite your equation from part **(iii)** so that it can be used to estimate the population of bacteria when the temperature is given in °F. [2]

10	In a factory, the average time taken by a machine to assemble a smartphone is 53 minutes. A new assembly process is trialled and the time taken to assemble a smartphone, x minutes, is recorded for a random sample of 60 smartphones. The total time taken was found to be 3129 minutes and the variance of the time was 18.35 minutes^2. The engineer wants to test whether the average time taken by a machine to assemble a smartphone has decreased, by carrying out a hypothesis test. **(i)** Explain why the engineer is able to carry out a hypothesis test without assuming anything about the distribution of the times taken to assemble a smartphone. [1] **(ii)** Find unbiased estimates of the population mean and variance and carry out the test at the 10% level of significance. [6] **(iii)** Explain, in the context of the question, the meaning of 'at 10% level of significance'. [1] After several trials, the engineer claims that the average time taken by a machine to assemble a smartphone is 45 minutes using the new assembly process. The internal control manager wishes to test whether the engineer's claim is valid. The population variance of the time taken to assemble a smartphone using the new assembly process may be assumed to be 9 minutes^2. A random sample of 50 smartphones is taken. **(iv)** Find the range of values of the mean time of this sample for which the engineer's claim would be rejected at the 10% significance level. [4] **[Question 11 is printed on the next page.]**

11	In the manufacture of child car seats, a resin made up of three ingredients is used. The ingredients are polymer A, polymer B and an impact modifier. The resin is prepared in batches and each ingredient is supplied by a separate feeder. The masses, in kg, of polymer A, polymer B and the impact modifier in each batch of resin are assumed to be normally distributed with means and standard deviations as shown in the table. The three feeders are also assumed to operate independently of each other.

	Mean	Standard deviation
Polymer A	2030	44.8
Polymer B	1563	22.7
Impact modifier	μ	σ

It is known that 3% of the batches of resin have less than 1350 kg of impact modifier and 30% of the batches of resin have more than 1414 kg of impact modifier.

(i) Show that $\mu \approx 1400$ and $\sigma \approx 26.6$. [3]

(ii) Given that polymer A costs \$2.20 per kg, polymer B costs \$2.80 per kg and the impact modifier costs \$1.50 per kg, find the probability that the total cost of 2 batches of resin exceeds \$22,000. [4]

(iii) A random sample of n batches of resin is chosen. If the probability that at most 6 batches of resin has more than 1414 kg of impact modifier is less than 0.001, find the least value of n. [3]

(iv) Each batch of resin is used to make a large number of car seats. It is found that the tensile strength (N/m^2) of resin for a car seat has mean 125 and standard deviation 17. A random sample of 50 car seats is selected. Find the probability that the average tensile strength of resin for these 50 car seats is less than 130 N/m^2. [3]

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YISHUN JUNIOR COLLEGE

Mathematics Department

PRELIM Solution

Subject : JC2 H2 MATHEMATICS 9758 P2 **Date** :

Qn	Solution
1(i)	$\sum_{r=1}^{n} a_r = \sum_{r=1}^{n}\left(T_r - T_{r-1}\right)$ $= \cancel{T_1} - T_0$ $+ \cancel{T_2} - \cancel{T_1}$ $+ \cancel{T_3} - \cancel{T_2}$ $\vdots$ $+ \cancel{T_{n-1}} - \cancel{T_{n-2}}$ $+ T_n - \cancel{T_{n-1}}$ $= T_n - T_0$ $= T_n$
(ii)	Let $T_r = r^2\pi^{-r}$ Note $T_0 = 0$ $T_r - T_{r-1} = r^2\pi^{-r} - (r-1)^2\pi^{-r+1}$ $= \pi^{-r}\left[r^2 - (r^2-2r+1)\pi\right]$ $= \pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$ $= a_r$ $\therefore$ From (i), $\sum_{r=1}^{n}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right] = \sum_{r=1}^{n} a_r$ $= T_n = n^2\pi^{-n}$
(iii)	$\sum_{r=4}^{20}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$ $= \sum_{r=1}^{20}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right] - \sum_{r=1}^{3}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$ $= 400\pi^{-20} - 9\pi^{-3}$
2(i)	Largest $\alpha = -3$ Let $y = \mathrm{g}(x) = x^2 + 6x + 8$ $= (x+3)^2 - 1$ $(x+3)^2 = y + 1$ $x + 3 = \pm\sqrt{y+1}$

Qn	Solution
	$x=-3\pm\sqrt{y+1}$ Since $x\le -3$, $x=-3-\sqrt{y+1}$ $\mathrm{g}^{-1}: x \mapsto -3-\sqrt{x+1}$, $x\in[-1,\infty)$ A reflection about the line $y=x$ will transform the curve $y=\mathrm{g}(x)$ onto the curve $y=\mathrm{g}^{-1}(x)$.
(ii)	Since $\mathrm{R_g}=[-1,\infty)\subseteq(-2,\infty)=\mathrm{D_h}$, the composite function hg exists. $\mathrm{R_{hg}}=\left(-\infty,-\dfrac{1}{\mathrm{e}}\right]$
3	By Conjugate Root Theorem, $z=1-\mathrm{i}$ is also a root. $\left[z-(1+\mathrm{i})\right]\left[z-(1-\mathrm{i})\right]=\left[(z-1)-\mathrm{i}\right]\left[(z-1)+\mathrm{i}\right]$ $=(z-1)^2-\mathrm{i}^2$ $=z^2-2z+2$ $\left(z^2-2z+2\right)\left(Az^2+Bz+C\right)=2z^4+az^3+7z^2+bz+2$ By observation, $A=2$, $C=1$. i.e. $\left(z^2-2z+2\right)\left(2z^2+Bz+1\right)=2z^4+az^3+7z^2+bz+2$ Coeff. of z^2 : $1-2B+4=7\Rightarrow B=-1$ Coeff. of z^3 : $B-4=a\Rightarrow a=-5$ Coeff. of z : $-2+2B=b\Rightarrow b=-4$ $2z^2-z+1=0$ $z=\dfrac{1\pm\sqrt{1-4(2)}}{2(2)}$ $z=\dfrac{1\pm\sqrt{7}\mathrm{i}}{4}$ Hence other roots are $1-\mathrm{i}$, $\dfrac{1\pm\sqrt{7}\mathrm{i}}{4}$. $2z^4+bz^3+7z^2+az+2=0$ $2+b\dfrac{1}{z}+7\dfrac{1}{z^2}+a\dfrac{1}{z^3}+2\dfrac{1}{z^4}=0$ Hence $z=\dfrac{1}{1+\mathrm{i}},\quad \dfrac{1}{1-\mathrm{i}},\quad \dfrac{4}{1+\sqrt{7}\mathrm{i}},\quad \dfrac{4}{1-\sqrt{7}\mathrm{i}}$ $z=\dfrac{1-\mathrm{i}}{2},\quad \dfrac{1+\mathrm{i}}{2},\quad \dfrac{1-\sqrt{7}\mathrm{i}}{2},\quad \dfrac{1+\sqrt{7}\mathrm{i}}{2}$

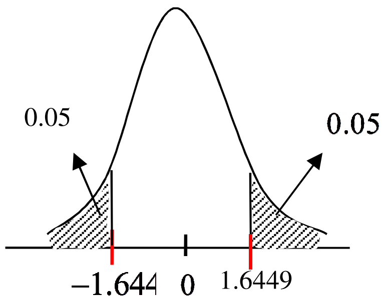
Qn	Solution
4(i)	$x=t^2$, $y=t-t^3$. $\frac{dx}{dt}=2t$, $\frac{dy}{dt}=1-3t^2$ $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{1-3t^2}{2t}$ At P, $x=p^2$, $y=p-p^3$, $\frac{dy}{dt}=\frac{1-3p^2}{2p}$ Equation of tangent at P: $\frac{y-(p-p^3)}{x-p^2}=\frac{1-3p^2}{2p}$ $\Rightarrow 2py-2p(p-p^3)=\left(x-p^2\right)\left(1-3p^2\right)$ $\Rightarrow 2py-2p^2+2p^4=x\left(1-3p^2\right)-p^2+3p^4$ $\Rightarrow 2py=x\left(1-3p^2\right)+p^2+p^4$ (shown) -------(1)
(ii)	At A, substitute $x=6$, $y=5$ into eqn (1) $2p(5)=6\left(1-3p^2\right)+p^2+p^4$ $10p=6-18p^2+p^2+p^4$ $p^4-17p^2-10p+6=0$ From GC, $p=4.35$ (rejected) or $p=-3.7261$ or $p=-1$ or $p=0.370$ (rejected) Hence coordinates of P: (1,0) and (13.9, 48.0)
(iii)	Required area = $-\int_0^1 y\ dx$ $=-\int_0^{-1}\left(t-t^3\right)(2t)\ dt$ $=0.267\text{ unit}^2$
5(i)	If p_1 and p_2 meet at l, then $\mathbf{m}$ is perpendicular to $\mathbf{n_2}$. $\mathbf{m}\bullet\mathbf{n_2}=0\Rightarrow\begin{pmatrix}2\\1\\1\end{pmatrix}\bullet\begin{pmatrix}m\\n\\2\end{pmatrix}=0$ $2m+n=-2$ Since $(2,-0.5,0)$ lies on p_2, $2m-0.5n=1$ $m=0$ $n=-2$
(ii)	Let θ be the acute angle between p_1 and p_2. $\cos\theta=\frac{\left\|\begin{pmatrix}1\\-4\\8\end{pmatrix}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}\right\|}{\sqrt{1+16+64}\sqrt{1+4+4}}$ $=\frac{1}{3}$ Therefore $\theta=70.5°$

Qn	Solution
(iii)	Let $B \equiv (1, b, 5)$. Observe $A_1(4,0,0)$ lies on p_1 $\overrightarrow{A_1B} = \begin{pmatrix}1\\b\\5\end{pmatrix} - \begin{pmatrix}4\\0\\0\end{pmatrix} = \begin{pmatrix}-3\\b\\5\end{pmatrix}$ Shortest distance of B from p_1 $= \dfrac{\left\|\begin{pmatrix}-3\\b\\5\end{pmatrix} \bullet \begin{pmatrix}1\\-4\\8\end{pmatrix}\right\|}{\sqrt{1+16+64}} = \dfrac{\|37-4b\|}{9}$ Observe $A_2(1,0,0)$ lies on p_2 $\overrightarrow{A_2B} = \begin{pmatrix}1\\b\\5\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\b\\5\end{pmatrix}$ Shortest distance of B from p_1 $= \dfrac{\left\|\begin{pmatrix}0\\b\\5\end{pmatrix} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix}\right\|}{\sqrt{1+4+4}} = \dfrac{\|10+2b\|}{3}$ $\dfrac{37-4b}{9} = \dfrac{2b+10}{3}$ or $\dfrac{37-4b}{9} = -\dfrac{2b+10}{3}$ $b = \dfrac{7}{10}$ $\quad$ $b = -\dfrac{67}{2}$
6(a)(i) (ii) (iii) (b)	No. of ways = $\dfrac{4!}{2!} = 12$ No. of ways = $\dfrac{3! \times {}^4C_3}{2!} = 12$ Case 1: Ending with "T" No. of ways $= \dfrac{5!}{3!} = 20$ Case 2: Ending with "R" No. of ways $= \dfrac{5!}{3!2!} = 10$ Total no. of ways $= 20 + 10 = 30$ Choose the two sizes that have the same colour: ${}^3C_2 = 3$ Choose colour that is same for two sizes: ${}^4C_1 = 4$ Choose colour of remaining size: ${}^3C_1 = 3$ No. of ways = ${}^3C_2 \times {}^4C_1 \times {}^3C_1 = 36$

Qn	Solution
7(i)	P(score of 4) $=$P(obtain 1 and 3 for the first 2 cards, and obtain 2 for the third card) $=\left(\frac{1}{4}\times\frac{1}{3}\times 2\right)\times\frac{1}{2}=\frac{1}{12}$
(ii)	<table><tr><td>s</td><td>1</td><td>3</td><td>4</td><td>5</td><td>7</td><td>9</td><td>16</td></tr><tr><td>P($S=s$)</td><td>$\frac{1}{12}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{12}$</td><td>$\frac{4}{12}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td></tr></table>Expected score $=1\left(\frac{1}{12}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{12}\right)+5\left(\frac{4}{12}\right)+7\left(\frac{1}{6}\right)+9\left(\frac{1}{12}\right)+16\left(\frac{1}{12}\right)$ $=\frac{35}{6}$
(iii)	P(score < 5 | draws three cards) $=\frac{\text{P(score <5 and draws three cards)}}{\text{P(draws three cards)}}$ $=\frac{\text{P(score 4 and 3 cards)+P(score 1 and 3 cards)}}{\text{P(obtain 1,3 or 2,4 for first two cards)}}$ $=\frac{\frac{1}{12}+\frac{1}{12}}{\left(\frac{1}{4}\times\frac{1}{3}\times 2\right)\times 2}$ $=\frac{1}{2}$
8(i)	Let X be the random variable 'number of rocks that contain fossils out of 25 rocks' $X\sim \mathrm{B}(25,0.1)$ $\mathrm{P}(4<X\le 10)=\mathrm{P}(X\le 10)-\mathrm{P}(X\le 4)$ ≈ 0.0979819403 ≈ 0.0980 (3 sig fig)
(ii)	$\frac{\mathrm{P}(X=k+1)}{\mathrm{P}(X=k)}=\frac{{}^{25}C_{k+1}(0.1)^{k+1}(0.9)^{25-k-1}}{{}^{25}C_{k}(0.1)^{k}(0.9)^{25-k}}$ $=\frac{\frac{25!}{(k+1)!(25-k-1)!}(0.1)^{k+1}(0.9)^{25-k-1}}{\frac{25!}{k!(25-k)!}(0.1)^{k}(0.9)^{25-k}}$ $=\frac{(25-k)(0.1)}{(k+1)(0.9)}=\frac{25-k}{9(k+1)}$ for k = 0, 1,2 …, 24 (shown) $\mathrm{P}(X=k+1)>\mathrm{P}(X=k)$ $\frac{\mathrm{P}(X=k+1)}{\mathrm{P}(X=k)}=\frac{(25-k)}{9(k+1)}>1$ $25-k>9k+9$ $10k<16$

Qn	Solution
	$k < 1.6$ $\Rightarrow\ k = 0$ or 1 for $\mathrm{P}(X = k+1) > \mathrm{P}(X = k)$ Since $\mathrm{P}(X=2) > \mathrm{P}(X=1) > \mathrm{P}(X=0)$, most probable value of $X = 2$
	Let Y be the 'number of rocks that contain fossils out of 20 rocks in the new area' $Y \sim \mathrm{B}(20,\ \frac{p}{100})$ $\mathrm{P}(Y=2) = 0.17$ Using g.c. $\frac{p}{100} = 0.045473$ or $\frac{p}{100} = 0.1815827$ Since $p > 10$, $p = 18.16$ (2 d.p)
9(i)	$b = -129.39 + 4.7529t$ From GC, $\bar{t} = 29.5$ $\bar{b} = -129.39 + 4.7529\bar{t}$ $\bar{b} = -129.39 + 4.7529(29.5)$ $= 10.82055$ $\frac{1.31+2.1+3.65+5.8+\alpha+19.56+31.2}{7} = 10.82055$ $\alpha = 12.124 = 12.12$ (2 dp)
(ii)	
(iii)	From (ii), the scatter diagram shows that as t increases, b increases at an increasing rate which would not be the case if the data follows a linear model. Hence the model $b = kt^3 + l$ is a better model. $b = -37.370 + 0.0018516t^3$ $= -37.4 + 0.00185t^3$ (3s.f.)

Qn	Solution
(iv)	When $t = 33$, $b = -37.370 + 0.0018516(33)^3$ $= 29.171$ $= 29.2\,(3\text{s.f.})$ The population of the bacteria is 29.2 millions. Since the estimate is obtained via extrapolation, the estimate is not reliable.
(v)	$b = -37.370 + 0.0018516\left(\frac{T-32}{1.8}\right)^3$ $= -37.370 + (3.1749\times10^{-4})(T-32)^3$ $= -37.4 + (3.17\times10^{-4})(T-32)^3 \ \ (3\text{ s.f.})$
10(i)	Since n is large, by Central Limit Theorem, the sample mean time for 60 smartphones is approximately normal. Hence the assumption that the time taken by a machine to assembly a smartphone is not necessary.
(ii)	Unbiased estimate for population mean μ is $\bar{x}$ $= \frac{3129}{60} = 52.15$ Unbiased estimate for population variance σ^2 is s^2 $= \frac{60}{59}(18.35)$ $= 18.661$ $= 18.7$ (3sf) $H_0 : \mu = 53$ $H_1 : \mu < 53$ Under H_0, the test statistic $Z = \frac{\overline{X}-\mu}{S/\sqrt{n}} \sim N(0,1)$ approx. by CLT, where $\mu = 53, s = \sqrt{18.661}, \bar{x} = 52.15,\ n = 60$. By GC, p-value $= 0.0637\,(3\text{ s.f.})$. Since p-value < 0.1, we reject H_0 and conclude at 10% level that there is sufficient evidence that average time taken by a machine to assembly a smartphone has reduced.
(iii)	There is a probability of 0.1 of concluding that the average time taken by a machine to assembly a smartphone has decreased when the average time taken by a machine to assembly a smartphone is 53 minutes.

Qn	Solution
(iv)	$H_0 : \mu = 45$ $H_1 : \mu \neq 45$ Under H_0, the test statistic $Z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$ approx. by CLT, where $\mu = 45, \sigma = \sqrt{9},\ n = 50$.  Since H_0 is rejected, $\dfrac{\overline{x} - 45}{\sqrt{9}/\sqrt{50}} < -1.6449$ or $\dfrac{\overline{x} - 45}{\sqrt{9}/\sqrt{50}} > 1.6449$ $\overline{x} < 44.3021$ $\overline{x} > 45.698$ $\overline{x} < 44.3\,(3\text{ s.f.})$ $\overline{x} > 45.7\,(3\text{ s.f.})$ Range of values of $\overline{x}$: $\overline{x} < 44.3\,(3\text{ s.f.})$ or $\overline{x} > 45.7\,(3\text{ s.f.})$
11(i)	Let X be the random variable 'amount (in kg) of impact modifier in a batch of resin' $X \sim N(\mu, \sigma^2)$ $P(X < 1350) = 0.03$ $P(Z < \dfrac{1350 - \mu}{\sigma}) = 0.03$ $\dfrac{1350 - \mu}{\sigma} = -1.88079361$ $\mu - 1.88079361\sigma = 1350 \text{ --(1)}$ $P(X > 1414) = 0.3$ $P(Z < \dfrac{1414 - \mu}{\sigma}) = 0.7$ $\dfrac{1414 - \mu}{\sigma} = 0.5244005101$ $\mu + 0.5244005101\sigma = 1414 \text{ ---(2)}$ Solve (1) and (2), $\mu = 1400.046 = 1400$ (shown) $\sigma = 26.609 = 26.6$ (shown)

Qn	Solution
(ii)	Let Y be the random variable 'amount (in kg) of Polymer A in a batch of resin' Let W be the random variable 'amount (in kg) of Polymer B in a batch of resin' $Y \sim N(2030, 44.8^2)$, $W \sim N(1563, 22.7^2)$ Total cost of a batch, $T = 2.20Y + 2.80W + 1.50X \sim N(10942.4, 15345.9572)$ Total cost of 2 batches, $T_1 + T_2 \sim N(21884.8, 30715.9144)$ $P(T_1 + T_2 > 22000) = 0.255$ (3.s.f.)
(iii)	Let H be the r.v.' number of batches of resin with more than 1414 kg of impact modifier out of n batches.' $H \sim B(n, 0.3)$ $\mathrm{P}(H \le 6) < 0.001$ Using GC. When n = 53, $\mathrm{P}(H \le 6) = 0.00120 > 0.001$ When n = 54, $\mathrm{P}(H \le 6) = 9.44 \times 10^{-4} < 0.001$ Therefore, least n = 54
(iv)	Let S be the r.v.' tensile strength (in N/m^2) of resin in a car seat' E(S) = 125, Var(S) =17^2 $\overline{S} = \dfrac{S_1 + S_2 + ... + S_{50}}{50}$ $\overline{S} \sim N\left(125, \dfrac{17^2}{50}\right)$ approx by Central Limit Thm $\mathrm{P}(\overline{S} < 130) = 0.981$ (3 s.f.)