ANGLO-CHINESE JUNIOR COLLEGE
MATHEMATICS DEPARTMENT

MATHEMATICS
Higher 2

9740 / 01

Paper 1

18 August 2016

JC 2 PRELIMINARY EXAMINATION

Time allowed: **3 hours**

Additional Materials: List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Index number, Form Class, graphic and/or scientific calculator model/s on the cover page.
Write your Index number and full name on all the work you hand in.
Write in dark blue or black pen on your answer scripts.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in the question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **5** printed pages.

Anglo-Chinese Junior College

[Turn Over

1 Without the use of a calculator, solve the inequality $\dfrac{2w-5}{w^2-3}>0$. [2]

Hence solve $\dfrac{(2|y|-5)\sin x}{y^2-3}\le 0$, given that $\pi < x \le \dfrac{3\pi}{2}$. [3]

2 The equation of the curve C is given by $y=\ln x$. The line ℓ with the equation $y=\dfrac{x}{\mathrm{e}}$ is tangential to the curve C at the point $(\mathrm{e},1)$. The region R is bounded by the curve C, the line ℓ and the x-axis. The solid S is formed by rotating the region R through 2π radians about the x-axis. Find the exact volume of the solid S in terms of π and e. [5]

3 **(i)** Every year Warren Gate's net worth increases by 100% of the previous year. His net worth was estimated to be \$1 000 000 on 31st December 1993. In what year will his fortune first surpass 2.5 billion dollars (1 billion = 10^9). [3]

(ii) On 1st January 2005, Warren Gates deposits \$100 000 in an investment account and receives an interest of \$1000 on 31st December 2005. After that the amount of interest earned at the end of the year is 1.5 times the amount of interest earned in the previous year. Taking year 2005 as the first year, find the amount of savings that Warren Gates has in his account at the end of the 15th year giving your answer to the nearest integer.

[2]

4 **(a)** It is given that $\mathrm{g}(x)=\dfrac{1}{\cos(\frac{\pi}{4}+x)\cos(\frac{\pi}{4}-x)}$ where x is sufficiently small for x^3 and higher powers of x to be neglected.

Show that $\mathrm{g}(x)\approx 2+ax+bx^2$, where a and b are constants. [3]

Comment on the value of m for this expression $\displaystyle\int_{-m}^{m}\mathrm{g}(x)\,\mathrm{d}x \approx \int_{-m}^{m}(2+ax+bx^2)\,\mathrm{d}x$ to be valid. [1]

(b) Find the first four non-zero terms of the expansion of $\left(1-x^2\right)^{-\frac{1}{2}}$ in ascending powers of x where $|x|<1$. [2]

Hence find the first four non-zero terms of the Maclaurin's series for $\cos^{-1}x$ in ascending powers of x. [3]

[Turn Over

5 A sequence $u_1,\ u_2,\ u_3,\ \ldots$ is given by

$$u_1 = 1 \text{ and } \frac{1}{3^r}\left(\frac{u_{r+1}}{3} - u_r\right) = 2r \text{ for all } r \geq 1.$$

Use the method of differences to prove that $u_{n+1} = 3^n(3n^2 + 3n + 1)$ for all $n \geq 1$. [4]

6 An investor deposits $\$K$ in a bank account. The bank offers an annual interest rate of 5% compounded continuously. No further deposits are made. The amount of money in the account at time t years is denoted by M. Both M and t are taken to be continuous variables. Money is withdrawn at a continuous rate of $4000 per year. Set up a differential equation and show that for $t > 0$, $\frac{\mathrm{d}M}{\mathrm{d}t} = aM + b$, where a and b are constants to be determined. [1]

For $t > 0$, find M in terms of t and K. [4]

On a single clearly labelled diagram, show the graph of M against t for

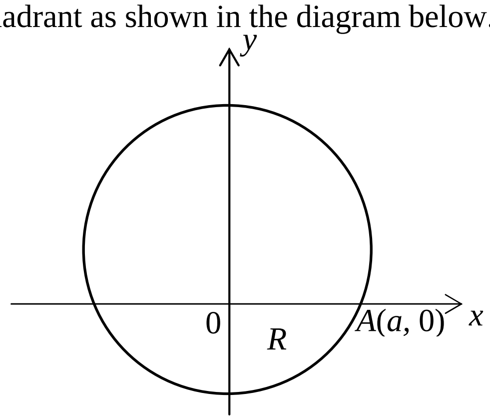
(i) $K > 80\,000$, [1]

(ii) $K < 80\,000$. [1]

Hence state the condition for which the money deposited initially will be completely withdrawn in a finite period of time. [1]

7 **(i)** Use the substitution $x = 5\sin\theta$ to find $\int \sqrt{25 - x^2}\,\mathrm{d}x$. [5]

(ii) The circle with equation $x^2 + (y - b)^2 = 25$ where $0 < b < 5$, cuts the positive x-axis at $A(a,\ 0)$. The region R is bounded by the x and y axes, and the part of the circle lying in the fourth quadrant as shown in the diagram below.



Use your result in **(i)** to find the area of the region R in terms of a. [3]

8 The complex number z is given by $z = k + \mathrm{i}$ where k is a non-zero real number.

(i) Find the possible values of k if $z = k + \mathrm{i}$ satisfies the equation $z^3 - \mathrm{i}z^2 - 2z - 4\mathrm{i} = 0$. [3]

(ii) For the complex number z found in part **(i)** for which $k > 0$, find the smallest integer value of n such that $\left|z^n\right| > 100$ and z^n is real. [3]

9 Use the method of mathematical induction to prove that

$$\sum_{r=1}^{n} \frac{3r+1}{r(r+1)(r+2)} = \frac{n(7n+9)}{4(n+1)(n+2)}, \text{ for all } n \geq 1.$$ [5]

(i) Show that $\dfrac{n(7n+9)}{4(n+1)(n+2)} < \dfrac{7}{4}$. [2]

(ii) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{3r}{(r+1)^3} < \frac{7}{4}$. [2]

10 The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2 - 4k^2}{x - k}$. It is given that k is a constant and $x \neq k$.

Find the set of possible values that y can take. [3]

For the case $k > 1$,

(i) Sketch the graph of C, stating in terms of k, the coordinates of any points of intersection with the axes and equations of any asymptotes. [3]

(ii) Hence find $\displaystyle\int_{-1}^{1} \mathrm{f}\left(|x|\right) \mathrm{d}x$ in terms of k. [3]

(iii) The graph of curve C is transformed by a scaling of factor 2 parallel to the x-axis, followed by a translation of $2k$ units in the negative x-direction. Find the equation of the new curve. You need not simplify your answer. [2]

11 Referred to the origin O, the position vectors of the points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Given that $\mathbf{a} \times \mathbf{b} = 4\mathbf{a} \times \mathbf{c}$, where $\mathbf{b} \neq 4\mathbf{c}$ and $\mathbf{a}$ is a non-zero vector,

(i) show that $\mathbf{b} - 4\mathbf{c} = \alpha\mathbf{a}$ where α is a scalar. [1]

(ii) Hence evaluate $\left|\mathbf{b} \times \mathbf{c}\right|$, given that the area of triangle OAB is $\sqrt{126}$ and $\alpha = \sqrt{3}$. [2]

(iii) Give the geometrical meaning of $\left|\mathbf{b} \times \mathbf{c}\right|$. [1]

It is also given that $\mathbf{b}$ is a unit vector, $|\mathbf{a}| = 5$, $|\mathbf{c}| = 2$ and $\mathbf{b} - 4\mathbf{c} = \sqrt{3}\mathbf{a}$.

(iv) By considering $(\mathbf{b} - 4\mathbf{c}) \cdot (\mathbf{b} - 4\mathbf{c})$, find the angle between $\mathbf{b}$ and $\mathbf{c}$. [3]

[Turn Over

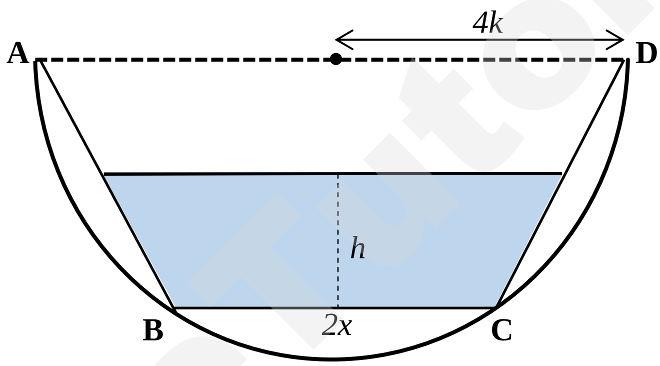
12 **(i)** Solve the equation $z^5 - \mathrm{i} = 0$, giving the roots in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. Show the roots z_1 , z_2 , z_3 , z_4 and z_5 on an Argand diagram where $-\pi < \arg(z_1) < \arg(z_2) < \arg(z_3) < \arg(z_4) < \arg(z_5) \le \pi$. [5]

(ii) Find the exact cartesian equation of the locus of all points z such that $|z - z_2| = |z - z_3|$ and sketch this locus on an Argand diagram. Find the least possible value of $|z - z_1|$. [4]

(iii) Sketch on the same Argand diagram in **(ii)**, the locus $\arg(z - z_1) = \arg(z_4)$. [1]

(iv) Find the complex number z that satisfy the 2 equations $|z - z_2| = |z - z_3|$ and $\arg(z - z_1) = \arg(z_4)$, giving your answer in the form $a + \mathrm{i}b$. [2]

13



The diagram shows the cross-section of a container. It is in the shape of a semicircle of fixed radius 4k metres with a hole in the shape of a trapezium $ABCD$.

(i) If $BC = 2x$ metres, show that the area S of the trapezium $ABCD$ is given by $S = (x + 4k)\sqrt{16k^2 - x^2}$. [2]

(ii) Use differentiation to show that the area of the trapezium is maximum when $x = 2k$ metres. [4]

It is given that $x = 2k$ metres and the length of the container is given as 3 metres. This container is filled with water at a constant rate of 0.2 m^3/s. At time t seconds the depth of water in the container is h metres as shown.

(iii) Show that the volume V of water in the container is given by $V = 3h\left(4k + \frac{h}{\sqrt{3}}\right)$. [2]

(iv) Find, in terms of k, the rate at which the depth is increasing at the instant when the depth is $k\sqrt{3}$ metres. [3]

- End of Paper -

ANGLO-CHINESE JUNIOR COLLEGE
MATHEMATICS DEPARTMENT

MATHEMATICS
Higher 2

9740 / 02

Paper 2

22 August 2016

JC 2 PRELIMINARY EXAMINATION

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[Turn Over

Section A: Pure Mathematics [40 marks]

1 A triangular region R is drawn on a large sheet of graphing paper marked in 1mm squares. The region R is bounded by the x-axis, the line $y = \frac{1}{20}x$ and the line $y = -\frac{1}{20}x + 40$. The scales on both the axes are such that 1 mm represents 1 unit.

By using the table below or otherwise, find the number of complete 1mm squares which lie inside region R. [3]

Range of x	Number of complete 1 mm squares which lies inside region R
$20 \le x \le 40$	
$40 \le x \le 60$	
$60 \le x \le 80$	
$\vdots$	

2 The parametric equations of curve C are

$$x = a\cos^3 t, \quad y = a\sin^3 t \quad \text{for } 0 \le t \le \pi,$$

where a is a positive constant.

(i) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis and the coordinates of the points at which the tangent is parallel to the y-axis. [5]

(ii) Hence sketch the curve C. [1]

(iii) If the cartesian equation of curve C is $y = \mathrm{f}(x)$, use the curve in part **(ii)** to sketch the graphs of

(a) $y = \dfrac{1}{\mathrm{f}(x)}$, **(b)** $y = \mathrm{f}'(x)$,

stating in each case, the equations of any asymptotes and the coordinates of any points of intersection with the axes. [4]

(iv) The point P on the curve has parameter p. Show that the equation of tangent at P is $x\sin p + y\cos p = a\sin p\cos p$. [2]

(v) The tangent at P is perpendicular to the tangent at another point Q, on the curve. If $p = \dfrac{\pi}{3}$, find the value of the parameter t at point Q. [2]

[Turn Over

3 The functions h and g are defined by

$$\mathrm{h} : x \mapsto \mathrm{e}^{|2x+1|} + 1,\ x \in \mathbb{R},\ x \le k\,,$$

$$\mathrm{g} : x \mapsto \begin{cases} 2x & \text{for } 0 \le x \le \frac{1}{2}, \\ 2 - 2x & \text{for } \frac{1}{2} \le x \le 1. \end{cases}$$

(i) Given that the function h has an inverse, state the greatest value of k. Find $\mathrm{h}^{-1}(x)$ and write down the domain of h^{-1}. [4]

(ii) Explain why the composite function gg exist. [1]

(iii) Sketch the graph of $y = \mathrm{gg}(x)$. [2]

(iv) Given that $\mathrm{f} : x \mapsto \mathrm{e}^{|2x+1|} + 1,\ x \in \mathbb{R},$ find the range of fg exactly. [1]

4 The equations of three planes $p_1,\ p_2,\ p_3$ are

$$2x + 3y - 6z = 10,$$
$$-2x - 3y + 6z = a,$$
$$x + y + bz = 5,$$

respectively, where a, b are constants.

The planes p_1 and p_3 intersect in the line l with cartesian equation $\dfrac{5-x}{3} = \dfrac{y}{4} = z$.

(i) Show that $b = -1$. [2]

(ii) The point S lies on p_1 and the point R has coordinates $(-2,\ 4,\ 1)$. Given that RS is perpendicular to p_3, find the coordinates of S. [4]

The planes p_1 and p_2 are $\dfrac{8}{7}$ units apart.

(iii) Given that $a < 0$, find the possible values of a. [4]

(iv) The point P with coordinates $(5,\ 2,\ c)$ lies on p_1. Find the value of c. [1]

(v) The point F is the foot of the perpendicular from P to the line l. The point Q is the reflection of F in the plane p_2. Find the distance PF and hence find the area of triangle FPQ. [4]

Section B: Statistics [60 marks]

5 Florida fitness club wants to carry out a survey to find out from their members the facilities that the club can improve on. The club has a list of all the 15000 members' names.

(i) Describe how to obtain a systematic sample of 500 members from the list to take part in the survey. [2]

(ii) State one disadvantage of using a systematic sample in this context. [1]

6 A factory manufactures rectangular glass panels. The length and breadth of each panel, in cm, are modelled as having independent normal distributions with means and standard deviations as shown in the table.

Glass Panel	Mean (cm)	Standard Deviation (cm)
Length	300	0.5
Breadth	150	0.2

The probability that the total perimeter of 2 randomly selected glass panels exceeds the mean length of n randomly selected glass panels by more than 1501cm is less than 0.2576. Find the least value of n. [5]

7 The mean number of guests checking into a hotel in an hour is 3.6 and can be modelled by a Poisson distribution.

(i) Find the probability that not more than 4 guests checked into the hotel in a given hour.[1]

(ii) Given that three non-overlapping one-hour blocks are chosen at random, find the probability that one of the blocks has not more than 4 guests checking into the hotel and the remaining two blocks have no guests checking into the hotel. [2]

(iii) Given that each day consists of 24 non-overlapping one-hour blocks. Use a suitable approximation, to find the probability that between 85 and 90 guests checked into the hotel in a particular day. State the parameters of the distribution that you use. [3]

(iv) The probability of at least n one-hour blocks in a day of 24 non-overlapping one-hour blocks has not more than 4 guests checking into the hotel is less than 0.124. Find the least value of n. [3]

(v) Explain why the Poisson distribution may not be a good model for the number of guests checking into the hotel in a year. [1]

[Turn Over

8 Tandao Café has an outlet at North Vista and another outlet at South Parc. On weekdays, the waiting time during lunch periods in each outlet follows a normal distribution with mean μ minutes.

(i) Hono has lunch regularly at the North Vista outlet. On 10 randomly selected weekdays, his waiting times per visit were recorded, in minutes, as follows:

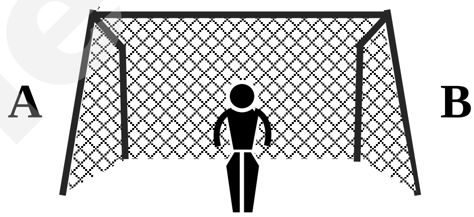
$$49, \ 38, \ 43, \ 70, \ 45, \ 51, \ 57, \ 85, \ 39, \ 44$$

Test, at the 10% significance level, whether the mean waiting time is less than one hour. [4]

(ii) Lulu has lunch regularly at the South Parc outlet. On 56 randomly selected weekdays, her waiting times per visit were recorded, in minutes. It was found that the sample mean waiting time is $\overline{t}$ minutes and the sample variance is 69.8 minutes^2. A test is to be carried out at the 5% level of significance to determine whether the average waiting time at the South Parc outlet is not one hour.

Find the range of values of $\overline{t}$ for which the result of the test would be that the null hypothesis is rejected, leaving your answers in 2 decimal places. (Answers obtained by trial and improvement from a calculator will obtain no marks.) [4]

9



(a) The diagram above shows a goalkeeper and a front view of a goal post labelled A on one side and B on the other side. Aaron is a goalkeeper in a football club. Based on his past experiences as a goalkeeper in penalty shoot-outs, the probability that he dives to side B is 0.72. In a particular match, Aaron's team went into a penalty shoot-out. The probability that a penalty kicker kicks the ball to side B is p, where $0 < p < 1$. Assume that Aaron's choice of direction to dive is independent of the penalty kicker's choice of direction to kick the ball.

(i) Show that the probability Aaron dives in the same direction as the ball is kicked is $0.44p + 0.28$. [1]

(ii) If Aaron dives in the same direction as the ball is kicked, the probability that he saves the ball is 0.4. Find, in terms of p, the probability that Aaron fails to save the ball. [3]

Hence find the values between which this probability must lie. [1]

(b) In any football match, the expected number of saves made by Aaron is 4.8, with a standard deviation of 1.2. Find the probability that in 50 matches, the average number of saves per match made by Aaron is less than 5. [2]

(c) Aaron can either be a goalkeeper or a forward in a match, depending on the strategy that the team uses. In the team, there is another goalkeeper, 6 defenders, 5 midfielders and 5 forwards. Altogether, there are 18 members in the team.

(i) Find the number of ways to select 4 defenders, 4 midfielders, 2 forwards and a goalkeeper from the team, given that Aaron is selected. [3]

After a match, the team stands in a straight line to take a photo.

(ii) Find the number of ways such that all the defenders are standing alternately with all the midfielders. [2]

The team is then asked to sit in a circle for a debrief.

(iii) Find the probability that the two goalkeepers sit opposite each other given that a group consisting of 2 particular midfielders and 6 defenders are seated together. [3]

10 **(i)** Sketch a scatter diagram that might be expected when h and s are related approximately as given in each of the models (A) and (B) below. In each model, your diagram should include 6 points, approximately equally spaced with respect to h, and all h- and s-values positive. The letters a, b, c and d represent constants.

(A) $s = a + b\ln h$, where a is negative and b is positive.

(B) $s = c + \dfrac{d}{h}$, where c is positive and d is negative. [2]

A company recently launched a new product in Singapore and wanted to know more about the relationship between the number of promoters, h, and the product's monthly sales, s, in Singapore dollars. They collected data for the past 9 months and the results are given in the table.

h	50	60	70	80	90	100	110	120	130
s	40 000	47 000	52 000	55 000	57 800	60 000	61 500	62 500	63 000

(ii) Draw a scatter diagram for these values, labelling the axes. [1]

(iii) Comment on whether a linear model would be appropriate, referring to both the scatter diagram and the context of the question. [2]

(iv) It is required to estimate the number of promoters needed to achieve a monthly sales of \$75,000. Using an appropriate model in part **(i)** to find the equation of the suitable regression line, correct to 3 decimal places. Use your equation to find the required estimate, correct to the nearest integer. [2]

(v) Comment on the reliability of your estimate. [1]

(vi) Given that 1 US dollar = 1.34 Singapore dollars, re-write your equation from part **(iv)**, correct to 3 decimal places, so that it can be used to estimate the number of promotors when the product's monthly sales is given in US dollars. [1]

11 The manager of a car show room wants to study the number of cars sold by the 2 car salesman under his charge. The number of potential car-buyers that they meet in a particular week and the average probabilities that each salesman is successful in closing a deal with each customer is given in the table below.

Salesman	Number of potential car-buyers	Probability of closing a deal
X	60	0.2
Y	50	0.3

(i) It is assumed that the deals closed are independent of one another. State, in context, another assumption needed for the number of deals closed by a car salesman to be well modelled by a binomial distribution. [1]

(ii) Explain why the assumption that the deals closed are independent of one another may not hold in this context. [1]

Assume now that the assumptions stated in part **(i)** do in fact hold and the deals closed by salesman X is independent of the deals closed by salesman Y.

(iii) Use suitable approximations to find the probability that both salesmen collectively closed a total of more than 20 deals in a particular week. State the parameters of the distributions that you use. [4]

A new salesman joined the company. During his probation week, he met 60 potential car-buyers. The number of car deals he closed during his probation week is denoted by C with the distribution $\mathrm{B}(60, p)$.

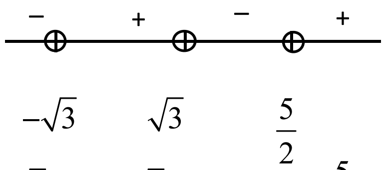
(iv) Given that $\mathrm{P}(C = 30) = 0.03014$. Find an equation for p. Hence find the value of p, correct to 1 decimal place, given that $p < 0.5$. [2]

(v) Given that $p = 0.05$, use a suitable approximation, which should be stated, to find the probability that he sold more than 4 cars. [2]

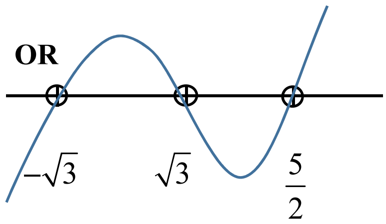
- End of Paper -

<table>
<tr><th>Qn</th><th>Solution</th><th></th></tr>
<tr><td>1</td><td>

$$\frac{2w-5}{w^2-3}>0$$



OR



$-\sqrt{3}<w<\sqrt{3}$ or $w>\frac{5}{2}$

$$\frac{(2|y|-5)\sin x}{y^2-3}\le 0,$$

Since $\sin x<0$ for $\pi<x\le\frac{3\pi}{2}$, $\frac{2|y|-5}{y^2-3}\ge 0$

From above, $-\sqrt{3}<|y|<\sqrt{3}$ or $|y|\ge\frac{5}{2}$

$0\le|y|<\sqrt{3}$ or $|y|\ge\frac{5}{2}$

$-\sqrt{3}<y<\sqrt{3}$ or $y\ge\frac{5}{2}$ or $y\le-\frac{5}{2}$

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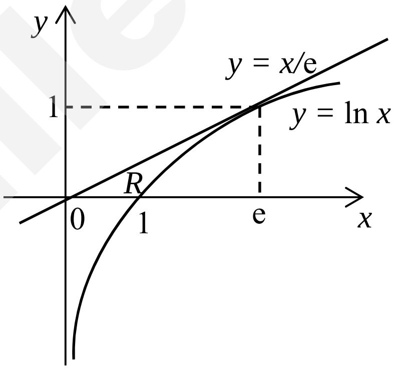
<tr><td>2</td><td>



Method 1 - Integration

Volume of solid S

$$=\pi\int_0^e\left(\frac{x}{e}\right)^2 dx-\pi\int_1^e(\ln x)^2\,dx$$

$$=\frac{\pi}{e^2}\int_0^e x^2dx-\pi\int_1^e(\ln x)^2\,dx$$

$$=\frac{\pi}{e^2}\left[\frac{x^3}{3}\right]_0^e-\pi\int_1^e(\ln x)^2\,dx$$

$$=\frac{\pi}{e^2}\left[\frac{e^3}{3}\right]-\pi\int_1^e(\ln x)^2\,dx$$

Method 2 – Volume of Cone

Volume of solid S

$$=\text{Vol of cone}-\pi\int_1^e(\ln x)^2\,dx$$

$$=\frac{1}{3}\pi(1)^2(e)-\pi\int_1^e(\ln x)^2\,dx$$

$$=\frac{1}{3}\pi e-\pi\int_1^e(\ln x)^2\,dx$$

</td><td></td></tr>
</table>

	$=\frac{1}{3}\pi e-\pi\int_1^e (\ln x)^2\,dx$ $\int_1^e (\ln x)^2\,dx$ $=\left[x(\ln x)^2\right]_1^e-2\int_1^e \ln x\,dx$ $\quad u_1=(\ln x)^2 \quad \frac{dv_1}{dx}=1$ $=\left[x(\ln x)^2\right]_1^e-2\left(\left[x\ln x\right]_1^e-\int_1^e 1\,dx\right)$ $\quad \frac{du_1}{dx}=2(\ln x)(\frac{1}{x}) \quad v_1=x$ $=\left[x(\ln x)^2\right]_1^e-2\left[x\ln x\right]_1^e+2\int_1^e 1\,dx$ $\quad u_2=\ln x \quad \frac{dv_2}{dx}=1$ $=\left[x(\ln x)^2\right]_1^e-2\left[e\ln e-\ln 1\right]_1^e+2\left[x\right]_1^e$ $\quad \frac{du_2}{dx}=\frac{1}{x} \quad v_2=x$ $=\left[x(\ln x)^2\right]_1^e-2\left[e\ln e-\ln 1\right]+2\left[e-1\right]$ $=e-2e+2e-2$ $=e-2$ Hence, volume of solid S $=\frac{1}{3}\pi e-\pi(e-2)$ $=\frac{1}{3}\pi e-\pi e+2\pi$ $=2\pi-\frac{2}{3}\pi e$ $=\frac{2}{3}\pi(3-e)$	
3 (i)	$2\,500\,000\,000 \le 1000\,000(2)^{n-1}$ $2^{n-1}\ge 2500$ $n-1\ge \frac{\ln 2500}{\ln 2}=11.2877$ $n\ge 12.2877$ His net worth will first exceed 2.5 billion when $n=13$ The year 1993 + (13 -1)(1) = 2005 or 1993 + 13 – 1 = 2005	
3 (ii)	$100\,000+1000+1000(1.5)+1000(1.5^2)+\ldots\ldots$(15 terms) = $100\,000+\frac{1000(1.5^{15}-1)}{1.5-1}=$ \$ 973 787.7808 = \$ 973 788	
4 (a)	$g(x)=\frac{1}{\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)}=\frac{1}{\left(\frac{1}{2}\cos^2 x-\frac{1}{2}\sin^2 x\right)}=$ $\frac{1}{\frac{1}{2}(\cos^2 x-\sin^2 x)}=\frac{2}{\cos 2x}\approx\frac{2}{(1-\frac{(2x)^2}{2})}=\frac{2}{1-2x^2}=2(1-2x^2)^{-1}$ $\approx 2(1+2x^2)=2+4x^2$ m must be sufficiently small for $g(x)\approx 2+ax+bx^2$ to be used	

4 (b)	$(1-x^2)^{-1/2} = 1 + \frac{\left(-\frac{1}{2}\right)}{1}\left(-x^2\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-x^2\right)^2 +$ $\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-x^2\right)^3 = 1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+.....$ $\cos^{-1}x = \int \frac{-1}{\sqrt{1-x^2}}\,dx = -\int (1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+....)\,dx$ $\approx -(x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7)+C$ When $x=0$, $\cos^{-1}0=\frac{\pi}{2}=C$ $\cos^{-1}(x) = -(x+\frac{1}{6}x^3+\frac{3}{40}x^5)+\frac{\pi}{2}$	
5	$\frac{1}{3^r}\left(\frac{u_{r+1}}{3}-u_r\right)=2r$ $\sum_{r=1}^{n}\left(\frac{u_{r+1}}{3^{r+1}}-\frac{u_r}{3^r}\right)=2\sum_{r=1}^{n}r$ $\cancel{\frac{u_2}{3^2}}-\frac{u_1}{3^1}$ $+\cancel{\frac{u_3}{3^3}}-\cancel{\frac{u_2}{3^2}}$ $+\cancel{...}$ $+\cancel{\frac{u_n}{3^n}}-\cancel{\frac{u_{n-1}}{3^{n-1}}}$ $+\frac{u_{n+1}}{3^{n+1}}-\cancel{\frac{u_n}{3^n}}=2\left(\frac{n}{2}(1+n)\right)$ $\frac{u_{n+1}}{3^{n+1}}-\frac{1}{3}=n(n+1)$ $u_{n+1}=3^{n+1}\left(n(n+1)+\frac{1}{3}\right)=3^n\left(3n^2+3n+1\right)$	
6	Rate of change = rate of growth – rate of decrease Rate of change = rate of earning interest – rate of withdrawal $\frac{dM}{dt}=0.05M-4000$ $\int\frac{1}{0.05M-4000}\,dM=\int 1\,dt$ $\frac{1}{0.05}\ln\lvert 0.05M-4000\rvert=t+C$ $\ln\lvert 0.05M-4000\rvert=\frac{t}{20}+\frac{C}{20}$	

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$\left|0.05M-4000\right|=e^{\frac{t}{20}+\frac{C}{20}}$

$0.05M-4000=\ Ae^{\frac{t}{20}}$ where $A=\pm e^{\frac{C}{20}}$

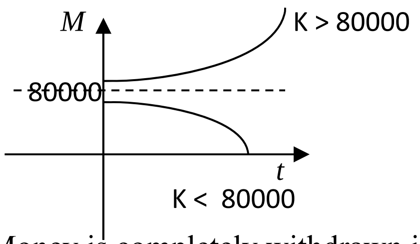
$M=80000+20Ae^{\frac{t}{20}}$

When $t=0$, $M=K$

$K=80000+20A$

$20A=K-80000$

Hence $M=80000+(K-80000)e^{\frac{t}{20}}$



Money is completely withdrawn if $K<80000$

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<tr><td>7(i)</td><td>

Let $x=5\sin\theta$

$\frac{dx}{d\theta}=5\cos\theta$

$\int\sqrt{25-x^2}\,dx$

$=\int\sqrt{25-(5\sin\theta)^2}(5\cos\theta)d\theta$

$=\int\sqrt{25-25\sin^2\theta}(5\cos\theta)d\theta$

$=\int\sqrt{25(1-\sin^2\theta)}(5\cos\theta)d\theta$

$=\int\sqrt{25(\cos^2\theta)}(5\cos\theta)d\theta$

$=\int(5\cos\theta)^2d\theta$

$=25\int\cos^2\theta d\theta$

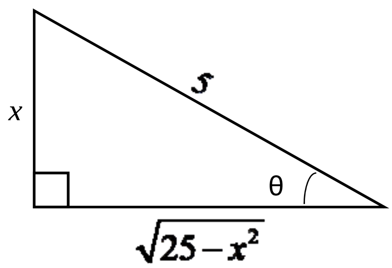
$=\frac{25}{2}\int1+\cos2\theta d\theta$

$=\frac{25}{2}\left(\theta+\frac{1}{2}\sin2\theta\right)+c$

$=\frac{25}{2}\left(\theta+\sin\theta\cos\theta\right)+c$

$=\frac{25}{2}\left(\sin^{-1}\frac{x}{5}+\left(\frac{x}{5}\right)\left(\frac{\sqrt{25-x^2}}{5}\right)\right)+c$

$=\frac{25}{2}\sin^{-1}\frac{x}{5}+\frac{1}{2}x\sqrt{25-x^2}+c$



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Let $x=5\sin\theta$

$\sin\theta=\frac{x}{5}$

$\theta=\sin^{-1}\frac{x}{5}$

$\cos\theta=\frac{1}{5}\sqrt{25-x^2}$

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<tr><td>7(ii)</td><td>

$x^2+(y-b)^2=25$

$(y-b)^2=25-x^2$

Since $y<0$, $y-b<0$,

$y-b=-\sqrt{25-x^2}$

$y=b-\sqrt{25-x^2}$

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Area of region R

$$= \left| \int_0^a b - \sqrt{25-x^2}\, dx \right|$$

$$= \left| \int_0^a b\ dx - \int_0^a \sqrt{25-x^2}\ dx \right|$$

$$= \left| \left[bx\right]_0^a - \left[\frac{25}{2}\sin^{-1}\frac{x}{5} + \frac{1}{2}x\sqrt{25-x^2} \right]_0^a \right|$$

$$= \left| ab - \frac{25}{2}\sin^{-1}\frac{a}{5} - \frac{1}{2}a\sqrt{25-a^2} \right|$$

$$= \left| a\sqrt{25-a^2} - \frac{1}{2}a\sqrt{25-a^2} - \frac{25}{2}\sin^{-1}\frac{a}{5} \right|$$

$$= \left| \frac{1}{2}a\sqrt{25-a^2} - \frac{25}{2}\sin^{-1}\frac{a}{5} \right| = \frac{25}{2}\sin^{-1}\frac{a}{5} - \frac{1}{2}a\sqrt{25-a^2}.$$

Substituting $x = a$ and $y = 0$ into the equation

$x^2 + (y-b)^2 = 25$,

we have

$a^2 + (0-b)^2 = 25$

$b = \sqrt{25-a^2}$

8 (i)

$z = k + i$

$z^2 = (k+i)^2 = k^2 + 2(k)(i) + (i)^2 = (k^2-1) + (2k)i$

$z^3 = (k+i)^3 = k^3 + 3(k)^2(i) + 3(k)(i)^2 + (i)^3$

$\quad = (k^3 - 3k) + (3k^2 - 1)i$

$z^3 - iz^2 - 2z - 4i = 0$

$[(k^3-3k) + (3k^2-1)i] - i[(k^2-1) + (2k)i] - 2[k+i] - 4i = 0$

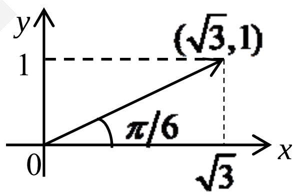
$[(k^3-3k) + 2k - 2k] + i[(3k^2-1) - (k^2-1) - 2 - 4] = 0$

$(k^3-3k) + i(2k^2-6) = 0$

$k(k^2-3) = 0$ and $2k^2 - 6 = 0$

$(k = 0$ or $k = \pm\sqrt{3})$ and $k = \pm\sqrt{3}$

Hence, $k = \pm\sqrt{3}$



8 (ii)

$z = \sqrt{3} + i \quad (\because k > 0)$

$|z| = \sqrt{1+3} = 2$

$\arg(z) = \frac{\pi}{6}$

<u>Method 1: By Polar Form & Trigonometry</u>

$z = 2e^{i\pi/6} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$z^n = 2^n e^{in\pi/6} = 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right)$

z^n is real $\Leftrightarrow \sin\frac{n\pi}{6} = 0$

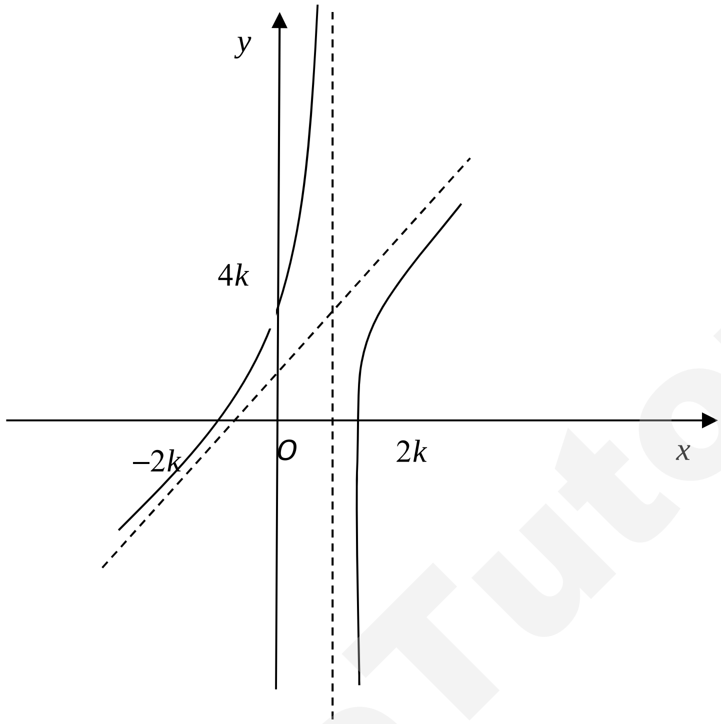
$\Leftrightarrow \frac{n\pi}{6} = k\pi$, where $k \in \mathbb{Z}$

$\Leftrightarrow n = 6k$, where $k \in \mathbb{Z}$

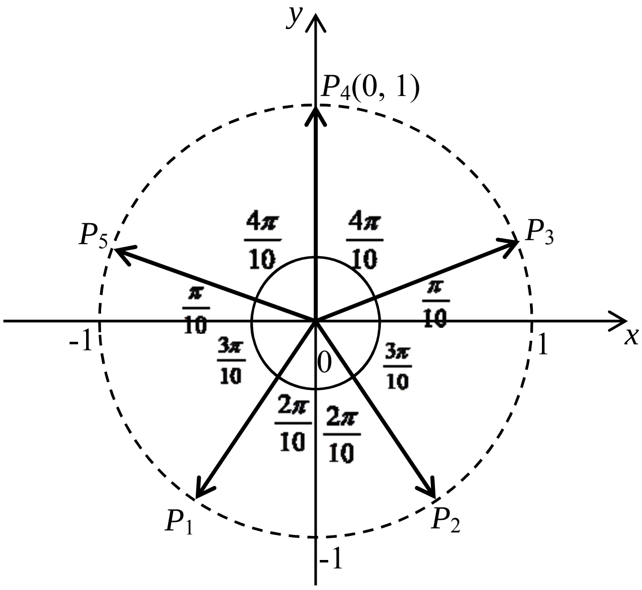
Hence, $n = 0,\ \pm 6,\ \pm 12,\ \pm 18,\ \ldots$

	<u>Method 2: By Properties of arg(z)</u> $\arg(z^n)=n\arg(z)=\dfrac{n\pi}{6}$ z^n is real, the point representing z^n on the Argand diagram is on the *x*-axis. Thus, $\arg(z^n)=\dfrac{n\pi}{6}=k\pi,$ where $k\in\mathbb{Z}$ $\therefore n=6k,$ where $k\in\mathbb{Z}$ i.e. $n=0,\ \pm6,\ \pm12,\ \pm18,\ \ldots$ Given $\left\|z^n\right\|>100$. $\left\|z^n\right\|=\left\|z\right\|^n=2^n$ Hence, $2^n>100$ But *n* is a multiple of 6. We then have $2^6=64<100$ $2^{12}=4096>100$ The least value of *n* is 12.	
9	Let P_n be the statement $\displaystyle\sum_{r=1}^{n}\frac{3r+1}{r(r+1)(r+2)}=\frac{n(7n+9)}{4(n+1)(n+2)}$ for all integers $n\geq1$. When n = 1, LHS = $\displaystyle\sum_{r=1}^{1}\frac{3r+1}{r(r+1)(r+2)}=\frac{3+1}{1(2)(3)}=\frac{2}{3}$ RHS = $\dfrac{7+9}{4(2)(3)}=\dfrac{2}{3}=$ LHS $\quad\therefore\ \mathrm{P}_1$ is true. Assume that P_k is true for **some** positive integer *k*, $k\geq1$, i.e. $\displaystyle\sum_{r=1}^{k}\frac{3r+1}{r(r+1)(r+2)}=\frac{k(7k+9)}{4(k+1)(k+2)}$ Need to prove P_{k+1} is true, i.e. $\displaystyle\sum_{r=1}^{k+1}\frac{3r+1}{r(r+1)(r+2)}=\frac{(k+1)(7k+16)}{4(k+2)(k+3)}$. LHS of P_{k+1} $\displaystyle=\sum_{r=1}^{k+1}\frac{3r+1}{r(r+1)(r+2)}$ $\displaystyle=\sum_{r=1}^{k}\frac{3r+1}{r(r+1)(r+2)}+\frac{3(k+1)+1}{(k+1)(k+2)(k+3)}$ $\displaystyle=\frac{k(7k+9)}{4(k+1)(k+2)}+\frac{3k+4}{(k+1)(k+2)(k+3)}$ $\displaystyle=\frac{7k^3+30k^2+39k+16}{4(k+1)(k+2)(k+3)}$	

	$= \dfrac{(7k^2+23k+16)(k+1)}{4(k+1)(k+2)(k+3)}$ $= \dfrac{(k+1)(7k+16)}{4(k+2)(k+3)} = \text{RHS}$ $\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by the Principle of Mathematical Induction, $\sum_{r=1}^{n}\dfrac{3r+1}{r(r+1)(r+2)} = \dfrac{n(7n+9)}{4(n+1)(n+2)}$ is true for all integers $n \geq 1$	
9 (i)	$\dfrac{n(7n+9)}{4(n+1)(n+2)} = \dfrac{7}{4} - \dfrac{6n+7}{2(n+1)(n+2)}$ Since n is a positive integer, $\dfrac{6n+7}{2(n+1)(n+2)} > 0$ $\therefore \sum_{r=1}^{n}\dfrac{3r+1}{r(r+1)(r+2)} = \dfrac{n(7n+9)}{4(n+1)(n+2)} = \dfrac{7}{4} - \dfrac{6n+7}{2(n+1)(n+2)} < \dfrac{7}{4}$	
9 (ii)	$(r+1)^3 = r^3+3r^2+3r+1$ $r(r+1)(r+2) = r^3+3r^2+2r \quad \therefore \quad (r+1)^3 > r(r+1)(r+2)$ *So* $\dfrac{1}{(r+1)^3} < \dfrac{1}{r(r+1)(r+2)}$ $\sum_{r=1}^{n}\dfrac{3r}{(r+1)^3} < \sum_{r=1}^{n}\dfrac{3r}{r(r+1)(r+2)} < \sum_{r=1}^{n}\dfrac{3r+1}{r(r+1)(r+2)} < \dfrac{7}{4}$.	
10	$y = \dfrac{x^2-4k^2}{x-k}$ where k is a constant such that $k \neq 0$ $xy - ky = x^2 - 4k^2$ $x^2 - xy + (ky - 4k^2) = 0$ x is real $\Rightarrow$ discriminant ≥ 0 $y^2 - 4(ky - 4k^2) \geq 0$ $y^2 - 4ky + 16k^2 \geq 0$ $(y-2k)^2 + 12k^2 \geq 0$ This inequality is true for all values of y. Therefore y can take the set of all real numbers. **Alternative Method:** $\dfrac{dy}{dx} = \dfrac{x^2-2xk+4k^2}{(x-k)^2} = \dfrac{(x-k)^2+3k^2}{(x-k)^2} > 0$ y is increasing, so $y \in \mathbb{R}$.	

10 (i)	$y=\dfrac{x^2-4k^2}{x-k}=x+k-\dfrac{3k^2}{x-k}$ Asymptotes: $y=x+k$ *and* $x=k$ Points of intercept with axes: $(0,4k)$, $(-2k,0)$, $(2k,0)$  	
10 (ii)	$\int_{-1}^{1} \mathrm{f}(\lvert x\rvert)\,\mathrm{d}x = 2\int_0^1 \mathrm{f}(x)dx = 2\int_0^1 x+k-\dfrac{3k^2}{x-k}\,\mathrm{d}x$ $=2\left[\dfrac{x^2}{2}+kx-3k^2\ln\lvert x-k\rvert\right]_0^1$ $=2\left[\dfrac{1}{2}+k-3k^2\ln\lvert 1-k\rvert+3k^2\ln\lvert -k\rvert\right]$ $=1+2k+6k^2\ln\dfrac{k}{k-1}.$	
10 (iii)	$y=\dfrac{x^2-4k^2}{x-k}\rightarrow y=\dfrac{\left(\frac{x}{2}\right)^2-4k^2}{\left(\frac{x}{2}\right)-k}$ (scaling) translation $\rightarrow y=\dfrac{\left(\frac{x+2k}{2}\right)^2-4k^2}{\left(\frac{x+2k}{2}\right)-k}=\dfrac{(x+2k)^2-16k^2}{2x}$	

11 (i)	$\mathbf{a}\times\mathbf{b}=4\mathbf{a}\times\mathbf{c}$ $(\mathbf{a}\times\mathbf{b})-(4\mathbf{a}\times\mathbf{c})=\mathbf{0}$ $(\mathbf{a}\times\mathbf{b})-(\mathbf{a}\times 4\mathbf{c})=\mathbf{0}$ $\mathbf{a}\times(\mathbf{b}-4\mathbf{c})=\mathbf{0}$ $\Rightarrow$ $\mathbf{a}=\mathbf{0}$ **or** $(\mathbf{b}-4\mathbf{c})=\mathbf{0}$ **or** $\mathbf{a}$ is parallel to $\boldsymbol{b}-4\boldsymbol{c}$ But we are given $\boldsymbol{b}\neq 4\boldsymbol{c}$ and $\mathbf{a}\neq\mathbf{0}$, hence $\boldsymbol{b}-4\boldsymbol{c}=\alpha\,\boldsymbol{a}$	
(ii)	$\frac{1}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert=\sqrt{126}$ $\frac{1}{2}\lvert 4\mathbf{a}\times\mathbf{c}\rvert=\sqrt{126}$ $\lvert\mathbf{a}\times\mathbf{c}\rvert=\frac{\sqrt{126}}{2}$ $\left\lvert\left(\frac{\mathbf{b}-4\mathbf{c}}{\sqrt{3}}\right)\times\mathbf{c}\right\rvert=\frac{\sqrt{126}}{2}$ $\lvert(\mathbf{b}\times\mathbf{c})-(4\mathbf{c}\times\mathbf{c})\rvert=\frac{\sqrt{3}\sqrt{126}}{2}$ $\lvert(\mathbf{b}\times\mathbf{c})\rvert=\frac{\sqrt{378}}{2}$	**c×c= 0**
(iii)	Area of parallelogram with adjacent sides OB and OC.	
(iv)	$(\boldsymbol{b}-4\boldsymbol{c})\,.\,(\boldsymbol{b}-4\boldsymbol{c})=3\lvert\boldsymbol{a}\rvert^2$ $\lvert\boldsymbol{b}\rvert^2-8\,\boldsymbol{b}\,.\,\boldsymbol{c}+16\lvert\boldsymbol{c}\rvert^2=3\lvert\boldsymbol{a}\rvert^2$ $\boldsymbol{b}\,.\,\boldsymbol{c}=-\frac{10}{8}$ $\cos\theta=\frac{\boldsymbol{b}\,.\,\mathrm{c}}{\lvert\boldsymbol{b}\rvert\lvert\boldsymbol{c}\rvert}=\frac{-\frac{10}{8}}{1(2)}$ $\theta=128.7^\circ$	
12 (i)	$z^5-i=0$ $z^5=i$ $z^5=e^{i\pi/2}=e^{i(2k\pi+\frac{\pi}{2})}$, where $k\in\mathbb{Z}$ $z=e^{i(\frac{2k\pi}{5}+\frac{\pi}{10})}$ Putting $n=-2,-1,0,1,2$, $z=e^{-\frac{7\pi}{10}i},e^{-\frac{3\pi}{10}i},e^{\frac{\pi}{10}i},e^{\frac{\pi}{2}i},e^{\frac{9\pi}{10}i}$ Given $-\pi<\arg(z_1)<\arg(z_2)<\arg(z_3)<\arg(z_4)<\arg(z_5)\leq\pi$. i.e. $z_1=e^{-\frac{7\pi}{10}i},z_2=e^{-\frac{3\pi}{10}i},z_3=e^{\frac{\pi}{10}i},z_4=e^{\frac{\pi}{2}i},z_5=e^{\frac{9\pi}{10}i}$ Let the points P_1, P_2, P_3, P_4 and P_5 on the Argand diagram represent the complex numbers $z_1,\ z_2,\ z_3,\ z_4,$ and z_5.	.

		
(ii)	The locus $\lvert z-z_2\rvert=\lvert z-z_3\rvert$ is a perpendicular bisector of the line segment P_2P_3 where $P_2 \equiv z_2$ and $P_3 \equiv z_3$. Since OP_2P_3 is an isosceles triangle, the perpendicular cuts through the origin and bisect the angle P_2OP_3. Thus the perpendicular bisector is inclined at angle $\frac{\pi}{10}$ radian below the positive x-axis. Cartesian equation of locus $\lvert z-z_2\rvert=\lvert z-z_3\rvert$ is $y=-x\tan(\frac{\pi}{10})$.	

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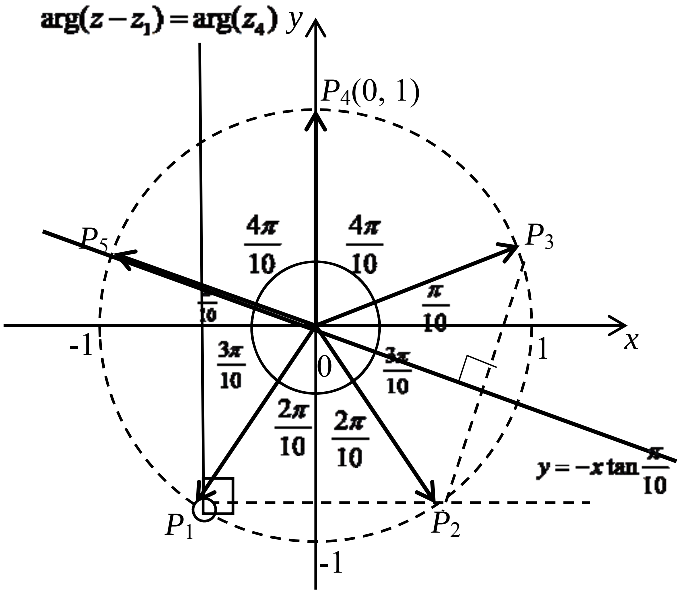

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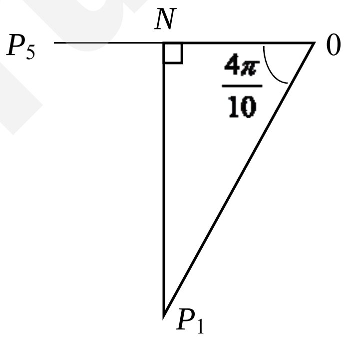
$z \equiv \overrightarrow{OP}$

$z_1 \equiv \overrightarrow{OP_1}$

$z - z_1 \equiv \overrightarrow{OP} - \overrightarrow{OP_1}$ i.e. $z - z_1 \equiv \overrightarrow{P_1P}$

The least value of $|z - z_1|$ is the shortest distance from P_1 to the perpendicular bisector (line OP_5).



$\frac{P_1N}{OP_1} = \sin\frac{4\pi}{10}$

$P_1N = OP_1 \sin\frac{4\pi}{10} = \sin\frac{2\pi}{5} = 0.951$

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<td>(iii)</td>
<td>The locus $\arg(z - z_1) = \arg(z_4)$ is a half-line with its initial point at P_1 and above and excluding P_1, parallel to the y-axis. (Refer to diagram.)</td>
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<td>12
(iv)</td>
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The intersection point between the 2 loci has the same x-coordinates as the point P_1, i.e. $x = \cos\frac{-7\pi}{10} = -\cos\frac{3\pi}{10}$.

Substituting $x = -\cos\frac{3\pi}{10}$ into equation of the perpendicular bisector $y = -x\tan(\frac{\pi}{10})$, we have

$y = -(-\cos\frac{3\pi}{10})\tan\frac{\pi}{10} = \cos\frac{3\pi}{10}\tan\frac{\pi}{10} = 0.191$

$x = -\cos\frac{3\pi}{10} = -0.588$

Coordinates of the intersection point are $(-0.588, 0.191)$.

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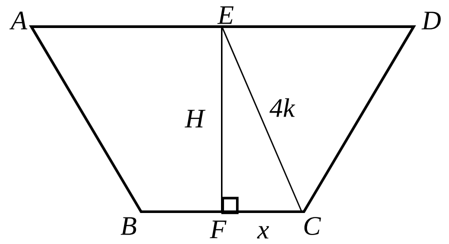
13 (i)



Let E be the centre of the circle and F be the foot of perpendicular from the point E to BC.
Since EC and ED are both radii of the circle, $ED = EC = 4k$.
Let H be the length of EF, which is the height of the trapezium.
By Pythagora's theorem,

$$H^2 + x^2 = (4k)^2$$
$$H^2 = 16k^2 - x^2$$
$$H = \sqrt{16k^2 - x^2}$$
$$S = \frac{1}{2}H(BC + AD) = \frac{1}{2}H\left(2x + 8k\right)$$
$$= (x+4k)\sqrt{16k^2 - x^2}\,(shown)$$

(ii)

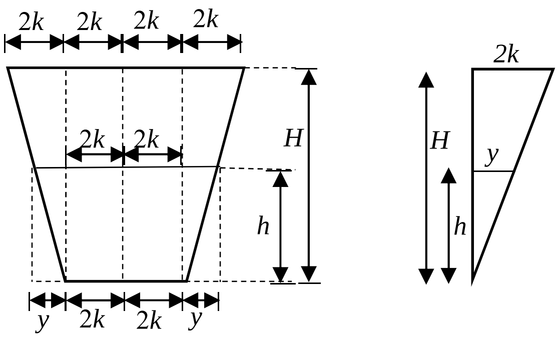
$$\frac{dS}{dx} = \sqrt{16k^2 - x^2} - \left(x + 4k\right)x\left(16k^2 - x^2\right)^{-\frac{1}{2}}$$
$$= \frac{-2x^2 - 4kx + 16k^2}{\sqrt{16k^2 - x^2}} = 0$$
$$x^2 + 2kx - 8k^2 = 0$$
$$\left(x - 2k\right)\left(x + 4k\right) = 0$$
$$\Rightarrow x = 2k \quad or \quad x = -4k \ \text{(N.A}, x > 0)$$
$$\frac{dS}{dx} = \frac{-2x^2 - 4kx + 16k^2}{\sqrt{16k^2 - x^2}} = \frac{-2(x^2 + 2kx - 8k^2)}{\sqrt{16k^2 - x^2}}$$
$$\frac{d^2S}{dx^2} = -2\left[\frac{\sqrt{16k^2 - x^2}\left(2x + 2k\right) - (x^2 + 2kx - 8k^2)\frac{1}{2}\left(16k^2 - x^2\right)^{-1/2}(-2x)}{16k^2 - x^2}\right]$$
$$= -2(16k^2 - x^2)^{-3/2}\left[2(x+k)(16k^2 - x^2) + x(x^2 + 2kx - 8k^2)\right]$$
$$= -2(16k^2 - x^2)^{-3/2}\left[2(-x^3 - kx^2 + 16k^2x + 16k^3) + (x^3 + 2kx^2 - 8k^2x)\right]$$
$$= -2(16k^2 - x^2)^{-3/2}\left[-x^3 + 24k^2x + 32k^3\right]$$

when $x = 2k$

$$\frac{d^2S}{dx^2} = -2(16k^2 - 4k^2)^{-3/2}\left[-8k^3 + 48k^3 + 32k^3\right] = -\frac{2(72k^3)}{12\sqrt{12}k^3} = -\sqrt{12} < 0$$

Area of trapezium is maximum when $x=2k$.

(iii)



Given $x = 2k$ and from (i) we have $H = \sqrt{16k^2 - x^2}$.

$\therefore H = \sqrt{16k^2 - x^2} = \sqrt{16k^2 - (2k)^2}$

$= \sqrt{16k^2 - 4k^2} = \sqrt{12k^2} = 2\sqrt{3}k$

Using similar triangles:

$$\frac{y}{2k} = \frac{h}{H}$$

$$y = \left(\frac{h}{2\sqrt{3}k}\right)2k = \frac{h}{\sqrt{3}}$$

$V =$ area of trapezium $\times$ length of container

$$= \frac{1}{2}(h)\left(4k + (4k + 2y)\right)(3)$$

$$= \frac{3}{2}(h)(8k + 2y)$$

$$= 3(h)(4k + y)$$

$$= 3h\left(4k + \frac{h}{\sqrt{3}}\right)(shown)$$

(iv)

$$\frac{dV}{dh} = \left(4k + \frac{h}{\sqrt{3}}\right)3 + 3h\left(\frac{1}{\sqrt{3}}\right)$$

$$= 12k + \frac{6h}{\sqrt{3}}$$

$When\ h = \sqrt{3}k,\ \frac{dV}{dh} = 18k$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$0.2 = 18k \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{90k} m/s$$

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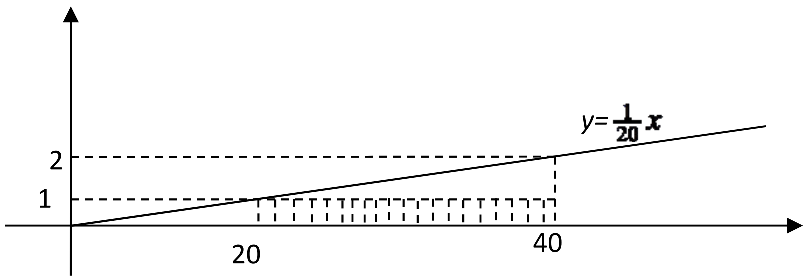
Qn	Solutions	Remarks

1



Number of squares for $20 \le x \le 40$ is 20

Column	Range of x	Number of squares
1	$20 \le x \le 40$	20
2	$40 \le x \le 60$	40
3	$60 \le x \le 80$	60
	.	
	.	
n	$380 \le x \le 400$	

Number of columns = $n = \dfrac{380}{20} = 19$ for $0 \le x \le 400$

OR 380 = 20 + (n-1)(20) $\Rightarrow n$ =19

OR 400 = 40 + (n-1)(20) $\Rightarrow n$ =19

AP sequence : 20, 40, 60, …….. 19 terms

Number of complete squares in region R for $0 \le x \le 400$ is

$$\frac{19}{2}\left[2(20)+18(20)\right] = 3800$$

Region R is symmetrical about the line x = 400.

Total number of squares in region R = 3800 x 2 = 7600

2(i)

$x = a\cos^3 t, \quad y = a\sin^3 t \quad for \ 0 \le t \le \pi$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

$Let \ \dfrac{dy}{dx} = 0$

$-\tan t = 0 \Rightarrow t = 0, \pi$

Points on curve where tangent is parallel to x-axis are (a,0) and ($-a$,0)

Let $\dfrac{dx}{dy} = 0 \Rightarrow \dfrac{1}{-\tan t} = 0 \Rightarrow t = \dfrac{\pi}{2}$

Point on curve where tangent is parallel to y-axis are (0,a)

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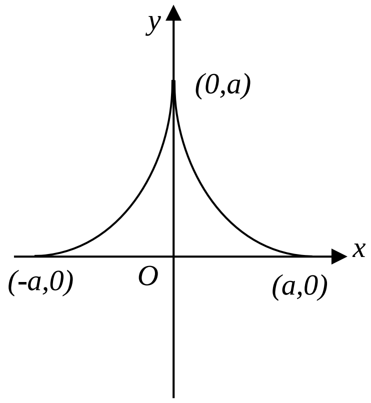
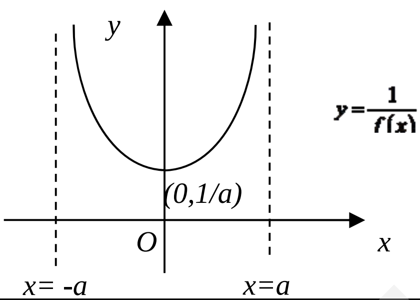
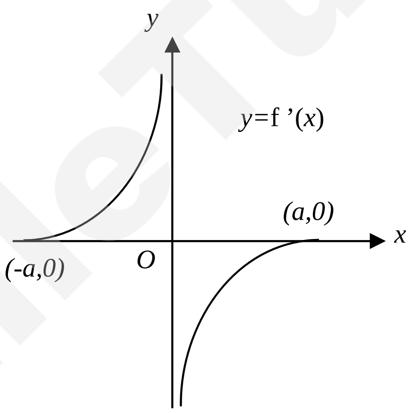
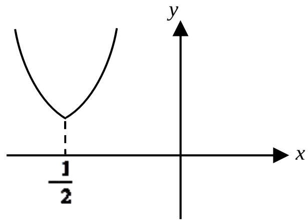
2 (ii)		
2 (iii) (a)		
2 (iii) (b)		
2 (iv)	Equation of tangent at point *P* with parameter *p:* $y-a\sin^3 p=-\tan p(x-a\cos^3 p)$ $y\cos p-a\sin^3 p\cos p=-\sin p(x-a\cos^3 p)$ $x\sin p+y\cos p=\left(a\sin p\cos p\right)(\cos^2 p+\sin^2 p)$ $x\sin p+y\cos p=a\sin p\cos p\ \ (\textit{shown})$	
2 (v)	Given $p=\frac{\pi}{3}$, gradient of tangent at P is $-\tan\frac{\pi}{3}=-\sqrt{3}$ $\therefore$ gradient of tangent at Q $=\frac{1}{\sqrt{3}}$ Value of parmeter at $Q: -\tan t=\frac{1}{\sqrt{3}}\Rightarrow t=\frac{5\pi}{6}$	

3

(i) Sketch y=f(x)



h(x) has an inverse if $x \le -\frac{1}{2}$.

Greatest value of k is $-\frac{1}{2}$

$y - 1 = e^{|2x+1|}$

$\ln(y-1) = |2x+1|$

$2x + 1 = \pm\ln(y-1)$

$x = -\frac{1}{2} \pm \frac{1}{2}\ln(y-1)$

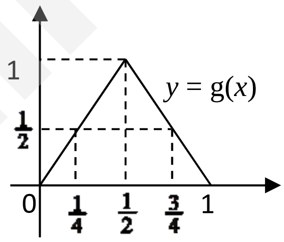
$x = -\frac{1}{2} - \frac{1}{2}\ln(y-1)$ since $x \le -\frac{1}{2}$

$h^{-1}(x) = -\frac{1}{2} - \frac{1}{2}\ln(x-1)$

$h(-\frac{1}{2}) = 2$

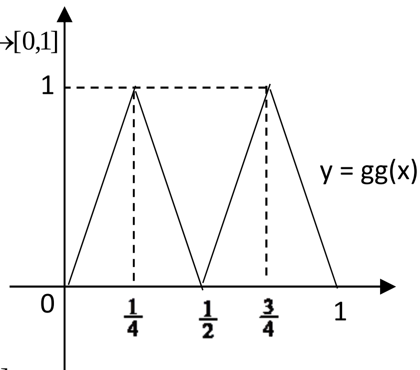
Domain of $h^{-1}(x)$ = Range of h(x) = $[2, \infty)$

(ii)



Composite function exist because
$R_g = [0,1] \subseteq D_g = [0,1]$

(iii) $[0,\frac{1}{4}] \xrightarrow{g} [0,\frac{1}{2}] \xrightarrow{g} [0,1]$



$[\frac{1}{4},\frac{1}{2}] \xrightarrow{g} [\frac{1}{2},1] \xrightarrow{g} [0,1]$

$[\frac{1}{2},\frac{3}{4}] \xrightarrow{g} [\frac{1}{2},1] \xrightarrow{g} [0,1]$

$[\frac{3}{4},1] \xrightarrow{g} [0,\frac{1}{2}] \xrightarrow{g} [0,1]$

	(iv) $[0,1] \xrightarrow{g} [0,1] \xrightarrow{f} \left[e+1, e^3+1\right]$ $R_{fg} = \left[e+1, e^3+1\right]$	
4 (i)	$\begin{pmatrix}1\\1\\b\end{pmatrix}.\begin{pmatrix}-3\\4\\1\end{pmatrix}=0$ $-3+4+b=0$ $b=-1$ **Alternatively,** $\begin{pmatrix}1\\1\\b\end{pmatrix}\times\begin{pmatrix}2\\3\\-6\end{pmatrix}=k\begin{pmatrix}-3\\4\\1\end{pmatrix}$ $\begin{pmatrix}-6-3b\\6+2b\\1\end{pmatrix}=k\begin{pmatrix}-3\\4\\1\end{pmatrix}$ Equating **k** component, $k=1.$ Equating **i** component, $-6-3b=-3k$ $\therefore -3b=-3(1)+6$ $b=-1$ (Shown)	
4 (ii)	Line RS: $\mathbf{r}=\begin{pmatrix}-2\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix},\ \lambda\in\mathbb{R}$ Substitute this line to p_1, $\begin{pmatrix}-2+\lambda\\4+\lambda\\1-\lambda\end{pmatrix}.\begin{pmatrix}2\\3\\-6\end{pmatrix}=10$ $\lambda=\frac{8}{11}$ $\overrightarrow{OS}=\begin{pmatrix}-2+\lambda\\4+\lambda\\1-\lambda\end{pmatrix}=\begin{pmatrix}-\frac{14}{11}\\\frac{52}{11}\\\frac{3}{11}\end{pmatrix}$	

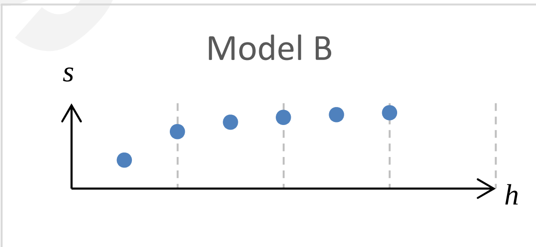
4 (iii)	$p_1: \quad r.\frac{\begin{pmatrix}2\\3\\-6\end{pmatrix}}{7}=\frac{10}{7}$ $p_2: \quad r.\frac{\begin{pmatrix}2\\3\\-6\end{pmatrix}}{7}=\frac{-a}{7}$ distance between two planes : $\frac{8}{7}=\frac{10-(-a)}{7}$ or $\frac{8}{7}=\frac{(-a)-10}{7}$ a = -2 or a = -18	
4 (iv)	$p_1: \quad \begin{pmatrix}5\\2\\c\end{pmatrix}.\begin{pmatrix}2\\3\\-6\end{pmatrix}=10$ $10+6-6c=10$ $c=1$	
4 (v)	Let point A be (5, 0, 0) which is a point on line l. $PF=\frac{\left\lvert \overrightarrow{AP}\times\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\rvert}=\frac{\left\lvert\begin{pmatrix}0\\2\\1\end{pmatrix}\times\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\rvert}=\frac{\left\lvert\begin{pmatrix}-2\\-3\\6\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\rvert}$ $=\frac{7}{\sqrt{26}}=1.3728$ $QF=2\left(\frac{8}{7}\right)=\frac{16}{7}$ Area PFQ = $\frac{1}{2}(1.3728)\left(\frac{16}{7}\right)=1.5689=1.57$	

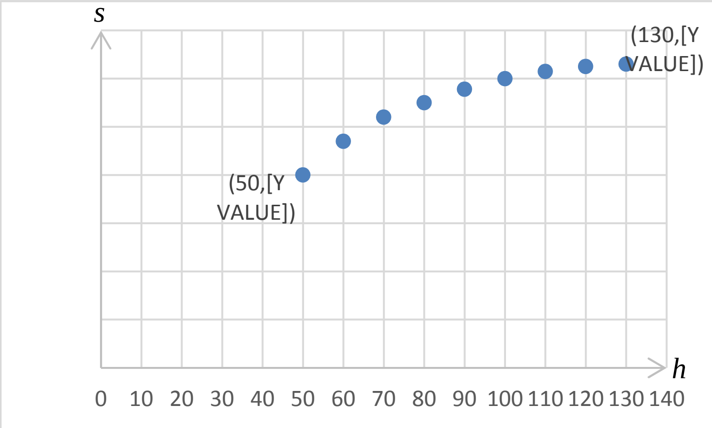
	Alternative method: $\overrightarrow{OF}=\begin{pmatrix}5-3\lambda\\4\lambda\\\lambda\end{pmatrix}$ $\overrightarrow{FP}=\overrightarrow{OP}-\overrightarrow{OF}=\begin{pmatrix}3\lambda\\2-4\lambda\\1-\lambda\end{pmatrix}$ Since $\overrightarrow{FP}\perp\begin{pmatrix}-3\\4\\1\end{pmatrix}$, $\begin{pmatrix}3\lambda\\2-4\lambda\\1-\lambda\end{pmatrix}.\begin{pmatrix}-3\\4\\1\end{pmatrix}=0$ $-9\lambda+8-16\lambda+1-\lambda=0$ $\lambda=\frac{9}{26}$ $\therefore\left\|\overrightarrow{FP}\right\|=\left\|\begin{pmatrix}(3)\left(\frac{9}{26}\right)\\2-(4)\left(\frac{9}{26}\right)\\1-\left(\frac{9}{26}\right)\end{pmatrix}\right\|=\left\|\begin{pmatrix}\frac{27}{26}\\\frac{8}{13}\\\frac{17}{26}\end{pmatrix}\right\|=1.3728$ $QF=2\left(\frac{8}{7}\right)=\frac{16}{7}$ Area PFQ $=\frac{1}{2}(1.3728)\left(\frac{16}{7}\right)=1.5689=1.57$	
5 (i)	Number the club members in order from 1 to 15000 according to the name list. (Alphabetical order) Since $k=\frac{15000}{500}=30$, select a member randomly from the name list. Thereafter, select every 30th member cycling to the start of the list if the end of list is reached until we form a sample of 500 members.	
5(ii)	The list is not representative since we have selected a group of all same gender or same age group who may have same request on the facilities.	
6	Let the random variable L be the length of a randomly chosen glass panel. Let the random variable B be the breadth of a randomly chosen glass panel.	

	Let $X = 2L_1 + 2B_1 + 2L_2 + 2B_2$ $\therefore X \sim N(1800, 2.32)$ Let $\bar{L} = \dfrac{L_1 + L_2 + L_3 + ... + L_n}{n}$ $\therefore \bar{L} \sim N\left(300, \dfrac{0.5^2}{n}\right)$ $X - \bar{L} \sim N\left(1500, 2.32 + \dfrac{0.5^2}{n}\right)$ $\mathrm{P}(X - \bar{L} > 1501) < 0.2576$ $\mathrm{P}\left(Z > \dfrac{1501-1500}{\sqrt{2.32 + \frac{0.5^2}{n}}}\right) < 0.2576$ $\mathrm{P}\left(Z < \dfrac{1}{\sqrt{2.32 + \frac{0.5^2}{n}}}\right) > 0.7424$ $\dfrac{1}{\sqrt{2.32 + \frac{0.5^2}{n}}} > 0.6507622042$ $\sqrt{2.32 + \dfrac{0.5^2}{n}} < 1.536659618$ $2.32 + \dfrac{0.5^2}{n} < 2.361322781$ $\dfrac{0.5^2}{n} < 0.0413227807$ $n > 6.049931679$ Least $n = 7$.	
7 (i)	Let the random variable X be the number of guests checking into the hotel in a given hour. $\mathrm{P}(X \le 4) = 0.7064384499 = 0.706$ (3s.f)	

7 (ii)	Required probability $=\left[P\left(X\le 4\right)\right]\times\left[P\left(X=0\right)\right]^2\times\frac{3!}{2!}$ $=0.0015822508$ $=0.00158$ (3s.f)	
7 (iii)	Let the random variable Y be the number of guests checking into the hotel in a day. $Y\sim P_O\left(86.4\right)$ Since $\lambda>10$, $Y\sim N\left(86.4,86.4\right)$ approx. $P\left(85<X<90\right)\overset{c.c}{\approx}P\left(85.5<X<89.5\right)=0.169$ (3s.f)	
7 (iv)	Let the random variable W be the number of one-hour blocks in a day, which has not more than 4 guests checking into the hotel. $W\sim B\left(24,0.7064384499\right)$ $P\left(W\ge n\right)<0.124$ $P\left(W\le n-1\right)>0.876$ Using G.C, $P\left(W\le 19\right)-0.876>-2.797\times10^{-4}$ $P\left(W\le 20\right)-0.876>0.7534$ $\therefore n-1=20$ Least $n=21$.	
7 (v)	This is because the **mean** number of guests checking into the hotel **per hour** is **unlikely to be constant throughout the year**. The number of guests checking into the hotel is likely to vary across different months in a year due to **seasonal fluctuations** caused by factors such as the **holiday seasons**, etc. Hence a Poisson distribution may not be a good model.	
8 (i)	Let random variable X be a randomly chosen Hono's lunch waiting time at North Vista outlet. To test $H_0:\ \mu=60$ Against $H_1:\ \mu<60$ at 10% level of significance. Under H_0, $T=\frac{\bar{X}-\mu_0}{s/\sqrt{n}}\sim t(n-1)$, i.e. $T=\frac{\bar{X}-60}{s/\sqrt{10}}\sim t(9)$	

	Value of test statistic: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$ $= \dfrac{52.1 - 60}{14.9328/\sqrt{10}} = -1.6729597$ $p\text{-}value = \; = 0.064332 < 0.1$ $\therefore$ Reject H_0 and conclude that there is sufficient evidence at 10% level of significance that the average waiting time during lunch periods at North Vista is less than one hour.	
8 (ii)	Unbiased estimate of population variance is $s^2 = \left(\dfrac{n}{n-1}\right)(\text{sample variance}) = \left(\dfrac{56}{55}\right)(69.8) = 71.06909$ To test $H_0: \; \mu = 60$ Against $H_1: \; \mu \neq 60$ at 5% level of significance. Under H_0, $Z = \dfrac{\bar{T} - 60}{s/\sqrt{56}} \sim N(0,1)\, approx.,$ Value of test statistic: $z = \dfrac{\bar{t} - 60}{\sqrt{71.06909}/\sqrt{56}} = \dfrac{\bar{t} - 60}{1.12654}$ Since the null hypothesis is rejected, $\dfrac{\bar{t} - 60}{1.12654} < -1.959964$ or $\dfrac{\bar{t} - 60}{1.12654} > 1.959964$ $\bar{t} < 57.792$ or $\bar{t} > 62.208$	
9 (a) (i)	$p\,(0.72) + (1 - p)\,(0.28) = 0.44p + 0.28$	
(ii)	$(0.44p + 0.28)(0.6) + 1 - (0.44p + 0.28)$ $= 0.888 - 0.176\text{p}$ Since $0 < p < 1$, $0 > -0.176p > -0.176$ $0.888 > 0.888 - 0.176p > 0.888 - 0.176$ $0.888 > 0.888 - 0.176p > 0.712$ $\therefore 0.712 < 0.888 - 0.176p < 0.888$	

(b)	Since n is large, by Central Limit Theorem, $\overline{X} = \dfrac{X_1 + X_2 + \ldots + X_{50}}{50} \sim N\left(4.8, \dfrac{1.44}{50}\right)$ approx. Required probability $= \mathrm{P}\left(\overline{X} < 5\right) = 0.8807035 = 0.881$ (3s.f)	
(ci)	If Aaron is the goalkeeper: ${}^6C_4\,{}^5C_4\,{}^5C_2 = 750$ If Aaron is the forward: ${}^6C_4\,{}^5C_4\,{}^5C_1 = 375$ Number of ways required = 1125	
(cii)	Number of ways required = $5!6!8!$ = 3483648000	
(ciii)	Required probability $= \dfrac{\mathrm{P}(\text{goalkeepers are opposite \& defenders with 2 particular midfielders are together})}{\mathrm{P}(\text{defenders with 2 particular midfielders are together})}$ $= \dfrac{8!2!8!}{(11-1)!8!} = \dfrac{2!}{10\times 9} = \dfrac{1}{45}$	
10 (i)	 	

10 (ii)		
10 (iii)	A Linear model will not be approriate. This is because the scatter diagram indicates that as h increases, s is increasing at a decreasing rate which is not a linear realtionship. Furthermore, a linear model will mean that the product's monthly sales in Singapore will increase indefinitely with the increase of the number of promoters. This is not realistic in the context of the question as the product's monthly sales will likely slow down and perhaps decrease due to market stauration.	
10 (iv)	Correlation coefficent of s on $\ln(h)$ = 0.981 Correlation coefficent of s on $1/h$ = -0.998 Since correlation coefficent of s on $1/h$ is stronger, hence use least square regression line of s on $1/h$. Least square regression line of s on $1/h$: $s = 78531.62777 - \frac{1896285.284}{h}$ $s = 78531.628 - \frac{1896285.284}{h}$ (3 d.p) when $s = 75000$, $75000 = 78531.62777 - \frac{1896285.284}{h}$ $h = 536.9437001 \approx 537$ (nearest integer).	
10 (v)	The estimate is not reliable as s=75000 does not lie within $40000 \le s \le 63000$. Hence, we are extrapolating.	

10 (vi)	$s = 78531.62777 - \dfrac{1896285.284}{h}$ $\dfrac{s}{1.34} = \dfrac{78531.62777}{1.34} - \dfrac{1896285.284}{1.34h}$ $\dfrac{s}{1.34} = 58605.692365671 - \dfrac{1415138.2716417}{h}$ $\Rightarrow\ u = 58605.692 - \dfrac{1415138.272}{h}$ (3 dp) where u = monthly sales in US dollars (i.e. $u = \dfrac{s}{1.34}$)	
11 (i)	The probability that each salesman is successful in closing a deal is assumed to be constant.	
11 (ii)	Assumption: Deals closed are independent of one another Deals closed may not be independent of one another as customers may collaborate to buy cars as a group for better bargaining power.	
11 (iii)	$X \sim \mathrm{B}(60, 0.2)$ Since $np = 12 > 5$ and $nq = 48 > 5$, $X \sim \mathrm{N}(12, 9.6)$ *approx.* $Y \sim \mathrm{B}(50, 0.3)$ Since $np = 15 > 5$ and $nq = 35 > 5$, $Y \sim \mathrm{N}(15, 10.5)$ *approx.* $X + Y \sim \mathrm{N}(27, 20.1)$ *approx.* $\mathrm{P}(X+Y > 20) \overset{c.c}{\approx} \mathrm{P}(X+Y > 20.5) = 0.926 (3\text{s.f})$	
11 (iv)	$\mathrm{P}(C = 30) = 0.03014$ ${}^{60}C_{30} \times p^{30} \times (1-p)^{30} = 0.03014$ $p^2 - p + 0.2399991029 = 0$ $\therefore\ p = 0.6$ or 0.3999955 Since $p < 0.5$, $p = 0.4$ (1 d.p)	
11 (v)	$C \sim \mathrm{B}(60, 0.05)$ Since $n = 60 > 50$ such that $np = 3 < 5$, $C \sim P_O(3)$ *approx.* $\mathrm{P}(C > 4) = 0.185$ (3s.f)	

CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC2 Preliminary Examination

MATHEMATICS 9740/01

Paper 1

24 Aug 2016
3 hours

Additional Materials: List of Formulae (MF15)

Name: ____________________________ **Class:** ________________

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

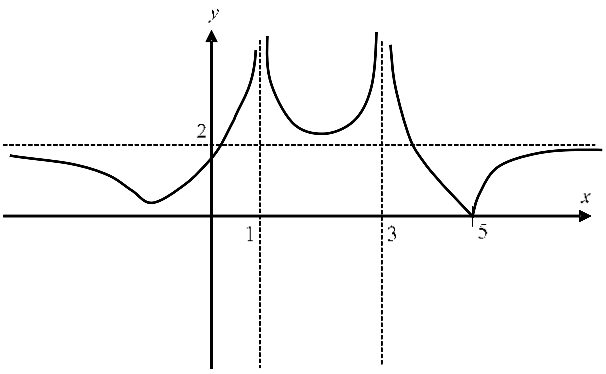
At the end of the examination, arrange your answers in NUMERICAL ORDER.

The number of marks is given in brackets [ ] at the end of each question or part question.

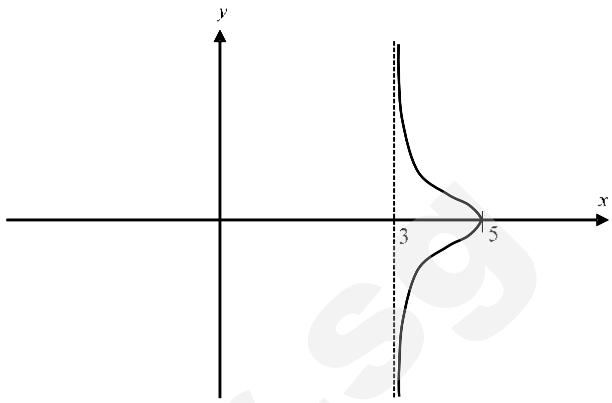
This document consists of **5** printed pages, including the cover page.

1 **[In this question, sketches of the given graphs are not drawn to scale]**

The graphs of $y=\left|\mathrm{f}\left(x\right)\right|$ and $y^2=\mathrm{f}\left(x\right)$ are given below.



$y=\left|\mathrm{f}\left(x\right)\right|$



$y^2=\mathrm{f}\left(x\right)$

On separate diagrams, draw sketches of the graphs of

(a) $y^2=\mathrm{f}\left(-2x\right)$, **[2]**

(b) $y=\mathrm{f}\left(x\right)$, **[3]**

stating the equations of any asymptotes and the coordinates of any points of intersection with the axes.

2 The vectors **a** and **b** are given by

$$\mathbf{a}=4\mathbf{i}+6p\mathbf{j}-8\mathbf{k} \text{ and } \mathbf{b}=2\mathbf{i}-3\mathbf{j}+4p\mathbf{k}, \text{ where } p>0.$$

It is given that $\left|\mathbf{a}\right|=2\left|\mathbf{b}\right|$.

(i) Find p. **[2]**

(ii) Give a geometrical interpretation of $\dfrac{1}{\left|\mathbf{b}\right|}\left|\mathbf{b}\cdot\mathbf{a}\right|$. **[1]**

(iii) Using the value of p found in part **(i)**, find the exact value of $\dfrac{1}{\left|\mathbf{b}\right|}\left|\mathbf{b}\cdot\mathbf{a}\right|$. **[2]**

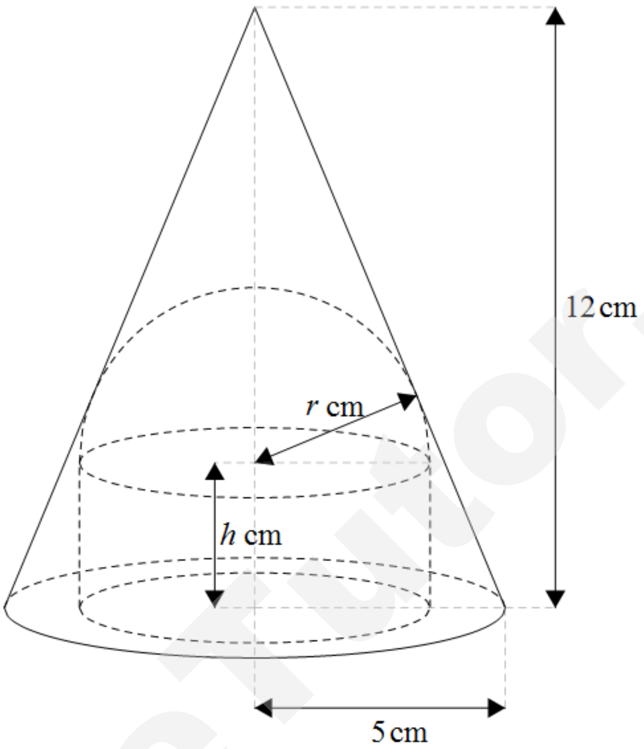
3 The cubic equation $x^3+ax^2+bx+c=0$, where a, b and c are constants, has roots 3 + i and 2.

(i) One JC2 student remarked that the third root is 3 − i. State a necessary assumption the student made in order that the remark is true. **[1]**

(ii) Given that the assumption in part **(i)** holds, find the values of a, b and c. **[4]**

4 A closed container is made up of a cylinder of base radius r cm and height h cm, and a hemispherical top with the same radius r.

It is inscribed within a fixed right circular cone of base radius 5 cm and height 12 cm, as shown in the diagram below.



(i) By using similar triangles, show that $h = 12 - \frac{13}{5}r$.

Determine the exact range of possible values of length r. **[3]**

(ii) Find the total volume V of the closed container in terms of r.
By differentiation, find the exact value of r that produces the maximum container volume V, as r varies.

[Volume of a sphere with radius R is $\frac{4}{3}\pi R^3$.] **[4]**

5 A sequence $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation $u_n = \frac{n}{(n-1)^2}u_{n-1}$, for $n \geq 2$.

(a) Given that $u_1 = 2$, use the method of mathematical induction to prove that $u_n = \frac{2n}{(n-1)!}$, for $n \geq 1$. **[4]**

(b) Given that $u_1 = a$, where a is any constant. Write down u_2, u_3 and u_4 in terms of a.
Hence or otherwise, find u_n in terms of a. **[3]**

6 **(i)** Given that $1-2r = A(r+1)+Br$, find the constants A and B. **[1]**

(ii) Use the method of differences to find $\sum_{r=1}^{n} \frac{1-2r}{3^r}$. **[3]**

(iii) Hence find the value of $\sum_{r=1}^{\infty} \frac{2-2r}{3^r}$. **[4]**

7 **(i)** Given that $y=\sqrt{1+\ln(1+x)}$, find the exact range of values of x for y to be well defined. **[1]**

(ii) Show that $2y\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{1-y^2}$. **[2]**

(iii) Hence, find the Maclaurin's series for $\sqrt{1+\ln(1+x)}$, up to and including the term in x^2. **[3]**

(iv) Verify that the same result is obtained using the standard series expansions given in the List of Formulae (MF15). **[3]**

8 **Do not use a calculator in answering this question.**

(i) It is given that complex numbers z_1 and z_2 are the roots of the equation $z^2-6z+36=0$ such that $\arg(z_1) > \arg(z_2)$. Find exact expressions of z_1 and z_2 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. **[4]**

(ii) Find the complex number $\frac{{z_1}^4}{\mathrm{i}z_2}$ in exact polar form. **[3]**

(iii) Find the smallest positive integer n such that ${z_2}^n$ is a positive real number. **[2]**

9 **(i)** By using the substitution $u=\sqrt{x+1}$, find $\int \frac{\sqrt{x+1}}{x-1}\,\mathrm{d}x$. **[5]**

(ii) The region R is bounded by the curve $y=\frac{\sqrt{x+1}}{x-1}$ and the lines $x = 8$ and $y = 1$.

Find

(a) the exact area of R, simplifying your answer in the form $A-\sqrt{2}\ln\left(\frac{B-\sqrt{2}}{B+\sqrt{2}}\right)$ where A and B are integers to be determined, **[5]**

(b) the volume of the solid generated when R is rotated 2π radians about the x-axis, giving your answer correct to 2 decimal places. **[3]**

10 The plane p passes through the points A, B and C with coordinates $(0, 1, 1)$, $(2, -1, 4)$ and $(-2, -1, 0)$ respectively.

(i) Show that a cartesian equation of the plane p is $2x - y - 2z = -3$. **[3]**

The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}$.

(ii) Find the acute angle between l and p. **[2]**

The point Q has position vector $5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

(iii) Show that Q lies on the line l. **[1]**

(iv) It is given that a variable point R lies on the plane p and is at a distance of $\sqrt{45}$ from the point Q. Find the foot of perpendicular from the point Q to the plane p and hence describe geometrically the locus of R. **[6]**

(v) Find a vector equation of the line which is a reflection of the line l in plane p. **[3]**

11 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{2x+k}{x-2}, x \in \mathbb{R}, \ x \neq 2,$$

where k is a positive constant.

(i) Sketch the graph of $y = \mathrm{f}(x)$, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the x and y axes. **[3]**

(ii) Describe fully a sequence of transformations which would transform the curve $y = \frac{1}{x}$ onto $y = \mathrm{f}(x)$. **[4]**

(iii) Find f^{-1} in a similar form and write down the range of f^{-1}. **[3]**

(iv) Hence or otherwise, find f^2.

Find the value of $\mathrm{f}^{2017}\left(\frac{1}{2}\right)$, leaving your answer in terms of k. **[4]**

The function g is defined by

$$\mathrm{g} : x \mapsto a + \sqrt{x-3}, \qquad x \in \mathbb{R}, \ x > 3,$$

where a is a real constant.

(v) Given that fg exists, write down an inequality for a and explain why gf does not exist. **[3]**

— THE END —

CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC2 Preliminary Examination

MATHEMATICS 9740/02

Paper 2

30 August 2016

3 hours

Additional Materials: List of Formulae (MF15)

Name: ____________________________ **Class:** ________________

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages, including the cover page.

Section A: Pure Mathematics [40 marks]

1 **(i)** Prove that the substitution $u = x^2 + y^2$ reduces the differential equation $y\dfrac{\mathrm{d}y}{\mathrm{d}x} + x = \sqrt{x^2 + y^2}$ to

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2\sqrt{u}\,.$$

Hence, show that the general solution of the differential equation $y\dfrac{\mathrm{d}y}{\mathrm{d}x} + x = \sqrt{x^2 + y^2}$ is given by $\sqrt{x^2 + y^2} = x + D$, where D is an arbitrary constant. **[4]**

(ii) The result in part **(i)** represents a family of curves. On a single diagram, sketch a non-linear member of the family which passes through the point $(-2, 0)$.

You should state the equation of the graph and axial intercepts clearly on the diagram. **[3]**

(iii) State an equation of the line of symmetry for the curve in part **(ii)**. **[1]**

2 The three distinct roots of the equation $x^3 - 1 = 0$ are denoted by $1, \omega$ and ω^2.

(a) Without first finding ω explicitly, show that $1 + \omega + \omega^2 = 0$. **[2]**

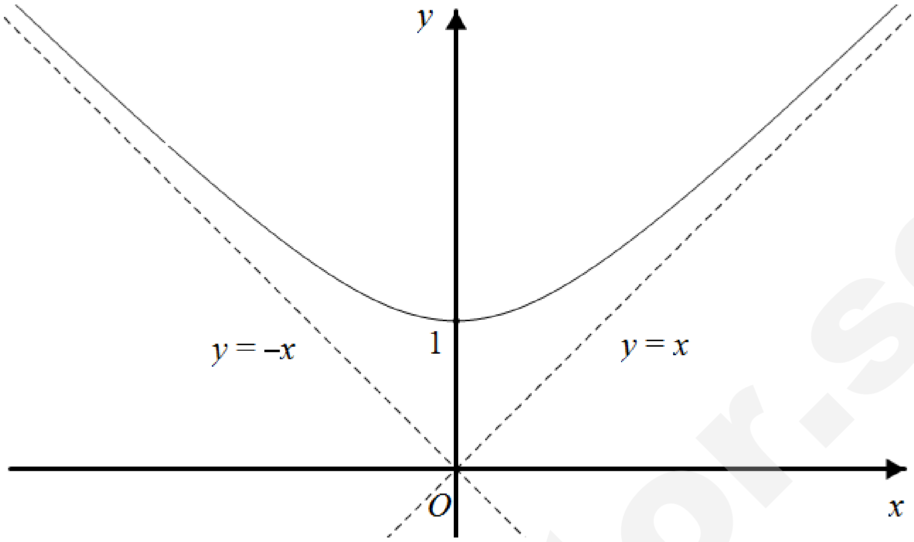
(b) Given now that $0 < \arg(\omega) < \pi$, sketch, on a single Argand diagram, the loci given by

(i) $|z - \omega| = |\omega|$ and **[3]**

(ii) $\arg(z + 1) = \pi + \arg(\omega^2)$. **[2]**

Hence, find the complex number that satisfies both loci, expressing your answer exactly in the form a + ib, where a and b are real numbers. **[2]**

3 The diagram shows the graph of curve C represented by $y = \mathrm{f}(x)$, with oblique asymptotes $y = x$ and $y = -x$.



(a) On a separate diagram, sketch a graph of $y = \mathrm{f}'(x)$, clearly indicating the equation(s) of the asymptote(s) and axial-intercept(s). **[2]**

(b) The above curve C is represented by the parametric equations

$$x = \tan\theta, \ y = \sec\theta, \text{ where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

(i) Show that the normal to the curve at point P, with coordinates $(\tan\theta, \ \sec\theta)$, for $0 < \theta < \frac{\pi}{2}$, is given by $y = -x\,\mathrm{cosec}\theta + 2\sec\theta$. **[2]**

(ii) The normal to the curve at point P intersects the x-axis at point N. Find the coordinates of the mid-point M of PN, in terms of θ. Hence find a cartesian equation of the locus of M, as θ varies. **[4]**

(iii) Taking O as the origin, show that the area of triangle OPN is $\tan\theta\sec\theta$.

Point P moves along the curve such that the rate of change of its parameter θ with respect to time t is given by $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \cos\theta$. Find the exact rate of change of the area of triangle OPN when $\theta = \frac{\pi}{6}$. **[4]**

4 Adam and Gregory signed up for a marathon. In preparation for this marathon, Adam and Gregory each planned a 15-week personalised training programme. Adam runs 2.4 km on the first day of Week 1, and on the first day of each subsequent week, the distance covered is increased by 20% of the previous week. Gregory also runs 2.4 km on the first day of Week 1, but on the first day of each subsequent week, the distance covered is increased by d km, where d is a constant. Assume Adam and Gregory only run on the first day of each week.

(i) Find, in terms of d, the total distance covered by Gregory in these 15 weeks. **[2]**

(ii) Adam targets to cover a total distance of 170 km in these 15 weeks. Can Adam achieve this target? You must show sufficient working to justify your answer. **[2]**

(iii) It is given that Adam covers a longer distance than Gregory on the first day of the 15^{th} week. Find the maximum value of d, correct to 2 decimal places.
Using this value of d, show that the difference in the distance covered by Adam and Gregory for their 15^{th} week training is 0.134 km correct to 3 significant figures. **[4]**

(iv) Due to unforeseen circumstances, Adam has to end his training programme early. In order for Adam to cover a total distance of 170 km by the end of the 13 weeks, the distance covered has to be increased by x % of the previous week on each subsequent week from Week 1. Find x. **[3]**

Section B: Statistics [60 marks]

5 A bag contains four red and eight blue balls of which two of the red balls and six of the blue balls have the number “0” printed on them. The remaining balls have the number “1” printed on them. Three balls are randomly drawn from the bag without replacement.

(i) Show that the probability that at least one blue ball is drawn is $\frac{54}{55}$. **[1]**

Find the probability that

(ii) at least one ball of each colour is drawn, **[2]**

(iii) the sum of the numbers on the balls drawn is at least two. **[3]**

6 Packets of a particular brand of potato chips are delivered to a supermarket in boxes of 60. On average, 1.8 packets in a box are underweight. The number of underweight packets from a randomly chosen box is the random variable X.

Assume that X has a binomial distribution.

(i) Use a suitable approximation to find the probability that two randomly chosen boxes of potato chips contain more than 6 packets of underweight potato chips. State the parameter(s) of the distribution that you use. **[4]**

A batch of 50 boxes of potato chips is delivered to the supermarket.

(ii) Use a suitable approximation to find the probability that the mean number of underweight packets per box is more than 2. **[3]**

7 A group of 9 friends, including Albert and Ben, are having dinner at Albert's house. They sit in two groups: a row of 4 on a couch and a group of 5 at a round dining table with 5 identical seats.

Find the number of ways they can sit if

(i) there are no restrictions, **[2]**

(ii) Albert and Ben sit beside each other, **[3]**

(iii) Albert and Ben both sit on the couch or both sit at the round table, but they do not sit beside each other. **[3]**

8 A factory manufactures a certain product for sale. The following table gives the quantity of product manufactured, x units in thousands, and its corresponding cost of production, y dollars in thousands. The data is recorded during different months of a certain year.

Quantity of product, x	2.0	2.4	3.0	3.8	4.8	6.0	7.2	8.2	9.4
Cost of production, y	10	19	35	47	58	35	78	80	81

(i) Draw a scatter diagram for the data. **[1]**

One of the values of y appears to be incorrect.

(ii) Indicate the corresponding point in your diagram by labelling it P. **[1]**

Remove P from the set of data.

(iii) By using the scatter diagram for the remaining points, explain whether $y = a + bx$ or $y = a + b\ln x$ is the better model for the relationship between x and y. **[1]**

(iv) Using the better model chosen in part **(iii)**, find the product moment correlation coefficient and the equation of a suitable regression line.
Explain what happens to the product moment correlation coefficient and the equation of the regression line if the factory decides to include a fixed cost of M thousand dollars for purchasing a packing machine to the cost of production, y. **[4]**

(v) Use the regression line found in part **(iv)** to estimate the cost of production when the quantity produced is 6000 units and comment on its reliability. **[2]**

9 The finishing times in a 10km race with a large number of runners follow a normal distribution. After 40 minutes, 10% of the runners have completed the race. After one hour, 35% of the runners have yet to complete the race. The first 20% of runners who finish the race receive a medal.

(i) Show that, correct to 1 decimal place, the runners have running times with mean 55.4 minutes and standard deviation 12.0 minutes. **[2]**

(ii) Find the maximum time a runner can take to finish the race in order to receive a medal. **[2]**

A random sample of 12 runners is selected.

(iii) Find the probability that more than four runners receive a medal. **[2]**

(iv) Given that none of the runners receives a medal, find the probability that the slowest runner completes the race in under one hour. **[3]**

10 **(a)** A Physical Education teacher wants to plan a volleyball training programme for all students in a secondary school, where each student has exactly one CCA. In order to check on the current fitness level of students in the school, he selects a sample of students by choosing the Captains of every sports team and the Presidents of every Club and Society in the school.

(i) Explain briefly why this may not provide a representative random sample of the student population. **[1]**

(ii) Name a more appropriate sampling method which would provide a representative random sample and explain how it can be carried out in this context. **[3]**

(b) The vertical jump heights of players from a volleyball team are normally distributed with mean 40 cm. The coach claims that a particular training regime is effective in improving the players' jump heights. After the regime is implemented for a period of time, a random sample of 7 players is taken and their jump heights are recorded.

The sample mean is 42.1 cm and the sample standard deviation is k cm.
A test is to be carried out at the 10% level of significance to determine whether the training regime has been effective.

(i) State appropriate hypotheses for the test. **[1]**

(ii) Find the set of values of k for which the result of the test would be to reject the null hypothesis. **[3]**

(iii) State the conclusion of the test in the case where the sample variance is 15. **[2]**

11 The average number of calls per hour received by telephone operators at the Call Centre of bank ECBC is being reviewed.

(i) State, in context, two assumptions that need to be made for the number of calls received by a telephone operator to be well modelled by a Poisson distribution. **[2]**

The Call Centre has only three telephone operators at any point in time. One handles calls pertaining to credit card queries, another handles calls pertaining to business banking queries while the last operator handles calls pertaining to personal banking queries, with the numbers of calls received in one hour assumed to have the independent distributions $\text{Po}(\mu)$, $\text{Po}(6)$ and $\text{Po}(7)$ respectively.

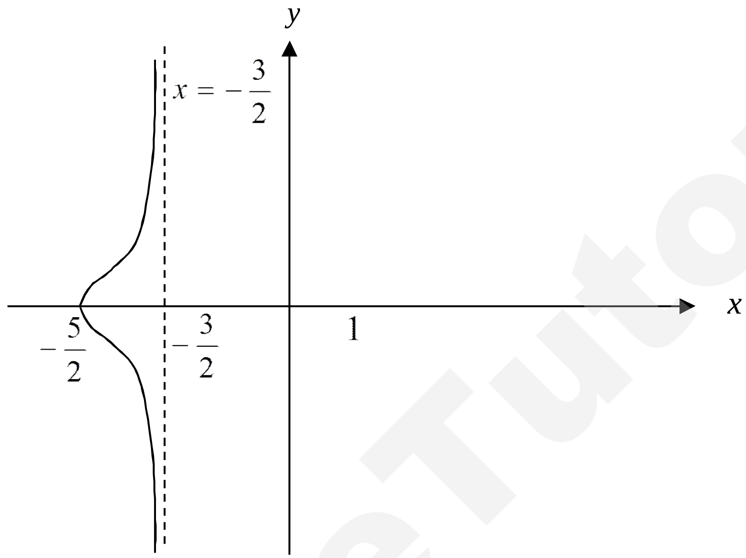
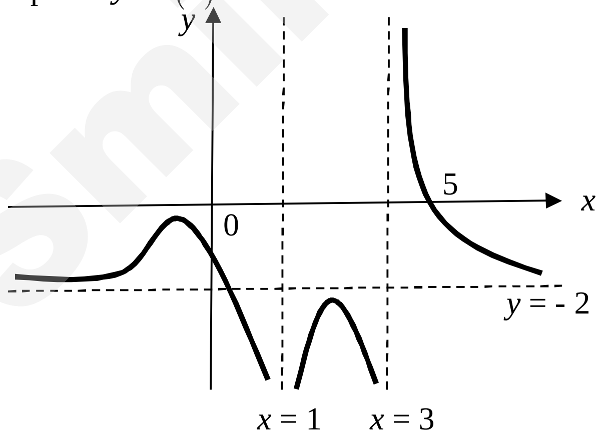
(ii) It is given that the probability of receiving two calls pertaining to credit card queries within an hour is eight times that of receiving two calls pertaining to credit card queries within four hours.

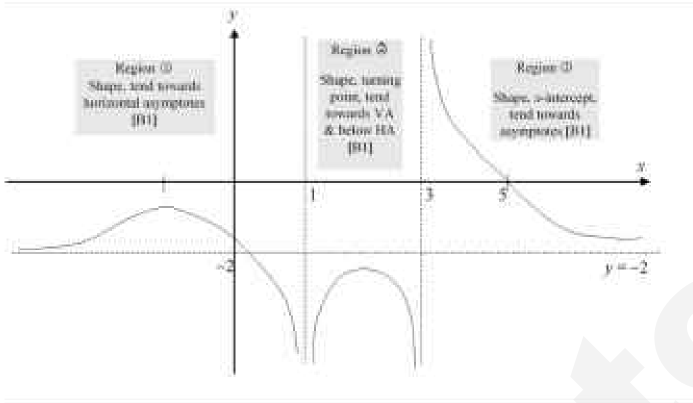
Find the exact value of μ, expressing your answer in the form $\frac{a}{b}\ln 2$ where a and b are two positive integers to be found. **[3]**

(iii) On a certain day, the Call Centre receives more than 50 calls from 1200 to 1400 hours. Find the probability that there are no calls pertaining to credit card queries during this period. **[3]**

(iv) Using suitable approximations, find the probability that there are more calls pertaining to business banking queries than personal banking queries within a two-hour period. **[3]**

— THE END —

1	Topic: Transformation of Curves					
			Solution			
			(a) Graph of $y^2 = \mathrm{f}\left(-2x\right)$ 			
			(b) Graph of $y = \mathrm{f}\left(x\right)$ 			

1	Topic: Transformation of Curves						
			Solution				
							

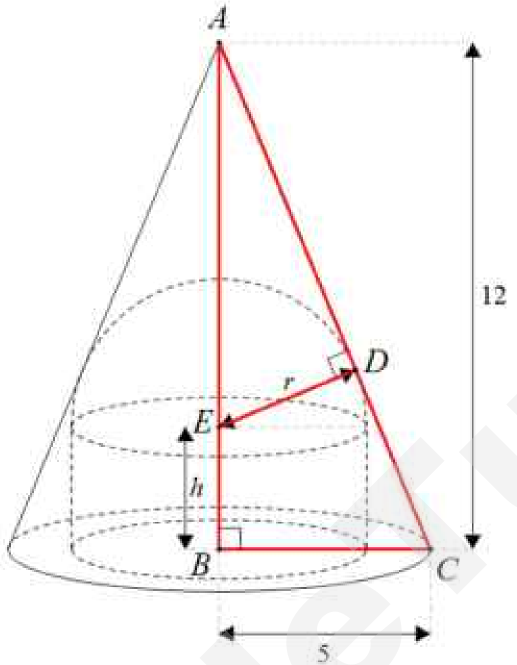
2	Topic: Vectors					
			Solution			
			(i) $\lvert\mathbf{a}\rvert = 2\lvert\mathbf{b}\rvert$ $\sqrt{16+36p^2+64} = 2\sqrt{4+9+16p^2}$ $80+36p^2 = 4\left(13+16p^2\right)$ $28p^2 = 28$ $p=1$ or $p=-1$ $\left(\text{Reject } \because p>0\right)$ $\therefore p=1$			
			(ii) Length of projection of **a** on **b**			
	.		**(iii)** $\dfrac{1}{\lvert\mathbf{b}\rvert}\lvert\mathbf{b}\cdot\mathbf{a}\rvert = \dfrac{\left\lvert\begin{pmatrix}2\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}4\\6\\-8\end{pmatrix}\right\rvert}{\sqrt{2^2+(-3)^2+4^2}} = \dfrac{42}{\sqrt{29}}$			

3	Topic: Complex Numbers						
				Solution			
			(i)	The assumption is that *a*, *b* and *c* are all real.			
			(ii)	Let $x^3+ax^2+bx+c=(x-(3+\mathrm{i}))(x-(3-\mathrm{i}))(x-2)$ $=(x^2-6x+10)(x-2)$ $=x^3-8x^2+22x-20$ By comparing coefficients, we have $a=-8$, $b=22$ and $c=-20$.			

4 Topic: Application of Differentiation

Solution

(i) Using the diagram provided,



Identify that ΔABC is similar to ΔADE.

$$\therefore \ \frac{AE}{DE} = \frac{AC}{BC}$$

Since $AC = \sqrt{5^2 + 12^2} = 13$, $BC = 5$, $DE = r$,

$\therefore \ \frac{AE}{r} = \frac{13}{5} \ \Rightarrow \ AE = \frac{13}{5}r$

$h = AB - AE = 12 - \frac{13}{5}r$

Since r and h are lengths,

$r \geq 0$ and $h \geq 0$ (i.e. $12 - \frac{13}{5}r \geq 0$, $r \leq \frac{60}{13}$).

$0 \leq r \leq \frac{60}{13}$

(ii) Volume of inscribed container,

4	Topic: Application of Differentiation						
				Solution			

$V = \pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$\quad = \pi r^2\left(12 - \frac{13}{5}r\right) + \frac{2}{3}\pi r^3$

$\quad = 12\pi r^2 - \frac{29}{15}\pi r^3$

Differentiating this,

$\frac{\mathrm{d}V}{\mathrm{d}r} = 24\pi r - \frac{29}{5}\pi r^2$.

Consider $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$.

i.e. $24\pi r - \frac{29}{5}\pi r^2 = 0$,

$\pi r(24 - \frac{29}{5}r) = 0$,

$r = 0$(rejected as $r \neq 0$) or $r = \frac{120}{29}$.

(Check that these stationary points are within the admissible range $0 \le r \le \frac{60}{13}$).

First derivative test

$\frac{\mathrm{d}V}{\mathrm{d}r} = 24\pi r - \frac{29}{5}\pi r^2$

r	$r = \frac{120}{29}^-$ e.g. $r = \frac{119}{29}$	$r = \frac{120}{29}$	$r = \frac{120}{29}^+$ e.g. $r = \frac{121}{29}$
$\frac{\mathrm{d}V}{\mathrm{d}r}$	$\frac{119}{145}\pi > 0$	0	$-\frac{121}{145}\pi < 0$

Maximum volume at $r = \frac{120}{29}$

Second derivative test

$\frac{\mathrm{d}V}{\mathrm{d}r} = 24\pi r - \frac{29}{5}\pi r^2$

4 Topic: Application of Differentiation

Solution

$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = 24\pi - 2\left(\tfrac{29}{5}\right)\pi r$$

$$= 24\pi - 2\left(\tfrac{29}{5}\right)\pi r$$

$$= 24\pi - \tfrac{58}{5}\pi\left(\frac{120}{29}\right)$$

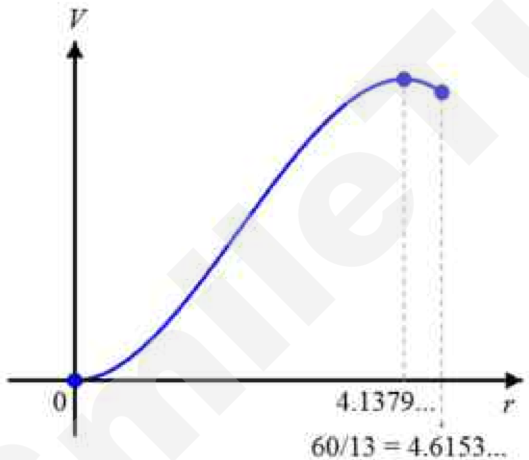
$$= -24\pi < 0 \text{ (maximum volume)}$$

<u>Alternative</u>:

Graphical method

Upon finding the stationary points,

inspect the graph of V against r, where $0 \le r \le \frac{60}{13}$.



Note that the maximum is attained at the stationary point where $r = \frac{120}{29} = 4.1379...$

5	Mathematical Induction					
			Solution			
			Let P_n be the statement that $u_n = \frac{2n}{(n-1)!}$ for $n \in \mathbb{Z}^+$, $n \geq 1$. When $n=1$, LHS= $u_1 = 2$ (given) RHS $= \frac{2}{(0)!} = 2$ = LHS, $\therefore P_1$ is true. Assume that P_k is true for some k, where $k \in \mathbb{Z}^+, k \geq 1$ i.e. $u_k = \frac{2k}{(k-1)!}$ To prove P_{k+1} is also true ,i.e., $u_{k+1} = \frac{2(k+1)}{k!}$ LHS = $u_{k+1} = \frac{k+1}{(k)^2} u_k$ $= \frac{k+1}{(k)^2} \frac{2k}{(k-1)!}$ $= \frac{2(k+1)}{k(k-1)!}$ $= \frac{2(k+1)}{k!}$ = RHS Since P_1 is true, and $P_k \text{ is true} \Rightarrow P_{k+1} \text{ is true}$, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.			

5	Mathematical Induction						
			Solution				
			(b) $u_2=\frac{2}{(1)^2}u_1=\frac{2}{1}a=2a=\frac{2}{1!}a$ $u_3=\frac{3}{(2)^2}u_2=\frac{3}{(2)^2}2a=\frac{3}{(2)}a=\frac{3}{2!}a$ $u_4=\frac{4}{(3)^2}u_3=\frac{4}{(3)^2}\frac{3}{(2)}a=\frac{4}{(3\times2)}a=\frac{4}{3!}a$				
			"Hence" approach $u_n=\frac{n}{(n-1)!}a$ by observation from part (ii) "Otherwise" approach $u_n=\frac{n}{(n-1)^2}u_{n-1}$ $=\frac{n}{(n-1)^2}\frac{n-1}{(n-2)^2}u_{n-2}$ $=\frac{n}{(n-1)^2}\frac{n-1}{(n-2)^2}\frac{n-2}{(n-3)^2}u_{n-3}$ $=\frac{n}{(n-1)^2}\frac{n-1}{(n-2)^2}\frac{n-2}{(n-3)^2}\cdots\frac{3}{(2)^2}\frac{2}{(1)^2}u_1$ $=\frac{n}{(n-1)^1}\frac{1}{(n-2)^1}\frac{1}{(n-3)^1}\cdots\frac{1}{(2)^1}\frac{1}{(1)^1}a$ $=\frac{n}{(n-1)!}a$				

6	Sigma Notation					
			Solution			
			(i) $1-2r=A(r+1)+Br$ $=(A+B)r+A$ $\therefore A=1, B=-3$			
			(ii) $\displaystyle\sum_{r=1}^{n}\frac{1-2r}{3^r}=\sum_{r=1}^{n}\frac{(r+1)-3r}{3^r}$ $\displaystyle=\sum_{r=1}^{n}\left(\frac{r+1}{3^r}-\frac{r}{3^{r-1}}\right)$ $\displaystyle=\left(\frac{2}{3^1}-\frac{1}{3^0}\right)+$ $\displaystyle\left(\frac{2}{3^2}-\frac{2}{3^1}\right)+$ $\vdots$ $\displaystyle\left(\frac{n}{3^{n-1}}-\frac{n-1}{3^{n-2}}\right)+$ $\displaystyle\left(\frac{n+1}{3^n}-\frac{n}{3^{n-1}}\right)$ $\displaystyle=\frac{n+1}{3^n}-1$			

6	Sigma Notation					
			Solution			
			(iii) $\sum_{r=1}^{\infty}\frac{2-2r}{3^r}=\sum_{r=1}^{\infty}\frac{1-2r+1}{3^r}$ $=\sum_{r=1}^{\infty}\left(\frac{1-2r}{3^r}+\frac{1}{3^r}\right)$ $=\sum_{r=1}^{\infty}\left(\frac{1-2r}{3^r}\right)+\sum_{r=1}^{\infty}\left(\frac{1}{3^r}\right)$ $=\lim_{n\to\infty}\left(\frac{n+1}{3^n}-1\right)+\frac{\frac{1}{3}}{1-\frac{1}{3}}$ $=-1+\frac{1}{2}$ $=-\frac{1}{2}$			

7 Topic: Maclaurin's Series

Solution

(i) For $y=\sqrt{1+\ln(1+x)}$ to be well-defined,

$1+x>0$ and $1+\ln(1+x)\geq 0$

$\ln(1+x)\geq -1$

$x>-1$ and $1+x\geq e^{-1}$

$x\geq e^{-1}-1$

$\therefore x\geq e^{-1}-1$

(ii) **By Implicit Differentiation,**

$y=\sqrt{1+\ln\left(1+x\right)}$

$y^2=1+\ln\left(1+x\right)$

$\Rightarrow \ln\left(1+x\right)=y^2-1$

$\Rightarrow 1+x=e^{y^2-1}$

Differentiate implicitly with respect to x,

$$2y\frac{dy}{dx}=\frac{1}{1+x}$$

$$=\frac{1}{e^{y^2-1}}$$

$$=e^{1-y^2} \text{ (shown)}$$

(iii) Differentiate the above results implicitly with respect to x,

$$2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=\frac{d}{dx}\left(1-y^2\right)\cdot e^{1-y^2}$$

$$2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=-2y\frac{dy}{dx}\cdot e^{1-y^2}$$

When $x=0$,

$y=\sqrt{1+\ln 1}=1$,

$\frac{dy}{dx}=\frac{1}{2}$,

7	Topic: Maclaurin's Series						
			Solution				
			$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{3}{4}$, $y = 1 + \frac{1}{2}x + \frac{-\frac{3}{4}}{2!}x^2 + \ldots$ $= 1 + \frac{1}{2}x - \frac{3}{8}x^2 + \ldots$				
			(iv) $y = \left(1 + \ln(1+x)\right)^{\frac{1}{2}}$ $= \left(1 + \left(x - \frac{x^2}{2} + \ldots\right)\right)^{\frac{1}{2}}$ $= 1 + \frac{1}{2}\left(x - \frac{x^2}{2} + \ldots\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(x - \frac{x^2}{2} + \ldots\right)^2 + \ldots$ $= 1 + \frac{1}{2}x - \frac{1}{4}x^2 - \frac{1}{8}\left(x - \frac{x^2}{2}\right)\left(x - \frac{x^2}{2}\right) + \ldots$ $= 1 + \frac{1}{2}x - \frac{1}{4}x^2 - \frac{1}{8}x^2 + \ldots$ $= 1 + \frac{1}{2}x - \frac{3}{8}x^2 + \ldots$				

8	Topic: Complex Numbers					
			Solution			
			(i) $z^2-6z+36=0 \Rightarrow z=\dfrac{6\pm\sqrt{36-4(1)(36)}}{2}=3\pm3\sqrt{3}\mathrm{i}$			
			Thus, $z_1=6e^{\mathrm{i}\frac{\pi}{3}}$ and $z_2=6e^{-\mathrm{i}\frac{\pi}{3}}$			
			(ii) $\dfrac{\left(6e^{i\frac{\pi}{3}}\right)^4}{\left(6e^{i\left(\frac{\pi}{2}-\frac{\pi}{3}\right)}\right)}=6^3e^{i\left(\frac{7\pi}{6}\right)}$ $=6^3\left[\cos\left(\dfrac{-5\pi}{6}\right)+i\sin\left(\dfrac{-5\pi}{6}\right)\right]$			
			(iii) $z_2=6e^{-\mathrm{i}\frac{\pi}{3}} \Rightarrow z_2{}^n=6^ne^{\mathrm{i}\left(-\frac{n\pi}{3}\right)}$ Since $z_2{}^n\in\mathbb{R}^+$, $-\frac{n\pi}{3}=2k\pi$ for some integer *k*. $n=-6k$. $n=\ldots,\ 12,\ 6,\ 0,\ -6,\ -12,\ \ldots$ Smallest positive integer n = 6.			

9	Topic: Techniques of Integration / Definite Integrals					
			Solution			
			(i) $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{x+1}}$ $=\frac{1}{2u}$ $\int\frac{\sqrt{x+1}}{x-1}dx=\int\frac{u}{u^2-2}.2u\mathrm{d}u$ $=2\int(1+\frac{2}{u^2-2})\mathrm{d}u$ $= 2\mathrm{u} + \frac{2}{\sqrt{2}}\ln\left\|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right\|+c$ $=2\sqrt{x+1}+\sqrt{2}\ln\left\|\frac{\sqrt{x+1}-\sqrt{2}}{\sqrt{x+1}+\sqrt{2}}\right\|+c$			
			(ii) (a) Area of R $=5(1)-\int_3^8\frac{\sqrt{x+1}}{x-1}dx$ $=5-\left[2\sqrt{x+1}+\sqrt{2}\ln\left\|\frac{\sqrt{x+1}-\sqrt{2}}{\sqrt{x+1}+\sqrt{2}}\right\|\right]_3^8$ $=5-\{2+\sqrt{2}\ln\left(\frac{3-\sqrt{2}}{3+\sqrt{2}}\right)-\sqrt{2}\ln\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\}$ $=3-\left(\sqrt{2}\ln\left(\frac{3-\sqrt{2}}{3+\sqrt{2}}.\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)\right)$			

9	**Topic: Techniques of Integration / Definite Integrals**					
			Solution			
			$=3-\left(\sqrt{2}\ln\left(\frac{4+\sqrt{2}}{4-\sqrt{2}}\right)\right)$ $=3-\sqrt{2}\ln\left(\frac{-4-\sqrt{2}}{-4+\sqrt{2}}\right)$			
			(ii) (b) Vol. generated $=\pi(1^2).5-\pi\int_3^8 \frac{x+1}{(x-1)^2}\,dx$ $=9.52830$ $=9.53 \text{ units}^3$			

10 Topic: Vectors

Solution

(i)

$$\overrightarrow{AB}=\begin{pmatrix}2\\-1\\4\end{pmatrix}-\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}2\\-2\\3\end{pmatrix}$$

$$\overrightarrow{AC}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}-\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}-2\\-2\\-1\end{pmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}2\\-2\\3\end{pmatrix}\times\begin{pmatrix}-2\\-2\\-1\end{pmatrix}=\begin{pmatrix}8\\-4\\-8\end{pmatrix}=4\begin{pmatrix}2\\-1\\-2\end{pmatrix}$$

Choose normal vector $\underset{\sim}{n}_p$ for plane p = $\begin{pmatrix}2\\-1\\-2\end{pmatrix}$.

$$p:\ \mathbf{r}\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-3$$

A cartesian equation of the plane p is $2x-y-2z=-3$

(ii) Let the acute angle between l and p be θ.

The angle between the normal vector $\underset{\sim}{n}_p$ (for plane p) and the direction vector $\underset{\sim}{m}_l$ (for line l),

$$\alpha=\cos^{-1}\frac{\begin{pmatrix}2\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}}{\left|\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right|\left|\begin{pmatrix}1\\0\\-2\end{pmatrix}\right|}=\cos^{-1}\frac{6}{3\sqrt{5}}=26.565^\circ$$

10	Topic: Vectors						
				Solution			
				$\therefore \theta = 90° - 26.565° = 63.4°\left(\text{to 1d.p.}\right)$ or 1.11 rad Alternative : Let the acute angle between *l* and *p* be θ. $$\sin\theta = \frac{\left\|\begin{pmatrix}2\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}\right\|}{\left\|\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right\|\left\|\begin{pmatrix}1\\0\\-2\end{pmatrix}\right\|} = \frac{6}{3\sqrt{5}}$$ $\theta = \sin^{-1}\frac{6}{3\sqrt{5}} = 63.4°\left(\text{to 1d.p.}\right)$ or 1.11 rad (3 s.f.)			
		2	**(iii)**	Assume Q lies on the line *l*. $$\begin{pmatrix}5\\-1\\-2\end{pmatrix} = \begin{pmatrix}2\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-2\end{pmatrix} \Rightarrow \begin{cases}5 = 2+\lambda\\-1=-1\\-2=4-2\lambda\end{cases} \Rightarrow \begin{cases}\lambda = 3\\ \\ \lambda = 3\end{cases}$$ Since $\lambda = 3$ is consistent throughout, Q lies on the line *l*. Alternative 1:			

10	Topic: Vectors						
				Solution			
				Since $\overrightarrow{OQ}=\begin{pmatrix}5\\-1\\-2\end{pmatrix}=\begin{pmatrix}2\\-1\\4\end{pmatrix}+\begin{pmatrix}3\\0\\-6\end{pmatrix}$ $=\begin{pmatrix}2\\-1\\4\end{pmatrix}+3\begin{pmatrix}1\\0\\-2\end{pmatrix}$ Q lies on the line l. Alternative 2: $l:\ \frac{x-2}{1}=\frac{z-4}{-2},\ y=-1$ $Q=(5,-1,-2)$, i.e. $x=5$, $y=-1$, $z=4$. $\frac{x-2}{1}=\frac{5-2}{1}=3$, $\frac{z-4}{-2}=\frac{-2-4}{-2}=3$. Hence, Q lies on the line l.			
			(iv)	Let F be the foot of perpendicular from the point Q to the plane p. $l_{QF}:\ \mathbf{r}=\begin{pmatrix}5\\-1\\-2\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}, \mu\in\mathbb{R}$ Since F lies on l_{QF}, $\overrightarrow{OF}=\begin{pmatrix}5\\-1\\-2\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}$, for some $\mu\in\mathbb{R}$.			

10	Topic: Vectors						
				Solution			
				Since F also lies on plane p, $\overrightarrow{OF}\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-3$. $\left[\begin{pmatrix}5\\-1\\-2\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-3$ $2(5+2\mu)-(-1-\mu)-2(-2-2\mu)=-3$ $15+9\mu=-3\Rightarrow\mu=-2$ $\therefore\overrightarrow{OF}=\begin{pmatrix}5\\-1\\-2\end{pmatrix}-2\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}$ The foot of perpendicular from the point Q to the plane p is $(1,1,2)$. <u>Alternative :</u> $\overrightarrow{QF}=(\overrightarrow{QA}\cdot\hat{\underset{\sim}{n}})\,\hat{\underset{\sim}{n}}$, where $\underset{\sim}{n}$ is a normal vector of p $=\left(\begin{pmatrix}-5\\2\\3\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right)\frac{1}{3}\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ $=-2\begin{pmatrix}2\\-1\\-2\end{pmatrix}$			

10	Topic: Vectors						
				Solution			
				$\therefore \overrightarrow{OF} = \overrightarrow{OQ} + \overrightarrow{QF} = \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ The foot of perpendicular from the point Q to the plane p is $(1,1,2)$. $\overrightarrow{QF} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$ $\lvert\overrightarrow{QF}\rvert = \sqrt{(-4)^2 + 2^2 + 4^2} = 6$ $RF = \sqrt{45-36} = 3$ The locus of R is a circle that lies in plane p with <u>centre</u> $(1,1,2)$ and <u>radius</u> 3.			
			(v)	Let Q' be the image of Q in plane p. $\overrightarrow{OF} = \frac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OQ'})$ $\overrightarrow{OQ'} = 2\overrightarrow{OF} - \overrightarrow{OQ}$ $= 2\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$ <u>Equivalent Methods</u> $\overrightarrow{OQ'} = \overrightarrow{OF} + \overrightarrow{QF}$ $\overrightarrow{OQ'} = \overrightarrow{OQ} + \overrightarrow{QQ'}$ $= \overrightarrow{OQ} + 2\overrightarrow{QF}$			

10 Topic: Vectors

Solution

$$\overrightarrow{BQ'} = \overrightarrow{OQ'} - \overrightarrow{OB}$$

$$= \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix}$$

<u>Alternative:</u>

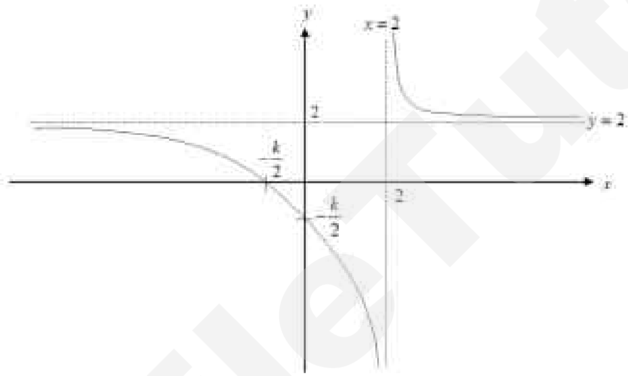
$$\overrightarrow{BQ'} = \overrightarrow{BQ} + \overrightarrow{QQ'}$$

$$= \left[\begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right] + 2\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} \quad = \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix}$$

A vector equation of the line which is a reflection of the line l in plane p is

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \gamma \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix}, \gamma \in \mathbb{R}$$

or $\mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix} + \gamma \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix}, \gamma \in \mathbb{R}$

11	**Topic: Graphing Techniques & Functions**					
			Solution			
			(i) $y=\dfrac{2x+k}{x-2}=2+\dfrac{(k+4)}{x-2}$			
			Vertical Asymptote: $x=2$ Horizontal Asymptote: $y=2$			
			x-intercept: $x=-\dfrac{k}{2}$ y-intercept: $y=-\dfrac{k}{2}$			
						

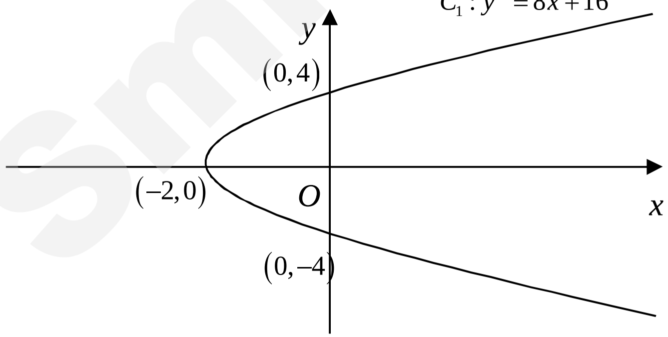
11	Topic: Graphing Techniques & Functions					
			Solution			
			$$\mathrm{f}(x)=\frac{1}{x}$$ $$\downarrow \quad \text{A}$$ $$\mathrm{f}(x-2)=\frac{1}{x-2}$$ $$\downarrow \quad \text{B}$$ $$(k+4)\mathrm{f}(x-2)=\frac{(k+4)}{x-2}$$ $$\downarrow \quad \text{C}$$ $$2+(k+4)\mathrm{f}(x-2)=2+\frac{(k+4)}{x-2}$$			
			A: Translation in the positive x direction by 2 units			
			B: Scaling parallel to the x direction (// to y-axis) by a factor of $(k+4)$			
			C: Translate in the positive y direction by 2 units			
		(iii)	Let $y=\frac{2x+k}{x-2}$			
			$y(x-2)=2x+k$ $yx-2y=2x+k$ $yx-2x=2y+k$ $x(y-2)=2y+k$			
			$x=\frac{2y+k}{y-2}$ $\mathrm{f}^{-1}(y)=\frac{2y+k}{y-2}$			

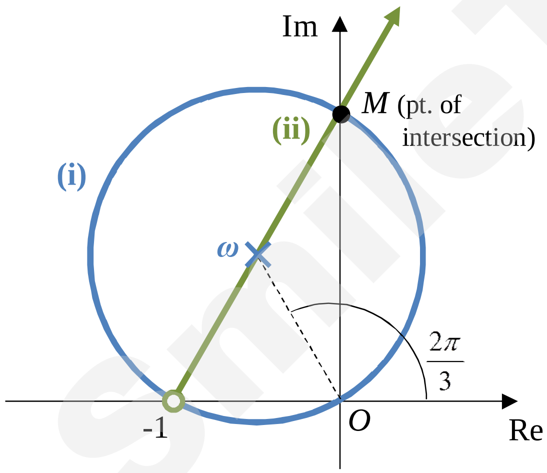
11	Topic: Graphing Techniques & Functions					
			Solution			
			$\mathrm{f}^{-1}(x)=\dfrac{2x+k}{x-2}$			
			$\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_\mathrm{f}=\mathrm{D}_\mathrm{f}=(-\infty,2)\cup(2,\infty)$			
			$\mathrm{f}^{-1}:x\mapsto\dfrac{2x+k}{x-2},\qquad x\in\mathbb{R},\ x\neq 2,$			
			$\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_\mathrm{f}=\mathrm{R}_\mathrm{f}=(-\infty,2)\cup(2,\infty)$			
			(iv) $\because \mathrm{f}(x)=\mathrm{f}^{-1}(x)$			
			$\therefore \mathrm{f}^2=\mathrm{ff}(x)=\mathrm{f}\left[\mathrm{f}^{-1}(x)\right]=x$			
			$\mathrm{D}_{\mathrm{f}^2}=\mathrm{D}_\mathrm{f}=(-\infty,2)\cup(2,\infty)$			
			$\therefore \mathrm{f}^{2017}\left(\dfrac{1}{2}\right)=\mathrm{f}\left(\dfrac{1}{2}\right)$			
			$=\dfrac{2\left(\frac{1}{2}\right)+k}{\left(\frac{1}{2}\right)-2}$ $=\dfrac{1+k}{-\frac{3}{2}}$ $=\dfrac{-2-2k}{3}$			
			(v) $\mathrm{f}(x)=\dfrac{2x+k}{x-2}$ $\mathrm{D}_\mathrm{f}=(-\infty,2)\cup(2,\infty)$ $\mathrm{R}_\mathrm{f}=(-\infty,2)\cup(2,\infty)$			

11	**Topic: Graphing Techniques & Functions**					
			Solution			
			$\mathrm{g}(x)=a+\sqrt{x-3}$			
			$\mathrm{D_g}=(3,\infty)$			
			$\mathrm{R_g}=(a,\infty)$			
			Since $\mathrm{fg}(x)$ exists, $\mathrm{R_g} \subseteq \mathrm{D_f}$			
			$\mathrm{R_g}=(a,\infty)$ $\mathrm{D_f}=(-\infty,2)\cup(2,\infty)$ $\therefore a\geq 2$			
			For $\mathrm{gf}(x)$ to exist, $\mathrm{R_f} \subseteq \mathrm{D_g}$ $\mathrm{R_f}=(-\infty,2)\cup(2,\infty)$ $\mathrm{D_g}=(3,\infty)$ Since $\mathrm{R_f} \not\subset \mathrm{D_g}$, $\mathrm{gf}(x)$ does not exist.			

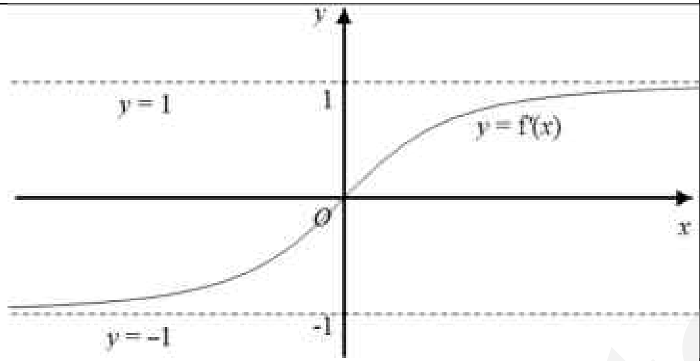
Section A: Pure Mathematics [40 marks]

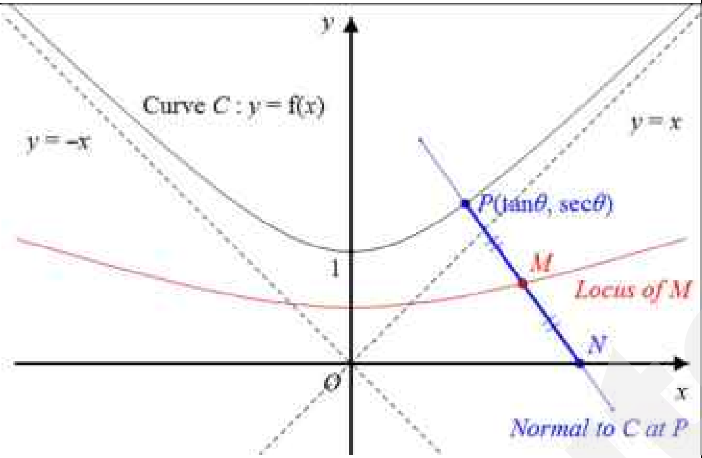
1		**Topic: Differential Equations**					
				Solution			
			(i)	Given $u = x^2 + y^2$, differentiate with respect to x: $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ $x + y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}x}$ (I) Substitute (I) & $u = x^2 + y^2$ and into D.E: $\frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}x} = \sqrt{u}$: $\frac{\mathrm{d}u}{\mathrm{d}x} = 2\sqrt{u}$ (shown) Hence, $\frac{1}{\sqrt{u}}\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ $u^{-\frac{1}{2}}\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ Integrate both sides with respect to x: $\int u^{-\frac{1}{2}}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int 2\,\mathrm{d}x$ $\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2x + C,$ where C is an arbitrary constant			

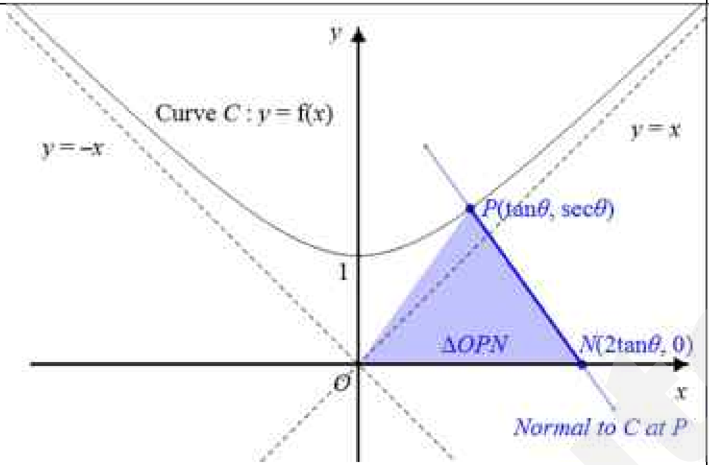
1	Topic: Differential Equations					
			Solution			
			$2u^{\frac{1}{2}} = 2x + C$ $\sqrt{u} = x + \frac{C}{2}$ $\sqrt{x^2 + y^2} = x + D,$ where $D = \frac{C}{2}$			
			(ii) $\sqrt{x^2 + y^2} = x + D$ $x^2 + y^2 = (x + D)^2$ $y^2 + x^2 = x^2 + 2Dx + D^2$ $y^2 = 2Dx + D^2$ $y = \pm\sqrt{2Dx + D^2}$ When $x = -2, y = 0$ $0 = -4D + D^2$ $D(D-4) = 0$ $D = 0$ (rej.) or $D = 4$ $y^2 = 8x + 16$ 			
			(iii) The equation of line of symmetry is $y = 0$.			

2	Topic: Complex Numbers					
			Solution			
			Using Geometric series, $1+\omega+\omega^2=\dfrac{\omega^3-1}{\omega-1}=0 \quad (\text{as } \omega^3=1)$ Or $\omega^3-1=0$ Since ω is a root of $x^3-1=0$, $(\omega-1)(\omega^2+\omega+1)=0$ Since $\omega\neq 1, \omega^2+\omega+1=0$ Or Any acceptable method			
			 Using Pythagoras Theorem, $OM=\sqrt{2^2-1^2}=\sqrt{3}$.			

2	Topic: Complex Numbers					
			Solution			
			$\therefore$ The complex number is $\sqrt{3}\mathrm{i}$.			

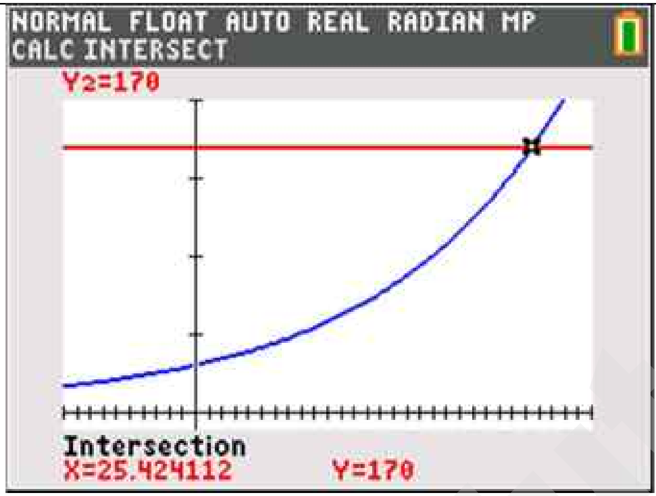
3		Topic: Application of Differentiation				
			Solution			
			(a) 			
			(b) (i) Given $x = \tan\theta$, $y = \sec\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$, $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \sec\theta\tan\theta$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \Big/ \frac{\mathrm{d}x}{\mathrm{d}\theta}$ $= \frac{\sec\theta\tan\theta}{\sec^2\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$ At point P, gradient of normal = $-\frac{1}{\sin\theta}$. Equation of the normal to the curve at P : $y - \sec\theta = -\frac{1}{\sin\theta}(x - \tan\theta)$, $y - \frac{1}{\cos\theta} = -x\frac{1}{\sin\theta} + \frac{1}{\cos\theta}$, $y = -x\operatorname{cosec}\theta + 2\sec\theta$. (shown)			

3		Topic: Application of Differentiation					
			Solution				
			(b) (ii)  x-intercept of the normal at P : $0 = -x \operatorname{cosec}\theta + 2\sec\theta$, $x = 2\frac{\sec\theta}{\operatorname{cosec}\theta} = 2\tan\theta$. $\therefore$ Point N is $(2\tan\theta,\ 0)$. Mid-point of PN is M $\left(\frac{x_P+x_N}{2},\ \frac{y_P+y_N}{2}\right) = \left(\frac{3}{2}\tan\theta,\ \frac{1}{2}\sec\theta\right)$. Locus of point M is given by the parametric equations $x = \frac{3}{2}\tan\theta$, $y = \frac{1}{2}\sec\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Since $\sec^2\theta - \tan^2\theta = 1$, and that $\sec\theta = 2y$, $\tan\theta = \frac{2}{3}x$, $\therefore\ (2y)^2 - (\frac{2}{3}x)^2 = 1$, a cartesian equation for the locus of M.				

3	Topic: Application of Differentiation				
			Solution		
			(b) (iii)  A, area of $\Delta OPN = \frac{1}{2}(ON)\left(\substack{\text{Height of } P \\ \text{w.r.t. } x\text{-axis}}\right)$ $= \frac{1}{2}(2\tan\theta)(\sec\theta) = \tan\theta\sec\theta$ ($ON = 2\tan\theta$, from (b)(ii) assuming $\theta > 0$.) $\frac{\mathrm{d}A}{\mathrm{d}\theta} = \left(\sec^2\theta\right)\sec\theta + \tan\theta\left(\sec\theta\tan\theta\right)$ $= \sec^3\theta + \sec\theta\tan^2\theta$ Rate of change of area of ΔOPN, $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$ $= (\sec^3\theta + \sec\theta\tan^2\theta) \times \cos\theta$ $= \sec^2\theta + \tan^2\theta$ Alternatively, Differentiating $A = \tan\theta\sec\theta$ implicitly with respect to time t,		

3	Topic: Application of Differentiation					
			Solution			
			$\frac{\mathrm{d}A}{\mathrm{d}t} = \left[\left(\sec^2\theta\right)\sec\theta + \tan\theta\left(\sec\theta\tan\theta\right)\right] \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$ $= (\sec^3\theta + \sec\theta\tan^2\theta) \times \cos\theta$ $= \sec^2\theta + \tan^2\theta$ When $\theta = \frac{\pi}{6}$, $\sec\theta = \frac{1}{\cos\theta} = \frac{2}{\sqrt{3}}$, $\tan\theta = \frac{1}{\sqrt{3}}$ $\therefore \ \frac{\mathrm{d}A}{\mathrm{d}t} = \sec^2\theta + \tan^2\theta = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$, rate of change of the area of ΔOPN when $\theta = \frac{\pi}{6}$.			

4		APGP				
			Solution			
			(i) S_{15} of B $= \frac{15}{2}\left(2(2.4)+14d\right)$ $= 36+105d$			
			(ii) S_{15} of A $= \frac{2.4\left((1.2)^{15}-1\right)}{1.2-1}$ $= 172.88$ > 170 Yes, Adam can achieve his target.			
			(iii) U_{15} of A $> U_{15}$ of B U_{15} of A $-$ U_{15} of B > 0 $2.4(1.2)^{14}-(2.4+14d)>0$ $d<2.02957$ $\max d = 2.02 (2\text{ dp})$			
			$2.4(1.2)^{14}-(2.4+14(2.02)) = 0.134$ (shown)			
			(iv) New S_{13} of A $= \frac{2.4\left(\left(1+\frac{r}{100}\right)^{13}-1\right)}{\left(1+\frac{r}{100}\right)-1} = 170$ From GC,			

4	APGP					
			Solution			
			 $x = 25.4\%$			

Section B: Statistics [60 marks]

5	Topic: Probability					
			Solution			
			(i) P(at least one blue ball) = 1 – P(no blue balls) = 1 – P(3 red balls) $= 1 - \frac{{}^{4}C_{3}}{{}^{12}C_{3}}$ $= 1 - \frac{4}{220}$ $= \frac{54}{55}$ (Shown)			
			<u>Alternative Method</u> P(at least one blue ball) = 1 – P(3 red balls) $= 1 - \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$ $= \frac{54}{55}$ (Shown)			
			<u>Alternative Method (Direct Method)</u> P(at least one blue ball) = P(1 blue and 2 red) + P(2 blue and 1 red) + P(3 blue) $= \frac{{}^{8}C_{1} \times {}^{4}C_{2}}{{}^{12}C_{3}} + \frac{{}^{8}C_{2} \times {}^{4}C_{1}}{{}^{12}C_{3}} + \frac{{}^{8}C_{3}}{{}^{12}C_{3}}$ $= \frac{48}{220} + \frac{112}{220} + \frac{56}{220}$ $= \frac{54}{55}$ (Shown)			

5	Topic: Probability					
			Solution			
			<u>Alternative Method</u> P(at least one blue ball) = P(1 blue and 2 red) + P(2 blue and 1 red) + P(3 blue) $=\left(\frac{8}{12}\times\frac{4}{11}\times\frac{3}{10}\times\frac{3!}{2!}\right)+\left(\frac{8}{12}\times\frac{7}{11}\times\frac{4}{10}\times\frac{3!}{2!}\right)+\left(\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\right)$ $=\frac{48}{220}+\frac{112}{220}+\frac{56}{220}$ $=\frac{54}{55}$ (Shown)			
			(ii) P(at least one of each colour drawn) = 1 – P(3 red) – P(3 blue) $= 1 - \frac{{}^{4}C_3}{{}^{12}C_3} - \frac{{}^{8}C_3}{{}^{12}C_3}$			
			$= 1 - \frac{4}{220} - \frac{56}{220}$ $=\frac{8}{11}$ or ≈ 0.727			
			<u>Alternative Method</u> P(at least one of each colour drawn) = P(1 blue and 2 red) + P(2 blue and 1 red) $=\frac{{}^{8}C_1\times{}^{4}C_2}{{}^{12}C_3}+\frac{{}^{8}C_2\times{}^{4}C_1}{{}^{12}C_3}$ $=\frac{48}{220}+\frac{112}{220}$ $=\frac{8}{11}$ or ≈ 0.727			
		1	**(iii)** There are 8 balls with “0” and 4 balls with “1” P(sum is at least two) = P(1, 1, 0) + P(1, 1, 1)			

5	Topic: Probability					
			Solution			
			$=\dfrac{{}^4C_2 \times {}^8C_1}{{}^{12}C_3}+\dfrac{{}^4C_3}{{}^{12}C_3}$			
			$=\dfrac{48}{220}+\dfrac{4}{220}$ $=\dfrac{13}{55}$ or ≈ 0.236			
			<u>Alternative Method</u> P(sum is at least two) = P(1, 1, 0) + P(1, 1, 1) $=\left(\dfrac{4}{12}\times\dfrac{3}{11}\times\dfrac{8}{10}\times\dfrac{3!}{2!}\right)+\left(\dfrac{4}{12}\times\dfrac{3}{11}\times\dfrac{2}{10}\right)$ $=\dfrac{13}{55}$ or ≈ 0.236			

6		Topic: Binomial Distribution				
			Solution			
			(i) $X \sim B(60, 0.03)$ Since $n = 60 > 50$ is large, $p = 0.03 < 0.1$ is small and $np = 1.8 < 5$, $X \sim \text{Po}(1.8)$ approximately $X_1 + X_2 \sim \text{Po}(3.6)$ approximately $P(X_1 + X_2 > 6) = 1 - P(X_1 + X_2 \le 6)$ ≈ 0.073273 $= 0.0733 \text{ (3s.f.)}$			
			(ii) Since sample size = 50 is large, $\bar{X} \sim \text{N}\left(1.8, \frac{1.746}{50}\right)$ approximately by Central Limit Theorem $P(\bar{X} > 2) \approx 0.14225$ $= 0.142 \text{ (3s.f.)}$			

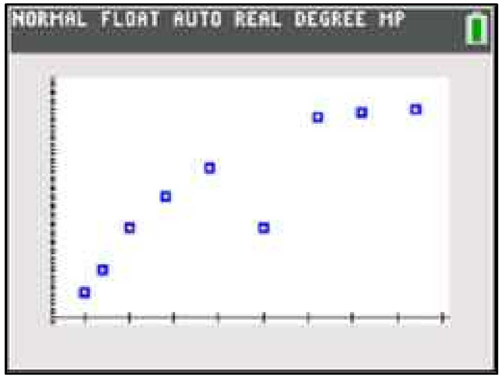
7	**Topic: Permutation and Combination**					
			(i) No. of ways with no restriction = (no. of ways to separate 9 people into a group of 4 and a group of 5) × (no. of ways to arrange the row of 4 people) × (no. of ways to arrange the circle of 5 people) $= {}^9C_4 \times 4! \times {}^5C_5 \times (5-1)!$ = 72 576			
			<u>Alternative Method</u> No. of ways with no restriction = (no. of ways to separate 9 people into a group of 5 and a group of 4) × (no. of ways to arrange the circle of 5 people) × (no. of ways to arrange the row of 4 people) $= {}^9C_5 \times (5-1)! \times {}^4C_4 \times 4!$ [Note: ${}^9C_4 = {}^9C_5$] = 72 576			
			(ii) No. of ways if Albert and Berta sit together = (Albert and Berta in the row) + (Albert and Berta in the circle) = (no. of ways to pick 2 remaining people for the row × arrange row people × A and B swap × arrange circle people) + (no. of ways to pick 3 remaining people for the circle × arrange circle people × A and B swap × arrange row people) $= ({}^7C_2 \times 3! \times 2 \times (5-1)!) + ({}^7C_3 \times (4-1)! \times 2! \times 4!)$			
			= 6048 + 10080 = 16128			
			(iii) No. of ways if Albert and Berta both sit on the couch or both sit at the table			

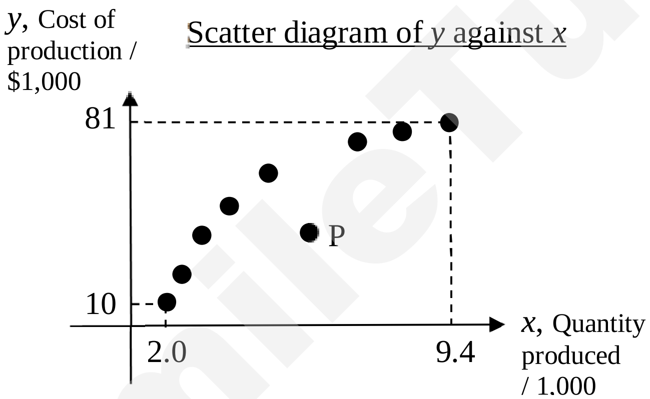
7	Topic: Permutation and Combination					
			= (Albert and Berta in the row) + (Albert and Berta in the circle) = (no. of ways to pick 2 remaining people for the row × arrange row people × arrange circle people) + (no. of ways to pick 3 remaining people for the circle × arrange row people × arrange circle people) $({}^7C_2 \times 4! \times (5-1)!) + ({}^7C_3 \times 4! \times (5-1)!)$ = 32256 No. of ways if Albert and Berta sit separately = 32256 – 16128 = 16128			
			<u>Alternative Method</u> No. of ways if Albert and Berta sit separately = (Albert and Berta in the row) + (Albert and Berta in the circle) = (no. of ways to pick 2 remaining people for the row × arrangement in row × arrangement in circle) + (no. of ways to pick 3 remaining people for the circle × arrangement in circle × arrangement in row) = (no. of ways to pick 2 remaining people for the row × [arrange remaining 2 people in row × slot in Albert and Berta] × arrangement in circle) + (no. of ways to pick 3 remaining people for the circle × [arrange remaining 3 people in circle × slot in Albert and Bert] × arrangement in row) $= ({}^7C_2 \times [\, 2! \times {}^3C_2 \times 2! \,] \times (5-1)!) + ({}^7C_3 \times [\, (3-1)! \times {}^3C_2 \times 2! \,) \times 4!)$			
			= 10080 + 6048 = 16128			

8 **Topic: Correlation and Regression**

Solution

(i) GC screenshot :



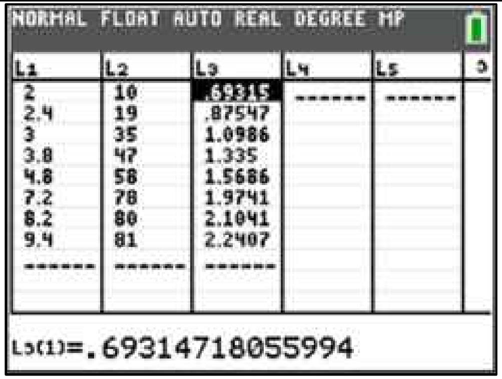


(ii) P = (6.0, 35).

(iii) The better model is $y = a + b \ln x$ since the set of data points in the scatter diagram exhibits a non-linear trend in which y increases at a decreasing rate as x increases, rather than a linear trend in which y increases at a constant rate with x.

Solution

(iv)



$r_{(\ln x),y} = 0.991$.

Line of regression y on $\ln x$.

$y = (47.73168413\ldots)\ln x + (-19.93984362\ldots)$

$\Rightarrow y = 47.7 \ln x - 19.9$

If a fixed cost is included in y, the value of $r_{(\ln x),y} = 0.991$ is unchanged, as there is no change in the strength of the linear relationship between ($\ln x$) and y, under translation.

However, as a constant value M is added to the y value of each data point, the regression line would be also translated in the direction of the positive y-axis by M units,

i.e. with resultant regression line equation being

8			Topic: Correlation and Regression			
			Solution			
			$y = 47.7 \ln x - 19.9 + M$ or $y - M = 47.7 \ln x - 19.9$			
			(v) When $x = 6$, $y = (47.73168413\ldots) \ln(6) - 19.93984362\ldots$ $= 65.5838534\ldots$ ≈ 65.6 Hence, the estimated cost of production is \$ 65,600 (to 3 s.f). This value is <u>reliable</u>, since <u>$r = 0.991$ is close to 1</u> indicating a strong positive linear relationship between y and $\ln x$, and <u>estimating y at $x = 6$ is an interpolation</u> as $x = 6$ is within the range of values of x in the data used to construct the regression line (i.e. $2.0 \le x \le 9.4$).			

9			Topic: Normal Distribution			
			Solution			
			(i) Let X minutes be the random variable denoting the finishing time of a randomly selected runner in the race. $X \sim N\left(\mu, \sigma^2\right)$. $P\left(X \le 40\right) = 0.1 \Rightarrow P\left(Z \le \dfrac{40-\mu}{\sigma}\right) = 0.1$ $\Rightarrow \dfrac{40-\mu}{\sigma} = -1.2816$ $\Rightarrow \mu - 1.2816\sigma = 40 \quad (1)$ $P\left(X > 60\right) = 0.35 \Rightarrow P\left(Z \le \dfrac{60-\mu}{\sigma}\right) = 0.65$ $\Rightarrow \dfrac{60-\mu}{\sigma} = 0.38532$ $\Rightarrow \mu + 0.38532\sigma = 60$ Solving (1) and (2), $\mu \approx 55.377 = 55.4(\text{3s.f.})$ and $\sigma \approx 11.998 = 12.0(\text{3s.f.})$			
			(ii) For $P\left(X \le a\right) = 0.2 \Rightarrow a \approx 45.300 = 45.3(\text{3s.f.})$ Maximal timing is 45.3 minutes. [Accept $a \approx 45.279$ for 5s.f. intermediate]			
			(iii) Let Y be the number of runners, out of 12, who receive a medal. $Y \sim B(12, 0.2)$ $P\left(Y > 4\right) = 1 - P\left(Y \le 4\right) \approx 0.072555 = 0.0726$			
			(iv) $P\big($slowest runner finishes within 1 hour | all do not re			

9			Topic: Normal Distribution			
			Solution			
			$=\dfrac{P\left(\text{slowest runner finishes within 1 hour and all do not}\right.}{P\left(\text{all do not receive medal}\right)}$ $=\dfrac{P\left(\text{all runners finish within 1 hour and do not receiv}\right.}{\left(1-0.2\right)^{12}}$ $=\dfrac{P\left(45.279 \le X_1 \le 60 \text{ and } 45.279 \le X_2 \le 60 \text{ and ... and } 45.\right.}{\left(1-0.2\right)^{12}}$ $=\dfrac{\left[0.65-0.2\right]^{12}}{\left(0.8\right)^{12}}$ $=\left(\dfrac{0.45}{0.8}\right)^{12}$ $\approx 0.0010034 = 0.00100 (3\text{s.f.})$ Method 2: Let A be the r.v. denoting the number of runners who finish in an hour and do not receive medal, out of 12. $A \sim B\left(12, 0.65-0.2\right)$ $A \sim B\left(12, 0.45\right)$ Required probability $=\dfrac{P(A=12)}{P(Y=0)}$			

10	Topic: Sampling Methods Hypothesis Testing						
				Solution			
			a(i)	This method is quota sampling and is not representative of the student population as there is no consideration of the size of each sports team and each club and society relative to the student population, and the student from each group is selected non-randomly i.e. only Captains and Presidents selected.			
			a(ii)	Stratified sampling. Divide school population by year/level (strata), calculate the proportion of each strata relative to the population and select the respective number randomly.			
			b(i)	Let X cm be the vertical jump height of a randomly selected volleyball player. $H_0 : \mu = 40$ vs $H_1 : \mu > 40$			
			b(ii)	Under H_0, test statistics: $T = \dfrac{\overline{X} - \mu}{s / \sqrt{n}} \sim t(n-1)$ Where $\mu = 40, \overline{x} = 42.1, n = 7, s^2 = \dfrac{7}{6}k^2$ To reject H_0, p-value < 0.1 $\Rightarrow P(T > t) < 0.1$ $\Rightarrow P\left(T > \dfrac{42.1 - 40}{\sqrt{7/6}\ \alpha / \sqrt{7}}\right) < 0.1$ $\Rightarrow P\left(T \le \dfrac{2.1}{k / \sqrt{6}}\right) > 0.9$ $\Rightarrow \dfrac{2.1}{k / \sqrt{6}} > 1.4398$			

10				Topic: Sampling Methods Hypothesis Testing			
				Solution			
				$\Rightarrow k < \dfrac{2.1\sqrt{6}}{1.4398} \approx 3.5728$ Required set is $\{k \in \mathbb{R} : 0 < k < 3.57\}$ or $(0, 3.57)$			
		b(iii)		Since $k^2 = 15 \Rightarrow k = \sqrt{15} > 3.57,$ we do not reject H_0 at 10% level of significance and conclude that there is insufficient evidence to support the claim that the training regime is effective.			

11				**Topic: Poisson Distribution**			
				Solution			
			(i)	Any two of the following 4 assumptions: (1) The calls are made independently of each other. (2) The probability of receiving two or more calls within a very short interval of time is negligible. OR Calls are received one at a time. (3) The average number of calls received over any time interval of the same duration within the day is constant. (4) The calls received occur randomly.			1.
			(ii)	Let W_n, X_n and Y_n be the random variables denoting the number of calls received in n hours pertaining to credit card queries, business banking queries and personal banking queries respectively. Then $W_n \sim \text{Po}(n\mu), X_n \sim \text{Po}(6n)$ and $Y_n \sim \text{Po}(7n)$ $8P(W_4 = 2) = P(W_1 = 2)$ where $W_1 \sim Po(\mu)$ and W $\Rightarrow \frac{8e^{-4\mu}(4\mu)^2}{2!} = \frac{e^{-\mu}\mu^2}{2!} \Rightarrow e^{3\mu} = 128$ $\Rightarrow \mu = \frac{1}{3}\ln 128 = \frac{7}{3}\ln 2$			
			(iii)	Since $W_2, X_2,$ and Y_2 are independent, $P(W_2 = 0 \mid W_2 + X_2 + Y_2 > 50) = \frac{P(W_2 = 0 \text{ and } W_2 +}{P(W_2 + X_2 +}$ $= \frac{P(W_2 = 0 \text{ and } X_2 +}{1 - P(W_2 + X_2 + Y_2}$			

11				Topic: Poisson Distribution			
				Solution			
				where $W_2 \sim \text{Po}\left(\frac{14}{3}\ln 2\right)$ and $W_2 + X_2 + Y_2 \sim \text{Po}\left(\frac{14}{3}\ln 2 + 26\right)$, $X_2 + Y_2 \sim \text{Po}(26)$ $= \frac{P(W_2 = 0)\cdot\left[1 - P(X_2 + Y_2 \le 50)\right]}{1 - P(W_2 + X_2 + Y_2 \le 50)}$ ≈ 0.0022359 $= 0.00224$ (3s.f.)			
			(iv)	$X_2 \sim \text{Po}(12)$ and $Y_2 \sim \text{Po}(14)$. Since $\lambda_1 = 12 > 10$ and $\lambda_2 = 14 > 10$, $X_2 \sim \text{N}(12, 12)$ approximately and $Y_2 \sim \text{N}(14, 14)$ approximately Since X_2 and Y_2 are independent, $X_2 - Y_2 \sim \text{N}(-2, 26)$ approximately $P(X_2 > Y_2) = P(X_2 - Y_2 > 0)$ $\approx P(X_2 - Y_2 > 0.5)$ by continuity correction ≈ 0.31196 $= 0.312$ (3s.f.)			

Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) 9740/01

Paper 1

14 September 2016

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Name, Index Number and Class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	**6**	**6**	**7**	**7**	**8**	**9**	**10**	**10**	**11**	**13**	**13**	**100**

1 **(i)** Given that $\int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ where c is an arbitrary constant and $n \neq -1$, find $\int x\sqrt{(4-x^2)} \, dx$. [2]

(ii) Hence find the exact volume of revolution when the region bounded by the curve $y = x^{\frac{3}{2}}(4-x^2)^{\frac{1}{4}}$, the lines $x = 0$, $x = 2$ and $y = 3$, is rotated completely about the x-axis. [4]

2 The complex number w is such that $kw^2 + kww^* + iw - iw^* - 1 = 0$, where w^* is the complex conjugate of w and k is a real and non-zero constant.

(i) For $w = a + bi$ where a and b are real numbers, obtain an expression for b in terms of a and k. Explain why w is either purely real or purely imaginary. [4]

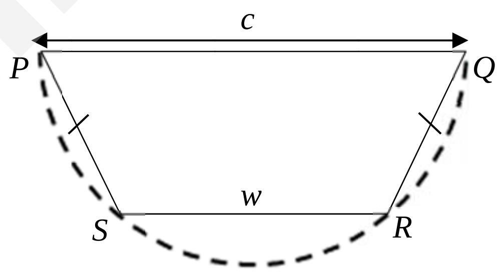
(ii) Using your result in part **(i)**, or otherwise, find the real roots of the equation $2w^2 + 2ww^* + iw - iw^* - 1 = 0$. [2]

3 **(i)** Without using a calculator, find the exact solution of the inequality

$$4 - x \geq \frac{4}{x+2}.$$ [4]

(ii) Hence solve $5 - |x| \geq \frac{4}{|x|+1}$. [3]

4



To travel along the River Nile, an adventurer decides to use a log with a semi-circular cross-section of constant diameter c metres to build a boat. The log is trimmed such that the uniform cross-section of the boat is an isosceles trapezium with base width w metres and $PS = QR$, as shown in the diagram above.

(i) Show that the cross-sectional area of the boat A metres^2 is given by

$$A = \tfrac{1}{4}(c+w)^{\frac{3}{2}}(c-w)^{\frac{1}{2}}.$$ [2]

(ii) Find the value of w, in terms of c, that gives the stationary value of A. Hence determine whether this stationary value is a maximum or a minimum. [5]

5 Given that $y = \ln(1 + \tan x)$,

(i) show that $\dfrac{d^2 y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + 2(1 - e^y)\dfrac{dy}{dx} = 0$, [3]

(ii) find the Maclaurin series for y up to and including the term in x^3, given that the value of $\dfrac{d^3 y}{dx^3}$ when $x = 0$ is 4. [2]

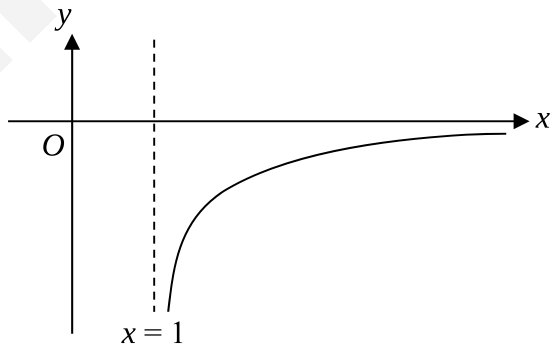
Hence find the first three terms in the series expansion of $\dfrac{\sec^2 x}{1 - \tan x}$. [3]

6 **(a)** Use the substitution $x = e^t$ to find $\displaystyle\int \frac{1}{2e^t + e^{-t}}\, dt$. [3]

(b) **(i)** Express $\dfrac{4 + x}{(1 - x)(4 + x^2)}$ in partial fractions. [2]

(ii) Evaluate $\displaystyle\int_2^n \frac{4 + x}{(1 - x)(4 + x^2)}\, dx$, giving your answer in the form $\frac{1}{2}\ln\left[\dfrac{f(n)}{8(n-1)^2}\right]$, where $f(n)$ is a function of n. [2]

The curve C has equation $y = \dfrac{4 + x}{(1 - x)(4 + x^2)}$. The diagram below shows the part of C for which $x > 1$.



Find the exact value of the area of the region between C and the positive x-axis for $x \geq 2$. [2]

7 A curve C has parametric equations

$$x = \frac{\theta}{\sqrt{(1-\theta^2)}}, \quad y = \sin^{-1}\theta, \quad \text{for } -1 < \theta < 1.$$

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 1-\theta^2$. What can be said about the tangents to C as $\theta \to \pm 1$? [4]

(ii) Sketch C, showing clearly its axial intercept and asymptotes. [2]

(iii) Find the equation of the tangent at the point where C has maximum gradient. By considering the intersection between C and an appropriate graph, find the set of positive values of k for which the equation $\sin^{-1} x - \frac{kx}{\sqrt{(1-x^2)}} = 0$ has at most one real root. [4]

8 A sequence of real numbers $u_0, u_1, u_2, \ldots$ satisfy the recurrence relation

$$u_n = u_{n-1} + \ln\left(\frac{n}{n+1}\right)$$

for $n \geq 1$ and $u_0 = 2$.

(i) Use the method of mathematical induction to prove that $u_n = 2 - \ln(n+1)$ for $n \geq 0$. [4]

(ii) By considering $u_r - u_{r-1}$, show how the result for u_n in part **(i)** can be obtained using the method of differences. [4]

(iii) Show that $\sum_{n=0}^{N} u_n > (N+1)(2-\ln(N+1))$. [2]

9 Joseph started a marathon race. After a while, his trainer, Sarah, starts to collect data on Joseph's speed and she realises that the rate of change of Joseph's speed is proportional to the difference between his speed and a constant a. If the speed of Joseph at time t hours after the start of collection of data is u kilometres per hour, it is found that $\frac{\mathrm{d}u}{\mathrm{d}t}=1$ when $u=14.5$ and $\frac{\mathrm{d}u}{\mathrm{d}t}=2$ when $u=14$.

(i) Show that $\frac{\mathrm{d}u}{\mathrm{d}t}=-2(u-15)$. [3]

(ii) Find the general solution of the equation in part **(i)**, expressing u in terms of t. [3]

(iii) Deduce the steady speed of Joseph eventually. [1]

The distance covered by Joseph, s kilometres, at time t hours after the start of collection of data can be modelled by

$$\frac{\mathrm{d}s}{\mathrm{d}t}=u.$$

(iv) Find s in terms of t. [2]

(v) The result in part **(iv)** can be represented by a family of solution curves. Sketch an appropriate non-linear member of the family of curves that has a linear asymptote that passes through the origin. [2]

10 A curve C has equation $y=\frac{x^2-5}{(x+1)^2-12}$.

(i) Determine the equations of the three asymptotes of C, giving each answer in an exact form. [2]

(ii) Prove algebraically that there are no values of x for which $\frac{1}{2}<y<\frac{5}{6}$. [3]

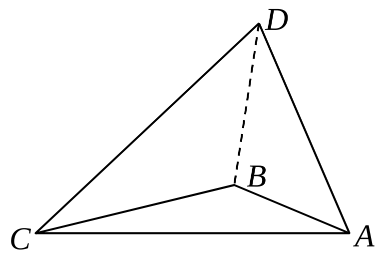
For parts **(iii)** and **(iv)**, you do not need to label the point where the graph cuts the y-axis.

(iii) Sketch C. [3]

(iv) Sketch the graph of $y=\frac{(x+1)^2-12}{x^2-5}$. [3]

(v) Describe a sequence of two transformations which transform C to the graph of $y=\frac{(x-1)^2-5}{(x-2)^2-12}$. [2]

11 The diagram below shows a tetrahedron *ABCD*. The equation of the plane *ABD* is $4x + y + 2z = 16$.



(i) Given that *A* is on the *x*-axis, find the coordinates of *A*. [1]

The equation of the plane *CBD* is $7x - 11y - 5z = -23$.

(ii) Find a vector equation of the line that passes through *B* and *D*. [2]

(iii) Given that *B* is on the *xy*-plane, find the coordinates of *B*. [2]

The cartesian equation of the line that passes through *A* and *D* is $\dfrac{4-x}{2} = \dfrac{y}{2} = \dfrac{z}{3}$.

(iv) Find the coordinates of *D*. [3]

The coordinates of *C* are $(-1, 1, 1)$.

(v) By considering the area of triangle *ABC*, find the exact volume of the tetrahedron *ABCD*. [5]

[Volume of tetrahedron = $\frac{1}{3} \times$ area of base $\times$ perpendicular height]

Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) 9740/02

Paper 2 **26 September 2016**

3 hours

Additional Materials: Answer Paper
Graph paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Name, Index Number and Class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
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Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	8	9	11	12	3	6	7	10	10	12	12	100

 This question paper consists of **6** printed pages (including this cover page).

Section A: Pure Mathematics [40 marks]

1 The function f is defined by

$$\mathrm{f} : x \mapsto \pi \sin\left(\tfrac{1}{2}x\right), \quad x \in \mathbb{R},\ 0 \le x \le a,$$

where a is a positive constant.

(i) State the largest exact value of a for which the function f^{-1} exists. [1]

For the rest of the question, use the value of a found in part (i).

(ii) Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$ and hence verify that 0 and π are solutions to the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

(iii) Find the area of the region bounded by the graphs of f and f^{-1}, giving your answer in terms of π. [3]

The function g is defined by

$$\mathrm{g} : x \mapsto |x-1|, \quad x \in \mathbb{R}.$$

(iv) Find the exact range of the composite function gf. [2]

2 The angle between two unit vectors **a** and **b** is $\cos^{-1}\frac{1}{4}$. Relative to the origin O, the position vector of a point P on a line l is given by $\overrightarrow{OP} = \mathbf{a} + \lambda(\mathbf{a} + 2\mathbf{b}),\ \lambda \in \mathbb{R}$ and the point C has position vector $\mathbf{a} - \mathbf{b}$.

(i) By considering scalar product, show that $CP^2 = 6\lambda^2 + \frac{9}{2}\lambda + 1$. [4]

(ii) Deduce the exact shortest distance of C to l and write down the position vector of the point F, the foot of the perpendicular from C to l, in terms of **a** and **b**. [3]

(iii) Find the equation of the plane that contains l and is perpendicular to CF in the form $\mathbf{r}.\mathbf{n} = d$ where $\mathbf{n}$ is expressed in terms of **a** and **b** and d is a constant. [2]

3 The number of bacteria (in millions) in Pond A at the start of the nth week, before any chemical treatment, is given by u_n. Pond A is treated at the start of each week with Chemical A, which kills 70% of all bacteria instantly. At the end of each week, 6 million new bacteria is reproduced.

(i) Write down a recurrence relation of the form $u_{n+1} = au_n + b,$ where a and b are constants to be determined. [1]

(ii) Show that $u_n = 0.3^{n-1}u_1 + \frac{60}{7}(1-0.3^{n-1}).$ [2]

The number of bacteria (in millions) in Pond B at the start of the nth week, before any chemical treatment, is given by v_n. Pond B is treated at the start of each week with Chemical B. It is known that v_n follows the recurrence relation

$$v_{n+1} = 0.01{v_n}^2 + 6.$$

It is given that if the sequence $v_1, v_2, v_3, \ldots$ converges to a limit, it converges to either α or β, where $\alpha < \beta$.

(iii) Find α and β. Explain whether v_n necessarily converges to α or β. [3]

(iv) If $u_1 = v_1 = 30,$ determine which chemical would be more effective in killing the bacteria in the long run. [2]

Pond C is treated with Chemical C. To account for the bacteria's increasing resistance to the chemical, the dosage of Chemical C is increased by 5 ml each week. The first dose is 20 ml.

(v) How many weeks does it take to finish the first 3 litres of chemical in the treatment of Pond C? [3]

4 **Do not use a graphic calculator in answering this question.**

(a) On a single Argand diagram, sketch the locus of z satisfying both inequalities $|z+1-2\mathrm{i}| \le 2$ and $\frac{1}{4}\pi \le \arg(z+1) \le \frac{1}{2}\pi.$ Hence find the range of $\arg(z)$. [5]

(b) Solve the equation

$$w^6 = 64,$$

giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta},$ where $r > 0$ and $-\pi < \theta \le \pi.$ [2]

(i) Hence write down the roots of the equation $(z-1-\mathrm{i}\sqrt{3})^6 = 64$ in the form $a + r\mathrm{e}^{\mathrm{i}\theta},$ where a is a complex number in cartesian form, $r > 0$ and $-\pi < \theta \le \pi.$ Show the roots on an Argand diagram. [3]

(ii) Of the roots found in part **(b)(i)**, find in cartesian form the root with the largest modulus. [2]

Section B: Statistics [60 marks]

5 The Land Transport Authority (LTA) wishes to gather feedback on the quality of train services at a new train station.

(i) The LTA decides to station a team of surveyors at the gantries to survey the first 100 commuters passing through the train station. State, with a reason, whether the method described is quota sampling. [1]

(ii) The LTA decides to obtain a random sample instead to survey 5% of the commuters on a particular day. Describe how a systematic sample can be carried out in this context. [2]

6 John plays for his school's soccer team. There is a probability of 0.15 that he scores in a game and a probability of 0.3 that his parents are present at a game. When he scores in a game, there is a probability of 0.2 that his parents are present.

(i) Show that the probability that he scores in a game when his parents are present is 0.1 [2]

(ii) State, with justification, whether his parents' presence at a game will affect his chances of scoring in the game. [1]

Games are equally likely to be home or away games. In a home game, there is a probability of 0.24 that John does not score and his parents are present.

(iii) Find the least and greatest values of the probability that a game is a home game and his parents are not present at the game. [3]

7 A committee decides to meet on four days in a span of four weeks. Find the probability that the committee meets on two Tuesdays and two Saturdays if

(i) committee meetings are equally likely to be held on any day in the four weeks, [2]

(ii) committee meetings are held once a week. The probability of holding a meeting on any day from Monday to Friday is $\frac{1}{9}$ and the probability of holding a meeting on either Saturday or Sunday is $\frac{2}{9}$. [2]

The committee of ten sits in a circle at a meeting.

(iii) Find the probability that the two committee vice-heads are seated together and they are not seated next to the committee head. [3]

8 A research is being conducted to study the growth of car population over time. The data below shows the population of the car, y millions after x years of study from the start of the research:

Years (x)	5	7	9	14	18	23	27
Car Population (y millions)	7.2	10.5	11.6	13.0	14.5	15.5	15.7

(i) Draw a scatter diagram for the data, labelling the axes. [1]

(ii) State, with a reason, which of the following models is appropriate:

$\mathbf{A}: y = a + bx^2$, $\quad\quad\quad\quad$ $\mathbf{B}: y = a + b\ln x$,

where b is positive. [2]

Based on the appropriate model chosen in part **(ii)**,

(iii) calculate the value of the product moment correlation coefficient. State, with a reason, whether this value would be different if y is recorded in thousands instead. [2]

(iv) calculate the least square estimates of a and b and write down the corresponding regression line. Obtain the value of the car population after 20 years of study. [3]

(v) give an interpretation of the value of a in the context of the question. Comment on the reliability of the value of a. [2]

9 In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters.

(a) The queuing time, in minutes, for flight passengers at the Economy and Business class check-in counters have independent normal distributions with means and standard deviations as shown in the table.

Check-in Counter	Mean queuing time	Standard deviation
Economy class	11.6	4.2
Business class	3.2	0.9

(i) Find the probability that the queueing time of a randomly chosen Economy class passenger is within 5 minutes of the total queueing time of 2 randomly chosen Business class passengers. [4]

(ii) The queueing time of 8 randomly chosen Business class passengers are taken. Find the probability that the shortest queuing time among all 8 passengers is no less than 2 minutes. [2]

(b) The probability that a passenger books a flight and does not turn up is 0.05. The airline decides to allow for over-booking by selling more tickets than the number of seats available.

For a particular flight with 350 available seats, n tickets were sold, where $n > 350$. By using a suitable approximation, show that if the flight is to have no more than 1% chance of having insufficient seats, the number of tickets sold must satisfy the approximate inequality

$$350.5 - 0.95n \geq 2.3263\sqrt{(0.0475n)}.$$ [4]

10 A manufacturer claims that ropes with a certain diameter produced by his factory have mean breaking strength of at least 169.7 kN. Recently, a new material is used to produce the ropes. A random sample of 8 coils of the rope made with the new material is taken and the breaking strength of each coil of rope, x kN, is measured as follows.

171.3	168.5	166.5	164.4	170.0	165.1	170.1	167.2

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) Stating a necessary assumption, test at the 5% significance level whether the manufacturer's claim is valid after the change in material. [5]

Instead of using the new material, the manufacturer decides to change the weaving process of the ropes. The manufacturer claims that the mean breaking strength is now μ_0 kN. The population variance is found to be 29.16 $(\text{kN})^2$. A random sample of 50 coils of the rope made using the new process is taken and the mean breaking strength, $\bar{y}$ kN, is found to be 171 kN.

(iii) Find the set of values of μ_0 for which the mean breaking strength does not differ from the claim when tested at the 1% significance level. [4]

(iv) Explain, in the context of the question, the meaning of 'at the 1% significance level'. [1]

11 **(a)** A restaurant has 15 tables consisting of 6 rectangular tables and 9 round tables. During the restaurant's opening hours, the rectangular tables are occupied, on average 80 percent of the time, and the round tables are occupied, on average 65 percent of the time. You may assume that the tables in the restaurant are occupied independently of each other.

(i) If a customer walks into the restaurant at a randomly selected time, what is the probability that 4 rectangular tables and 7 round tables are occupied? [2]

(ii) Give a reason in context why the assumption made above may not be valid. [1]

(b) A café sells both coffee and tea. The number of cups of coffee and tea sold in a randomly chosen 20-minute period have independent Poisson distributions with means 5 and 3.5 respectively.

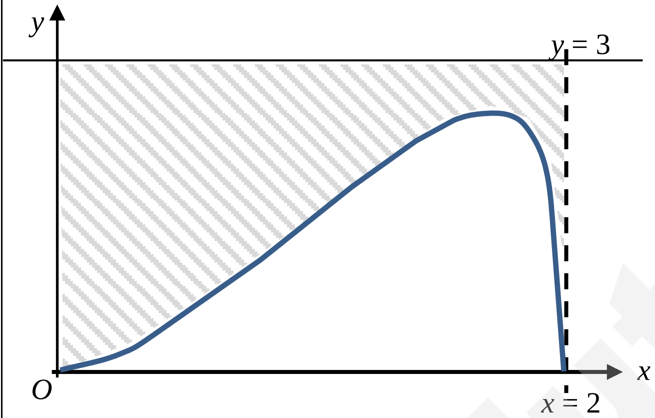
(i) In a particular 20-minute period, at least 7 cups of beverages are sold. Find the probability that at least 6 cups of tea are sold during the 20-minute period. [4]

(ii) Let p_k denote the probability that k cups of coffee are sold in a 20-minute period.

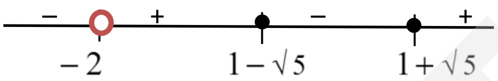
Show that $\dfrac{p_{k+1}}{p_k} = \dfrac{5}{k+1}$ and deduce that $p_{k+1} > p_k$, when $k < 4$. [3]

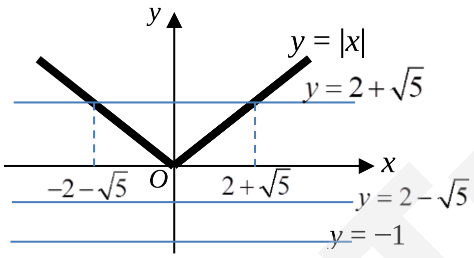
Hence find the most probable number(s) of cups of coffee sold in a 20-minute period.[2]

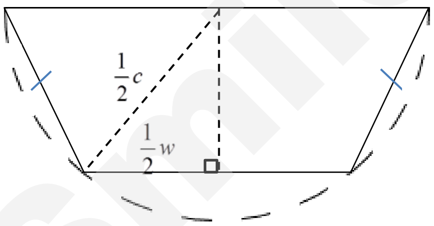
2016 Year 6 Prelim Paper 1 Suggested Solutions

Qn	Suggested Solution
1(i)	$\int x\sqrt{(4-x^2)}\,\mathrm{d}x = -\frac{1}{2}\int (-2x)\sqrt{(4-x^2)}\,\mathrm{d}x$ $= -\frac{1}{2}\left[\frac{2}{3}(4-x^2)^{\frac{3}{2}}\right]+C$ $= -\frac{1}{3}(4-x^2)^{\frac{3}{2}}+C$
	 Volume required $= \pi(3^2)(2) - \pi\int_0^2 \left(x^{\frac{3}{2}}(4-x^2)^{\frac{1}{4}}\right)^2 \mathrm{d}x$ $=18\pi - \pi\int_0^2 x^3(4-x^2)^{\frac{1}{2}}\mathrm{d}x$ $=18\pi - \pi\int_0^2 x^2\left[x(4-x^2)^{\frac{1}{2}}\right]\mathrm{d}x$ $u = x^2 \quad \frac{dv}{dx} = x(4-x^2)^{\frac{1}{2}}$ $\frac{du}{dx} = 2x \quad v = -\frac{1}{3}(4-x^2)^{\frac{3}{2}}$ $=18\pi$ $-\pi\left\{\left[-x^2\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right]_0^2 - \int_0^2 -\frac{1}{3}(4-x^2)^{\frac{3}{2}}(2x)\mathrm{d}x\right\}$ $=18\pi + \frac{\pi}{3}\left[\frac{2}{5}(4-x^2)^{\frac{5}{2}}\right]_0^2$ Use $\int f'(x)[f(x)]^n\,\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}+c$ $=18\pi + \frac{\pi}{3}\left[-\frac{2}{5}(4)^{\frac{5}{2}}\right]$ $=18\pi - \frac{64}{15}\pi$ $=\frac{206}{15}\pi$

Qn	Suggested Solution
2(i)	$kw^2 + kww^* + \mathrm{i}w - \mathrm{i}w^* - 1 = 0$ $kw(w + w^*) + \mathrm{i}(w - w^*) - 1 = 0$ $k(a+b\mathrm{i})(2a) + \mathrm{i}(2b\mathrm{i}) - 1 = 0$ $\left(2ka^2 - 2b\right) + 2abk\mathrm{i} = 1 ----(+)$ <u>Real part</u> $2ka^2 - 2b = 1 \Rightarrow b = \dfrac{2ka^2 - 1}{2}$ ---(1) <u>Im part</u> $ab = 0 \quad \because k \neq 0$ $\Rightarrow b = 0$ **or** $a = 0$ ie, w is either purely real or imaginary.
b(ii)	<u>Hence</u> Since w is real, $b = 0$. Using $k = 2$ and $b = 0$ From part **(i)**: $\dfrac{2(2)a^2 - 1}{2} = 0$ $4a^2 = 1 \Rightarrow a = \pm\sqrt{\dfrac{1}{4}}$ ie, $w = -\dfrac{1}{2}$ or $w = \dfrac{1}{2}$ <u>Otherwise</u> Since w is real, $b = 0$, ie, $w = a$ Using $k = 2$ and $w = a$ eqn becomes: $2a^2 + 2a^2 + \mathrm{i}a - \mathrm{i}a - 1 = 0$ $4a^2 = 1 \Rightarrow a = \pm\sqrt{\dfrac{1}{4}}$ ie, $w = -\dfrac{1}{2}$ or $w = \dfrac{1}{2}$

Qn	Suggested Solution
3 (i)	$4-x\geq\dfrac{4}{x+2}$ $\dfrac{4}{x+2}+x-4\leq 0$ $\dfrac{4+(x+2)(x-4)}{x+2}\leq 0$ $\dfrac{x^2-2x-4}{x+2}\leq 0$ $\dfrac{(x-1)^2-5}{(x+2)}\leq 0$ $\quad$ $a^2-b^2=(a+b)(a-b)$ $\dfrac{\left(x-[1-\sqrt{5}]\right)\left(x-[1+\sqrt{5}]\right)}{x+2}\leq 0$  $\therefore x<-2$ or $1-\sqrt{5}\leq x\leq 1+\sqrt{5}$

(ii)	$4-x\geq\dfrac{4}{x+2}$ $4-\left(\lvert x\rvert-1\right)\geq\dfrac{4}{\left(\lvert x\rvert-1\right)+2}$ $5-\lvert x\rvert\geq\dfrac{4}{\lvert x\rvert+1}$ Since $2-\sqrt{5}<0$, $\lvert x\rvert$ is always greater than $2-\sqrt{5}$ for all real x Hence, replace x with $\left(\lvert x\rvert-1\right)$ in earlier sol: $\therefore \lvert x\rvert-1<-2$ or $1-\sqrt{5}\leq \lvert x\rvert-1\leq 1+\sqrt{5}$ $\Rightarrow \lvert x\rvert<-1$ or $2-\sqrt{5}\leq \lvert x\rvert\leq 2+\sqrt{5}$ **No solution** as $\lvert x\rvert\geq 0$ for all real x — $2-\sqrt{5}\leq \lvert x\rvert$ and $\lvert x\rvert\leq 2+\sqrt{5}$ $x\in\mathbb{R}$ as $\lvert x\rvert\geq 0$ and $-(2+\sqrt{5})\leq x\leq 2+\sqrt{5}$ i.e. $-2-\sqrt{5}\leq x\leq 2+\sqrt{5}$ 

Qn	Suggested Solution
4(i)	 Height of cross-section of boat = $\sqrt{\frac{1}{4}c^2-\frac{1}{4}w^2}$ $A=\frac{1}{2}\sqrt{\frac{1}{4}c^2-\frac{1}{4}w^2}\left(c+w\right)$ $=\frac{1}{4}\sqrt{c^2-w^2}\left(c+w\right)$ $=\frac{1}{4}\sqrt{\left(c+w\right)\left(c-w\right)}\left(c+w\right)$ $=\dfrac{1}{4}(c+w)^{\frac{1}{2}}(c-w)^{\frac{1}{2}}\left(c+w\right)$ $=\dfrac{1}{4}(c+w)^{\frac{3}{2}}(c-w)^{\frac{1}{2}}$ (shown)

(ii)	$\frac{dA}{dw} = \frac{1}{4}\left(\frac{3}{2}\right)(c+w)^{\frac{1}{2}}(c-w)^{\frac{1}{2}} - \frac{1}{4}\left(\frac{1}{2}\right)(c+w)^{\frac{3}{2}}(c-w)^{-\frac{1}{2}}$ $= \frac{3(c+w)^{\frac{1}{2}}(c-w)-(c+w)^{\frac{3}{2}}}{8(c-w)^{\frac{1}{2}}}$ $= \frac{(c+w)^{\frac{1}{2}}[3(c-w)-(c+w)]}{8(c-w)^{\frac{1}{2}}}$ $= \frac{(c+w)^{\frac{1}{2}}[c-2w]}{4(c-w)^{\frac{1}{2}}}$ For stationary $A \Rightarrow \frac{dA}{dw}=0$ $\frac{1}{8}(c+w)^{\frac{1}{2}}(c-2w)=0$ $w=-c$ (reject as $w>0$) or $w=\frac{1}{2}c$ **<u>Alternative</u>** $A^2 = \frac{1}{16}(c+w)^3(c-w)$ $2A\frac{dA}{dw} = \frac{1}{16}\left[3(c+w)^2(c-w)-(c+w)^3\right]$ $= \frac{1}{16}(c+w)^2(3c-3w-c-w)$ $= \frac{1}{16}(c+w)^2(2c-4w)$ $A\frac{dA}{dw} = \frac{1}{16}(c+w)^2(c-2w)$ For stationary $A \Rightarrow \frac{dA}{dw}=0$ $w=-c$ (reject as $w>0$) or $w=\frac{1}{2}c$

w	$\frac{1}{2}c^-$, say $0.4c$	$\frac{1}{2}c$	$\frac{1}{2}c^+$, say $0.6c$
$\frac{dA}{dw}$	+ve	0	+ve
Tangent	/	–	\

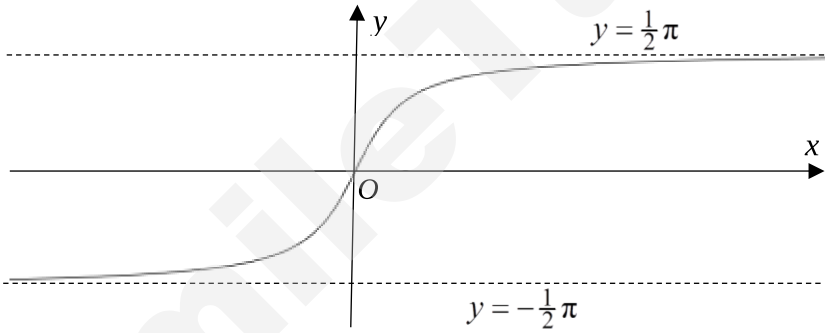
$\therefore$ When $w=\frac{c}{2}$, A is maximum.

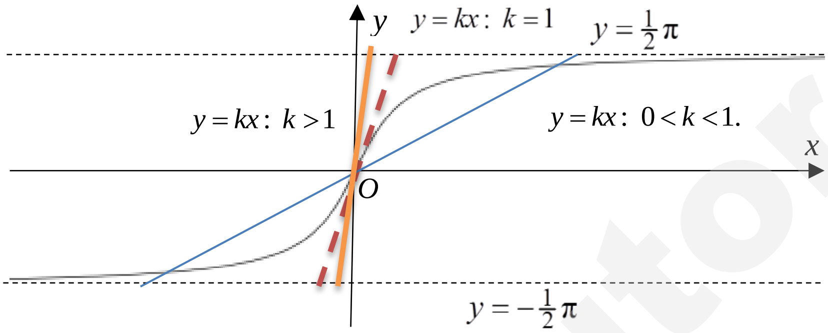
Qn	Suggested Solution
5(i)	**Method 1** $y=\ln(1+\tan x)$ $e^{y}=1+\tan x \qquad -----(1)$ Differentiate wrt x, $e^{y}\dfrac{dy}{dx}=\sec^{2}x \qquad -----(2)$ Differentiate wrt x, $e^{y}\dfrac{d^{2}y}{dx^{2}}+e^{y}\left(\dfrac{dy}{dx}\right)^{2}=2\sec x(\sec x\tan x)$ [from (1), $\tan x=e^{y}-1$] $e^{y}\dfrac{d^{2}y}{dx^{2}}+e^{y}\left(\dfrac{dy}{dx}\right)^{2}=2e^{y}\dfrac{dy}{dx}(e^{y}-1)$ [from (2), $\sec^{2}x=e^{y}\dfrac{dy}{dx}$] $\Rightarrow \dfrac{d^{2}y}{dx^{2}}+\left(\dfrac{dy}{dx}\right)^{2}+2\dfrac{dy}{dx}(1-e^{y})=0 \quad (\because e^{y}\neq 0)$ (shown) **Method 2 (Discouraged)** $y=\ln(1+\tan x)$ $\dfrac{dy}{dx}=\dfrac{\sec^{2}x}{1+\tan x}$ $\dfrac{d^{2}y}{dx^{2}}=\dfrac{(1+\tan x)2\sec^{2}x\tan x-(\sec^{2}x)(\sec^{2}x)}{(1+\tan x)^{2}}$ $=\dfrac{2\sec^{2}x\tan x}{1+\tan x}-\left(\dfrac{\sec^{2}x}{1+\tan x}\right)^{2}$ $=2\left(\dfrac{dy}{dx}\right)\tan x-\left(\dfrac{dy}{dx}\right)^{2}$ $\Rightarrow \dfrac{d^{2}y}{dx^{2}}=2\left(\dfrac{dy}{dx}\right)(e^{y}-1)-\left(\dfrac{dy}{dx}\right)^{2} \quad [\because e^{y}=1+\tan x]$ $\Rightarrow \dfrac{d^{2}y}{dx^{2}}+\left(\dfrac{dy}{dx}\right)^{2}+2\left(\dfrac{dy}{dx}\right)(1-e^{y})=0$ (shown)
(ii)	When $x=0$, $y=\ln(1+\tan 0)=0$ $e^{0}\dfrac{dy}{dx}=\sec^{2}0\Rightarrow \dfrac{dy}{dx}=1$ $\dfrac{d^{2}y}{dx^{2}}+(1)^{2}+2(1)(1-e^{0})=0\Rightarrow \dfrac{d^{2}y}{dx^{2}}=-1$ Given that $\dfrac{d^{3}y}{dx^{3}}=4$, $y=x+(-1)\dfrac{x^{2}}{2!}+(4)\dfrac{x^{3}}{3!}+\ldots=x-\dfrac{1}{2}x^{2}+\dfrac{2}{3}x^{3}+\ldots$

	$\frac{d}{dx}\ln(1+\tan x)=\frac{d}{dx}(x-\frac{1}{2}x^2+\frac{2}{3}x^3+...)$ $\Rightarrow \frac{\sec^2 x}{1+\tan x}=1-x+2x^2+...$ Replace x with (-x), $\frac{\sec^2(-x)}{1+\tan(-x)}=1-(-x)+2(-x)^2+...$ $\Rightarrow \frac{\sec^2(x)}{1-\tan(x)}=1+x+2x^2+...$ $(\because \tan(-x)=-\tan x$ and $\cos(-x)=\cos(x))$ **<u>Alternative</u>** Replace x with (-x), $\ln(1-\tan x)=\ln(1+\tan(-x))$ $=-x-\frac{1}{2}(-x)^2+\frac{2}{3}(-x)^3+...$ $=-x-\frac{1}{2}x^2-\frac{2}{3}x^3+...$ $\frac{d}{dx}\ln(1-\tan x)=\frac{d}{dx}(-x-\frac{1}{2}x^2-\frac{2}{3}x^3+...)$ $\frac{-\sec^2 x}{1-\tan x}=-1-x-2x^2+...$ $\frac{\sec^2 x}{1-\tan x}=1+x+2x^2+...$

Qn	Suggested Solution	
6(a)	$\int \frac{1}{2e^t + e^{-t}}\,dt$ $= \int \frac{1}{2x + \frac{1}{x}}\left(\frac{1}{x}\right)dx$ $= \int \frac{1}{2x^2+1}\,dx$ $= \frac{1}{2}\int \frac{1}{x^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\,dx$ $= \frac{\sqrt{2}}{2}\tan^{-1}(\sqrt{2}x) + C$ $= \frac{\sqrt{2}}{2}\tan^{-1}(\sqrt{2}e^t) + C$	$x = e^t$ $\frac{dx}{dt} = e^t$ $\text{"}dt = \frac{1}{x}dx\text{"}$
(b)	Let $\frac{4+x}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{4+x^2}$. Then $A(4+x^2) + (Bx+C)(1-x) = 4+x$ $x=1: A=1$ $x^2: A-B=0 \therefore B=1$ constant: $4A+C=4 \therefore C=0$ $\frac{4+x}{(1-x)(4+x^2)} = \frac{1}{1-x} + \frac{x}{4+x^2}$. $\int_2^n \frac{4+x}{(1-x)(4+x^2)}\,dx$ $= \int_2^n \left(\frac{1}{1-x} + \frac{x}{4+x^2}\right)dx$ $= \left[-\ln\|1-x\| + \frac{1}{2}\ln\left(4+x^2\right)\right]_2^n$ $= -\ln\|1-n\| + \frac{1}{2}\ln\left(4+n^2\right) - \frac{1}{2}\ln 8$ $= -\frac{1}{2}\ln\|1-n\|^2 + \frac{1}{2}\ln\left(4+n^2\right) - \frac{1}{2}\ln 8$ $= \frac{1}{2}\ln\left(\frac{4+n^2}{8(n-1)^2}\right)$	

	The required area A $= \lim_{n\to\infty}\left[-\frac{1}{2}\ln\left(\frac{4+n^2}{8(n-1)^2}\right)\right]$ $= \lim_{n\to\infty}\left[-\frac{1}{2}\ln\left(\frac{4+n^2}{8(n^2-2n+1)}\right)\right]$ $\lim_{n\to\infty}\left[-\frac{1}{2}\ln\left(\frac{\frac{4}{n^2}+1}{8(1-\frac{2}{n}+\frac{1}{n^2})}\right)\right]$ $=-\frac{1}{2}\ln\left(\frac{1}{8}\right)$ $\quad [\because \text{since } \frac{1}{n} \text{and} \frac{1}{n^2} \to 0 \text{ as } n \to \infty]$ $=\frac{3}{2}\ln 2$ **Alternative** The required area A $= \lim_{n\to\infty}\left[-\frac{1}{2}\ln\left(\frac{4+n^2}{8(n-1)^2}\right)\right]$ $= \lim_{n\to\infty}\left[-\frac{1}{2}\ln\left(\frac{\frac{4}{n^2}+1}{8(1-\frac{1}{n})^2}\right)\right]$ $=-\frac{1}{2}\ln\left(\frac{1}{8}\right)$ $\quad [\because \text{since } \frac{1}{n} \text{and} \frac{4}{n^2} \to 0 \text{ as } n \to \infty]$ $=\frac{3}{2}\ln 2$

Qn	Suggested Solution
7(i)	$x=\dfrac{\theta}{\sqrt{1-\theta^2}}$ $\Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\dfrac{(1-\theta^2)^{\frac{1}{2}}-\theta(\frac{1}{2})(-2\theta)(1-\theta^2)^{-\frac{1}{2}}}{(1-\theta^2)}$ $=\dfrac{(1-\theta^2)^{-\frac{1}{2}}\left[(1-\theta^2)+\theta^2\right]}{(1-\theta^2)}$ $=(1-\theta^2)^{-\frac{3}{2}}$ $y=\sin^{-1}\theta$ $\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\dfrac{1}{\sqrt{1-\theta^2}}=(1-\theta^2)^{-\frac{1}{2}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(1-\theta^2)^{-\frac{1}{2}}}{(1-\theta^2)^{-\frac{3}{2}}}$ $=1-\theta^2$ (shown) As $\theta\to\pm1,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}\to 0.$ The tangents becomes parallel to the x-axis as $\theta\to\pm1.$
(ii)	 Note: as $\theta\to\pm1,\ x\to\pm\infty$ and $y\to\pm\frac{1}{2}\pi.$
(iii)	Since $\theta^2\ge 0,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=1-\theta^2$ is maximum at $\theta=0.$ **<u>Alternative</u>** Let $\mathrm{g}(\theta)=\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-\theta^2.$ Set $\mathrm{g}'(\theta)=-2\theta=0,$ so $\theta=0.$ Then $\mathrm{g}''(\theta)=-2<0,$ and thus $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-\theta^2$ is maximum at $\theta=0.$ At $\theta=0,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=1,\ x=y=0.$ Equation of tangent at (0, 0): $y=x.$ Replacing x with θ, and rearranging the given equation, we get

	$\sin^{-1}\theta = k\dfrac{\theta}{\sqrt{(1-\theta^2)}}$. Since the curve C is described by $$C: x = \frac{\theta}{\sqrt{(1-\theta^2)}},\ y = \sin^{-1}\theta,$$ the number of real roots of the given equation is equal to the number of intersection points between the line $y = kx$ and the curve C. Since the maximum gradient value of C is 1, which is attained at O, thus the required set of positive constants k is given by $\{k \in \mathbb{R}^+ : k \geq 1\}$, in order for the given equation to have at most(exactly) one real root. 

Qn	Suggested Solution
8(i)	Let P(n) be the proposition $u_n = 2 - \ln(n+1)$ for $n \geq 0$. When n = 0, LHS of P(0) $= u_0 = 2$ (given) RHS of P(0) $= 2 - \ln(0+1) = 2 =$ LHS of P(0) $\therefore$ P(0) is true. Assuming that P(k) is true for some $k \geq 0$ i.e. $u_k = 2 - \ln(k+1)$, To show that P(k + 1) is true i.e. $u_{k+1} = 2 - \ln(k+2)$. LHS of P($k+1$) $= u_{k+1}$ $= u_k + \ln\left(\dfrac{k+1}{k+2}\right)$ $= 2 - \ln(k+1) + \ln\left(\dfrac{k+1}{k+2}\right)$ $= 2 - \ln(k+1) + \ln(k+1) - \ln(k+2)$ $= 2 - \ln(k+2)$ $=$ RHS $\therefore$ P(k) is true $\Rightarrow$ P($k+1$) is true. Since P(0) is true, and P(k) is true $\Rightarrow$ P($k+1$) is true, hence by mathematical induction, P(n) is true for all $n \geq 0$.

<table>
<tr><td></td><td></td></tr>
<tr><td>(ii)</td><td>
Consider

$$u_r - u_{r-1} = \ln\frac{r}{r+1} = \ln(r) - \ln(r+1)$$

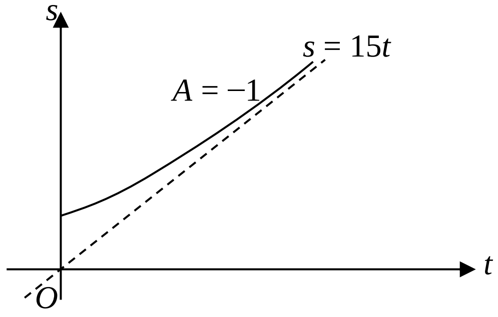
$$\sum_{r=1}^{n}\left(u_r - u_{r-1}\right) = \sum_{r=1}^{n}\left(\ln(r) - \ln(r+1)\right)$$

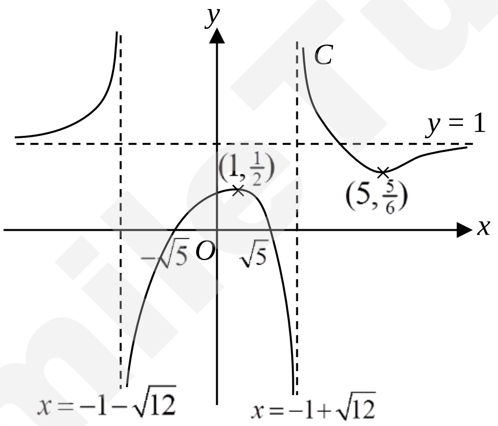
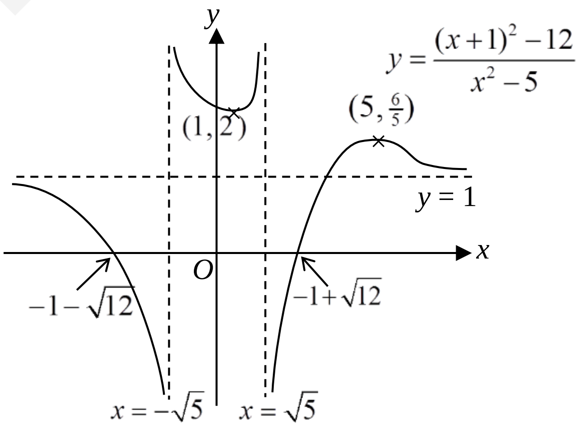
$$\begin{aligned}
&[\, u_1 - u_0 && = && [\ln(1) - \ln(2) \\
+\ & u_2 - u_1 && && + \ln(2) - \ln(3) \\
+\ & u_3 - u_2 && && + \ln(3) - \ln(4) \\
& \quad\vdots && && \quad\vdots \\
+\ & u_{n-1} - u_{n-2} && && + \ln(n-1) - \ln(n) \\
+\ & u_n - u_{n-1}\,] && && + \ln(n) - \ln(n+1)]
\end{aligned}$$

$$u_n - u_0 = \ln(1) - \ln(n+1)$$
$$u_n = 2 - \ln(n+1)$$

Alternative for RHS

$$\sum_{r=1}^{n}\left(\ln\frac{r}{r+1}\right)$$
$$= \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \ldots + \ln\frac{n}{n+1}$$
$$= \ln\left(\frac{1}{2}\frac{2}{3}\frac{3}{4}\ldots\frac{n}{n+1}\right)$$
$$= \ln\frac{1}{n+1}$$
$$= -\ln\,(n+1)$$
</td></tr>
<tr><td>(iv)</td><td>
$$\begin{aligned}
\sum_{n=0}^{N} u_n &= \sum_{n=0}^{N}[2 - \ln(n+1)] \\
&= (N+1)(2) - [\ln 1 + \ln 2 + \ldots + \ln(N+1)] \\
&> (N+1)(2) - [\ln(N+1) + \ln(N+1) + \ldots + \ln(N+1)] \\
&= (N+1)(2) - (N+1)\ln(N+1) \\
&= (N+1)[2 - \ln(N+1)]
\end{aligned}$$
</td></tr>
</table>

Qn	Suggested Solution
9i)	$\frac{du}{dt} = k(u-a)$, where k is a constant. Given $\frac{du}{dt} = 1$ when $u = 14.5$ and $\frac{du}{dt} = 2$ when $u = 14$, $1 = k(14.5-a)$ -----(1) $2 = k(14-a)$ ------(2) From GC, $k = -2$ and $ak = -30$ $\therefore a = 15$ $\therefore \frac{du}{dt} = -2(u-15)$ (shown)
v(ii)	$\int \frac{1}{u-15}\, du = -2\int dt$ $\ln\lvert u-15\rvert = -2t + C$ $\lvert u-15\rvert = e^{-2t+C}$ $u = 15 + Ae^{-2t}$ where $A = \pm e^{C}$
(iii)	As $t \to \infty$, $e^{-2t} \to 0$, $u \to 15$ Joseph will eventually reach a steady speed of 15 km/h. Alternatively, at steady speed, $\frac{du}{dt} \to 0$, $u \to 15$
(iv)	$s = \int u\, dt$ $= \int (15 + Ae^{-2t})\, dt$ $= 15t - \frac{A}{2}e^{-2t} + D$
9(v)	For graph to tends towards an asymptote that passes through the origin, D = 0. I.e. $s = 15t - \frac{A}{2}e^{-2t}$ For $A = -1$, $s = 15t + \frac{1}{2}e^{-2t}$ 

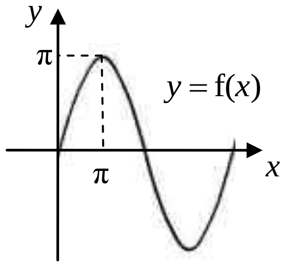
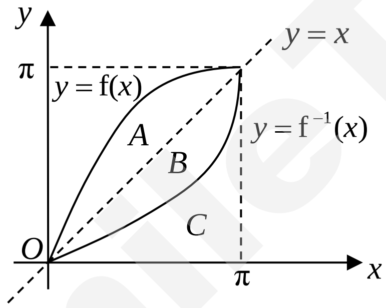
<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>10(i)</td><td>$(x+1)^2-12=0 \Rightarrow x=-1\pm\sqrt{12}$
Asymptotes are: $y=1,\ x=-1-\sqrt{12},\ x=-1+\sqrt{12}$</td></tr>
<tr><td>(ii)</td><td>$y=\dfrac{x^2-5}{(x+1)^2-12}=\dfrac{x^2-5}{x^2+2x-11}$
$y(x^2+2x-11)=x^2-5$
$(y-1)x^2+2yx+5-11y=0$
For no values of x, there are no real solutions for the above quadratic equation.
Discriminant $=4y^2+4(y-1)(11y-5)<0$
$12y^2-16y+5<0$
$(6y-5)(2y-1)<0$
$\therefore \frac{1}{2}<y<\frac{5}{6}$ (shown)</td></tr>
<tr><td>(iii)</td><td>At turning points of C,
When $y=\frac{1}{2}$, $x=1$;
When $y=\frac{5}{6}$, $x=5$.
Coordinates of turning points $(1,\frac{1}{2})$ and $(5,\frac{5}{6})$.

</td></tr>
<tr><td>(iv)</td><td>
</td></tr>
</table>

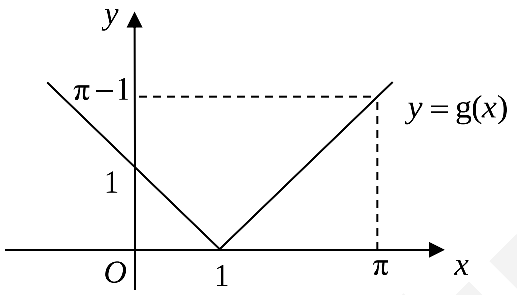
(v)	$y=\frac{(1-x)^2-5}{((1-x)+1)^2-12}=\frac{(x-1)^2-5}{(x-2)^2-12}$. Therefore x is replaced with $1-x$. $y=\mathrm{f}(x)\rightarrow y=\mathrm{f}(x+1)\rightarrow y=\mathrm{f}((-x)+1)=\mathrm{f}(1-x)$ Sequence of transformation: C is translated by 1 unit in the negative x-direction and then reflected in the y-axis OR $y=\mathrm{f}(x)\rightarrow y=\mathrm{f}(-x)\rightarrow y=\mathrm{f}(-(x-1))=\mathrm{f}(1-x)$ C is reflected in the y-axis and then translated 1 unit in the positive x-direction.
Qn	**Suggested Solution**
11(i)	Plane ABD: $4x+y+2z=16$ When A is on the x-axis, $y = z = 0$. $4x=16\Rightarrow x=4$ $A(4, 0, 0)$
(ii)	Plane CBD: $7x-11y-5z=-23$ Line BD is the line of intersection between planes ABD and CBD. From GC, $\mathbf{r}=\begin{pmatrix}3\\4\\0\end{pmatrix}+\lambda'\begin{pmatrix}-\frac{1}{3}\\-\frac{2}{3}\\1\end{pmatrix}$ $l_{BD}:\mathbf{r}=\begin{pmatrix}3\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-3\end{pmatrix}, \lambda\in\mathbb{R}$
(iii)	Equation of xy-plane : $\mathbf{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0\Rightarrow z=0$ Using $\overrightarrow{OB}=\begin{pmatrix}3\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-3\end{pmatrix}\Rightarrow 3\lambda=0\Rightarrow\lambda=0$ $\therefore\overrightarrow{OB}=\begin{pmatrix}3\\4\\0\end{pmatrix}$ $B(3, 4, 0)$ **<u>Alternative</u>** B is the point of intersection between planes ABD, CBD and xy-plane $4x+y+2z=16$ $7x-11y-5z=-23$ $z=0$ Using GC, $B(3, 4, 0)$
(iv)	$l_{AD}:\mathbf{r}=\begin{pmatrix}4\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\2\\3\end{pmatrix}$ D is the point of intersection between l_{AD} and plane CBD :

	$\left[\begin{pmatrix}4\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\2\\3\end{pmatrix}\right]\cdot\begin{pmatrix}7\\-11\\-5\end{pmatrix}=-23$ $28+\mu(-51)=-23$ $\mu=1$ $\overrightarrow{OD}=\begin{pmatrix}4\\0\\0\end{pmatrix}+\begin{pmatrix}-2\\2\\3\end{pmatrix}=\begin{pmatrix}2\\2\\3\end{pmatrix}$ $D(2, 2, 3)$ **<u>Alternative</u>** D is the point of intersection between l_{AD} and l_{BD}. $\begin{pmatrix}4\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\2\\3\end{pmatrix}=\begin{pmatrix}3\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-3\end{pmatrix}$ $\lambda+2\mu=1$ $2\lambda-2\mu=-4$ $-3\lambda-3\mu=0\Rightarrow\lambda=-\mu$ $\therefore\mu=1$
(v)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-1\\4\\0\end{pmatrix}$ $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}-5\\1\\1\end{pmatrix}$ Area of triangle ABC = $\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}\left\|\begin{pmatrix}-1\\4\\0\end{pmatrix}\times\begin{pmatrix}-5\\1\\1\end{pmatrix}\right\|=\frac{1}{2}\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|$ Distance from D to plane ABC $=\dfrac{\left\|\overrightarrow{AD}\cdot\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}{\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}=\dfrac{\left\|\begin{pmatrix}-2\\2\\3\end{pmatrix}\cdot\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}{\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}=\dfrac{51}{\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}$ **<u>Alternative</u>** $\therefore$ Plane $ABC:\mathbf{r}\cdot\begin{pmatrix}4\\1\\19\end{pmatrix}=16$

	Distance from D to plane $ABC = \dfrac{\left\lvert 16 - \begin{pmatrix}2\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4\\1\\19\end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix}4\\1\\19\end{pmatrix} \right\rvert} = \dfrac{\lvert 16-67 \rvert}{\left\lvert \begin{pmatrix}4\\1\\19\end{pmatrix} \right\rvert} = \dfrac{51}{\left\lvert \begin{pmatrix}4\\1\\19\end{pmatrix} \right\rvert}$ Volume of tetrahedron $OABC$ = $\dfrac{1}{3} \times \dfrac{1}{2} \left\lvert \begin{pmatrix}4\\1\\19\end{pmatrix} \right\rvert \dfrac{51}{\left\lvert \begin{pmatrix}4\\1\\19\end{pmatrix} \right\rvert} = \dfrac{51}{6} = \dfrac{17}{2}$ units^3

2016 Year 6 H2 Math Prelim Exam Paper 2 Solutions

Qn	Suggested Solution
1(i)	Largest $a = \pi$ 
(ii)	The line is $y = x$. $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ Since the points of intersection lies on $y = x$, $\mathrm{f}(x) = x$ $\pi\sin\left(\frac{1}{2}x\right) = x$ When $x = 0$: $\text{LHS} = \pi\sin(0) = 0 = \text{RHS}$ When $x = \pi$: $\text{LHS} = \pi\sin\left(\frac{1}{2}\pi\right) = \pi = \text{RHS}$ $\therefore$ 0 and π are solutions to the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$
(iii)	 Required area = $A + B = 2A$ (by symmetry) Area of $B + C = \frac{1}{2}\pi^2$ (area of triangle) Area of $A + B + C = \int_0^{\pi} \mathrm{f}(x)\,\mathrm{d}x$ $= \int_0^{\pi} \pi\sin\left(\frac{1}{2}x\right)\mathrm{d}x$ $= \left[-2\pi\cos\left(\frac{1}{2}x\right)\right]_0^{\pi}$ $= 2\pi$ $\therefore$ Area bounded by the graphs of f and f^{-1} $= 2\left(2\pi - \frac{1}{2}\pi^2\right) = 4\pi - \pi^2$

	Alternative Required area $= 2\int_0^{\pi} \mathrm{f}(x) - x \, \mathrm{d}x$ $= 2\int_0^{\pi} \pi\sin\left(\frac{1}{2}x\right) - x \, \mathrm{d}x$ $= 2\left[-2\pi\cos\left(\frac{1}{2}x\right) - \frac{x^2}{2}\right]_0^{\pi}$ $= 2\left(2\pi - \frac{\pi^2}{2}\right) = 4\pi - \pi^2$
(iv)	 $[0,\pi] \xrightarrow{\mathrm{f}} [0,\pi] \xrightarrow{\mathrm{g}} [0,\pi-1]$ $\therefore \mathrm{R}_{\mathrm{gf}} = [0,\pi-1]$

Qn	**Suggested Solution**
2(i)	$\overrightarrow{CP} = \mathbf{a} + \lambda(\mathbf{a}+2\mathbf{b}) - (\mathbf{a}-\mathbf{b}) = \mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})$ $CP^2 = [\mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})].[\mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})]$ $= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}+2\mathbf{b}).(\mathbf{a}+2\mathbf{b}) + 2\lambda\mathbf{b}.(\mathbf{a}+2\mathbf{b})$ $= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}.\mathbf{a} + 4\mathbf{a}.\mathbf{b} + 4\mathbf{b}.\mathbf{b}) + 2\lambda(\mathbf{b}.\mathbf{a} + 2\mathbf{b}.\mathbf{b})$ $= 1 + \lambda^2(1+1+4) + 2\lambda\left(\frac{1}{4}+2\right)$ as $\mathbf{a}.\mathbf{a} = \mathbf{b}.\mathbf{b} = 1$ and $\mathbf{a}.\mathbf{b} = \frac{1}{4}$ $= 6\lambda^2 + \frac{9}{2}\lambda + 1$ (shown)
(ii)	$CP^2 = 6\left[\lambda^2 + \frac{3}{4}\lambda\right] + 1$ $= 6\left(\lambda + \frac{3}{8}\right)^2 + 1 - \frac{54}{64} = 6\left(\lambda + \frac{3}{8}\right)^2 + \frac{5}{32}$ $CP = \sqrt{6\left(\lambda + \frac{3}{8}\right)^2 + \frac{5}{32}}$ The perpendicular distance from C to l occurs when P is nearest to l, that is when CP is least or $\lambda = -\frac{3}{8}$. Least CP is $\frac{\sqrt{10}}{8}$. P is F in this case. $\overrightarrow{OF} = \mathbf{a} - \frac{3}{8}(\mathbf{a}+2\mathbf{b}) = \frac{1}{8}(5\mathbf{a} - 6\mathbf{b})$

	Alternative to find minimum CP: CP is minimum when CP^2 is minimum: $\frac{\mathrm{d}}{\mathrm{d}x}(CP^2)=12\lambda+\frac{9}{2}$ When $\frac{\mathrm{d}}{\mathrm{d}x}(CP^2)=0, 12\lambda+\frac{9}{2}=0$ $\therefore \lambda=-\frac{3}{8}$. Since CP^2 is quadratic and coefficient of $\lambda^2>0$, CP^2 is minimum at $\lambda=-\frac{3}{8}$ $\therefore$ perpendicular distance from C to l occur when $\lambda=-\frac{3}{8}$.
(iii)	$\overrightarrow{CF}=\frac{1}{8}(5\mathbf{a}-6\mathbf{b})-(\mathbf{a}-\mathbf{b})=\frac{1}{8}(-3\mathbf{a}+2\mathbf{b})$ Equation of plane $\mathbf{r}.(-3\mathbf{a}+2\mathbf{b})=\mathbf{a}.(-3\mathbf{a}+2\mathbf{b})=-3+\frac{2}{4}=-\frac{5}{2}$

Qn	Suggested Solution
3(i)	$u_{n+1}=0.3u_n+6$
(ii)	$u_n=0.3u_{n-1}+6$ $=0.3(0.3u_{n-2}+6)+6$ $=0.3^2u_{n-2}+0.3(6)+6$ $=0.3^2(0.3u_{n-3}+6)+0.3(6)+6$ $=0.3^3u_{n-3}+0.3^2(6)+0.3(6)+6$ $\vdots$ $=0.3^{n-1}u_1+0.3^{n-2}(6)+0.3^{n-3}(6)+\ldots+0.3(6)+6$ $=0.3^{n-1}u_1+\frac{6(1-0.3^{n-1})}{1-0.3}$ $=0.3^{n-1}u_1+\frac{60}{7}(1-0.3^{n-1})$ **Alternative** $u_2=0.3u_1+6$ $u_3=0.3u_2+6$ $=0.3(0.3u_1+6)+6$ $=0.3^2u_1+0.3(6)+6$ $\vdots$ $u_n=0.3^{n-1}u_1+0.3^{n-2}(6)+0.3^{n-3}(6)+\ldots+0.3(6)+6$ $=0.3^{n-1}u_1+\frac{6(1-0.3^{n-1})}{1-0.3}$ $=0.3^{n-1}u_1+\frac{60}{7}(1-0.3^{n-1})$

(iii)

As $n\to\infty$, $v_{n+1}\to L$ and $v_n\to L$.

$\therefore L=0.01L^2+6$

$0.01L^2-L+6=0$

From G.C., $L=6.4110$ or 93.588

$\therefore \alpha=6.41,\ \beta=93.6$ (3 s.f.)

v_n may not necessarily converge to a limit as we do not know what is the value of its starting term v_1 (or initial number of bacteria).

(iv)

As $n\to\infty$,

$u_n\to 0u_1+\frac{60}{7}(1-0)=8.57$ (3 s.f.)

$v_1=30,\ v_2=0.01{v_1}^2+6=15,\ v_3=0.01{v_2}^2+6=8.25,\ v_4=0.01{v_3}^2+6=6.68,$

$v_5=0.01{v_4}^2+6=6.446,\ldots$

From GC, $v_n\to 6.41$ (3 s.f.)

Thus, Chemical B is more effective in the long run.

(v)

The amount of Chemical C used per week is an Arithmetic Progression with first term $a=20$, common difference $d=5$.

Want: $S_n\ge 3000$

$$\frac{n}{2}[2(20)+(n-1)(5)]\ge 3000$$

$n^2+7n-1200\ge 0$

From G.C.,

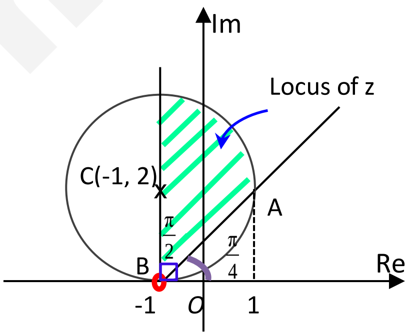
n	$n^2+7n-1200$
31	-22
32	48
33	120

$\therefore$ it takes 32 weeks

Alternative: From GC, $n\ge 31.3$(to 3 s.f.) $\Rightarrow$ It takes 32 wks

Qn	Suggested Solution

4(a)



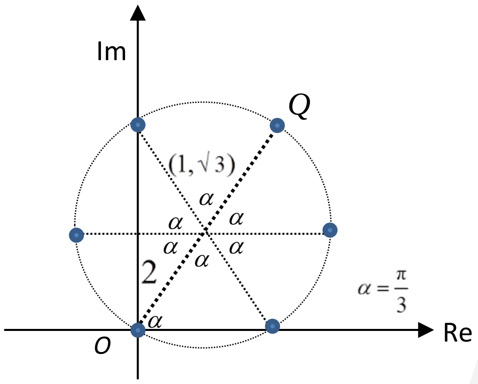
$|z+1-2\mathrm{i}|=|z-(-1+2\mathrm{i})|$

ie, $|z-(-1+2\mathrm{i})|\le 2$

Min arg (z) occurs at $A=\tan^{-1}\dfrac{2}{1}=\tan^{-1}2$

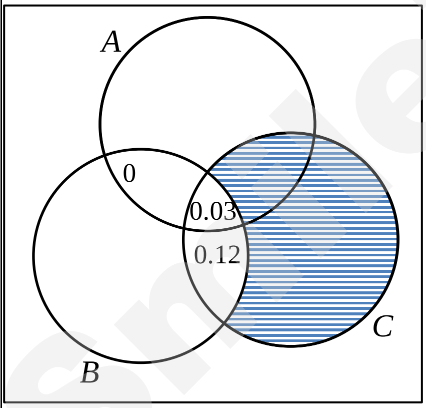
Max arg (z) occurs at B

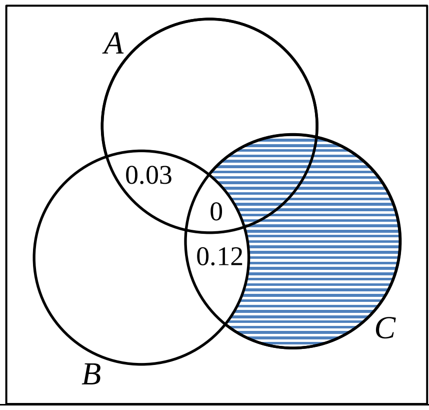
Hence, $\tan^{-1}2\le \arg(z)<\pi$

(b)	$w^6 = 2^6$ $\Rightarrow w^6 = 2^6 \mathrm{e}^{2k\pi \mathrm{i}}$ $\therefore w = 2\mathrm{e}^{\mathrm{i}\frac{k\pi}{3}}, \quad k = 0, \pm 1, \pm 2, 3$
(i)	$z = 1 + \mathrm{i}\sqrt{3} + 2\mathrm{e}^{\mathrm{i}\frac{k\pi}{3}}, \quad k = 0, \pm 1, \pm 2, 3$ 
(ii)	$\lvert z \rvert$ is maximum at Q, where Q is the point representing the root when k = 1 and OQ is the diameter of the circle $z = 1 + \mathrm{i}\sqrt{3} + 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ $= 1 + \mathrm{i}\sqrt{3} + 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$ $= 1 + \mathrm{i}\sqrt{3} + 1 + \mathrm{i}\sqrt{3}$ $= 2 + \mathrm{i}\,2\sqrt{3}$

Qn	**Suggested Solution**
5(i)	No, the people are surveyed without consideration of the stratum e.g. age group they belong to.
(ii)	To obtain a systematic sample of 5%, we can first randomly select the first commuter from the first 20 commuters entering the train station by stationing the surveyors at the gantries and thereafter select every 20th commuter thereafter entering the train station from the start to the end of that particular day.

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>6(i)</td><td>
Let A denote the event that John scores in a game.
Let B denote the event that John’s parents are present at a game.
Given that $\mathrm{P}(A)=0.15, \mathrm{P}(B)=0.3, \mathrm{P}(B\mid A)=0.2$,
$\mathrm{P}(A\mid B)$
$=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$
$=\dfrac{\mathrm{P}(B\mid A)\times \mathrm{P}(A)}{\mathrm{P}(B)}$
$=\dfrac{0.2\times 0.15}{0.3}=\dfrac{0.03}{0.3}$
$=0.1$ (shown)
</td></tr>
<tr><td>(ii)</td><td>
Since $\mathrm{P}(A\mid B)=0.1\neq \mathrm{P}(A)=0.15$ his parent’s presence in a game will affect his chances of scoring in the game.

Alternative
Since $\mathrm{P}(B\mid A)=0.2\neq \mathrm{P}(B)=0.3$ his parent’s presence in a game will affect his chances of scoring in the game.
</td></tr>
<tr><td>(iii)</td><td>
Let C denote the event that the game is a home game.
$\mathrm{P}(A'\cap B\cap C)=\mathrm{P}((A'\cap B)\mid C)\times \mathrm{P}(C)=0.24\times 0.5=0.12$


Least $\mathrm{P}(B'\cap C)$
$=0.5-0.03-0.12$
$=0.35$


Max $\mathrm{P}(B'\cap C)$
$=0.5-0.12$
$=0.38$
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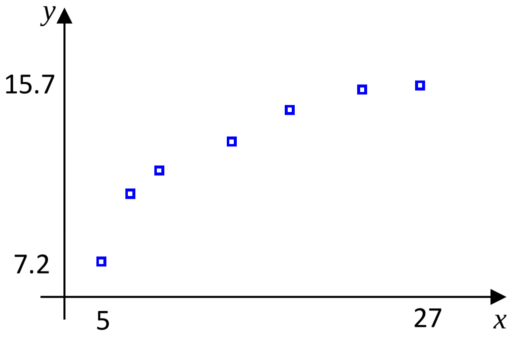
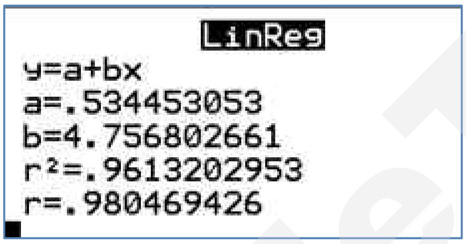
Qn	Suggested Solution
7(i)	Required probability $=\dfrac{\left({}^{4}\mathrm{C}_2\right)^2}{{}^{28}\mathrm{C}_4}$ $=\dfrac{4}{2275}$
(ii)	Required probability $=\left(\dfrac{1}{9}\right)^2\times\left(\dfrac{2}{9}\right)^2\times\dfrac{4!}{2!2!}$ $=\dfrac{8}{2187}$
(iii)	No. of ways to seat the remaining members $=(7-1)!$ $=720$ No. of ways to slot in the committee head and the 2 vice-heads as a pair $={}^{7}\mathrm{P}_2=42$ No. of ways to arrange the 2 vice-heads = 2 Required probability $=\dfrac{720\times42\times2}{(10-1)!}=\dfrac{1}{6}$
	Alternative No. of ways in which the vice-heads are seated together $=(9-1)!\times2!$ $=80640$ No of ways in which the vice-heads are seated together and the head is seated next to one of them $=(8-1)!\times2!\times2$ $=20160$ Required probability $=\dfrac{80640-20160}{(10-1)!}=\dfrac{1}{6}$
	Alternative No. of ways in which the vice-heads are seated together with the rest of the committee excluding the committee head $=(8-1)!\times2!$ No. of ways to slot in the committee head $={}^{6}C_1$ Required probability $=\dfrac{(8-1)!(2)(6)}{(10-1)!}=\dfrac{1}{6}$

Qn	Suggested Solution
8(i)	
(ii)	Model **B**: $y = a + b\ln x$ is appropriate but not Model *A*. From the scatter diagram, as *x* increases, *y* increases at a **decreasing rate,** which is consistent with Model B but not Model A which predicts an increasing rate of increase for *y*.
(iii)	Screenshot for reference:  $r = 0.98047$ (5 s.f.) ≈ 0.980 (3 s.f.) The value of *r* would not be different as it is unaffected when data is scaled
(iv)	$a = 0.534$, $b = 4.76$ $y = 0.53445 + 4.7568\ln x$ (5 s.f.) (or $y = 0.534 + 4.76\ln x$) When $x = 20$, $y = 0.53445 + 4.7568\ln(20)$ $= 14.785$ $= 14.8$ (3 s.f.) Car population is 14.8 millions
(v)	Value of *a* represents the predicted car population after 1 year of study. [If wrong model chosen: Value of *a* represents the predicted car population at the start of the study.] The value of *a* is unreliable (invalid) as it is an extrapolation (outside the data range Year 5 to Year 27) and the linear relationship may not hold.

Qn	Suggested Solution
9(a) (i)	Let A and B denote the queuing times of a randomly chosen passenger at Economy and Business class counters respectively. $A \sim \mathrm{N}(11.6, 4.2^2)$ $\quad$ $B \sim \mathrm{N}(3.2, 0.9^2)$ Find $\mathrm{P}\left(\lvert A-(B_1+B_2)\rvert < 5\right)$ $A-(B_1+B_2) \sim \mathrm{N}(11.6-2\times 3.2,\ 4.2^2+2\times 0.9^2)$ i.e. $A-(B_1+B_2) \sim \mathrm{N}(5.2,\ 19.26)$ $\therefore \mathrm{P}\left(\lvert A-(B_1+B_2)\rvert < 5\right) = \mathrm{P}\left(-5 < A-(B_1+B_2) < 5\right)$ ≈ 0.47177 $= 0.472$ (3 s.f.)
(ii)	Required probability $= \mathrm{P}(B_1 \geq 2)\times \mathrm{P}(B_2 \geq 2)\times \ldots \times \mathrm{P}(B_8 \geq 2)$ $= \left[\mathrm{P}(B \geq 2)\right]^8$ ≈ 0.46526 $= 0.465$ (3 s.f.)
(b)	Let X denote the number of passengers who <u>turned up</u> for their flight, out of n passengers who bought the n tickets. $X \sim \mathrm{B}(n, 0.95)$ Since $n > 350$ is sufficiently large such that $np = 0.95n > 5$ and $nq = 0.05n > 5$, $X \sim \mathrm{N}\left(0.95n, 0.0475n\right)$ approximately $\mathrm{P}\left(\text{Flight is overbooked}\right) \leq 0.01$ $\Rightarrow \mathrm{P}\left(X > 350\right) \leq 0.01$ $\Rightarrow \mathrm{P}\left(X > 350.5\right) \leq 0.01$ (continuity correction) $\Rightarrow \mathrm{P}\left(X < 350.5\right) \geq 0.99$ $\Rightarrow \mathrm{P}\left(Z < \dfrac{350.5-0.95n}{\sqrt{0.0475n}}\right) \geq 0.99$ $\Rightarrow \dfrac{350.5-0.95n}{\sqrt{0.0475n}} \geq 2.3263$ $\Rightarrow 350.5-0.95n \geq 2.3263\sqrt{0.0475n}$ (shown)

<u>Alternative to show approx inequality:</u>

Let X denote the number of passengers who <u>did not turn up</u> for their flight, out of n passengers who bought the n tickets.

$X \sim \mathrm{B}(n, 0.05)$

Since $n > 350$ is sufficiently large such that $np = 0.05n > 5$ and $nq = 0.95n > 5$,

$X \sim \mathrm{N}\left(0.05n, 0.0475n\right)$ approximately

$$\mathrm{P}\left(\text{Flight is overbooked}\right) \le 0.01$$

$$\Rightarrow \mathrm{P}\left(X < n - 350\right) \le 0.01$$

$$\Rightarrow \mathrm{P}\left(X \le n - 351\right) \le 0.01$$

$$\Rightarrow \mathrm{P}\left(X < n - 350.5\right) \le 0.01 \quad \text{(continuity correction)}$$

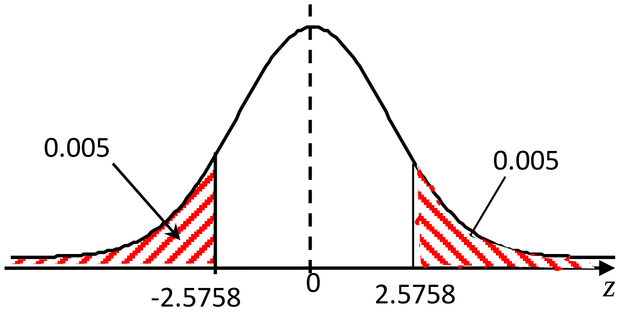
$$\Rightarrow \mathrm{P}\left(Z < \frac{n - 350.5 - 0.05n}{\sqrt{0.0475n}}\right) \le 0.01$$

$$\Rightarrow \frac{0.95n - 350.5}{\sqrt{0.0475n}} \le -2.3263$$

$$\Rightarrow 0.95n - 350.5 \le -2.3263\sqrt{0.0475n}$$

$$\Rightarrow 350.5 - 0.95n \ge 2.3263\sqrt{0.0475n} \quad \text{(shown)}$$

Qn	Suggested Solution
10(i)	Unbiased estimate of population mean, $\bar{x} = 167.89 = 168$ (3 s.f.) Unbiased estimate of population variance, $s^2 = 2.4988^2 = 6.24$ (3 s.f.)
(ii)	Assume that the breaking strength of each coil of rope is normally distributed. $\mathrm{H}_0 : \mu = 169.7$ $\mathrm{H}_1 : \mu < 169.7$ Perform 1-tail test at 5% significance level. Under H_0, $\dfrac{\bar{X} - 169.7}{S/\sqrt{8}} \sim t(8-1)$. Using t-test, $p\text{-value} = 0.039673$. Since $p\text{-value} \le 0.05$, we reject H_0 and conclude that there is sufficient evidence at the 5% significance level that the mean breaking strength is less than 169.7 kg, i.e. the manufacturer's claim is not valid.

(iii)	$H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$ Under H_0, $\bar{Y} \sim N(\mu_0, \frac{29.16}{50})$ approximately by Central Limit Theorem.  Given $\bar{y} = 171$, and H_0 is not rejected, $-2.5758 < \frac{171 - \mu_0}{\sqrt{\frac{29.16}{50}}} < 2.5758$ $\Rightarrow 169.03 < \mu_0 < 172.97$ $\therefore$ set of values of μ_0 is: $\{\mu_0 \in \mathbb{R} : 169 < \mu_0 < 173\}$
(iv)	Testing at the 1% significance level means that there is a <u>probability of 0.01 of concluding</u> that the <u>mean breaking strength differs from the claim</u> when it is actually <u>unchanged</u>.
Qn	**Suggested Solution**
11(a) **(i)**	Let X and Y be the number of rectangular tables and round tables that are occupied. $X \sim B(6, 0.8)$ $\quad Y \sim B(9, 0.65)$ Required probability = $P(X = 4)\, P(Y = 7)$ $= 0.24576 \times 0.21619$ $= 0.0531$ (3 s.f.)
(ii)	• Customers may arrive as a big group that requires them to be split into two separate tables next to each other. OR • The restaurant may choose to seat the customers at tables in a particular section first.
(b) **(i)**	Let C and T be the number of cups of coffee and tea sold in 20 minutes, respectively. $C \sim P_O(5)$ $\quad T \sim P_O(3.5)$ $C + T \sim P_O(8.5)$ $P(T \geq 6 \mid C + T \geq 7) = \frac{P(\{T \geq 6\} \cap \{C + T \geq 7\})}{P(C + T \geq 7)}$

$$=\frac{\mathrm{P}(T=6)\ \mathrm{P}(C\ge 1)+\mathrm{P}(T\ge 7)\mathrm{P}(C\ge 0)}{\mathrm{P}(C+T\ge 7)}$$

$$=\frac{\mathrm{P}(T=6)\,[1-\mathrm{P}(C=0)]+[1-\mathrm{P}(T\le 6)]}{1-\mathrm{P}(C+T\le 6)}$$

$$=\frac{0.077098(0.99326)+0.065288}{0.74382}$$

$$=\frac{0.14187}{0.74382}=0.191 \quad \text{(3 s.f.)}$$

(ii)

Using $p_k=\mathrm{e}^{-\lambda}\dfrac{\lambda^k}{k!}$ for $\lambda=5$,

$$\frac{p_{k+1}}{p_k}=\frac{\left(\mathrm{e}^{-5}\dfrac{5^{k+1}}{(k+1)!}\right)}{\left(\mathrm{e}^{-5}\dfrac{5^k}{k!}\right)}=\frac{5^{k+1}k!}{5^k(k+1)!}=\frac{5}{k+1} \quad \text{(shown)}$$

When $k<4$, $k+1<5$,

$$\Rightarrow \frac{5}{k+1}>1 \Rightarrow \frac{p_{k+1}}{p_k}>1 \Rightarrow p_{k+1}>p_k\,.$$

When $k<4$, i.e. $k=0,1,2,3$

$$p_{k+1}>p_k \Rightarrow p_4>p_3>p_2>p_1>p_0.$$

When $k>4$, i.e. $k=5,6,7,\ldots$

$$p_{k+1}<p_k \Rightarrow p_5>p_6>p_7\cdots.$$

When $k=4$, $p_{k+1}=p_k \Rightarrow p_4=p_5$.

From above, $p_0<p_1<\cdots<p_4=p_5>p_6>p_7>p_8>\cdots$

(Thus p_k is greatest when k = 4 and 5)

The most probable number of cups of coffee sold (i.e. the mode) are 4 and 5.

HWA CHONG INSTITUTION
2016 JC2 PRELIMINARY EXAMINATION

MATHEMATICS 9740/01
Higher 2

Paper 1

Wednesday **14 September 2016** **3 hours**

Additional materials: Answer paper
List of Formula (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT class on all the work you hand in, including the Cover Page which is found on Page 2.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.
Do not write anything on the List of Formula (MF15).

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question. At the end of the examination, place the completed cover page on top of your answer scripts and fasten all your work securely together with the string provided.

This question paper consists of 6 printed pages.

1. A sequence follows the recurrence relation

$$U_{n+1} - U_n = 2\cos\frac{(2n+1)x}{2}\sin\frac{x}{2}\ ,\ U_1 = \sin x \text{ for } n = 1, 2, 3, \ldots .$$

Prove by mathematical induction that $U_n = \sin(nx)$ for all positive integer n. [4]

2. Solve the inequality $\dfrac{2}{4(x+1)^2+1} > 1.$ [2]

Hence find $\displaystyle\int_{-1}^{\frac{\sqrt{3}-2}{2}} \left|1 - \frac{2}{4(x+1)^2+1}\right| \mathrm{d}x,$ leaving your answer in exact form. [3]

3. Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ with $\mathbf{a}$ not parallel to $\mathbf{b}$. The point P is on AB produced with $AP : AB = 3 : 1$ and the position vector of point Q is $2\mathbf{a}$.

(a) Find the position vector of the point of intersection of lines OB and PQ, giving your answer in terms of $\mathbf{b}$. [4]

(b) It is given that $\mathbf{a} \times \mathbf{b} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and the point $C(0,\ 3,\ 4)$ does not lie on the plane OAB. Find the foot of the perpendicular from C to the plane OAB. [4]

4. Prove that $\dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} = \sqrt{n^2+2n} - \sqrt{n^2-1}.$ [2]

Hence find $\displaystyle\sum_{n=1}^{N} \frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}.$ [3]

(a) Deduce the value of $\displaystyle\sum_{n=2}^{N} \frac{2n-1}{\sqrt{n^2-2n}+\sqrt{n^2-1}}.$ [3]

(b) Show that $\displaystyle\sum_{n=1}^{N} \frac{2n+1}{2n-1} > \sqrt{N^2+2N}$. [1]

[Turn over

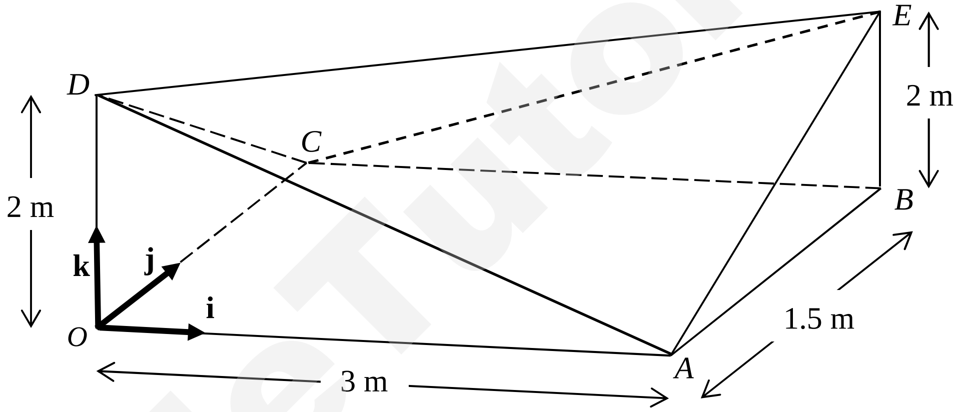
5. Sketch on a single Argand diagram, the loci defined by $-\frac{\pi}{4} < \arg(z+1+2\mathrm{i}) \le \frac{\pi}{4}$ and $|(2+\mathrm{i})w+5| \le \sqrt{5}$. [4]

(i) Find the minimum value of $\arg(w)$. [2]

(ii) Find the minimum value of $|z-w|$. [2]

(iii) Given that $\arg(z-w) < \theta,\ -\pi < \theta \le \pi$, state the minimum value of θ. [1]

6. A group of boys want to set up a camping tent. They lay down a rectangular tarp *OABC* on the horizontal ground with *OA* = 3 m and *AB* = 1.5 m and secure the points *D* and *E* vertically above *O* and *B* respectively, such that $OD = BE = 2$ m.



Assume that the tent takes the shape as shown above with 6 triangular surfaces and a rectangular base. The point *O* is taken as the origin and the unit vectors **i**, **j** and **k** are taken to be in the direction of $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively.

(i) Show that the line *DE* can be expressed as $\mathbf{r} = 2\mathbf{k} + \lambda(2\mathbf{i}+\mathbf{j}),\ \lambda \in \mathbb{R}$. [2]

(ii) Find the Cartesian equation of the plane *ADE*. [3]

(iii) Determine the acute angle between the planes *ADE* and *OABC*. Hence, or otherwise, find the acute angle between the planes *ADE* and *CDE*. [4]

7. The curve *C* has equation $y = \frac{x-2}{kx^2+x-2}$, where $k > 1$.

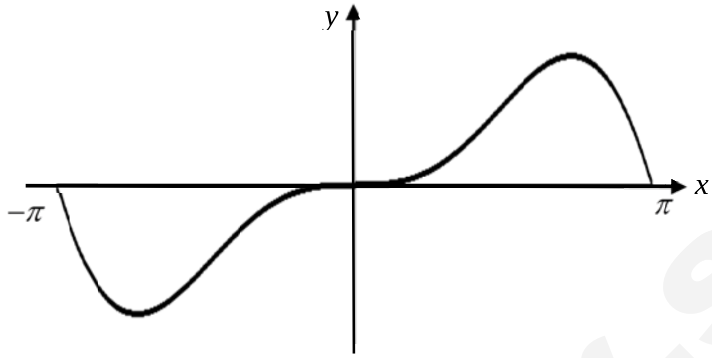
(i) Find the equation of the tangent at the point *A* where *C* cuts the *y*-axis. [2]

(ii) Sketch *C*, giving the equations of asymptotes, the coordinates of turning points and axial intercepts in terms of k, if any. [4]

(iii) Find the equation of the normal at the point *B* where *C* cuts the *x*-axis. Leave your answer in terms of k. [2]

(iv) Hence show that the value of the area bounded by the tangent at *A*, the normal at *B* and both the *x*- and *y*-axes is more than $\frac{15}{8}$ square units. [2]

8. The curve C (as shown in the diagram below) has equation $y = x^2 \sin x,\ -\pi \le x \le \pi$.



(i) Calculate the exact area of the region R enclosed by C and the x-axis. [4]

(ii) Sketch the curve with equation $(y+1)^2 - 4(x+2)^2 = 1$, showing clearly the coordinates of the turning points and the equation(s) of any asymptote(s). [2]

(iii) Hence find the volume of the solid generated when the region bounded by the 2 curves is rotated through 4 right angles about the x-axis. [4]

9. A manufacturer produces cylindrical containers using sheet metal of negligible thickness. The cylindrical container has an open top, and a base and curved sides made up of the sheet metal.

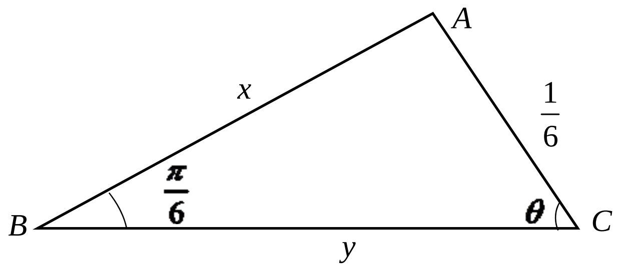
(a) **(i)** It is given that the volume of the cylindrical container is fixed at k cm^3. Show that when the amount of sheet metal used for the cylindrical container is a minimum, the ratio of its height to its radius is $1:1$. [5]

(ii) A product designer proposed a new design where the height of the cylindrical container is always 2.5 times that of its radius. Given that the radius of a cylindrical container produced using the new design equals the radius of the container produced in part **(a)(i)** with minimum sheet metal. Find the ratio of the amount of sheet metal used in this new design to the minimum amount of sheet metal used in part **(a)(i)**. [2]

(b) To reduce cost, plastic with negligible thickness, instead of sheet metal is used to manufacture the new design cylindrical containers in part **(a)(ii)** using *injection blow moulding* technology. In the injection blow moulding process, it is assumed that the cylindrical containers increase in size proportionately with the height to radius ratio remaining constant at 5:2 throughout the process. If the volume of the cylindrical container increases at a rate of 80 cm^3 per second, find the rate of change of the surface area of the cylindrical container when its height is 50 cm. [4]

[Turn over

10.



In the triangle ABC, $AB = x$, $BC = y$, $AC = \frac{1}{6}$, angle $ABC = \frac{\pi}{6}$ radians and angle $ACB = \theta$ radians (see diagram).

(a) **(i)** Show that $\frac{x}{y} = \frac{2\sin\theta}{\cos\theta + \sqrt{3}\sin\theta}$. [3]

(ii) Given that θ is sufficiently small, express $\frac{x}{y}$ as a cubic polynomial in θ. [3]

(b) **(i)** Show that $\theta = \sin^{-1}(3x)$. [1]

(ii) Find the Maclaurin series for θ, up to and including the term in x^3. [5]

11. The functions f and g are defined by

$$\mathrm{f} : x \mapsto \frac{1}{2}\mathrm{e}^{1-x^2},\ x \in \mathbb{R},\ x \le 1 \text{ and}$$

$$\mathrm{g} : x \mapsto \sqrt{1 - \ln x},\ x \in \mathbb{R},\ 0 < x \le \mathrm{e}.$$

(i) Show that gf exists, and find the range of gf. [4]

(ii) Justify, with a reason, whether f^{-1} exists. [2]

(iii) The domain of f is restricted to $(-\infty, b]$ such that b is the largest value for which the inverse function f^{-1} exists. State the value of b and define f^{-1} clearly. [4]

(iv) The graph of $y = \mathrm{h}(x)$ is obtained by transforming the graph of $y = \mathrm{g}(x)$ in the following 2 steps.

Step 1: Scale parallel to the x-axis by a factor of 2.

Step 2: Reflect in the x-axis.

Define h in a similar form. [3]

End of Paper

HWA CHONG INSTITUTION
2016 JC2 PRELIMINARY EXAMINATION

MATHEMATICS
Higher 2

9740/02

Paper 2

Tuesday **20 September 2016** **3 hours**

Additional materials: Answer paper
List of Formula (MF15)

READ THESE INSTRUCTIONS FIRST

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Section A: Pure Mathematics [40 marks]

1. On the first day of last month, the temperature of a machine in a manufacturing plant was found to be $90\ ^\circ\mathrm{C}$ and was deemed as too hot by the supervisor. On each day waste heat was produced by the machine as a by-product and caused the temperature of the machine to increase by $2\ ^\circ\mathrm{C}$. In an attempt to make it cooler, the supervisor decided to adjust the thermostat and decreased the temperature by 3% at the end of each day.

(i) Show that the temperature of the machine was 87.8 $^\circ\mathrm{C}$ at the end of the 3rd day. [2]

(ii) At the end of which day would the temperature of the machine first dropped below $70^\circ\mathrm{C}$? [3]

(iii) Will the temperature continue to drop indefinitely? Justify your answer. If not, what is the long term temperature of the machine? [3]

2. (a) Given that $z_1 = -\dfrac{\mathrm{i}}{2}$ is a root of the equation $2z^3+(\mathrm{i}-8)z^2+az+13\mathrm{i}=0$, find the complex number a and solve the equation, giving your answer in Cartesian form $x+\mathrm{i}y$. [4]

Hence, find in Cartesian form the roots of the equation

$$2w^3+(1+8\mathrm{i})w^2-aw-13=0. \quad [2]$$

(b) Solve the equation $z^6+729=0$, expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [2]

Given that z_1 is a root of the above equation and $0<\arg z_1<\dfrac{\pi}{2}$.

If $\dfrac{z_1{}^n}{z_1{}^*}$ is a positive real number, find the smallest positive integer n. [2]

3. (i) Use the substitution $x = 3\sin\theta+1,\ 0<\theta<\dfrac{\pi}{2}$ to find $\displaystyle\int \frac{x}{\sqrt{9-(x-1)^2}}\,\mathrm{d}x$. [4]

(ii) A curve has parametric equations

$$x=\frac{1}{\sqrt{9-(t-1)^2}},\quad y=t^2,\quad 0<t<4.$$

(a) Sketch the curve, indicating the end point and the equation of the asymptote. [3]

(b) Using the result in part **(i)**, find the exact area of the region bounded by the curve, the lines $y=1$, $y=\dfrac{25}{4}$ and $x=\dfrac{8}{7}$. [4]

4. In harvesting of renewable natural resources, it is desirable that policies are formulated to allow maximal harvest of the natural resources, and yet not deplete the resources below a sustainable level. A simple harvesting model devised for the rate of change of the population of wild salmon in a particular region in the Pacific Ocean is given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(4-P) - h,$$

where P is the population of wild salmon in millions at time t years and h is the constant harvest rate in millions.

(i) Sketch a graph of $\frac{\mathrm{d}P}{\mathrm{d}t}$ against P, expressing the turning point in terms of h. [2]

(ii) The *Maximum Sustainable Yield (MSY)* is the largest harvest rate h that allows for a sustainable harvest of wild salmon without long-term depletion. State the *MSY* for the wild salmon. [1]

(iii) It is given that the population of wild salmon in that region was 3.2 million in 2015 and the constant harvest rate is 3 million. Find an expression for P at any time t. [5]

Hence find the population of the wild salmon in that region in 2016. [2]

(iv) State one assumption you made in your calculation. [1]

Section B: Statistics [60 marks]

5. In the last election, there were speculations from unofficial sources before the counting of votes is completed. For the current election, to prevent unnecessary speculations, the election office of Sunny Island will be conducting a sample count in each electoral division after voting is done. Each electoral division has a different number of registered voters and a sample of 400 votes will to be sampled from each electoral division.

(i) Identify and describe an appropriate method to obtain the sample. [3]

(ii) State an advantage and a disadvantage of the sampling method used in part **(i)**. [2]

[Turn over

6. The random variable X has a binomial distribution $\mathrm{B}(n, p)$, where $0 < p < 1$, and n is an integer. Show that $\dfrac{\mathrm{P}(X = r)}{\mathrm{P}(X = r-1)} = \left(\dfrac{n-r+1}{r}\right)\left(\dfrac{p}{1-p}\right)$. [3]

Hence find a condition relating n and p such that X has two values for its mode, and determine these two values, giving your answer in terms of n and p. [3]

7. A group of ten people consists of four single women, two single men and 2 couples. The ten people are arranged randomly in a circle.

(i) Find the probability that the four single women are all separated. [2]

(ii) Find the probability that either the four single women are next to one another or the two single men are next to each other but not both. [3]

One of the ten people left the group and the remaining nine decided to sit at a round table with ten identical chairs equally spaced around the table. The chairs are decorated such that every alternate chair is tied with an identical chair sash. Given that the nine people have no preference to which seat to take, find the number of possible seating arrangements. [2]

8. In an examination, the score, X, for paper 1 of a student is found to follow a normal distribution with mean 62 and standard deviation σ, and the score, Y, for paper 2 of a student is found to follow a normal distribution with mean 71 and standard deviation 8. The final score of a student for the examination is the average score of the 2 papers and it is assumed that X and Y are independent random variables.

(i) Find the probability that for two randomly selected students A and B taking the examination for paper 2, A has at most 2 marks less than the marks of B. [2]

(ii) Given that 15% of the students have at least a final score of 75, find σ. [4]

(iii) Using the value of σ found in part **(ii)**, find the probability that a randomly selected student performs better in her Paper 1 than in her Paper 2.

(iv) Comment on the validity of the answer obtained in part **(ii)** and **(iii).** [3]

9. To reduce the number of speeding incidents on the road, traffic police in Country S set up traffic cameras at 3 busy traffic Junctions A, B and C to monitor the speeds of vehicles passing through these junctions. The average number of speeding vehicles caught by the camera at Junctions A, B and C are 2 in every 3 hours, 5 in every 4 hours and λ in every hour respectively. It is assumed that the number of speeding vehicles caught by the cameras at the three junctions followed Poisson distributions.

(a) Find the probability that there are at least 2 speeding vehicles caught at Junctions A and B in an hour. [2]

(b) Given that there are 2 speeding vehicles caught at the three junctions in an hour, find the probability that at least one speeding vehicle caught is at Junction C. Leave your answer in terms of λ. [3]

(c) Given that the traffic cameras are in operation 24 hours in a day, using a suitable approximation, find the probability that there will be more speeding vehicles caught at Junction A than at Junction B in a day. State an assumption for the calculation to be valid. [5]

10. A nutritionist claims that the mean number of calories in an energy bar is 350 cal. The nutritionist collected and measured the number of calories of a random sample of 15 energy bars. The mean and variance of the sample was 347.2 cal and 20.74 cal^2 respectively.

(i) The nutritionist wishes to carry out a hypothesis test on his claim. Explain why t-test instead of z-test is to be used. State an assumption for the test to be valid. [2]

(ii) Test at 5% level of significance, whether the mean number of calories in an energy bar is 350 cal, defining any symbols that you use. [4]

(iii) Suppose the nutritionist uses a different test in part (**ii**). Without further calculation, explain and state whether the conclusion will be different. [2]

The manufacturer of the energy bar refines the manufacturing process and the new energy bars follow a normal distribution with mean μ cal and variance 20.74 cal^2. The manufacturer then provides the nutritionist with another sample of 15 energy bars.

[Turn over

(iv) Find the range of mean number of calories, $\overline{x}$, of the second sample of 15 energy bar so that the null hypothesis in part **(ii)** is not rejected at 5% level of significance. Leave your answer correct to one decimal place. [3]

11. A group of scientists is interested to find out the correlation between the number of species and the size of the natural habitat. The scientists sampled non-overlapping lands of different areas (x) in square kilometres, and noted the corresponding number of species (y) found. The results are shown in the table below.

Area (x)	300	400	500	600	700	800	900	1000	1100	1200
Number of species (y)	12	15	18	21	k	25	26	27	27	28

(i) Given that the equation of the regression line is $y = 0.01758x + 9.018$, show that the value of k to the nearest whole number is 23. [3]

Take k to be 23.

(ii) Draw the scatter diagram for the given data, labelling the axes clearly. [2]

(iii) Calculate the product moment correlation coefficient r. With reference to both the scatter diagram and r, explain why a linear model is not appropriate. [2]

(iv) The following models are suggested for the data.

(A) $y = a + bx^2$, (B) $y = ax^b$, where $a > 0$ and $0 < b < 1$.

Use a graphical approach to determine which model is more appropriate. [2]

(v) Use the more appropriate model to estimate the area of the natural habitat when the number of species found is 24. Comment on the reliability of your estimation. [3]

End of Paper

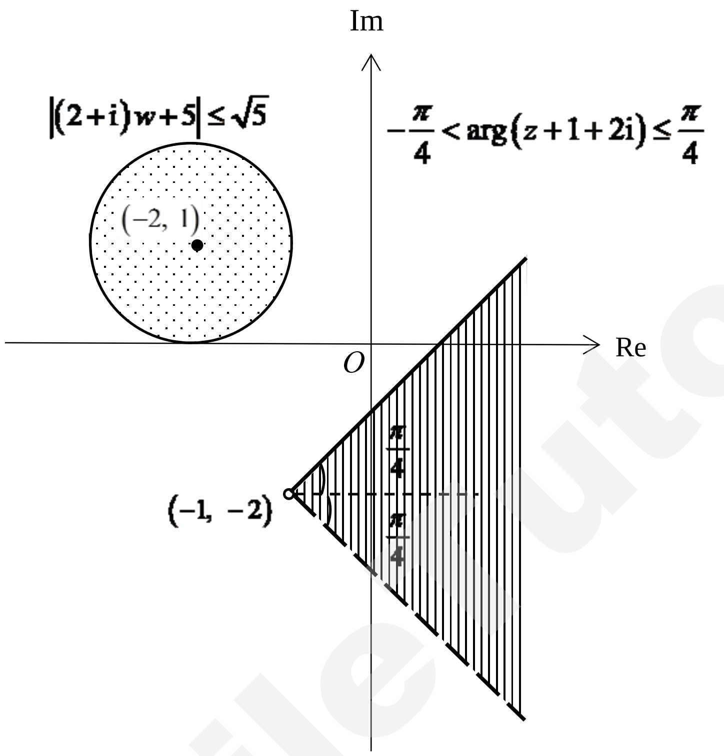
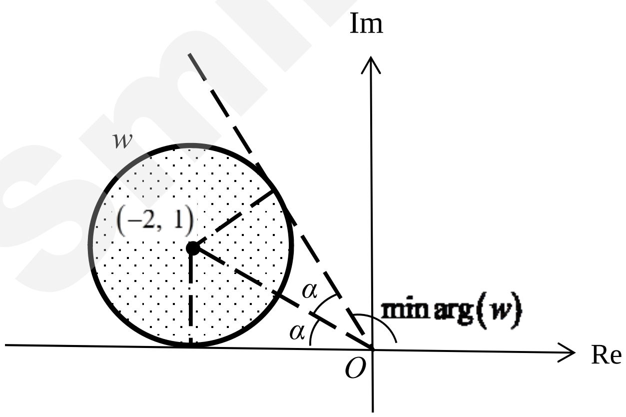
2016 HCI Prelim Paper 1 Solutions

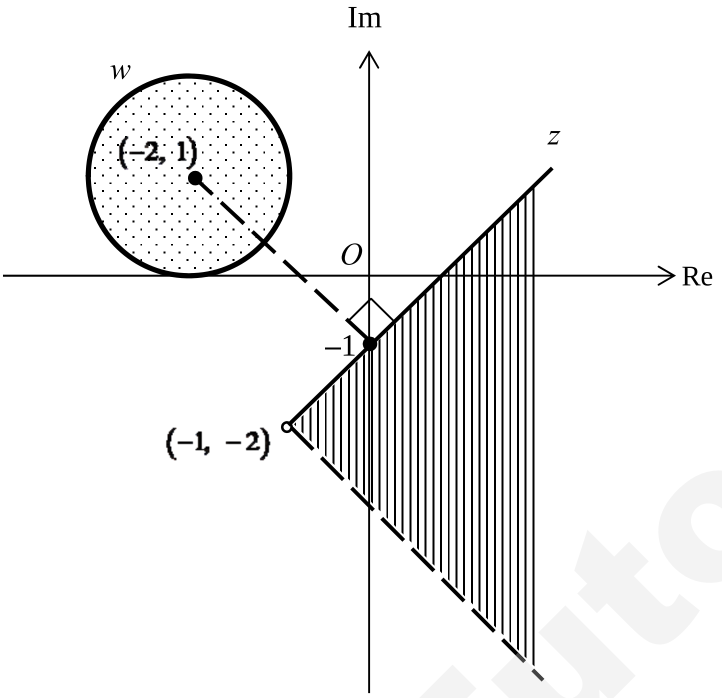
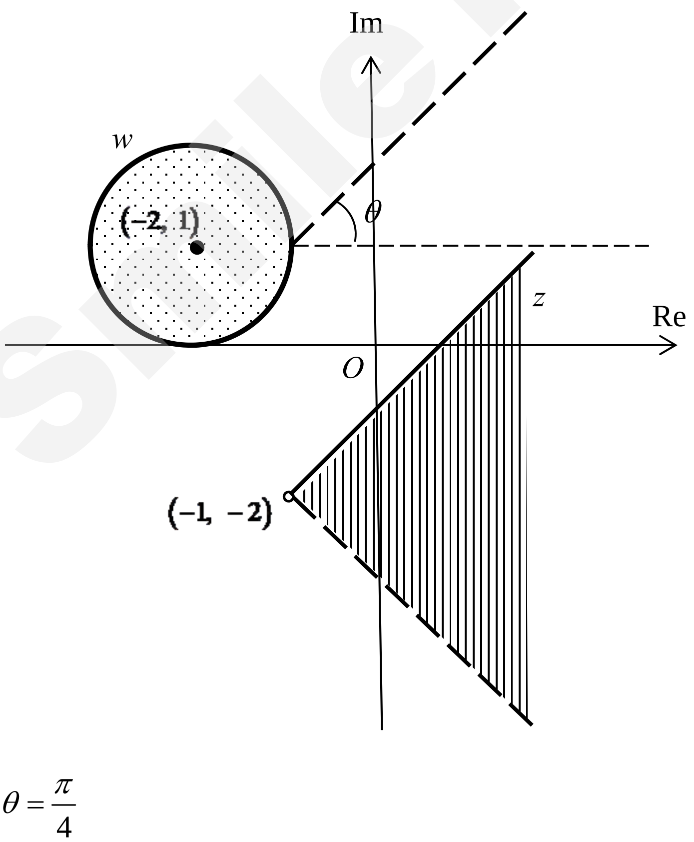
<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>1</td><td>* Let P_n be statement $U_n = \sin(nx)$ for all $n \in \mathbb{Z}^+$.
When $n=1$, LHS $= U_1 = \sin x$, RHS $= \sin x$ $\therefore P_1$ is true.
* Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $U_k = \sin(kx)$.
Want to prove that P_{k+1} is true, i.e. $U_{k+1} = \sin(k+1)x$.
LHS
$= U_{k+1}$
$= U_k + 2\cos\frac{(2k+1)x}{2}\sin\frac{x}{2}$
$= \sin(kx) + 2\cos\left(\frac{2k+1}{2}\right)x\sin\left(\frac{1}{2}\right)x$
$= \sin(kx) + \sin(k+1)x - \sin(kx)$
$= \sin(k+1)x = \text{RHS}$
*Since P_1 is true, P_k is true implies P_{k+1} is true, by MI P_n is true for all $n \in \mathbb{Z}^+$.</td></tr>
<tr><td>2</td><td>$\frac{2}{4(x+1)^2+1} > 1$
$\frac{-(2x+1)(2x+3)}{4(x+1)^2+1} > 0$
Since $4(x+1)^2+1>0$ for all x,
$(2x+1)(2x+3)<0$
$\therefore -\frac{3}{2} < x < -\frac{1}{2}$
$\int_{-1}^{\frac{\sqrt{3}}{2}-1}\left|1-\frac{2}{4(x+1)^2+1}\right|\mathrm{d}x$
$= \int_{-1}^{-\frac{1}{2}}\left(-1+\frac{2}{4(x+1)^2+1}\right)\mathrm{d}x + \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}-1}\left(1-\frac{2}{4(x+1)^2+1}\right)\mathrm{d}x$
$= \left[-x+\tan^{-1}(2x+2)\right]_{-1}^{-\frac{1}{2}} + \left[x-\tan^{-1}(2x+2)\right]_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}-1}$
$= \left[\frac{1}{2}+\tan^{-1}1-1\right] + \left[\frac{\sqrt{3}}{2}-1-\tan^{-1}\sqrt{3}+\frac{1}{2}+\tan^{-1}1\right]$
$= \frac{\pi}{6}+\frac{\sqrt{3}}{2}-1$</td></tr>
</table>

3 (a)	$\overrightarrow{OP} = \underset{\sim}{a} + 3\overrightarrow{AB} = \underset{\sim}{a} + 3\left(\underset{\sim}{b} - \underset{\sim}{a}\right) = 3\underset{\sim}{b} - 2\underset{\sim}{a}$ $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 2\underset{\sim}{a} - \left(3\underset{\sim}{b} - 2\underset{\sim}{a}\right) = 4\underset{\sim}{a} - 3\underset{\sim}{b}$ $l_{PQ} : \underset{\sim}{r} = 2\underset{\sim}{a} + \lambda\left(4\underset{\sim}{a} - 3\underset{\sim}{b}\right),\ \lambda \in \mathbb{R}$ $l_{OB} : \underset{\sim}{r} = \mu\underset{\sim}{b},\ \mu \in \mathbb{R}$ At point of intersection, $2\underset{\sim}{a} + \lambda\left(4\underset{\sim}{a} - 3\underset{\sim}{b}\right) = \mu\underset{\sim}{b}$ Comparing coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\lambda = -\frac{1}{2},\ \mu = \frac{3}{2}$ $\therefore$ position vector of the point of intersection $= \frac{3}{2}\underset{\sim}{b}$
(b)	$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \underset{\sim}{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ Let F be the foot of perpendicular. <u>Method 1</u> $l_{FC} : \underset{\sim}{r} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},\ s \in \mathbb{R}$, $\Pi_{OAB} : \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$ $\left[\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$ $-6 + 4 + s\left(1 + 4 + 1\right) = 0$ $s = \frac{1}{3}$ $\therefore \overrightarrow{OF} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0+1 \\ 9-2 \\ 12+1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 7 \\ 13 \end{pmatrix}$ $\therefore F\left(\frac{1}{3}, \frac{7}{3}, \frac{13}{3}\right)$

	Method 2 $\overrightarrow{FC}=\left(\overrightarrow{OC}\cdot\hat{\underset{\sim}{n}}\right)\hat{\underset{\sim}{n}}$ $=\left(\begin{pmatrix}0\\3\\4\end{pmatrix}\cdot\frac{\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|}\right)\frac{\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|}$ $=\frac{-6+4}{\sqrt{6}}\frac{\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\sqrt{6}}$ $=-\frac{1}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix}$ $\overrightarrow{OF}=\overrightarrow{OC}+\overrightarrow{CF}$ $=\begin{pmatrix}0\\3\\4\end{pmatrix}+\frac{1}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}0+1\\9-2\\12+1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1\\7\\13\end{pmatrix}$ $\therefore F\left(\frac{1}{3},\frac{7}{3},\frac{13}{3}\right)$
4	Method 1 $\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}\times\frac{\sqrt{n^2+2n}+\sqrt{n^2-1}}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $=\frac{(2n+1)\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)}{\left(n^2+2n\right)-\left(n^2-1\right)}$ $=\sqrt{n^2+2n}-\sqrt{n^2-1}$

	Method 2 $\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)\left(\sqrt{n^2+2n}+\sqrt{n^2-1}\right)$ $=\left(n^2+2n-\left(n^2-1\right)\right)$ $=2n+1$ $\therefore \dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\sqrt{n^2+2n}-\sqrt{n^2-1}$
	$\sum_{n=1}^{N}\dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $=\sum_{n=1}^{N}\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)$ $=\left[\begin{array}{c}\sqrt{3}-\sqrt{0}\\ +\sqrt{8}-\sqrt{3}\\ \dots\\ +\sqrt{N^2+2N}-\sqrt{N^2-1}\end{array}\right]$ $=\sqrt{N^2+2N}$
(a)	Replace n by $n+1$, $\sum_{n=2}^{N}\dfrac{2n-1}{\sqrt{n^2-1}+\sqrt{n(n-2)}}$ $=\sum_{n=1}^{N-1}\dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $=\sqrt{(N-1)^2+2(N-1)}$ $=\sqrt{N^2-1}$
(b)	Notice that $\sqrt{n^2+2n}>n$ and $\left(\sqrt{n^2-1}\right)^2-(n-1)^2=2n-2\geq 0$. $\Rightarrow \sqrt{n^2-1}\geq n-1$ $\Rightarrow \sqrt{n^2+2n}+\sqrt{n^2-1}>2n-1$ $\Rightarrow \dfrac{1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}<\dfrac{1}{2n-1}$ $\therefore \sum_{n=1}^{N}\dfrac{2n+1}{2n-1}>\sum_{n=1}^{N}\dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\sqrt{N^2+2N}$

5	$\lvert(2+\mathrm{i})w+5\rvert \le \sqrt{5}$ $\lvert 2+\mathrm{i}\rvert \left\lvert w+\dfrac{5}{2+\mathrm{i}}\right\rvert \le \sqrt{5}$ $\lvert w+2-\mathrm{i}\rvert \le 1 \Rightarrow$ circle centre $(-2,\ 1)$, radius 1 
(i)	 $\alpha = \sin^{-1}\dfrac{1}{\sqrt{(-2)^2+1^2}} = \sin^{-1}\dfrac{1}{\sqrt{5}}$ $\min\arg(w) = \pi - 2\alpha = 2.2143 = 2.21\,(3\text{sf})$

(ii)	 $\min\lvert z-w\rvert = 2\sqrt{2}-1$
(iii)	 $\theta = \dfrac{\pi}{4}$

6 (i)	$\overrightarrow{OD}=\begin{pmatrix}0\\0\\2\end{pmatrix},\ \overrightarrow{OE}=\begin{pmatrix}3\\1.5\\2\end{pmatrix}$ $\overrightarrow{DE}=\begin{pmatrix}3\\1.5\\2\end{pmatrix}-\begin{pmatrix}0\\0\\2\end{pmatrix}=\begin{pmatrix}3\\1.5\\0\end{pmatrix}=1.5\begin{pmatrix}2\\1\\0\end{pmatrix}$ $l_{DE}: \underset{\sim}{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix},\ \lambda\in\mathbb{R}$
(ii)	$\overrightarrow{AD}=\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}=\begin{pmatrix}-3\\0\\2\end{pmatrix}$ $\overrightarrow{DE}\times\overrightarrow{AD}=1.5\begin{pmatrix}2\\1\\0\end{pmatrix}\times\begin{pmatrix}-3\\0\\2\end{pmatrix}=1.5\begin{pmatrix}2\\-4\\3\end{pmatrix}\Rightarrow\underset{\sim}{n}=\begin{pmatrix}2\\-4\\3\end{pmatrix}$ $\Pi_{ADE}: \underset{\sim}{r}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}=\begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}=6$ $\therefore 2x-4y+3z=6$
(iii)	$\underset{\sim}{n}_{OABC}=\begin{pmatrix}0\\0\\1\end{pmatrix},\ \underset{\sim}{n}_{ADE}=\begin{pmatrix}2\\-4\\3\end{pmatrix}$ angle between planes $=\cos^{-1}\dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}\right\|}{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}2\\-4\\3\end{pmatrix}\right\|}=\cos^{-1}\dfrac{3}{\sqrt{4+16+9}}$ $=\cos^{-1}\dfrac{3}{\sqrt{29}}=56.1°(1\text{dp})$ Angle $=180°-2\cos^{-1}\dfrac{3}{\sqrt{29}}\ =67.7°$ (1 d.p.)

7 (i)

$$\frac{dy}{dx}=\frac{(kx^2+x-2)-(x-2)(2kx+1)}{(kx^2+x-2)^2}=\frac{-kx^2+4kx}{(kx^2+x-2)^2}$$

When $x=0$, $\frac{dy}{dx}=\frac{0}{(-2)^2}=0$ and $y=\frac{-2}{-2}=1$

Hence required equation of tangent is $y=1$.

(ii)

For axial intercepts, when $y=0$, $x=2$.

when $x=0$, $y=1$.

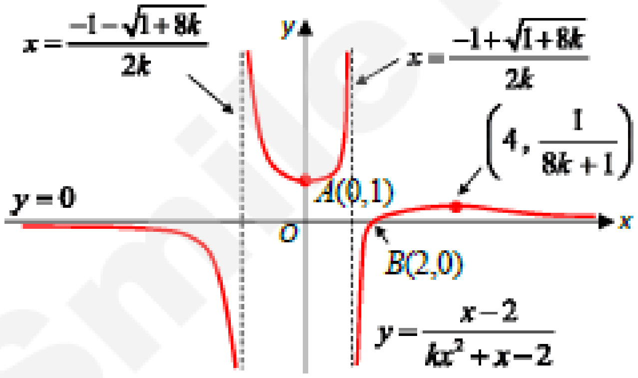
For vertical asymptotes, $kx^2+x-2=0$

$$\therefore\ x=\frac{-1\pm\sqrt{1+8k}}{2k}$$

For turning points, $\frac{dy}{dx}=0$

$$-kx^2+4kx=0$$
$$-kx(x-4)=0$$

$\therefore\ x=0$ or $x=4$



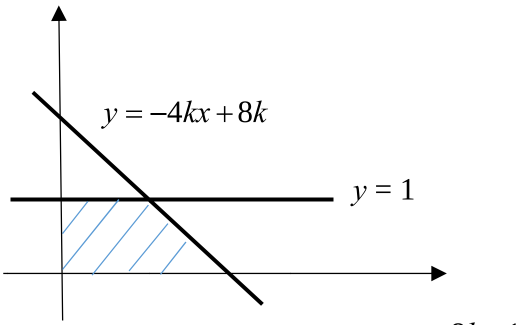
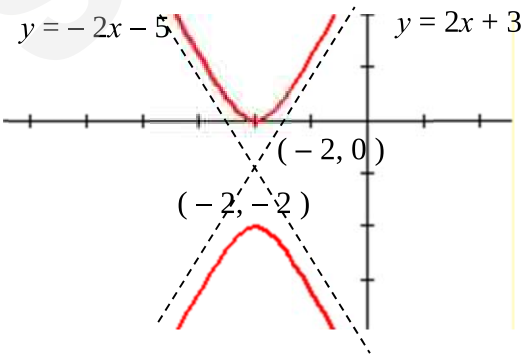
(iii)

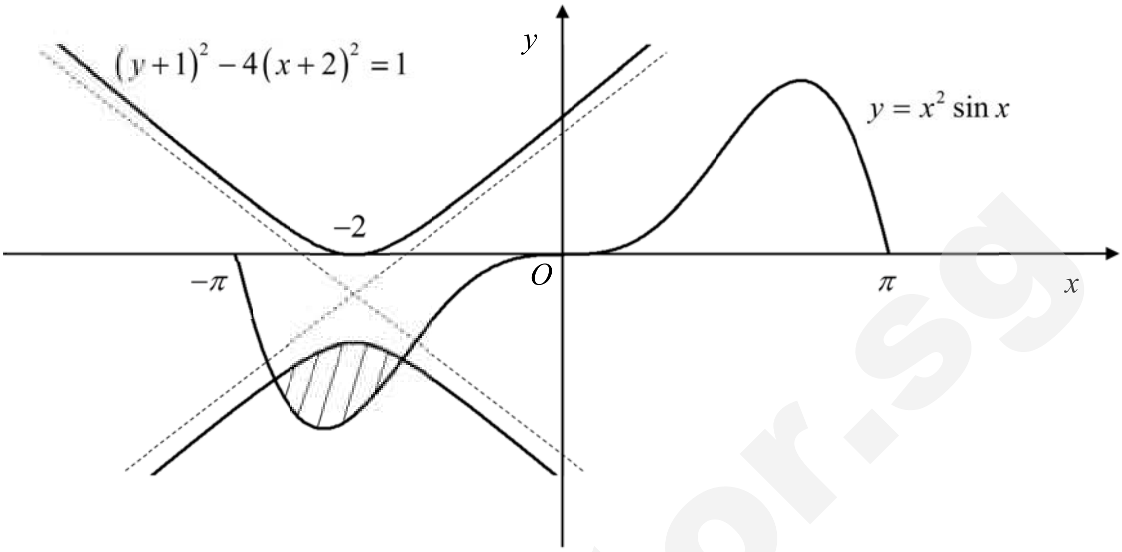
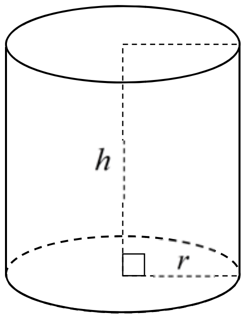
At $x=2$, $\frac{dy}{dx}=\frac{-4k+8k}{(4k)^2}=\frac{4k}{16k^2}=\frac{1}{4k}$

$\therefore$ gradient of normal $=-4k$

Hence required equation of normal is $y-0=-4k(x-2)$

$$y=-4kx+8k$$

(iv)	 When $y=1$, $1=-4kx+8k \Rightarrow x=\dfrac{8k-1}{4k}$ $\therefore$ required area $=\dfrac{1}{2}\left(\dfrac{8k-1}{4k}+2\right)(1)$ $=\dfrac{16k-1}{8k}$ $=2-\dfrac{1}{8k}$ $>2-\dfrac{1}{8}$ (since $k>1$) $>\dfrac{15}{8}$
8 (i)	$\text{Area}=2\int_0^{\pi} x^2\sin x\,\mathrm{d}x$ $=2\left[\left[-x^2\cos x\right]_0^{\pi}+\int_0^{\pi}2x\cos x\,\mathrm{d}x\right]$ $=2\left[\pi^2+2(\left[x\sin x\right]_0^{\pi}-\int_0^{\pi}\sin x\,\mathrm{d}x)\right]$ $=2\left[\pi^2+2\left[\cos x\right]_0^{\pi}\right]$ $=2(\pi^2-4)\text{ units}^2$
(ii)	

(iii)	 Coordinates of the points of intersections of the 2 curves are $(-1.5374,-2.3623)$ and $(-2.7626,-2.8238)$. Volume of solid generated $=\pi\int_{-2.7626}^{-1.5374}(x^2\sin x)^2\,dx-\pi\int_{-2.7626}^{-1.5374}(-1-\sqrt{1+4(x+2)^2})^2\,dx=26.8$ units^3
9 (i)	Let A cm^2 be the surface area of the cylindrical container. Let r cm and h cm be the radius and height of the cylindrical container respectively.  Volume $=\pi r^2 h=k$ $\therefore\ h=\dfrac{k}{\pi r^2}$ $A=2\pi rh+\pi r^2$ $=2\pi r\left(\dfrac{k}{\pi r^2}\right)+\pi r^2$ $=\dfrac{2k}{r}+\pi r^2$ Can also express r in terms of h and find A in terms of h, then let $\dfrac{dA}{dh}=0$ to obtain h and subsequently r. Hence $\dfrac{dA}{dr}=-\dfrac{2k}{r^2}+2\pi r$ When $\dfrac{dA}{dr}=0$,

	$-\dfrac{2k}{r^2}+2\pi r=0$ $r^3=\dfrac{k}{\pi}$ $r=\sqrt[3]{\dfrac{k}{\pi}}$ $\therefore\ h=\dfrac{k}{\pi r^2}=\dfrac{k}{\pi\left[\left(\frac{k}{\pi}\right)^{\frac{1}{3}}\right]^2}=\sqrt[3]{\dfrac{k}{\pi}}$ Hence $h:r=\sqrt[3]{\dfrac{k}{\pi}}:\sqrt[3]{\dfrac{k}{\pi}}=1:1$ (shown) $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2}=\dfrac{4k}{r^3}+2\pi>0$ since $p>0$ and $k>0$ Hence A is a minimum when $r=\sqrt[3]{\dfrac{k}{\pi}}$	
(ii)	From (i), $h:r=1:1$ Hence $A=2\pi rh+\pi r^2=2\pi r(r)+\pi r^2=3\pi r^2$ For new design, $h:r=5:2$ Hence new $A=2\pi rh+\pi r^2=2\pi r\left(\dfrac{5}{2}r\right)+\pi r^2=6\pi r^2$ $\therefore$ required ratio is $6\pi r^2:3\pi r^2=2:1$	
(b)	Method 1 Let V cm^3 be the volume of the cylindrical container. $V=\pi r^2h=\pi r^2(\dfrac{5}{2}r)=\dfrac{5}{2}\pi r^3$ $A=2\pi rh+\pi r^2=2\pi r\left(\dfrac{5}{2}r\right)+\pi r^2=6\pi r^2$ $\dfrac{\mathrm{d}V}{\mathrm{d}r}=\dfrac{15}{2}\pi r^2$ $\dfrac{\mathrm{d}A}{\mathrm{d}r}=12\pi r$ $\dfrac{\mathrm{d}A}{\mathrm{d}t}=\dfrac{\mathrm{d}A}{\mathrm{d}r}\times\dfrac{\mathrm{d}r}{\mathrm{d}V}\times\dfrac{\mathrm{d}V}{\mathrm{d}t}$ $=12\pi r\times\dfrac{2}{15\pi r^2}\times 80=\dfrac{128}{r}$ When $h=50$, $r=\dfrac{2}{5}(50)=20$ Hence $\dfrac{\mathrm{d}A}{\mathrm{d}t}=\dfrac{128}{20}=6.4$ cm^2/s	Can also find $\dfrac{\mathrm{d}V}{\mathrm{d}h}$ and $\dfrac{\mathrm{d}A}{\mathrm{d}h}$ and use $\dfrac{\mathrm{d}A}{\mathrm{d}t}=\dfrac{\mathrm{d}A}{\mathrm{d}h}\times\dfrac{\mathrm{d}h}{\mathrm{d}V}\times\dfrac{\mathrm{d}V}{\mathrm{d}t}$

	Method 2 $A = 6\pi r^2 \quad \therefore\ r = \sqrt{\dfrac{A}{6\pi}}$ (reject $r = -\sqrt{\dfrac{A}{6\pi}}$ since $r \geq 0$) Hence $V = \pi r^2 h = \pi r^2 \left(\dfrac{5}{2}r\right) = \dfrac{5}{2}\pi r^3$ $= \dfrac{5}{2}\pi\left(\sqrt{\dfrac{A}{6\pi}}\right)^3$ $= \dfrac{5A^{\frac{3}{2}}}{2(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}$ $\dfrac{\mathrm{d}V}{\mathrm{d}A} = \dfrac{15A^{\frac{1}{2}}}{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}$ When $h = 50$, $r = \dfrac{2}{5}(50) = 20$ $\therefore\ A = 6\pi(20)^2 = 2400\pi$ Hence $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \dfrac{\mathrm{d}A}{\mathrm{d}V} \times \dfrac{\mathrm{d}V}{\mathrm{d}t}$ $= \dfrac{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}{15A^{\frac{1}{2}}} \times 80$ $= \dfrac{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}{15(2400\pi)^{\frac{1}{2}}} \times 80$ $= 6.4 \quad \text{cm}^2/\text{s}$
10 (a) (i)	$\dfrac{\sin\theta}{x} = \dfrac{\sin\left(\pi - \frac{\pi}{6} - \theta\right)}{y}$ $\dfrac{x}{y} = \dfrac{\sin\theta}{\sin\left(\frac{5\pi}{6} - \theta\right)}$ $\dfrac{x}{y} = \dfrac{\sin\theta}{\sin\frac{5\pi}{6}\cos\theta - \sin\theta\cos\frac{5\pi}{6}}$ $\dfrac{x}{y} = \dfrac{\sin\theta}{\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta} = \dfrac{2\sin\theta}{\cos\theta + \sqrt{3}\sin\theta}$ (shown)
(a) (ii)	$\dfrac{x}{y} = \dfrac{2\sin\theta}{\cos\theta + \sqrt{3}\sin\theta}$

	$\frac{x}{y}=\frac{2\left(\theta-\frac{\theta^3}{3!}+...\right)}{1+\sqrt{3}\theta-\frac{\theta^2}{2}+...}$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(1+\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)\right)^{-1}$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(1+(-1)\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)+\frac{(-1)(-2)}{2!}\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)^2\right)$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(1-\sqrt{3}\theta+\frac{\theta^2}{2}+3\theta^2\right)$ $\frac{x}{y}\approx 2\theta-2\sqrt{3}\theta^2+\frac{20}{3}\theta^3$
(b) (i)	Using sine rule, $\frac{\sin\theta}{x}=\frac{\sin\frac{\pi}{6}}{\frac{1}{6}}=3 \quad \therefore \theta=\sin^{-1}3x$
(b) (ii)	<u>Method 1</u> $\sin\theta=3x$ $\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}x}=3$ ---- (1) $\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}-\sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2=0$ --- (2) $\cos\theta\frac{\mathrm{d}^3\theta}{\mathrm{d}x^3}-\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}-2\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}-\cos\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^3$ $=0---(3)$ When $x=0$, $\theta=0$, $\frac{\mathrm{d}\theta}{\mathrm{d}x}=3$, $\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}=0$, $\frac{\mathrm{d}^3\theta}{\mathrm{d}x^3}=27$ $\theta=3x+\frac{27}{3!}x^3+...=3x+\frac{9}{2}x^3+...$

Method 2

$\theta = \sin^{-1}(3x)$

$$\frac{d\theta}{dx} = \frac{3}{\sqrt{1-9x^2}} = 3\left(1-9x^2\right)^{-\frac{1}{2}}$$

$$\frac{d^2\theta}{dx^2} = 3\left(-\frac{1}{2}\right)\left(1-9x^2\right)^{-\frac{3}{2}}(-18x) = 27x\left(1-9x^2\right)^{-\frac{3}{2}}$$

$$\frac{d^3\theta}{dx^3} = -\frac{81}{2}x(-18x)\left(1-9x^2\right)^{-\frac{5}{2}} + 27\left(1-9x^2\right)^{-\frac{3}{2}}$$

$$= 729x^2\left(1-9x^2\right)^{-\frac{5}{2}} + 27\left(1-9x^2\right)^{-\frac{3}{2}}$$

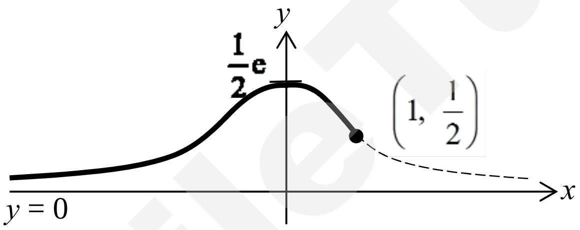
When $x = 0$,

$$\theta = 0,\ \frac{d\theta}{dx} = 3,\ \frac{d^2\theta}{dx^2} = 0,\ \frac{d^3\theta}{dx^3} = 27$$

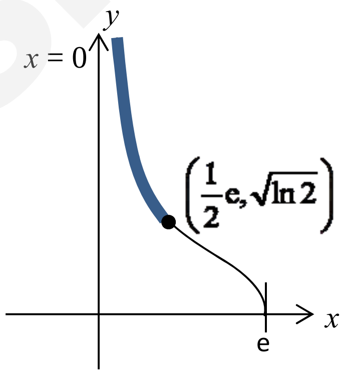
$$\theta = 3x + \frac{27}{3!}x^3 + \ldots = 3x + \frac{9}{2}x^3 + \ldots$$

11

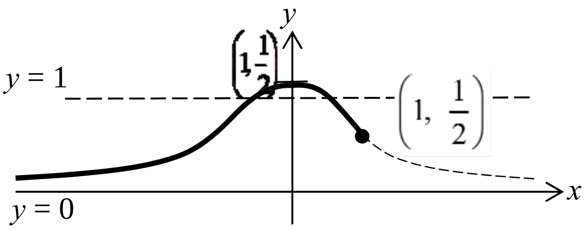
(i)



Since $R_f = \left(0, \frac{1}{2}e\right] \subseteq D_g = (0, e]$, $R_f \subseteq D_g$ and gf exists.

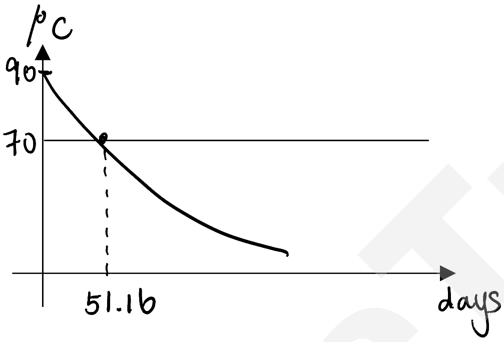


$R_{gf} = \left[\sqrt{\ln 2}, \infty\right)$

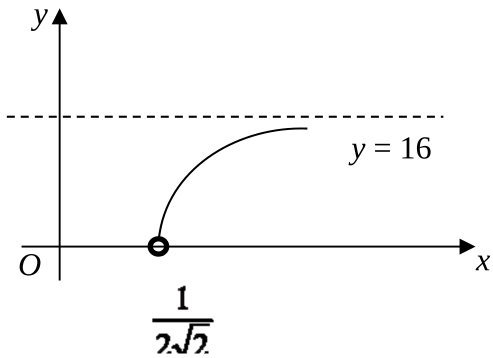
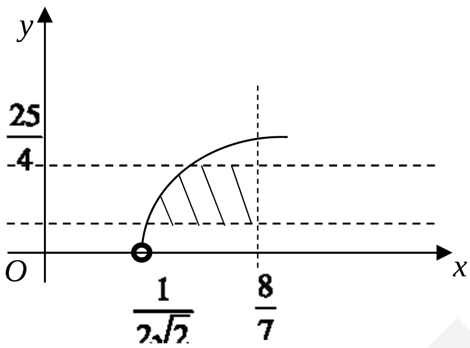
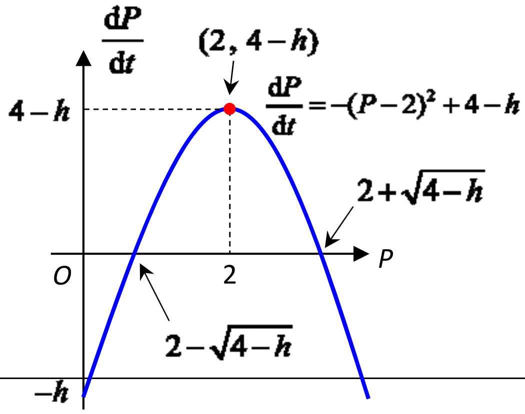
<table>
<tr><td>(ii)</td><td>

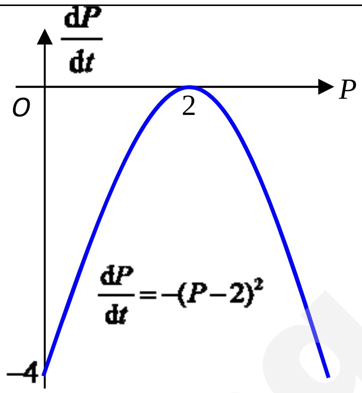
Since a horizontal line y = 1 cuts the graph of y = f(x) twice, f is not a one-to-one function and f^{-1} does not exist.</td></tr>
<tr><td>(iii)</td><td>$b = 0$
Let $y = \mathrm{f}(x)$
$y = \dfrac{1}{2}\mathrm{e}^{1-x^2}$
$\ln(2y) = 1 - x^2$
$x = \pm\sqrt{1-\ln(2y)}$
Since $x \le 0,\ x = -\sqrt{1-\ln(2y)}$
$\mathrm{f}^{-1} : x \mapsto -\sqrt{1-\ln(2x)},\ x \in \mathbb{R},\ 0 < x \le \dfrac{1}{2}\mathrm{e}$</td></tr>
<tr><td>(iv)</td><td>$y = \sqrt{1-\ln x} \xrightarrow{\text{Step 1}} y = \sqrt{1-\ln\left(\dfrac{x}{2}\right)}$
$y = \sqrt{1-\ln\left(\dfrac{x}{2}\right)} \xrightarrow{\text{Step 2}} y = -\sqrt{1-\ln\left(\dfrac{x}{2}\right)}$
$0 < x \le \mathrm{e} \to 0 < \dfrac{x}{2} \le \mathrm{e}$
$\therefore 0 < x \le 2\mathrm{e}$
$\mathrm{h} : x \mapsto -\sqrt{1-\ln\left(\dfrac{x}{2}\right)},\ x \in \mathbb{R},\ 0 < x \le 2\mathrm{e}$</td></tr>
</table>

2016 Prelim Paper 2 Solutions

Qn	Solution
1 (i)	1^{st} day: $0.97(92)$ 2^{nd} day: $0.97\left(0.97(92)+2\right)$ 3^{rd} day: $0.97\left(0.97\left(0.97(92)+2\right)+2\right)$ $=0.97^3(92)+0.97^2(2)+0.97(2)$ $=87.788$ $=87.8\ {}^\circ\text{C}$ (3 s.f.)
(ii)	n^{th} day: $=0.97^n(92)+0.97^{n-1}(2)+...+0.97(2)$ $=0.97^n(92)+2\left(\dfrac{0.97\left(1-0.97^{n-1}\right)}{1-0.97}\right)$ <u>Method 1</u>  <u>Method 2</u> Using GC, When $n=51$, temperature $=70.025°\text{C}$ When $n=52$, temperature $=69.865°\text{C}$ $\therefore$ 52 days
(iii)	No, temperature will not drop infinitely since $r=0.97<1$. As $n\to\infty$, temperature approaches $0+\dfrac{2\times0.97}{1-0.97}=64.7\ °\text{C}$ in the long run.
2 (a)	$2\left(-\dfrac{\text{i}}{2}\right)^3+(\text{i}-8)\left(-\dfrac{\text{i}}{2}\right)^2+a\left(-\dfrac{\text{i}}{2}\right)+13\text{i}=0$ $\dfrac{1}{4}\text{i}+2-\dfrac{1}{4}\text{i}-\dfrac{a\text{i}}{2}+13\text{i}=0$ $a=26-4\text{i}$ $2z^3+(\text{i}-8)z^2+(26-4\text{i})z+13\text{i}=(2z+\text{i})(z^2+bz+13)$ Comparing the coefficient of z^2, $\text{i}-8=2b+\text{i}$ $\therefore b=-4$ $(2z+\text{i})(z^2-4z+13)=0 \quad \therefore z=-\dfrac{\text{i}}{2}$ or $z=2\pm3\text{i}$.

	Replace z with iw, $2(\mathrm{i}w)^3+(\mathrm{i}-8)(\mathrm{i}w)^2+a(\mathrm{i}w)+13\mathrm{i}=0$ $-2\mathrm{i}w^3+(8-\mathrm{i})w^2+a\mathrm{i}w+13\mathrm{i}=0$ Dividing throughout by $-\mathrm{i}$, $2w^3+(1+8\mathrm{i})w^2-aw-13=0.$ $\mathrm{i}w=-\frac{\mathrm{i}}{2}$ or $\mathrm{i}w=2\pm3\mathrm{i}.$ $\therefore w=-\frac{1}{2}$ or $w=\pm3-2\mathrm{i}$
(b)	$z^6=-729=729\mathrm{e}^{\mathrm{i}\pi}=729\mathrm{e}^{(\pi+2k\pi)\mathrm{i}}$ $z=3\mathrm{e}^{\left(\frac{\pi}{6}+\frac{2k\pi}{6}\right)\mathrm{i}}, k=0,\pm1,\pm2,-3$ OR $z=3\mathrm{e}^{-\frac{5\pi}{6}\mathrm{i}},3\mathrm{e}^{\frac{5\pi}{6}\mathrm{i}},3\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}},3\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},3\mathrm{e}^{-\frac{\pi}{6}\mathrm{i}},3\mathrm{e}^{\frac{\pi}{6}\mathrm{i}}$ $z_1=3\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ $\arg\left(\frac{z_1{}^n}{z_1{}^*}\right)=n\arg z_1+\arg z_1=(n+1)\frac{\pi}{6}$ Positive real number $(n+1)\frac{\pi}{6}=2k\pi$ $\therefore (n+1)\frac{\pi}{6}=2\pi$ Minimum $n=11.$
3 (i)	$x=3\sin\theta+1$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=3\cos\theta$ $\int\frac{x}{\sqrt{9-(x-1)^2}}\,\mathrm{d}x,$ $=\int\frac{3\sin\theta+1}{\sqrt{9-(3\sin\theta)^2}}(3\cos\theta)\,\mathrm{d}\theta$ $=\int(3\sin\theta+1)\,\mathrm{d}\theta$ $=-3\cos\theta+\theta+C$ $=-\sqrt{9-(x-1)^2}+\sin^{-1}\frac{x-1}{3}+C$

(ii) (a)	
(ii) (b)	 $\text{Area} = -\int_1^{\frac{25}{4}} x\,\mathrm{d}y + \left(\frac{25}{4} - 1\right) \times \frac{8}{7}$ $= -\int_1^{\frac{5}{2}} \frac{2t}{\sqrt{9-(t-1)^2}}\,\mathrm{d}t + 6$ $= -2\left[-\sqrt{9-(t-1)^2} + \sin^{-1}\frac{t-1}{3}\right]_1^{\frac{5}{2}} + 6$ $= -2\left[(-\sqrt{\frac{27}{4}} + \sin^{-1}\frac{1}{2}) - (-\sqrt{9} + \sin^{-1}0)\right] + 6$ $= 3\sqrt{3} - \frac{\pi}{3}$
4(i)	$\frac{\mathrm{d}P}{\mathrm{d}t} = P(4-P) - h$ $= -P^2 + 4P - h$ $= -(P^2 - 4P) - h$ $= -(P-2)^2 + 4 - h$ 

(ii)	From graph in (i), when $h=4$, $\frac{dP}{dt}=-(P-2)^2=0$ at $P=2$. $\therefore$ largest $h=4$ where the population P remains constant at $P=2$. Hence $MSY=4$ million 
(iii)	$\frac{dP}{dt}=P(4-P)-3=-\left[(P^2-4P)+3\right]=-\left[(P-2)^2-1\right]$ <u>Method 1</u> $\int\frac{1}{(P-2)^2-1}\,dP=-\int 1\,dt$ $\frac{1}{2}\ln\left\|\frac{(P-2)-1}{(P-2)+1}\right\|=-t+C$ $\frac{P-3}{P-1}=\pm e^{-2t+2C}=Ae^{-2t}$ where $A=\pm e^{2C}$ In 2015, let $t=0$, $P=3.2$; hence $A=\frac{1}{11}$ $\therefore\ P-3=\frac{1}{11}e^{-2t}(P-1)$ $11P-33=Pe^{-2t}-e^{-2t}$ Hence $P=\frac{33-e^{-2t}}{11-e^{-2t}}=\frac{33e^{2t}-1}{11e^{2t}-1}$ <u>Method 2</u> $\frac{dP}{dt}=1-(P-2)^2$ $\int\frac{1}{1-(P-2)^2}\,dP=\int 1\,dt$ $\frac{1}{2}\ln\left\|\frac{1+(P-2)}{1-(P-2)}\right\|=t+C$ $\frac{P-1}{3-P}=\pm e^{2t+2C}=Ae^{2t}$ where $A=\pm e^{2C}$ In 2015, let $t=0$, $P=3.2$; hence $A=-11$ $P-1=-11e^{2t}(3-P)$ $P-1=-33e^{2t}+11Pe^{2t}$ $P=\frac{1-33e^{2t}}{1-11e^{2t}}$ or $P=\frac{33e^{2t}-1}{11e^{2t}-1}$

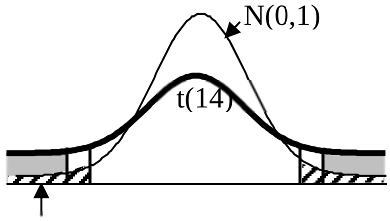
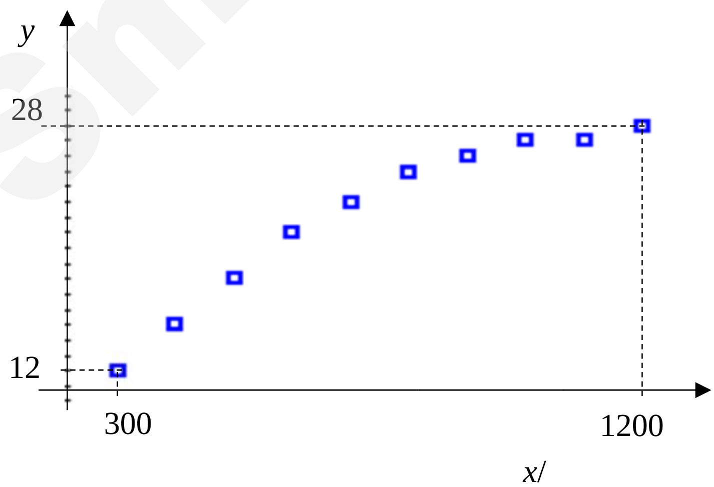
	Method 3 $-\int \frac{1}{P^2-4P+3}\,\mathrm{d}P = \int 1\,\mathrm{d}t$ $-\int \frac{1}{(P-3)(P-1)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$ $-\int \frac{1}{2(P-3)}\,\mathrm{d}P + \int \frac{1}{2(P-1)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$ (using partial fractions) $\frac{1}{2}\ln\left\|\frac{P-1}{P-3}\right\| = t + C$ $\frac{P-1}{P-3} = \pm e^{2t+2C} = Ae^{2t}$ In 2015, let $t=0$, $P=3.2$; hence $A=11$ $P-1 = 11e^{2t}(P-3)$ $P-1 = 11Pe^{2t} - 33e^{2t}$ $P = \frac{1-33e^{2t}}{1-11e^{2t}}$ or $P = \frac{33e^{2t}-1}{11e^{2t}-1}$ In 2016, $t=1$ Hence $P = \frac{33e^2-1}{11e^2-1} = 3.02$ $\therefore$ the population of wild salmon is 3.02 million in 2016.
(iv)	There are no external factors such as marine pollution or climate change that drastically affect the population of wild salmon in that region.

5 (i)	Simple Random Sampling: Using a random number generator to generate 400 numbers and use select the voting slips corresponding to these 400 numbers Systematic Sampling: Consider N registered voter in the electoral division such that the sampling interval $\frac{N}{400}$ is an integer. Using a random number generator, select a number from 1 to k and take every kth number thereafter until a sample of 400 is obtained. Choose the voting slips corresponding to the numbers. Stratified Sampling: Use each polling station as the stratum. The number of votes in each stratum is calculated by $\frac{\text{no. of voters polling at the station}}{\text{total number of voters in the electoral division}} \times 400$. The votes is then obtained from each stratum using simple random sampling.
(ii)	Simple random sampling: Advantage: The sample obtained is free from bias The sampling procedures are easy to follow Disadvantage The sample obtained might not be a good representation of the electoral division

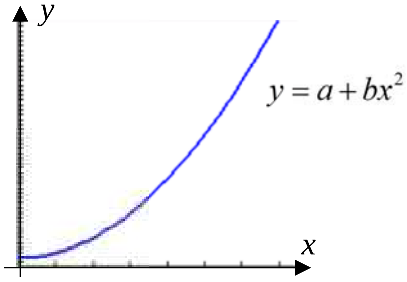
	Systematic Sampling: Advantage: It is easy execute because only the first number needs to be chosen The electoral division will be evenly sampled as the voting slips is chosen at regular intervals Disadvantage: If there is a periodic trend like every *k*th voters are of the same gender, systematic sampling may produce a biased sample Stratified Sampling: Advantage: Stratified sampling will provide a sample of voter that is representative of electoral division The results in each polling station can be analysed separately. Easy to conduct as the sampling frame (registered voters) is known. Disadvantage: It is time consuming to carry out stratified sampling
6	$\mathrm{P}(X=r-1)=\mathrm{P}(X=r)$ $\binom{n}{r-1}p^{r-1}(1-p)^{n-r+1}=\binom{n}{r}p^{r}(1-p)^{n-r}$ $\frac{n!}{(r-1)!(n-r+1)!}p^{r-1}(1-p)^{n-r+1}=\frac{n!}{(r)!(n-r)!}p^{r}(1-p)^{n-r}$ $\frac{r!(n-r)!}{(r-1)!(n-r+1)!}=\frac{p^{r}(1-p)^{n-r}}{p^{r-1}(1-p)^{n-r+1}}$ $\frac{r}{n-r+1}=\frac{p}{1-p}$ $\frac{\mathrm{P}(X=r)}{\mathrm{P}(X=r-1)}=\left(\frac{n-r+1}{r}\right)\frac{p}{1-p}$ (shown) $r(1-p)=p(n-r+1)$ $r-pr=np-pr+p$ $r=(n+1)p$ X will have two modes when $(n+1)p$ is a positive integer. $r=(n+1)p$, $r=(n+1)p-1$
7 (i)	No of ways that the single women are all separated $= {}^{6}C_{4}\times 4!\ \times(6-1)!$ =43200 Probability = $\frac{43200}{9!}=\frac{5}{42}=0.119$
(ii)	Probability that the single women are next to one another $=\ \mathrm{P}(S)=\frac{(7-1)!\ \times\ 4!}{9!}=\frac{1}{21}$ Probability that the single men are next to each other $=\ \mathrm{P}(B)=\frac{(9-1)!\ \times\ 2!}{9!}=\frac{2}{9}$

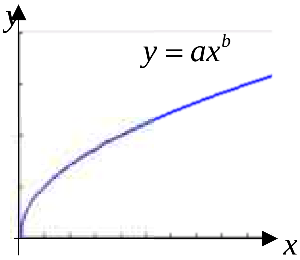
	Probability that the single women are next to one another and the single men are next to each other $= \mathrm{P}(S \cap B) = \dfrac{(6-1)!\ \times\ 2!\ \times\ 4!}{9!} = \dfrac{1}{63}$ Therefore probability $= \mathrm{P}(S) + \mathrm{P}(B) - 2\mathrm{P}(S \cap B) = \dfrac{1}{21} + \dfrac{2}{9} - \dfrac{2}{63} = \dfrac{5}{21} = 0.238$
	No of ways $= 9! \times 2! = 725760$
8 (i)	$A, B \sim \mathrm{N}(71, 8^2)$ $A - B \sim \mathrm{N}(0, 128)$ $\mathrm{P}(0 \le A - B \le 2) = 0.0702$ (3.s.f)
(ii)	$X \sim \mathrm{N}(62, \sigma^2)$, $Y \sim \mathrm{N}(71, 8^2)$ Let $M = \dfrac{X+Y}{2} \sim \mathrm{N}(66.5,\ \dfrac{\sigma^2 + 8^2}{4})$ $\mathrm{P}(M \ge 75) = 0.15$ $\mathrm{P}(M \le 75) = 0.85$ $\mathrm{P}(Z \le \dfrac{75 - 66.5}{\sqrt{\dfrac{\sigma^2 + 64}{4}}}) = 0.85$ $\dfrac{8.5}{\sqrt{\dfrac{\sigma^2 + 64}{4}}} = 1.03643338$ $\sigma = 14.319$ $\sigma = 14.3$
(iii)	$X - Y \sim \mathrm{N}(-9, 269.0389)$ $\mathrm{P}(X > Y) = \mathrm{P}(X - Y > 0)$ $= 0.292$ (3 s.f.)
(iv)	Not valid because X and Y are not be independent for the same student.
9(a)	Let X be the total number of speeding incidents caught at Junctions A and B in an hour. $X \sim \mathrm{Po}(\frac{23}{12})$ $\mathrm{P}(X \ge 2) = 1 - \mathrm{P}(X \le 1) = 0.571$
(b)	$A + B + C \sim\sim \mathrm{Po}\left(\dfrac{23}{12} + \lambda\right)$ Required Probability $= \mathrm{P}\left(C \ge 1 \mid A + B + C = 2\right)$ $= \dfrac{\mathrm{P}(A+B=1)\mathrm{P}(C=1) + \mathrm{P}(A+B=0)\mathrm{P}(C=2)}{\mathrm{P}(A+B+C=2)}$ $= \dfrac{\left(\dfrac{23}{12}\mathrm{e}^{-\frac{23}{12}}\right)\left(\lambda \mathrm{e}^{-\lambda}\right) + \left(\mathrm{e}^{-\frac{23}{12}}\right)\left(\mathrm{e}^{-\lambda}\dfrac{\lambda^2}{2}\right)}{\left(\mathrm{e}^{-\frac{23}{12}-\lambda}\right)\dfrac{\left(\dfrac{23}{12} + \lambda\right)^2}{2}}$

	$= \dfrac{\dfrac{e^{-\frac{23}{12}-\lambda}}{2}\left(\dfrac{23}{6}\lambda+\lambda^2\right)}{\dfrac{e^{-\frac{23}{12}-\lambda}}{2}\left(\dfrac{23}{12}+\lambda\right)^2}$ $= \dfrac{\dfrac{\lambda}{6}(23+6\lambda)}{\dfrac{1}{144}(23+12\lambda)^2}$ $= \dfrac{24\lambda(23+6\lambda)}{(23+12\lambda)^2}$
(c)	Let A be the number of speeding incidents caught at Junctions A, and B be the number of speeding incidents caught at Junction B in a day $A \sim \text{Po}(16)$, $B \sim \text{Po}(30)$ Since both 16 and 30 are greater than 10, $A \sim \text{N}(16,16)$ and $B \sim \text{N}(30,30)$ approximately $\Rightarrow A-B \sim \text{N}(-14,46)$ $\text{P}(A-B>0) \xrightarrow{c.c} \text{P}(A-B>0.5) = 0.0163$ The occurrence of speeding incidents caught at Junction A and Junction B are independent of each other.
10 (i)	Since n is small and population variance unknown, the nutritionist should use t-test. It is assumed that the calories count of the energy bar follows a normal distribution.
(ii)	$s^2 = \dfrac{15}{14}(20.74) = 22.221$ Let H_0 be the null hypothesis, H_1 be the alternative hypothesis. Let μ be the population mean number of calories in an energy bar and $\overline{X}$ be the sample mean. $\text{H}_0 : \mu = 350$ $\text{H}_1 : \mu \neq 350$ Under H_0, Test statistic, $T = \dfrac{\overline{X}-350}{\sqrt{\dfrac{22.221}{15}}} \sim t_{14}$ p value = 0.0373 < 0.05 Since the p – value < 0.05, reject H_0. There is sufficient evidence at 5% level of significance to conclude that the mean number of calories in an energy bar is not 350.
(iii)	Since H_0 is rejected for t-test, $p_t < 0.05$ and since $p_z < p_t < 0.05$. Therefore H_0 will be rejected under z test. So the conclusion will not be different.

	Alternative method  H_0 is rejected under t test $\Rightarrow$ test statistic is inside critical region for t-test $\Rightarrow$ test statistic is inside critical region for Z-test $\Rightarrow$ H_0 is rejected under Z test The conclusion would be the same.
(iv)	Since H_0 is not rejected, $\left\lvert \dfrac{\bar{x}-350}{\sqrt{1.3827}} \right\rvert < 1.95996$ $\Rightarrow -1.95996 < \dfrac{\bar{x}-350}{\sqrt{1.3827}} < 1.95996$ $-2.3047 < \bar{x}-350 < 2.3047$ $347.7 < \bar{x} < 352.3$
11(i)	$\bar{x} = 750$ $\bar{y} = 0.01758(750)+9.018$ $\quad = 22.203$ $22.203(10) = k+199$ $\therefore k = 23.03 \approx 23$
(ii)	
(iii)	$r = 0.958$. Though r is close to 1, the shape of the scatter plot is curvilinear therefore a linear model is not appropriate

(iv)





Model B is a more appropriate model as graph concave downwards like the scatter diagram

$\ln y = -0.96684 + 0.61722 \ln x$

$\ln y = -0.967 + 0.617 \ln x$

$\ln(24) = -0.96684 + 0.61722 \ln x$

$0.61722 \ln x = \ln(24) + 0.96684$

$x = 825$

Since $r = 0.982$ is close to 1 and $y = 24$ is within the data range, the prediction is appropriate

INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

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INDEX NUMBER	

Mathematics **9740/01**

Paper 1 **22 August 2016**

3 hours

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

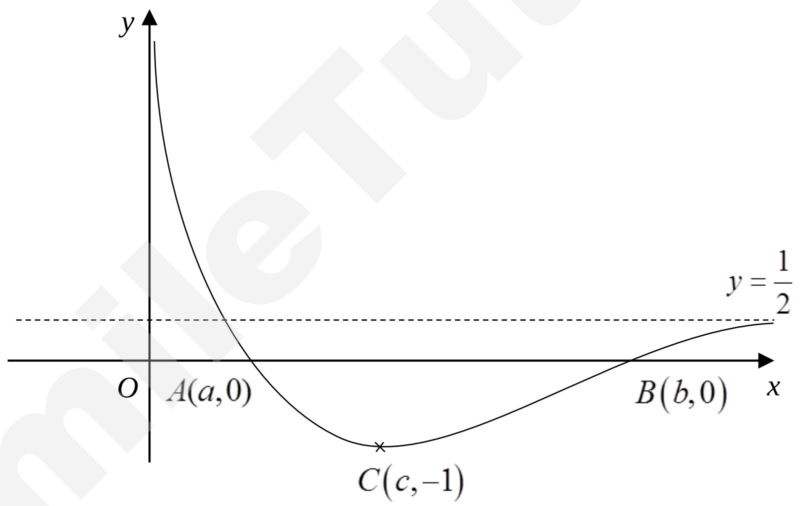
This document consists of **6** printed pages.

1 A theme park sells tickets at different prices according to the age of the customer. The age categories are senior citizen (ages 60 and above), adult (ages 13 to 59) and child (ages 4 to 12). Four tour groups visited the theme park on the same day. The numbers in each category for three of the groups, together with the total cost of the tickets for each of these groups, are given in the following table.

Group	Senior Citizen	Adult	Child	Total cost
A	2	19	9	\$1982
B	0	10	3	\$908
C	1	7	4	\$778

Find the total cost of the tickets for Tour Group *D*, which consists of four senior citizens, five adults and one child. [4]

2



The diagram shows the curve $y = \mathrm{f}(x)$. The curve passes through the point $A(a,0)$ and the point $B(b,0)$, has a turning point at $C(c,-1)$ and asymptotes $y = \frac{1}{2}$ and $x = 0$.

Sketch, on separate diagrams, the graphs of

(a) $y = 3 - |\mathrm{f}(x)|$, [3]

(b) $y = \dfrac{2}{\mathrm{f}(x)}$. [3]

Label the graph in each case clearly and indicate the equations of the asymptotes and the coordinates of the points corresponding to *A*, *B* and *C*.

3 In the triangle ABC, AB = 1, BC = 4 and angle ABC = θ radians. Given that θ is a sufficiently small angle, show that

$$AC \approx \left(9+4\theta^2\right)^{\frac{1}{2}} \approx a+b\theta^2 \ ,$$

for constants a and b to be determined. [5]

4 [It is given that the volume of a pyramid is $\frac{1}{3}\times(\text{base area})\times(\text{height})$.]

A right pyramid of vertical height h m has a square base with side of length $2x$ m and volume $\frac{8}{3}$ m^3.

(i) Express h in terms of x. [1]

(ii) Show that the surface area S m^2 of the pyramid is given by

$$S = 4x^2\left[\ 1+\sqrt{\left(1+\frac{4}{x^6}\right)}\ \right].$$ [3]

(iii) Use differentiation to find the value of x, correct to 2 decimal places, that gives a stationary value of S. [3]

5 Referred to the origin O, the points A and B are such that $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. The point C on OA is such OC : OA = 1 : 3. The line l passes through the points A and B. It is given that angle $BOA = 60^\circ$ and $|\mathbf{a}| = 3|\mathbf{b}|$.

(i) By considering $(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})$, or otherwise, express $|\mathbf{b}-\mathbf{a}|$ in the form $k|\mathbf{b}|$, where k is a constant to be found in exact form. [3]

(ii) Find, in terms of $|\mathbf{b}|$, the shortest distance from C to l. [5]

6 A curve has parametric equations

$$x = \cos^2 \theta, \quad y = \sin 2\theta, \quad \text{for } -\frac{\pi}{2} < \theta \le \frac{\pi}{2}.$$

(i) Sketch the curve. [2]

The region enclosed by the curve is denoted by R. The part of R above the x-axis is rotated through 2π radians about the x-axis.

(ii) Show that the volume of the solid formed is given by

$$\pi \int_a^b \sin^3 2\theta \ \mathrm{d}\theta,$$

for limits a and b to be determined. [3]

Use the substitution $u = \cos 2\theta$ to find this volume, leaving your answer in exact form. [4]

7 The equation of a curve C is given by

$$3y^3 - 8y^2 + 10y = 4 - 5x.$$

(i) Find the equation of the tangent at the point where $x = \frac{4}{5}$. [5]

(ii) Find the Maclaurin series for y, up to and including the term in x^2. [4]

(iii) State the equation of the tangent to the curve C at the point where $x = 0$. [1]

8 **(a)** The complex number w is given by $(\sqrt{3}) + k\mathrm{i}$, where $k < 0$.

Given that w^5 is real, find the possible values of k in the form $k = (\sqrt{3})\tan(n\pi)$, where n is a constant to be determined. [4]

(b) **(i)** If $z = \cos\theta + \mathrm{i}\sin\theta$, where $0 \le \theta \le \frac{\pi}{2}$, show that

$$1 - z^2 = 2\sin\theta(\sin\theta - \mathrm{i}\cos\theta).$$ [2]

(ii) Hence find $|1 - z^2|$ and $\arg(1 - z^2)$ in terms of θ. [3]

9 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{1}{x^2 - x - 6} + 2, \quad x \in \mathbb{R},\ x \neq -2,\ x \neq 3.$$

(i) Explain why the function f^{-1} does not exist. [2]

(ii) Find, algebraically, the set of values of x for which f is decreasing. [3]

In the rest of the question, the domain of f is further restricted to $x \leq \frac{1}{2}$.

The function g is defined by

$$\mathrm{g} : x \mapsto 2 - x,\ x \in \mathbb{R}.$$

(iii) Find an expression for $\mathrm{gf}(x)$ and hence, or otherwise, find $(\mathrm{gf})^{-1}\left(\frac{1}{4}\right)$. [3]

10 A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = \frac{1}{2}$ and

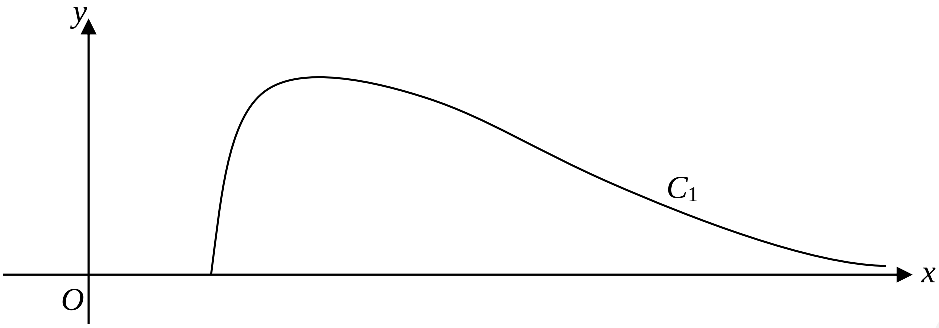
$$u_{n+1} = u_n - \frac{n^2 + n - 1}{(n+2)!}, \text{ for all } n \geq 1.$$

(i) Use the method of mathematical induction to prove that $u_n = \frac{n}{(n+1)!}$. [5]

(ii) Hence find $\sum_{n=1}^{N} \frac{n^2 + n - 1}{(n+2)!}$. [3]

(iii) Explain why $\sum_{n=1}^{\infty} \frac{n^2 + n - 1}{(n+2)!}$ is a convergent series, and state the value of the sum to infinity. [2]

11



The diagram above shows the curve C_1 with equation $y=\frac{\ln x}{x^2}$, where $x \geq 1$.

(i) Show that the exact coordinates of the turning point on C_1 are $\left(\sqrt{\mathrm{e}}, \frac{1}{2\mathrm{e}}\right)$. [3]

(ii) The curve C_2 has equation $\left(x-\sqrt{\mathrm{e}}\right)^2+\left(2\mathrm{e}y\right)^2=1$, where $y \geq 0$. Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes. [3]

(iii) Write down an integral that gives the area of the smaller region bounded by the two curves, C_1 and C_2, and the x-axis. Evaluate this integral numerically. [4]

12 **(a)** **(i)** Solve the equation

$$z^6-2\mathrm{i}=0,$$

giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [4]

(ii) Show the roots on an Argand diagram. [2]

(iii) The points A, B, C, D, E and F represent the roots z_1, z_2, z_3, z_4, z_5 and z_6 respectively in the Argand diagram. Find the perimeter of the polygon $ABCDEF$, leaving your answer to 3 decimal places. [2]

(b) The complex number w satisfies the relations

$$|w+5-12\mathrm{i}|\leq 13 \text{ and } 0\leq\arg\left(w+18-12\mathrm{i}\right)<\frac{\pi}{4}.$$

(i) On an Argand diagram, sketch the region in which the points representing w can lie. [4]

(ii) State the maximum and minimum possible values of $|w+10|$. [2]

H

INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME	

CIVICS GROUP		INDEX NUMBER	

Mathematics **9740/02**

Paper 2 **24 August 2016**

3 hours

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

Section A: Pure Mathematics [40 marks]

1 Without using a calculator, solve the inequality

$$\frac{3}{4x+3} \leq \frac{x}{x+1}.$$ [5]

Hence, or otherwise, solve the inequality

$$\frac{3}{4e^x+3} > \frac{e^x}{e^x+1}.$$ [2]

2 Analysts estimate that when a viral video is posted online, the video attracts comments in such a way that at the end of every hour, the number of comments added for the video is thrice the number of comments at the start of that hour.

In a particular instance, a viral video was posted online and there was one comment immediately after the video was posted. Using the above model proposed by analysts, there will be 3 additional comments by the end of the first hour, 12 additional comments by the end of the second hour, and so on.

(i) Find the number of complete hours for the total number of comments posted online to exceed 200 000. [3]

When the number of these comments posted online reaches 200 000 exactly, Software X is immediately activated to remove the comments. Software X works in such a way that it removes x comments at the start of each day. Once Software X is activated, it is also known that the number of comments at the end of the day is 2% more than the number of comments at the start of the day.

(ii) Show that the number of comments at the end of day n is

$$1.02^n(200\,000) - 51x\left(1.02^n - 1\right),$$

where day 1 is the day that the number of comments is exactly 200 000. [3]

(iii) Hence find the range of values of x such that all comments are removed by the end of day 30. Leave your answer to the nearest integer. [2]

Software Y is able to remove comments at the following rate.

- Day 1: 15 000 comments removed
- Subsequent Day: 90% of the number of comments removed on the preceding day

Without using Software X, explain whether Software Y alone is able to remove all 200 000 comments eventually. [2]

3 A team of naturalists is studying the change in population of wild boars on an island. It is suggested that the population of wild boars, x hundred, at time t years, can be modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{10}x(5-x).$$

(i) Find an expression for x in terms of t, given that $x = 1$ when $t = 0$. [7]

(ii) Find the exact time taken for the population of wild boars to reach 200. [2]

(iii) Explain in simple terms what will eventually happen to the population of wild boars on the island using this model. [1]

4 The line l has equation $\frac{x-1}{-2} = y = \frac{z+7}{4}$, and the plane p has equation $x - z = 2$.

(i) Find the acute angle between l and p. [3]

(ii) Find the coordinates of the point at which l intersects p. [3]

(iii) The perpendicular to p from the point with coordinates $(1, 0, -7)$ meets p at the point N. Find the position vector of N. [4]

(iv) Find a vector equation of the line which is a reflection of l in p. [3]

Section B: Statistics [60 marks]

5 A company wants to find out the transportation habits of their employees. On one particular workday, the interviewer selects a sample of employees to interview from those walking into the company building by

- standing at the entrance of building and choosing at random one of the first 10 employees who walks into the building,
- then choosing every 10th employee after the first employee is chosen.

(i) What is this type of sampling method called? [1]

(ii) State, in this context, a disadvantage of the sampling method stated in part **(i)**. [1]

(iii) Explain briefly how the interviewer could select a sample of 30 employees using quota sampling. [2]

6 Historical data shows that the number of goals scored per match at European Football Championships has a mean of 1.93 and a variance of 1.4. A large random sample of n matches is taken. Find the least value of n such that the probability that the average number of goals scored per match exceeds 2 goals is less than 0.24. [5]

7 A class of twenty four pupils consists of 11 girls and 13 boys. To form the class committee, four of the pupils are chosen at random as "Chairperson", "Vice Chairperson", "Treasurer" and "Secretary".

(i) Find the probability that the committee will consist of at least one girl and at least one boy. [3]

(ii) Find the probability that the "Treasurer" and "Secretary" are both girls. [3]

8 Under normal continuous use, the average battery life of a PI-99 calculator is claimed to be k hours . A random sample of 13 calculators were obtained, and the battery life, x hours, of each calculator was measured. The results are summarised by

$$\sum x = 573.39 \quad \text{and} \quad \sum (x - \bar{x})^2 = 42.22.$$

(i) Find unbiased estimates of the population mean and variance. [2]

A test is to be carried out at the 5% level of significance to determine if the claim made is valid.

(ii) State a necessary assumption to carry out the test. [1]

(iii) State the appropriate hypotheses for the test, defining any symbols that you use. [2]

(iv) Find the set of values of k for which the result of the test would be that the null hypothesis is not rejected. Leave all numerical answers in 2 decimal places. [3]

9 A roller-coaster ride has two separate safety systems to detect faults on the track and on the roller-coaster train itself. Over a long period of time, it is found that the average number of faults detected per day by the systems are 0.25 for the track and 0.15 for the train. Assume that the faults detected on the track are independent of those detected on the train.

(i) State, in this context, a condition that must be met for a Poisson distribution to be a suitable model for the number of faults occurring on a randomly chosen day. [1]

(ii) Find the probability that a total of at most 4 faults is detected by the two systems in a period of 10 days. [2]

(iii) Find the smallest number of days for which the probability that no fault is detected by the two systems is less than 0.05. [2]

(iv) Find the probability that, in a randomly chosen period of 10 days, there are at least 3 faults detected on the track, given that there are a total of at most 4 faults detected by the two systems. [3]

10 Alex and Ben play with each other a set of ten games at table tennis and for each game, the probability that Ben loses is 0.7.

(i) State, in this context, an assumption needed to use a binomial distribution to model the number of games that Ben loses. [1]

Assume that the assumption made in part **(i)** holds.

(ii) Find the probability that Ben loses more than half of the games. [2]

In order to improve his skills at table tennis, Ben attends an intensive training programme. After completing the training, Ben decides to play another set of n games with Alex. Assume that the number of games Ben loses, out of these n games, has the distribution $\mathrm{B}(n, 0.3)$.

(iii) Find the greatest value of n such that the probability that Ben loses more than 8 games is at most 0.01. [3]

(iv) Given that $n = 50$, use a suitable approximation to find the probability that the number of games Ben loses is between 10 and 20 inclusive. State the parameters of the distribution that you use. [3]

11 Research is being carried out into how the concentration of a drug in the bloodstream varies with time, measured from when the drug is given. Observations at successive times give the data shown in the following table.

Time (t minutes)	20	40	70	100	130	190	250
Concentration (m micrograms per litre)	85	62	51	33	29	14	6

(i) Draw a scatter diagram of these values, labelling the axes. Explain how you know from your diagram that the relationship between m and t should not be modelled by an equation of the form $m = a + bt$. [2]

It is thought that the concentration of the drug in the bloodstream at different times can be modelled by one of the formulae

$$m = ct^2 + d \quad \text{or} \quad m = e\ln t + f$$

where c, d, e and f are constants.

(ii) Find, correct to 4 decimal places, the product moment correlation coefficient between

(a) t^2 and m,

(b) $\ln t$ and m. [2]

(iii) Explain which of $m = ct^2 + d$ or $m = e\ln t + f$ is the better model and find the equation of a suitable regression line for this model. [3]

(iv) Use the equation of your regression line to estimate the concentration of the drug in the bloodstream when $t = 150$, correct to 2 decimal places. Comment on the reliability of the estimate obtained. [2]

12 Min Ho has just learnt how to use two different methods to mow a piece of lawn in his house garden.

Method A: This is a two-stage process that involves cutting the grass with a strimmer and then collecting the grass by raking it up. The time, X minutes, taken to cut the grass has the distribution $\mathrm{N}(30, 4.8^2)$. Once the grass is cut, the time, Y minutes, taken to collect the grass has the distribution $\mathrm{N}(20, 3.1^2)$.

Method B: This method uses a mower with a rechargeable battery that will cut and collect the grass at the same time. The time, S minutes, taken to do this has the distribution $\mathrm{N}(38, 2.6^2)$. In addition to this, the battery has to be recharged once before the cut, and this time is fixed at 15 minutes.

(i) Find the probability that Min Ho takes more than 45 minutes to mow the lawn using Method A. [3]

(ii) Find the probability that using Method A to mow the lawn is faster than using Method B by more than 5 minutes. [4]

Assume that Min Ho mows the piece of lawn in his house garden on a weekly basis. Over a particular period of ten consecutive weeks, Min Ho uses Method A for the first four weeks and Method B for the next six weeks. Find the probability the average time taken to mow the lawn in a week is greater than 50 minutes. [4]

ANNEX B

IJC H2 Preliminary Examination (Paper 1)

Qn/No	Topic Set	Answers
1	System of linear equations	\$570
2	Further Curve Sketching	------
3	Small angle approximation, Binomial expansion	$AC \approx 3+\frac{2}{3}\theta^2$, where $a=3$ and $b=\frac{2}{3}$
4	Application of differentiation (Stationary value)	(i) $h=\frac{2}{x^2}$ (iii) 0.89
5	Vectors (scalar and cross-product)	(i) $\lvert\mathbf{b}-\mathbf{a}\rvert=\sqrt{7}\lvert\mathbf{b}\rvert$ (ii) $\sqrt{\frac{3}{7}}\lvert\mathbf{b}\rvert$
6	Application of Integration (Volume of revolution)	(ii) $a=0,\quad b=\frac{\pi}{2}$ $\frac{2}{3}\pi$ units^3
7	Application of differentiation (Tangent & Normal), Maclaurin Series	(i) $y=-\frac{1}{2}x+\frac{2}{5}$ (ii) $y=\frac{2}{3}-\frac{3}{2}x+\frac{27}{20}x^2+\ldots$ (iii) $y=\frac{2}{3}-\frac{3}{2}x$
8	Complex numbers	(a) $k=\sqrt{3}\tan\left(-\frac{\pi}{5}\right)$ or $k=\sqrt{3}\tan\left(-\frac{2\pi}{5}\right)$ (b)(ii) $2\sin\theta$; $\theta-\frac{\pi}{2}$
9	Functions	(ii) $\{x\in\mathbb{R}:\ x\geq 0.5,\ x\neq 3\}$ (iii) $\mathrm{gf}(x)=-\frac{1}{x^2-x-6},\quad x\leq\frac{1}{2}$; -1
10	Mathematical Induction, Sequence & Series (M.O.D.)	(ii) $\frac{1}{2}-\frac{N+1}{(N+2)!}$ (iii) $\sum_{n=1}^{\infty}\frac{n^2+n-1}{(n+2)!}\to\frac{1}{2}$ which is a constant, hence it is a convergent series. $S_\infty=\frac{1}{2}$

11	Application of differentiation (Stationary point), Curve Sketching, Application of Integration (Area)	(iii) $\int_{\sqrt{e}-1}^{\sqrt{e}} \frac{\sqrt{1-\left(x-\sqrt{e}\right)^2}}{2e}\,dx - \int_1^{\sqrt{e}} \frac{\ln x}{x^2}\,dx$; 0.0543
12	Complex numbers (including Loci)	(a)(i) $2^{\frac{1}{6}}e^{i\frac{-11\pi}{12}}, 2^{\frac{1}{6}}e^{i\frac{-7\pi}{12}}, 2^{\frac{1}{6}}e^{i\frac{-\pi}{4}}, 2^{\frac{1}{6}}e^{i\frac{\pi}{12}}, 2^{\frac{1}{6}}e^{i\frac{5\pi}{12}}, 2^{\frac{1}{6}}e^{i\frac{3\pi}{4}}$ (a)(iii) 6.735 (b)(ii) Maximum $\lvert w+10\rvert = 26$; Minimum $\lvert w+10\rvert = 12$

ANNEX B

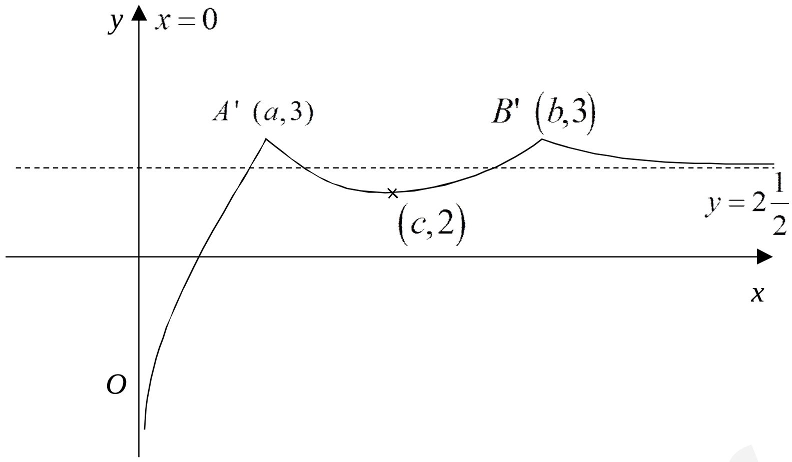
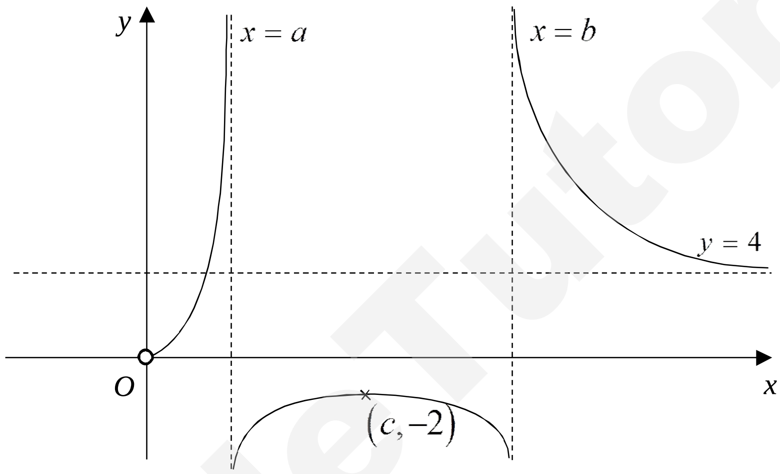
IJC H2 Preliminary Examination (Paper 2)

Qn/No	Topic Set	Answers
1	Inequalities	$x < -1$ or $-\frac{\sqrt{3}}{2} \le x < -\frac{3}{4}$ or $x \ge \frac{\sqrt{3}}{2}$; $x < \ln\left(\frac{\sqrt{3}}{2}\right)$
2	AP and GP	(i)9 (iii) $x \ge 8755$ (last part) Software Y is unable to remove all the comments because eventually it is only able to remove 150 000 comments.
3	Differential Equations	(i) $x = \frac{5e^{\frac{1}{2}t}}{4 + e^{\frac{1}{2}t}}$ (ii) $2\ln\left(\frac{8}{3}\right)$ years (iii) The population of wild boars will increase and stabilise at 500 eventually.
4	Vectors (Lines and Planes)	(i) $\theta = 67.8°$ (ii) $(-1,1,-3)$ (iii) $\begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}$ (iv) $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}, \alpha \in \mathbb{R}$
5	Sampling Methods	(i) Systematic sampling (ii)(slower, more difficult to collect) Systematic sampling is a more tedious process to select the employees, whereas quota sampling is quick and easy.
6	Sampling distribution (Central Limit Theorem)	143
7	Probability	(i) 0.902 (ii) 0.199
8	Hypothesis Testing	(i) 44.1; 3.52 (ii) The battery life of a PI-99 calculator is assumed to be normally distributed.

		(iii) $H_0: \quad \mu = k$ $H_1: \quad \mu \neq k$ (iv) $\{k \in \mathbb{R} : 42.97 < k < 45.24\}$
9	Poisson Distribution	(i) The average number of faults detected by each system (for the track and the train) is constant from one day to another. (ii) 0.629 (iii) 8 (iv) 0.237
10	Binomial Distribution	(i) Ben's performance (i.e. whether he loses or wins) in a game is independent of any other games that he plays with Alex. (ii) 0.850 (iii) 14 (iv) 0.910
11	Correlation & Regression	(ii)(a) -0.8454 (ii)(b) -0.9961 (iii) m and $\ln t$ is the better model; $m = 179 - 31.2\ln t$ (iv) 22.61 micrograms per litre; The estimate obtained is reliable, because the given value of t = 150 lies within the given sample data range for t and the product moment correlation coefficient between m and $\ln t$ is very close to -1, hence indicating a strong negative linear correlation between the variables m and $\ln t$.
12	Normal Distribution	(i) 0.809 (ii) 0.375 (last part) 0.916

Innova Junior College
H2 Mathematics
JC2 Preliminary Examinations Paper 1
Solutions

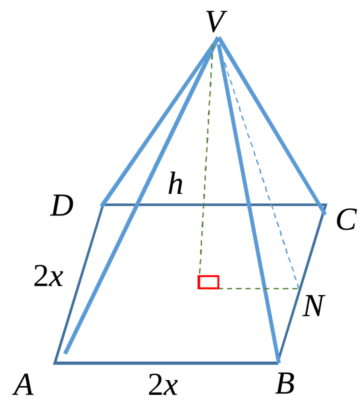
1	**Solution**
	Let \$$x$, \$$y$ and \$$z$ be the cost of a ticket for a senior citizen, adult and child respectively. $2x+19y+9z=1982$ $10y+3z=908$ $x+7y+4z=778$ Using GC, $x=36$ $y=74$ $z=56$ Thus, the cost of a ticket for a senior citizen is \$36, for an adult is \$74 and for a child is \$56. 4(36) + 5(74) + 1(56) = 570 Therefore, the total cost for Group *D* = \$570

2	**Solution**
(a)	
(b)	

3	**Solution**
	Using cosine rule, $AC^2 = 1^2 + 4^2 - 2(1)(4)\cos\theta$ $= 1 + 16 - 8\cos\theta$ $= 17 - 8\cos\theta$ $\approx 17 - 8\left(1 - \frac{\theta^2}{2}\right)$ $= 9 + 4\theta^2$ $AC \approx \left(9 + 4\theta^2\right)^{\frac{1}{2}} \quad (\because\ AC > 0)$ $AC \approx \left(9 + 4\theta^2\right)^{\frac{1}{2}}$ $= 9^{\frac{1}{2}}\left(1 + \frac{4}{9}\theta^2\right)^{\frac{1}{2}}$ $= 3\left(1 + \frac{1}{2}\left(\frac{4}{9}\theta^2\right) + \ldots\right)$ $\approx 3\left(1 + \frac{2}{9}\theta^2\right)$ $= 3 + \frac{2}{3}\theta^2$ Therefore, $a = 3$ and $b = \frac{2}{3}$

4	Solution

(i)



Volume of the pyramid $= \dfrac{8}{3}$

$$\Rightarrow \frac{1}{3}(2x)^2 h = \frac{8}{3}$$

$$\Rightarrow h = \frac{2}{x^2}$$

(ii)

In triangle VBC, height $= VN = \sqrt{h^2 + x^2}$

Area of the triangle $VBC = \dfrac{1}{2}(2x)\sqrt{h^2 + x^2}$

$$= x\sqrt{\frac{4}{x^4} + x^2} = x\sqrt{x^2\left(\frac{4}{x^6} + 1\right)}$$

$$= x^2\sqrt{1 + 4x^{-6}}$$

Hence total surface area of the pyramid,

$$S = \text{base area} + 4 \times \text{area of triangle } VBC$$

$$= (2x)^2 + 4\left(x^2\sqrt{1 + 4x^{-6}}\right)$$

$$S = 4x^2\left[1 + \sqrt{\left(1 + \frac{4}{x^6}\right)}\right] \quad \text{(shown)}$$

(iii)

$$S = 4x^2\left[1 + \sqrt{(1 + 4x^{-6})}\right]$$

$$\frac{dS}{dx} = 4x^2\left(\frac{1}{2}\left(1 + 4x^{-6}\right)^{-\frac{1}{2}}\left(-24x^{-7}\right)\right) + (8x)\left[1 + \sqrt{(1 + 4x^{-6})}\right]$$

$$= -48x^{-5}\left(1 + 4x^{-6}\right)^{-\frac{1}{2}} + 8x\left[1 + \sqrt{(1 + 4x^{-6})}\right]$$

$$= -8x\left[6x^{-6}\left(1 + 4x^{-6}\right)^{-\frac{1}{2}} - 1 - \sqrt{(1 + 4x^{-6})}\right]$$

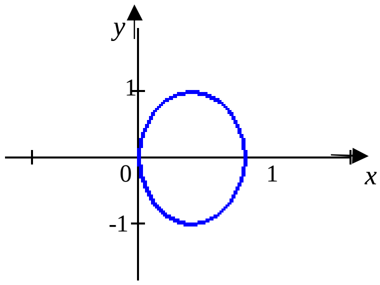
At the stationary value of S, $\dfrac{dS}{dx} = 0$.

	$\therefore\quad -8x\left[6x^{-6}\left(1+4x^{-6}\right)^{-\frac{1}{2}}-1-\sqrt{(1+4x^{-6})}\right]=0$ By G.C., $x=0.89090=0.89$ (to 2dp)

5	**Solution**
(i)	$(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \lvert\mathbf{b}\rvert^2 + \lvert\mathbf{a}\rvert^2 - 2\mathbf{a}.\mathbf{b}$ $= \lvert\mathbf{b}\rvert^2 + 9\lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 60^\circ$ $\lvert\mathbf{b}-\mathbf{a}\rvert^2 = 10\lvert\mathbf{b}\rvert^2 - 2\left(3\lvert\mathbf{b}\rvert\right)\lvert\mathbf{b}\rvert\frac{1}{2}$ $\lvert\mathbf{b}-\mathbf{a}\rvert = \sqrt{7}\,\lvert\mathbf{b}\rvert$ Therefore, $k=\sqrt{7}$.
(ii)	$\mathbf{c} = \frac{1}{3}\mathbf{a}$ $\overrightarrow{CA} = \frac{2}{3}\mathbf{a}$ Shortest distance of C to l = $\frac{\left\lvert\frac{2}{3}\mathbf{a}\times(\mathbf{b}-\mathbf{a})\right\rvert}{\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \frac{\left\lvert\frac{2}{3}\mathbf{a}\times\mathbf{b} - \frac{2}{3}\mathbf{a}\times\mathbf{a}\right\rvert}{\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \frac{2\lvert\mathbf{a}\times\mathbf{b}\rvert}{3\lvert\mathbf{b}-\mathbf{a}\rvert} \qquad \because \mathbf{a}\times\mathbf{a} = \mathbf{0}$ $= \frac{2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin 60^\circ}{3\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \frac{6\lvert\mathbf{b}\rvert^2\frac{\sqrt{3}}{2}}{3\sqrt{7}\,\lvert\mathbf{b}\rvert}$ $= \frac{\sqrt{3}\lvert\mathbf{b}\rvert}{\sqrt{7}}$ $= \sqrt{\frac{3}{7}}\lvert\mathbf{b}\rvert$

<table>
<tr><td>6</td><td>Solution</td></tr>
<tr><td>(i)</td><td>

$$x=\cos^2\theta,\quad y=\sin 2\theta,\quad \text{for } -\frac{\pi}{2}<\theta\le\frac{\pi}{2}.$$



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<tr><td>(ii)</td><td>

$x=\cos^2\theta$

$\mathrm{d}x=-2\cos\theta\sin\theta\,\mathrm{d}\theta$

When $y=0,\quad \sin 2\theta=0$

$2\theta=0,\pi$

$\theta=0,\dfrac{\pi}{2}$

When $\theta=0,\quad x=1;$ When $\theta=\dfrac{\pi}{2},\quad x=0$

Volume of the solid formed

$=\pi\int_0^1 y^2\,\mathrm{d}x$

$=\pi\int_{\frac{\pi}{2}}^{0}(\sin 2\theta)^2(-2\cos\theta\sin\theta\,\mathrm{d}\theta)$

$=\pi\int_{\frac{\pi}{2}}^{0}(\sin 2\theta)^2(-\sin 2\theta\,\mathrm{d}\theta)$

$=\pi\int_0^{\frac{\pi}{2}}\sin^3 2\theta\,\mathrm{d}\theta$ (shown)

$\therefore\ a=0,\quad b=\dfrac{\pi}{2}.$

Let $u=\cos 2\theta$.

$\therefore \mathrm{d}u=-2\sin 2\theta\,\mathrm{d}\theta$

When $\theta=0,\quad u=1;$

When $\theta=\dfrac{\pi}{2},\quad u=-1$

Volume of the solid formed

$=\pi\int_0^{\frac{\pi}{2}}\sin^3 2\theta\,\mathrm{d}\theta$

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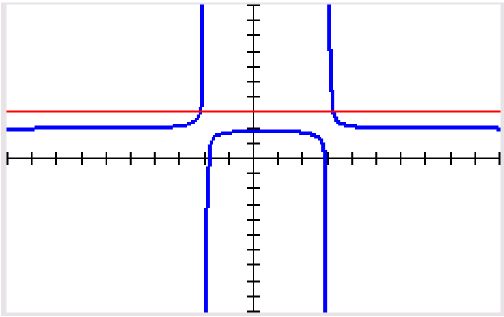
	$=\frac{1}{2}\pi\int_0^{\frac{\pi}{2}}\sin^2 2\theta\left(2\sin 2\theta\right)\mathrm{d}\theta$ $=\frac{1}{2}\pi\int_0^{\frac{\pi}{2}}\left(1-\cos^2 2\theta\right)\left(2\sin 2\theta\,\mathrm{d}\theta\right)$ $=\frac{1}{2}\pi\int_1^{-1}\left(1-u^2\right)\left(-\mathrm{d}u\right)$ $=-\frac{1}{2}\pi\left[u-\frac{u^3}{3}\right]_1^{-1}$ $=-\frac{1}{2}\pi\left[\left(-1+\frac{1}{3}\right)-\left(1-\frac{1}{3}\right)\right]$ $=-\frac{1}{2}\pi\left(-\frac{4}{3}\right)$ $=\frac{2}{3}\pi \text{ units}^3$

7	**Solution**
(i)	$3y^3-8y^2+10y=4-5x$ Differentiate wrt x, $\left(9y^2-16y+10\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-5$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-5}{9y^2-16y+10}$ When $x=\frac{4}{5}$, $3y^3-8y^2+10y=0$. $\therefore y=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}$. Eqn of tangent : $y-0=-\frac{1}{2}\left(x-\frac{4}{5}\right)$, ie $y=-\frac{1}{2}x+\frac{2}{5}$
(ii)	$\left(9y^2-16y+10\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-5$ Differentiate wrt x, $\left(9y^2-16y+10\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(18y-16\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$. When $x=0$, $y=\frac{2}{3},\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3}{2},\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{27}{10}$. $\therefore \quad y=\frac{2}{3}-\frac{3}{2}x+\frac{27}{20}x^2+\ldots$
(iii)	$y=\frac{2}{3}-\frac{3}{2}x$

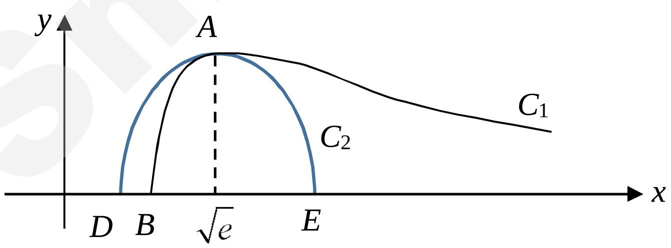
8	**Solution**
(a)	$\arg\left(w^5\right)=5\arg\left(w\right)=0,\pm\pi,\pm2\pi\ldots$ $\arg\left(w\right)=0,\frac{\pi}{5},-\frac{\pi}{5},\frac{2\pi}{5},\frac{-2\pi}{5},\ldots$ Since $k<0$, $\arg\left(w\right)=-\frac{\pi}{5}$ or $-\frac{2\pi}{5}$. $\frac{k}{\sqrt{3}}=\tan\left(-\frac{\pi}{5}\right)$ or $\frac{k}{\sqrt{3}}=\tan\left(-\frac{2\pi}{5}\right)$ $k=\sqrt{3}\tan\left(-\frac{\pi}{5}\right)$ or $k=\sqrt{3}\tan\left(-\frac{2\pi}{5}\right)$ $n=-\frac{1}{5}$ or $-\frac{2}{5}$
(bi)	**Method 1** $1-z^2=1-\left(\cos\theta+\mathrm{i}\sin\theta\right)^2$ $=1-\left(\cos^2\theta+2\mathrm{i}\cos\theta\sin\theta+\left(\mathrm{i}\sin\theta\right)^2\right)$ $=1-\left(1-\sin^2\theta+2\mathrm{i}\sin\theta\cos\theta-\sin^2\theta\right)$ $=1-1+2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$ **Method 2** $1-z^2=1-\left(\cos\theta+\mathrm{i}\sin\theta\right)^2$ $=1-\left(\cos2\theta+\mathrm{i}\sin2\theta\right)$ $=1-\cos2\theta-\mathrm{i}\sin2\theta$ $=1-\left(1-2\sin^2\theta\right)-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$
(bii)	**Method 1** $\left\|1-z^2\right\|=\left\|2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)\right\|$ $=2\sin\theta\sqrt{\sin^2\theta+\cos^2\theta}$ $=2\sin\theta$ Given that $0\le\theta\le\frac{\pi}{2}$

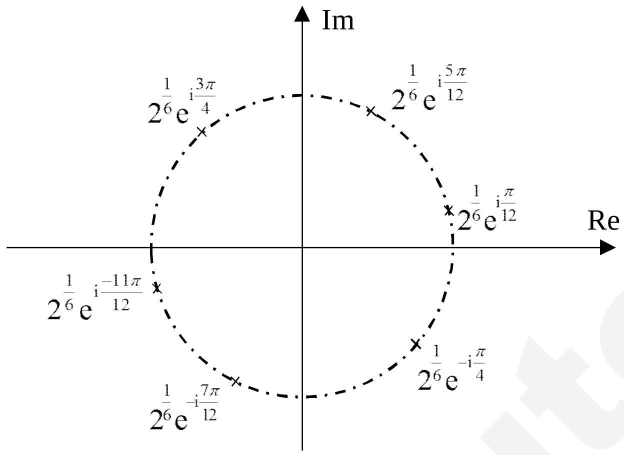
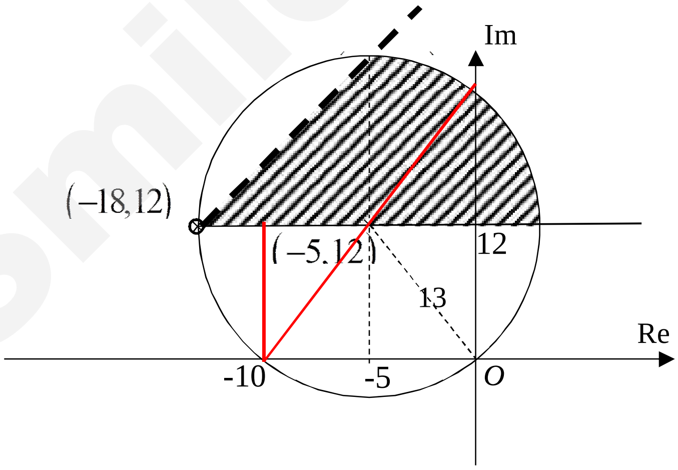
	$\arg\left(1-z^2\right)=\arg\left[2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)\right]$ $=\arg\left(2\sin\theta\right)+\arg\left(\sin\theta-\mathrm{i}\cos\theta\right)$ $=0-\tan^{-1}\left(\dfrac{\cos\theta}{\sin\theta}\right)$ $=-\tan^{-1}\left(\tan\left(\dfrac{\pi}{2}-\theta\right)\right)$ $=-\left(\dfrac{\pi}{2}-\theta\right)$ $=\theta-\dfrac{\pi}{2}$ **Method 2** $1-z^2=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$ $=2\sin\theta\left(-\mathrm{i}\right)\left(\cos\theta+\mathrm{i}\sin\theta\right)$ $=\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}$ $\left\|1-z^2\right\|=\left\|\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}\right\|$ $=2\sin\theta$ $\arg\left(1-z^2\right)=\arg\left(\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}\right)$ $=\arg\left(-2\mathrm{i}\sin\theta\right)+\arg\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ $=-\dfrac{\pi}{2}+\theta$ **Method 3** $1-z^2=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$ $=2\sin\theta\left(\cos\left(\dfrac{\pi}{2}-\theta\right)-\mathrm{i}\sin\left(\dfrac{\pi}{2}-\theta\right)\right)$ $=2\sin\theta\left(\cos\left(\theta-\dfrac{\pi}{2}\right)+\mathrm{i}\sin\left(\theta-\dfrac{\pi}{2}\right)\right)$ $\left\|1-z^2\right\|=2\sin\theta$ $\arg\left(1-z^2\right)=\theta-\dfrac{\pi}{2}$

9	Solution
(i)	 From graph, the horizontal line $y = 3$ cuts the graph at two points. Hence f is not a **one-one** function, hence f^{-1} does not exist.
(ii)	$f(x) = \frac{1}{x^2 - x - 6} + 2$ $f'(x) = -(x^2 - x - 6)^{-2}(2x - 1)$ $= \frac{1 - 2x}{(x^2 - x - 6)^2}$ For the function to be decreasing, $f'(x) \leq 0$. $1 - 2x \leq 0$ $1 \leq 2x$ $x \geq 0.5$ $\{x \in \mathbb{R}: x \geq 0.5, x \neq 3\}$
(iii)	$gf(x) = 2 - \frac{1}{x^2 - x - 6} - 2$ $= -\frac{1}{x^2 - x - 6}, \quad x \leq \frac{1}{2}$ $(gf)^{-1}\left(\frac{1}{4}\right) = x$ $gf(x) = \frac{1}{4}$ $-\frac{1}{x^2 - x - 6} = \frac{1}{4}$ $x^2 - x - 6 = -4$ $x^2 - x - 2 = 0$ $(x - 2)(x + 1) = 0$ $x = 2$ (rejected) or $x = -1$ $x = -1$

10	**Solution**
(i)	Let P_n be the statement $u_n = \dfrac{n}{(n+1)!}$ for $n \in \mathbb{Z}^+$. P_1 is true since $u_1 = \dfrac{1}{2!} = \dfrac{1}{2}$. Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \dfrac{k}{(k+1)!}$ Consider P_{k+1}: i.e. $u_{k+1} = \dfrac{k+1}{(k+2)!}$ $u_{k+1} = \dfrac{k}{(k+1)!} - \dfrac{k^2+k-1}{(k+2)!}$ $= \dfrac{k(k+2)-(k^2+k-1)}{(k+2)!}$ $= \dfrac{k^2+2k-k^2-k+1}{(k+2)!}$ $= \dfrac{k+1}{(k+2)!}$ Thus, P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
(ii)	$\displaystyle\sum_{n=1}^{N} \frac{n^2+n-1}{(n+2)!} = \sum_{n=1}^{N} (u_n - u_{n+1})$ $[\, u_1 - \cancel{u_2}$ $+ \cancel{u_2} - \cancel{u_3}$ $+ \cancel{u_3} - \cancel{u_4}$ $+ \cancel{\ldots}$ $+ \cancel{u_{N-1}} - \cancel{u_N}$ $+ \cancel{u_N} - u_{N+1} \,]$ $= u_1 - u_{N+1} = \dfrac{1}{2} - \dfrac{N+1}{(N+2)!}$

(iii)	As $N \to \infty$, $\dfrac{N+1}{(N+2)!} \to 0$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n-1}{(n+2)!} \to \frac{1}{2}$ which is a constant, hence it is a convergent series.

11	**Solution**
(i)	$y=\dfrac{\ln x}{x^2},\ x\ge 1$ $\dfrac{dy}{dx}=\dfrac{x^2\cdot\frac{1}{x}-\ln x\cdot 2x}{x^4}$ $=\dfrac{x-2x\ln x}{x^4}$ When $\dfrac{dy}{dx}=0$ and since $x\neq 0$, $1-2\ln x=0$ $2\ln x=1$ $\ln x=\dfrac{1}{2}$ $x=e^{\frac{1}{2}}=\sqrt{e}$ When $x=\sqrt{e}$, $y=\dfrac{\ln\sqrt{e}}{\left(\sqrt{e}\right)^2}$ $=\dfrac{1}{2e}$ Hence the coordinates of A is $\left(\sqrt{e},\dfrac{1}{2e}\right)$.
(ii)	$B(1,0)$, $D\left(\sqrt{e}-1,0\right)$ and $E\left(\sqrt{e}+1,0\right)$ 
(iii)	Area $=\displaystyle\int_{\sqrt{e}-1}^{\sqrt{e}}\frac{\sqrt{1-\left(x-\sqrt{e}\right)^2}}{2e}\,dx-\int_1^{\sqrt{e}}\frac{\ln x}{x^2}\,dx$ $=0.14446942-0.09020401$ $=0.05426541$ $=0.0543$ (correct to 3 s.f.)

12	Solution
(ai)	$z^6 - 2\mathrm{i} = 0$ $z^6 = 2\mathrm{i}$ $z^6 = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2k\pi\right)},\ k = 0, \pm 1, \pm 2, -3$ $z = 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{-11\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{-7\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{-\pi}{4}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{5\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$
(aii)	
(aiii)	Since *ABCDEF* is a regular hexagon, the triangles *OAB*, *OBC* … are equilateral triangles. Perimeter of the polygon $= 6 \times 2^{\frac{1}{6}}$ $= 6.735$ (to 3 d.p.)
(bi)	
(bii)	Minimum $\lvert w+10 \rvert = 12$ Maximum $\lvert w+10 \rvert = 26$ (diameter of circle)

Innova Junior College
H2 Mathematics
JC2 Preliminary Examinations Paper 2
Solutions

<table>
<tr><th>1</th><th>Solution</th></tr>
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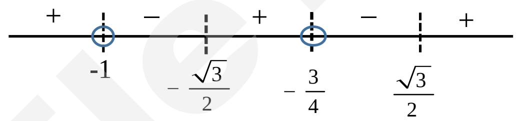
$$\frac{3}{4x+3} \le \frac{x}{x+1}$$
$$\frac{3}{4x+3} - \frac{x}{x+1} \le 0$$
$$\frac{3x+3-4x^2-3x}{(4x+3)(x+1)} \le 0$$
$$\frac{-4x^2+3}{(4x+3)(x+1)} \le 0 \quad ---(*)$$
$$\frac{4x^2-3}{(4x+3)(x+1)} \ge 0$$
$$\frac{\left(2x-\sqrt{3}\right)\left(2x+\sqrt{3}\right)}{(4x+3)(x+1)} \ge 0$$



Hence $x < -1$ or $-\frac{\sqrt{3}}{2} \le x < -\frac{3}{4}$ or $x \ge \frac{\sqrt{3}}{2}$

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For $\frac{3}{4e^x+3} > \frac{e^x}{e^x+1}$, making use of the result in above part,

$-1 < e^x < -\frac{\sqrt{3}}{2}$ or $-\frac{3}{4} < e^x < \frac{\sqrt{3}}{2}$

(no solns since e^x is always positive)

Hence, $e^x < \frac{\sqrt{3}}{2} \Rightarrow x < \ln\left(\frac{\sqrt{3}}{2}\right)$

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2 **Solution**

(i)

Hour	Start of hour	End of hour
1	1	1 + 3 = 4
2	4	4 + 12 = 16
3	16	16 + 48 = 64
4	…	…

$$4(4)^{n-1} > 200\,000$$
$$(4)^{n-1} > 50000$$
$$n-1 > 7.80482$$
$$n > 8.80842$$

Number of complete hours = 9

<u>Alternative Solution</u>

$$4(4)^{n-1} > 200\,000$$

n	Total
8	65536 < 200 000
9	262144 > 200 000
10	11048576 > 200 000

Number of complete hours = 9

(ii)

Day	Start of day	End of day
1	$200000-x$	$1.02(200000-x)$
2	$1.02(200000-x)$ $-x$	$1.02[1.02(200000-x)$ $-x]$ $=1.02^2(200000)$ $-1.02x-1.02^2x$
3	…	…

At the end of day n, the number comments

$=1.02^n(200000)-\left(1.02x+1.02^2x+...+1.02^nx\right)$ --- (*)

	$=1.02^n(200000)-x\left(\dfrac{1.02(1.02^n-1)}{0.02}\right)$ $=1.02^n(200000)-51x(1.02^n-1)$
(iii)	$1.02^{30}(200\,000)-51x(1.02^{30}-1)<0$ $x>\dfrac{1.02^{30}(200\,000)}{51(1.02^{30}-1)}$ $x\geq 8755$ (to nearest integer)
	Day 1: no. of comments removed $=15000$ Day 2: no. of comments removed $=15000(0.9)$ Day 3: no. of comments removed $=15000(0.9)^2$ As $n\to\infty$, no. of comments removed $=\dfrac{15000}{1-0.9}=150\,000$ Software Y is unable to remove all the comments because eventually it is only able to remove 150 000 comments.

3	**Solution**
(i)	Method 1: $\displaystyle\int \frac{1}{x(5-x)}\,\mathrm{d}x = \int \frac{1}{10}\ \mathrm{d}t \quad ---(*)$ Doing partial fractions $\displaystyle\frac{1}{x(5-x)} = \frac{A}{x} + \frac{B}{5-x}$ $\displaystyle= \frac{A(5-x)+B(x)}{x(5-x)}$ $A = \frac{1}{5}$ $B = \frac{1}{5}$ $\displaystyle\int \frac{1}{5x} + \frac{1}{5(5-x)}\,\mathrm{d}P = \int \frac{1}{10}\ \mathrm{d}t$ $\displaystyle\frac{1}{5}\left[\ln\lvert x\rvert - \ln\lvert 5-x\rvert\right] = \frac{1}{10}t + c$ $\displaystyle\ln\left\lvert \frac{x}{5-x} \right\rvert = \frac{1}{2}t + c$ $\displaystyle\frac{x}{5-x} = A\mathrm{e}^{\frac{1}{2}t}$, where $A = \pm \mathrm{e}^{c}$ Given $x = 1$ when $t = 0$, $\displaystyle\frac{1}{5-1} = A\mathrm{e}^{0} \Rightarrow A = \frac{1}{4}$ $\displaystyle x = \frac{5}{4}\mathrm{e}^{\frac{1}{2}t} - \frac{1}{4}x\mathrm{e}^{\frac{1}{2}t}$ $x(4+\mathrm{e}^{\frac{1}{2}t}) = 5\mathrm{e}^{\frac{1}{2}t}$ $\displaystyle x = \frac{5\mathrm{e}^{\frac{1}{2}t}}{4+\mathrm{e}^{\frac{1}{2}t}}$

(i)	Method 2: $$\int \frac{1}{x(5-x)}\,\mathrm{d}x = \int \frac{1}{10}\ \mathrm{d}t \quad ---(*)$$ $$\int \frac{1}{\frac{25}{4}-(x-\frac{5}{2})^2}\,\mathrm{d}x = \int \frac{1}{10}\ \mathrm{d}t$$ $$\frac{1}{2\left(\frac{5}{2}\right)}\ln\left\|\frac{\frac{5}{2}+(x-\frac{5}{2})}{\frac{5}{2}-(x-\frac{5}{2})}\right\| = \frac{1}{10}t+c$$ $$\ln\left\|\frac{x}{5-x}\right\| = \frac{1}{2}t+c$$ $$\frac{x}{5-x} = A\mathrm{e}^{\frac{1}{2}t}\text{ , where } A=\mathrm{e}^{\pm c}$$ Given $x=1$ when $t=0$, $\frac{1}{5-1}=A\mathrm{e}^0 \Rightarrow A=\frac{1}{4}$ $$x=\frac{5}{4}\mathrm{e}^{\frac{1}{2}t}-\frac{1}{4}x\mathrm{e}^{\frac{1}{2}t}$$ $$x(4+\mathrm{e}^{\frac{1}{2}t})=5\mathrm{e}^{\frac{1}{2}t}$$ $$x=\frac{5\mathrm{e}^{\frac{1}{2}t}}{4+\mathrm{e}^{\frac{1}{2}t}} \quad \text{or} \quad x=\frac{5}{4\mathrm{e}^{-\frac{1}{2}t}+1}$$
(ii)	When x = 2, $\frac{5\mathrm{e}^{\frac{1}{2}t}}{4+\mathrm{e}^{\frac{1}{2}t}}=2$ $$8+2\mathrm{e}^{\frac{1}{2}t}=5\mathrm{e}^{\frac{1}{2}t}$$ $$3\mathrm{e}^{\frac{1}{2}t}=8$$ $$\mathrm{e}^{\frac{1}{2}t}=\frac{8}{3}$$ $$t=2\ln\left(\frac{8}{3}\right)$$ It takes $t=2\ln\left(\frac{8}{3}\right)$ years.
(iii)	As $t\to\infty$, $x\to 5$. $\therefore$ The population of wild boars will increase and stabilise at 500 eventually.

4	**Solution**
(i)	$l:\ \mathbf{r}=\begin{pmatrix}1\\0\\-7\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\4\end{pmatrix}$, where λ is a real parameter. $p:\ \mathbf{r}\bullet\begin{pmatrix}1\\0\\-1\end{pmatrix}=2$ $\sin\theta=\dfrac{\left\|\begin{pmatrix}-2\\1\\4\end{pmatrix}\bullet\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}-2\\1\\4\end{pmatrix}\right\|\left\|\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\|}=\dfrac{6}{\sqrt{21}\sqrt{2}}$ $\therefore\theta=67.8°$ (1 dec pl)
(ii)	For the point of intersection between *l* and *p*, $\begin{pmatrix}1-2\lambda\\\lambda\\-7+4\lambda\end{pmatrix}.\begin{pmatrix}1\\0\\-1\end{pmatrix}=2$ $1-2\lambda+7-4\lambda=2$ $\lambda=1$ The position vector of point of intersection is $\begin{pmatrix}1\\0\\-7\end{pmatrix}+\begin{pmatrix}-2\\1\\4\end{pmatrix}=\begin{pmatrix}-1\\1\\-3\end{pmatrix}$. Coordinates of point of intersection are $(-1,1,-3)$.
(iii)	The line perpendicular to *p* passing through $(1,0,-7)$ is $\mathbf{r}=\begin{pmatrix}1\\0\\-7\end{pmatrix}+\mu\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \mu\in\mathbb{R}$ $\begin{pmatrix}1+\mu\\0\\-7-\mu\end{pmatrix}.\begin{pmatrix}1\\0\\-1\end{pmatrix}=2$ $1+\mu+7+\mu=2$

	$2\mu = -6$ $\mu = -3$ $\overrightarrow{ON} = \begin{pmatrix} 1-3 \\ 0 \\ -7+3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}$
(iv)	Method 1: Let the coordinates of A be $(1,0,-7)$. Let A' be the reflected point of A in p. Using ratio theorem, $\overrightarrow{ON} = \dfrac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$ $\Rightarrow \overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA} = 2\begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ -1 \end{pmatrix}$ The reflected line contains the point A' and point of intersection between l and p. The direction vector of the reflected line is $\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ $\therefore \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}, \alpha \in \mathbb{R}$
(iv)	Method 2: Let the coordinates of A be $(1,0,-7)$.Let A' be the reflected point of A in p. Let the coordinates of B be $(-1,1,-3)$. $\overrightarrow{BN} = \dfrac{\overrightarrow{BA} + \overrightarrow{BA'}}{2}$ $\overrightarrow{BA'} = 2\overrightarrow{BN} - \overrightarrow{BA}$ $= 2\begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$ $\therefore \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \alpha \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}, \alpha \in \mathbb{R}$

5	**Solution**
(i)	Systematic sampling
(ii)	(slower, more difficult to collect) Systematic sampling is a more tedious process to select the employees, whereas quota sampling is quick and easy. Another possible reason: might miss out a certain group of people due to different reporting times.
(iii)	The interviewer could consider transport mode of the employees as the stratum. A possible quota for each stratum is as follows: <table><tr><td>By private transport</td><td>By public transport</td><td>By walking</td></tr><tr><td>10</td><td>10</td><td>10</td></tr></table> The interviewer can then stand at the entrance of the building and select the sample until the above quota is met.

6	**Solution**
	$\mathrm{E}(X)=1.93 \quad \mathrm{Var}(X)=1.4$ Since n is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(1.93, \frac{1.4}{n}\right)$ approximately. Given that $\mathrm{P}\left(\overline{X}>2\right)<0.24$ --- (*)
	<u>Method 1: Using GC to set up table</u> when $n=142$, $\mathrm{P}\left(\overline{X}>2\right)=0.24041$ (>0.24) when $n=143$, $\mathrm{P}\left(\overline{X}>2\right)=0.23964$ (<0.24) when $n=144$, $\mathrm{P}\left(\overline{X}>2\right)=0.23887$ (<0.24) $\therefore$ least n is 143.
	<u>Method 2: Using algebraic method via standardization</u> $\mathrm{P}\left(\overline{X}\le 2\right)>0.76$ $\mathrm{P}\left(Z\le \frac{2-1.93}{\sqrt{1.4/n}}\right)>0.76$

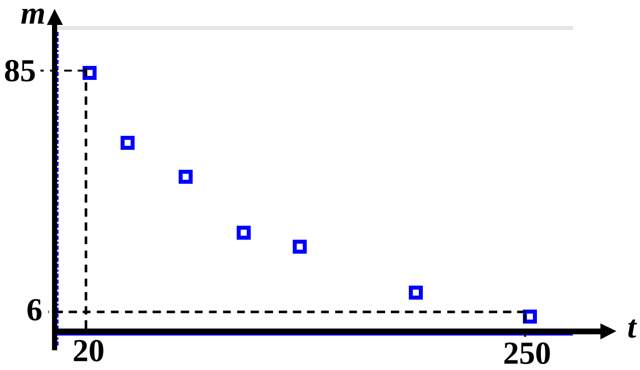
	From GC, $$\frac{2-1.93}{\sqrt{1.4/n}} > 0.70630 \quad ---(**)$$ $$\sqrt{n} > \frac{0.70630}{0.07}\sqrt{1.4}$$ $$\sqrt{n} > 11.939$$ $$n > 142.53$$ $\therefore$ least n is 143.

7	**Solution**
(i)	*Method 1:* Required probability $=1-\frac{^{13}C_4}{^{24}C_4}-\frac{^{11}C_4}{^{24}C_4}=0.902$ (3 sig fig) *Method 2:* Required probability $=1-\frac{13\times12\times11\times10}{24\times23\times22\times21}-\frac{11\times10\times9\times8}{24\times23\times22\times21}=0.902$ (3 sig fig) *Method 3:* Required probability $=\frac{^{11}C_1\times{}^{13}C_3+{}^{11}C_2\times{}^{13}C_2+{}^{11}C_3\times{}^{13}C_1}{^{24}C_4}$ $=0.902$ (3 sig fig)
(ii)	*Method 1:* Required probability $=\frac{^{11}C_2\times2!\times{}^{22}C_2\times2!}{^{24}C_4\times4!}=0.199$ (3 sig fig) *Method 2:* Required probability $=\frac{11\times10\times22\times21}{24\times23\times22\times21}=0.199$ (3 sig fig)

8	Solution
(i)	Unbiased estimate of the population mean $\bar{x} = \frac{573.39}{13} = 44.10692308 = 44.1$ (3 s.f.) Unbiased estimate of the population variance $s^2 = \frac{42.22}{12} = 3.518333333 = 3.52$ (3 s.f.)
(ii)	The battery life of a PI-99 calculator is assumed to be normally distributed.
(iii)	Let X be the r.v. denoting the battery life of a randomly chosen PI-99 calculator. Let μ be the population mean battery life of the PI-99 calculators. $H_0: \quad \mu = k$ $H_1: \quad \mu \neq k$ where H_0 is the null hypothesis and H_1 is the alternative hypothesis.
(iv)	To test at 5% level of significance. Under H_0 , the test statistic is $T = \frac{\bar{X} - k}{\frac{S}{\sqrt{13}}} \sim t_{(12)}$. Since the null hypothesis is not rejected, t-value falls outside critical region. $\therefore \quad -2.178812 < t - \text{value} < 2.178812$ $-2.178812 < \frac{\bar{x} - k}{s/\sqrt{13}} < 2.178812$ --- (*) $\bar{x} - 2.178812\left(\frac{s}{\sqrt{13}}\right) < k < \bar{x} + 2.178812\left(\frac{s}{\sqrt{13}}\right)$ where $\bar{x} = 44.10692$ and $s = \sqrt{3.51833}$ $\therefore \quad 42.97 < k < 45.24$ The required set is $\{k \in \mathbb{R}: 42.97 < k < 45.24\}$

9	**Solution**
(i)	The average number of faults detected by each system (for the track and the train) is constant from one day to another.
(ii)	Let X be the r.v. denoting the total number of faults detected by the two systems in a periods of 10 days. $X \sim \text{Po}\left((0.25+0.15)\times 10\right)$, i.e. $X \sim \text{Po}(4)$ $\therefore\ \text{P}\left(X \le 4\right) = 0.6288369 = 0.629$ (3 sig fig)
(iii)	Let Y be the r.v. denoting the total number of faults detected by the two systems in a period of n days. $Y \sim \text{Po}(0.4n)$ Given $\text{P}\left(Y=0\right) < 0.05$, *Method 1: Algebraic method* $e^{-0.4n} < 0.05$ **(o.e.** $\left(e^{-0.25n}\right)\left(e^{-0.15n}\right) < 0.05$**)** $n > 7.489$ $\therefore$ the smallest number of days required is 8. *Method 2: GC table* When $n=7$, $\text{P}\left(Y=0\right) = 0.06081$ (>0.05) When $n=8$, $\text{P}\left(Y=0\right) = 0.04076$ (<0.05) When $n=9$, $\text{P}\left(Y=0\right) = 0.02732$ (<0.05) $\therefore$ the smallest number of days required is 8.
(iv)	Let W and V be the r.v. denoting the number of faults detected on the track and on the track in a period of 10 days respectively. $W \sim \text{Po}(2.5)$ and $V \sim \text{Po}(1.5)$ Required probability $= \text{P}\left(W \ge 3 \mid V+W \le 4\right)$ $= \dfrac{\text{P}\left(W \ge 3 \cap V+W \le 4\right)}{\text{P}\left(V+W \le 4\right)}$ $= \dfrac{\text{P}(W=3)\text{P}(V=0)+\text{P}(W=3)\text{P}(V=1)+\text{P}(W=4)\text{P}(V=0)}{\text{P}\left(V+W\le 4\right)}$ $= \dfrac{\text{P}(W=3)\text{P}(V\le 1)+\text{P}(W=4)\text{P}(V=0)}{\text{P}\left(V+W \le 4\right)}$ $= 0.237$ (3 sig fig)

10	**Solution**
(i)	Ben's performance (i.e. whether he loses or wins) in a game is independent of any other games that he plays with Alex.
(ii)	Let X be the r.v. denoting the number of games that Ben loses out of 10 games. $X \sim \mathrm{B}(10, 0.7)$ $\mathrm{P}(X > 5) = 1 - \mathrm{P}(X \le 5)$ $= 0.84973$ ≈ 0.850 (3 sig fig)
(iii)	Let Y be the r.v. denoting the number of games that Ben loses out of n games. $Y \sim \mathrm{B}(n, 0.3)$ $\mathrm{P}(Y > 8) \le 0.01$ $1 - \mathrm{P}(Y \le 8) \le 0.01$ Using GC, When $n = 13$, $\mathrm{P}(Y > 8) = 0.00403$ (< 0.01) When $n = 14$, $\mathrm{P}(Y > 8) = 0.00829$ (< 0.01) When $n = 15$, $\mathrm{P}(Y > 8) = 0.01524$ (> 0.01) $\therefore$ the greatest value of n is 14.
(iv)	Let W be the r.v. denoting the number of games that Ben loses out of 50 games. $W \sim \mathrm{B}(50, 0.3)$ As n = 50 is large, $np = 15\,(> 5)$ and $nq = 35\,(> 5)$, $\therefore W \sim \mathrm{N}(15, 10.5)$ approximately $\mathrm{P}(10 \le W \le 20) = \mathrm{P}(9.5 \le W \le 20.5)$ $---(*)$ $= 0.910$ (3 sig fig)

11	**Solution**
(i)	 From the scatter diagram, a curvinlinear correlation is observed between m and t (i.e. as t increases, m decreases at a decreasing rate), and hence a linear model with equation of the form $m = a + bt$ cannot be used to model the relationship between m and t.
(ii) (a)	Product moment correlation coefficient between m and $t^2 = -0.8454$.
(b)	Product moment correlation coefficient between m and $\ln t = -0.9961$.
(iii)	Since the absolute value of the correlation coefficient between m and $\ln t$ (i.e. case (b)) is **<u>closer</u>** to 1, this indicates that the linear correlation between the variables m and $\ln t$ is **<u>stronger</u>** as compared to that between the variables for case (a). $\therefore$ case (b) is the better model for the relationship between m and t. $m = 179.026 - 31.2175 \ln t$ $\Rightarrow m = 179 - 31.2 \ln t$ (3 sig fig)
(iv)	When $t = 150$, $m = 179.026 - 31.2175 \ln 150 = 22.61$ (2 dec pl) The estimate obtained is reliable, because the given value of $t = 150$ lies within the given sample data range for t and the product moment correlation coefficient between m and $\ln t$ is very close to -1, hence indicating a strong negative linear correlation between the variables m and $\ln t$.

12	Solution
(i)	$X+Y \sim \mathrm{N}(50, 32.65)$ $\mathrm{P}(X+Y>45)=0.809224=0.809$ (3 sig fig)
(ii)	$\mathrm{E}(X+Y-S)=12$ $\mathrm{Var}(X+Y-S)=39.41$ $\therefore\ X+Y-S \sim \mathrm{N}(12, 39.41)$ P(method A is faster than method B by more than 5 mins) $=\mathrm{P}(S+15-(X+Y)>5)$ $=\mathrm{P}(X+Y-S<10)$ $=0.375020=0.375$ (3 sig fig)
(iii)	Let $A=X+Y$ and $B=S+15$. $A \sim \mathrm{N}(50, 32.65)$ and $B \sim \mathrm{N}(53, 2.6^2)$ Let $W=\dfrac{A_1+A_2+A_3+A_4+B_1+...+B_6}{10}$ $\therefore\quad \mathrm{E}(W)=\dfrac{50\times4+53\times6}{10}=51.8$ & $\quad \mathrm{Var}(W)=\dfrac{32.65\times4+2.6^2\times6}{10^2}=1.7116$ $\therefore W \sim \mathrm{N}(51.8, 1.7116)$ Required probability $=\mathrm{P}(W>50)$ $=0.915566=0.916$ (3 sig fig)

JURONG JUNIOR COLLEGE

Preliminary Examinations

MATHEMATICS
Higher 2

9740 / 1

30 August 2016

3 hours

Additional materials: Answer Paper
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READ THESE INSTRUCTIONS FIRST

Write your name and civics class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together, with the cover page in front.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

[Turn over

1 In the finals of a General Knowledge Quiz, a team is required to answer 25 questions. Each question that is correctly answered scores 5 points, while a question that is wrongly answered is deducted 3 points. If the answer is partially correct, the team scores 2 points.

After 24 questions, the results are shown in the following table.

Number of Questions			
Correct	**Partially Correct**	**Wrong**	**Points**
a	b	c	79

If the team answers the last question wrongly, then the total number of questions answered correctly and partially correct is four times the number of questions answered wrongly. By forming a system of linear equations, find the values of a, b and c. [4]

2 A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = \frac{1}{3}$ and

$$u_{r+1} = u_r - \frac{1}{(2r-1)(2r+3)}, \quad \text{for all } r \geq 1.$$

(i) Use the method of mathematical induction to prove that $u_n = \dfrac{n}{4n^2-1}$. [4]

(ii) Hence prove that the sum of the first n terms of the series

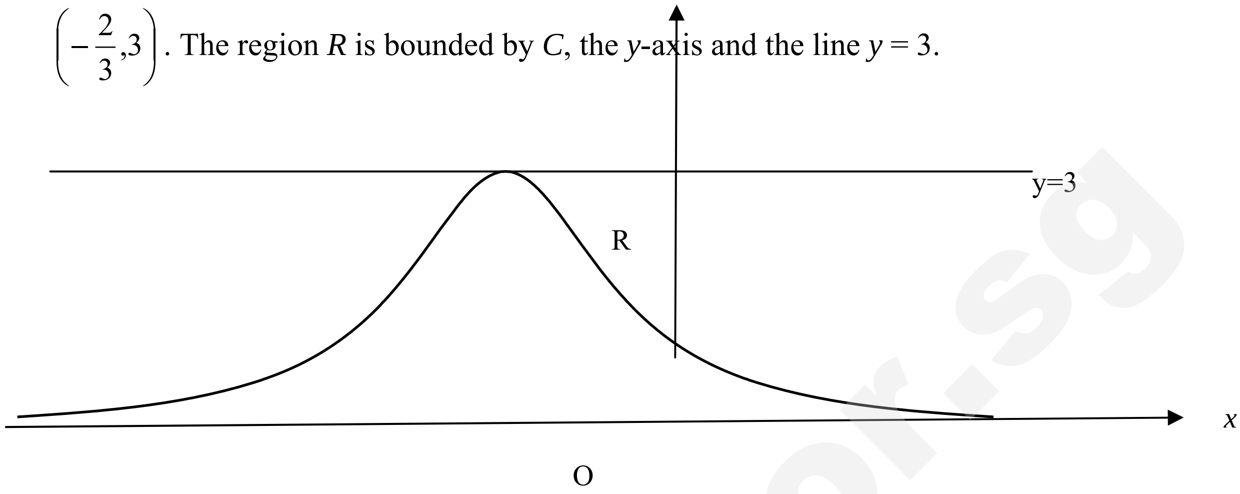
$$\frac{1}{5\times9}+\frac{1}{7\times11}+\frac{1}{9\times13}+\cdots$$

is $\dfrac{3}{35}-\dfrac{n+3}{4(n+3)^2-1}$. [3]

(iii) Give a reason why the series in part **(ii)** is convergent and state the sum to infinity. [2]

[Turn ove

3 The diagram shows the curve C with equation $y = \frac{12}{(3x+2)^2+4}$ which has a turning point at $\left(-\frac{2}{3}, 3\right)$. The region R is bounded by C, the y-axis and the line $y = 3$.



(i) Find the exact area of R. [5]

(ii) R is rotated through 2π radians about the y-axis. Find the volume of the solid of revolution formed, giving your answer to 4 decimal places. [3]

4 Let $y = \tan\left(2\tan^{-1} x + \frac{\pi}{4}\right)$.

(i) Show that $\left(1+x^2\right)\frac{dy}{dx} = 2\left(1+y^2\right)$. [2]

(ii) Hence find the Maclaurin series for y, up to and including the term in x^2. [4]

Denote the answer to part **(ii)** of the Maclaurin series by g(x) and f(x)= $\tan\left(2\tan^{-1} x + \frac{\pi}{4}\right)$.

(iii) Find, for $-0.4 \le x \le 0.4$, the set of values of x for which the value of g(x) is within ±0.5 of the value of f(x). [2]

[Turn over

5 A curve C has parametric equations

$$x = 2\sin 2t, \quad y = \cos 2t, \quad \text{for } 0 \le t < \pi.$$

(i) Show that the equation of the normal to C at the point P with parameter θ is

$$(2\cos 2\theta)x - (\sin 2\theta)y = m\sin 2\theta\cos 2\theta,$$

where m is an integer to be determined. [3]

(ii) The normal to C at the point P cuts the x-axis and y-axis at points A and B respectively. By finding the mid-point of AB, determine a cartesian equation of the locus of the mid-point of AB as θ varies. [5]

6 A function f is said to be self-inverse if $f(x) = f^{-1}(x)$ for all x in the domain of f.
The function g is defined by

$$g : x \mapsto \sqrt{\frac{x^2+2}{x^2-1}}, \quad x > 1.$$

(i) Sketch the curve $y = g(x)$, stating the equations of the asymptotes clearly. [2]

(ii) Define g^{-1} in a similar form and show that g is self-inverse. [4]

(iii) Show that $g^2(x) = x$ and that $g^3(x) = g(x)$. Hence find the values of x for which

$$4 - g^{50}(x) = \left[g^{51}(x)\right]^2.$$ [4]

7 **(a)** Find $\int \frac{\cos(\ln x)}{x^2}\, dx$. [4]

(b) Using the substitution $u = \sqrt{x+3}$, find $\int_1^6 \frac{x-2}{x\sqrt{x+3}}\, dx$, giving your answer in the form

$$a + \frac{b}{\sqrt{3}}\ln\left(\frac{c-\sqrt{d}}{c+\sqrt{d}}\right),$$

where a, b, c and d are constants to be determined. [6]

8 The complex number z satisfies the following inequalities:

$$|z| \leq 4 \text{ and } -\frac{\pi}{6} \leq \arg(z + \sqrt{3} - \mathrm{i}) \leq 0$$

.

(i) On an Argand diagram, sketch the region R in which the point representing z can lie. [4]

(ii) Find exactly the minimum and maximum possible values of $|z - 2\mathrm{i}|$. [3]

(iii) Determine the number of roots of the equation $z^{100} = 2^{100}$ that lie in the region R. [3]

9 It is given that $\mathrm{f}(x) = x + \dfrac{m^2}{x-2}$, where $0 < m < 1$.

(i) Sketch the graph of $y = \mathrm{f}(x)$, showing clearly the coordinates of the turning points and the equation(s) of any asymptote(s). [5]

(ii) By inserting a suitable graph to your sketch in **(i)**, find the set of values of k, in terms of m, for which the equation $x^2 - (2+k)x + (m^2 + 2k) = 0$ has two distinct positive roots. [4]

(iii) The curve $y = \mathrm{f}(x)$ undergoes the transformations A, B and C in succession:

A: A translation of -2 units in the direction of x-axis,

B: A stretch parallel to the x-axis with scale factor of $\dfrac{1}{2}$, and

C: A translation of -2 units in the direction of y-axis.

Given that the resulting curve is $y = 2x + \dfrac{1}{8x}$, find the value of m. [2]

10 The point A has position vector $\begin{pmatrix}4\\-3\\0\end{pmatrix}$ and the line l has equation $\mathbf{r} = \begin{pmatrix}2\\-3\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$, where $\lambda \in \mathbb{R}$.

(i) Find the position vector of the foot of the perpendicular from A to l. [4]

(ii) Show that a cartesian equation of the plane π_1 which contains A and l is

$$x + y + 2z = 1.$$ [2]

The equation of the plane π_2 is $x + 7z = c$, where c is a constant.

(iii) Given that π_1 and π_2 intersect in a line L, show that a vector equation of L is

$$\mathbf{r} = \begin{pmatrix}c\\1-c\\0\end{pmatrix} + \mu\begin{pmatrix}-7\\5\\1\end{pmatrix}, \quad \mu \in \mathbb{R}.$$ [2]

Another plane π_3 has equation $2x - y + dz = 5$, where d is a constant.

(iv) Find the values of c and/or d if all three planes π_1, π_2 and π_3

(a) meet in the line L, [3]

(b) have only one point in common. [1]

11 At the beginning of May 2016, Sam borrowed \$50 000 from a bank that charges him a special rate of 0.2% interest at the end of every month. Sam pays back \$1 000 for every instalment at the beginning of every month, starting from June 2016.

(i) Show that the total amount with interest that Sam still owes the bank at the end of the month after the nth instalment is paid is

$$\$\left[50\,000(1.002^{n+1}) - 501\,000(1.002^{n} - 1)\right].$$ [4]

(ii) Find the number of instalments required for Sam to settle all the amount owed. [2]

(iii) How much does he pay on his last instalment? [2]

(iv) If Sam wishes to settle all the amount owed after paying 19 instalments, what is the minimum amount (to the nearest dollar) he should pay each month? [2]

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Preliminary Examinations

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This document consists of **7** printed pages and **1** blank page.

[Turn over

Section A: Pure Mathematics [40 marks]

1 **(a)** Given that x and y are related by $\frac{dy}{dx} = \sec^2 y$ and that $y = 0$ when $x = 1$, find x in terms of y. [4]

(b) A medical researcher is investigating the rate of spread of a virus in a group of people of size n at time t weeks. He suggests that n and t are related by the differential equation $\frac{d^2n}{dt^2} = e^{-t/5}$.

.

(i) Find the general solution of the differential equation, giving your answer in the form $n = \mathrm{f}(t)$. [2]

(ii) Explain why all solution curves of the differential equation are concave upwards. [1]

(iii) It is given that initially, the number of people infected with the virus is 50. Sketch on a single diagram, two distinct solution curves for the differential equation to illustrate the following two cases for large values of t :

I. the population of infected people increases indefinitely,

II. the population of infected people stabilizes at a certain positive number. [3]

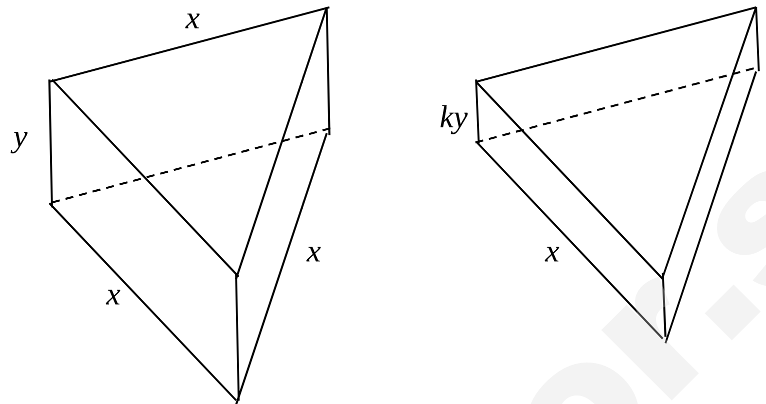
2 **(a)** A parallelogram has two adjacent sides defined by the vectors **a** and 2**a** + 3**b**. Given that the magnitudes of **a** and **b** are 4 and 5 respectively and the angle between **a** and **b** is 30° , find the area of the parallelogram. [4]

(b) A point P has coordinates (2, −1, −2) and a line l has equation $\frac{x-1}{2} = 1 - z, y = 3$.

(i) Find the perpendicular distance from P to l. [4]

(ii) Find the acute angle between l and the line L that is parallel to the z-axis. [2]

3 A box with volume 250 cm^3 is made of cardboard of negligible thickness. It has a height of y cm and an equilateral triangular base of side x cm. Its lid has depth ky cm, where $0 < k \le 1$ (see diagram).



(i) Show that the total external surface area of the box and lid can be expressed as

$$\frac{1000\sqrt{3}(1+k)}{x}+\frac{\sqrt{3}}{2}x^2 .$$ [4]

(ii) Use differentiation to find, in terms of k, the value of x that gives a minimum total external surface area of the box and lid. [3]

(iii) Find the ratio $\frac{y}{x}$ in this case, in terms of k, simplifying your answer. [2]

(iv) Find the values for which $\frac{y}{x}$ must lie. [2]

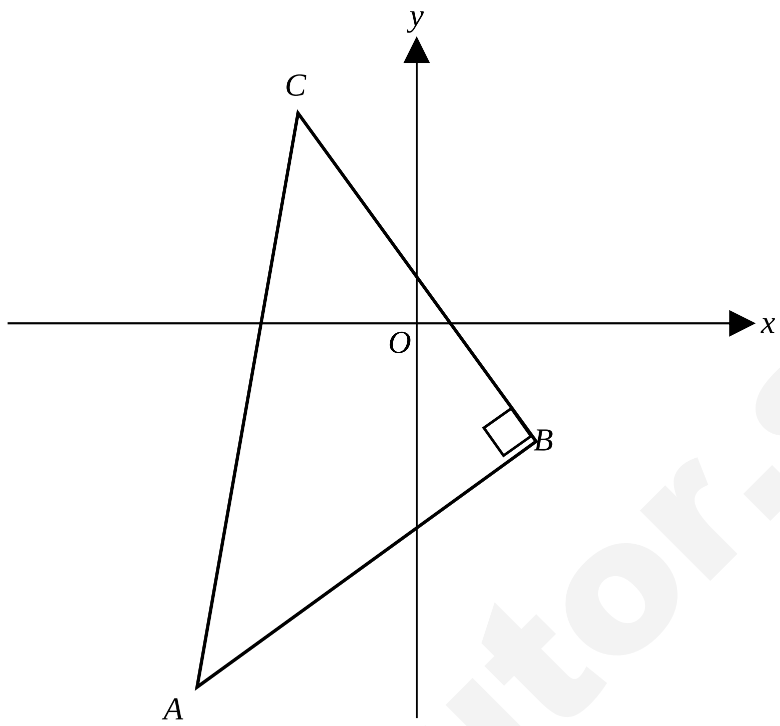
4 The complex numbers a and b are given by $a = -(1+\sqrt{3}\,\mathrm{i})$ and $b=\frac{1}{2}(1-\mathrm{i})$.

(i) Without using a calculator, find the value of a^2b in the form $x + \mathrm{i}y$. [2]

(ii) By using the moduli and arguments of a and b, find the modulus and argument of a^2b. [3]

(iii) Use your answers to parts **(i)** and **(ii)** to show that $\sin\frac{5\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}$. [2]

(iv) The diagram below shows an isosceles right triangle ABC, where the points A, B and C represent the complex numbers a, b and c respectively. Find the exact value of c. [2]



Section B: Statistics [60 marks]

5 A group of 11 people consists of 6 men and 5 women, 3 of whom are sisters. A committee consisting of six people is to be selected. Find the number of ways the committee can be formed if

(i) it consists of exactly two men, [1]

(ii) it includes at least one of the sisters. [2]

Given that the chosen committee consists of 2 sisters, Sue and Suzy, together with 3 other men, Muthu, Mark, Michael and 1 other woman, Wina. They are seated at a round table meant for six people. Find the number of possible arrangements if

(iii) one of the men is to be seated between the two sisters, [2]

(iv) the two sisters are sitting directly opposite each other. [2]

6 The table below shows the number of male and female students studying Chemistry, Physics and Biology at a private school.

	Chemistry	**Physics**	**Biology**
Male	200	130	70
Female	250	300	50

One of the students is chosen at random. Events C, B and M are defined as follows:

C : The student chosen is studying Chemistry.

B : The student chosen is studying Biology.

M : The student chosen is a male.

Find

(i) $\mathrm{P}(C\,|M\,)$, [1]

(ii) $\mathrm{P}(M \cup C)$, [1]

(iii) $\mathrm{P}(M'\,)\,B')$. [1]

Determine whether C and M are independent. [2]

It is given that 20% of Chemistry students, 30% of Physics students and 5% of Biology students are international students.

(iv) One of the students selected at random is an international student. What is the probability that this student studies Chemistry? [2]

(v) Three students are chosen at random. Find the probability that there is exactly one international student who studies Physics. [2]

[Turn over

7 In order to investigate whether there is a correlation between rainfall and crop yields, the total rainfall, x mm, and the weights of a particular crop per square metre, y kg, were recorded in a number of fields. The data are shown below.

x	36	72	44	74	64	50
y	2.2	8.4	1.8	7.4	4.3	2.2

(i) Draw a scatter diagram to illustrate the data. [2]

(ii) Calculate the value of the product moment correlation coefficient, and explain why its value does not necessarily mean that the best model for the relationship between x and y is $y = a + bx$. [2]

(iii) By comparing the product moment correlation coefficients, explain whether $y = a + bx$ or $y = c + dx^2$ is a better model. [2]

(iv) Using a suitable regression line, estimate the yield of crop per square metre when the total rainfall is 55mm. Comment on the reliability of your estimation. [3]

8 It is known that 8% of the population of a large city use a particular web browser called Voyager. A researcher wishes to interview people from the city who use Voyager and selects people at random, one at a time.

(i) Find the probability that the first person that he finds uses Voyager is the third person selected. [2]

A random sample of n people is now selected.

(ii) State two conditions needed for the number of people in the sample who use Voyager to be well modelled by binomial distribution. [2]

(iii) Given that $n = 80$, use a suitable approximation to find the probability that, fewer than 10 people use Voyager. [3]

(iv) Find the least value of n such that the probability of at least 10 people use Voyager is more than 0.2. [3]

9 A supermarket sells boxes of a particular brand of biscuits in two flavours, chocolate and strawberry. The mean number of boxes of chocolate biscuits sold in a day is 2.2.

(i) Find the probability that in a day, no boxes of chocolate biscuits were sold. [1]

(ii) In a week of 7 days, find the expected number of days that no boxes of chocolate biscuits were sold. [2]

The mean number of boxes of strawberry biscuits sold in a day is denoted by λ.

(iii) Given that the probability of less than 2 boxes of strawberry biscuits sold in a day is 0.6, write down an equation for the value of λ, and find λ numerically, correct to 1 decimal place. [3]

(iv) Find the probability that in a week of 7 days, the total number of boxes of chocolate and strawberry biscuits sold exceeds 25 boxes. [2]

(v) Use a suitable approximation to find the probability that, in a month of 30 days, the number of boxes of chocolate biscuits sold is more than the number of boxes of strawberry biscuits. [4]

10 A researcher is running a trial of a new variety of potato. A field contains 20 rows of the new variety of potato plants, with 80 plants in each row. A researcher intends to dig up 8 plants and measure the mass of potatoes produced by each plant.

(i) Describe how he could choose a systematic sample of 8 plants from a single row of 80 plants and state the advantage of this sampling method. [3]

The researcher claims that the average mass of the new variety of potato is at least 150g. The mass of a new variety of potato is denoted by X grams. The masses of a random sample of 80 new variety potatoes are summarized by

$$\sum(x-150)=-160, \quad \sum(x-150)^2=5520.$$

(ii) Calculate the unbiased estimates of the population mean and variance. [2]

(iii) Test at the 1% significance level, whether the researcher's claim is valid. [4]

(iv) Explain what you understand by the phrase "at the 1% significance level" in the context of this question. [1]

Another random sample of 8 potatoes was chosen with mean mass 148.5g and standard deviation k g. Find the range of values that k can take such that at 1% level of significance, this sample would indicate that the researcher's claim is invalid. [3]

Jurong Junior College
2016 JC2 H2 Mathematics Prelim Paper 1 Solutions

Qn	Solution
1	$a + b + c = 24$ $5a + 2b - 3c = 79$ $a + b = 4(c + 1)$ $a = 17, b = 3, c = 4$
2(i)	Let P_n be the statement that $u_n = \dfrac{n}{4n^2 - 1}$ for all $n \in \mathbb{Z}^+$ When $n = 1$, LHS = $u_1 = \dfrac{1}{3}$ (given) RHS = $\dfrac{1}{4 \times 1^2 - 1} = \dfrac{1}{3}$ Hence P_1 is true. Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \dfrac{k}{4k^2 - 1}$. We want to prove that P_{k+1} is true, i.e. $u_{k+1} = \dfrac{k+1}{4(k+1)^2 - 1}$ LHS $= u_{k+1}$ $= u_k - \dfrac{1}{(2k-1)(2k+3)}$ $= \dfrac{k}{(2k-1)(2k+1)} - \dfrac{1}{(2k-1)(2k+3)}$ $= \dfrac{k(2k+3) - (2k+1)}{(2k-1)(2k+1)(2k+3)}$ $= \dfrac{2k^2 + k - 1}{(2k-1)(2k+1)(2k+3)}$ $= \dfrac{(2k-1)(k+1)}{(2k-1)(2k+1)(2k+3)}$ $= \dfrac{k+1}{(2k+1)(2k+3)}$ $= \dfrac{k+1}{4k^2 + 8k + 3}$

	$=\frac{k+1}{4(k+1)^2-1}=$ RHS Hence P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true & P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
(ii)	Sum of 1st n terms of $\frac{1}{5\times 9}+\frac{1}{7\times 11}+\frac{1}{9\times 13}+\cdots\cdots\cdots+\frac{1}{(2n+3)(2n+7)}$ $=\sum_{r=3}^{n+2}\frac{1}{(2r-1)(2r+3)}$ $=\sum_{r=3}^{n+2}[u_r-u_{r+1}]$ $= u_3-\cancel{u_4}$ $+\cancel{u_4}-\cancel{u_5}$ $+\cancel{u_5}-\cancel{u_6}$ $+\cdots\cdots\cdots$ $+\cdots\cdots\cdots$ $+\cancel{u_{n+2}}-u_{n+3}$ $=u_3-u_{n+3}$ $=\frac{3}{35}-\frac{n+3}{4(n+3)^2-1}$.
(iii)	As $n\to\infty, \frac{n+3}{4(n+3)^2-1}\to 0$ Hence, the series is convergent and $\sum_{r=3}^{\infty}\frac{1}{(2r-1)(2r+3)}=\frac{3}{35}$

3 (i)	Required Area $= \int_{-\frac{2}{3}}^{0} \left(3 - \frac{12}{(3x+2)^2+4}\right) \mathrm{d}x$ $= \int_{-\frac{2}{3}}^{0} \left(3 - \frac{12}{(3x+2)^2+2^2}\right) \mathrm{d}x$ $= \left[3x - 2\tan^{-1}\left(\frac{3x+2}{2}\right)\right]_{-\frac{2}{3}}^{0}$ $= 0 - 2\left(\frac{\pi}{4}\right) - [-2-0]$ $= 2 - \frac{\pi}{2}$
(ii)	$y = \frac{12}{(3x+2)^2+4} \quad \Rightarrow \quad 3x = -2 \pm \sqrt{\frac{12}{y} - 4} = -2 \pm \sqrt{\frac{12-4y}{y}}$ Since $x \geq -\frac{2}{3}$, $x = -\frac{2}{3} + \frac{1}{3}\sqrt{\frac{12-4y}{y}}$ Required volume = $\pi \int_{\frac{3}{2}}^{3} x^2 \ \mathrm{d}y$ $= \pi \int_{\frac{3}{2}}^{3} \left(-\frac{2}{3} + \frac{1}{3}\sqrt{\frac{12-4y}{y}}\right)^2 \ \mathrm{d}y$ = 0.5125
4(i)	$y = \tan\left(2\tan^{-1}x + \frac{\pi}{4}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec^2\left(2\tan^{-1}x + \frac{\pi}{4}\right)\frac{2}{1+x^2}$ $\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(1 + \tan^2\left(2\tan^{-1}x + \frac{\pi}{4}\right)\right)$ $\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(1+y^2\right)$
	<u>Alternative Method</u> $\tan^{-1} y = 2\tan^{-1} x + \frac{\pi}{4}$ Differentiate with respect to *x*, $\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{1+x^2}$ $\Rightarrow \left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(1+y^2\right)$

(ii)

Differentiate with respect to *x*,

$$\Rightarrow \left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}=4y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow \left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(2x-4y)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

When $x=0$, $y=\tan\frac{\pi}{4}=1$

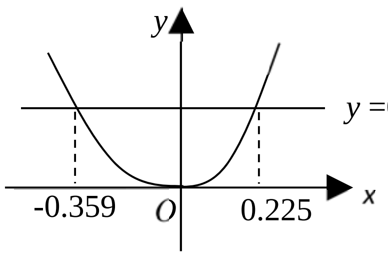
$$(1+0)\frac{\mathrm{d}y}{\mathrm{d}x}=2(1+1)\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=4$$

$$\left(1+0\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(0-4)(4)=0\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=16$$

$$y=\tan\left[2\tan^{-1}x+\frac{\pi}{4}\right]$$

$$=1+4x+\frac{16}{2!}x^2+.....$$

$$=1+4x+8x^2+....$$

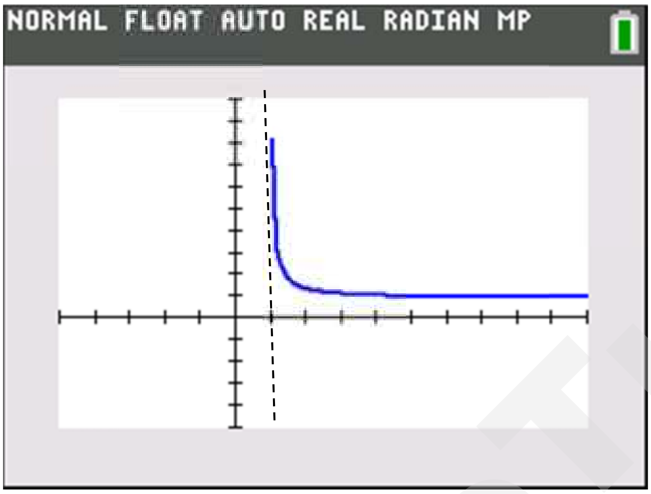


(iii)

Sketch $y=\left|\mathrm{f}(x)-\mathrm{g}(x)\right|$ and $y_1=0.5$

For $\left|\mathrm{f}(x)-\mathrm{g}(x)\right|<0.5,\quad -0.359<x<0.225$

5(i)	$x = 2\sin 2t,\qquad y = \cos 2t,\qquad \text{for } 0 \le t < \pi.$ $\frac{dx}{dt} = 4\cos 2t \qquad \frac{dy}{dt} = -2\sin 2t$ $\frac{dy}{dx} = -\frac{\sin 2t}{2\cos 2t}$ Equation of normal at *P*, $t = \theta$: $y - \cos 2\theta = \frac{2\cos 2\theta}{\sin 2\theta}\left(x - 2\sin 2\theta\right)$ $\left(\sin 2\theta\right) y - \cos 2\theta \sin 2\theta = \left(2\cos 2\theta\right) x - 4\cos 2\theta \sin 2\theta$ $\left(2\cos 2\theta\right) x - \left(\sin 2\theta\right) y = 3\cos 2\theta \sin 2\theta$ (shown) i.e. $m = 3$
(ii)	At the *x*-axis, $y = 0$ $\left(2\cos 2\theta\right) x = 3\sin 2\theta \cos 2\theta$ $x = \frac{3}{2}\sin 2\theta$ i.e. $A\left(\frac{3}{2}\sin 2\theta,\ 0\right)$ At the *y*-axis, $x = 0$ $-\left(\sin 2\theta\right) y = 3\sin 2\theta \cos 2\theta$ $y = -3\cos 2\theta$ i.e. $B\left(0,\ -3\cos 2\theta\right)$ mid-point of *AB*: $\left(\frac{3}{4}\sin 2\theta, -\frac{3}{2}\cos 2\theta\right)$ $x = \frac{3}{4}\sin 2\theta \Rightarrow \sin 2\theta = \frac{4}{3}x$ $y = -\frac{3}{2}\cos 2\theta \Rightarrow \cos 2\theta = -\frac{2}{3}y$ Cartesian equation of the locus of the mid-point of *AB*: $\sin^2 2\theta + \cos^2 2\theta = 1$ $\frac{16x^2}{9} + \frac{4y^2}{9} = 1$ i.e. $16x^2 + 4y^2 = 9$

Qn	Solution
6(i)	
(ii)	Let $y = g(x) = \sqrt{\dfrac{x^2+2}{x^2-1}}, \quad x>1$ Then $y^2 = \dfrac{x^2+2}{x^2-1} = 1+\dfrac{3}{x^2-1}$ $y^2-1 = \dfrac{3}{x^2-1}$ $x^2-1 = \dfrac{3}{y^2-1}$ $x^2 = 1+\dfrac{3}{y^2-1} = \dfrac{y^2+2}{y^2-1}$ $x = g^{-1}(y) = \sqrt{\dfrac{y^2+2}{y^2-1}}$ since $x>1>0$ $\Rightarrow g^{-1} : x \mapsto \sqrt{\dfrac{x^2+2}{x^2-1}}$, $x>1$ g is self-inverse as $g(x) = g^{-1}(x)$ and $D_{g^{-1}} = D_g$

(iii)	$g^2(x) = gg(x) = gg^{-1}(x) = x.$ $g^3(x) = gg^2(x) = g(x).$ (shown) It follows that $g^{50}(x) = x$ and $g^{51}(x) = g(x)$, $x > 1$ For $4 - g^{50}(x) = \left[g^{51}(x)\right]^2$ Then $4 - x = \dfrac{x^2+2}{x^2-1}, \quad x > 1$ $(4-x)(x^2-1) = x^2+2, \quad x>1$ $x^3 - 3x^2 - x + 6 = 0, \quad x > 1$ $(x-2)(x^2-x-3) = 0$ $x = 2$ or $x = \dfrac{1 \pm \sqrt{1+12}}{2}$ since $x > 1$, $\Rightarrow x = 2$ or $x = \dfrac{1+\sqrt{13}}{2}$ (ans)

Qn	Solution
7 (a)	$u = \cos(\ln x) \qquad \dfrac{dv}{dx} = \dfrac{1}{x^2}$ $\dfrac{du}{dx} = -\dfrac{\sin(\ln x)}{x} \qquad v = -\dfrac{1}{x}$ $\int \dfrac{\cos(\ln x)}{x^2}\,dx = -\dfrac{\cos(\ln x)}{x} + \int \dfrac{\sin(\ln x)}{x^2}\,dx$ $= -\dfrac{\cos(\ln x)}{x} + \dfrac{\sin(\ln x)}{x} - \int \dfrac{\cos(\ln x)}{x^2}\,dx$
	® $2\int \dfrac{\cos(\ln x)}{x^2}\,dx = \dfrac{\sin(\ln x)}{x} - \dfrac{\cos(\ln x)}{x}$ 4 $\int \dfrac{\cos(\ln x)}{x^2}\,dx = \dfrac{1}{2x}\left[\sin(\ln x) - \cos(\ln x)\right] + c$

(b)

$u=\sqrt{x+3}$ ® $u^2=x+3$

Differentiating w.r.t. x, $2u\dfrac{\mathrm{d}u}{\mathrm{d}x}=1$

When x = 1, u = 2; When x = 6, u = 3

$$\int_1^6 \frac{x-2}{x\sqrt{x+3}}\,\mathrm{d}x = \int_2^3 \frac{u^2-5}{(u^2-3)u}\,(2u\ \mathrm{d}u)$$

$$= 2\int_2^3 \left(1-\frac{2}{u^2-3}\right)\mathrm{d}u$$

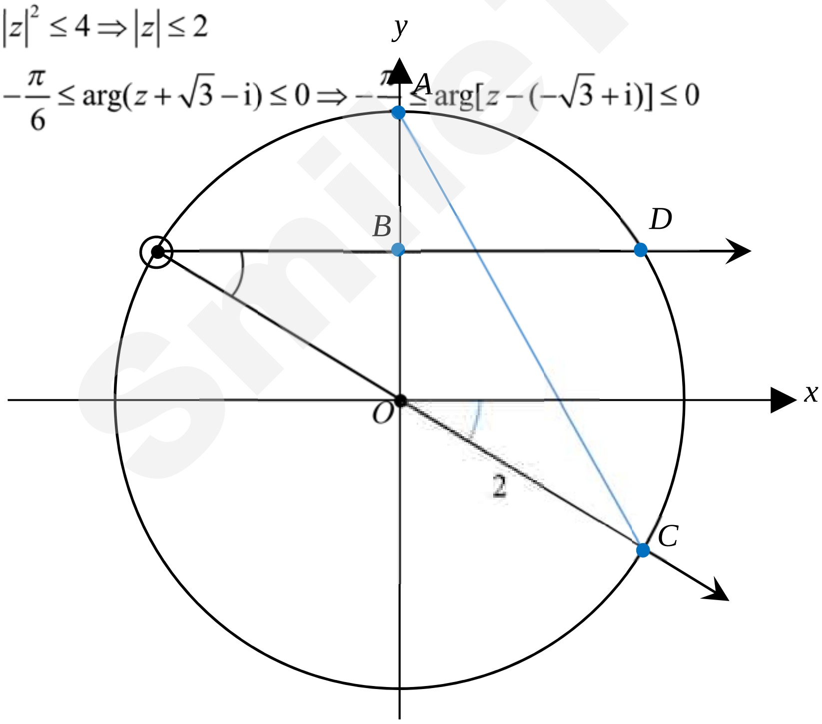
$$= \left[2u-\frac{4}{2\sqrt{3}}\ln\left(\frac{u-\sqrt{3}}{u+\sqrt{3}}\right)\right]_2^3$$

$$=2(3)-\frac{2}{\sqrt{3}}\ln\left(\frac{3-\sqrt{3}}{3+\sqrt{3}}\right)-2(2)+\frac{2}{\sqrt{3}}\ln\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)$$

$$=2+\frac{2}{\sqrt{3}}\ln\left(\frac{3-\sqrt{3}}{3+\sqrt{3}}\right)$$

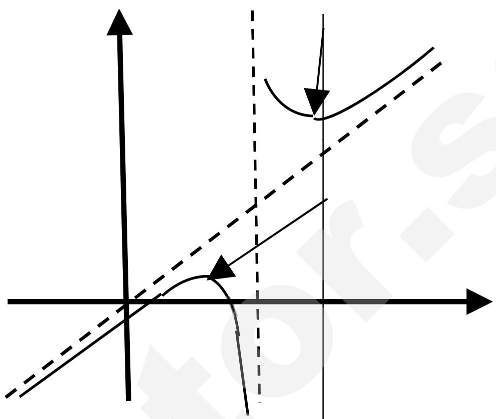
i.e. $a = b = 2$, $c = d = 3$

Qn	Solution
8 (i)	$\lvert z\rvert^2 \le 4 \Rightarrow \lvert z\rvert \le 2$ $-\dfrac{\pi}{6} \le \arg(z+\sqrt{3}-\mathrm{i}) \le 0 \Rightarrow -\dfrac{\pi}{6} \le \arg[z-(-\sqrt{3}+\mathrm{i})] \le 0$



(ii)	Minimum value of $\lvert z-2\mathrm{i}\rvert$ $= AB$ $=1$ Maximum value of $\lvert z-2\mathrm{i}\rvert$ $= AC$ $=\sqrt{2^2+2^2-2(2)(2)\cos\frac{2\pi}{3}}$ $=2\sqrt{3}$
(iii)	$z^{100}=2^{100}=2^{100}\mathrm{e}^{\mathrm{i}0}$ $\Rightarrow z=2\mathrm{e}^{\mathrm{i}\left(\frac{0+2k\pi}{100}\right)},\ k=0,\ \pm1,\ \pm2,\ \ldots,\ \pm49,\ 50$ $\Rightarrow z=2\mathrm{e}^{\mathrm{i}\frac{k\pi}{50}}$ Roots are found in region *R* (along the minor arc *CD*) if $-\frac{\pi}{6}\le\frac{k\pi}{50}\le\frac{\pi}{6}$. $\Rightarrow -8\frac{1}{3}\le k\le 8\frac{1}{3}$ $\Rightarrow k=-8,\ -7,\ -6,\ \ldots,\ 8$ $\therefore$ Number of roots found in region $R=17$.

Qn	Solution
9(i)	$\mathrm{f}(x)=x+\frac{m^2}{x-2}$ Let $\frac{\mathrm{df}}{\mathrm{d}x}=1-\frac{m^2}{(x-2)^2}=0$ $(x-2)^2-m^2=0$ $x=2\pm m$ When $x=2+m$, $\mathrm{f}(x)=2+m+\frac{m^2}{2+m-2}=2+2m$ When $x=2-m$, $\mathrm{f}(x)=2-m+\frac{m^2}{2-m-2}=2-2m$ The stationary points are $(2+m, 2+2m)$ and $(2-m, 2-2m)$

	$y = \mathrm{f}(x) = x + \dfrac{m^2}{x-2}$ 
(ii)	$\mathrm{f}(x) = \dfrac{x(x-2)+m^2}{x-2}$ when $x = 0$, $\mathrm{f}(x) = -\dfrac{m^2}{2}$ $x^2 - (2+k)x + (m^2 + 2k) = 0$ $x^2 - 2x + m^2 = k(x-2)$ $x + \dfrac{m^2}{x-2} = k$ By inserting a horizontal line $y = k$ on the graph of *C*, by observation, to have two distinct positive roots, then $-\dfrac{m^2}{2} < k < 2 - 2m \quad \text{or} \quad k > 2 + 2m$ (ans)

(iii)	$y = \mathrm{f}(x) = x + \dfrac{m^2}{x-2}$ After A: $y = \mathrm{f}(x+2) = x + 2 + \dfrac{m^2}{x}$ After B: $y = \mathrm{f}(2x+2) = 2x + 2 + \dfrac{m^2}{2x}$ After C: $y = \mathrm{f}(2x+2) - 2 = 2x + \dfrac{m^2}{2x}$ Given that $2x + \dfrac{m^2}{2x} = 2x + \dfrac{1}{8x}$ $\Rightarrow \dfrac{m^2}{2} = \dfrac{1}{8}$ $\Rightarrow m = \dfrac{1}{2}$ since $0 < m < 1$ (ans)

Qn	Solution
10 (i)	Let F be the foot of the perpendicular. $\overrightarrow{AF} \bullet \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$ $\left[\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}\right] \bullet \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$ $\Rightarrow -2 + 3\lambda - 1 = 0$ $\Rightarrow \lambda = 1$ $\overrightarrow{OF} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$.
(ii)	Let B be $(2,\ -3,\ 1)$. $\overrightarrow{BA} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ A cartesian equation of π_1 is $x + y + 2z = 1$.

(iii)	$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix}1\\1\\2\end{pmatrix} \times \begin{pmatrix}1\\0\\7\end{pmatrix} = \begin{pmatrix}7\\-5\\-1\end{pmatrix}$ A direction vector of L is $\begin{pmatrix}-7\\5\\1\end{pmatrix}$. π_1: $x+y+2z=1$ π_2: $x+7z=c$ Let $z=0$. Then $x=c$ and $y=1-c$. A point on L is $(c,\ 1-c,\ 0)$. $\therefore$ A vector equation of L is $\mathbf{r} = \begin{pmatrix}c\\1-c\\0\end{pmatrix} + \mu\begin{pmatrix}-7\\5\\1\end{pmatrix}$, where $\mu \in \mathbb{R}$.
(iv) (a)	For the 3 planes to meet in the line L, $\begin{pmatrix}7\\-5\\1\end{pmatrix} \bullet \begin{pmatrix}2\\-1\\d\end{pmatrix} = 0$ and $\begin{pmatrix}c\\1-c\\0\end{pmatrix} \bullet \begin{pmatrix}2\\-1\\d\end{pmatrix} = 5$. $\Rightarrow -14-5+d=0$ and $2c-1+c=5$ $\Rightarrow d=19$ and $c=2$
(b)	For the 3 planes to have only one point in common, $\begin{pmatrix}7\\-5\\1\end{pmatrix} \bullet \begin{pmatrix}2\\-1\\d\end{pmatrix} \neq 0$. $\Rightarrow d \neq 19$

Qn	Solution

11 (i)

Instalment	Outstanding amount at the beginning of month	Total amount with interest owed at the end of month
	50000	50000×1.002
1	$50000\times1.002-1000$	$50000\times1.002^2-1000\times1.002$
2	$50000\times1.002^2-1000\times1.002-1000$	$50000\times1.002^3-1000\times1.002^2-1000\times1.002$
……..	…………………… ……….	……………………………. ………

Amount owed at the end of n instalments

$$=50000\times1.002^{n+1}-1000\times1.002^{n}-1000\times1.002^{n-1}-\dots-1000\times1.002$$

$$=50000\times1.002^{n+1}-\frac{1000\times1.002\times\left[1.002^{n}-1\right]}{1.002-1}$$

$$=50000\times1.002^{n+1}-1000\times501\times(1.002^{n}-1)\quad\text{------------(*)}$$

$$=50000\times1.002^{n+1}-501000\times(1.002^{n}-1)\ \text{(shown)}$$

(ii)

$$50000\times1.002^{n+1}-501000\times(1.002^{n}-1)\le0$$

By using G.C., least integer $n=53$

i.e. no of instalments required = 53

(iii)

Amount paid at the 53th instalment

= Amount owed at the end of 52 instalments

$$=50000\times1.002^{53}-501000\times\left(1.002^{52}-1\right)$$

=733.12 (to 2 d.p.)

(iv)

Let the amount need to be paid for each instalment be k.

Then from (*) in (i)

$$50000\times1.002^{20}-k\times501\left[1.002^{19}-1\right]\le0$$

$$k\ge\frac{50000\times1.002^{20}}{501\left[1.002^{19}-1\right]}=2684.53$$

Least value of k = 2685 (to the nearest dollars)

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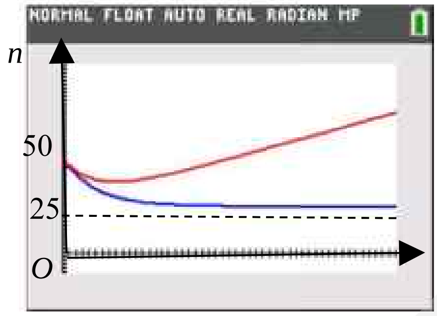
2016 JC2 H2 Mathematics Prelim Paper 2 Solutions

Qn	Solution
1(a)	$\frac{dy}{dx} = \sec^2 y$ $\int \cos^2 y \, dy = \int 1 \, dx$ $\int \frac{\cos 2y + 1}{2} \, dy = \int 1 \, dx$ $\frac{1}{2}\left[\frac{\sin 2y}{2} + y\right] = x + c$ When $y = 0$, $x = 1 \Rightarrow c = -1$ $\therefore \ x = \frac{1}{4}\sin 2y + \frac{1}{2}y + 1$
1(b) (i)	$\frac{d^2n}{dt^2} = e^{-\frac{t}{5}}$ $\frac{dn}{dt} = \int e^{-\frac{t}{5}} dt = -5e^{-\frac{t}{5}} + C$ $n = 25e^{-\frac{t}{5}} + Ct + D$
1(b) (ii)	$\frac{d^2n}{dt^2} = e^{-\frac{t}{5}} > 0$ for all values of t. Solution curves are concave upwards.

1(b) (iii)	When $t = 0, n = 50$ $C = 1$ $50 = 25e^0 + C(0) + D$ $D = 25$ $C = 0$ $n = 25e^{-\frac{t}{5}} + Ct + 25$ When $C = 0$, $n = 25e^{-\frac{t}{5}} + 25$. As $t \to \infty, n \to 25$ When $C = 1$, $n = 25e^{-\frac{t}{5}} + t + 25$. As $t \to \infty, n \to \infty$ 

Qn	Solution
2(a)	Area of parallelogram $= \lvert \mathbf{a} \times (2\mathbf{a} + 3\mathbf{b}) \rvert$ $= \lvert 2(\mathbf{a} \times \mathbf{a}) + 3(\mathbf{a} \times \mathbf{b}) \rvert$ $= 3\lvert \mathbf{a} \times \mathbf{b} \rvert$ $= 3\lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \sin 30°$ $= 3(4)(5)\frac{1}{2}$ $= 30$

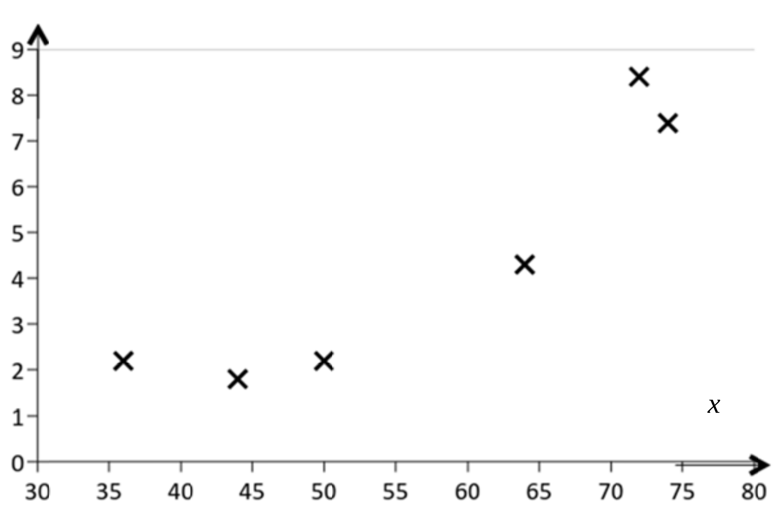
(b) **(i)**	A vector equation of l is $\mathbf{r}=\begin{pmatrix}1\\3\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-1\end{pmatrix}$. Let A be the point $(1,\ 3,\ 1)$ on l. Perpendicular distance from P to l= $\dfrac{\left\lvert\overrightarrow{AP}\times\begin{pmatrix}2\\0\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}2\\0\\1\end{pmatrix}\right\rvert}$ $=\dfrac{1}{\sqrt{5}}\left\lvert\begin{pmatrix}1\\-4\\-3\end{pmatrix}\times\begin{pmatrix}2\\0\\-1\end{pmatrix}\right\rvert$ $=\dfrac{1}{\sqrt{5}}\left\lvert\begin{pmatrix}4\\-5\\8\end{pmatrix}\right\rvert$ $=\dfrac{\sqrt{105}}{\sqrt{5}}\ =\sqrt{21}$
(ii)	Acute angle between l and L= $\cos^{-1}\dfrac{\left\lvert\begin{pmatrix}2\\0\\-1\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}2\\0\\-1\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}$ $=\cos^{-1}\dfrac{1}{\sqrt{5}}$ $=63.4°$

Qn	Solution
3(i)	Area of equilateral ⊗ = $\frac{1}{2}x^2\sin 60° = \frac{\sqrt{3}}{4}x^2$ Given that the volume of the box is 250 cm^3 $V = \frac{\sqrt{3}}{4}x^2 y = 250$ $y = \frac{1000}{\sqrt{3}x^2}$ Surface Area $A = 3xy + 3kxy + 2\left(\frac{\sqrt{3}}{4}x^2\right)$ $= 3xy(1+k) + \frac{\sqrt{3}}{2}x^2$ $= 3x(1+k)\frac{1000}{\sqrt{3}x^2} + \frac{\sqrt{3}}{2}x^2$ $= \frac{1000\sqrt{3}(1+k)}{x} + \frac{\sqrt{3}}{2}x^2$ (shown)
(ii)	For stationary points, $\frac{dA}{dx} = -\frac{1000\sqrt{3}(1+k)}{x^2} + \sqrt{3}x = 0$ $x^3 = 1000(1+k)$ $x = 10(1+k)^{\frac{1}{3}}$ $\frac{d^2A}{dx^2} = \frac{2000\sqrt{3}(1+k)}{x^3} + \sqrt{3} > 0$ Thus, $x = 10(1+k)^{\frac{1}{3}}$ gives a minimum surface area.

(iii)	Since $y = \frac{1000}{\sqrt{3}x^2}$ $\frac{y}{x} = \frac{1000}{\sqrt{3}x^3} = \frac{1000}{\sqrt{3}(1000)(1+k)}$ $= \frac{1}{\sqrt{3}(1+k)}$
(iv)	Since $0 < k \le 1$ $1 < 1 + k \le 2$ $\frac{1}{2} \le \frac{1}{1+k} < 1$ $\frac{1}{2\sqrt{3}} \le \frac{1}{\sqrt{3}(1+k)} < \frac{1}{\sqrt{3}}$ i.e. $\frac{1}{2\sqrt{3}} \le \frac{y}{x} < \frac{1}{\sqrt{3}}$
Qn	**Solution**
4(i)	$a^2 b = \frac{1}{2}(1+\sqrt{3}\mathrm{i})^2(1-\mathrm{i})$ $= \frac{1}{2}(1+2\sqrt{3}\mathrm{i}-3)(1-\mathrm{i})$ $= (-1+\sqrt{3}\mathrm{i})(1-\mathrm{i})$ $= (\sqrt{3}-1)+(\sqrt{3}+1)\mathrm{i}$
(ii)	$\lvert a^2 b\rvert = \lvert a\rvert^2 \lvert b\rvert$ $= 2^2\left(\frac{1}{\sqrt{2}}\right)$ $= 2\sqrt{2}$ $\arg(a^2 b) = 2\arg(a) + \arg(b)$ $= 2\left(-\frac{2\pi}{3}\right) - \frac{\pi}{4}$ $= -\frac{19\pi}{12}$ $\therefore \quad \arg(a^2 b) = -\frac{19\pi}{12} + 2\pi = \frac{5\pi}{12}$.
(iii)	Considering the imaginary part of $a^2 b$, we have

	$2\sqrt{2}\sin\frac{5\pi}{12}=\sqrt{3}+1$ $\Rightarrow \sin\frac{5\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}$
(iv)	Vector *BA* can be obtained by rotating vector *BC* through $90°$ in the anticlockwise direction about *B*. $\text{i}(c-b)=a-b$ $\Rightarrow c=-\text{i}(a-b)+b$ $=-\text{i}a+b(1+\text{i})$ $=\text{i}(1+\sqrt{3}\text{i})+\frac{1}{2}(2)$ $=(1-\sqrt{3})+\text{i}$
5 (i)	${}^{6}C_2\times{}^{5}C_4=75$ ways
(ii)	Number of ways if at least one of the sisters are included = number of ways without restriction – number of ways if none of the sisters is included $={}^{11}C_6-{}^{8}C_6$ $=434$
	<u>Alternative Method</u> ${}^{3}C_1\times{}^{8}C_5+{}^{3}C_2\times{}^{8}C_4+{}^{3}C_3\times{}^{8}C_3=434$
(iii)	Select a man to be between the 2 sisters and group the 3 of them as one unit and arrange 4 units round a table Number of ways = ${}^{3}C_1\times3!\times2$ $=36$
(iv)	First arrange the other 4 persons round the table. There are 4 ways to insert the sisters. Number of ways = $3!\times4$ $=24$

Qn	Solution
6 (i)	$\mathrm{P}(C\,\vert M) = \dfrac{\mathrm{P}(C \cap M)}{\mathrm{P}(M)}$ $= \dfrac{200}{400} = \dfrac{1}{2}$
(ii)	$\mathrm{P}(M \cup C) = \mathrm{P}(M) + \mathrm{P}(C) - \mathrm{P}(M \cap C)$ $= \dfrac{400}{1000} + \dfrac{450}{1000} - \dfrac{200}{1000}$ $= \dfrac{650}{1000} = \dfrac{13}{20}$
(iii)	$\mathrm{P}(M'\ )\ B') = \dfrac{250+300}{1000} = \dfrac{11}{20}$ $P(C) = \dfrac{9}{20}$ $P(C\vert M) = \dfrac{1}{2} \neq P(C)$ C and M are not independent.
(iv)	No. of international studens in the sample $= 0.2(200+250) + 0.3(130+300) + 0.05(120) = 225$ $\mathrm{P}(C\vert \text{international student}) = \dfrac{\mathrm{P}(C \cap \text{international student})}{\mathrm{P}(\text{international student})}$ $= \dfrac{\frac{(200+250)0.2}{1000}}{\frac{225}{1000}}$ $= 0.4$
(v)	Number of international students studying Physics $= 0.3(430) = 129$ P(exactly one international student studying Physics) $= \dfrac{{}^{129}C_1\ {}^{871}C_2}{{}^{1000}C_3}$ $= 0.294$
	<u>Alternative method</u> Required Probability $= \dfrac{129}{1000}\dfrac{871}{999}\dfrac{870}{998} \times 3$ $= 0.294$

7 **(i)**	
(ii)	$r = 0.914099 \approx 0.914$ (to 3 s.f) Though the value of *r* shows a strong positive linear correlation, from the scatter diagram, it is possible that *x* and *y* may have a curvilinear relationship.
(iii)	For $y = c + dx^2$, $r = 0.93986 \approx 0.940$ Since the value of *r* for $y = c + dx^2$ is closer to the value of 1, $y = c + dx^2$ is a better model.
(iv)	$y = -0.88934 + 0.0015441x^2$ When *x* = 55, $y = -0.88934 + 0.0015441(55)^2$ $y = 3.7816 \approx 3.8$ (to 1 d.p)
	Since $x = 55$ is within the range of data given and $r \approx 0.940$ is close to 1, the estimation is reliable.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>8
(i)</td><td>P(first person that uses Voyager is the third person selected)

$= 0.92 \times 0.92 \times 0.08$

= 0.067712</td></tr>
<tr><td>(ii)</td><td>1. Whether a person uses Voyager is independent of another person.
2. The probability that a person uses Voyager is constant for every person in the sample.</td></tr>
<tr><td>(iii)</td><td>Let Y be the number of people who use Voyager out of 80 people.

$Y \sim \mathrm{B}(80, 0.08)$

Since $n = 80 > 50, \ \ np = 6.4 > 5, \ \ nq = 73.6 > 5$,

$Y \sim \mathrm{N}(6.4, 5.888)\,approx$

$\mathrm{P}(Y < 10) \xrightarrow{c.c} \mathrm{P}(Y \le 9.5)$

= 0.899295

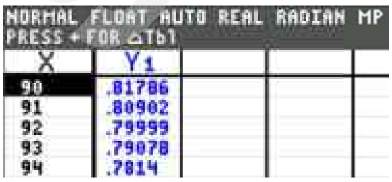
= 0.899 (to 3 s.f.)</td></tr>
<tr><td>(iv)</td><td>Let V be the number of people who use Voyager out of n people.

$V \sim \mathrm{B}(n, 0.08)$

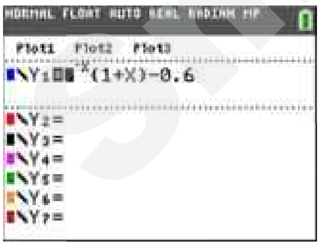
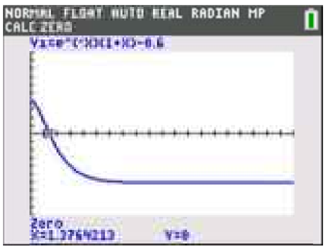
$\mathrm{P}(V \ge 10) > 0.2$

$1 - \mathrm{P}(V \le 9) > 0.2$
$\mathrm{P}(V \le 9) < 0.8$

Using GC,


NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR ΔTbl
X | Y1
90 | .81786
91 | .80902
92 | .79999
93 | .79078
94 | .7814

Least value of n = 92</td></tr>
</table>

9 (i)	Let C be the number of boxes of chocolate biscuits sold in a day. $C \sim \text{Po}(2.2)$ $\text{P}(C=0) = 0.11080$ $= 0.111$ (to 3 s.f.)
(ii)	Let D be the number of days that no boxes of chocolate biscuits were sold out of 7 days. $D \sim \text{B}(7, 0.11080)$ $\text{E}(D) = 7 \times 0.11080$ $= 0.77562$ $= 0.776$
(iii)	Let S be the number of boxes of strawberry biscuits sold in a day. $S \sim \text{Po}(\lambda)$ $\text{P}(S<2) = 0.6$ $\text{P}(S=0)+\text{P}(S=1) = 0.6$ $\text{e}^{-\lambda}\left(\frac{\lambda^0}{0!}\right)+\text{e}^{-\lambda}\left(\frac{\lambda^1}{1!}\right) = 0.6$ $\text{e}^{-\lambda}(1+\lambda) = 0.6$
	Using GC,   $\lambda = 1.376$ $= 1.4$ (to 1 d.p)

(iv)	Let T be the total number of boxes of chocolate and strawberry biscuits sold in 7 days. $T \sim \text{Po}\left(7\times 2.2+7\times 1.376\right)=\text{Po}(25.032)$ $\text{P}(T>25)=1-\text{P}(T\le 25)$ $= 0.44962$ $= 0.450$ (to 3 s.f)
(v)	Let X be number of boxes of chocolate biscuits sold in 30 days. $X \sim \text{Po}\left(30\times 2.2\right)=\text{Po}(66)$ Since $\lambda = 66 > 10, \quad X \sim \text{N}(66, 66)\text{approx}$ Let Y be number of boxes of strawberry biscuits sold in 30 days. $Y \sim \text{Po}\left(30\times 1.376\right)=\text{Po}(41.28)$ Since $\lambda = 41.28 > 10, \quad Y \sim \text{N}(41.28, 41.28)\text{approx}$ $X - Y \sim \text{N}(24.72, 107.28)\text{approx}$ $\text{P}\left(X-Y>0\right)\xrightarrow{c.c}\text{P}\left(X-Y>0.5\right)$ $= 0.99032$ $= 0.990$ (to 3 s.f.)

10 (i)	Choose a plant randomly from the first 10 plants, say the 5th plant. Choose every 10th plant thereafter until 8 plants are selected i.e. 5th, 15th, 25th, … The 8 plants selected will be *evenly spread out* across the row of 80 plants.
(ii)	Unbiased estimate of the population mean, $\hat{\mu}$ $=\frac{\sum(x-150)}{80}+150$ $=-\frac{160}{80}+150$ $=148$ Unbiased estimate of the population variance, s^2 $=\frac{1}{80-1}\left[\sum(x-150)^2-\frac{\left(\sum(x-150)\right)^2}{80}\right]$ $=\frac{1}{79}\left[5520-\frac{(-160)^2}{80}\right]$ $=\frac{5200}{79}$

(iii)	$H_0 : \mu = 150$ $H_1 : \mu < 150$ Under H_0, since n =80 >50, by Central Limit Theorem, $\overline{X} \sim N\left(150, \frac{5200}{79(80)}\right)$ approx. Test statistic $Z = \frac{\overline{X} - 150}{\sqrt{\frac{5200}{79(80)}}} \sim N(0,1)$ approx. From GC, p–value = 0.013731 = 0.0137 (to 3 s.f.) $\alpha = 0.01$ Since p–value = 0.0137 > $\alpha = 0.01$, we do not reject H_0 at 1% level of significance and conclude that there is insufficient evidence that the researcher's claim is invalid.
(iv)	It means that there is a probability of 0.01 of concluding that the population mean mass of a new variety of potato is less than 150g given that the population mean mass of a new variety of potato is in fact 150g.

	Unbiased estimate of the population variance $=\frac{8}{7}k^2$ $H_0 : \mu = 150$ $H_1 : \mu < 150$ Under H_0, test statistic $T = \frac{\overline{X} - 150}{\sqrt{\frac{S^2}{8}}} \sim t(7)$ $\alpha = 0.01$ Researcher's claim is invalid at 1% level of significance $\Rightarrow H_0$ is rejected at 1% level of significance $\Rightarrow t \le -2.9980$ $\Rightarrow \frac{148.5 - 150}{\sqrt{\frac{k^2}{7}}} \le -2.9980$ $\Rightarrow k \le 1.3238$ $\therefore k \le 1.32$ (to 3 s.f)

MERIDIAN JUNIOR COLLEGE

JC2 Preliminary Examination

Higher 2

H2 Mathematics 9740/01

Paper 1 **13 September 2016**

3 Hours

Additional Materials: Writing paper

List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **8** printed pages.

1 A graphic calculator is not to be used for this question.

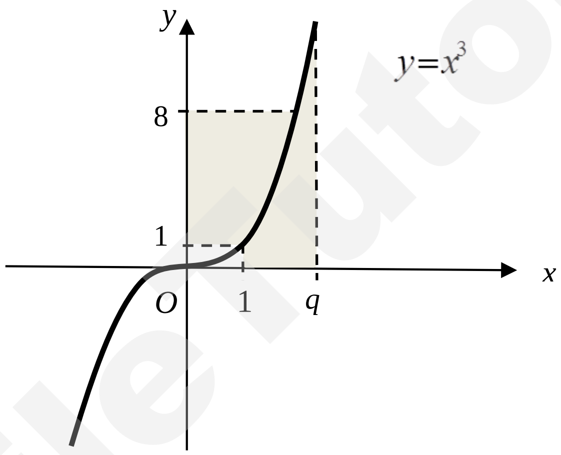
Show algebraically that $x^2 - 2x + 5$ is always positive for $x \in \mathbb{R}$, and solve the inequality

$$\frac{x}{x^2 - 2x + 5} \le \frac{x+2}{x^3 - 2x^2 + 5x}.$$ [4]

Hence solve the inequality $\dfrac{\mathrm{e}^x}{\mathrm{e}^{2x} - 2\mathrm{e}^x + 5} \ge \dfrac{\mathrm{e}^x + 2}{\mathrm{e}^{3x} - 2\mathrm{e}^{2x} + 5\mathrm{e}^x}$. [2]

2 **(a)** Find, in terms of p, $\int_1^p \ln(x)\,\mathrm{d}x$, where $p > 1$. [2]

(b)



The diagram shows the curve with the equation $y = x^3$. The area of the region bounded by the curve, the lines $x = 1$, $x = q$ and the x-axis is equal to the area of the region bounded by the curve, $y = 1$, $y = 8$ and the y-axis, where $q > 1$. Find the exact value of q in the form $a^{\frac{1}{b}}$, where a and b are integers. [4]

3 Prove by mathematical induction that

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = 2\left(1 - \frac{1}{n+1}\right), \; n \in \mathbf{Z}^+.$$ [5]

Hence state the value of the infinite series $1 + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \ldots + \dfrac{1}{1+2+3+\ldots+n} + \ldots$.

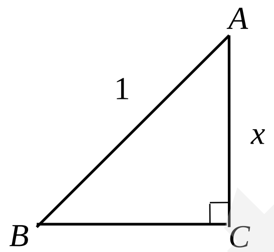
[1]

4 Let $\mathrm{f}(x) = \cos^{-1} x$, where $-1 < x < 1$ and $0 < \mathrm{f}(x) < \pi$. Show that

$$\left(1-x^2\right)\mathrm{f}''(x) = x\,\mathrm{f}'(x).$$ [2]

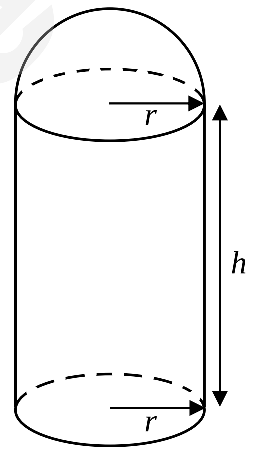
By further differentiation of this result, or otherwise, find the first three non-zero terms in the expansion of $\mathrm{f}(x)$ in ascending powers of x. [3]

The diagram shows a triangle *ABC*. Given that the lengths of *AB* and *AC* are 1 and x units respectively, show that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$. [1]



Hence find the series expansion of $\sin^{-1} x$ in ascending powers of x, up to and including the term in x^3. [1]

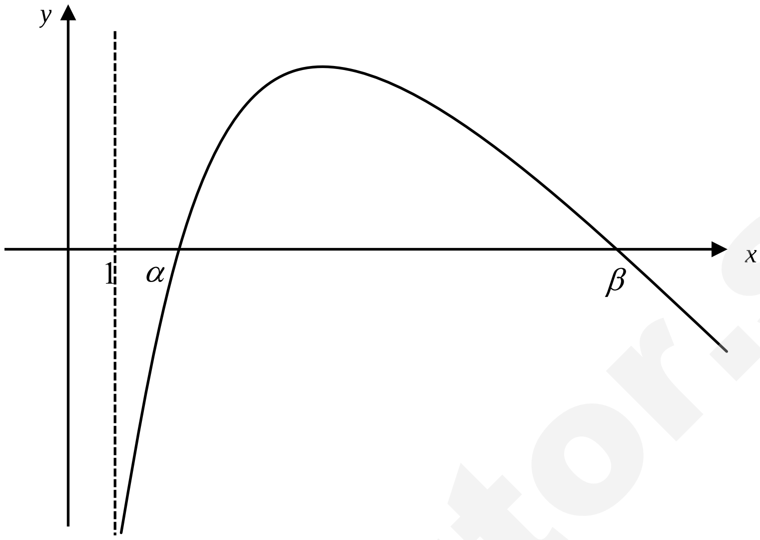
5 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]



A rice farmer wants to build a new grain silo to store his rice grains. The cylindrical section has height h m and the hemispherical roof has radius r m. After building the grain silo, the farmer will be painting its rooftop and the external curved surface. The time needed to paint the grain silo will be 20 minutes per square metre for the curved surface area of the cylinder and 35 minutes per square metre for the hemispherical roof. Given that a total time of 60 000 minutes is taken to paint the grain silo, find, using differentiation, the value of r which gives a grain silo of maximum volume. [8]

6 The diagram below shows the graph of $y = 2\ln\left(x-1\right)+4-x$.

The two roots of the equation $2\ln\left(x-1\right)+4-x=0$ are denoted by α and β, where $\alpha<\beta$.



(i) Find the values of α and β, correct to 3 decimal places. [2]

A sequence of real numbers $x_1, x_2, x_3, \ldots$ where $x_n > 1$, satisfies the recurrence relation

$$x_{n+1} = \ln(x_n - 1)^2 + 4 \text{ for } n \geq 1.$$

(ii) Prove algebraically that if the sequence converges, it must converge to either α or β. [2]

(iii) Use a calculator to determine the behaviour of the sequence for each of the cases $x_1 = 3$, $x_1 = 12$. [2]

(iv) By considering $x_{n+1} - x_n$ and the graph above, prove that

$$x_{n+1} > x_n \text{ if } \alpha < x_n < \beta,$$

$$x_{n+1} < x_n \text{ if } 1 < x_n < \alpha \text{ or } x_n > \beta.$$ [2]

7 The equations of three planes are

$$x + 2y + z = 60$$
$$4x + 5y + 10z = 180$$
$$2x + 3y + 4z = 100$$

(i) It is given that all three planes meet in the line l. Find a vector equation of l. [2]

(ii) Find a cartesian equation of the plane which contains l and the origin. [3]

A technology company specialises in manufacturing circuit boards that are used for space exploration. It manufactures only 3 types of circuit boards (A, B and C). Each circuit board requires particular amounts of different raw materials for manufacturing. The amounts of raw material (in units) required for each type of circuit board and the total amounts of raw material available to the company are shown in the following table.

	Copper	Lead	Fibreglass
Circuit Board A	1	4	2
Circuit Board B	2	5	3
Circuit Board C	1	10	4
Total amount of material available (in units)	60	180	100

The company is required to use all the materials available to manufacture its circuit boards.

The vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is defined such that variables x, y, and z represent the number of circuit boards A, B and C that are manufactured respectively.

(iii) With the aid of your answer in part **(i)**, solve for $\mathbf{r}$. Leave your answer clearly in the form of $\mathbf{a} + \mu\mathbf{b}$ and state the possible values for μ. [2]

(iv) Explain, in context, why your vector equation in part **(i)** is not an appropriate answer for part **(iii)**. [2]

8 Functions f and g are defined by

$$\mathrm{f} : x \mapsto \left(\frac{1}{x+1}\right)^2, \quad x > -1,$$

$$\mathrm{g} : x \mapsto \ln x, \quad x > 0.$$

(i) Show that gf exists and express gf in a similar form. [3]

(ii) Sketch, in a single diagram, the graphs of g and gf, labelling each graph clearly. Write down the range of gf. [3]

(iii) Describe a sequence of transformations which maps the graph of g onto the graph of gf. [4]

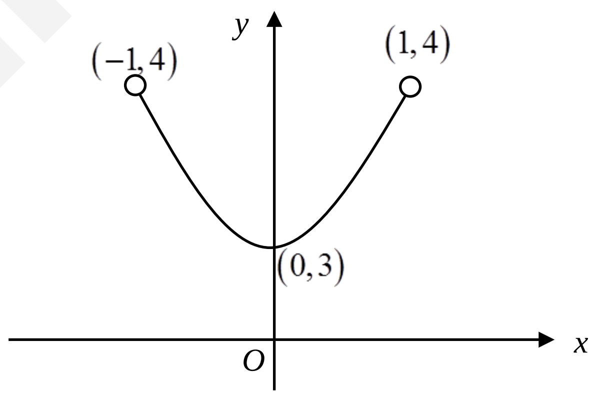
9 It is given that

$$\mathrm{f}(x) = \frac{x}{\sqrt{(1-x^2)}}, \text{ where } -1 < x < 1.$$

(i) Show by differentiation that f is strictly increasing. [3]

(ii) Sketch the graph of $y = \mathrm{f}(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

The diagram below shows the graph of $y = \mathrm{g}(x)$, which is continuous and differentiable on $(-1, 1)$. It has a minimum turning point at $(0, 3)$.



(iii) It is given that $\mathrm{w}(x) = \mathrm{g}(x)\mathrm{f}(x)$, where $-1 < x < 1$. By finding $\mathrm{w}'(x)$ and using your earlier results in **(i)** and **(ii)**, determine the number of stationary points on the graph of w. [4]

10 **(a)** Solve the simultaneous equations

$$z = w + 2\mathrm{i} - 1 \text{ and } z^2 - \mathrm{i}w + \frac{5}{2} = 0,$$

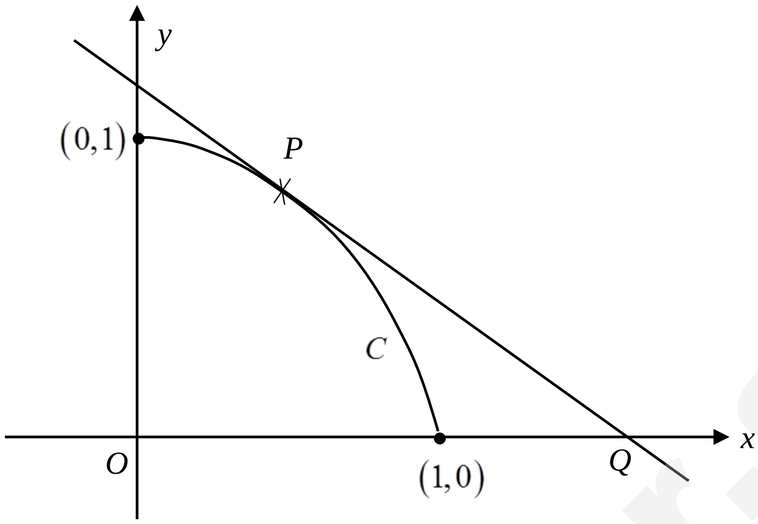
giving z and w in the form $x + y\mathrm{i}$ where x and y are real. [5]

(b) **(i)** Given that $z = w - \frac{1}{w}$ where $w = 2(\cos\theta + \mathrm{i}\sin\theta)$, $-\pi < \theta \le \pi$, express the real and imaginary parts of z in terms of θ. [3]

(ii) Hence show that locus of z on an Argand diagram lies on the curve with cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are constants. [3]

(iii) Sketch this locus on an Argand diagram, indicating clearly the points of intersection with the axes. [2]

11



It is given that curve C has parametric equations

$$x = t^3, \quad y = \sqrt{(1-t^2)} \quad \text{for } 0 \le t \le 1.$$

The diagram shows the curve C and the tangent to C at P. The tangent at P meets the x-axis at Q.

(i) The point P on the curve has parameter p. Show that the equation of the tangent at P is $3p(1-p^2) - 3py\sqrt{(1-p^2)} = x - p^3$. [3]

(ii) Given further that the line $y = (4\sqrt{3})x$ meets the curve at point P, find the exact coordinates of P. [3]

(iii) Hence find the exact coordinates of Q. [2]

(iv) Show that the area of the region bounded by C, the tangent to C at P, and the x-axis is given by $\dfrac{9\sqrt{3}}{32} - \displaystyle\int_{\frac{1}{2}}^{1} 3t^2\sqrt{(1-t^2)}\,\mathrm{d}t$. [3]

Show that the substitution $t = \sin u$ transforms the above integral to $\dfrac{9\sqrt{3}}{32} - \dfrac{3}{8}\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 - \cos 4u \,\mathrm{d}u$. Hence, evaluate this area exactly. [6]

END OF PAPER

MERIDIAN JUNIOR COLLEGE
JC2 Preliminary Examination
Higher 2

H2 Mathematics — 9740/02

Paper 2 — **21 September 2016**

3 Hours

Additional Materials: Writing paper
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

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Answer **all** the questions.
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At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

Section A: Pure Mathematics [40 marks]

1 Referred to the origin O, points A and B have position vectors **a** and **b** respectively, where **a** and **b** are non-zero vectors that are neither perpendicular nor parallel to each other. The length of projection of **a** onto **b** and the length of projection of **b** onto **a** are equal. Show that $|\mathbf{a}| = |\mathbf{b}|$. [2]

Hence state the geometrical interpretation of $|\mathbf{a} \times \mathbf{b}|$. [1]

It is further given that $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + p\mathbf{k}$, where $p < 0$.

(i) Find the exact value of p. [2]

(ii) A circle with centre O passes through A and B. Find the area of the minor sector OAB. [4]

2 **(a)** The roots of the equation $z^3 - z^2 - z - 15 = 0$ are denoted by z_1, z_2 and z_3 where $\arg(z_1) = 0$, and $\arg(z_2) > \arg(z_3)$. Find z_1, z_2 and z_3 and show these roots on an Argand diagram. [3]

Explain why the locus of all points z such that $|z+1| = 2$ passes through the roots represented by z_2 and z_3 Draw this locus on the same Argand diagram. [3]

(b) **(i)** Show that $1 + e^{i\theta} = 2e^{i\frac{\theta}{2}} \cos\frac{\theta}{2}$. [2]

(ii) Hence find, in trigonometric form, the imaginary part of the complex number $w = \dfrac{e^{i\theta}}{1+e^{i\theta}}$. [2]

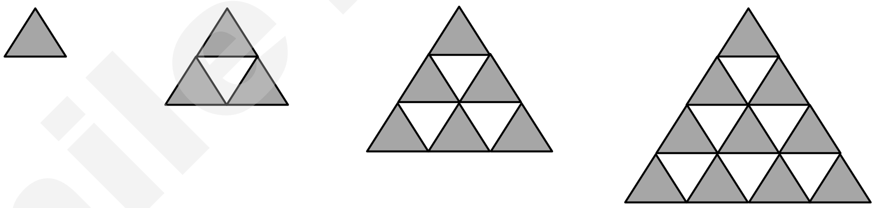
3 Newton's law of cooling states that the rate of decrease of temperature of a hot body is proportional to the difference in temperature between the body and its surroundings. Using t for time in minutes, θ for temperature of the body in °C and α for the temperature of the surroundings (assumed constant), express the law in the form of a differential equation. [1]

(i) Show that the general solution of the differential equation may be expressed in the form $\theta = \alpha + A\mathrm{e}^{-kt}$ where A and k are constants. [3]

(ii) Given that $\theta = 9\alpha$ when $t = 0$ and that $\theta = 5\alpha$ when $t = T$, find, in terms of T, the value of t when $\theta = 2\alpha$. [3]

(iii) State what happens to θ for large values of t and sketch the solution curve of θ against t. [3]

4 **(a)** Judith is making a pattern consisting of rows of matchstick triangles as shown. She uses three matchsticks to complete a triangle. She adds two more triangles in the second row, three more triangles in the third row and four more triangles in the fourth row.



Judith has completed $n-1$ rows in the pattern. How many matchsticks does she need in order to form the n^{th} row? [1]

Show that the total number of matchsticks used in making a pattern with n rows is $\dfrac{3n(n+1)}{2}$. Hence find the maximum number of complete rows she is able to make with two thousand matchsticks. [4]

(b) A geometric progression has first term a and second term b, where a and b are non-zero constants. Given that the sum to infinity of the series is $a+2b$, find the common ratio. [2]

The sum of the first n terms is denoted by G_n. Find G_n in terms of a and n. Hence show that $\sum_{n=1}^{N} G_n = 2aN - G_N$. [4]

Section B: Statistics [60 marks]

5 **(i)** Describe what is meant by 'systematic sampling'. [2]

(ii) A bakery wishes to gather feedback on what residents in the neighbourhood think of its new salted egg lava buns. A surveyor is hired to survey a sample of 150 residents who visit the bakery during the evening rush hour using systematic sampling. State, in this context, one advantage and one disadvantage of this procedure. [2]

6 A box consists of a very large number of balls, of which 20% are red and 80% are white.

A game consists of a player drawing n balls at random from the box and counting the number of red balls drawn. If at most one red ball is drawn, the player wins. If more than two red balls are drawn, the player loses. If exactly two red balls are drawn, the player draws another n balls and if none of these n balls drawn are red, the player wins. Otherwise, the player loses.

Show that the probability that a randomly chosen player wins is P where

$$P = (0.8 + 0.2n)(0.8)^{n-1} + \binom{n}{2}(0.2)^2(0.8)^{2n-2}.$$ [3]

(i) Given that the probability that a randomly chosen player wins is less than 0.1, write down an inequality in terms of n to represent this information. Hence find the least possible value of n. [3]

(ii) Given instead that $P = 0.3$, find the probability that out of 100 games played, at least 40 games are won. [2]

7 An overseas study revealed that school children sleep an average of 6.5 hours each night. Ms Patricia believes that the children in her school sleep even fewer than that. She took a random sample of 8 children from her school. The number of hours of sleep each child gets at night was reported as:

5.9 6 6.1 6.2 6.3 6.5 6.7 6.9

Test, at the 8% level of significance, whether this evidence supports Ms Patricia's belief, stating clearly any assumption made. [5]

Ms Patricia conducted a further study involving a random sample of 15 children from another school and the number of hours of sleep each child gets at night is recorded. The sample mean is $\bar{x}$ and the sample variance is 0.849. Find the set of values of $\bar{x}$ for which the null hypothesis would be rejected at the 8% level of significance. [3]

8 The mass of a randomly chosen bar of body soap manufactured by a factory has a normal distribution with mean 110 grams and standard deviation 1.5 grams.

(i) Find the probability that the difference in sample means between any two random samples of 20 bars of body soap each, is within 0.5 grams. [4]

(ii) Five randomly chosen bars of body soap are liquefied and separated into four equal portions, which are each placed into a bottle. Find the probability that the mass of liquid body soap in a randomly chosen bottle exceeds 140 grams. [4]

The factory ventured into the manufacturing of coconut oil soap as its new product and the mass of a randomly chosen bar of coconut oil soap has a normal distribution. A random sample of 15 bars of coconut oil soap is taken and the mass, u grams, of each bar is measured. The results are summarised by $\sum u = 1590$, $\sum u^2 = 169046$.

(iii) Find unbiased estimates of population mean and variance. [2]

9 (a) For events A and B, it is given that $\mathrm{P}(A)=\frac{1}{4}$ and $\mathrm{P}(B)=\frac{1}{2}$.

(i) Given that $\mathrm{P}(A'|B)=\frac{3}{4}$, determine whether events A and B are independent and calculate $\mathrm{P}(A\cup B)$. [3]

(ii) For a third event C, it is given that $\mathrm{P}(C|A)=\frac{2}{3}$. Find the value of $\mathrm{P}(A\cap C)$. [2]

(b) Find the number of ways in which the word EVERYDAY can be arranged if

(i) all the vowels (A, E) must be together and the two 'Y's must be separated, [3]

(ii) the repeated letters E and Y must appear symmetrical about the centre of the word (e.g. EVRYYDAE, YVERDEAY). [2]

10 (i) A bakery sells cookies in tins and keeps track of the number of tins sold per week. State two conditions under which a Poisson distribution would be a suitable probability model for the number of tins sold in a week. [2]

(ii) Two types of cookies, chocolate and raisin, are sold. The mean number of tins for chocolate cookies sold in a week is 2.4. The mean number of tins for raisin cookies sold in a week is 1.8. Use a Poisson distribution to find the probability that in a given week, the total number of tins sold is more than 9. [3]

(iii) Use a normal approximation to the Poisson distribution to find the probability that the total number of tins sold in 4 weeks is at least 15 but not more than 25. [4]

(iv) Explain why the Poisson distribution may not be a good model for the number of cookies sold in a year. [1]

11 The table gives the population y, in thousands, for a particular species of mammal over 10 years.

Year, x	1	2	3	4	5	6	7	8	9	10
Population, y (in thousands)	10.8	8.7	6.9	5.5	4.4	3.5	2.8	2.3	1.8	1.4

(i) Find the equation of the regression line of y on x, giving your answer to 3 decimal places. [1]

(ii) Let Y be the value obtained by substituting a value of x into the equation of the regression line of y on x found in **(i)**. Find $\sum(y-Y)^2$. [1]

(iii) For each of the values of x, Y' is given by $Y' = A + Bx$, where A and B are any constants. What can you say about the value of $\sum(y-Y')^2$? [1]

(iv) Draw a scatter diagram to illustrate the data. [2]

An animal conservationist suggested the model $\ln y = c + dx$ for this set of data.

(v) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) x and y,

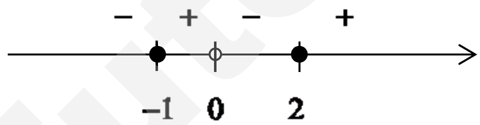
(b) x and $\ln y$. [2]

(vi) Use your answers to parts **(iv)** and **(v)** to explain which of $y = a + bx$ or $\ln y = c + dx$ is the better model. [1]

(vii) Using the better model found in **(vi)**, predict the population of this species in the 20th year. [2]

END OF PAPER

2016 H2 MATH (9740/01) JC 2 PRELIM EXAMINATION SOLUTIONS

Qn	Solution
1	**Inequalities**
(i)	$x^2-2x+5=(x-1)^2-1+5$ $\qquad\qquad\;=(x-1)^2+4\;>0$ for all real x
(ii)	$\dfrac{x}{x^2-2x+5}\le\dfrac{x+2}{x^3-2x^2+5x},\ x>0$ $\dfrac{x}{x^2-2x+5}-\dfrac{x+2}{x\left(x^2-2x+5\right)}\le 0$ $\dfrac{x^2-x-2}{x\left(x^2-2x+5\right)}\le 0$ Since $x^2-2x+5>0$ for all real x, $\dfrac{(x-2)(x+1)}{x}\le 0$ $x\le -1$ or $0<x\le 2$ 
(iii)	$\dfrac{e^x}{e^{2x}-2e^x+5}\ge\dfrac{e^x+2}{e^{3x}-2e^{2x}+5e^x}$ Replace x with e^x. $-1\le e^x<0$ (rej. $\because e^x>0$) or $e^x\ge 2$ $\qquad\qquad\qquad\qquad\qquad x\ge\ln 2$

Qn	Solution
2	**Definite Integrals**
(a)	$\int_1^p \ln(x)\mathrm{d}x = \left[x\ln x\right]_1^p - \int_1^p \left(\frac{1}{x}\right)x\ \mathrm{d}x$ $= \left[\left(p\ln p - 0\right) - \left(p-1\right)\right]$ $= p\ln p - p + 1$
(b)	$\int_1^q x^3\ \mathrm{d}x = \int_1^8 \sqrt[3]{y}\ \mathrm{d}y$ $\left[\frac{x^4}{4}\right]_1^q = \left[\frac{3y^{\frac{4}{3}}}{4}\right]_1^8$ $\frac{q^4}{4} - \frac{1}{4} = \left(\frac{3}{4}\right)\left(8^{\frac{4}{3}} - 1\right)$ $\frac{q^4}{4} - \frac{1}{4} = \left(\frac{3}{4}\right)(16-1)$ $q^4 = 46$ $q = 46^{\frac{1}{4}}$ $\therefore a = 46,\ b = 4$

Qn	Solution
3	**Mathematical Induction + APGP**
(i)	Let P_n be the statement $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = 2\left(1 - \frac{1}{n+1}\right), n \in \mathbb{Z}^+$. When $n = 1$, LHS = 1. RHS $= 2\left(1 - \frac{1}{1+1}\right) = 1 = \text{LHS}$ $\therefore P_1$ is true. Assume that P_k is true for some $k \in \mathbb{Z}^+$ i.e. $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k} = 2\left(1 - \frac{1}{k+1}\right)$ To show that P_{k+1} is also true i.e. $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k+1} = 2\left(1 - \frac{1}{(k+1)+1}\right)$ When $n = k+1$, LHS

	$=1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+k+1}$ $=2\left(1-\frac{1}{k+1}\right)+\frac{1}{1+2+3+...+k+1}$ $=2\left(1-\frac{1}{k+1}\right)+\frac{1}{\frac{k+1}{2}(1+k+1)}$ $=2\left(1-\frac{1}{k+1}\right)+\frac{2}{(k+1)(k+2)}$ $=2+\frac{-2(k+2)+2}{(k+1)(k+2)}$ $=2+\frac{-2k-2}{(k+1)(k+2)}$ $=2+\frac{-2(k+1)}{(k+1)(k+2)}$ $=2+\frac{-2}{(k+2)}$ $=2\left(1-\frac{1}{((k+1)+1)}\right)$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n\in\mathbb{Z}^+$.
	$1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+n}=2\left(1-\frac{1}{n+1}\right)$ As $n\to\infty, \frac{1}{n+1}\to 0 \;\therefore 2\left(1-\frac{1}{n+1}\right)\to 2$ $\therefore 1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+n}+...=2$

Qn	Solution
4	**Maclaurin Series**
	$\mathrm{f}(x)=\cos^{-1}x$ $\mathrm{f}'(x)=-\frac{1}{\sqrt{1-x^2}}$ $\mathrm{f}''(x)=\frac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}}(-2x)$ $\left(1-x^2\right)\mathrm{f}''(x)=-x\left(1-x^2\right)^{-\frac{1}{2}}$ $\left(1-x^2\right)\mathrm{f}''(x)=x\,\mathrm{f}'(x)$ (shown)

$\left(1-x^2\right)\mathrm{f}''(x)=x\,\mathrm{f}'(x)$

$\left(1-x^2\right)\mathrm{f}'''(x)-2x\,\mathrm{f}''(x)=x\,\mathrm{f}''(x)+\mathrm{f}'(x)$

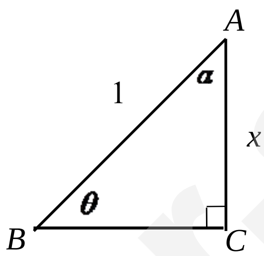
$\left(1-x^2\right)\mathrm{f}'''(x)=3x\,\mathrm{f}''(x)+\mathrm{f}'(x)$

$\mathrm{f}(0)=\dfrac{\pi}{2},\ \mathrm{f}'(0)=-1,\ \mathrm{f}''(0)=0,\quad \mathrm{f}'''(0)=-1$

$\cos^{-1}x=\dfrac{\pi}{2}-x-\dfrac{x^3}{6}+\ldots$

Let $\sin^{-1}x=\theta$

Let $\cos^{-1}x=\alpha$



$\theta+\alpha=\dfrac{\pi}{2}$

$\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$ (shown)

$\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$

$\sin^{-1}x=\dfrac{\pi}{2}-\cos^{-1}x=\dfrac{\pi}{2}-\left(\dfrac{\pi}{2}-x-\dfrac{x^3}{6}+\ldots\right)$

$\sin^{-1}x=x+\dfrac{x^3}{6}+\ldots$

Qn	Solution
5	**Maximum/Minimum Problem**

Let volume of silo be V

$V=\pi r^2h+\dfrac{2}{3}\pi r^3$

Time needed to paint the silo $=20\left(2\pi rh\right)+35\left(2\pi r^2\right)$

$$60000=40\pi rh+70\pi r^2$$

$$h=\frac{60000-70\pi r^2}{40\pi r}$$

$V=\pi r^2\left(\dfrac{60000-70\pi r^2}{40\pi r}\right)+\dfrac{2}{3}\pi r^3$

$\quad=1500r-\dfrac{7}{4}\pi r^3+\dfrac{2}{3}\pi r^3$

$\quad=1500r-\dfrac{13}{12}\pi r^3$

$\dfrac{\mathrm{d}V}{\mathrm{d}r}=1500-\dfrac{13}{4}\pi r^2$

For maximum V, $\dfrac{\mathrm{d}V}{\mathrm{d}r}=0$

$\Rightarrow 1500 - \frac{13}{4}\pi r^2 = 0$

$\frac{13}{4}\pi r^2 = 1500$

$r^2 = \frac{6000}{13\pi}$

$r = \pm\sqrt{\frac{6000}{13\pi}}$

Since $r > 0$, $\therefore r = \sqrt{\frac{6000}{13\pi}} = 12.1207 = 12.1 \quad (3 \text{ s.f.})$.

$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -\frac{13}{2}\pi r < 0$, for $r = 12.1207$.

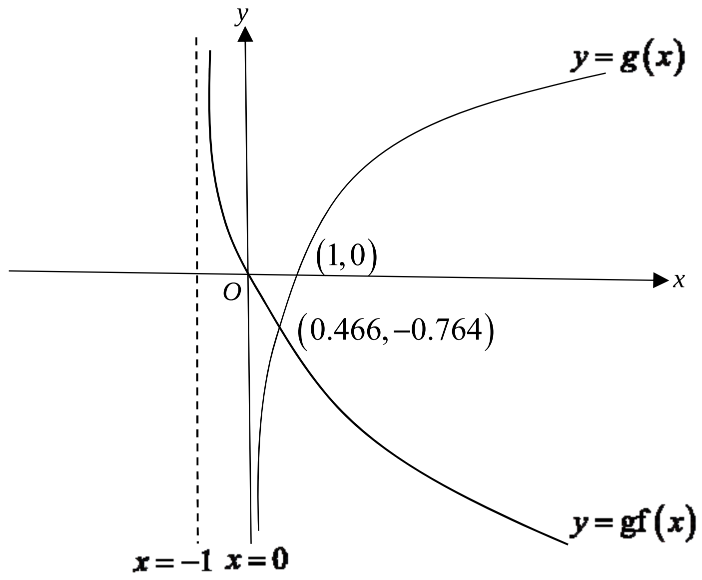
Alternative:

r	$\left(\sqrt{\frac{6000}{13\pi}}\right)^-$	$\sqrt{\frac{6000}{13\pi}}$	$\left(\sqrt{\frac{6000}{13\pi}}\right)^+$
$\frac{\mathrm{d}V}{\mathrm{d}r}$	$+$ /	0 —	$-$ \

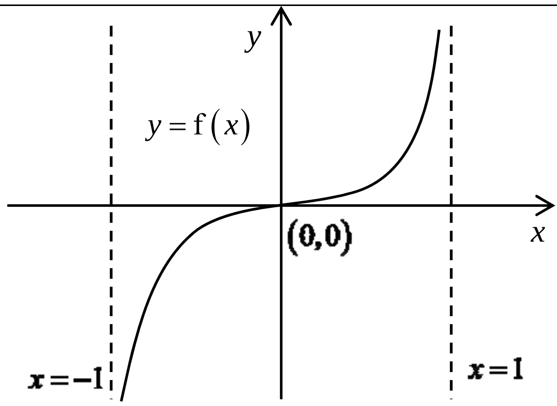
Qn	Solution
6	**Recurrence Relations**
(i)	Using GC, roots of equation are $\alpha = 1.253$, $\beta = 7.848$.
(ii)	As $n \to \infty, \quad x_n \to L, \quad x_{n+1} \to L$ $\therefore L = \ln(L-1)^2 + 4 \Rightarrow 2\ln(L-1) + 4 - L = 0$ Since equation is identical to $2\ln(x-1) + 4 - x = 0$ $\therefore L = 1.253 = \alpha$ or $L = 7.848 = \beta$ Hence the sequence converges to either α or β.
(iii)	Using GC, it can be observed that when $x_1 = 3$, the sequence increases and converges to $7.848 = \beta$. when $x_1 = 12$, the sequence decreases and converges to $7.848 = \beta$.
(iv)	$x_{n+1} - x_n = \ln(x_n - 1)^2 + 4 - x_n$ From graph, if $\alpha < x_n < \beta$, $2\ln(x_n - 1) + 4 - x_n > 0 \Rightarrow \ln(x_n - 1)^2 + 4 > x_n \Rightarrow x_{n+1} > x_n$ if $1 < x_n < \alpha$ or $x_n > \beta$, $2\ln(x_n - 1) + 4 - x_n < 0 \Rightarrow \ln(x_n - 1)^2 + 4 < x_n \Rightarrow x_{n+1} < x_n$.

Qn	Solution
7	**Vectors**
(i)	Using GC, $x = 20 - 5z$ $y = 20 + 2z$ $z = z$ $l : \mathbf{r} = \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$
(ii)	Normal Vector $\begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} \times \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ -20 \\ 140 \end{pmatrix}$ $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 0$ Cartesian Equation: $x - y + 7z = 0$ or equivalent
(iii)	$\mathbf{r} = \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$ where $\mu = 0, 1, 2, 3, 4$
(iv)	The vector equation in **(i)** allows for *x*, *y* and *z* to be real numbers. But the circuit boards produced is a physical quantity and must minimally be an integer.

Qn	Solution
8	**Functions & Transformation of Graphs**
(i)	Since $\mathrm{R_f} = (0, \infty) \subseteq \mathrm{D_g} = (0, \infty)$, gf exists. $\mathrm{gf} : x \mapsto \ln\left(\frac{1}{x+1}\right)^2, \quad x > -1.$

(ii)	 $R_{gf} = (-\infty,\infty) = \mathbb{R}$
(iii)	Since $x > -1, x+1 > 0$, $gf(x) = -2\ln(x+1)$ From $y = \ln x$ to $y = -2\ln(x+1)$: 1. Translation of 1 unit in the negative *x*-direction 2. Reflection in the *x*-axis 3. Scaling parallel to *y*-axis by a factor of 2 [Accept any other possible correct sequence such as 1-3-2]

Qn	Solution
9	**Curve sketching and differentiation**
(i)	$f(x) = \dfrac{x}{\sqrt{(1-x^2)}}$ $f'(x) = \dfrac{\sqrt{(1-x^2)}\cdot 1 - x\cdot\frac{1}{2}\cdot(1-x^2)^{-\frac{1}{2}}\cdot(-2x)}{(1-x^2)}$ $= \dfrac{1}{\sqrt{(1-x^2)}} + \dfrac{x^2}{(1-x^2)^{\frac{3}{2}}}$ $= \dfrac{1}{(1-x^2)^{\frac{3}{2}}} > 0 \left(\because -1 < x < 1, \therefore (1-x^2)^{\frac{3}{2}} > 0 \right)$ Since $f'(x) > 0$ for $-1 < x < 1$, f is strictly increasing.

(ii)	
(iii)	$\mathrm{w}'(x)=\mathrm{g}(x)\mathrm{f}'(x)+\mathrm{g}'(x)\mathrm{f}(x)$ From **(i)**, $\mathrm{f}'(x)>0$ for $-1<x<1$. From **(ii)**, $\mathrm{f}(x)=\begin{cases}<0 & \text{for } -1<x<0\\ =0 & \text{for } x=0\\ >0 & \text{for } 0<x<1\end{cases}$ From the given graph, $\mathrm{g}(x)>0$ for $-1<x<1$ and $\mathrm{g}'(x)=\begin{cases}<0 & \text{for } -1<x<0\\ =0 & \text{for } x=0\\ >0 & \text{for } 0<x<1\end{cases}$ Therefore, When $-1<x<0$, $\mathrm{w}'(x)>0$. . When $x=0$, $\mathrm{w}'(x)>0$. When $0<x<1$, $\mathrm{w}'(x)>0$. Since $\mathrm{w}'(x)>0\neq 0$ for $-1<x<1$, there are no stationary points in the graph of w.

Qn	Solution
10	**Complex Numbers**
(a)	$z=w+2\mathrm{i}-1$ --- (1) $z^2-\mathrm{i}w+\dfrac{5}{2}=0$ --- (2) **<u>Method 1</u>** From (1): $w=z-2\mathrm{i}+1$ --- (3) Substitute (3) into (2): $z^2-\mathrm{i}(z-2\mathrm{i}+1)+\dfrac{5}{2}=0$ $z^2-\mathrm{i}z-\mathrm{i}+\dfrac{1}{2}=0$ $z=\dfrac{-(-\mathrm{i})\pm\sqrt{(-\mathrm{i})^2-4(1)\left(-\mathrm{i}+\dfrac{1}{2}\right)}}{2(1)}$ $=\dfrac{\mathrm{i}\pm\sqrt{-3+4\mathrm{i}}}{2}$

$$=\frac{\mathrm{i}\pm(1+2\mathrm{i})}{2}$$

$$z=\frac{1}{2}+\frac{3}{2}\mathrm{i},\ w=\frac{3}{2}-\frac{1}{2}\mathrm{i}, \quad \text{or} \quad z=-\frac{1}{2}-\frac{1}{2}\mathrm{i},\ w=\frac{1}{2}-\frac{5}{2}\mathrm{i},$$

<u>Method 2</u>

Substitute (1) into (2):

$$(w+2\mathrm{i}-1)^2-\mathrm{i}w+\frac{5}{2}=0$$

$$w^2+(2\mathrm{i}-1)^2+2(2\mathrm{i}-1)w-\mathrm{i}w+\frac{5}{2}=0$$

$$w^2+w(3\mathrm{i}-2)-\frac{1}{2}-4\mathrm{i}=0$$

$$w=\frac{-(3\mathrm{i}-2)\pm\sqrt{(3\mathrm{i}-2)^2-4(1)\left(-\frac{1}{2}-4\mathrm{i}\right)}}{2(1)}$$

$$w=\frac{-(3\mathrm{i}-2)\pm(1+2\mathrm{i})}{2}$$

$$w=\frac{3}{2}-\frac{1}{2}\mathrm{i},\ z=\frac{1}{2}+\frac{3}{2}\mathrm{i} \quad \text{or} \quad w=\frac{1}{2}-\frac{5}{2}\mathrm{i},\ z=-\frac{1}{2}-\frac{1}{2}\mathrm{i}$$

(i)

$$z=w-\frac{1}{w}=2\cos\theta+2\mathrm{i}\sin\theta-\left(\frac{1}{2}\cos\theta-\frac{1}{2}\mathrm{i}\sin\theta\right)=\frac{3}{2}\cos\theta+\frac{5}{2}\mathrm{i}\sin\theta$$

$$\mathrm{Re}(z)=\frac{3}{2}\cos\theta, \quad \mathrm{Im}(z)=\frac{5}{2}\sin\theta$$

(ii)

$$x=\frac{3}{2}\cos\theta, \quad y=\frac{5}{2}\sin\theta$$

$$\therefore \cos\theta=\frac{2}{3}x, \quad \sin\theta=\frac{2}{5}y$$

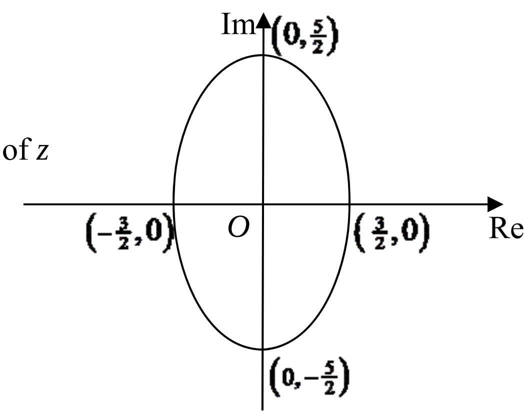
Since $\cos^2\theta+\sin^2\theta=1$

$$\left(\frac{2}{3}x\right)^2+\left(\frac{2}{5}y\right)^2=1$$

$$\frac{x^2}{\left(\frac{3}{2}\right)^2}+\frac{y^2}{\left(\frac{5}{2}\right)^2}=1$$

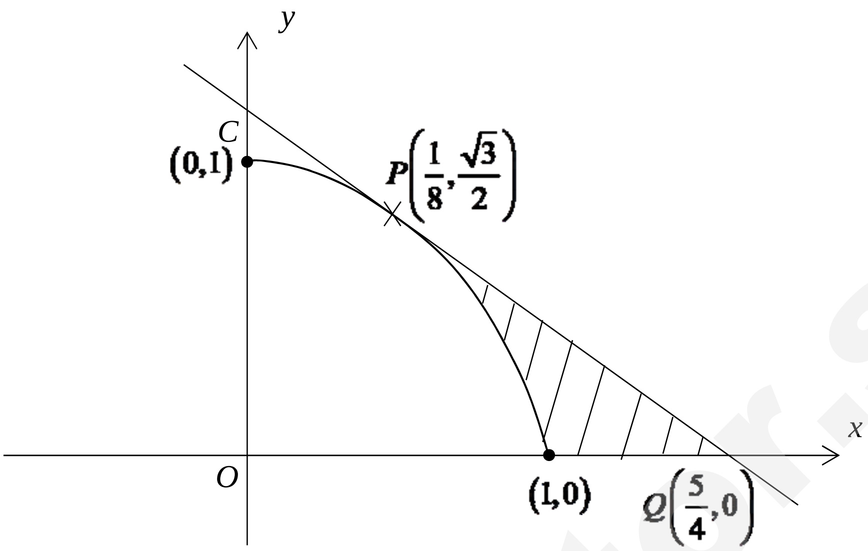
$$\therefore a=\frac{3}{2}, \quad b=\frac{5}{2}$$

Locus of z



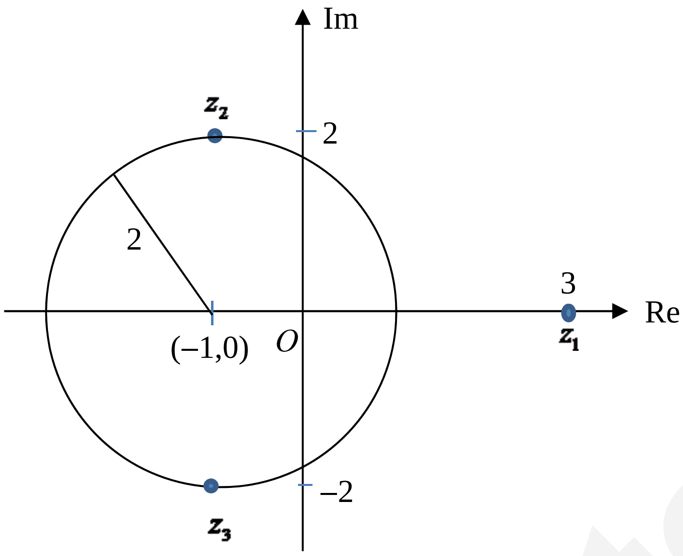
Qn	Solution
11	**Parametric Equations + Applications of Differentiation and Integration**
(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=3t^2, \quad \frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{t}{\sqrt{1-t^2}}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$ $=-\frac{t}{\sqrt{1-t^2}}\times\frac{1}{3t^2}$ $=-\frac{1}{3t\sqrt{1-t^2}}$ At point P, $x=p^3$, $y=\sqrt{1-p^3}$. Equation of tangent at P: $y-\sqrt{1-p^2}=-\frac{1}{3p\sqrt{1-p^2}}\left(x-p^3\right)$ $3p\left(1-p^2\right)-3py\sqrt{\left(1-p^2\right)}=x-p^3$ (Shown)
(ii)	Substitute $x=p^3$, $y=\sqrt{1-p^2}$ into $y=4\sqrt{3}\ x$ $\sqrt{1-p^2}=4\sqrt{3}\ p^3$ $1-p^2=48p^6$ $48p^6+p^2-1=0$ Using GC, $p=-\frac{1}{2}$ or $p=\frac{1}{2}$ (N.A. since $0\le p\le 1$) $\therefore$ Exact coordinates of p are $\left(\frac{1}{8},\frac{\sqrt{3}}{2}\right)$.

(iii)	Equation of tangent to C at P: $3\left(\frac{1}{2}\right)\left[1-\left(\frac{1}{2}\right)^2\right]-\frac{3}{2}y\sqrt{1-\left(\frac{1}{2}\right)^2}=x-\left(\frac{1}{2}\right)^3$ $\frac{9}{8}-\frac{3y}{2}\sqrt{\frac{3}{4}}=x-\frac{1}{8}$ When $y=0$, $\frac{9}{8}=x-\frac{1}{8}$ $x=\frac{5}{4}$

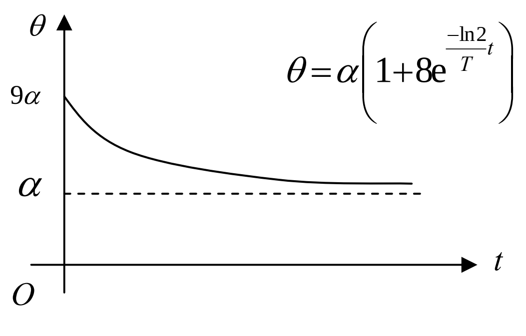
	$\therefore$ Exact coordinates of Q are $\left(\frac{5}{4}, 0\right)$.
(iv)	 When $x = \frac{1}{8}$, $t = \frac{1}{2}$. When $x = 1$, $t = 1$. Area of shaded region = Area of required region $= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{5}{4} - \frac{1}{8}\right) - \int_{\frac{1}{8}}^{1} y_C \, dx$ $= \frac{9\sqrt{3}}{32} - \int_{\frac{1}{2}}^{1} \sqrt{1-t^2} \left(\frac{dx}{dt}\right) dt$ $= \frac{9\sqrt{3}}{32} - \int_{\frac{1}{2}}^{1} 3t^2\sqrt{1-t^2} \, dt$ (Shown) Let $t = \sin u$, $\frac{dt}{du} = \cos u$. When $t = \frac{1}{2}$, $u = \frac{\pi}{6}$. When $t = 1$, $u = \frac{\pi}{2}$.

	$$\frac{9\sqrt{3}}{32}-\int_{\frac{1}{2}}^{1} 3t^2\sqrt{1-t^2}\ \mathrm{d}t=\frac{9\sqrt{3}}{32}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 u\sqrt{1-\sin^2 u}\ \cos u\ \mathrm{d}u$$ $$=\frac{9\sqrt{3}}{32}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 u\cos^2 u\ \mathrm{d}u$$ $$=\frac{9\sqrt{3}}{32}-\frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(2\sin u\cos u\right)^2\ \mathrm{d}u$$ $$=\frac{9\sqrt{3}}{32}-\frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 2u\ \mathrm{d}u$$ $$=\frac{9\sqrt{3}}{32}-\frac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1-\cos 4u\ \mathrm{d}u$$ $$\frac{9\sqrt{3}}{32}-\frac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1-\cos 4u\ \mathrm{d}u=\frac{9\sqrt{3}}{32}-\frac{3}{8}\left[u-\frac{1}{4}\sin 4u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\mathrm{d}u$$ $$=\frac{9\sqrt{3}}{32}-\frac{3}{8}\left[\left(\frac{\pi}{2}-\frac{1}{4}\sin 2\pi\right)-\left(\frac{\pi}{6}-\frac{1}{4}\sin\frac{2\pi}{3}\right)\right]$$ $$=\frac{9\sqrt{3}}{32}-\frac{3}{8}\left[\frac{\pi}{2}-\frac{\pi}{6}+\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right]$$ $$=\frac{9\sqrt{3}}{32}-\frac{3}{8}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right]$$ $$=\frac{9\sqrt{3}}{32}-\frac{3\sqrt{3}}{64}-\frac{\pi}{8}$$ $$=\frac{15\sqrt{3}}{64}-\frac{\pi}{8}$$

Qn	Solution
1	**Vectors**
	$\frac{\lvert\mathbf{a}\bullet\mathbf{b}\rvert}{\lvert\mathbf{b}\rvert}=\frac{\lvert\mathbf{b}\bullet\mathbf{a}\rvert}{\lvert\mathbf{a}\rvert}$ $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert$ as $\lvert\mathbf{a}\bullet\mathbf{b}\rvert\neq 0$ (as **a** and **b** are not perpendicular) $\lvert\mathbf{a}\times\mathbf{b}\rvert$ is the area of rhombus with adjacent sides OA and OB.
(i)	$\lvert\mathbf{a}\rvert=\lvert\mathbf{i}-\mathbf{j}+3\mathbf{k}\rvert=\sqrt{11}$ $\qquad$ $\lvert\mathbf{b}\rvert=\lvert-2\mathbf{i}+\mathbf{j}+p\mathbf{k}\rvert=\sqrt{5+p^2}$ As $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert$, $\Rightarrow 11=5+p^2$ $\Rightarrow p^2=6$ Since $p<0$, $p=-\sqrt{6}$
(ii)	Let θ denote angle AOB. $\cos\theta=\frac{\mathbf{a}\bullet\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ $=\frac{\begin{pmatrix}1\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\-\sqrt{6}\end{pmatrix}}{\left(\sqrt{11}\right)\left(\sqrt{11}\right)}$ $=\frac{-3-3\sqrt{6}}{11}$ $\theta=\cos^{-1}\left(\frac{-3-3\sqrt{6}}{11}\right)=2.79569$ rad or 160.18126 degrees Area of minor sector OAB $=\frac{1}{2}r^2\theta$ $=\frac{1}{2}(11)(2.79569)$ $=15.4\text{ units}^2$ OR $=\left(\frac{\theta}{360}\right)\left(\pi r^2\right)$ $=\left(\frac{160.18126}{360}\right)(11\pi)$ $=15.4\text{ units}^2$ OR $=\left(\frac{2.79569}{2\pi}\right)\left(\pi r^2\right)$ $=\left(\frac{2.79569}{2\pi}\right)(11\pi)$ $=15.4\text{ units}^2$

Qn	Solution
2	**Complex Numbers**
(a)	Using GC, $z_1 = 3,\ z_2 = -1+2\text{i},\ z_3 = -1-2\text{i}.$  The locus of points given by $\lvert z+1\rvert = 2$ passes through the roots represented by z_2 and z_3 since $\lvert z_2+1\rvert = \lvert -1+2\text{i}+1\rvert = \lvert 2\text{i}\rvert = 2$ and $\lvert z_3+1\rvert = \lvert -1-2\text{i}+1\rvert = \lvert -2\text{i}\rvert = 2$.
(b) (i)	$1+\text{e}^{\text{i}\theta} = \text{e}^{\text{i}\frac{\theta}{2}}\left(\text{e}^{-\text{i}\frac{\theta}{2}} + \text{e}^{\text{i}\frac{\theta}{2}}\right)$ $= \text{e}^{\text{i}\frac{\theta}{2}}\left(\cos\frac{\theta}{2} - \text{i}\sin\frac{\theta}{2} + \cos\frac{\theta}{2} + \text{i}\sin\frac{\theta}{2}\right)$ $= \text{e}^{\text{i}\frac{\theta}{2}}\left(2\cos\frac{\theta}{2}\right) = 2\text{e}^{\text{i}\frac{\theta}{2}}\cos\frac{\theta}{2}$
(ii)	$w = \dfrac{\text{e}^{\text{i}\theta}}{1+\text{e}^{\text{i}\theta}}$ $= \dfrac{\text{e}^{\text{i}\theta}}{2\text{e}^{\text{i}\frac{\theta}{2}}\cos\frac{\theta}{2}} = \dfrac{\text{e}^{\text{i}\frac{\theta}{2}}}{2\cos\frac{\theta}{2}}$ $= \dfrac{\cos\frac{\theta}{2} + \text{i}\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = \dfrac{1}{2} + \dfrac{1}{2}\text{i}\tan\dfrac{\theta}{2}$ $\therefore \text{Im}(w) = \dfrac{1}{2}\tan\dfrac{\theta}{2}$

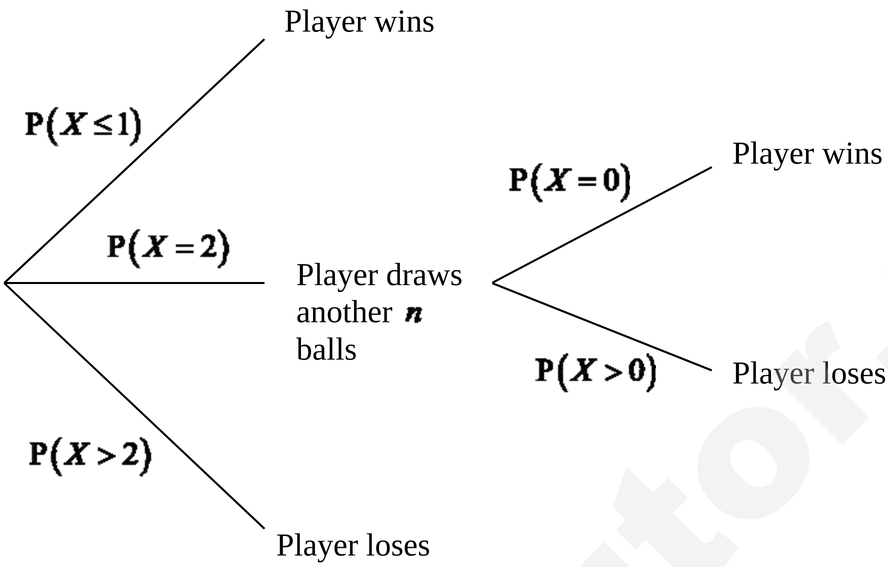
Qn	Solution
3	**Differential Equations**
	$\dfrac{\text{d}\theta}{\text{d}t} = -k(\theta-\alpha), k$ is a positive constant

(i)	$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-k(\theta-\alpha), k>0$ $\int \frac{1}{\theta-\alpha}\,\mathrm{d}\theta=\int -k\,\mathrm{d}t$ $\ln(\theta-\alpha)=-kt+c$ since $\theta>\alpha$ $\theta-\alpha=\mathrm{e}^{-kt+c}$ $\theta-\alpha=A\mathrm{e}^{-kt}, A=\mathrm{e}^{c}$ $\theta=\alpha+A\mathrm{e}^{-kt}$ (shown)
(ii)	When $t=0,\theta=9\alpha$ $\therefore 9\alpha=\alpha+A \quad \therefore A=8\alpha$ When $t=T,\theta=5\alpha$ $5\alpha=\alpha+8\alpha\mathrm{e}^{-kT}$ $\mathrm{e}^{-kT}=\frac{1}{2}$ $kT=\ln 2$ $k=\frac{\ln 2}{T}$ $\theta=\alpha+8\alpha\mathrm{e}^{\frac{-\ln 2}{T}t}$ $\theta=\alpha\left(1+8\mathrm{e}^{\frac{-\ln 2}{T}t}\right)$ When $\theta=2\alpha$ $2\alpha=\alpha\left(1+8\mathrm{e}^{\frac{-\ln 2}{T}t}\right)$ $\mathrm{e}^{\frac{-\ln 2}{T}t}=\frac{1}{8}$ $-\frac{\ln 2}{T}t=-\ln 8$ $t=\frac{\ln 8}{\ln 2}T=3T$
(iii)	For large values of t, $\mathrm{e}^{\frac{-\ln 2}{T}t}\to 0, \theta\to\alpha$, θ decreases and approaches to α. 

Qn	Solution
4	**APGP + Summation**
(a)	1st row: number of matches = 3 2nd row: number of matches = 6 3rd row: number of matches = 9 n^{th} row: number of matches = $3+(n-1)(3)=3n$
	1 row: total number of matches = 3 2 rows: total number of matches = 3 + 6 3 rows: total number of matches = 3 + 6 + 9 n rows: total number of matches $=\frac{n}{2}(3+3n)$ $=\frac{3n(n+1)}{2}$ (shown) OR $=\frac{n}{2}\left[2(3)+(n-1)(3)\right]$ $=\frac{3n(n+1)}{2}$ (shown) $\frac{3n(n+1)}{2}\le 2000$ Using GC, When $n=36$, $\frac{3n(n+1)}{2}=1998<2000$ When $n=37$, $\frac{3n(n+1)}{2}=2109>2000$ Maximum number of complete rows $=36$.
(b)	Let $r=\frac{b}{a}$ $\frac{a}{1-r}=a+2b$ $\frac{1}{1-r}=1+2r$ $1=1+r-2r^2$ $2r^2-r=0$ $r(2r-1)=0$ $r=0$ (rejected $\frac{b}{a}\neq 0$) or $r=\frac{1}{2}$ $\therefore$ common ratio = $\frac{1}{2}$ $G_n=\frac{a\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ $G_n=2a\left(1-\left(\frac{1}{2}\right)^n\right)$

	$\displaystyle\sum_{n=1}^{N} G_n = 2a\sum_{n=1}^{N}\left(1-\left(\frac{1}{2}\right)^{n}\right)$ $\displaystyle = 2a\left[N-\sum_{n=1}^{N}\left(\frac{1}{2}\right)^{n}\right]$ $\displaystyle = 2a\left[N-\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{N}\right)}{1-\frac{1}{2}}\right]$ $\displaystyle = 2a\left(N-\left(1-\left(\frac{1}{2}\right)^{N}\right)\right)$ $\displaystyle = 2aN-2a\left(1-\left(\frac{1}{2}\right)^{N}\right)$ $= 2aN-G_N$

Qn	Solution
5	**Sampling Methods**
(i)	Systematic sampling is a sampling method in which the **entire population is listed in some order**. The population is **divided into sampling intervals of *k* members**. After obtaining **a random starting point from the first *k* members, every k^{th} member is chosen** from the list until the required number is achieved.
(ii)	Possible Advantages: • It is easy to conduct the survey as the members of the sample are easily accessible. • It is easy to conduct as the surveyor does not need the list of all the residents in the neighbourhood. Possible Disadvantages: • It is a biased sample as only residents who visit the bakery during the evening rush hour is surveyed. Hence the sample may not be representative. • It is a biased sample as some people may visit the bakery multiple times during the evening rush hours increasing their chances to be selected. • It may not be easy to get residents to visit the bakery in sequence so selection of every k^{th} resident in this case may be difficult.

Qn	Solution
6	**Binomial Distribution**
	Let X be the number of red balls drawn, out of n balls. $X \sim \mathrm{B}(n, p)$  $P = \mathrm{P}(\text{player wins})$ $= \mathrm{P}(X \le 1) + \mathrm{P}(X = 2)\mathrm{P}(X = 0)$ $= \mathrm{P}(X = 0) + \mathrm{P}(X = 1) + \mathrm{P}(X = 2)\mathrm{P}(X = 0)$ $= \binom{n}{0}(0.2)^0(0.8)^n + \binom{n}{1}(0.2)^1(0.8)^{n-1} + \left[\binom{n}{2}(0.2)^2(0.8)^{n-2}\right]\left[\binom{n}{0}(0.2)^0(0.8)^n\right]$ $= (0.8)^n + n(0.2)(0.8)^{n-1} + \left[\binom{n}{2}(0.2)^2(0.8)^{n-2}\right](0.8)^n$ $= (0.8 + 0.2n)(0.8)^{n-1} + \binom{n}{2}(0.2)^2(0.8)^{2n-2}$ (Shown)
(i)	$P = (0.8 + 0.2n)(0.8)^{n-1} + \binom{n}{2}(0.2)^2(0.8)^{2n-2} < 0.1$ Using G.C., When $n = 18$, $P = 0.10218 > 0.1$ When $n = 19$, $P = 0.08509 < 0.1$ $\therefore$ least $n = 19$
(ii)	Let Y be the number of games won, out of 100 games played. $Y \sim \mathrm{B}(100, 0.3)$.

	Required probability $= \mathrm{P}(Y \geq 40)$ $= 1 - \mathrm{P}(\mathrm{Y} \leq 39)$ $= 0.020988$ ≈ 0.0210 (3 s.f.)

Qn	Solution
7	**Hypothesis Testing**
	Let X denote the number of hours of sleep each child gets at night. Let μ denote the population mean hours of sleep each child gets at night. Assumption: $X \sim \mathrm{N}(\mu, \sigma^2)$ $\therefore \overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$. $\mathrm{H_0}$: $\mu = 6.5$ $\mathrm{H_1}$: $\mu < 6.5$ Test statistic : $T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$ Level of Significance: 8% Reject $\mathrm{H_0}$ if p-value < 0.08 Under $\mathrm{H_0}$, using GC, p-value = 0.0998 Since p-value = 0.0998 > 0.08, we do not reject $\mathrm{H_0}$ and conclude that there is insufficient evidence at 8% level of significance, that supports Ms Patricia's claim. $s^2 = \frac{15}{14}(0.849) = 0.90964$ (5 s.f) Level of Significance: 8% Reject $\mathrm{H_0}$ if t-value < -1.48389 $\frac{\overline{x} - 6.5}{\sqrt{0.90964}/\sqrt{15}} < -1.48389$ $\overline{x} < 6.13458$ $\therefore$ set of values required $0 < \overline{x} < 6.13$

Qn	Solution
8	**Normal Distribution**
(i)	Let X be the mass of a randomly chosen bar of body soap in grams. Let $\overline{X}$ be the sample mean mass of 20 randomly chosen bars of body soaps in grams. $X \sim \mathrm{N}(110, 1.5^2)$ $\overline{X} \sim \mathrm{N}\left(110, \frac{1.5^2}{20}\right)$ $\overline{X}_1 - \overline{X}_2 \sim \mathrm{N}\left(0, 2\left(\frac{1.5^2}{20}\right)\right)$

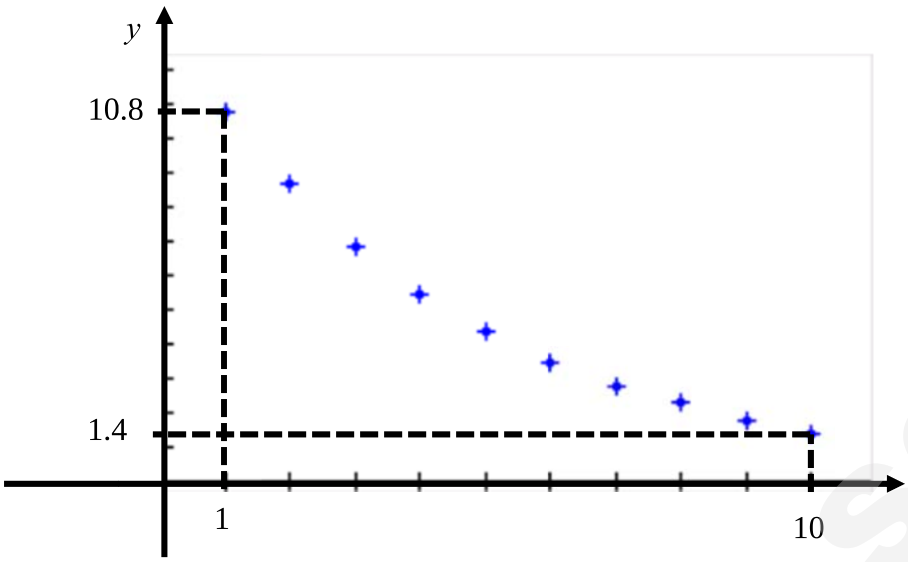
	$P\left(\left\lvert\bar{X}_1-\bar{X}_2\right\rvert\le 0.5\right)=P\left(-0.5\le\bar{X}_1-\bar{X}_2\le 0.5\right)$ $=0.708\ (3\text{ s.f.})$
(ii)	Let W be the mass of a portion of liquefied soap. $W=\dfrac{X_1+X_2+X_3+X_4+X_5}{4}$ $W\sim\mathrm{N}\left(\dfrac{(5)(110)}{4},\dfrac{(5)(1.5^2)}{4^2}\right)$ $W\sim\mathrm{N}\left(\dfrac{275}{2},\dfrac{45}{64}\right)$ $\mathrm{P}(W>140)=0.00143\ (3\text{ s.f.})$
(iii)	Unbiased estimate of population mean, $\bar{u}=\dfrac{\sum u}{n}=\dfrac{1590}{15}=106$ Unbiased estimate of population variance, $s^2=\dfrac{1}{n-1}\left(\sum u^2-\dfrac{\left(\sum u\right)^2}{n}\right)$ $=\dfrac{1}{15-1}\left(169046-\dfrac{(1590)^2}{15}\right)$ $=36.1\ (3\text{ s.f.})$ OR $\dfrac{253}{7}$

Qn	Solution
9	**Probability**
(ai)	Since $\mathrm{P}(A'\mid B)=\dfrac{3}{4}=\mathrm{P}(A')$, A' and B are independent events $\Rightarrow A$ and B are independent events $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ $=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A)\,\mathrm{P}(B)$ (Since A and B are independent) $=\dfrac{1}{4}+\dfrac{1}{2}-\dfrac{1}{4}\cdot\dfrac{1}{2}$ $=\dfrac{5}{8}$
(ii)	$\mathrm{P}(C\mid A)=\dfrac{\mathrm{P}(C\cap A)}{\mathrm{P}(A)}=\dfrac{2}{3}$ $\Rightarrow\mathrm{P}(A\cap C)=\dfrac{2}{3}\mathrm{P}(A)$ $=\dfrac{2}{3}\cdot\dfrac{1}{4}=\dfrac{1}{6}$
(bi)	Required number of ways $=4!\times\dfrac{3!}{2!}\times{}^5C_2$ $=720$ Choose 2 out of 5 slots to put 'Y's Arrange vowels. AEE V R D Group vowels together, and arrange these 4 groups.

(ii)	Required number of ways $= {}^4C_2 \times 2! \times 4!$ $= 288$ 3. The 4 remaining letters can permute themselves within the 4 remaining blanks. _ E Y _ _ Y E _ 1. Choose 2 out of the 1st 4 blanks to put E & Y. **×2!** Because E & Y can switch positions. 2. Correspondingly, the remaining E & Y would take their respective positions in the next 4 blanks.

Qn	Solution
10	**Poisson Distribution**
(i)	The average number of tins sold per week is constant. The sale of one tin is independent of another throughout the week.
(ii)	Let X be the number of tins for chocolate cookies sold in a week. $X \sim \text{Po}(2.4)$ Let Y be the number of tins for raisin cookies sold in a week. $Y \sim \text{Po}(1.8)$ $X+Y \sim \text{Po}(4.2)$ $P(X+Y>9)=1-P(X+Y\le 9)=0.0111$ (3 s.f.)
(iii)	Let W be the total number of tins sold in 4 weeks. $W \sim \text{Po}(16.8)$ Since $\lambda = 16.8 > 10$, $\therefore W \sim \text{N}(16.8, 16.8)$ approximately. $\text{P}(15 \le W \le 25) = \text{P}(14.5 < W < 25.5)$ after continuity correction $= 0.69576 = 0.696$ (3 s.f.)
(iv)	The mean number of tins sold per week might not be constant from one week to another because of seasonal fluctuations such as sales and holidays.

Qn	Solution
11	**Correlation and Regression**
(i)	Using GC, $y = 10.30667 - 0.99939x$ $y = 10.307 - 0.999x$
(ii)	Using GC, $\sum (y-Y)^2 = 6.3689 = 6.37$ (to 3 s.f.) [In GC, define L3=10.30667 – 0.99939L1 and L4 = (L2 – L3)^2 and use 1-var Stat to find the summation]
(iii)	$\sum (y-Y')^2 \ge 6.37$

	
(v)	For x and y, $r = -0.9635$ For x and ln$y$, $r = -0.9999$
(vi)	Since the scatter diagram shows that the population decreases at a decreasing rate as the years pass and the r value for the model $\ln y = c + dx$ is closer to –1 than that of $y = a + bx$, $\ln y = c + dx$ is the better model.
(vii)	Using GC, when $x = 20, \ln y = -1.89587 \Rightarrow y = 0.150187$. The population in 20th year will be 150.

NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2 PRELIMINARY EXAMINATION
Higher 2

MATHEMATICS 9740/01

Paper 1

25 August 2016

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in the brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

National Junior College

1 A local café, Toast Rox, sells its coffee in three sizes (regular, medium and large). Toast Rox customers get a 12.5% discount on their total bill if they buy at least 12 cups of coffee, regardless of size. The number of cups of coffee bought by three particular customers and the total amount they paid are shown in the following table.

Customer	Regular	Medium	Large	Amount paid
A	5	3	2	\$20.90
B	3	4	1	\$17.10
C	2	8	4	\$28.00

Find the original price of each of the 3 sizes of coffee drink. [3]

2 **(i)** By using an algebraic method, solve the inequality

$$\frac{3x^2+14}{(x+1)(x+2)} \geq 2.$$ [4]

(ii) Hence, showing all your working clearly, solve the inequality

$$\frac{3x^2+14}{(|x|-1)(|x|-2)} \geq 2.$$ [2]

3 Referred to the origin O, points A and B have position vectors **a** and **b** respectively. Point C lies on the line segment AB, such that AC: CB = 2 : 1. Find, in terms of **a** and **b**, the position vector of C. [1]

If the angle between **a** and **b** is 60°, show that the length of projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is

$$\frac{1}{3}\left(|\mathbf{a}|+|\mathbf{b}|\right).$$ [4]

4 **(a)** Two complex numbers z and w are such that

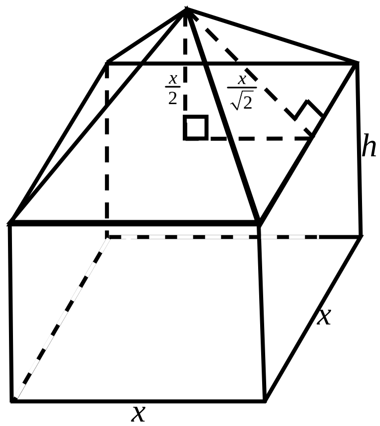
$$2w - z = 6\mathrm{i} \quad \text{and} \quad wz = \frac{13}{2}.$$

Find w and z, giving each answer in the form $a + b\mathrm{i}$, where a and b are real numbers. [4]

(b) The points P and Q represent the fixed complex numbers p and q respectively. It is given that $0 < \arg p < \arg q < \frac{\pi}{2}$, $|p| = 1$, $|q| = 2$, and $\arg q = 2\arg p$.

In a single Argand diagram, sketch and label the points P, Q, and the points R and S representing q^* and $q^* + 2p^2$ respectively, showing clearly any geometrical relationships. Identify the shape of the quadrilateral $OQSR$, where O is the origin. [4]

5 [It is given that volume of pyramid $= \frac{1}{3} \times \text{base area} \times \text{height}$.]



A model of a house is made up of the following parts.

- The roof is modelled by a pyramid with a square base of sides x cm and height $\frac{x}{2}$ cm. For each triangular side of the prism, the length of the perpendicular from the vertex to the base is $\frac{x}{\sqrt{2}}$ cm.
- The walls are modelled by rectangles with sides x cm and h cm as shown in the diagram.
- The base is a square with sides x cm.

All the parts are joined together as shown in the diagram. The model is made of material of negligible thickness. It is given that the volume of the model is a fixed value V cm^3 and the external surface area is at a minimum value, A cm^2. Use differentiation to find

(i) x, in the form $pV^{\frac{1}{3}}$, and

(ii) A, in the form $qV^{\frac{2}{3}}$,

leaving the values of p and q correct to 3 decimal places. [6]

6 A curve C has parametric equations

$$x = t^3 - kt, \qquad y = 3\left(t^2 - k\right),$$

where k is a positive constant and t is a real parameter.

(i) Sketch C, labelling clearly the coordinates of any points of intersection with the axes. [2]

(ii) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t and k. [2]

(iii) Find, in terms of k, the exact equation of the tangent to C at the point where $t = -\sqrt{\frac{k}{3}}$. [3]

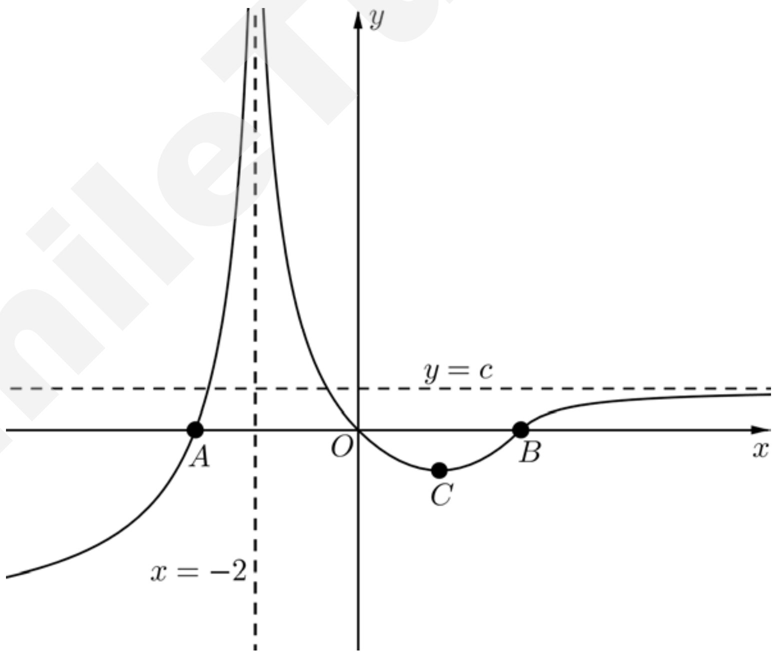
(iv) Given that the tangent found in part **(iii)** intersects C again at the point $\left(\frac{2}{3}k, k\right)$, find the value of k. [2]

7 **(i)** Given that $y = \ln(\sec x)$, show that $\frac{d^3y}{dx^3} = 2\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)$. [2]

(ii) Hence, by further differentiation, find the first two non-zero terms in the Maclaurin's series for y. [3]

(iii) The equation $\frac{1}{12}x^2 + \ln(\sec x) = \cos 2x$ has a positive root α close to zero. Use the result in part **(ii)** and the first three terms of the Maclaurin series for $\cos 2x$ to obtain an approximation to α, leaving your answer in surd form. [3]

8 The diagram below shows the curve with equation $y = \mathrm{f}(x)$. The curve passes through the origin O, crosses the x-axis at the points A and B, and has a turning point at C. The coordinates of A, B and C are $(-4, 0)$, $(4, 0)$ and $\left(a, -\frac{b}{2}\right)$ respectively, where a and b are positive constants such that $a > 1$. The curve also has asymptotes $x = -2$ and $y = c$, where $c > 1$.



On separate diagrams, sketch the following curves, labelling clearly any asymptotes, axial intercepts and turning points in terms of a, b and c whenever necessary.

(a) $y = \mathrm{f}(1-2x)$ [3]

(b) $y^2 = \mathrm{f}(x)$ [3]

(c) $y = \frac{1}{\mathrm{f}(x)}$ [4]

9 The gradient of a curve at the point (x, y) is given by the differential equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}-1=\frac{x-2}{y}.$$

(i) By using the substitution $y = z - x$, find the equation of the curve such that it has a minimum point at $(1,1)$. [6]

(ii) Sketch the curve, indicating clearly the axial intercept(s) and the minimum point. [2]

10 **(a)** Using partial fractions, find $\int \frac{5x^2-2x+7}{(1-x)\left(2x^2+3\right)}\,\mathrm{d}x$. [6]

(b) **(i)** Differentiate $\sin\left(\mathrm{e}^{-x}\right)$ with respect to x. [1]

(ii) Obtain a formula for $\int_0^n \mathrm{e}^{-2x}\cos\left(\mathrm{e}^{-x}\right)\mathrm{d}x$ in terms of n, where $n > 0$. [3]

(iii) Hence find $\int_0^\infty \mathrm{e}^{-2x}\cos\left(\mathrm{e}^{-x}\right)\mathrm{d}x$ exactly. [2]

11 **(a)** Prove by the method of mathematical induction that

$$\frac{2}{1^2\times3^2}+\frac{3}{2^2\times4^2}+\cdots+\frac{n+1}{n^2(n+2)^2}=\frac{5}{16}-\frac{1}{4(n+1)^2}-\frac{1}{4(n+2)^2}.$$ [5]

(b) **(i)** By expressing $\frac{4n+5}{n(n+1)}$ in partial fractions, show that

$$\sum_{n=1}^{N}\left[\frac{4n+5}{n(n+1)}\left(\frac{1}{5^{n+1}}\right)\right]=a+\frac{b}{(N+1)5^{N+1}},$$

for some real constants a and b to be determined exactly. [3]

(ii) State the sum to infinity of the series in part **(b)(i)**. [1]

(iii) Use your answer to part **(b)(i)** to find $\sum_{n=2}^{N-2}\left[\frac{4n+1}{n(n-1)}\left(\frac{1}{5^{n}}\right)\right]$ in terms of N. [2]

12 The curves C_1 and C_2 have equations $x^2+16(y-1)^2=16$ and $x^2-16(y-1)^2=16$ respectively.

(i) Verify that the point $(4,1)$ lies on both C_1 and C_2. [1]

(ii) Sketch C_1 and C_2 on the same diagram, labelling clearly any points of intersection with the axes and the equations of any asymptotes. [4]

(iii) The region R is bounded by the two curves C_1, C_2 and the positive x-axis. Find the numerical value of the volume of revolution formed when R is rotated completely about the x-axis. [3]

S is the region bounded by C_1.

(iv) Using the substitution $y=1+\cos\theta$, where $-\pi<\theta\leq\pi$, evaluate $\int_0^2 \sqrt{1-(y-1)^2}\,\mathrm{d}y$ exactly. [4]

(v) Hence find the exact area of S. [2]

– END OF PAPER –

NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2 PRELIMINARY EXAMINATION
Higher 2

MATHEMATICS **9740/02**

Paper 2 **14 September 2016**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in the brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

National Junior College

Section A: Pure Mathematics [40 marks]

1 **(a)** Kenny took a loan of \$9600 from a friend, and arranged to pay his loan fully in a period of exactly 48 months. To fulfil this arrangement, he paid \$$a$ on the last day of the first month, and on the last day of each subsequent month, he paid \$$d$ more than in the previous month. However, due to financial difficulties, Kenny stopped his payments after his 40th payment, and as a result he still had exactly \$2400 left unpaid.

In which month did Kenny first pay at least \$130 on the last day of that month? [5]

(b) **(i)** Explain why the series $1+e^{-2x}+e^{-4x}+\cdots$ converges for any positive real number x, and express the sum to infinity in terms of x. [2]

(ii) Given that $x = 10$, find the least value of n such that $S-S_n < S(10^{-100})$, where S and S_n represent the sum to infinity and the sum of the first n terms of the series respectively. [3]

2 The functions f and g are defined by

$$\mathrm{f}: x \mapsto x^2-4x+3, \text{ for } x \le a \text{ and}$$

$$\mathrm{g}: x \mapsto \tan^{-1}(2x+1), \text{ for } x > -2,$$

where a is a constant.

(a) If $a = 2$, solve the equation $\mathrm{f}(x) = x$ exactly. [2]

(b) If $a = 3$,

(i) give a reason why f has no inverse. [2]

(ii) Prove that the composite function gf exists and state the rule, domain and exact range of the composite function. [6]

3 The point A has coordinates $(2q, 0, 2)$, where q is a constant, and the planes p_1, p_2 have equations $x+y=4$ and $3x+2y-5z=7$ respectively.

(i) Find the coordinates of the foot of perpendicular from A to p_1. Express your answer in terms of q. [3]

(ii) The point B is the mirror image of A in p_1. If B lies in p_2, find the value of q. [4]

(iii) p_1 and p_2 intersect in a line l. Find a vector equation of l. [1]

Another plane p_3 has equation $\lambda x + z = \mu$, where λ and μ are constants.

(iv) Given that the three planes have no point in common, what can be said about the values of λ and μ? [2]

4 The complex number z satisfies the relation $|z-3|=5$.

(i) Illustrate this relation in an Argand diagram. [2]

(ii) Find the largest possible value of $\arg(z+3-3\text{i})$. [3]

It is further given that z also satisfies the relation $|z-4\text{i}|=|z-6+4\text{i}|$.

(iii) Illustrate this relation in the same diagram as your sketch in part **(i)**. Find the possible values of z exactly. [5]

Section B: Statistics [60 marks]

5 A school comprises a large number of students. A sample comprising 2% of the student population is to be selected to take part in a survey on their opinions about the school facilities.

(a) Describe briefly how this sample can be obtained via systematic sampling. [2]

(b) Give one advantage and one disadvantage of quota sampling in this context. [2]

6 The continuous random variable X has the distribution $\text{N}(\mu, \sigma^2)$. It is known that $\text{P}(X<17.7)=0.15$ and $\text{P}(X>21.9)=0.2$. Calculate the values of μ and σ. [4]

7 A group of 15 student councillors comprises 6 from the House Committee, 5 from the Liaison Committee and 4 from the Welfare Committee. Two particular student councillors, Louis and Lionel, are from the House Committee and the Liaison Committee respectively.

The group stand in a circle to have a meeting. Find the number of possible arrangements if

(i) no two student councillors from the House Committee stand next to each other. [2]

(ii) student councillors from the same committee must stand next to one another and Louis and Lionel must stand next to each other. [2]

The group is to form a Task Force of 10 student councillors to organise a school activity. Find the number of possible ways the Task Force may be formed if the Task Force must include at least 1 student councillor from each of the 3 committees. [3]

8 The table below shows the ages of teak trees, x years, with trunk diameters, y inches. It can be assumed that the diameters of teak trees depend on their ages.

Age x (years)	11	15	28	45	52	57	75	81	88	97
Diameter y (inches)	7.5	11.5	16	19	20.5	21	21.5	21.9	22.2	22.22

(i) Draw a scatter diagram for these values, labelling the axes. [2]

(ii) It is desired to predict the diameters of very old trees (of over hundred years old). Explain why, in this context, neither a linear nor a quadratic model is likely to be appropriate. [2]

(iii) Fit a model of the form $y = a - \frac{b}{x}$ to the data, and calculate the least squares estimates of a and b. Find the product moment correlation coefficient for this model. Use the equation that you have obtained to estimate the diameter of a 40 year-old teak tree, and comment on the reliability of your answer. [4]

9 It has been estimated that only 8% of the world's population has blue eyes. A group of 60 people are randomly selected from all over the world. The number of people in this group who have blue eyes is the random variable Y.

(i) State, in the context of this question, one assumption needed to model Y by a binomial distribution. [1]

Assume now that Y indeed follows a binomial distribution.

(ii) Find the probability that at least 5 but less than 21 people in the group will have blue eyes. [2]

(iii) Use a suitable approximation to find the probability that more than 9 people in the group have blue eyes. You should state the parameters of the distribution you have used. [3]

10 **(i)** Suppose a fair die is tossed twice. Calculate the probabilities that

(a) the sum of the scores of the two tosses is at least 8, and [1]

(b) the absolute difference between the scores of the two tosses is at least 4. [1]

In one round of a game, a player is to draw a ball, without replacement, from a box that contains 3 red balls and 4 white balls. If a red ball is drawn, the player will add the scores obtained from tossing a fair die twice. If a white ball is drawn, the player will take the absolute difference of the scores obtained from tossing a die twice.

The game ends if the sum of the scores is at least 8 or the absolute difference of the scores is at least 4. Else, the player will proceed to the second round of the game where the process of picking a ball from the box and tossing the die twice repeats.

(ii) Find the probability that the game ends at the first round. [2]

(iii) Suppose the game ends at the first round. Find the probability that a red ball is drawn.[2]

(iv) Find the probability that there are a total of 3 rounds of game played and exactly 2 white balls are selected. [3]

11 An accountant believes that the figures provided by a particular company for the amount of loans borrowed by its clients, \$$x$, are too low. He carries out an online survey for clients of this company. The responses from a random sample of 20 clients are summarised by

$$\sum x = 21350, \ \sum (x-\bar{x})^2 = 345900.$$

(i) Calculate unbiased estimates of the population mean and variance of the amount of loans borrowed by each client, correct to 1 decimal place. [2]

The company claims that its clients will borrow \$1000 on average.

(ii) Stating a necessary assumption, carry out a test at the 5% level of significance to determine whether the company has understated the mean amount of loans received by its clients. [5]

(iii) Explain, in the context of the question, the meaning of 'at the 5% level of significance'. [1]

The responses from another random sample of n clients are collected. The sample mean value for this sample is the same as the sample mean value for the previously collected sample.

(iv) Given that the standard deviation of X is 250, and that the assumption you have made in part **(ii)** holds, calculate the range of values of n for which the null hypothesis would not be rejected at the 5% level of significance. [3]

12 Cars join an immigration checkpoint queue in a 1-hour period, such that no two cars join the queue at the same instant in time.

(i) State, in the context of this question, an assumption needed for the number of cars joining an immigration checkpoint queue in a 1-hour period to be well modelled by a Poisson distribution. [1]

Assume now that the number of cars joining an immigration checkpoint queue in a 1-hour period is a random variable with the distribution Po(23). It is further given that the number of cars leaving the same immigration checkpoint queue in a 1-hour period is a random variable with the distribution Po(27).

(ii) It is given that in a period of n minutes, the probability that at least one car leaves the queue exceeds 0.9. Write down an inequality in n. Hence find the least integer value of n. [4]

(iii) At 0900 on a certain morning there are 19 cars in the queue. Use appropriate approximations to find the probability that by 1100 there are at most 12 cars in the queue, stating the parameters of any distributions that you use. (You may assume that the queue does not become empty during this period.) [5]

(iv) Explain why a Poisson model for the number of cars joining an immigration checkpoint queue would probably not be valid if applied to a time period of several hours. [1]

– END OF PAPER –

NANYANG JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

MATHEMATICS **9740/01**

Paper 1 **15th September 2016**

3 Hours

Additional Materials: Cover Sheet
Answer Papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 The nth term of a sequence is given by $u_n = \dfrac{4^n n^2}{(n+1)(n+2)}$, for $n \geq 1$.

The sum of the first n terms is denoted by S_n. Use the method of mathematical induction to show that $S_n = \left(\dfrac{n-1}{n+2}\right)\left(\dfrac{4^{n+1}}{3}\right) + \dfrac{2}{3}$ for all positive integers n. **[5]**

2 Using partial fractions, find $\displaystyle\int_{-2}^{2} \frac{17x^2+23x+12}{(3x+4)(x^2+4)}\,dx$, leaving your answer in exact form. **[6]**

3 A curve C has parametric equations

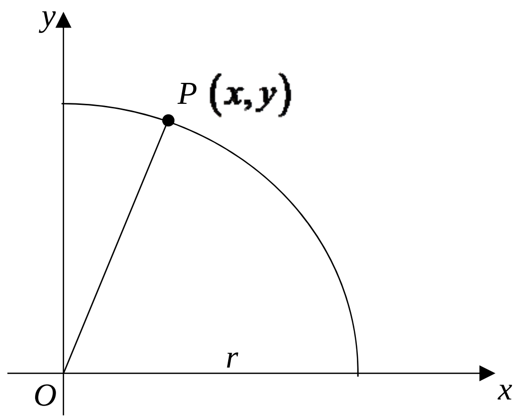
$$x = \frac{1}{2}(\sin t \cos t + t), \quad y = \frac{1}{2}t - \frac{1}{4}\sin 2t, \quad \text{for } -\frac{\pi}{2} < t \leq 0.$$

The tangent to the curve at the point P has gradient 1. Find the equation of the normal at P. The region bounded by this normal, the curve C and the x-axis is rotated through 2π radians about the x-axis. Find, to 5 decimal places, the volume of the solid obtained.
What can be said about the tangents to the curve as t approaches 0? **[7]**

4 Referred to the origin, the points A and B have position vectors **a** and **b** respectively. A point C is such that $OACB$ forms a parallelogram. Given that M is the mid-point of AC, find the position vector of point N if M lies on ON produced such that $OM:ON$ is in the ratio 3:2. Hence show that A, B and N are collinear. **[4]**

Point P is on AB is such that MP is perpendicular to AB. Given that angle AOB is 60^o, $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 3$, find the position vector of P in terms of **a** and **b**. **[4]**

5



A particle P moves along the curve with equation $x^2 + y^2 = r^2$, where $x \geq 0, y \geq 0,$ and r is a constant. By letting $m = \tan\left(\sin^{-1}\dfrac{y}{r}\right)$, find an expression for $\dfrac{\mathrm{d}m}{\mathrm{d}y}$ in terms of y and r.

Given that the rate of change of y with respect to time t is 0.1% of r, show that $\dfrac{\mathrm{d}m}{\mathrm{d}t} = \left(\dfrac{r}{10\sqrt{r^2 - y^2}}\right)^3$.

State the geometrical meaning of $\dfrac{\mathrm{d}m}{\mathrm{d}t}$. **[7]**

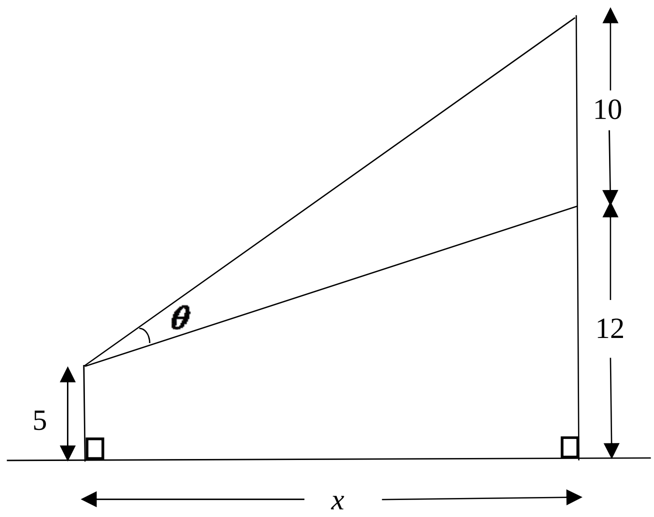
6 **(i)** Show that $\dfrac{r^2 + r - 1}{(r+2)!} = \dfrac{A}{r!} + \dfrac{B}{(r+1)!} + \dfrac{C}{(r+2)!}$, where A, B and C are constants to be determined. **[2]**

(ii) Hence find $\displaystyle\sum_{r=1}^{n} \frac{r^2 + r - 1}{(r+2)!}$ in terms of n. (There is no need to express your answer as a single algebraic function.) **[3]**

(iii) Explain why $\displaystyle\sum_{r=1}^{n} \frac{r^2 - 1}{(r+2)!} < \frac{1}{2}$. **[2]**

(iv) Use your answer to part **(ii)** to find $\displaystyle\sum_{r=4}^{n} \frac{r^2 - 3r + 1}{r!}$ in terms of n. **[3]**

7



A 10 feet tall statue is mounted on a 12 feet tall pedestal. A boy is standing x feet away from the pedestal. His eyes are 5 feet above ground level, and the angle subtended by the statue from the boy's eyes is θ radians (see diagram).

Prove that

$$\tan\theta = \frac{10x}{119+x^2} \ .$$

Hence, or otherwise, find the exact value of x for which θ is maximum and justify that this value of x gives the maximum value of θ.

Deduce, to the nearest degree, the maximum angle subtended by the statue from the boy's eyes. **[9]**

8 **(i)** Find the fourth roots of $-1+\sqrt{3}\,\mathrm{i}$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. **[3]**

(ii) Hence, or otherwise, write down the roots of the equation $(1+z)^4+1-\mathrm{i}\sqrt{3}=0$ and show the roots Z_i, $i=1,2,3,4$ on an Argand diagram. **[3]**

(iii) Illustrate, using the same Argand diagram, the locus of a point Q representing the complex number v, where $\left|v+1-4\sqrt{3}-4\mathrm{i}\right|=2$.

Hence find the exact greatest and least possible values of Z_iQ . **[4]**

9 Two biologists are investigating the growth of a certain bacteria of size x hundred thousand at time t days. It is known that the number of bacteria initially is 20% of a, where a is a positive constant.

(i) One biologist believes that x and t are related by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = x(a-x)$. Given that the number of bacteria increases to 50% of a when $t = \ln 2$ days, show that $x = \frac{2}{4\mathrm{e}^{-2t}+1}$. **[7]**

(ii) Another biologist believes that x and t are related by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 10 - 9t^2$. Find the general solution of this differential equation and sketch three members of the family of solution curves. **[5]**

10 **(a)** **(i)** Express $\sin x + \sqrt{3}\cos x$ in the form $R\sin(x+\alpha)$ where R and α are exact positive constants to be found. **[1]**

The function f is defined by $\mathrm{f} : x \mapsto \sin x + \sqrt{3}\cos x,\ \frac{\pi}{6} \le x \le k.$

(ii) Find the largest exact value of k such that f has an inverse. Hence define f^{-1} in similar form and write down the set of values of x for which $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$. **[5]**

(b) The function g is defined by $\mathrm{g} : x \mapsto 2 - \frac{5x}{1+x^2},\ x \in \mathbb{R}.$

(i) Use an algebraic method to find the range of g. **[3]**

(ii) State a sequence of transformations which transform the graph of $y = \mathrm{g}(x)$ to the graph of $y = \frac{10x}{4+x^2}$. **[3]**

11 The line l_1 passes through the point A with coordinates (1, 2, 1) and is parallel to the vector $\mathbf{i} + a\mathbf{j} + 2\mathbf{k}$, where $a \in \mathbb{R}$. The line l_2 has equation $x - 3 = \frac{y}{2} = \frac{z-5}{3}$. It is given that l_1 and l_2 intersect at point B.

(i) Find the value of a. **[4]**

(ii) The plane p_1 contains the point A and is perpendicular to l_2. Find the exact shortest distance from point B to p_1. Hence find the acute angle between l_1 and p_1. **[5]**

(iii) Find a cartesian equation of plane p_2 that is perpendicular to p_1 and contains l_1. **[3]**

(iv) Find the acute angle between p_2 and the xy-plane. **[2]**

NANYANG JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

MATHEMATICS **9740/02**

Paper 2 **22nd September 2016**

3 Hours

Additional Materials: Cover Sheet
Answer Papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
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You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

Section A: Pure Mathematics [40 marks]

1 The first four terms of a sequence of numbers are 10, 6, 5 and 7. S_n is the sum of the first n terms of this sequence. Given that S_n is a cubic polynomial in n, find S_n in terms of n. **[4]**

Show that $U_n = \frac{3}{2}n^2 - \frac{17}{2}n + 17$, where U_n denotes the n^{th} term of the sequence. **[2]**

Find the set of values of n for which $S_n < 3U_n$. **[2]**

2 On separate diagrams, draw sketches of the graphs of

(i) $y = \dfrac{x^2(3-x)}{1+x}$,

(ii) $y^2 = \dfrac{x^2(3-x)}{1+x}$,

including the coordinates of the points where the graphs cross the axes and the equations of any asymptotes. You should show the features of the graphs at the points where it crosses the x-axis clearly.

Show that the area of the region enclosed by the graph in **(ii)** may be expressed in the form

$$2\int_0^3 \frac{3x - x^2}{\sqrt{(4-(x-1)^2)}}\,dx \ .$$

By using the substitution $x - 1 = 2\sin\theta$, evaluate this area exactly. **[10]**

3 **(a)** Solve $z^3 - 2(2-\mathrm{i})z^2 + (8-3\mathrm{i})z - 5 + \mathrm{i} = 0$, given that one of the three roots is real. **[5]**

(b) The complex number u is given by $u = \cos\theta + \mathrm{i}\sin\theta$, where $0 < \theta < \frac{\pi}{2}$.

(i) Show that $1 - u^2 = -2\mathrm{i}u\sin\theta$ and hence find the modulus and argument of $1-u^2$ in terms of θ. **[4]**

(ii) Given that $(1-u^2)^{10}$ is real and negative, find the possible values of θ in terms of π. **[3]**

4 [In this question, you may use the result that for a circle with radius r, a sector with angle θ has arc length $r\theta$ and area $\frac{1}{2}r^2\theta$.]

(a) A circle of radius r is divided into 16 sectors of decreasing arc length. Let L_n and A_n be the arc length and the area of the nth sector respectively. Suppose L_n is an arithmetic sequence with first term r and common difference d.

(i) Show that $d = \left(\frac{\pi - 8}{60}\right)r$. **[2]**

(ii) Show that A_n is an arithmetic sequence. **[3]**

(b) Let G_n be the area of a sector of a circle with radius a. Suppose that G_n is a geometric sequence with first term a and common ratio r, where $0 < r < 1$.

(i) If N sectors are needed to form the circle, show that r satisfies the equation

$$r^N - \pi ar + (\pi a - 1) = 0.$$ **[3]**

(ii) If an infinite number of sectors are needed to form the circle, find r in terms of a. **[2]**

Section B: Statistics [60 marks]

5 A company sells a certain brand of baby milk powder and would like to gather feedback on their product. Explain why quota sampling is appropriate in this situation and describe briefly how a sample of 50 could be chosen using quota sampling. **[3]**

The company wishes to randomly reward 5 customers with free milk vouchers through a lucky draw. Suppose that 2000 customers qualify for the draw, show that there will be equal probability of a particular customer being the first to be selected or the third to be selected for the free milk vouchers. **[2]**

6 The mass, in grams, of an ice-cube has the distribution $\mathrm{N}(\mu, \sigma^2)$. The mean mass of a random sample of n ice-cubes is denoted by $\bar{X}$. It is given that $\mathrm{P}(\bar{X} < 35.0) = 0.97725$ and $\mathrm{P}(\bar{X} \geq 20.0) = 0.84134$.

(i) Obtain an expression for σ in terms of n. **[3]**

(ii) Find $\mathrm{P}(\bar{X} > 32)$. **[2]**

Assume now that the mass of an ice-cube has the distribution $\mathrm{N}(25, 50)$.

An ice dispenser discharges 15 ice cubes each time into a cup. State the distribution of the mass of a discharge of 15 ice cubes. **[1]**

(iii) Find the mass exceeded by 10% of these discharges, correct to 1 decimal place. **[2]**

(iv) Find the probability that the mass of the first discharge of ice-cubes is more than the second discharge. **[2]**

7 A team of 5 men and 5 women is to be picked from 8 men and 9 women such that two of the 9 women, Ann and Lucy, must both be selected or not at all. Find the number of ways in which this can be done. **[2]**

Assume now the team is selected and Ann, Carrie and Lucy are included.

(i) The selected team is to form a queue. Find the number of possible arrangements if Ann and Lucy are to occupy both the second and the sixth positions and no two people of the same gender are to stand next to each other. **[3]**

(ii) On another occasion, the selected team is required to be seated at a round table with 10 chairs of different colours. If only Carrie can be seated between Ann and Lucy, find the number of possible arrangements. **[3]**

8 Two teams, the Ramblers and the Strollers, meet annually for a quiz which always has a winner. If the Ramblers wins the quiz, the probability of them winning the following year is 0.7. If the Strollers wins the quiz, the probability of them winning the following year is 0.5. The Ramblers won the quiz in 2015.

(i) Find the probability that the Strollers will win in 2018. **[2]**

(ii) If the Strollers were to win in 2018, what is the probability that it will be their first win for at least three years since 2015?

[2]

(iii) Assuming that the Strollers wins in 2018, find the smallest value of n such that the probability of the Ramblers winning the quiz for n consecutive years after 2018 is less than 5%. **[3]**

9 It is believed that the probability p of a randomly chosen pregnant woman giving birth to a Down Syndrome child is related to the woman's age x, in years. The table gives observed values of p for 6 different values of x.

x	20	25	30	35	40	45
p	0.00023	0.00067	0.00125	0.00333	0.01000	0.03330

(i) Sketch the scatter diagram for the given data. **[1]**

(ii) Find, correct to 4 decimal places, the product moment correlation coefficient between

(a) p and x ,

(b) $\ln p$ and x ,

(c) p and x^2 . **[2]**

(iii) Using the most appropriate case from part **(ii)**, find the equation which best models the probability of a pregnant woman giving birth to a Down Syndrome child at different ages. **[2]**

(iv) Hence, estimate the expected number of children with Down Syndrome that will be born to 5000 randomly chosen pregnant women of age 32. **[2]**

10 At an early stage in analysing the marks, x, scored by a large number of candidates in an examination paper, the Examination Board takes the scores from a random sample of 250 candidates. The results are summarised as follows:

$$\sum x = 11872 \quad \text{and} \quad \sum x^2 = 646193$$

(i) Calculate unbiased estimates of the population mean and variance to 3 decimal places. **[2]**

(ii) In a 1-tail test of the null hypothesis $\mu = 49.5$, the alternative hypothesis is accepted. State the alternative hypothesis and find an inequality satisfied by the significance level of the test. **[4]**

(iii) It is subsequently found that the population mean and standard deviation for the examination paper are 45.292 and 18.761 respectively. Find the probability that in a random sample of size 250, the sample mean is at least as high as the one found in the sample above. **[2]**

11 On a typical weekday morning, customers arrive at the post office independently and at a rate of 3 per 10 minute period.

(i) State, in context, a condition needed for the number of customers who arrived at the post office during a randomly chosen period of 30 minutes to be well modelled by a Poisson distribution. **[1]**

(ii) Find the probability that no more than 4 customers arrive between 11.00 a.m. and 11.30 a.m. **[2]**

(iii) The period from 11.00 a.m. to 11.30 a.m. on a Tuesday morning is divided into 6 periods of 5 minutes each. Find the probability that no customers arrive in at most one of these periods. **[2]**

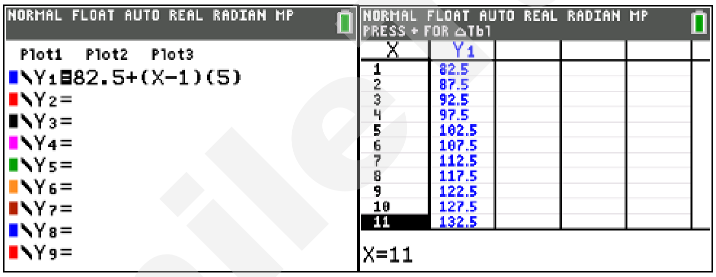
The post office opens for 3.5 hours each in the morning and afternoon and it is noted that on a typical weekday afternoon, customers arrive at the post office independently and at a rate of 1 per 10 minute period. Arrivals of customers take place independently at random times.

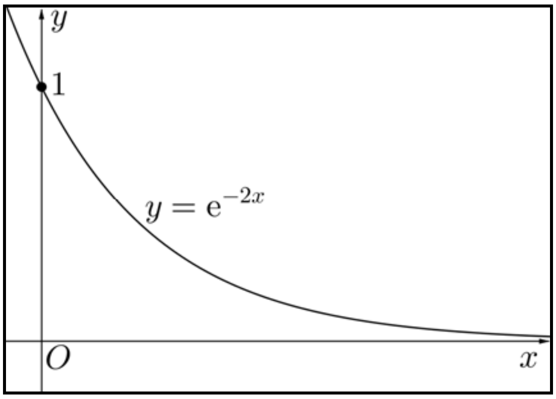
(iv) Show that the probability that the number of customers who arrived in the afternoon is within one standard deviation from the mean is 0.675, correct to 3 decimal places. **[3]**

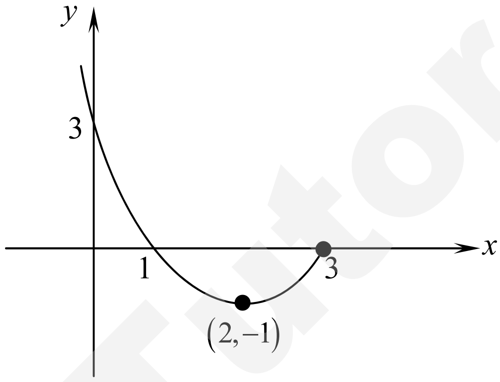
(v) Find the probability that more than 38 customers arrived in a morning given that a total of 40 customers arrived in a day. **[4]**

(vi) Using a suitable approximation, estimate the probability that more than 100 customers arrive at the post office in a day. **[3]**

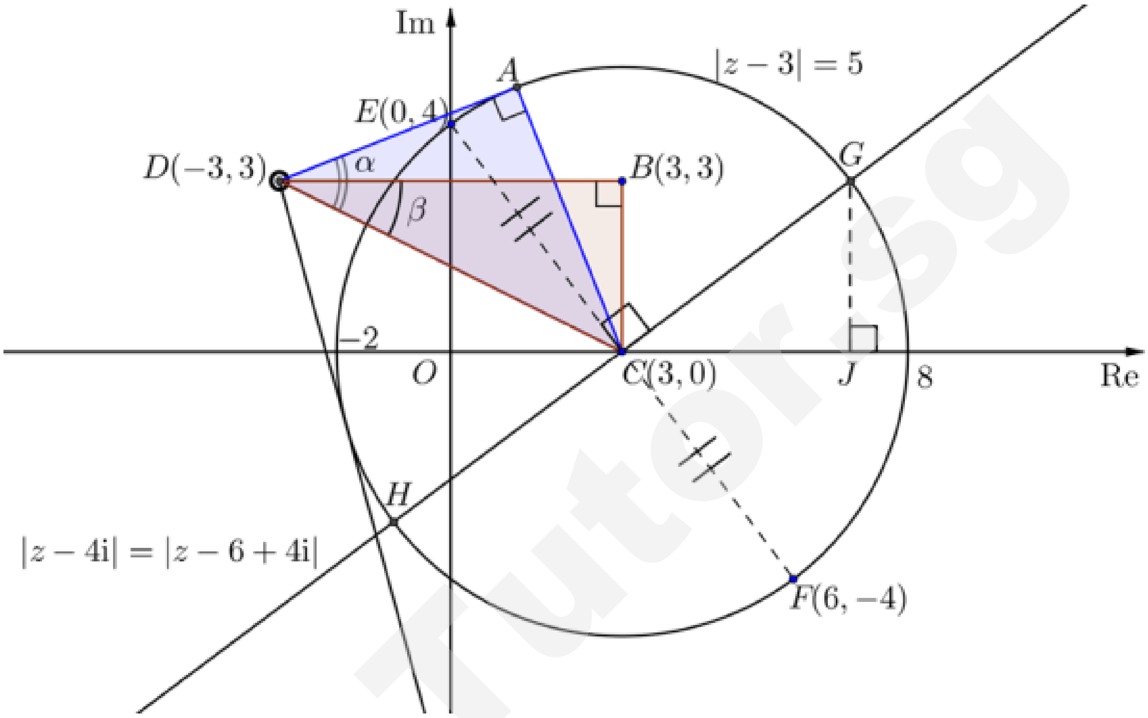
2016 SH2 H2 Mathematics Preliminary Examination Paper 2 Suggested Solutions

Qn No.	Solution
1 (a)	Since Kenny would have paid up his loan in full exactly in the 48th month, $S_{48} = \frac{48}{2}\left[2a+(48-1)d\right]$ $9600 = 24(2a+47d)$ $400 = 2a+47d \cdots\cdots (1)$ Since Kenny had an outstanding payment of \$2400 after the 40th month, total amount paid by the 40th month = \$9600 – 2400 = \$7200. Therefore, $S_{40} = \frac{40}{2}\left[2a+(40-1)d\right]$ $7200 = 20(2a+39d)$ $360 = 2a+39d \cdots\cdots (2)$ Solving (1) & (2), $a = 82.5$, $d = 5$. On the last day of the n^{th} month (for $1 \le n \le 40$), the amount paid by Kenny = \$82.5 + $(n-1)(5)$.  Therefore, amount paid on last day of 10th month = \$127.5 < \$130, amount paid on last day of 11th month = \$132.5 > \$130. Therefore Kenny first paid at least \$130 on the last day of the **11th** month.

Qn No.	Solution
1 (b) (i)	Common ratio of $1+e^{-2x}+e^{-4x}+\cdots$ is $r=e^{-2x}$.  For $x>0$, $0<e^{-2x}<1$ (see above sketch). Therefore, the geometric series converges (since e^{-2x} is the common ratio). **OR** As $n\to\infty$, $e^{-2nx}\to 0$ (for $x>0$). Therefore $S_n=\dfrac{1-e^{-2nx}}{1-e^{-2x}}\to\dfrac{1}{1-e^{-2x}}$, i.e. the series is convergent. Sum to infinity $=\dfrac{1}{1-e^{-2x}}$
1 (b) (ii)	For $x=10$, $S_n=\dfrac{1-e^{-20n}}{1-e^{-20}}$, $S=\dfrac{1}{1-e^{-20}}$. $S-S_n<\dfrac{S}{10^{100}}$ $S_n>S-\dfrac{S}{10^{100}}$ $S_n>S\left(1-\dfrac{1}{10^{100}}\right)$ $\dfrac{1-e^{-20n}}{1-e^{-20}}>\dfrac{1}{1-e^{-20}}\left(1-\dfrac{1}{10^{100}}\right)$ $1-e^{-20n}>1-\dfrac{1}{10^{100}}$ $-e^{-20n}>-\dfrac{1}{10^{100}}$ $e^{-20n}<\dfrac{1}{10^{100}}$ $-20n<\ln\dfrac{1}{10^{100}}=-100\ln 10$ $n>5\ln 10=11.513$ Therefore, least value of $n=$ **12**

Qn No.	Solution
2(a)	$\mathrm{f}(x)=x$ $x^2-4x+3=x$ $x^2-5x+3=0$ $x=\dfrac{-(-5)\pm\sqrt{(-5)^2-4(1)(3)}}{2}$ $=\dfrac{5\pm\sqrt{13}}{2}$ Since $x\le 2$, we have $x=\dfrac{5-\sqrt{13}}{2}$
2(b)(i)	 There exist a horizontal line $y=k$ where $k\in(-1,0]$ cuts the graph of $y=\mathrm{f}(x)$ twice. Thus f is not one-one and hence its inverse does not exist. **OR** $\mathrm{f}(3)=\mathrm{f}(1)=0$ but $3\ne 1$. Thus f is not one-one and its inverse does not exist.
2(b) (ii)	From the graph, $\mathrm{R_f}=[-1,\infty)$. Moreover, $\mathrm{D_g}=(-2,\infty)$. Since $\mathrm{R_f}\subset\mathrm{D_g}$, gf exists. $\mathrm{gf}(x)=\mathrm{g}\left(x^2-4x+3\right)$ $=\tan^{-1}\left(2x^2-8x+7\right)$ Note that $\mathrm{D_{gf}}=\mathrm{D_f}$. Thus, $\mathrm{gf}:x\mapsto\tan^{-1}\left(2x^2-8x+7\right),\quad x\le 3.$ $\mathrm{R_{gf}}=\left[-\dfrac{\pi}{4},\dfrac{\pi}{2}\right)$

Qn No.	Solution
3 (i)	Let the foot of perpendicular be N. Equation of the line that passes through A and perpendicular to p_1 is $l_A : \mathbf{r} = \begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \gamma \in \mathbb{R}$. Since N lies on l_A, $\overrightarrow{ON} = \begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ for some $\gamma \in \mathbb{R}$. $\left[\begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 4 \Rightarrow 2q + 2\gamma = 4$ $\Rightarrow \gamma = 2 - q$ $\therefore \overrightarrow{ON} = \begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2-q \\ 2-q \\ 0 \end{pmatrix} = \begin{pmatrix} 2+q \\ 2-q \\ 2 \end{pmatrix}$ Hence, N is the point $(2+q, 2-q, 2)$.
3 (ii)	Let $\mathbf{b}$ be the position vector of point B. By Ratio Theorem, $\begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} + \mathbf{b} = 2 \begin{pmatrix} 2+q \\ 2-q \\ 0 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 4 \\ 4-2q \\ 2 \end{pmatrix}$ Since B lies in p_2, $\begin{pmatrix} 4 \\ 4-2q \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} = 7$ $12 + 8 - 4q - 10 = 7$ $q = 0.75$ **or** $\frac{3}{4}$
3 (iii)	Using GC, $l : \mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + \theta \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}, \theta \in \mathbb{R}$.
3 (iv)	$\begin{pmatrix} \lambda \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix} = 0 \Rightarrow \lambda = -\frac{1}{5}$.

	$\begin{pmatrix}-1\\5\\0\end{pmatrix}\cdot\begin{pmatrix}\lambda\\0\\1\end{pmatrix}\neq\mu\Rightarrow\mu\neq\frac{1}{5}.$
Qn No.	**Solution**
4 (i), (iii) (1st part)	
4 (ii)	$CD=\sqrt{(3-(-3))^2+(0-3)^2}$ $=\sqrt{45}$ Largest value of $\arg(z+3-3\text{i})$ $=\alpha-\beta$ $=\sin^{-1}\left(\frac{AC}{CD}\right)-\tan^{-1}\left(\frac{BC}{BD}\right)$ $=\sin^{-1}\left(\frac{5}{\sqrt{45}}\right)-\tan^{-1}\left(\frac{1}{2}\right)$ $=0.37742$ $=0.377$ rad (to 3 s.f.)

4 (iii) (2nd part)

Method 1: Applying relationship between gradient of a line and the angle it makes with the positive horizontal axis

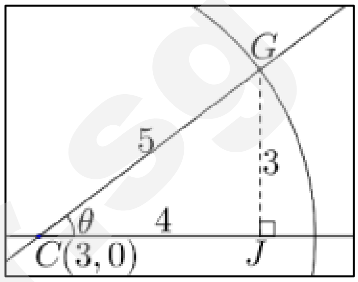
Gradient of perpendicular bisector $=-\dfrac{1}{\left(-\frac{4}{3}\right)}=\dfrac{3}{4}$

Angle that perpendicular bisector makes with positive real axis, $\theta=\tan^{-1}\left(\dfrac{3}{4}\right)$

Hence,

$\dfrac{GJ}{5}=\sin\theta=\dfrac{3}{5}\Rightarrow GJ=3$

$\dfrac{CJ}{5}=\cos\theta=\dfrac{4}{5}\Rightarrow CJ=4$



So G and H represent the complex numbers

$z = (3 + 4) + (0 + 3)\text{i} = 7 + 3\text{i}$ (corresponding to G), and

$z = (3 – 4) + (0 – 3)\text{i} = –1 – 3\text{i}$ (corresponding to H) resp.

Qn No.	Solution

4 (iii) (2nd part)

Method 2: Using Similar Triangles

$$\angle GCE = 90^\circ \Rightarrow \angle OCE + 90^\circ + \angle GCJ = 180^\circ$$
$$\Rightarrow \angle OCE = 90^\circ - \angle GCJ$$

Also, $\angle OEC = 90^\circ - \angle OCE$

$$= 90^\circ - \left(90^\circ - \angle GCJ\right)$$
$$= \angle GCJ$$

Furthermore, $\angle COE = \angle GJC = 90^\circ$.

Therefore, $\triangle COE \sim \triangle GJC$. Hence,

$\dfrac{CO}{CE}=\dfrac{GJ}{GC}\Rightarrow\dfrac{3}{5}=\dfrac{GJ}{5}\Rightarrow GJ=3$, and

$\dfrac{OE}{CE}=\dfrac{CJ}{GC}\Rightarrow\dfrac{4}{5}=\dfrac{CJ}{5}\Rightarrow CJ=4$

So coordinates of G are (3 + 4, 0 + 3), i.e. (7, 3) and similarly, coordinates of H are (3 – 4, 0 – 3), i.e. (–1, –3). Therefore, possible values of z are $7 + 3\text{i}$ and $–1 –3\text{i}$.

Method 3: Using Cartesian Equations

Equation of circle: $(x-3)^2+y^2=5^2$

Equation of perpendicular bisector:

$$y-0=-\frac{1}{\left(-\frac{8}{6}\right)}(x-3)\Rightarrow y=\frac{3}{4}(x-3)$$

Substituting, $(x-3)^2+\left(\dfrac{3}{4}(x-3)\right)^2=5^2$

$$\left(1+\left(\frac{3}{4}\right)^2\right)(x-3)^2=5^2$$

$$\frac{25}{16}(x-3)^2 = 25$$
$$(x-3)^2 = 16$$
$$x-3 = \pm 4$$
$$x = 7 \text{ or } -1.$$

When $x = 7$, $y = \frac{3}{4}(7-3) = 3$

When $x = -1$, $y = \frac{3}{4}(-1-3) = -3$

Therefore, possible values of z are $7 + 3\text{i}$ and $-1 - 3\text{i}$.

Qn No.	Solution

5 (a)

Assign a number from 1 to N to each of the students, where N represents the student population size OR obtain a list of the students from the administration office in order of their identification numbers or registration numbers.

Next, determine the sampling interval size $k = \frac{1}{0.02} = 50$.

Randomly select any student from the list, say the 1st student. Select every 50th student thereafter (i.e. 51st, 101th,…) until the required sample is obtained.

5 (b)

Advantages:

- **<u>Representativeness of Sample</u>**

Quota sampling allows the survey to capture the responses that represent various groups of students (e.g. different PM classes, or **1st** CCAs); this may be preferred as certain homeroom or sports facilities may not be in as good a condition as others, and the representation of each group will ensure that the results will not be biased towards those who are often using these less functional facilities or towards those who are often using the more functional facilities.

- **<u>Efficiency of Collecting the Sample</u>**

Quota sampling may be more efficient as systematic sampling in this case requires the surveyor to identify the selected respondents and to contact them, which can be time consuming (e.g. student selected may be on MC on day of survey, selected students do not respond to online survey etc).

Disadvantages:

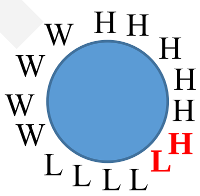
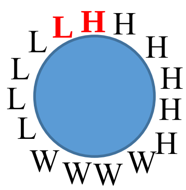
- **<u>Non-randomness/Selection Bias</u>**

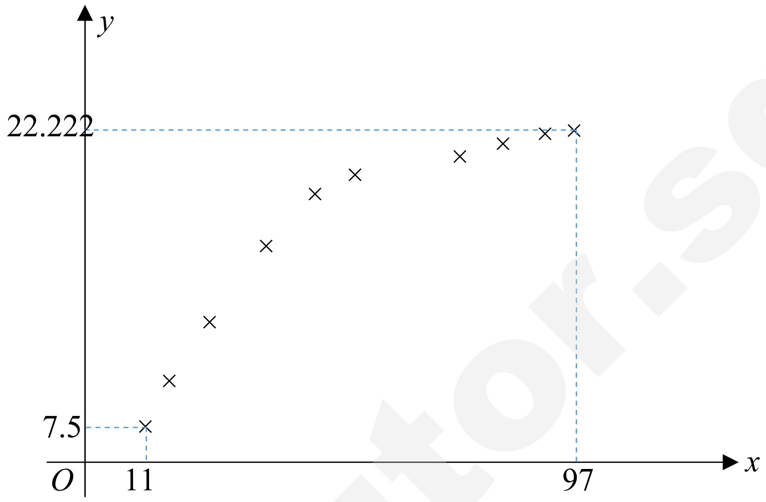
Quota sampling is non-random and may contain selection bias, where the surveyor chooses people who may appear friendlier or choose students in the canteen only at a selected time period. This results in certain students having no chance of being selected at all, which may affect the validity of the survey results.

- **<u>Non-representativeness of Sample</u>**

Quota sampling may result in a group (e.g. one entire cohort, or people coming later to the canteen etc.) being excluded entirely from the selection, which may result in the data collected being an inaccurate representation of the entire school population.

Qn No.	Solution
6	$X \sim \mathrm{N}(\mu, \sigma^2)$ $\mathrm{P}(X < 17.7) = 0.15$ $\mathrm{P}\left(Z < \frac{17.7-\mu}{\sigma}\right) = 0.15$ $\frac{17.7-\mu}{\sigma} = -1.03643$ --- (1) $\mathrm{P}(X > 21.9) = 0.2$ $\mathrm{P}(X < 21.9) = 0.8$ $\mathrm{P}\left(Z < \frac{21.9-\mu}{\sigma}\right) = 0.8$ $\frac{21.9-\mu}{\sigma} = 0.841621$ --- (2) Solving simultaneous equations (1) and (2): From (1): $\mu = 1.03643\sigma + 17.7$ From (2): $\mu = -0.841621\sigma + 21.9$ Using GC, $\mu = 20.0$ (3s.f) and $\sigma = 2.24$ (3s.f)

Qn No.	Solution
7 (i)	No. of ways $= \left(\frac{9!}{9}\right)\binom{9}{6}6! = 2\,438\,553\,600$
7 (ii)	No. of ways $= \left(\frac{3!}{3}\right)(5!)(4!)(4!)(1)$ $= 138\,240$ OR **Case 1** No. of ways $= (5!)(4!)(4!)$ $= 138\,240 / 2$  **Case 2** No. of ways $= (5!)(4!)(4!)$ $= 138\,240 / 2$ 
7 (last part)	**Case 1** : None from Liaison Committee No of ways $= 1$ **Case 2** : None from Welfare Committee No of ways $= \binom{6}{6}\binom{5}{4} + \binom{6}{5}\binom{5}{5} = 11$ or No of ways $= \binom{11}{10} = 11$

	Total Number of ways to select at least 3 men and 3 women $=\binom{15}{10}-1-11$ $=2991$
Qn No.	**Solution**
8 (i)	
8 (ii)	**<u>Unsuitability of a Linear Model</u>** A linear model predicts the average diameter will keep increasing indefinitely without any limit. Therefore a linear model is not appropriate. **<u>Unsuitability of a Quadratic Model</u>** A quadratic model predicts that the average diameter will eventually attain a maximum value, and thereafter decrease as the age increases, till it eventually takes on negative values. This is not possible, and therefore a quadratic model is not appropriate.
8 (iii)	Using the suggested model, the least square regression line is $y=23.886-\dfrac{185.346}{x}=23.9-\dfrac{185}{x}$ (to 3 s.f.) r-value = –0.994 (to 3 s.f.) When $x=40$, $y=23.886-\dfrac{185.346}{40}=19.3$ (to 3 s.f.) Since x = 40 is within the range of values of x, [11, 97] and the product moment correlation coefficient, –0.994, has an absolute value that is close to 1, suggesting a strong linear correlation between the variables y and $1/x$, therefore the estimate is reliable.

Qn No.	Solution
9 (i)	The eye colour of a person is independent of that of another person. **OR** Every person is equally likely to have blue eyes.
9 (ii)	$Y \sim \mathrm{B}(60, 0.08)$ Then $\mathrm{P}(5 \le Y < 21)$ $= \mathrm{P}(Y \le 20) - \mathrm{P}(Y \le 4)$ $= 0.530$ (to 3 s.f.)
9 (iii)	Since $n = 60$ (> 50) is large and $np = 4.8 < 5$, $Y \sim \mathrm{Po}(4.8)$ approximately. $\mathrm{P}(Y > 9) = 1 - \mathrm{P}(Y \le 9) = 0.0251$ (to 3 s.f.)

Qn No.	Solution
10 (i) (a)	Let E and F denote the event that the sum of scores is at least 8 and the event that the absolute difference between the scores is at most 4 respectively.

Sum of Scores

1st die \ 2nd die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

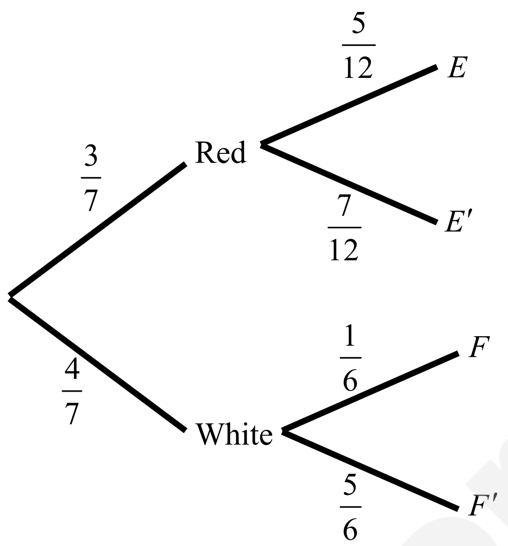
$$\mathrm{P}(E) = \frac{1}{6} \times \frac{1}{6} \times 15 = \frac{5}{12}$$

10 (i) (b)

Absolute Difference between Scores

1st die \ 2nd die	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

$$\mathrm{P}(F) = \frac{1}{6} \times \frac{1}{6} \times 6 = \frac{1}{6}$$

Qn No.	Solution
10 (ii)	 $\text{P}(\text{game ends at 1st round})$ $= \text{P}(\text{red and game ends at 1st round})$ $+\text{P}(\text{white and game ends at 1st round})$ $= \left(\frac{3}{7}\right)\left(\frac{5}{12}\right)+\left(\frac{4}{7}\right)\left(\frac{1}{6}\right)$ $= \frac{23}{84}$
10 (iii)	P(red ball selected | game ends at the first round) $= \frac{\text{P(red ball selected} \cap \text{game ends at the first round)}}{\text{P(game ends at the first round)}}$ $= \frac{\left(\frac{3}{7}\right)\left(\frac{5}{12}\right)}{\frac{23}{84}} = \frac{15}{23}$
10 (iv)	P(total of 3 rounds of game & exactly 2 white balls selected) $= \text{P(WWR)} + \text{P(WRW)} + \text{P(RWW)}$ $= \left(\frac{4}{7}\right)\left(\frac{5}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{3}{5}\right)\left(\frac{5}{12}\right)+\left(\frac{4}{7}\right)\left(\frac{5}{6}\right)\left(\frac{3}{6}\right)\left(\frac{7}{12}\right)\left(\frac{3}{5}\right)\left(\frac{1}{6}\right)$ $+\left(\frac{3}{7}\right)\left(\frac{7}{12}\right)\left(\frac{4}{6}\right)\left(\frac{5}{6}\right)\left(\frac{3}{5}\right)\left(\frac{1}{6}\right)$ $= \frac{25}{504}+\frac{1}{72}+\frac{1}{72}$ $= \frac{13}{168}$

<table>
<tr><th>Qn No.</th><th>Solution</th></tr>
<tr><td>11 (i)</td><td>Unbiased estimate of μ, $\bar{x}=\frac{\sum x}{n}=\frac{21350}{20}=1067.5$
Unbiased estimate of σ^2, $s^2=\frac{1}{19}(345900)$
$=18205.3$ (to 1 d.p.)</td></tr>
<tr><td>11 (ii)</td><td>H_0: $\mu=1000$
H_1: $\mu>1000$

Assume that the amounts of loans borrowed by the bank's clients follow a normal distribution.

OR

Assume that the amount of loans borrowed by each client follows a normal distribution.

Level of Significance: 5% (upper-tailed)

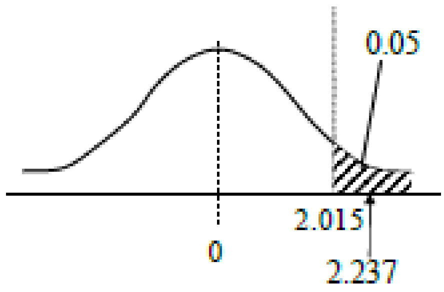
Under H_0, $T=\frac{\bar{X}-1000}{S/\sqrt{20}}\sim t(19)$
Test Statistic: $t=\frac{\bar{x}-1000}{s/\sqrt{20}}$

<u>Method 1: Using critical region and observed test statistic, t</u>

Critical region: $t>2.015$

$t=\frac{1067.5-1000}{s/\sqrt{20}}\approx 2.237$ $\left(s=\sqrt{18205.3}\right)$

Since $t=2.237>2.015$, we reject H_0.



<u>Method 2: Using p-value</u>

p-value $=0.0187$

Since p-value $=0.0187<0.05$, we reject H_0.

We conclude that there is sufficient evidence at 5% level of significance that the company has understated the mean amount of loans borrowed by its clients.</td></tr>
<tr><td>11 (iii)</td><td>The meaning of 'at the 5% significance level' is that there is a probability of 0.05 that it was wrongly concluded that the company had understated the mean amount of loans borrowed by its clients.</td></tr>
</table>

Qn No.	Solution
11 (iv)	Test Statistic: $z = \dfrac{\overline{x}-1000}{250/\sqrt{n}}$ To not reject $\mathrm{H_0}$, $z \le 1.6449$ $\dfrac{1067.5-1000}{250/\sqrt{n}} \le 1.6449$ $\sqrt{n} \le 6.092$ $n \le 37.1$ Since $n \in \mathbb{Z}^+$, $n \le 37$.

Qn No.	Solution
12 (i)	Any one of the following: [*Constant mean rate*] The mean number of cars joining the immigration checkpoint queue for any subinterval of the same length of time within 1 hour (e.g. minute) is constant. **OR** [*Independence of occurrence of event*] Cars join the immigration queue independently of one another, throughout the entire hour.
12 (ii)	Let X denote the random variable for the number of cars leaving an immigration checkpoint queue in a period of n minutes. $X \sim \mathrm{Po}\left(\dfrac{27}{60}n\right)$ i.e. $X \sim \mathrm{Po}(0.45n)$ $\mathrm{P}(X \ge 1) > 0.9$ $1-\mathrm{P}(X=0) > 0.9$ $\mathrm{P}(X=0) < 0.1$ $\dfrac{\mathrm{e}^{-0.45n}(0.45n)^0}{0!} < 0.1$ $\mathrm{e}^{-0.45n} < 0.1$ $-0.45n < \ln(0.1)$ $n > \dfrac{\ln(0.1)}{-0.45} \approx 5.11$ Therefore, least integer n is 6.

Qn No.	**Solution**
12 (iii)	Let J and L denote the random variables for the number of cars joining and leaving an immigration checkpoint queue respectively in a 2-hour period. Then J~Po(46) and L~Po(54) Since $46 > 10$, $J \sim \mathrm{N}(46, 46)$ **approximately**. Since $54 > 10$, $L \sim \mathrm{N}(54, 54)$ **approximately**. Let W denote the number of people in the queue at 1100. Then $W = 19 + J - L$. $\mathrm{E}(W) = \mathrm{E}(19 + J - L)$ $= 19 + \mathrm{E}(J) - \mathrm{E}(L) = 19 + 46 - 54 = 11$, and $\mathrm{Var}(W) = \mathrm{Var}(19 + J - L)$ $= \mathrm{Var}(J) + \mathrm{Var}(L) = 46 + 54 = 100$ Therefore, $W = 19 + J - L \sim \mathrm{N}(11, 100)$ **approximately.** $\mathrm{P}\left(19 + J - L \le 12\right) = \mathrm{P}\left(19 + J - L \le 12.5\right)$ (by c.c.) $= 0.560$ (to 3.s.f) **<u>Alternatively,</u>** $J - L \sim \mathrm{N}(-8, 100)$ **approximately.** $\mathrm{P}\left(J - L \le -7\right) = \mathrm{P}\left(J - L \le -6.5\right)$ $= 0.560$ (to 3.s.f) **<u>Or equivalently,</u>** $L - J \sim \mathrm{N}(8, 100)$ **approximately.** $\mathrm{P}\left(L - J \ge 7\right) = \mathrm{P}\left(L - J \ge 6.5\right)$ $= 0.560$ (to 3.s.f)
12 (iv)	The mean number of cars joining the immigration checkpoint queue every hour may not be constant due to peak periods as there may be more cars heading to or returning from work.

Solutions to P1 Prelim 2016

1	Let P_n denote the proposition $S_n=\left(\frac{n-1}{n+2}\right)\left(\frac{4^{n+1}}{3}\right)+\frac{2}{3}$ for n=1,2,3,…. When $n=1$, $\text{LHS}=S_1=u_1=\frac{4^1(1)^2}{(2)(3)}=\frac{2}{3}$. $\text{RHS}=\left(\frac{1-1}{1+2}\right)\left(\frac{4^2}{3}\right)+\frac{2}{3}=\frac{2}{3}=\text{LHS}.$ $\therefore \mathrm{P}_1$ is true. Assume P_k is true for some k=1,2,3.., i.e. $S_k=\left(\frac{k-1}{k+2}\right)\left(\frac{4^{k+1}}{3}\right)+\frac{2}{3}$. To prove P_{k+1} is also true, i.e. $S_{k+1}=\left(\frac{k}{k+3}\right)\left(\frac{4^{k+2}}{3}\right)+\frac{2}{3}$. $\text{LHS}=S_{k+1}$ $=S_k+u_{k+1}$ $=\left(\frac{k-1}{k+2}\right)\left(\frac{4^{k+1}}{3}\right)+\frac{2}{3}+\frac{4^{k+1}(k+1)^2}{(k+2)(k+3)}$ $=\left(\frac{4^{k+1}}{3(k+2)}\right)\left[k-1+\frac{3(k+1)^2}{k+3}\right]+\frac{2}{3}$ $=\left(\frac{4^{k+1}}{3(k+2)}\right)\left[\frac{(k-1)(k+3)+3(k+1)^2}{k+3}\right]+\frac{2}{3}$ $=\left(\frac{4^{k+1}}{3(k+2)}\right)\left[\frac{k^2+2k-3+3k^2+6k+3}{k+3}\right]+\frac{2}{3}$ $=\left(\frac{4^{k+1}}{3(k+2)}\right)\left[\frac{4k^2+8k}{k+3}\right]+\frac{2}{3}$ $=\left(\frac{4^{k+2}}{3(k+2)}\right)\left[\frac{k(k+2)}{k+3}\right]+\frac{2}{3}=\left(\frac{k}{k+3}\right)\left(\frac{4^{k+2}}{3}\right)+\frac{2}{3}=\text{RHS}.$ Since P_1 is true and P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true, by Mathematical Induction, P_n is true for all n =1, 2, 3,….

2	$\dfrac{17x^2+23x+12}{(3x+4)(x^2+4)} \equiv \dfrac{A}{3x+4}+\dfrac{Bx+C}{x^2+4}$ $17x^2 + 23x + 12 = A(x^2 + 4) + (Bx + C)(3x + 4)$ Solving, $A = 2$, $B = 5$ and $C = 1$ $\displaystyle\int_{-2}^{2}\frac{17x^2+23x+12}{(3x+4)(x^2+4)}\,dx = \int_{-2}^{2}\frac{2}{3x+4}+\frac{5x+1}{x^2+4}\,dx$ $\displaystyle=\int_{-2}^{2}\frac{2}{3x+4}+\frac{5}{2}\left(\frac{2x}{x^2+4}\right)+\frac{1}{x^2+4}\,dx$ $=\dfrac{2}{3}\Big[\ln\lvert 3x+4\rvert\Big]_{-2}^{2}+\dfrac{5}{2}\Big[\ln(x^2+4)\Big]_{-2}^{2}+\left[\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)\right]_{-2}^{2}$ $=\dfrac{2}{3}\ln 5+\dfrac{\pi}{4}$
3	$x=\dfrac{1}{2}(\sin t\cos t+t)=\dfrac{1}{2}t+\dfrac{1}{4}\sin 2t$ and $y=\dfrac{1}{2}t-\dfrac{1}{4}\sin 2t$ $\dfrac{dx}{dt}=\dfrac{1}{2}\cos 2t+\dfrac{1}{2}$ and $\dfrac{dy}{dt}=\dfrac{1}{2}-\dfrac{1}{2}\cos 2t$ $\dfrac{dy}{dx}=\dfrac{1-\cos 2t}{1+\cos 2t}=\dfrac{1-(1-2\sin^2 t)}{(2\cos^2 t-1)+1}=\tan^2 t$ When $\dfrac{dy}{dx}=1 \Rightarrow \tan^2 t=1 \Rightarrow t=\pm\dfrac{\pi}{4}$ Since $t<0$, $t=-\dfrac{\pi}{4}$, and $x=-\dfrac{1}{4}-\dfrac{\pi}{8}$ and $y=\dfrac{1}{4}-\dfrac{\pi}{8}$ Equation of normal is $y-\left(\dfrac{1}{4}-\dfrac{\pi}{8}\right)=-\left[x-\left(\dfrac{1}{4}-\dfrac{\pi}{8}\right)\right]$ $y=-x-\dfrac{\pi}{4}$ When $y=0$, $x=-\dfrac{\pi}{4}$. Volume required is $\displaystyle=\frac{1}{3}\pi\left(\frac{1}{4}-\frac{\pi}{8}\right)^2\left(\frac{\pi}{8}-\frac{1}{4}\right)+\pi\int_{-\frac{\pi}{4}}^{0}\left(\frac{1}{2}t-\frac{1}{4}\sin 2t\right)^2\left(\frac{1}{2}\cos 2t+\frac{1}{2}\right)dt$ $=0.00759$ (5 d.p.) $\dfrac{dy}{dx}=\tan^2 t\approx t^2\to 0$ as t approaches 0. Therefore the tangents are parallel to the x-axis.
4	$\overrightarrow{OM}=\dfrac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OC}\right)=\dfrac{1}{2}(2\mathbf{a}+\mathbf{b})$ $\overrightarrow{ON}=\dfrac{2}{3}\overrightarrow{OM}=\dfrac{2}{3}\times\dfrac{1}{2}(2\mathbf{a}+\mathbf{b})=\dfrac{1}{3}(2\mathbf{a}+\mathbf{b})$

$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \frac{1}{3}(2\mathbf{a}+\mathbf{b}) - \mathbf{a} = \frac{1}{3}(\mathbf{b}-\mathbf{a}) = \frac{1}{3}\overrightarrow{AB}$

Since $\overrightarrow{AN}$ is parallel to $\overrightarrow{AB}$ and A is the common point, hence A, B and N are collinear. B1

Since P is on AB, $\overrightarrow{OP} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}), where\ \lambda \in \mathbb{R}$

$\overrightarrow{MP} \bullet \overrightarrow{AB} = 0$

$\left[\mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) - \frac{1}{2}(2\mathbf{a}+\mathbf{b})\right] \bullet (\mathbf{b}-\mathbf{a}) = 0$

$\left[\left(\lambda - \frac{1}{2}\right)\mathbf{b} - \lambda\mathbf{a}\right] \bullet (\mathbf{b}-\mathbf{a}) = 0$

$\left(\lambda - \frac{1}{2}\right)|\mathbf{b}|^2 - \left(\lambda - \frac{1}{2}\right)\mathbf{a}\bullet\mathbf{b} - \lambda\mathbf{a}\bullet\mathbf{b} + \lambda|\mathbf{a}|^2 = 0$

But $\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos AOB = 2\times 3\cos 60^o = 3$

Hence, $9\left(\lambda - \frac{1}{2}\right) - 3\left(\lambda - \frac{1}{2}\right) - 3\lambda + 4\lambda = 0$

$$\lambda = \frac{3}{7}$$

$\overrightarrow{OP} = \mathbf{a} + \frac{3}{7}(\mathbf{b}-\mathbf{a}) = \frac{1}{7}(4\mathbf{a}+3\mathbf{b})$

<u>Alternative method:</u>

$\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos AOB = 2\times 3\cos 60^o = 3$

Using cosine formula $|\mathbf{b}\text{-}\mathbf{a}| = \sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2 \mathbf{-2}|\mathbf{a}||\mathbf{b}|\cos 60} = \sqrt{7}$

$$\overrightarrow{AP} = \left(\overrightarrow{AM} \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}\right)\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$$

$$= \left(\frac{1}{2}\mathbf{b} \cdot \frac{(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|}\right)\frac{\mathbf{b}-\mathbf{a}}{|\mathbf{b}-\mathbf{a}|}$$

$$= \frac{1}{2}\left(\frac{\mathbf{b}\cdot\mathbf{b}\text{-}\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}-\mathbf{a}|^2}\right)(\mathbf{b}-\mathbf{a})$$

$$= \frac{1}{2}\cdot\frac{6}{7}(\mathbf{b}-\mathbf{a})$$

$$= \frac{3}{7}(\mathbf{b}-\mathbf{a})$$

$\overrightarrow{OP} = \mathbf{a} + \frac{3}{7}(\mathbf{b}-\mathbf{a}) = \frac{1}{7}(4\mathbf{a}+3\mathbf{b})$

5

1. Let $\theta = \sin^{-1}\dfrac{y}{r} \Rightarrow \sin\theta = \dfrac{y}{r} \Rightarrow \cos\theta = \sqrt{1-\dfrac{y^2}{r^2}}$,

Diff wrt y : $\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}y} = \dfrac{1}{r} \Rightarrow \dfrac{\mathrm{d}\theta}{\mathrm{d}y} = \dfrac{1}{r\cos\theta}$

$$\therefore m = \tan\theta \Rightarrow \frac{\mathrm{d}m}{\mathrm{d}y} = \sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}y} = \frac{1}{r\cos^3\theta}$$

$$\text{ie, } \frac{\mathrm{d}m}{\mathrm{d}y} = \frac{1}{r\left(\dfrac{r^2-y^2}{r^2}\right)^{\frac{3}{2}}}$$

$$= \frac{r^2}{\left(r^2-y^2\right)^{\frac{3}{2}}}$$

Using $\dfrac{\mathrm{d}m}{\mathrm{d}t} = \dfrac{\mathrm{d}m}{\mathrm{d}y}\times\dfrac{\mathrm{d}y}{\mathrm{d}t}$, we have $\dfrac{\mathrm{d}m}{\mathrm{d}t} = \dfrac{r^2}{\left(r^2-y^2\right)^{\frac{3}{2}}}\times\dfrac{r}{1000} = \dfrac{r^3}{10^3\left(\sqrt{r^2-y^2}\right)^3}$

$$= \left(\frac{r}{10\sqrt{r^2-y^2}}\right)^3$$

$\dfrac{\mathrm{d}m}{\mathrm{d}t}$ is the rate of change of the gradient of the line OP

Alternate method 1:

$$m = \tan\left(\sin^{-1}\frac{y}{r}\right)$$

$$\frac{\mathrm{d}m}{\mathrm{d}y} = \sec^2\left(\sin^{-1}\frac{y}{r}\right)\left[\frac{1}{\sqrt{1-\left(\dfrac{y}{r}\right)^2}}\left(\frac{1}{r}\right)\right]$$

$$= \frac{1}{\cos^2\left(\sin^{-1}\dfrac{y}{r}\right)}\left[\frac{1}{\sqrt{r^2-y^2}}\right]$$

$= \dfrac{1}{\cos^2\theta}\left[\dfrac{1}{\sqrt{r^2-y^2}}\right]$ where $\theta = \sin^{-1}\dfrac{y}{r} \Rightarrow \sin\theta = \dfrac{y}{r}$

$$= \frac{1}{(x/r)^2}\left[\frac{1}{\sqrt{r^2-y^2}}\right] = \frac{r^2}{x^2}\left[\frac{1}{\sqrt{r^2-y^2}}\right]$$

$$= \frac{r^2}{r^2 - y^2}\left[\frac{1}{\sqrt{r^2 - y^2}}\right]$$

$$= \frac{r^2}{\left(r^2 - y^2\right)^{3/2}}$$

Using $\dfrac{\mathrm{d}m}{\mathrm{d}t} = \dfrac{\mathrm{d}m}{\mathrm{dy}} \times \dfrac{\mathrm{dy}}{\mathrm{d}t}$

$$= \frac{r^2}{\left(r^2 - y^2\right)^{3/2}} \times \frac{r}{1000}$$

$$= \frac{r^3}{10^3\left(r^2 - y^2\right)^{3/2}}$$

$$= \left(\frac{r}{10\sqrt{r^2 - y^2}}\right)^3$$

$\dfrac{\mathrm{d}m}{\mathrm{d}t}$ is the rate of change of the gradient of the line OP.

Alternate method 2:

$$m = \tan\left(\sin^{-1}\frac{y}{r}\right)$$

$$\tan^{-1} m = \sin^{-1}\frac{y}{r}$$

$$\left(\frac{1}{1+m^2}\right)\frac{\mathrm{d}m}{\mathrm{d}y} = \frac{1}{\sqrt{1-\left(\frac{y}{r}\right)^2}}\left(\frac{1}{r}\right)$$

$$\frac{\mathrm{d}m}{\mathrm{d}y} = \frac{1+m^2}{\sqrt{r^2 - y^2}}$$

$$= \frac{1+\tan^2\theta}{\sqrt{r^2 - y^2}} \qquad \text{where } \theta = \sin^{-1}\frac{y}{r} \quad \Rightarrow \quad \sin\theta = \frac{y}{r}$$

$$= \frac{1+\left(y/x\right)^2}{\sqrt{r^2 - y^2}}$$

$$= \frac{1+\frac{y^2}{x^2}}{\sqrt{r^2 - y^2}}$$

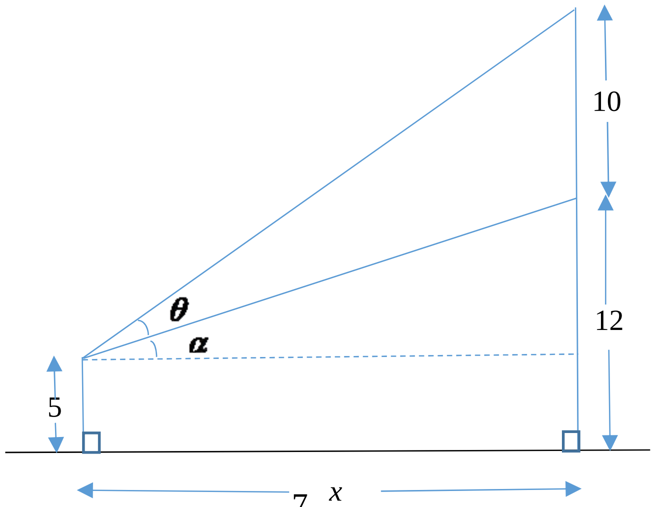
$$= \frac{1+\frac{y^2}{r^2 - y^2}}{\sqrt{r^2 - y^2}}$$

	$$=\frac{\left(r^2-y^2\right)+y^2}{\left(r^2-y^2\right)\sqrt{r^2-y^2}}$$ $$=\frac{r^2}{\left(r^2-y^2\right)^{3/2}}$$ Second part is similar to the above.
	Alternate method 3: $m=\tan\left(\sin^{-1}\frac{y}{r}\right)=\tan\theta$, where $\theta=\sin^{-1}\frac{y}{r}$ $$m=\frac{y}{x}=\frac{y}{\sqrt{r^2-y^2}}$$ $$\frac{\mathrm{d}m}{\mathrm{d}y}=\frac{\sqrt{r^2-y^2}-y\left(\frac{1}{2}\right)\left(r^2-y^2\right)^{-1/2}(-2y)}{\left(r^2-y^2\right)}$$ $$=\frac{\sqrt{r^2-y^2}+y^2\left(r^2-y^2\right)^{-1/2}}{\left(r^2-y^2\right)}$$ $$=\frac{\left(r^2-y^2\right)+y^2}{\left(r^2-y^2\right)^{1/2}\left(r^2-y^2\right)}$$ $$=\frac{r^2}{\left(r^2-y^2\right)^{3/2}}$$ Second part is similar to the above.
6(i)	$$\frac{r^2+r-1}{(r+2)!}=\frac{A}{r!}+\frac{B}{(r+1)!}+\frac{C}{(r+2)!}$$ $$\frac{r^2+r-1}{(r+2)!}=\frac{A(r+1)(r+2)+B(r+2)+C}{(r+2)!}$$ By comparing coefficients, $A=1, B=-2, C=1$ $$\therefore\frac{r^2+r-1}{(r+2)!}=\frac{1}{r!}-\frac{2}{(r+1)!}+\frac{1}{(r+2)!}$$

(ii)	$\begin{aligned} \sum_{r=1}^{n} \frac{r^2+r-1}{(r+2)!} &= \sum_{r=1}^{n}\left(\frac{1}{r!}-\frac{2}{(r+1)!}+\frac{1}{(r+2)!}\right) \\ &= \left[\frac{1}{1!}-\frac{2}{2!}+\cancel{\frac{1}{3!}}\right. \\ &\quad +\frac{1}{2!}-\cancel{\frac{2}{3!}}+\cancel{\frac{1}{4!}} \\ &\quad +\cancel{\frac{1}{3!}}-\cancel{\frac{2}{4!}}+\frac{1}{5!} \\ &\quad +\cancel{\frac{1}{4!}}-\frac{2}{5!}+\frac{1}{6!} \\ &\quad \vdots \\ &\quad +\cancel{\frac{1}{(n-2)!}}-\cancel{\frac{2}{(n-1)!}}+\cancel{\frac{1}{n!}} \\ &\quad +\cancel{\frac{1}{(n-1)!}}-\cancel{\frac{2}{n!}}+\frac{1}{(n+1)!} \\ &\quad \left.+\cancel{\frac{1}{n!}}-\frac{2}{(n+1)!}+\frac{1}{(n+2)!}\right] \\ &= \frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!} \end{aligned}$
(iii)	Since $r^2-1<r^2+r-1$ for $r>0$, so we have $\begin{aligned} \sum_{r=1}^{n}\frac{r^2-1}{(r+2)!} < \sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!} &= \frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!} \\ &= \frac{1}{2}-\left[\frac{n+1}{(n+2)!}\right] < \frac{1}{2} \end{aligned}$ $\left(\because \frac{n+1}{(n+2)!}>0 \text{ as } n\in\mathbb{Z}^+\right)$
(iv)	$\sum_{r=4}^{n}\frac{r^2-3r+1}{r!}$ Replace r with $(k+2)$ $=\sum_{k=2}^{n-2}\frac{(k+2)^2-3(k+2)+1}{(k+2)!}$

	$$=\sum_{k=2}^{n-2}\frac{k^2+k-1}{(k+2)!}$$ $$=\sum_{k=1}^{n-2}\frac{k^2+k-1}{(k+2)!}-\left(\frac{1}{6}\right)$$ $$=\frac{1}{2}-\frac{1}{(n-1)!}+\frac{1}{n!}-\frac{1}{6}$$ $$=\frac{1}{3}-\frac{1}{(n-1)!}+\frac{1}{n!}$$ Alternatively, consider $\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}$ and sub. $r=k-2$. So we have $$\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}=\sum_{k=3}^{n+2}\frac{k^2-3k+1}{k!}.$$ $$\Rightarrow\sum_{k=3}^{n+2}\frac{k^2-3k+1}{k!}=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$$ $$\Rightarrow\sum_{k=4}^{n+2}\frac{k^2-3k+1}{k!}=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}-\left(\frac{3^2-3(3)+1}{3!}\right)$$ $$=\frac{1}{3}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$$ $$\therefore\sum_{k=4}^{n}\frac{k^2-3k+1}{k!}=\frac{1}{3}-\frac{1}{(n-1)!}+\frac{1}{n!}$$

7



From diagram, $\tan\alpha = \dfrac{7}{x}$

and $\tan(\theta+\alpha) = \dfrac{17}{x}$

$$\tan\theta = \tan((\theta+\alpha)-\alpha)$$

$$= \frac{\tan(\theta+\alpha)-\tan\alpha}{1-\tan(\theta+\alpha)\tan\alpha}$$

$$= \frac{\frac{17}{x}-\frac{7}{x}}{1-\frac{17}{x}\left(\frac{7}{x}\right)}$$

$$= \frac{\frac{10}{x}}{1-\frac{119}{x^2}} = \frac{10x}{x^2+119} \quad \text{(shown)}$$

Differentiating wrt x,

$$\sec^2\theta\frac{d\theta}{dx} = \frac{(119+x^2)(10)-10x(2x)}{(119+x^2)^2}$$

$$= \frac{1190-10x^2}{(119+x^2)^2}$$

For maximum angle,

$$\frac{d\theta}{dx} = 0 \Rightarrow 10x^2 = 1190$$

Since $x>0$, $x=\sqrt{119}$

Since $\sec^2\theta > 0$,

Using first derivative test at vicinity of $x=\sqrt{119}$, it can be shown that the angle is maximum.

Therefore, $\tan\theta=\dfrac{10\sqrt{119}}{119+119}=\dfrac{5\sqrt{119}}{119}\Rightarrow\theta\approx25^{o}$

	$\left(\sqrt{119}\right)^{-}$	$\left(\sqrt{119}\right)$	$\left(\sqrt{119}\right)^{+}$
Sign of $\dfrac{d\theta}{dx}$	+	0	-
Slope			

8 *(i)* $w^4=-1+\sqrt{3}\,i$

$$w^4=2e^{i\frac{2\pi}{3}}$$

$$w=2^{\frac{1}{4}}e^{i\frac{1}{4}\left(\frac{2\pi}{3}+2k\pi\right)}\qquad k=-2,-1,0,1,$$

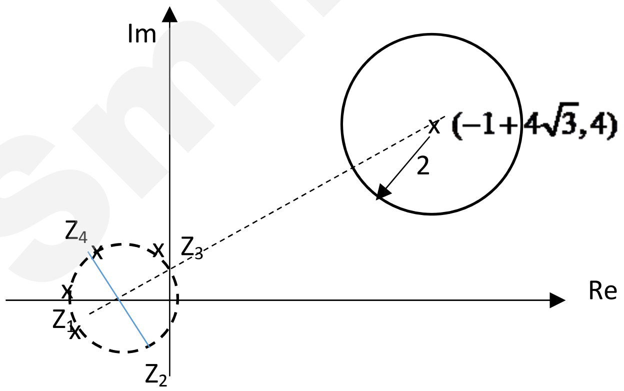
$$w=2^{\frac{1}{4}}e^{i\left(-\frac{5\pi}{6}\right)},\ 2^{\frac{1}{4}}e^{i\left(-\frac{\pi}{3}\right)},\ 2^{\frac{1}{4}}e^{i\left(\frac{\pi}{6}\right)},\ 2^{\frac{1}{4}}e^{i\left(\frac{2\pi}{3}\right)}$$

(ii) $(1+z)^4+1-i\sqrt{3}=0\ \Rightarrow\ (1+z)^4=-1+i\sqrt{3}$

$$z=w-1$$

$$z=2^{\frac{1}{4}}e^{i\left(-\frac{5\pi}{6}\right)}-1,\ 2^{\frac{1}{4}}e^{i\left(-\frac{\pi}{3}\right)}-1,\ 2^{\frac{1}{4}}e^{i\left(\frac{\pi}{6}\right)}-1,\ 2^{\frac{1}{4}}e^{i\left(\frac{2\pi}{3}\right)}-1$$

$$z_1=2^{\frac{1}{4}}e^{i\left(-\frac{5\pi}{6}\right)}-1,\ z_2=2^{\frac{1}{4}}e^{i\left(-\frac{\pi}{3}\right)}-1,\ z_3=2^{\frac{1}{4}}e^{i\left(\frac{\pi}{6}\right)}-1,\ z_4=2^{\frac{1}{4}}e^{i\left(\frac{2\pi}{3}\right)}-1$$

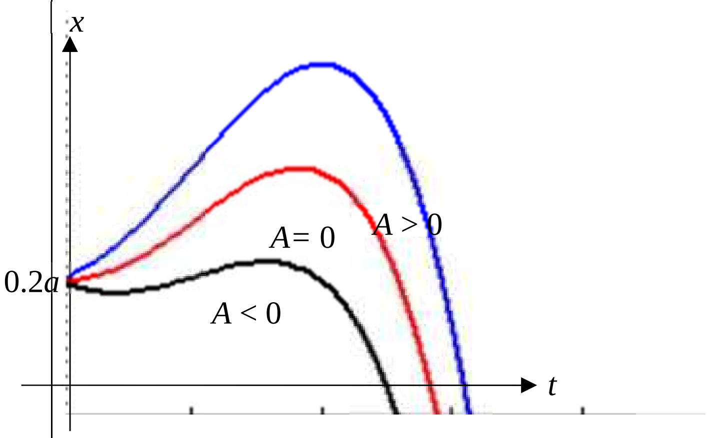
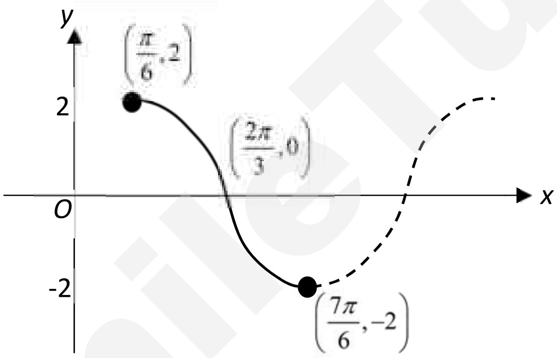


(iii)

Least possible $Z_iQ=Z_3Q=8-2-2^{1/4}=6-2^{1/4}$

Greatest possible $Z_iQ=Z_1Q=8+2+2^{1/4}=10-2^{1/4}$

9	(i) $\frac{dx}{dt}=x(a-x)$ $\int\frac{dx}{x(a-x)}=\int dt$ $\int\frac{\frac{1}{a}}{x}+\frac{\frac{1}{a}}{a-x}\,dx=\int dt$ $\frac{1}{a}\ln\|x\|-\frac{1}{a}\ln\|a-x\|=t+C$ $\frac{1}{a}\ln\left\|\frac{x}{a-x}\right\|=t+C$ $\ln\left\|\frac{x}{a-x}\right\|=at+aC$ $\frac{x}{a-x}=Ae^{at}$, where $A=\pm e^{aC}$ When t = 0, x = 0.2a $\frac{0.2a}{0.8a}=A$ $A=\frac{1}{4}$ When t = ln 2, x = 0.5a $\frac{0.5a}{0.5a}=\frac{1}{4}e^{a\ln 2}$ $4=e^{a\ln 2}$ $2^a=4$ $a=2$ Subst. values of A and a, $\frac{x}{2-x}=\frac{1}{4}e^{2t}$ $4x=2e^{2t}-xe^{2t}$ $x\left(4+e^{2t}\right)=2e^{2t}$ $x=\frac{2e^{2t}}{\left(4+e^{2t}\right)}=\frac{2}{4e^{-2t}+1}$ (shown) (ii) $\frac{d^2x}{dt^2}=10-9t^2$ $\frac{dx}{dt}=10t-3t^3+A$ $x=5t^2-\frac{3}{4}t^4+At+B$ When t = 0, x = 0.2a $B=0.2a$ $x=5t^2-\frac{3}{4}t^4+At+0.2a$

	
10 (a)(i)	$\sin x+\sqrt{3}\cos x=2\sin\left(x+\frac{\pi}{3}\right)$
(ii)	 For f to have an inverse, f must be one-to-one. Hence largest $k=\frac{7\pi}{6}$. Consider $y=2\sin\left(x+\frac{\pi}{3}\right)\Rightarrow x=\sin^{-1}\left(\frac{y}{2}\right)-\frac{\pi}{3}$ So $\mathrm{f}^{-1}:x\mapsto\sin^{-1}\left(\frac{x}{2}\right)-\frac{\pi}{3},\quad -2\le x\le 2.$ For $\mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)$, we must have $D_{\mathrm{f}}\cap D_{\mathrm{f}^{-1}}$. So the solution set is $x\in\left[\frac{\pi}{6},2\right]$.

(b)(i)	Consider $y=2-\frac{5x}{1+x^2}$ $\Rightarrow 2-y=\frac{5x}{1+x^2}$ $\Rightarrow (2-y)\left(1+x^2\right)=5x$ $\Rightarrow (2-y)x^2-5x+(2-y)=0$ $\mathrm{D}=(-5)^2-4(2-y)(2-y)\geq 0$ $25-4(2-y)^2\geq 0$ $\left(5-2(2-y)\right)\left(5+2(2-y)\right)\geq 0$ $(1+2y)(9-2y)\geq 0$ $\therefore -\frac{1}{2}\leq y\leq \frac{9}{2}$ So range of g $=\left[-\frac{1}{2},\frac{9}{2}\right]$
(ii)	$\mathrm{g}(x)=2-\frac{5x}{1+x^2}$ $\mathrm{g}\left(-\frac{x}{2}\right)=2-\frac{5\left(-\frac{x}{2}\right)}{1+\left(-\frac{x}{2}\right)^2}=2+\frac{10x}{4+x^2}$ $\mathrm{g}\left(-\frac{x}{2}\right)-2=\frac{10x}{4+x^2}$ <u>Scale the graph</u> of g <u>by factor 2</u> <u>parallel to the *x*-axis</u> followed by a <u>reflection</u> <u>in</u> the <u>*y*-axis</u> followed by a <u>translation</u> of <u>-2</u> <u>units</u> <u>in the direction of *y*-axis</u>. Or $\mathrm{g}(x)=2-\frac{5x}{1+x^2} \quad \rightarrow -\mathrm{g}\left(\frac{x}{2}\right)=-\left[2-\frac{5\left(\frac{x}{2}\right)}{1+\left(\frac{x}{2}\right)^2}\right]=-2+\frac{10x}{4+x^2}$ <u>Scale the graph</u> of g $\rightarrow -\mathrm{g}\left(\frac{x}{2}\right)+2=\frac{10x}{4+x^2}$

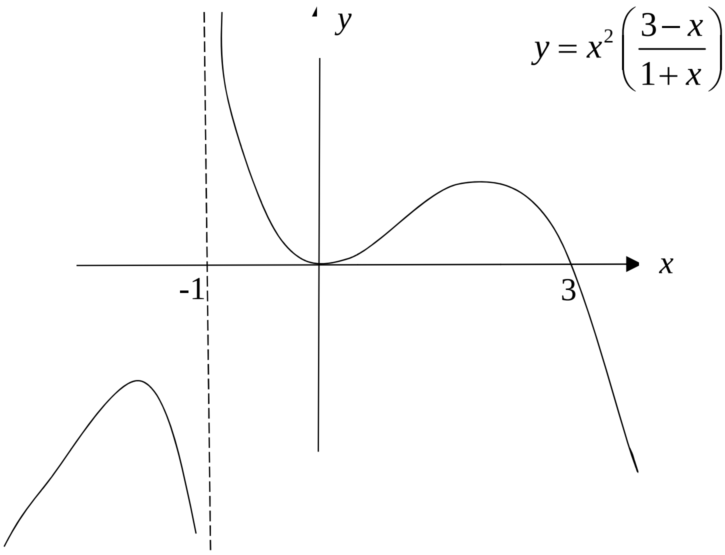
	by factor 2 parallel to the x-axis followed by a reflection in the x-axis followed by a translation of 2 units in the direction of y-axis.
11	(i) $l_1 : \mathbf{r} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\a\\2\end{pmatrix}, \lambda \in \mathbb{R}$ $l_2 : \mathbf{r} = \begin{pmatrix}3\\0\\5\end{pmatrix} + \mu\begin{pmatrix}1\\2\\3\end{pmatrix}, \mu \in \mathbb{R}$ Equating the equations of the 2 lines, $1+\lambda = 3+\mu$ $2+a\lambda = 2\mu$ $1+2\lambda = 5+3\mu$ Solving, $\mu = 0, \lambda = 2$ $\therefore a = -1$ (ii) Coordinates of B is (3, 0, 5) Shortest distance from B to p_1 $= \dfrac{\left\lvert \overrightarrow{AB} \bullet \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rvert}{\sqrt{14}}$ $= \dfrac{\left\lvert \left[\begin{pmatrix}3\\0\\5\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix}\right] \bullet \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rvert}{\sqrt{14}} = \dfrac{\left\lvert \begin{pmatrix}2\\-2\\4\end{pmatrix} \bullet \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rvert}{\sqrt{14}} = \dfrac{10}{\sqrt{14}}$ **Alternative solution 1:** Equation of plane $p_1 : \boldsymbol{r} \bullet \dfrac{\begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \dfrac{\begin{pmatrix}1\\2\\2\end{pmatrix} \bullet \begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \dfrac{8}{\sqrt{14}}$ Equation of plane parallel to p_1 and containing point B is $\boldsymbol{r} \bullet \dfrac{\begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \dfrac{\begin{pmatrix}3\\0\\5\end{pmatrix} \bullet \begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \dfrac{18}{\sqrt{14}}$ Shortest distance from B to p_1 is $\dfrac{18}{\sqrt{14}} - \dfrac{8}{\sqrt{14}} = \dfrac{10}{\sqrt{14}}$ **Alternative solution 2:** Let F be the foot of perpendicular from point B to plane p_1 Line l_2 meets plane p_1 at point F

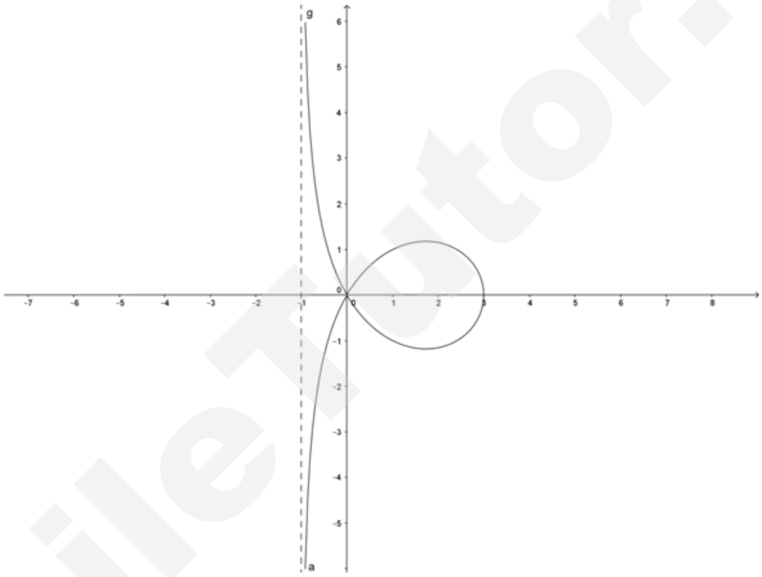
	$\overrightarrow{OF} = \begin{pmatrix}3\\0\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$ for some $\lambda \in \mathbb{R}$ $\left[\begin{pmatrix}3\\0\\5\end{pmatrix} + \mu\begin{pmatrix}1\\2\\3\end{pmatrix}\right]\bullet\begin{pmatrix}1\\2\\3\end{pmatrix} = 8$ $3+15+\mu(1+4+9) = 8$ $\mu = -\frac{5}{7}$ Shortest distance from B to p_1 is $\left\lvert\overrightarrow{BF}\right\rvert = \left\lvert -\frac{5}{7}\begin{pmatrix}1\\2\\3\end{pmatrix}\right\rvert = \frac{5}{7}\sqrt{1+4+9} = \frac{5}{7}\sqrt{14}$ $\left\lvert\overrightarrow{AB}\right\rvert = \sqrt{2^2+(-2)^2+4^2} = \sqrt{24}$ $\sin\theta = \frac{10}{\sqrt{14}}\frac{1}{\sqrt{24}}$ $\theta \approx 33.1^0$ (iii)A normal vector to p_2 is $\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}7\\1\\-3\end{pmatrix}$ Equation of p_2 is $r\bullet\begin{pmatrix}7\\1\\-3\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}7\\1\\-3\end{pmatrix} = 6$ Cartesian equation of p_2 is $7x+y-3z=6$ (iv) Equation of x-y plane is $z = 0$ Let α be the acute angle between p_2 and x-y plane. $\cos\alpha = \frac{\left\lvert\begin{pmatrix}7\\1\\-3\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\sqrt{59}} = \frac{3}{\sqrt{59}}$ $\alpha = 67.0^\circ$

Solutions to P2 Prelim 2016:

1.	$S_n = an^3 + bn^2 + cn + d$ $10 = a + b + c + d$ $16 = 8a + 4b + 2c + d$ $21 = 27a + 9b + 3c + d$ $28 = 64a + 16b + 4c + d$ $a = \frac{1}{2},\ b = -\frac{7}{2},\ c = 13,\ d = 0$ $S_n = \frac{1}{2}n^3 - \frac{7}{2}n^2 + 13n$ $U_n = S_n - S_{n-1}$ $= \frac{1}{2}n^3 - \frac{7}{2}n^2 + 13n - \left(\frac{1}{2}(n-1)^3 - \frac{7}{2}(n-1)^2 + 13(n-1)\right)$ $= \frac{3}{2}n^2 - \frac{17}{2}n + 17$ $S_n < 3U_n$ $\frac{1}{2}n^3 - \frac{7}{2}n^2 + 13n < 3\left(\frac{3}{2}n^2 - \frac{17}{2}n + 17\right)$ $\frac{1}{2}n^3 - 8n^2 + \frac{77}{2}n - 51 < 0$ From GC n : $\frac{1}{2}n^3 - 8n^2 + \frac{77}{2}n - 51$ 1 : -20 2 : -2 3 : 6 4 : 7 5 : 4 6 : 0 7 : -2 8 : 1 From GC, $\left\{n : n \in \mathbb{Z}^+, n = 1, 2, 7\right\}$

2





$$y^2 = x^2\left(\frac{3-x}{1+x}\right)$$

Area enclosed by loop $= 2\int_0^3 x\sqrt{\dfrac{3-x}{1+x}}\,\mathrm{d}x$

$$= 2\int_0^3 x\sqrt{\frac{(3-x)(3-x)}{(1+x)(3-x)}}\,\mathrm{d}x$$

$$= 2\int_0^3 \frac{x(3-x)}{\sqrt{3+2x-x^2}}\,\mathrm{d}x$$

$$= 2\int_0^3 \frac{3x-x^2}{\sqrt{(4-(x-1)^2)}}\,\mathrm{d}x$$

Using $x-1 = 2\sin\theta$, we have $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$

Also, when $x = 0$, $\sin\theta = -\dfrac{1}{2} \Rightarrow \theta = -\dfrac{\pi}{6}$

And when $x = 3, \sin\theta = 1 \Rightarrow \theta = \dfrac{\pi}{2}$

Therefore area $= 2\int_0^3 \dfrac{3x-x^2}{\sqrt{(4-(x-1)^2)}}\,\mathrm{d}x$

	$= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3(2\sin\theta+1)-(2\sin\theta+1)^2}{\sqrt{(4-(2\sin\theta)^2)}}(2\cos\theta)\mathrm{d}\theta$ $= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \left(2\sin\theta+2-4\sin^2\theta\right)\mathrm{d}\theta$ $=4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\sin\theta+\left(1-2\sin^2\theta\right)\right)\mathrm{d}\theta$ $= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\sin\theta+\cos 2\theta\right)\mathrm{d}\theta$ $= 4\left[-\cos\theta+\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$ $= 4\left[0-\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{4}\right)\right]=3\sqrt{3}$
3(a)	$z^3-2\left(2-\mathrm{i}\right)z^2+\left(8-3\mathrm{i}\right)z-5+\mathrm{i}=0$ Let $z=x$ be the real root. $x^3-2\left(2-\mathrm{i}\right)x^2+\left(8-3\mathrm{i}\right)x-5+\mathrm{i}=0$ $x^3-4x^2+2\mathrm{i}x^2+8x-3\mathrm{i}x-5+\mathrm{i}=0$ $\left(x^3-4x^2+8x-5\right)+\left(2x^2-3x+1\right)\mathrm{i}=0$ Since $z=x$ is a root, $x^3-4x^2+8x-5=0$ and $2x^2-3x+1=0$ From GC: $x=1$ Therefore, the real root is $z=1$ $z^3-2\left(2-\mathrm{i}\right)z^2+\left(8-3\mathrm{i}\right)z-5+\mathrm{i}=0$ $\left(z-1\right)\left(z^2+Az+\left(5-\mathrm{i}\right)\right)=0$ $\left(z-1\right)\left(z^2+\left(-3+2\mathrm{i}\right)z+\left(5-\mathrm{i}\right)\right)=0$ $z=1$ or $z^2+\left(-3+2\mathrm{i}\right)z+\left(5-\mathrm{i}\right)=0$ $z=\frac{-\left(-3+2\mathrm{i}\right)\pm\sqrt{\left(-3+2\mathrm{i}\right)^2-4\left(5-\mathrm{i}\right)}}{2}$ $=\frac{-\left(-3+2\mathrm{i}\right)\pm\left(1-4\mathrm{i}\right)}{2}$ $=2-3\mathrm{i}$ or $1+\mathrm{i}$ Roots: 1, $2-3\mathrm{i}$, $1+\mathrm{i}$

3(b)

$$
\begin{aligned}
1-u^2 &= 1-\left(\cos\theta+\mathrm{i}\sin\theta\right)^2 \\
&= 1-\cos^2\theta+\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta \\
&= 2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta \\
&= 2\sin\theta\left(\sin\theta-2\mathrm{i}\cos\theta\right) \\
&= -2\mathrm{i}\sin\theta\left(\cos\theta+\mathrm{i}\sin\theta\right) \\
&= -2\mathrm{i}u\sin\theta
\end{aligned}
$$

Alternative

$$u=\cos\theta+\mathrm{i}\sin\theta=\mathrm{e}^{\theta\mathrm{i}}$$

$$
\begin{aligned}
1-u^2 &= 1-\mathrm{e}^{2\theta\mathrm{i}} \\
&= \mathrm{e}^{\theta\mathrm{i}}\left(\mathrm{e}^{-\theta\mathrm{i}}-\mathrm{e}^{\theta\mathrm{i}}\right) \\
&= u\left(\cos\theta-\mathrm{i}\sin\theta-\mathrm{i}\sin\theta-\cos\theta\right) \\
&= u\left(-2\mathrm{i}\sin\theta\right) \\
&= -2\mathrm{i}u\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
\left|1-u^2\right| &= \left|-2\mathrm{i}u\sin\theta\right| = \left|-2\sin\theta\right| \text{ i } \left|u\right| \\
&= 2\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
\arg\left(1-u^2\right) &= \arg\left(-2\mathrm{i}u\sin\theta\right) \\
&= \arg\left(-2\mathrm{i}\sin\theta\right)+\arg\left(u\right) \\
&= -\frac{\pi}{2}+\theta
\end{aligned}
$$

$\left(1-u^2\right)^{10}$ is real and negative: $\arg\left(1-u^2\right)^{10}=10\arg\left(1-u^2\right)=\left(2k+1\right)\pi, k\in\mathbb{Z}$

$$
\begin{aligned}
10\left(-\frac{\pi}{2}+\theta\right) &= \left(2k+1\right)\pi \\
-5\pi+10\theta &= \left(2k+1\right)\pi \\
\theta &= \frac{\left(2k+6\right)\pi}{10}
\end{aligned}
$$

$0<\theta<\dfrac{\pi}{2}$: $\theta=\dfrac{1}{5}\pi, \dfrac{2}{5}\pi$

Alternative

$$
\begin{aligned}
\left(1-u^2\right)^{10} &= \left(2\sin\theta\mathrm{e}^{-\frac{\pi}{2}+\theta}\right)^{10} \\
&= \left(2^{10}\sin^{10}\theta\right)\left(\cos\left(-5\pi+10\theta\right)+\mathrm{i}\sin\left(-5\pi+10\theta\right)\right)
\end{aligned}
$$

Since $\left(1-u^2\right)^{10}$ is real and negative, and $2^{10}\sin^{10}\theta>0$,

$\sin\left(-5\pi+10\theta\right)=0$ and $\cos\left(-5\pi+10\theta\right)<0$

$$
\begin{aligned}
-5\pi+10\theta &= k\pi \quad, k\in\mathbb{Z} \\
\theta &= \frac{\left(k+5\right)\pi}{10}
\end{aligned}
$$

	$0<\theta<\frac{\pi}{2}$: $\theta=\frac{1}{10}\pi,\frac{1}{5}\pi,\frac{3}{10}\pi,\frac{2}{5}\pi$ Only when $\theta=\frac{1}{5}\pi,\frac{2}{5}\pi$ will $\cos(-5\pi+10\theta)<0$. Therefore, $\theta=\frac{1}{5}\pi,\frac{2}{5}\pi$.
4(a)(i)	Since $S_{16}=2\pi r$, thus $\frac{16}{2}(2r+15d)=2\pi r$ $\Rightarrow 2r+15d=\frac{\pi r}{4}$ $\Rightarrow 15d=\left(\frac{\pi-8}{4}\right)r$ $\Rightarrow d=\left(\frac{\pi-8}{60}\right)r$
(a)(ii)	Since $L_n=r\theta_n$ and $A_n=\frac{1}{2}r^2\theta_n$, thus $A_n=\frac{1}{2}rL_n$. Hence $A_{n+1}-A_n=\frac{1}{2}rL_{n+1}-\frac{1}{2}rL_n$ $=\frac{1}{2}r\left(L_{n+1}-L_n\right)$ $=\frac{1}{2}rd=\text{constant}$ for all $n=2,\ldots,15$ Thus A_n is an arithmetic sequence.
(b)(i)	Since $S_N=\pi a^2$, we have $\frac{a(r^N-1)}{r-1}=\pi a^2$ $\Rightarrow r^N-1=\pi ar-\pi a$ $\Rightarrow r^N-\pi ar+(\pi a-1)=0$
(b)(ii)	Since $0<r<1$, we have $r^N\to 0$ as $N\to\infty$. Thus $-\pi ar+(\pi a-1)=0$ $\Rightarrow r=\frac{\pi a-1}{\pi a}$ Alternative: Since $S_\infty=\pi a^2$, we have $\frac{a}{1-r}=\pi a^2$ $\Rightarrow 1-r=\frac{1}{\pi a}$ $\Rightarrow r=\frac{\pi a-1}{\pi a}$

5	No sampling frame or entire list of people consuming this brand is not available Station an interviewer at the exits of a local supermarket store during peak hours and he is free to choose 25 male and 25 female customers who buy the products. P(a particular consumer is the first to be selected) = $\frac{1}{2000}$ P(a particular consumer is the third to be selected) $= \frac{1999}{2000}\frac{1998}{1999}\frac{1}{1998} = \frac{1}{2000}$ (shown)
6	(i) $\mathrm{P}(\bar{X} < 35.0) = 0.97725$ $\mathrm{P}(Z < \frac{35-\mu}{\sigma/\sqrt{n}}) = 0.97725$ $\frac{35-\mu}{\sigma/\sqrt{n}} = 2 -----[1]$ $\mathrm{P}(\bar{X} < 20.0) = 0.15866$ $\mathrm{P}(Z < \frac{20-\mu}{\sigma/\sqrt{n}}) = 0.15866$ $\frac{20-\mu}{\sigma/\sqrt{n}} = -1 ----[2]$ Eqn [1] –[2]: $\frac{3\sigma}{\sqrt{n}} = 15$ $\sigma = 5\sqrt{n}$ (ii) $\mu = 25$, $\bar{X} \sim N(25, 5^2)$ since $\frac{\sigma}{\sqrt{n}} = 5$ $\mathrm{P}(\bar{X} > 32) = 0.0808$ let M be the mass of a randomly chosen discharge of 15 ice cubes. $M \sim N(375, 750)$ (iii) $P(M > a) = 0.1$ $P(M \le a) = 0.9$ $a = 410.1$ (iv) $M_1 - M_2 \sim N(0, 1500)$ $P(M_1 > M_2) = P(M_1 - M_2 > 0) = 0.5$
7	5M, [3W, A, L]: No. of selections = $\left({}^8\mathrm{C}_5\right)\left({}^7\mathrm{C}_3\right)$ = 1960 5M, 5W (exclude A and L): No. of selections = $\left({}^8\mathrm{C}_5\right)\left({}^7\mathrm{C}_5\right)$ = 1176 Total number of selections $= \left({}^8\mathrm{C}_5\right)\left({}^7\mathrm{C}_3\right) + \left({}^8\mathrm{C}_5\right)\left({}^7\mathrm{C}_5\right)$ = 1960 + 1176= 3136

<table>
<tr><td>(i)</td><td>

M_1 A M_2 M_3 L M_4 M_5

W W W

Arrange 5M: No. of ways = 5! = 120

Arrange A and L: No. of ways = 2! = 2

Arrange 3W: No. of ways = 1(3!) = 6

Total no. of arrangements $= (5!)\left[(2!)\times(3!)\right]$

$= 1440$

</td></tr>
<tr><td>(ii)</td><td>

No of arrangements $(8-1)!(2)(10) = 100800$

</td></tr>
<tr><td>8(i)</td><td>

P(Strollers will win in 2018)

$= (0.7)(0.7)(0.3) + (0.7)(0.3)(0.5)$
$\quad + (0.3)(0.5)(0.3) + (0.3)(0.5)(0.5)$
$= 0.372$

</td></tr>
<tr><td>(ii)</td><td>

Let event A denotes “Strollers first win for at least three years” and event B denotes “Strollers win in 2018”

$$\mathrm{P}(A\,|\,B) = \frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$$

$$= \frac{(0.7)(0.7)(0.3)}{0.372}$$
$$= 0.39516$$
$$= 0.395 \quad (3\text{ s.f.})$$

</td></tr>
<tr><td>(iii)</td><td>

$(0.5)(0.7)^{n-1} < 0.05$

$(0.7)^{n-1} < 0.1$

$n-1 > \dfrac{\ln(0.1)}{\ln(0.7)} = 6.4557$

$n > 7.4557$

Hence, the smallest value is n = 8.

Alternative method:

$(0.5)(0.7)^{n-1} < 0.05$

Using GC:

<table>
<tr><td>n</td><td>probability</td></tr>
<tr><td>7</td><td>0.05582</td></tr>
<tr><td>8</td><td>0.04418 < 0.05</td></tr>
<tr><td>9</td><td>0.02882</td></tr>
</table>

Hence, the smallest value is n = 8.

Or

</td></tr>
</table>

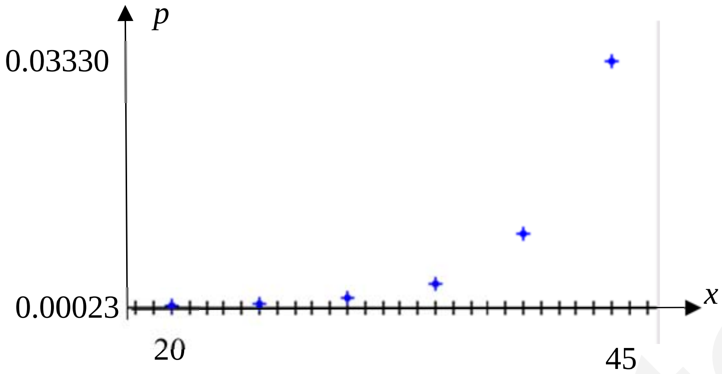
$(0.7)^{n-1} < 0.1$

n	probability
7	0.11765
8	0.08235 < 0.01
9	0.05465

Hence, the smallest value is n = 8.

9

(i)



(ii)(a) r = 0.8130 (b) r = 0.9960 (c) $r = 0.8667$

(iii)Since |r| is closest to 1 for model (b), equation would be

$\ln p = 0.19409x - 12.322$

(iv)When x = 32,

$p = 0.0022181$

Expected number = 5000(0.0022181) = 11.1

10

(i) Unbiased estimate of pop mean =

$$\bar{x} = \frac{\sum x}{n} = \frac{11872}{250} = 47.488$$

Unbiased estimate of population variance, $s^2 = \frac{1}{249}(646193 - \frac{11872^2}{250}) = 330.986$

(i) To test

$H_o : \mu = 49.5$

$H_1 : \mu < 49.5$

At $\alpha\%$ significance level

Since n= 250 is large, by Central limit Theorem, $\bar{X} \sim N(49.5, \frac{330.986}{250})$ approx.

Since H_0 is rejected,

p-value $= P(\bar{X} < 47.488) = 0.040179 < \frac{\alpha}{100}$

$\alpha > 4.02$

(ii) Since n =250 is large, by Central Limit Theorem,

$\bar{X} \sim N(45.292, \frac{18.761^2}{250})$ approximately

$P(\bar{X} \geq 47.488) \approx 0.0321$

11(i)	The mean number of customers who arrived at the village post office during a random chosen 30 minutes period must be a constant.
(ii)	Let X be the random variable denoting the number of customers who arrive at the village post office between 11.00 a.m. and 11.30 a.m. i.e. $X \sim \mathrm{P}_o(9)$ $\mathrm{P}(X \le 4) = 0.054964 = 0.0550 \quad (3\text{ s.f.})$
(iii)	Let Y be the random variable denoting the number of customers who arrive at the village post office in 5 minutes i.e. $Y \sim \mathrm{P}_o(1.5)$ $\mathrm{P}(Y = 0) = 0.22313$ Let W be the random variable denoting the number of periods (of 5 minutes each) out of 6 where $Y = 0$ i.e. $W \sim \mathrm{B}(6, 0.22313)$ $\mathrm{P}(W \le 1) = 0.59867 = 0.599 \quad (3\text{ s.f.})$
(iv)	Let U be the random variable denoting the number of customers who arrive at the village post office in 3.5 hours in the afternoon i.e. $U \sim \mathrm{Po}(21)$ $P(\mu - \sigma < U < \mu + \sigma) = P(16.4 < U < 25.6)$ $= P(U \le 25) - P(U \le 16)$ $= 0.675$
(v)	Let T be the random variable denoting the number of customers who arrive at the village post office in 3.5 hours in the morning $T + U \sim \mathrm{Po}(84)$ $P(T > 38 \mid T + U = 40) = \dfrac{\mathrm{P}(T > 38 \text{ and } T + U = 40)}{\mathrm{P}(T + U = 40)}$ $= \dfrac{\mathrm{P}(T = 39)\mathrm{P}(U = 1) + \mathrm{P}(T = 40)\mathrm{P}(U = 0)}{\mathrm{P}(T + U = 40)}$ $= 1.44 \times 10^{-4}$
(vi)	$T + U \sim \mathrm{Po}(84)$ Since $\lambda = 84 > 10$, hence use the normal distribution for approximation i.e. $T + U \sim \mathrm{N}\left(126, \left(\sqrt{126}\right)^2\right)$ approximately $\mathrm{P}(T + U > 100) \xrightarrow{c.c.} \mathrm{P}(T + U > 100.5) = 0.0359 \quad (3\text{ s.f.})$

1 The police wish to crack a 3-digit passcode. The sum of the digits is 14. When the digits in the number are reversed, the new number becomes 495 more than the original number. The digit in the tens position is 3 more than the digit in the hundreds position. What is the passcode? [4]

2

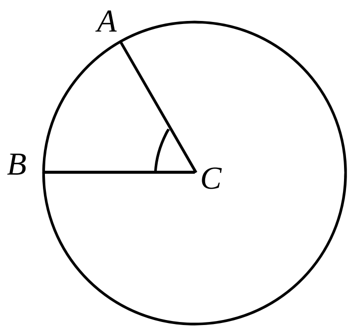


Fig. 1

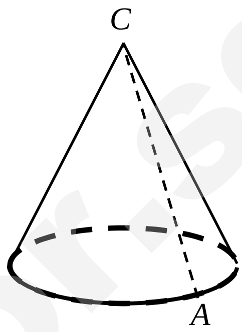


Fig. 2

Fig. 1 shows a circular card with centre *C*. A sector *CAB* is removed from the card, and the remaining card is folded such that *AC* and *BC* meet without overlapping to form a cone, as shown in Fig. 2 (*A* will meet *B*). Use differentiation to find the angle *ACB* exactly such that the volume of the cone is as large as possible. [6]

[It is given that a cone with radius *r* and height *h* has volume $\frac{1}{3}\pi r^2 h$ and curve surface area πrl where *l* is the slant height.]

3 **(i)** Show that $\frac{4}{4r^2+12r+5}$ can be expressed as $\frac{A}{2r+1}+\frac{B}{2r+5}$, where *A* and *B* are constants to be determined. [2]

(ii) Hence, find an expression for $\sum_{r=1}^{n-1}\frac{2}{4r^2+12r+5}$ in terms of *n*. [3]

(iii) Hence, find the smallest value of *n* for which $\sum_{r=1}^{n-1}\frac{2}{4r^2+12r+5}$ is at least 99% of its sum to infinity. [3]

4 A curve C has parametric equations

$$x = 2a\cos^3\theta, \quad y = a\sin^3\theta,$$

where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and a is a positive constant.

~~A point P lies on C.~~

~~**(i)** Find, in terms of a, the exact coordinates of P, whose tangent is parallel to the line $2y = -x$.~~ ~~[4]~~

~~**(ii)** The tangent at P meets the y-axis at a point Q. Find the cartesian equation of the locus of the mid-point of PQ as θ varies.~~ ~~[4]~~

(i) A point P lies on C. Find, in terms of a, the exact coordinates of P, whose tangent is parallel to the line $2y = -x$. [4]

(ii) The tangent to C at the point $Q\left(2a\cos^3 t,\ a\sin^3 t\right)$, where $0 < t < \frac{\pi}{2}$, meets the x- and y-axes at R and S respectively. Find a cartesian equation of the locus of the mid-point of RS as t varies. [4]

5 The sum, S_n, of the first n terms of a sequence is given by

$$S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}.$$

(i) Find the values of S_1, S_2, S_3 and S_4. [2]

(ii) By expressing S_n in the form $[1 - \mathrm{f}(n)]$ for $n = 1, 2, 3, 4$, find a conjecture for S_n in terms of n. [2]

(iii) Hence prove by mathematical induction the result of S_n for all positive integers n. [4]

[It is given that $n! = n(n-1)(n-2)\ldots(3)(2)(1)$]

6 Referred to the origin O, points A and B have position vectors **a** and **b** respectively. Point C lies on OA produced such that $OA : OC = 2 : 5$. Point D is on AB, between A and B such that $AD : DB = 4 : 1$.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of **a** and **b**. [2]

(ii) Find a vector equation of line CD. [2]

(iii) Point E lies on CD produced, and it is also on OB, between O and B. Find $\overrightarrow{OE}$ and the ratio $OE: EB$. [5]

7 Newton's law of cooling states that the rate of cooling in t minutes is proportional to the difference between the body temperature $T\,^{\circ}C$ and its immediate surrounding temperature $T_o\,^{\circ}C$. Show that $T = T_o + Ae^{-kt}$, where A and k are positive constants.

[3]

Nurul is the chef of a dessert shop and she leaves her work place at 9pm daily. Before she leaves, she is required to cook a big pot of dessert and leave it to cool, before placing it in the refrigerator for the next business day. She takes 30 minutes to cook the pot of dessert to $100\,^{\circ}C$, and then leaves it to cool. After 15 minutes, the pot of dessert cools to $70\,^{\circ}C$.

The room temperature of the kitchen is $30\,^{\circ}C$, and the refrigerator can only accommodate items with temperature of at most $35\,^{\circ}C$. By what time, correct to the nearest minute, must Nurul start to cook the pot of dessert so that she will be able to leave her work place on time? [5]

8 A lion eyes its prey which is k m away. The lion starts his chase of its prey with a leap of 2.5 m. Each subsequent leap of the lion is shorter than his preceding leap by 0.05 m. Its prey notices the lion's chase and runs away with a first leap of 1.5 m, with each subsequent leap 5% less than the previous leap. You may assume that the lion and the prey start running at the same moment and they complete the same number of leaps after the first leap.

(i) Find the total distance covered by the lion after n leaps. [2]

(ii) Find the total distance covered by the prey after n leaps. Deduce that the distance covered by the prey can never be greater than 30 m. [3]

(iii) Given $k = 25$, find the least number of leaps the lion needs to take to catch its prey. [3]

(iv) Assuming that the lion can cover a maximum of 30 leaps, find the least integer k, so that the prey will survive the hunt. [3]

9 **(a)** **(i)** If $t = \tan\frac{\theta}{2}$, show that $\sin\theta = \frac{2t}{1+t^2}$. [2]

(ii) Use the substitution $t = \tan\frac{\theta}{2}$ to find the exact value of

$$\int_0^{\frac{\pi}{2}} \left(\frac{\tan\frac{\theta}{2}+1}{\sin\theta+1} \right) \mathrm{d}\theta .$$ [5]

(b) Find $\int \mathrm{e}^{2v}\cos 3v \,\mathrm{d}v$. [4]

10 The point A has coordinates $(18, 2, 0)$. The plane p_1 has the equation $x+3y+z=a$, where a is a constant. It is given that p_1 contains the line l_1 with equation $\frac{x-1}{2} = y = \frac{z-1}{-5}$.

(i) Show that $a = 2$. [2]

(ii) Find the coordinates of the foot of perpendicular from the point A to p_1.

[3]

(iii) B is given to be a general point on l_1. Find an expression for the distance between the point A and B. Hence find the position vector of B that is nearest to A. [4]

The planes p_2 and p_3 have the equations $x+z=1$ and $2x+by+z=4$ respectively, where b is a constant.

(iv) Given that p_2 and p_3 intersect at l_2, show that l_2 is parallel to the vector $\begin{pmatrix} -b \\ 1 \\ b \end{pmatrix}$. By finding a point that lies on both planes, find a vector equation of l_2.

[3]

11 **(a)** The complex number w is such that $w = re^{i\theta}$, where $r > 0$ and $0 < \theta \le \frac{\pi}{2}$. The complex conjugate of w is denoted by w^*. Given that $\frac{w^2}{w^*} = -3$, find the exact values of r and θ. Hence find the three smallest positive integer n for which w^n is a real number. [5]

(b) The complex number z is such that $z^5 - 1 - i = 0$.

(i) Find the modulus and argument of each of the possible values of z. [5]

(ii) Two of these values are z_1 and z_2, where $\frac{\pi}{2} < \arg z_1 < \pi$ and $-\pi < \arg z_2 < -\frac{\pi}{2}$. Find the exact value of $\arg(z_1 - z_2)$ in terms of π and illustrate the locus $\arg(z - z_1) = \arg(z_1 - z_2)$ on an Argand diagram. [5]

2016 JC 2 Preliminary Examination Paper 1 Solution

1

Let the passcode be *xyz*.

$x+y+z=14$........(1)

$100z+10y+x=100x+10y+z+495$

$99z-99x=495$......(2)

$y-x=3$.................(3)

Using the GC, $x=2, y=5, z=7$

$\therefore$ the passcode is 257

2

Let the radius and height of the cone be *r* and *h* respectively.
Let the radius of the circular card be *x* and angle *ACB* be θ.
By Pythagoras Theorem,

$x^2=r^2+h^2 \Rightarrow r^2=x^2-h^2$

$V=\frac{1}{3}\pi r^2 h \quad =\frac{1}{3}\pi\left(x^2-h^2\right)h \quad =\frac{1}{3}\pi\left(x^2 h-h^3\right)$

$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}\pi\left(x^2-3h^2\right)=0$

$h^2=\frac{x^2}{3} \Rightarrow r^2=\frac{2}{3}x^2$

Consider the circumference of the circle without sector:

$$2\pi r=\frac{2\pi-\theta}{2\pi}(2\pi x)$$

$$2\pi\sqrt{\frac{2}{3}}x=(2\pi-\theta)(x)$$

$$\theta=2\left(1-\sqrt{\frac{2}{3}}\right)\pi$$

Alternatively, consider the curve surface area of the cone,

$\pi x^2\left(\frac{2\pi-\theta}{2\pi}\right)=\pi r x$

$\pi x^2-\pi x^2\left(\frac{\theta}{2\pi}\right)=\pi\sqrt{\frac{2}{3}x^2}\,x$

$1-\left(\frac{\theta}{2\pi}\right)=\sqrt{\frac{2}{3}}$

$\theta=2\left(1-\sqrt{\frac{2}{3}}\right)\pi$

$\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\frac{1}{3}\pi\left(-6h\right)=-2\pi h<0$ (Max)

3

(i) $$\frac{4}{4r^2+12r+5}=\frac{4}{(2r+1)(2r+5)}=\frac{A}{2r+1}+\frac{B}{2r+5}$$

$$4=A(2r+5)+B(2r+1)$$

when $r=-\frac{5}{2}$: $\quad 4=B\left[2\left(-\frac{5}{2}\right)+1\right] \quad \Rightarrow B=-1$

when $r=-\frac{1}{2}$: $\quad 4=A\left[2\left(-\frac{1}{2}\right)+5\right] \quad \Rightarrow A=1$

$$\therefore \frac{4}{4r^2+12r+5}=\frac{1}{2r+1}-\frac{1}{2r+5}$$

(ii) $$\sum_{r=1}^{n-1}\frac{2}{4r^2+12r+5}=\frac{1}{2}\sum_{r=1}^{n-1}\frac{4}{4r^2+12r+5}$$

$$\begin{aligned}
=\frac{1}{2}\Bigg\{ & \frac{1}{3}-\frac{1}{7} \\
+\ & \frac{1}{5}-\frac{1}{9} \\
+\ & \frac{1}{7}-\frac{1}{11} \\
+\ & \frac{1}{9}-\frac{1}{13} \\
+\ & \ldots\ldots\ldots \\
+\ & \frac{1}{2n-5}-\frac{1}{2n-1} \\
+\ & \frac{1}{2n-3}-\frac{1}{2n+1} \\
+\ & \frac{1}{2n-1}-\frac{1}{2n+3}\Bigg\}
\end{aligned}$$

$$=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2n+1}-\frac{1}{2n+3}\right)=\frac{1}{2}\left(\frac{8}{15}-\frac{4n+4}{4n^2+8n+3}\right)=\frac{4}{15}-\frac{2(n+1)}{4n^2+8n+3}$$

(iii) $S_{n-1}\geq 0.99S_\infty$

$$\frac{2n+2}{4n^2+8n+3}\leq\left(\frac{1}{100}\right)\left(\frac{4}{15}\right)$$

$$\frac{1500(2n+2)-4(4n^2+8n+3)}{(1500)(4n^2+8n+3)}\leq 0$$

$$\frac{-4n^2+742n+747}{(1500)(4n^2+8n+3)}\leq 0$$

$$\frac{(-n+186.501)(n+1.001)}{1500(2n+1)(2n+3)}\leq 0$$

$-n+186.501\leq 0$ since $(2n+1)>0,(2n+3)>0,(n+1.001)>0$

$n\geq 186.501$

Alternatively

$$\frac{4}{15}-\frac{2n+2}{4n^2+8n+3}\geq 0.99\left(\frac{4}{15}\right)$$

$$\frac{2n+2}{4n^2+8n+3}\leq\left(\frac{1}{100}\right)\left(\frac{4}{15}\right)$$

$$4n^2+8n+3\geq 750n+750 \quad \text{(Since } n \text{ is positive integer)}$$

$$4n^2-742n-747\geq 0$$

$$n\geq 186.5$$

minimum n = 187 (Alternative solution)

4 (i)

$x=2a\cos^3\theta$ $\qquad$ $y=a\sin^3\theta$

$\frac{dx}{dt}=2a(2\cos^2\theta)(-\sin\theta)$ $\qquad$ $\frac{dy}{dt}=a(3\sin^2\theta)(\cos\theta)$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}$$

$$=\frac{3a\cos\theta\sin^2\theta}{-6a\cos^2\theta\sin\theta}$$

$$=-\frac{1}{2}\tan\theta$$

$$\frac{dy}{dx}=-\frac{1}{2}$$

$$\theta=\frac{\pi}{4}$$

Point P: $\left(2a\cos^3\left(\frac{\pi}{4}\right),a\sin^3\left(\frac{\pi}{4}\right)\right)=\left(\frac{a\sqrt{2}}{2},\frac{a\sqrt{2}}{4}\right)$

(ii) The equation of tangent at Q is

$$y-a\sin^3 t=-\frac{1}{2}\tan t(x-2a\cos^3 t)$$

$$y=-\left(\frac{1}{2}\tan t\right)x+a\sin t\cos^2 t+a\sin^3 t$$

$$y=-\left(\frac{1}{2}\tan t\right)x+a\sin t$$

$R(2a\cos t, 0), S(0, a\sin t)$

Midpoint of $RS = \left(a\cos t, \frac{1}{2}a\sin t\right)$

$x = a\cos t \Rightarrow \cos t = \frac{x}{a}$

$y = \frac{1}{2}a\sin t \Rightarrow \sin t = \frac{2y}{a}$

$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{a}\right)^2 + \left(\frac{2y}{a}\right)^2 = 1$

$x^2 + 4y^2 = a^2$

Since $0 < t < \frac{\pi}{2}, 0 < x < a$ or $0 < y < \frac{a}{2}$

5 (i) $S_1 = \frac{1}{2!} = \frac{1}{2}$

$S_2 = \frac{1}{2!} + \frac{2}{3!} = \frac{5}{6}$

$S_3 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{23}{24}$

$S_4 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} = \frac{119}{120}$

(ii)

$S_1 = \frac{1}{2} = 1 - \frac{1}{2}$

$S_2 = \frac{5}{6} = 1 - \frac{1}{6}$

$S_3 = \frac{23}{24} = 1 - \frac{1}{24}$

$S_4 = \frac{119}{120} = 1 - \frac{1}{120}$

$S_n = 1 - \frac{1}{(n+1)!}$

(iii)

Let P_n be the statement $S_n = 1 - \frac{1}{(n+1)!}$ for $n = 1, 2, 3, ...$

when $n = 1$

$\text{LHS} = S_1 = \frac{1}{2}$

$\text{RHS} = 1 - \frac{1}{(1+1)!} = \frac{1}{2}$

$\therefore P_1$ is true

Assume P_k is true for some $k = 1, 2, 3, ...$

$$S_k = 1 - \frac{1}{(k+1)!}$$

We want to prove that P_{k+1} is also true

$$S_{k+1} = 1 - \frac{1}{(k+2)!}$$

$$\begin{aligned}
\text{LHS} &= S_{k+1} \\
&= S_k + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \left[\frac{(k+2)-(k+1)}{(k+2)!}\right] \\
&= 1 - \frac{1}{(k+2)!} \\
&= \text{RHS}
\end{aligned}$$

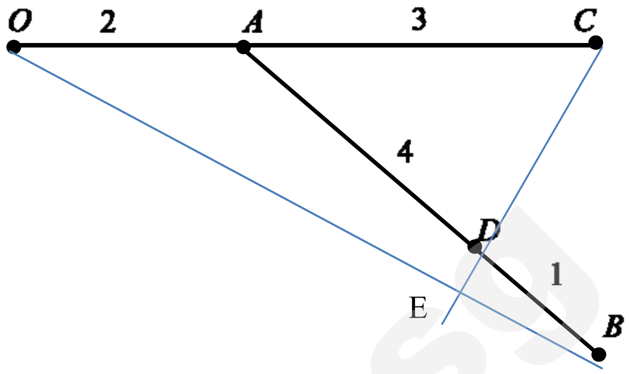
$\therefore P_{k+1}$ is true

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true ,

by mathematical induction P_n is true for all $n = 1, 2, 3, ...$

6

(i) $\dfrac{OA}{OC}=\dfrac{2}{5}$



$\dfrac{OC}{OA}=\dfrac{5}{2}$

$\overrightarrow{OC}=\dfrac{5}{2}\overrightarrow{OA}=\dfrac{5}{2}\mathbf{a}$

By ratio theorem,

$\overrightarrow{OD}=\dfrac{\mathbf{a}+4\mathbf{b}}{5}$

$\overrightarrow{OD}=\dfrac{1}{5}\mathbf{a}+\dfrac{4}{5}\mathbf{b}$

(ii) $\overrightarrow{CD}=\dfrac{1}{5}\mathbf{a}+\dfrac{4}{5}\mathbf{b}-\dfrac{5}{2}\mathbf{a}=-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}$

$l_{CD}:\mathbf{r}=\dfrac{5}{2}\mathbf{a}+\lambda\left(-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}\right)\quad \lambda\in\mathbf{R}$

(iii) Since E is a point on CD produced,

$\overrightarrow{OE}=\dfrac{5}{2}\mathbf{a}+\lambda\left(-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}\right)\quad \lambda\in\mathbf{R}$

Since E is a point on OB,

$\overrightarrow{OE}=\alpha\mathbf{b}\quad \alpha\in\mathbf{R}$

$\dfrac{5}{2}\mathbf{a}+\lambda\left(-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}\right)=\alpha\mathbf{b}$

$\left(\dfrac{5}{2}-\dfrac{23}{10}\lambda\right)\mathbf{a}+\dfrac{4}{5}\lambda\mathbf{b}=\alpha\mathbf{b}$

$\dfrac{5}{2}-\dfrac{23}{10}\lambda=0\Rightarrow\lambda=\dfrac{25}{23}$

$\dfrac{4}{5}\lambda=\alpha\Rightarrow\alpha=\dfrac{20}{23}$

$\therefore\overrightarrow{OE}=\dfrac{20}{23}\mathbf{b}$

$OE: EB=20: 3$

7

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k\left(T - T_{\mathrm{o}}\right), \quad k > 0$$

Note that no modulus required since T > T_0

$$\int \frac{1}{T - T_{\mathrm{o}}} \mathrm{d}T = \int -k \mathrm{d}t$$

$\ln\left(T - T_{\mathrm{o}}\right) = -kt + C$, where C is an arbitary constant

$T - T_{\mathrm{o}} = \mathrm{e}^{-kt+C}$

$T - T_{\mathrm{o}} = \mathrm{e}^{-kt}\mathrm{e}^{C}$

$T - T_{\mathrm{o}} = A\mathrm{e}^{-kt}$, where $A = \mathrm{e}^{C}$

$\therefore T = T_{\mathrm{o}} + A\mathrm{e}^{-kt}$ (shown)

$T_{\mathrm{o}} = 30^{\circ}C$

At $t = 0$: $100 = 30 + A\mathrm{e}^{-k(0)}$

$A = 70$

At $t = 15$: $70 = 30 + 70\mathrm{e}^{-15k}$

$40 = 70\mathrm{e}^{-15k}$

$$\mathrm{e}^{-15k} = \frac{4}{7}$$

$$k = -\frac{1}{15}\ln\frac{4}{7} \approx 0.0373077$$

To find time taken for pot of dessert to cool to at most $35^{\circ}C$:

$30 + 70\mathrm{e}^{-kt} \leq 35$

$70\mathrm{e}^{-kt} \leq 5$

$$\mathrm{e}^{-kt} \leq \frac{5}{70}$$

$$-kt \leq \ln\frac{5}{70}$$

$$t \geq \frac{\ln(5/70)}{-\frac{1}{15}\ln(4/7)}$$

$t \geq 70.74$

$t = 71$ minutes

It takes at least 71 minutes for the pot of dessert to cool to $35^{\circ}C$ and 30 minutes to cook.
Hence Nurul must start preparing the pot of dessert at 7.19pm the latest.

8

(i) Let L be the distance covered by the lion.

$a = 2.5$ and $d = -0.05$

$$L = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$= \frac{n}{2}\left[2(2.5) + (n-1)(-0.05)\right]$$
$$= -\frac{1}{40}n^2 + \frac{101}{40}n$$

(ii) Let P be the distance covered by the prey.
$a = 1.5$ and $r = 0.95$

$$P = \frac{1.5\left(1-0.95^n\right)}{1-0.95}$$
$$= 30(1-0.95^n)$$

When $n \to \infty$, $P \to 30$
So the distance covered by the prey can never exceed 30m

(iii) In order for the lion to catch its prey,
$L \geq P+25$

$$-\frac{1}{40}n^2 + \frac{101}{40}n \geq 30(1-0.95^n) + 25$$
$$-\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) \geq 55$$
$$n = 24, -\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) = 54.96 < 55$$
$$n = 25, -\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) = 55.822 > 55$$
$$n = 26, -\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) = 56.556 > 55$$

least $n = 25$
Hence, the lion will need at least 25 leaps to catch its prey.

(iv) Let the initial distance be k
In order for the prey to escape the hunt,
$P+k \geq L$

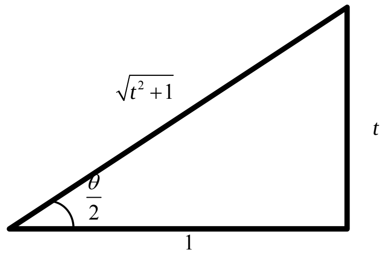
$$30(1-0.95^{30}) + k \geq -\frac{1}{40}(30^2) + \frac{101}{40}(30)$$
$$23.561 + k \geq 53.25$$
$$k \geq 29.689$$

$\therefore$ the shortest distance is 30 m.

9

(a) **(i)** $t=\tan\dfrac{\theta}{2}$

$$\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=\frac{2t}{1-t^2}$$



by triangle rule:

$$\sin\theta=\frac{2t}{1+t^2}\quad\text{(shown)}$$

Alternatively RHS $=\dfrac{2t}{1+t^2}=\dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\dfrac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}=\dfrac{2\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\times\cos^2\dfrac{\theta}{2}=\sin\theta$ =LHS

Alternatively

Use double angle formula: $\sin\theta=2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}=2\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cos^2\dfrac{\theta}{2}=\dfrac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}=\dfrac{2t}{1+t^2}$

(ii)

$$\int_0^{\frac{\pi}{2}}\frac{\tan\frac{\theta}{2}+1}{\sin\theta+1}\,\mathrm{d}\theta$$
$$=\int_0^1\frac{\mathrm{t}+1}{\frac{2t}{1+t^2}+1}\left(\frac{2}{1+t^2}\mathrm{dt}\right)$$
$$=\int_0^1\frac{\mathrm{t}+1}{\frac{2t+1+t^2}{1+t^2}}\left(\frac{2}{1+t^2}\mathrm{dt}\right)$$
$$=\int_0^1\frac{2(\mathrm{t}+1)}{2t+1+t^2}\mathrm{dt}$$
$$=\int_0^1\frac{2t+2}{t^2+2t+1}\mathrm{dt}=\int_0^1\frac{2}{t+1}\mathrm{dt}$$
$$=2\left[\ln(t+1)\right]_0^1$$
$$=2\ln 2$$

$t=\tan\dfrac{\theta}{2}$

when $\theta=\dfrac{\pi}{2}$: $t=\tan\dfrac{\pi/2}{2}=1$

when $\theta=0$: $t=\tan\dfrac{0}{2}=0$

$$\tan^{-1}t=\frac{\theta}{2}$$
$$\frac{1}{1+t^2}\frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{1}{2}$$
$$\frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{1+t^2}{2}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{2}{1+t^2}$$

(b)

$$\int e^{2v}\cos 3v\,dv$$

$$= \frac{1}{3}e^{2v}\sin 3v - \int \frac{2}{3}e^{2v}\sin 3v\,dv$$

$$\boxed{\begin{array}{ll} u = e^{2v} & \dfrac{dy}{dv} = \cos(3v) \\ \dfrac{du}{dv} = 2e^{2v} & y = \dfrac{1}{3}\sin(3v) \end{array}}$$

$$= \frac{1}{3}e^{2v}\sin 3v - \frac{2}{3}\left[-\frac{1}{3}e^{2v}\cos(3v) + \int \frac{2}{3}e^{2v}\cos(3v)\,dv\right]$$

$$\boxed{\begin{array}{ll} u = e^{2v} & \dfrac{dy}{dv} = \sin(3v) \\ \dfrac{du}{dv} = 2e^{2v} & y = -\dfrac{1}{3}\cos(3v) \end{array}}$$

$$= \frac{1}{3}e^{2v}\sin 3v + \frac{2}{9}e^{2v}\cos(3v) - \int \frac{4}{9}e^{2v}\cos(3v)\,dv$$

$$\frac{13}{9}\int e^{2v}\cos 3v\,dv = \frac{1}{3}e^{2v}\sin 3v + \frac{2}{9}e^{2v}\cos(3v)$$

$$\int e^{2v}\cos 3v\,dv = \frac{3}{13}e^{2v}\sin 3v + \frac{2}{13}e^{2v}\cos(3v) + c$$

Alternatively

$$\int e^{2v}\cos 3v\,dv$$

$$= \frac{1}{2}e^{2v}\cos 3v + \int \frac{3}{2}e^{2v}\sin 3v\,dv$$

$$\boxed{\begin{array}{ll} u = \cos(3v) & \dfrac{dy}{dv} = e^{2v} \\ \dfrac{du}{dv} = -3\sin(3v) & y = \dfrac{1}{2}e^{2v} \end{array}}$$

$$= \frac{1}{2}e^{2v}\cos 3v + \frac{3}{2}\left[\frac{1}{2}e^{2v}\sin(3v) - \int \frac{3}{2}e^{2v}\cos(3v)\,dv\right]$$

$$\boxed{\begin{array}{ll} u = \sin(3v) & \dfrac{dy}{dv} = e^{2v} \\ \dfrac{du}{dv} = 3\cos(3v) & y = \dfrac{1}{2}e^{2v} \end{array}}$$

$$= \frac{1}{2}e^{2v}\cos 3v + \frac{3}{4}e^{2v}\sin(3v) - \frac{9}{4}\int e^{2v}\cos(3v)\,dv$$

$$\frac{13}{4}\int e^{2v}\cos 3v\,dv = \frac{1}{2}e^{2v}\cos 3v + \frac{3}{4}e^{2v}\sin(3v)$$

$$\int e^{2v}\cos 3v\,dv = \frac{3}{13}e^{2v}\sin 3v + \frac{2}{13}e^{2v}\cos(3v) + c$$

10

(i) $\ell_1 : \mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-5\end{pmatrix}, \quad \lambda \in \mathbf{R}$

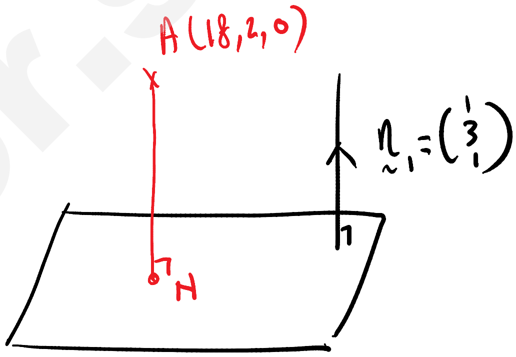
Since (1, 0, 1) is on ℓ_1 and p_1

$$\begin{pmatrix}1\\0\\1\end{pmatrix}\bullet\begin{pmatrix}1\\3\\1\end{pmatrix} = a$$

$1+0+1=a$

$a = 2$ (shown)

(ii) Let N be the foot of perpendicular from A to p_1



$$\ell_{AN} : \mathbf{r} = \begin{pmatrix}18\\2\\0\end{pmatrix} + \alpha\begin{pmatrix}1\\3\\1\end{pmatrix}, \quad \alpha \in \mathbf{R}$$

let $\overrightarrow{ON} = \begin{pmatrix}18+\alpha\\2+3\alpha\\\alpha\end{pmatrix}$ for some value of α

Since N is a point on p_1

$$\begin{pmatrix}18+\alpha\\2+3\alpha\\\alpha\end{pmatrix}\bullet\begin{pmatrix}1\\3\\1\end{pmatrix} = 2$$

$18+\alpha+6+9\alpha+\alpha=2$

$24+11\alpha=2$

$11\alpha=-22$

$\alpha=-2$

$$\overrightarrow{ON} = \begin{pmatrix}18-2\\2-6\\-2\end{pmatrix} = \begin{pmatrix}16\\-4\\-2\end{pmatrix} \therefore N(16,-4,-2)$$

(iii) Since B is on ℓ_1

$$\overrightarrow{OB} = \begin{pmatrix}1+2\lambda\\\lambda\\1-5\lambda\end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix}1+2\lambda\\\lambda\\1-5\lambda\end{pmatrix} - \begin{pmatrix}18\\2\\0\end{pmatrix} = \begin{pmatrix}-17+2\lambda\\-2+\lambda\\1+5\lambda\end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(-17+2\lambda)^2+(-2+\lambda)^2+(1+5\lambda)^2}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(289-68\lambda+4\lambda^2)+(4-4\lambda+\lambda^2)+(1+10\lambda+25$$

$$\left|\overrightarrow{AB}\right| = \sqrt{294-62\lambda+30\lambda^2}$$

There is a typo in Q11, so the correct

$$\overrightarrow{OB} = \begin{pmatrix}1+2\lambda\\\lambda\\1-5\lambda\end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix}1+2\lambda\\\lambda\\1-5\lambda\end{pmatrix} - \begin{pmatrix}18\\2\\0\end{pmatrix} = \begin{pmatrix}-17+2\lambda\\-2+\lambda\\1-5\lambda\end{pmatrix}$$

$$\left|\overrightarrow{AB}\right|^2 = 294-72\lambda+30\lambda^2$$

$\left|\overrightarrow{AB}\right|^2$ must be minimum

$$\therefore \left|\overrightarrow{AB}\right|^2 = 30\lambda^2-72\lambda+294$$

$$2\left|\overrightarrow{AB}\right|\frac{\mathrm{d}\left|\overrightarrow{AB}\right|^2}{\mathrm{d}\lambda} = 60\lambda-72$$

$$\frac{\mathrm{d}\left|\overrightarrow{AB}\right|^2}{\mathrm{d}\lambda} = 0$$

$60\lambda-72=0$

$$\lambda = \frac{6}{5}$$

$$\overrightarrow{OB} = \begin{pmatrix}1+2\lambda\\\lambda\\1+5\lambda\end{pmatrix} = \begin{pmatrix}17/5\\6/5\\7\end{pmatrix} = \frac{1}{5}\begin{pmatrix}17\\6\\35\end{pmatrix}$$

$$\left|\overrightarrow{AB}\right|^2 = 294 - 62\lambda + 30\lambda^2$$

For shortest distance from A to ℓ_1

$\left|\overrightarrow{AB}\right|^2$ must be minimum

$$\therefore \left|\overrightarrow{AB}\right|^2 = 30\lambda^2 - 62\lambda + 294$$

$$2\left|\overrightarrow{AB}\right|\frac{\mathrm{d}\left|\overrightarrow{AB}\right|^2}{\mathrm{d}\lambda} = 60\lambda - 62$$

$$\frac{\mathrm{d}\left|\overrightarrow{AB}\right|^2}{\mathrm{d}\lambda} = 0$$

$$60\lambda - 62 = 0$$

$$\lambda = \frac{31}{30}$$

$$\overrightarrow{OB} = \begin{pmatrix} 1+2\lambda \\ \lambda \\ 1+5\lambda \end{pmatrix} = \begin{pmatrix} 46/15 \\ 31/30 \\ 37/6 \end{pmatrix} \text{ or } \frac{1}{30}\begin{pmatrix} 92 \\ 31 \\ 185 \end{pmatrix}$$

(iv) direction vector of $\ell_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ -(1-2) \\ b \end{pmatrix} = \begin{pmatrix} -b \\ 1 \\ b \end{pmatrix}$

To find a common point between p_2 and p_3 by letting $y = 0$:

$x + z = 1$ --- (1)

$2x + z = 4$ --- (2)

Solve (1) and (2):

$x = 3,\ \ z = -2$

Hence ℓ_2: $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} -b \\ 1 \\ b \end{pmatrix},\ \ \mu \in \mathbf{R}$ (shown)

11

(i) $w = re^{i\theta}$

$w^* = re^{-i\theta}$

$$\frac{w^2}{w^*} = \frac{\left(re^{i\theta}\right)^2}{re^{-i\theta}}$$

$$= \frac{r^2e^{i2\theta}}{re^{-i\theta}}$$

$$= re^{i3\theta} = -3 = 3e^{i\pi}$$

$$3\theta = \pi \Rightarrow \theta = \frac{\pi}{3} (0 < \theta \le \frac{1}{2}\pi)$$

$r = 3$

$w = 3e^{i\frac{\pi}{3}}$, $w^n = 3^n e^{i\frac{n\pi}{3}}$

w^n is real $, \Rightarrow \frac{n\pi}{3} = 0, \pi, 2\pi, \ldots$, so n = 3,6, 9,…

(b) **(i)** $z^5 = 1 + i$

$$= \sqrt{2}e^{\left(2k\pi + \frac{\pi}{4}\right)i}$$

$$z = 2^{\frac{1}{10}}e^{\left(\frac{2k\pi}{5} + \frac{\pi}{20}\right)i}, \ k = 0 \ , \pm 1 \ , \pm 2$$

$$z = 2^{\frac{1}{10}}e^{\frac{\pi}{20}i}, \ 2^{\frac{1}{10}}e^{\frac{9\pi}{20}i}, \ 2^{\frac{1}{10}}e^{-\frac{7\pi}{20}i}, \ 2^{\frac{1}{10}}e^{\frac{17\pi}{20}i}, \ 2^{\frac{1}{10}}e^{-\frac{3\pi}{4}i}$$

So $|z| = 2^{\frac{1}{10}}$ for all z

$$\arg(z) = \frac{\pi}{20}, \frac{9\pi}{20}, -\frac{7\pi}{20}, \frac{17\pi}{20}, -\frac{3\pi}{4}$$

(ii) $z_1 = 2^{\frac{1}{10}}e^{\frac{17\pi}{20}i}$

$z_2 = 2^{\frac{1}{10}}e^{-\frac{15\pi}{20}i}$

Let Point A and B represent z_1 and z_2 respectively.

$|z_1| = |z_2| \Rightarrow OAB$ is an isosceles triangle.

$$\measuredangle AOB = \frac{2\pi}{5}$$

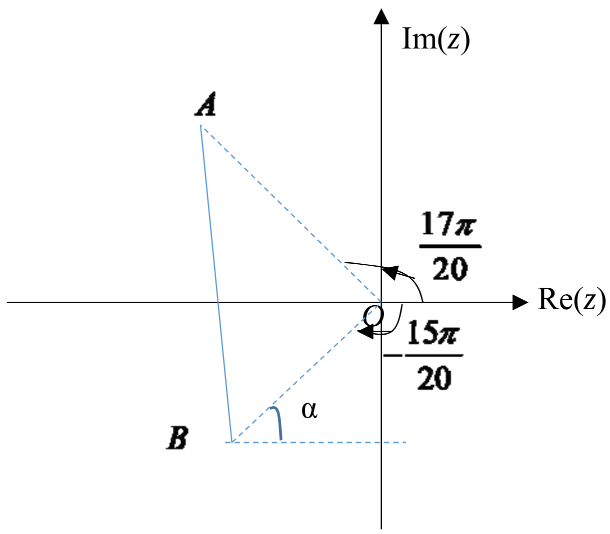
$$\measuredangle OAB = \measuredangle OBA = \frac{1}{2}[\pi - \measuredangle AOB]$$

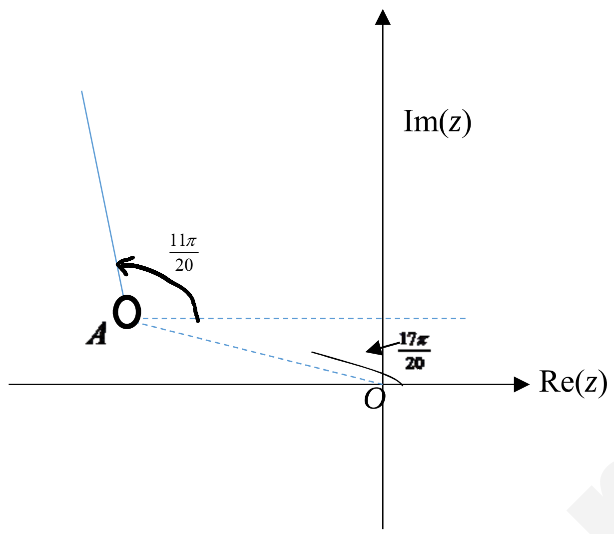
$$= \frac{1}{2}\left[\pi - \frac{2\pi}{5}\right] = \frac{3\pi}{10}$$

$$\arg(z_1 - z_2) = \alpha + \measuredangle OBA$$

$$= \left(\pi - \frac{15\pi}{20}\right) + \frac{3\pi}{10}$$

$$= \frac{11\pi}{20}$$




Im(z)
$\frac{11\pi}{20}$
A
$\frac{17\pi}{20}$
Re(z)
O

Section A: Pure Mathematics [40 marks]

1 Two complex numbers a and b are given by $2+3\mathrm{i}$ and $-4-5\mathrm{i}$ respectively.

(i) On a single Argand diagram, sketch the loci

(a) $|2z-a-b|=|a-b|$,

(b) $0 \le \arg(z-b) \le \arg(a-b)$. [4]

(ii) Find range of $\arg(z)$ where z is the complex number satisfies the relations in part **(i)**. [4]

2 A curve C has equation $y=\dfrac{ax}{x-1}$ where $a>0$.

(i) By writing the equation of C as $y=A+\dfrac{B}{x-1}$, state a sequence of transformations which transform the graph of $y=\dfrac{1}{x}$ to C. [3]

(ii) Sketch C, giving the equations of any asymptotes and the coordinates of any points of intersection with the axes. [2]

(iii) The region R is bounded by C, the lines $x=2$, $x=4$ and $y=a$. Find the exact volume in terms of a when R is rotated through 2π radians about the x-axis. [3]

(iv) The region S is bounded by $y=\dfrac{a}{x}$, the lines $x=1$, $x=3$ and $y=0$. State the exact volume in terms of a when S is rotated through 2π radians about the line $y=-a$. [1]

3 The function f is defined by $\mathrm{f}(x)=\begin{cases}\dfrac{x}{2} & \text{if } x\le 0,\\ 2\sin x & \text{if } 0<x\le 4.\end{cases}$

(i) Sketch the graph of $y=\mathrm{f}(x)$. [2]

(ii) If the domain of f is restricted to $x\le k$, state the largest value of k, in exact form, for which the function f^{-1} exist. [1]

(iii) Using the domain from part **(ii)**, define f^{-1} in a similar form. [4]

(iv) Solve $\mathrm{f}^{-1}(x)=\mathrm{f}(x)$. [2]

In the rest of the question, the domain of f is as originally defined.

The function g is defined by $\mathrm{g}: x\mapsto -x^3, x\in\mathbb{R}, x>0$.

(v) Find an expression for $\mathrm{fg}(x)$. [2]

4 **(i)** Differentiate $\tan^{-1}\left(\dfrac{\sqrt{3}}{2}x\right)$ with respect to *x*. [2]

(ii) Find the binomial expansion for $\dfrac{1}{3x^2+4}$ up to and including the term in x^6, giving the coefficients as exact fractions in their simplest form. Find the set of values of *x* for which the expansion is valid. [5]

(iii) Hence, find the first four non-zero terms of the Maclaurin series for $\tan^{-1}\left(\dfrac{\sqrt{3}}{2}x\right)$. Give the coefficients as exact fractions in their simplest form.

[5]

Section B: Statistics [60 marks]

5 A pharmaceutical company has invented a new drug for diabetic patients and wishes to carry out a trial of the new drug involving 5% of the patients from a local hospital.

(i) Explain how a systematic sample could be carried out. [2]

(ii) State one disadvantage of systematic sampling in this context and name a more appropriate sampling method. [2]

6 Given that $\mathrm{P}(A \mid B') = 3\mathrm{P}(A \mid B)$ and $\mathrm{P}(B') = 4\mathrm{P}(B)$.

(i) Show that $\mathrm{P}(B') = \frac{4}{5}$. [2]

(ii) Using $\mathrm{P}(A \cap B') = \mathrm{P}(A) - \mathrm{P}(A \cap B)$, find $\mathrm{P}(B' \mid A)$. [3]

7 The sales department of a company consists of 3 teams led by Mrs Wong, Miss Tan and Mr Lim. Each team is made up of 1 team leader and 5 sales executives. The number of male and female sales executives within each team is given in the table below:

	Team A	Team B	Team C
Team Leader	Mrs Wong	Miss Tan	Mr Lim
Number of Male Executive(s)	3	4	0
Number of Female Executive(s)	2	1	5

A taskforce is to be formed by selecting 7 representatives from the 18 members of the department. Find the number of different taskforces that can be formed if the taskforce must include

(i) Miss Tan and 1 other team leader, [2]

(ii) more females than males, [2]

(iii) at least 1 representative from each team. [3]

8 In this question you should state clearly the values of the parameters of any normal distribution you use.

The mass in kilograms of an Atlantic salmon is a normally distributed continuous random variable X with mean μ and standard deviation σ.

(i) It is known that $\mathrm{P}(X < 22) = 0.159$ and $\mathrm{P}(X > 31) = 0.106$. Show that $\mu = 26.0$ and $\sigma = 4.01$. [3]

(ii) In a random sample of 40 Atlantic salmon, estimate the probability that at least 35 of them have a mass of at most 31 kilograms. [3]

It is also known that the mass in kilograms of the Bluefin tuna has the distribution $\mathrm{N}(380, 10^2)$.

(iii) Find the probability that the average mass of 2 randomly chosen Bluefin tuna and 3 randomly chosen Atlantic salmon is at most 170 kg. [2]

9 The table gives the world record time, in seconds, for the 100 metres free style swimming event at the Olympic Games in the past years.

Year, x	1908	1920	1956	1968	1976	2000
Time, t	65.60	60.40	55.40	52.20	49.99	48.18

(i) Draw a scatter diagram to illustrate the data. [1]

(ii) Comment on whether a linear model would be appropriate, referring both to the scatter diagram and the context of the question. [2]

(iii) It is thought that the time can be modelled by one of the formulae $\ln t = a + bx$ or $\frac{1}{t} = a + bx$. Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) $\ln t$ and x,

(b) $\frac{1}{t}$ and x. [2]

(iv) Use your answers to part **(iii)** to explain which of $\ln t = a + bx$ or $\frac{1}{t} = a + bx$ is the better model. [1]

(v) The time corresponding to 1964 was added to obtain the equation with appropriate model chosen in part **(iv)** where $a = -0.09836$ and $b = 5.96846 \times 10^{-5}$. Find the time in 1964. [2]

10 A car manufacturer launches a new car model "Green Leaf" that is marketed to be environmentally friendly. It is claimed that the carbon emission of the "Green Leaf" is at most 80 g/km. The transport authority suspects that the figure is understated, and requests the manufacturer to submit test data from 20 units of the "Green Leaf". The test data submitted is as follows.

Carbon Emission (g/km)	78	79	80	81	82
No. of units	2	3	6	4	5

(i) Calculate unbiased estimates of the population mean and variance. [2]

(ii) Stating a necessary assumption, test at the 10% level of significance whether there is any evidence to doubt the manufacturer's claim. [4]

The transport authority subsequently decides to conduct their own test, and invites 10 owners of the "Green Leaf" to form a sample. The mean and variance of this sample is found to be 80.6 g/km and m^2 g^2/km^2 respectively.

(iii) Find the set of values of m for which the result of the test would be to reject the manufacturer's claim, at the 1% significance level. [3]

11 There are 2 main types of T-cells in the human body. T4-cells are "helper" cells that lead attacks against infections in the human body, while T8-cells are "suppressor" cells that kill cancer and virus infected cells in the human body. It is to be assumed that the number of T4-cells per 0.01 mm^3 of blood can be modelled by the distribution Po(5) and the number of T8-cells per 0.01 mm^3 of blood can be modelled by the independent distribution Po(1.5).

A patient is considered healthy if he or she has at least 4 T4-cells and at least 1 T8-cells in 0.01 mm^3 of blood.

(i) Find the probability that a randomly selected patient is healthy. [3]

(ii) Find the probability that only 1 out of 3 randomly selected patients is healthy. [2]

A patient is susceptible to infections if his or her T4-cells count falls below 3 per 0.01 mm^3 of blood.

(iii) Use a suitable approximation, which should be stated, to find the probability that, in 100 randomly selected patients, the number of patients susceptible to infections is between 20 and 50 inclusive. [4]

12 Mr Ouyang, a car manufacturer, finds that on average, 2% of his cars have faulty gearboxes. On a particular occasion, he selects n cars randomly for inspection, and the number of cars with faulty gearbox is denoted by the random variable C.

(i) State in context of this question, what must be assumed for C to be well modelled by a binomial distribution. [2]

(ii) Given that $n = 20$, find the probability that C is between 2 and 6. [2]

(iii) The probability that there are less than 2 cars with faulty gearbox in a sample of n cars is at most 0.95. Write down an inequality in terms of n, and find the least possible value of n. [3]

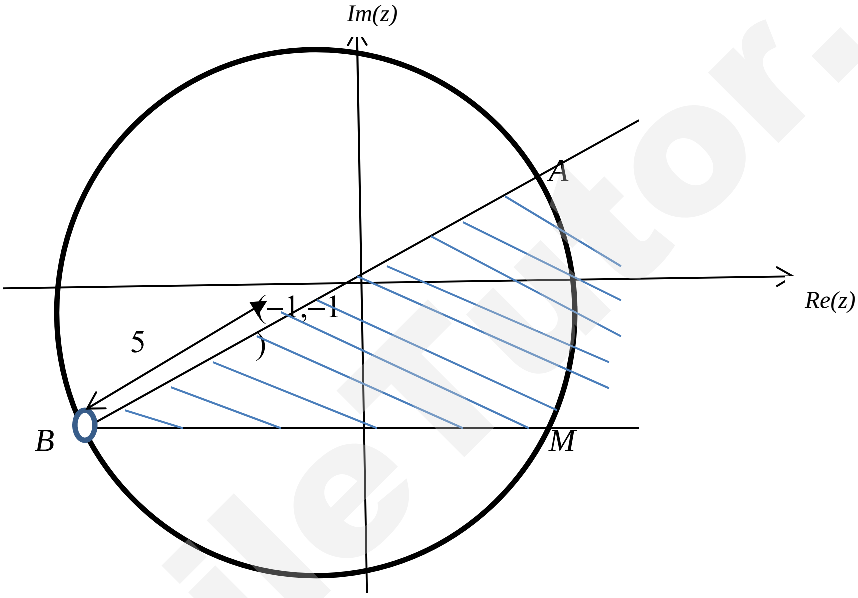
(iv) Mr Ouyang selects 100 batches of 20 cars. Estimate the probability that the average number of cars with faulty gearbox per batch is at least 0.3. [3]

Pioneer Junior College
H2 Mathematics
JC 2 Preliminary Examination Paper 2 Solution
1

(i) $|2z-a-b|=|a-b|$

Centre of circle C $= \dfrac{(2+3\mathrm{i})+(-4-5\mathrm{i})}{2} = -1-\mathrm{i}$

Radius of circle C $= \dfrac{\sqrt{(2+4)^2+(3+5)^2}}{2} = 5$



(ii)*AM* is the common region satisfies both (i) and (ii).
$\angle$AMB = 90° since AB is diameter and angle in semicircle is a right angle
Max arg(z) = arg(a) = $\tan^{-1}(3/2)$ = 0.983 radians
Min arg(z) = arg(2 – 5i) = $-\tan^{-1}(5/2)$ = −1.19 radians
So range required is $-1.19 \le \arg(z) \le 0.983$

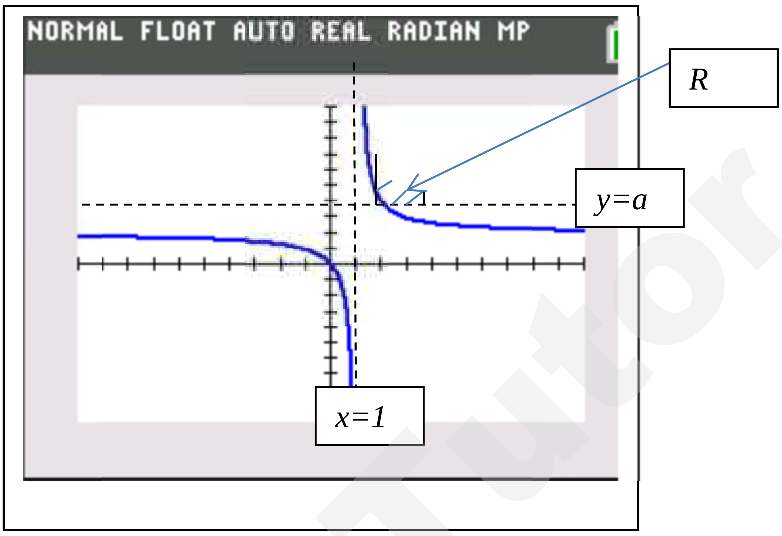
2

(i) $y=\dfrac{ax}{x-1}=a+\dfrac{a}{x-1}$

$y=\dfrac{1}{x}$ is translated 1 unit in the direction of x-axis, followed by a scaling of a units parallel to the y axis and is translated a units in the direction of y-axis

(ii) The equations of asymptotes are $x = 1$ and $y = a$

The intercepts are (0,0)



(iii) The volume required $=\pi\int_2^4\left(a+\dfrac{a}{x-1}\right)^2 \mathrm{d}x-\pi(a)^2(2)$

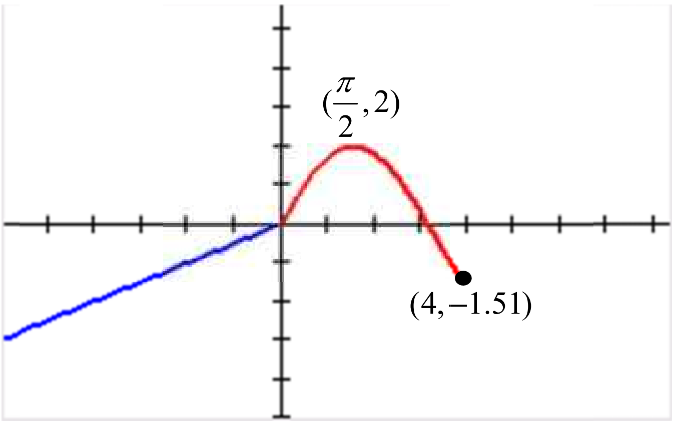
$$= a^2\pi\int_2^4\left(1+\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)\mathrm{d}x-2\pi a^2$$

$$= a^2\pi\left[x+2\ln|x-1|-\frac{1}{x-1}\right]_2^4-2\pi a^2$$

$$=\left(\frac{2}{3}+2\ln 3\right)\pi a^2$$

(iv) Using part (i), the area S is the same as the area R found in (iii). To rotate S about the line $y = -a$ is the same as to rotate R about the x-axis. So the volume obtained is $\left(\dfrac{2}{3}+2\ln 3\right)\pi a^2$

3
(i)



(ii) From the graph, take $k=\frac{\pi}{2}$

(iii)

$$f(x)=\begin{cases}\frac{x}{2} & \text{if } x\le 0\\ 2\sin x & \text{if } 0<x\le\frac{\pi}{2}\end{cases}$$

Let $y_1=\frac{x}{2}$

$x=2y_1$

Let $y_2=2\sin x$

$x=\sin^{-1}\left(\frac{y_2}{2}\right)$

$$f^{-1}(x)=\begin{cases}2x & \text{if } x\le 0\\ \sin^{-1}\left(\frac{x}{2}\right) & \text{if } 0<x\le 2\end{cases}$$

(iv) $f^{-1}(x)=f(x)$ is the same as solving $f(x)=x$

$\frac{x}{2}=x \Rightarrow x=0 \quad \text{if } x\le 0$

$2\sin x=x$, $x=1.90>\frac{\pi}{2}$, so only solution is $x=0$

(v) Using $R_g=(-\infty,0)$, $fg(x)=\frac{-x^3}{2}$

4

(i)

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{\sqrt{3}}{2}x\right)\right)=\frac{1}{1+\left(\frac{\sqrt{3}}{2}x\right)^2}\left(\frac{\sqrt{3}}{2}\right)$$

$$=\left(\frac{\sqrt{3}}{2}\right)\frac{1}{1+\frac{3}{4}x^2}$$

$$=\left(\frac{\sqrt{3}}{2}\right)\frac{4}{3x^2+4}$$

$$=\frac{2\sqrt{3}}{3x^2+4}$$

(ii)

$$\frac{1}{3x^2+4}=(3x^2+4)^{-1}=4^{-1}\left(1+\frac{3x^2}{4}\right)^{-1}$$

$$=\frac{1}{4}\left[1+(-1)\left(\frac{3x^2}{4}\right)+\frac{-1(-2)}{2}\left(\frac{3x^2}{4}\right)^2+\frac{-1(-2)(-3)}{6}\left(-\frac{3x^2}{4}\right)^3+\ldots\right]$$

$$=\frac{1}{4}\left(1-\frac{3}{4}x^2+\frac{9}{16}x^4-\frac{27}{64}x^6+\ldots\right)$$

$$\approx\frac{1}{4}-\frac{3}{16}x^2+\frac{9}{64}x^4-\frac{27}{256}x^6$$

Range of validity: $\left|\frac{3x^2}{4}\right|<1\Rightarrow x^2<\frac{4}{3}\Rightarrow -\frac{2}{\sqrt{3}}<x<\frac{2}{\sqrt{3}}$

(iii)

$$y=\tan^{-1}\left(\frac{\sqrt{3}}{2}x\right)=2\sqrt{3}\int\frac{1}{3x^2+4}\,dx$$

$$=2\sqrt{3}\int\left(\frac{1}{4}-\frac{3}{16}x^2+\frac{9}{64}x^4-\frac{27}{256}x^6\right)dx$$

$$=2\sqrt{3}\left(\frac{1}{4}x-\frac{1}{16}x^3+\frac{9}{320}x^5-\frac{27}{1792}x^7\right)+C$$

when $x=0$, $y=0$, $C=0$

$$\therefore \tan^{-1}\left(\frac{\sqrt{3}}{2}x\right)=\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{8}x^3+\frac{9\sqrt{3}}{160}x^5-\frac{27\sqrt{3}}{896}x^7$$

5 (i) Number the list of patients from 1 to N. $k = N/0.05N = 20$ (Randomly select a number from 1 to 20, and let every 20th patient after first patient chosen try the new drug. For example, if a number 5 is chosen, then survey every 5th, 25th, 45th patient and so on, until the sample size of 5% patients is obtained.

(ii) Disadvantage: The sample is not representative of the population of diabetic patients as age and gender may affect the drug. More appropriate method is stratified sampling.

6

(i) $P\left(B^{'}\right)=4P\left(B\right)$

$P\left(B^{'}\right)=4\left[1-P\left(B^{'}\right)\right]$

$P\left(B^{'}\right)=4-4P\left(B^{'}\right)$

$5P\left(B^{'}\right)=4$

$P\left(B^{'}\right)=\frac{4}{5}$ (shown)

Alternatively,

$1-P\left(B\right)=4P\left(B\right)$

$5P\left(B\right)=1$

$P\left(B\right)=\frac{1}{5}$

$P\left(B^{'}\right)=\frac{4}{5}$ (shown)

(ii) $P\left(A\mid B^{'}\right)=3P\left(A\mid B\right)$

$$\frac{P\left(A\cap B^{'}\right)}{P\left(B^{'}\right)}=\frac{3P\left(A\cap B\right)}{P\left(B\right)}$$

$$\frac{P\left(A\cap B^{'}\right)}{P\left(B^{'}\right)}=\frac{3P\left(A\cap B\right)}{1-P\left(B^{'}\right)}$$

$$\frac{P\left(A\cap B^{'}\right)}{4/5}=\frac{3P\left(A\cap B\right)}{1/5}$$

$P\left(A\cap B^{'}\right)=12P\left(A\cap B\right)----(*)$

$P\left(A\right)-P\left(A\cap B\right)=12P\left(A\cap B\right)$

$P\left(A\right)=13P\left(A\cap B\right)----(**)$

$$P\left(B^{'}\mid A\right)=\frac{P\left(A\cap B^{'}\right)}{P\left(A\right)}$$

$$=\frac{12P\left(A\cap B\right)}{13P\left(A\cap B\right)}$$

$=\frac{12}{13}$ or 0.923 (3 s.f.)

7 (i)Number of teams $={}^{2}C_{1}\,{}^{15}C_{5}=6006$

(i) 4 cases: 4F3M, 5F2M , 6F1M and 7F

Number of teams $={}^{10}C_{4}\,{}^{8}C_{3}+{}^{10}C_{5}\,{}^{8}C_{2}+{}^{10}C_{6}\,{}^{8}C_{1}+{}^{10}C_{7}=20616$

(ii) Total $-$ teams from A and B $-$ teams from B and C $-$ teams from A and C

$={}^{18}C_{7}-{}^{12}C_{7}\times 3$

$=29448$

8 (i) X – mass in kilograms of an Atlantic salmon

$X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$\mathrm{P}\left(X < 22\right) = 0.159$

$\mathrm{P}\left(Z < \dfrac{22-\mu}{\sigma}\right) = 0.159$

$\dfrac{22-\mu}{\sigma} = -0.99858$

$\mu - 0.99858\sigma = 22$ ---(1)

$\mathrm{P}\left(X > 31\right) = 0.106$

$\mathrm{P}\left(Z > \dfrac{31-\mu}{\sigma}\right) = 0.106$

$\dfrac{31-\mu}{\sigma} = 1.2481$

$\mu + 1.2481\sigma = 31$ ---(2)

Solving (1) and (2):

$\mu = 26.00022 \approx 26.0$ (shown)

$\sigma = 4.00591 \approx 4.01$ (shown)

(ii) Let W be the number of Atlantic salmon with more than 31 kg, out of 40

$W \sim \mathrm{B}(40, 0.106)$

$n = 40$ large, $np = 40(0.106) = 4.24 < 5$

so $W \sim \mathrm{Po}(4.24)$ approx.

Required prob = $\mathrm{P}(W \leq 5) = 0.74659 \approx 0.747$

(iii) Y – mass in kilograms of an Bluefin tuna.

$Y \sim \mathrm{N}\left(380, 10^2\right)$

Let T be the mass of 2 Bluefin tuna and 3 Atlantic salmon

$T = X_1 + X_2 + X_3 + Y_1 + Y_2 \sim \mathrm{N}\left(838, 248.2403\right)$

$\dfrac{X_1 + X_2 + X_3 + Y_1 + Y}{5} = \overline{T} \sim \mathrm{N}\left(\dfrac{838}{5}, \dfrac{248.2403}{25}\right)$ exactly

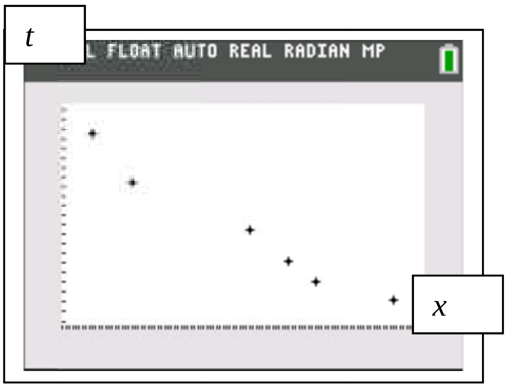
$\mathrm{P}\left(\overline{T} \leq 170\right) = 0.777$ (3 s.f.)

Alternatively

$\mathrm{P}(T \leq 170 \times 5) = 0.777$

9

(i)



(ii) The time for swimming cannot decrease forever as there is a limit on how fast a swimmer can swim and from the scatter diagram, as x increases, t decreases with decreasing amount, so linear model is not appropriate.

(iii)

$\ln t = a + bx$: $r = -0.9851$

$\frac{1}{t} = a + bx$: $r = 0.9877$

(iv) Since $|r|$ for $\frac{1}{t} = a + bx$ is higher than that of $\ln t = a + bx$, $\frac{1}{t} = a + bx$ is the preferred model.

(v) Let the timing be t

$$\frac{1}{t} = -0.09836 + \left(5.96846 \times 10^{-5}\right)x$$

Only value that satisfies the equation is $\left(\bar{x}, \overline{\left(\frac{1}{t}\right)}\right)$.

$$\overline{\frac{1}{t}} = -0.09836 + \left(5.96846 \times 10^{-5}\right)\bar{x} = -0.09836 + 0.0000596846(1956) = 0.018383$$

So $\frac{1}{7}\left(\frac{1}{65.6} + \frac{1}{60.4} + \frac{1}{55.4} + \frac{1}{52.2} + \frac{1}{49.99} + \frac{1}{48.18} + \frac{1}{t}\right) = 0.018383$ =

$t = 52.87 \approx 52.9$

So the timing at 1964 is 52.9 second

10

(i) Let X be the carbon emission of "Green Leaf".

From GC, unbiased estimate of population mean $= \bar{x} = 80.35$,

Unbiased estimate of population variance $= s^2 = (1.3089)^2 = 1.7132$

(ii) Since n is small and σ^2 is unknown, we use the t-test.

Assumption: The carbon emission of the "Green Leaf" is normally distributed.

$H_0 : \mu = 80$ vs $H_1 : \mu > 80$

Test Statistic, $t = 1.1959$

From GC, p-value = 0.12323 > 0.1

Since the p-value is more than the level of significance, we do not reject H_0. There is insufficient evidence, at the 10% level, to indicate that the manufacturer's claim is not true.

(iii) Since n is small and σ^2 is unknown, we use the t-test.

For H_0 to be rejected, Test Statistic > 2.8214

Unbiased estimate of population variance $s^2 = \frac{10}{9}m^2$

Test Statistic, $t = \frac{80.6-80}{m/3} > 2.8214$

$m < 0.638$

11 (i) X – number of T4-cells in 0.01 mm^3 of blood

$X \sim \text{Po}(5)$

Y – number of T8-cells in 0.01 mm^3 of blood

$Y \sim \text{Po}(1.5)$

$$\begin{aligned}\text{P}(\text{healthy}) &= \text{P}(X \geq 4)\text{P}(Y \geq 1)\\ &= [1-\text{P}(X \leq 3)][1-\text{P}(Y=0)]\\ &= 0.57098 \approx 0.571 \text{ (3 s.f.)}\end{aligned}$$

(ii) Req prob $= [\text{P}(\text{healthy})][\text{P}(\text{unhealthy})]^2 \times \frac{3!}{2!}$

$= (0.57098)(1-0.57098)^2 \times \frac{3!}{2!} = 0.315$ (3 s.f.)

Alternatively,

A – number of patients who are healthy out of 3 patients

$A \sim \text{B}(3, 0.57098)$

$\text{P}(A=1) = 0.31528 \approx 0.315$ (3 s.f.)

(iii) $\text{P}(\text{susceptible})$

$= \text{P}(X < 3) = \text{P}(X \leq 2)$

$= 0.12465$

W – number of patients who are susceptible to infection out of 100 patients

$W \sim \text{B}(100, 0.12465)$

Since n is large and

$np = (100)(0.12465) = 12.465 > 5$ and $n(1-p) = (100)(1-0.12465) = 87.535 > 5$

$W \sim \text{N}(12.465, 10.911)$ approx

$\text{P}(20 \leq W \leq 50)$

$= \text{P}(19.5 < W < 50.5)$ (c.c)

$= 0.016595 \approx 0.0166$ (3 s.f.)

12 (i) The occurrences of faulty gearbox must be independent of one another

The probability of a faulty gearbox is constant

(ii) C – number of cars that has gearbox issues out of 20 cars

$C \sim \text{B}(20, 0.02)$

$\text{P}(2 < C < 6) = \text{P}(C \leq 5) - \text{P}(C \leq 2)$

$= 0.0070667 = 0.00707$ (3 s.f.)

Alternatively,

$\text{P}(2 < C < 6) = \text{P}(C=3) + \text{P}(C=4) + \text{P}(C=5)$

$= 0.0070667 = 0.00707$ (3 s.f.)

(iii) $C \sim B(n, 0.02)$

$\mathrm{P}(C < 2) \le 0.95$

$\mathrm{P}(C \le 1) \le 0.95$

$\mathrm{P}(C = 0) + \mathrm{P}(C = 1) \le 0.95$

${}^{n}C_0 (0.02)^0 (0.98)^{n-0} + {}^{n}C_1 (0.02)^1 (0.98)^{n-1} \le 0.95$

$(0.98)^n + n\left(\frac{1}{49}\right)(0.98)^n \le 0.95$

$(0.98)^n \left(1 + \frac{n}{49}\right) \le 0.95$

$(0.98)^n \left(1 + \frac{n}{49}\right) - 0.95 \le 0$

$n \ge 18.0977$

Hence the least number of cars Mr Ouyang has to sample is 19

Alternatively, By GC using table

n	$(0.98)^n\left(1+\frac{n}{49}\right)$
17	0.95541
18	0.95049
19	0.94538

Hence the least number of cars Mr Ouyang has to sample is 19.

(iv) C – number of cars that has gearbox issues out of 20 cars

$C \sim \mathrm{B}(20, 0.02)$

$\mathrm{E}(C) = 0.4$ $\mathrm{Var}(C) = 0.392$

Since $n = 100$ large, by CLT, $\overline{C} \sim N\left(0.4, \frac{0.392}{100}\right)$ approx.

$\mathrm{P}(\overline{C} \ge 0.3) = 0.945$

RAFFLES INSTITUTION

2016 YEAR 6 PRELIMINARY EXAMINATION

MATHEMATICS PAPER 1 **9740/01**

Higher 2 **19 SEPTEMBER 2016**

Total Marks: 100 **3 hours**

Additional materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the test, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 It is given that $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants.

The graph of $y = \mathrm{f}(x)$ passes through the point with coordinates $(-1, -27)$ and has a turning point at $(2, 27)$. Given also that $\mathrm{f}''(0) = 0$, find $\mathrm{f}(x)$. **[4]**

2 In a laboratory experiment, an empty 10-litre tank is transported back and forth between station A and station B by a machine.

Starting at station A, 1000ml of water is added to the tank and on subsequent visits, 90% of the amount of water added previously is added to the tank, that is, 900ml on the 2^{nd} visit, 810 ml on the 3^{rd} visit and so on.

At station B, the machine removes 100ml of water from the tank and on subsequent visits, it removes 50ml more water than the previous visit, that is, 150ml on the 2^{nd} visit, 200ml on the 3^{rd} visit and so on.

(i) Show that the amount of water in the tank after the 3^{rd} visit to station B is 2260ml. **[1]**

The machine stops when the amount of water to be removed exceeds the amount of water present in the tank.

(ii) Determine the amount of water in the tank when the machine stops.
Leave your answer to the nearest millilitre. **[3]**

3 The even positive integers, starting at 2, are grouped into sets containing 1, 3, 5, 7, ... integers, as indicated below, so that the number of integers in each set is two more than the number of integers in the previous set.

$$\{2\}, \{4,6,8\}, \{10,12,14,16,18\}, \{20,22,24,26,28,30,32\}, \ldots$$

Find, in terms of r, an expression for

(i) the number of integers in the r^{th} set, **[1]**

(ii) the last integer in the r^{th} set. **[2]**

Given that the n^{th} set contains the integer 2016, find n. **[2]**

4 **(i)** Use integration by parts to show that for any real constant a,

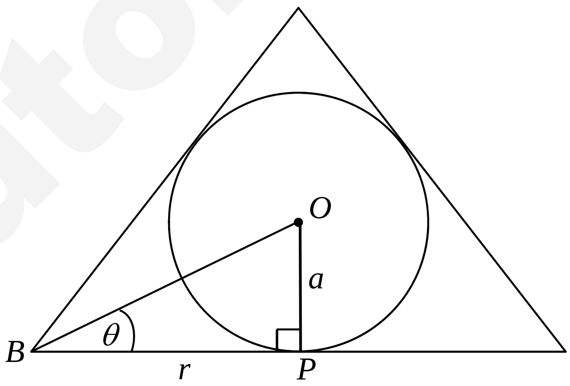
$$\int e^x \sin ax \, dx = \frac{e^x(\sin ax - a\cos ax)}{1+a^2} + c,$$

where c is an arbitrary constant. **[4]**

(ii) By expressing $\sin 2x \cos x$ in the form $A(\sin Px + \sin Qx)$, for real constants A, P and Q, find $\int e^x \sin 2x \cos x \, dx$. **[2]**

5 [A right circular cone with base radius r, height h and slant height l has curved surface area πrl.]

A right circular cone of base radius r is designed to contain a sphere of fixed radius a. The sphere touches both the curved surface and the base of the cone. (See diagram for a cross-sectional view.)



The point O is the centre of the sphere, the point B is on the circumference of the base of the cone, the point P is the centre of the circular base of the cone and θ is the angle OB makes with the base.

(i) Show that $\cos 2\theta = \dfrac{r^2 - a^2}{r^2 + a^2}$. **[2]**

(ii) Use differentiation to find, in terms of a, the minimum total surface area of the cone (consisting of the curved surface area and the base area), proving that it is a minimum. **[6]**

6 **Do not use a calculator in answering this question.**

(i) For $y = 2\cos\left(\frac{2}{3}\cos^{-1} x\right)$, show that $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -\frac{4}{9}y$. Hence find the Maclaurin series for y, up to and including the term in x^2. **[4]**

(ii) Given that the first three terms found in part **(i)** are equal to the first three terms in the series expansion of $(1+bx)^n$, find the values of the constants b and n. **[4]**

[Turn over

7 The complex numbers z and w are such that

$$z = 1 - \mathrm{i}\sqrt{3} \text{ and } w = -\sqrt{2} + \mathrm{i}c$$

where c is real and positive. It is given that $\left|\frac{z}{w}\right| = 1$.

(i) Find the exact value of c. **[2]**

(ii) Show that $\arg\left(\frac{z}{w}\right) = \frac{11\pi}{12}$. **[2]**

(iii) Express $\frac{z}{w}$ in the form $x + \mathrm{i}y$, where x and y are real, giving the exact values of x and y in non-trigonometrical form. **[2]**

(iv) Hence, by considering the complex number $\frac{z}{w}$ on an Argand diagram, show that $\tan\left(\frac{5\pi}{12}\right) = 2 + \sqrt{3}$. **[2]**

8 The function f is defined by

$$\mathrm{f} : x \to x^2 + \lambda x + 7, \quad x \in \mathbb{R},\ x \le 3,$$

where λ is a constant.

(i) State the range of values that λ can take if f^{-1} exists. **[1]**

It is given that $\lambda = -6$.

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. **[3]**

(iii) Sketch on the same diagram the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$. **[3]**

(iv) Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$. Show algebraically that the solution to $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ satisfies the equation $x^2 - 7x + 7 = 0$.
Hence find the exact value of x that satisfies the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. **[3]**

9 A curve C has parametric equations

$$x = t^2, \quad y = 1 + 2t \quad \text{for } t > 0.$$

(i) Sketch C. **[2]**

(ii) Find the equations of the tangent and the normal to C at the point $P(p^2, 1+2p)$. **[4]**

(iii) The tangent and normal at P meet the y-axis at T and N respectively. Show that $\frac{PT^2}{TN} = p$. **[4]**

10 The curve C has equation $y = \frac{a(x-1)(x-2)}{2-3x}$.

(i) The curve C is scaled by a factor of 3 parallel to the x-axis to get the curve C'. Given that the point $(4,1)$ lies on C' show that $a = 9$. **[2]**

For the rest of the question, use $a = 9$.

(ii) Obtain the equations of the two asymptotes of C. **[2]**

(iii) Sketch C, stating the coordinates of any turning points and of the points where the curve crosses the axes. **[4]**

(iv) Without using a calculator, find the range of values of λ for which the line $y = 9x + \lambda$ and C have at least one point in common. **[3]**

11 The line l_1 passes through the point A, whose position vector is $-\mathbf{i} + 2\mathbf{j}$, and is parallel to the vector $\mathbf{i} + \mathbf{k}$. The line l_2 passes through the point B, whose position vector is $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, and is parallel to the vector $\mathbf{j} + \mathbf{k}$.

(i) Show that the lines l_1 and l_2 are skew. **[2]**

(ii) Find the position vector of the point N on l_2 such that AN is perpendicular to l_2. **[3]**

The plane Π contains l_2 and is perpendicular to AN.

(iii) Find a vector equation for Π in the form $\mathbf{r} = \mathbf{u} + \alpha\mathbf{v} + \beta\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are perpendicular vectors. **[3]**

(iv) The point X varies in such a way that the mid-point of AX is always in Π. Find a vector equation for the locus of X. Describe this locus and state its geometrical relationship with the plane Π. **[4]**

[Turn over

12 **(a)** By using the substitution $y = 2ux^2$, find the general solution of the differential equation $2x^2 \frac{\mathrm{d}y}{\mathrm{d}x} - 4xy + y^2 = 0$, where $x > 0$. **[4]**

(b) A glass of water is taken from a refrigerator and placed in a room where the temperature is a constant 32°C. As the water warms up, the rate of increase of its temperature θ°C after t minutes is proportional to the temperature difference $(32-\theta)$ °C. Initially the temperature of the water is 4°C and the rate of increase of the temperature is 2°C per minute.

By setting up and solving a differential equation, show that $\theta = 32 - 28\mathrm{e}^{-\frac{1}{14}t}$. **[6]**

(i) Find the time, to the nearest minute, it takes the water to reach a temperature of 20°C. **[1]**

(ii) State what happens to θ for large values of t. **[1]**

(iii) Sketch a graph of θ against t. **[2]**

******* **End of Paper** *******

RAFFLES INSTITUTION
2016 YEAR 6 PRELIMINARY EXAMINATION

MATHEMATICS PAPER 2 **9740/02**

Higher 2 **21 SEPTEMBER 2016**

Total Marks: 100 **3 hours**

Additional materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
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This document consists of **7** printed pages.

Section A: Pure Mathematics [40 Marks]

1 **(a)** Use the method of mathematical induction to prove that $\sum_{r=1}^{n} 4r^3 = \left[n(n+1)\right]^2$. **[4]**

(b) It is given that $\mathrm{f}(r) = r^4 + 2r^3 + 2r^2 + r$.

Show that $\mathrm{f}(r) - \mathrm{f}(r-1) = ar^3 + 2r$, where a is a real constant to be determined. Hence find a formula for $\sum_{r=1}^{n} r\left(2r^2 + 1\right)$. **[4]**

2 Referred to the origin *O*, points *A* and *B* have position vectors **a** and **b** respectively. Point *C* lies on *OA*, between *O* and *A*, such that *OC* : *CA* = 3 : 2. Point *D* lies on *OB*, between *O* and *B*, such that *OD* : *DB* = 1 : μ.

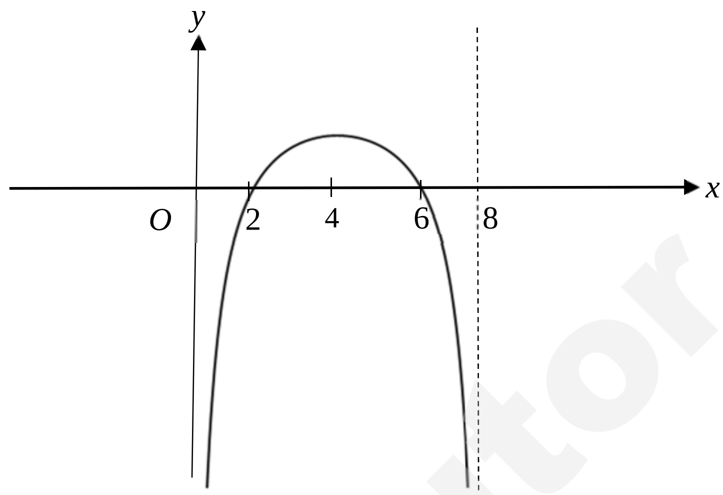
(i) It is given that the area of triangle *ABD* is twice the area of triangle *ABC*. Find μ. **[4]**

(ii) Show that the vector equation of the line *BC* can be written as $\mathbf{r} = \frac{3}{5}s\,\mathbf{a} + (1-s)\mathbf{b}$, where s is a parameter. By writing down the vector equation of the line *AD* in a similar form, in terms of a parameter *t*, find, in terms of **a** and **b**, the position vector of the point *E* where the lines *BC* and *AD* meet. **[3]**

It is further given that the angle *AOB* is $45°$ and *O* lies on the perpendicular bisector of the line segment *AB*.

(iii) Find the length of projection of **a** on **b**, giving your answer in terms of $|\mathbf{b}|$. Hence find the position vector of the foot of the perpendicular from *A* to *OB*. **[3]**

3 The diagram shows the curve with equation $y = 2 + \frac{24}{x(x-8)}$, for $0 \le x \le 8$. The curve crosses the x-axis at $x = 2$ and $x = 6$, has a maximum turning point at $x = 4$, and asymptotes at $x = 0$ and $x = 8$.



For $0 \le x \le 8$, the region bounded by the curves with equations $y = \left| 2 + \frac{24}{x(x-8)} \right|$ and $y = 2 + \frac{4}{x-8}$ is denoted by S.

(i) On the same diagram, for $0 \le x \le 8$, sketch the curves with equations

$$y = \left| 2 + \frac{24}{x(x-8)} \right| \text{ and } y = 2 + \frac{4}{x-8}.$$

Indicate clearly on your diagram the region S, the coordinates of the points of intersection of the two curves and the equation(s) of any asymptote(s). **[3]**

(ii) Find the exact area of S.
Express your answer in the form $A + B\ln 2 + C\ln 3 + D\ln 7$ where A, B, C and D are constants to be determined. **[5]**

(iii) Find the volume of the solid of revolution formed when S is rotated about the x-axis through $360°$. **[2]**

[Turn over

4 **(a)** Solve the equation

$$z^4 + 4\text{i} = 0,$$

giving the roots in the form $r\text{e}^{\text{i}\alpha}$, where $r > 0$ and $-\pi < \alpha \le \pi$. **[3]**

These roots correspond to four points on an Argand diagram. Identify the quadrilateral that has these four points as vertices. **[1]**

(b) The point A represents the fixed complex number a, which has modulus r and argument θ, where $0 < \theta < \dfrac{\pi}{2}$.

(i) On a single Argand diagram, sketch the loci

(a) $|z - a| = 2r$,

(b) $\arg(z + a) = \theta$,

making clear the relationship between the loci and the point A. **[3]**

(ii) Hence, or otherwise, find exactly the complex number(s) z that satisfy both equations in part **(i)**, giving your answer(s) in terms of r and θ. **[2]**

(iii) Given instead that $|z - a| \le 2r$ and $-\pi < \arg(z + a) \le \theta$, find exactly the minimum and maximum possible values of $|z - \text{i}a|$, giving your answers in terms of r. **[3]**

Sections B: Statistics [60 Marks]

5 The recreational committee of a large company is organizing a family day and would like to conduct a survey with 5% of its employees about their preferences for an outdoor or indoor based carnival as well as the activities involved.

Describe how the committee could obtain a sample using

(i) systematic sampling, **[2]**

(ii) quota sampling. **[2]**

6 A market stall sells rice in packets which have masses that are normally distributed. The stall owner claims that the mean mass of the packet of rice is at least 5 kg. Jane buys a random selection of 10 packets of rice from the stall. The 10 packets have masses, in kg, as follows:

4.9 4.7 5.1 4.8 4.5 5.3 5.0 4.8 4.6 5.2

Find unbiased estimates of the population mean and population variance of the mass of rice packets. **[2]**

A test, at λ% significance level, shows that there is insufficient evidence for Jane to doubt the stall owner. Find the set of possible values for λ. **[4]**

7 Simon owns a diecast car display case which has 4 shelves and 8 individual compartments on each shelf. Each of these compartments can only hold one diecast car. He arranged his collection of 32 different diecast cars in the display case.

Find the number of different selections that can be made by

(i) taking two cars, both from the same shelf, **[2]**

(ii) taking a total of six cars from the display case, **[1]**

(iii) taking a total of six cars from the display case with at least one from each shelf. **[3]**

8 **(a)** S and W are independent random variables with the distributions $\mathrm{N}(20, 25)$ and $\mathrm{N}(\mu, \sigma^2)$ respectively. It is known that $\mathrm{P}(W < 10) = \mathrm{P}(W > 13)$ and $\mathrm{P}(S > 2W) = 0.43$. Calculate the values of μ and σ correct to three significant figures. **[4]**

(b) A small hair salon has two hairstylists Joe and Joan attending to customers wanting an express haircut. For Joe, the time taken to attend to a customer follows a normal distribution with mean 10 minutes and standard deviation 42 seconds. For Joan, the time taken to attend to a customer follows a normal distribution with mean 10.2 minutes and standard deviation 45 seconds.

(i) Find the probability that among three randomly chosen customers attended to by Joe, one took more than 10.5 minutes while the other two each took less than 10 minutes. **[2]**

(ii) Joe and Joan each attended to two customers. Find the probability that the difference in the total time taken by Joe and Joan to attend to their two customers respectively is more than 3 minutes. State any assumption(s) that you have used in your calculation. **[4]**

[Turn over

9 The following table shows the marks (x) obtained in a mid-year examination and the marks (y) obtained in the year-end examination by a group of eight students. The year-end mark of the eighth student was accidentally deleted from the records after the marks were analyzed, and this is indicated by m below.

Mid-year mark (x)	70	31	68	73	46	78	79	55
Year-end mark (y)	80	39	70	80	48	94	98	m

It is given that the equation of the regression line of y on x is $y = 1.2x - 4$.
Show that $m = 59$. **[2]**

(i) Draw the scatter diagram for these values, labelling the axes clearly. Find the value of the product moment correlation coefficient between x and y. **[2]**

(ii) It is thought that a model of the form $\ln y = a + bx$ may also be a suitable fit to the data. Calculate least square estimates of a and b and find the value of the product moment correlation coefficient between x and $\ln y$. **[3]**

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of

$$y = 1.2x - 4 \text{ or } \ln y = a + bx$$

is the better model.

Hence, estimate the mark that a student who obtained a mark of 75 in the mid-year examination but was absent from the year-end examination would have obtained in the year-end examination. **[3]**

10 For events A and B, it is given that $\mathrm{P}(A) = \frac{5}{8}$ and $\mathrm{P}(B) = \frac{2}{3}$.

(i) Find the greatest and least possible values of $\mathrm{P}(A \cap B)$. **[2]**

It is given in addition that $\mathrm{P}(A' \mid B') = \frac{3}{8}$.

(ii) Find $\mathrm{P}(A' \cap B')$. **[1]**

(iii) Find $\mathrm{P}(A \cup B)$. **[2]**

(iv) Determine if A and B are independent events. **[3]**

(v) Given another event C such that $\mathrm{P}(C) = \frac{3}{8}$, $\mathrm{P}(A \cap C) = \mathrm{P}(B \cap C) = \frac{1}{4}$ and $\mathrm{P}(A \cup B \cup C) = \frac{11}{12}$, find $\mathrm{P}(A \cap B \cap C)$. **[2]**

11 A chocolate shop puts gift vouchers at random into 7% of all their packets of mini chocolates produced. A customer must collect 3 vouchers to exchange for a gift.

(i) Adeline buys 8 packets of the mini chocolates. Find the probability that she gets exactly 2 gift vouchers. **[2]**

(ii) Aileen buys 31 packets of the mini chocolates. Find the probability that she is able to exchange for at least one gift. **[2]**

(iii) Angelina and Angeline buy 60 packets of the mini chocolates altogether. Use a suitable approximation to estimate the probability of them being able to exchange for exactly two gifts. **[4]**

(iv) Ashley buys n packets of the mini chocolates. Given that she already has 2 unused vouchers from her previous purchase, find the value of n for which the probability of her being able to exchange for exactly one gift is greatest. **[3]**

(v) The shopkeeper observes that the number of gifts exchanged in a day has a mean of 10 and variance of 25. Estimate the number of gifts the shop needs to stock if there is to be no more than a 5% chance of running out of gifts in a 40-day period. **[3]**

********* End of Paper *********

Prelim Paper 1 Solutions

Qn.	Solution
1	$\mathrm{f}(x) = ax^3 + bx^2 + cx + d$ $\mathrm{f}'(x) = 3ax^2 + 2bx + c$ $\mathrm{f}''(x) = 6ax + 2b$ $\mathrm{f}''(0) = 0 + 2b = 0 \Rightarrow b = 0$ $\mathrm{f}(-1) = -27 \quad \Rightarrow \quad -a - \ c + d = -27$ $\mathrm{f}(2) = 27 \quad \Rightarrow \quad 8a + 2c + d = 27$ $\mathrm{f}'(2) = 0 \quad \Rightarrow \quad 12a + \ c \quad = 0$ Use GC to obtain $a = -2$, $c = 24$ and $d = -5$. So $\mathrm{f}(x) = -2x^3 + 24x - 5$.

Qn.	Solution
2(i)	$(1000 - 100) + (900 - 150) + (810 - 200) = 2260$
(ii)	**Solution 1** Find the least n when $\frac{1000(1 - 0.9^n)}{1 - 0.9} < \frac{n}{2}(2(100) + (n-1)50)$ $10000(1 - 0.9^n) < 25n(n+3)$

X	Y1	Y2
10	6513.2	3250
11	6861.9	3850
12	7175.7	4500
13	7458.1	5200
14	7712.3	5950
15	7941.1	6750
16	8147	7600
17	8332.3	8500
18	8499.1	9450
19	8649.1	10450
20	8784.2	11500

X=17

Least n=17,

Amount of water is $8332.3 - 7600 = 732$ (nearest ml)

OR Solution 2

After nth visit to Station A, (so $n-1$ visits to Station B) the amount of water in tank is

$$\frac{1000(1 - 0.9^n)}{1 - 0.9} - \frac{n-1}{2}(2(100) + (n-2)50)$$

$$= 10000(1 - 0.9^n) - 25(n-1)(n+2)$$

Amount that is to be removed at nth visit to Station B is $100 + (n-1)50 = 50(n+1)$

NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl

X	Y1	Y2
10	3813.2	550
11	3611.9	600
12	3325.7	650
13	2958.1	700
14	2512.3	750
15	1991.1	800
16	1397	850
17	732.28	900
18	-.9464	950
19	-800.9	1000
20	-1666	1050

X=17

Amount of water in the tank when machine stops is 732 ml.

OR Solution 3

Let u_n be the amount of water after nth visit to Station B

$u_n = u_{n-1} + 1000 \times 0.9^{n-1} - 50(n+1),\ u_0 = 0$

(i) $u_1 = 900,\ u_2 = 1650,\ u_3 = 2260$

(ii) From GC, $u_{16} = 546.98,\ u_{17} = -167.7$

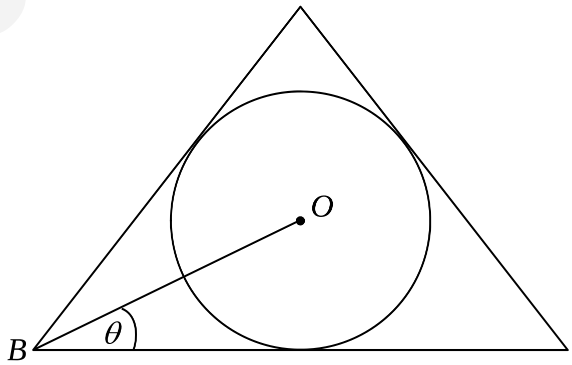
The required amount of water $= u_{16} + 1000 \times 0.9^{17-1} = 732\,\mathbf{ml}$

Qn.	Solution
3(a)(i)	Number of integers in rth set $= 1 + (r-1)2 = 2r - 1$
(ii)	Total number of integers in r sets $= \dfrac{1 + 2r - 1}{2}(r) = r^2$ Last integer in rth set $= 2 + (r^2 - 1)2 = 2r^2$
(iii)	$2(n-1)^2 < 2016 \le 2n^2$ $(n-1)^2 < 1008 \le n^2$ $n = 32$

Qn.	Solution
4(i)	$\int e^x \sin ax \, dx = e^x \sin ax - a\int e^x \cos ax \, dx$ $= e^x \sin ax - a\left(e^x \cos ax + a\int e^x \sin ax \, dx\right)$ $= e^x \sin ax - ae^x \cos ax - a^2 \int e^x \sin ax \, dx$ $(1 + a^2)\int e^x \sin ax \, dx = e^x \sin ax - ae^x \cos ax$ $\int e^x \sin ax \, dx = \dfrac{e^x(\sin ax - a\cos ax)}{1 + a^2} + c$ (shown) **OR:** $\int e^x \sin ax \, dx = -\dfrac{e^x}{a}\cos ax + \dfrac{1}{a}\int e^x \cos ax \, dx$ $= -\dfrac{e^x}{a}\cos ax + \dfrac{1}{a}\left(\dfrac{e^x}{a}\sin ax - \dfrac{1}{a}\int e^x \sin ax \, dx\right)$ $= -\dfrac{e^x}{a}\cos ax + \dfrac{e^x}{a^2}\sin ax - \dfrac{1}{a^2}\int e^x \sin ax \, dx$ $a^2\int e^x \sin ax \, dx = -ae^x \cos ax + e^x \sin ax - \int e^x \sin ax \, dx$

	$\left(1+a^2\right)\int e^x \sin ax \, dx = e^x \sin ax - ae^x \cos ax$ $\int e^x \sin ax \, dx = \dfrac{e^x\left(\sin ax - a\cos ax\right)}{1+a^2}+c$ (shown)
(ii)	$\sin 2x \cos x = \frac{1}{2}\left(\sin 3x + \sin x\right)$ i.e. $A=\frac{1}{2}, P=3, Q=1$ $\int e^x \sin 2x \cos x \, dx = \frac{1}{2}\int e^x\left(\sin 3x + \sin x\right) dx$ $= \frac{1}{2}\int\left(e^x \sin 3x + e^x \sin x\right) dx$ $= \frac{1}{2}\left[\dfrac{e^x\left(\sin 3x - 3\cos 3x\right)}{1+3^2}+\dfrac{e^x\left(\sin x - \cos x\right)}{1+1^2}\right]+c$ $= \dfrac{e^x}{2}\left(\dfrac{\sin 3x - 3\cos 3x}{10}+\dfrac{\sin x - \cos x}{2}\right)+c$ $= \dfrac{e^x}{20}\left(\sin 3x - 3\cos 3x + 5\sin x - 5\cos x\right)+c$

Qn.	Solution



5(i)	$\cos 2\theta = 2\cos^2\theta - 1$ $= 2\left(\dfrac{r}{\sqrt{r^2+a^2}}\right)^2 - 1 = \dfrac{2r^2}{r^2+a^2} - \dfrac{r^2+a^2}{r^2+a^2} = \dfrac{r^2-a^2}{r^2+a^2}$ [*shown*] *OR* $\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\dfrac{r}{\sqrt{r^2+a^2}}\right)^2 - \left(\dfrac{a}{\sqrt{r^2+a^2}}\right)^2 = \dfrac{r^2-a^2}{r^2+a^2}$ [*shown*]
(ii)	Let T be the total surface area of the cone.

$$T = \pi r l + \pi r^2$$

$$= \pi r\left(\frac{r}{\cos 2\theta}\right) + \pi r^2$$

$$= \pi r^2\left(\frac{r^2 + a^2}{r^2 - a^2} + 1\right) = \frac{2\pi r^4}{r^2 - a^2}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{8\pi r^3\left(r^2 - a^2\right) - 2\pi r^4\left(2r\right)}{\left(r^2 - a^2\right)^2}$$

$$= \frac{4\pi r^3\left(r^2 - 2a^2\right)}{\left(r^2 - a^2\right)^2} \quad or \quad \frac{4\pi r^3}{\left(r^2 - a^2\right)^2}\left(r - \sqrt{2}a\right)\left(r + \sqrt{2}a\right)$$

For $\frac{\mathrm{d}T}{\mathrm{d}r} = 0$, $r = \sqrt{2}a$ since $r > 0$

By the First Derivative test, since $\frac{4\pi r^3}{\left(r^2 - a^2\right)^2} > 0$, we have

r	$\left(\sqrt{2}a\right)^-$	$\left(\sqrt{2}a\right)$	$\left(\sqrt{2}a\right)^+$
r^2	$\left(2a^2\right)^-$	$\left(2a^2\right)$	$\left(2a^2\right)^+$
$r^2 - a^2$	0^-	0	0^+
$\frac{\mathrm{d}T}{\mathrm{d}r}$	-ve	0	+ve

Alternatively, by the Second Derivative Test,

$$\left.\frac{\mathrm{d}^2T}{\mathrm{d}r^2}\right|_{r=\sqrt{2}a} = 4\pi \left.\frac{(r^2 - a^2)^2(5r^4 - 6a^2r^2) - r^3(r^2 - 2a^2)[2(r^2 - a^2)(2r)]}{(r^2 - a^2)^4}\right|_{r=\sqrt{2}a}$$

$$= 4\pi\frac{(a^2)^2(20a^4 - 12a^4) - 0}{(a^2)^4} = 32\pi > 0$$

Hence $r = \sqrt{2}a$ gives minimum $T = \frac{2\pi\left(\sqrt{2}a\right)^4}{\left(\sqrt{2}a\right)^2 - a^2} = 8\pi a^2$.

Qn.	**Solution**
6(i)	

$$y = 2\cos\left(\frac{2}{3}\cos^{-1}x\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\sin\left(\frac{2}{3}\cos^{-1}x\right)\frac{2}{3}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\sqrt{1-x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{3}\sin\left(\frac{2}{3}\cos^{-1}x\right)$$

$$\sqrt{1-x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{1}{2\sqrt{1-x^2}}(-2x)\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4}{3}\cos\left(\frac{2}{3}\cos^{-1}x\right)\frac{2}{3}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{4}{9}y$$

OR "implicit differentiation"

$$\cos^{-1}\frac{y}{2}=\frac{2}{3}\cos^{-1}x$$

$$\frac{-1}{\sqrt{1-\left(\frac{y}{2}\right)^2}}\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{3}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\left(1-x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{16}{9}\left(1-\left(\frac{y}{2}\right)^2\right)$$

$$2\left(1-x^2\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)-2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{16}{9}\left(-\frac{y}{2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left(1-x^2\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-\frac{4}{9}y$$

At $x=0$, $y=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\sqrt{3}}{3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{4}{9}$

The series is $y=1+\frac{2\sqrt{3}}{3}x-\frac{2}{9}x^2+\cdots$

(ii)

$$(1+bx)^n=1+nbx+\frac{n(n-1)}{2}b^2x^2+\cdots$$

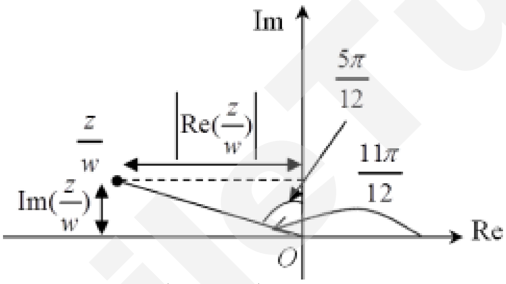
x: $\quad nb=\frac{2\sqrt{3}}{3} \quad ---(1)$

x^2: $\quad \frac{n(n-1)}{2}b^2=-\frac{2}{9} \quad ---(2)$

$\frac{(2)}{(1)}$: $\quad \frac{n-1}{2}b=-\frac{\sqrt{3}}{9} \quad ---(3)$

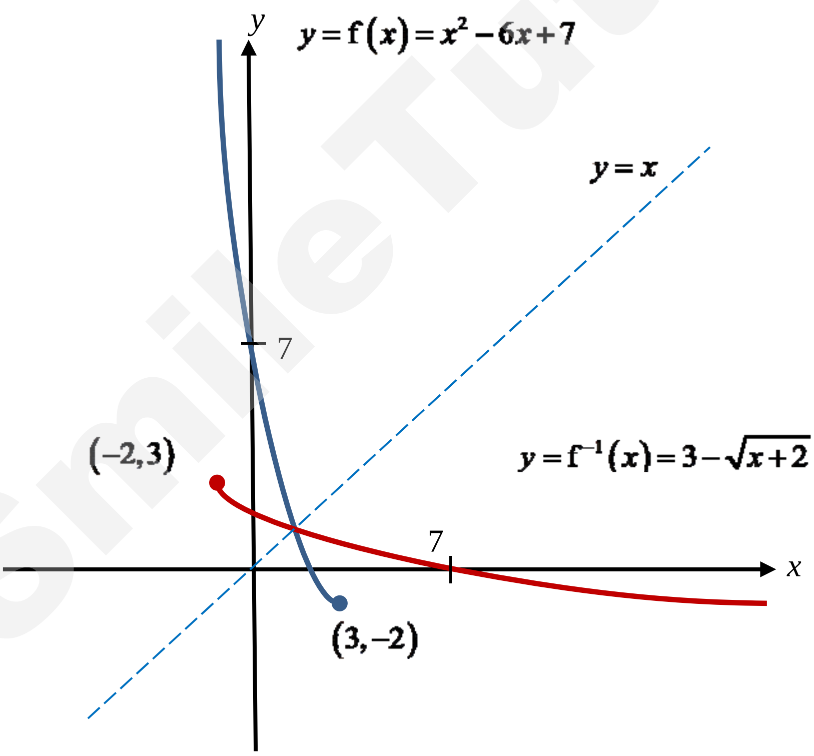
$\frac{(1)}{(3)}$: $\quad \frac{2n}{n-1}=-6 \quad \Rightarrow \quad n=\frac{3}{4}$

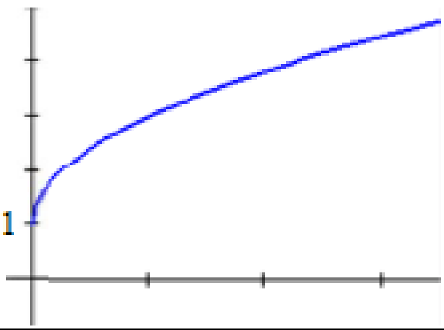
(1): $\quad b=\frac{2\sqrt{3}}{3}\cdot\frac{4}{3}=\frac{8\sqrt{3}}{9}$

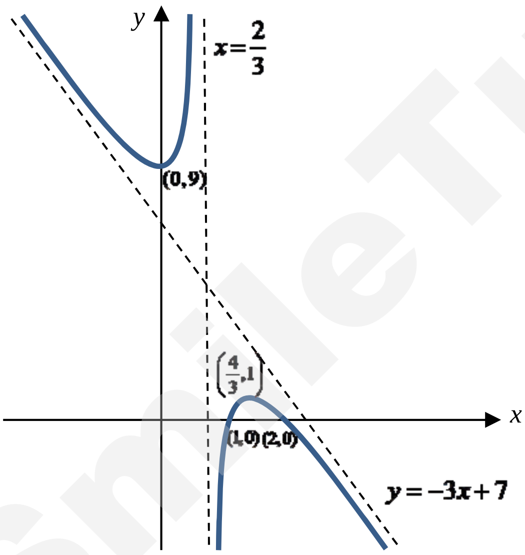
<table>
<tr><th>Qn.</th><th>Solution</th></tr>
<tr><td>7(i)</td><td>Given $\left|\frac{z}{w}\right|=1$,
$\frac{2}{\sqrt{2+c^2}}=1$
$c^2=2 \Rightarrow c=\sqrt{2}$ ($\because c$ is positive)</td></tr>
<tr><td>(ii)</td><td>$\arg\left(\frac{z}{w}\right)=\arg(z)-\arg(w)+2\pi$
$=-\frac{\pi}{3}-\frac{3\pi}{4}+2\pi$
$=\frac{11\pi}{12}$ (shown)</td></tr>
<tr><td>(iii)</td><td>$\frac{z}{w}=\frac{1-\mathrm{i}\sqrt{3}}{\sqrt{2}(-1+\mathrm{i})}\times\frac{(-1-\mathrm{i})}{(-1-\mathrm{i})}=\frac{-1-\sqrt{3}+\mathrm{i}(\sqrt{3}-1)}{2\sqrt{2}}$
$=\frac{-(\sqrt{3}+1)}{2\sqrt{2}}+\mathrm{i}\frac{(\sqrt{3}-1)}{2\sqrt{2}}$</td></tr>
<tr><td>(iv)</td><td>From the diagram,


$\tan\left(\frac{5\pi}{12}\right)=\frac{\left|\mathrm{Re}(\frac{z}{w})\right|}{\mathrm{Im}(\frac{z}{w})}$
$=\frac{\sqrt{3}+1}{2\sqrt{2}}\div\frac{\sqrt{3}-1}{2\sqrt{2}}$
$=\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$
$=\frac{4+2\sqrt{3}}{2}$
$=2+\sqrt{3}$ (shown)</td></tr>
</table>

Qn.	Solution
8(i)	$\mathrm{f}(x)=x^2+\lambda x+7$ $\mathrm{f}'(x)=2x+\lambda=0 \;\Rightarrow\; x=-\frac{\lambda}{2}$ is the x-coodinate of the min. point Hence, f^{-1} exists when $-\frac{\lambda}{2}\geq 3 \Rightarrow \lambda \leq -6$

	Alternative solution: $\mathrm{f}(x)=x^2+\lambda x+7=\left(x+\frac{\lambda}{2}\right)^2-\frac{\lambda^2}{4}+7$ Inverse of f exists when $-\frac{\lambda}{2}\geq 3 \Rightarrow \lambda\leq -6.$
(ii)	Given that $\lambda=-6$, $\mathrm{f}:x\to x^2-6x+7,\quad x\in\mathbb{R},\ x\leq 3$ $R_f=[-2,\infty)$ For $\mathrm{f}^{-1}(x)$, let $y=x^2-6x+7=(x-3)^2-2$ $\Rightarrow y+2=(x-3)^2 \Rightarrow x=3\pm\sqrt{y+2}$ Since $D_\mathrm{f}=(-\infty,3]=R_{\mathrm{f}^{-1}}$ $\mathrm{f}^{-1}(x)=3-\sqrt{x+2}$
(iii)	
(iv)	$y=x$ To solve $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, we can use $\mathrm{f}(x)=x$ with $x\leq 3$ $\Rightarrow x^2-6x+7=x \Rightarrow x^2-7x+7=0$ [AG] [shown] $x=\frac{7\pm\sqrt{49-28}}{2}$ $\because x\leq 3,\ x=\frac{7-\sqrt{21}}{2}$

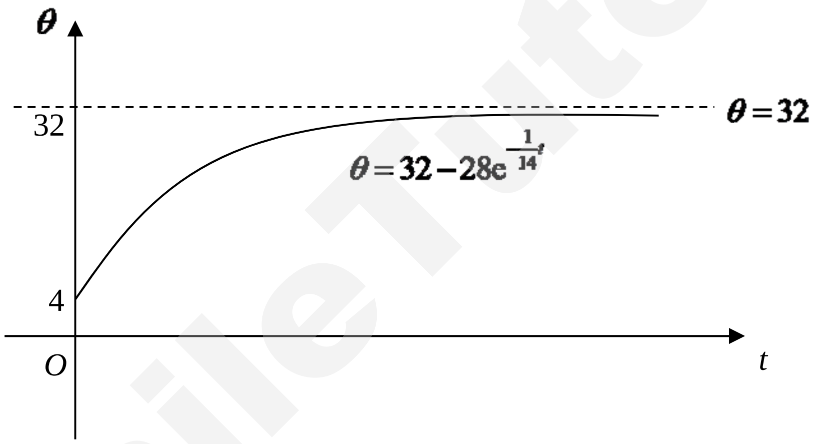
Qn.	Solution
9(i)	
(ii)	$x=t^2 \Rightarrow \frac{dx}{dt}=2t \qquad y=1+2t \Rightarrow \frac{dy}{dt}=2$ $\frac{dy}{dx}=\frac{dy}{dt}/\frac{dx}{dt}=\frac{2}{2t}=\frac{1}{t}$ Equation of tangent at $P(p^2,1+2p)$, $y-(1+2p)=\frac{1}{p}(x-p^2)$ $y-1-2p=\frac{1}{p}x-p$ $y=\frac{1}{p}x+1+p$ Equation of normal at $P(p^2,1+2p)$: $y-(1+2p)=-p(x-p^2)$ $y-1-2p=-px+p^3$ $y=-px+p^3+1+2p$
(iii)	Subs. $x=0$ into $y=\frac{1}{p}x+1+p$, we have $y=1+p$ $\therefore T$ is $(0,1+p)$ Subs. $x=0$ into $y=-px+p^3+1+2p, y=p^3+1+2p$ $\therefore N$ is $(0, p^3+1+2p)$ Given that P is $(p^2,1+2p)$ $PT^2=(p^2-0)^2+(1+2p-1-p)^2$ $\quad =p^4+p^2$ $TN=\lvert(p^3+1+2p)-(1+p)\rvert$ $\quad =\lvert p^3+p\rvert$ $\quad =p^3+p \quad (p\geq 0)$ $\frac{PT^2}{TN}=\frac{p^4+p^2}{p^3+p}=\frac{p^2(p^2+1)}{p(p^2+1)}=p \qquad$ (shown)

Qn.	Solution
10(i)	If the point $(4,1)$ lies on the curve which is C transformed by a stretch with scale factor 3 parallel to the x-axis, then $\left(\frac{4}{3},1\right)$ is on C. Subs. $x=\frac{4}{3}$ and $y=1$ into C, $1=\dfrac{a\left(\frac{4}{3}-1\right)\left(\frac{4}{3}-2\right)}{2-3\left(\frac{4}{3}\right)} \Leftrightarrow 1=\dfrac{a\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{-2}$ $a=9$
(ii)	$y=\dfrac{9x^2-27x+18}{2-3x}=-3x+7+\dfrac{4}{2-3x}$ Vertical asymptote, $x=\frac{2}{3}$ Oblique asymptote, $y=-3x+7$
(iii)	
(iv)	$\dfrac{9(x-1)(x-2)}{2-3x}=9x+\lambda$ $9x^2-27x+18=18x+2\lambda-27x^2-3\lambda x$ $36x^2+3(\lambda-15)x+2(9-\lambda)=0$ Since the line $y=9x+\lambda$ and C have at least one point in common, $b^2-4ac\geq 0$ $[3(\lambda-15)]^2-4(36)(2(9-\lambda))\geq 0$ $\lambda^2-30\lambda+225-288+32\lambda\geq 0$ $\lambda^2+2\lambda-63\geq 0$ $(\lambda+9)(\lambda-7)\geq 0$ $\lambda\leq -9$ or $\lambda\geq 7$

Qn.	Solution
11(i)	$l_1 : \mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ s \in \mathbb{R} \qquad l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ t \in \mathbb{R}$ If l_1 and l_2 intersect, $-1+s=1 \quad \ldots(1)$ $2=1+t \quad \ldots(2)$ $s=3+t \quad \ldots(3)$ from (1), s = 2 from (2), t = 1 but for (3), LHS = 2 $\neq$ 3 + 1 = 4 = RHS Hence l_1 and l_2 are non-intersecting. Since $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \neq k\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ for any $k \in \mathbb{R}$, l_1 and l_2 are non-parallel. Thus the lines l_1 and l_2 are skew.
(ii)	Since N is on l_2, $\overrightarrow{ON} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ for some $t \in \mathbb{R}$. $\overrightarrow{AN} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1+t \\ 3+t \end{pmatrix}$ $\overrightarrow{AN}.\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$ $\Rightarrow \begin{pmatrix} 2 \\ -1+t \\ 3+t \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$ $\Rightarrow -1+t+3+t=0$ $\Rightarrow t=-1$ $\therefore \overrightarrow{ON} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$
(iii)	Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

	$\mathbf{w}$ is perpendicular to $\mathbf{v}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ and also to $\overrightarrow{AN}=\begin{pmatrix}2\\-2\\2\end{pmatrix}$. Consider $\begin{pmatrix}0\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$. Let $\mathbf{w}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$. $\therefore \Pi:\mathbf{r}=\begin{pmatrix}1\\1\\3\end{pmatrix}+\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}+\beta\begin{pmatrix}2\\1\\-1\end{pmatrix},\quad \alpha,\beta\in\mathbb{R}$
(iv)	Let M be the mid-point of AX. By ratio theorem, $\overrightarrow{OM}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OX}\right)$. Since M lies on Π, $\overrightarrow{OM}=\begin{pmatrix}1\\1\\3\end{pmatrix}+\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}+\beta\begin{pmatrix}2\\1\\-1\end{pmatrix}$ for some $\alpha,\beta\in\mathbb{R}$. $\frac{1}{2}\left[\begin{pmatrix}-1\\2\\0\end{pmatrix}+\overrightarrow{OX}\right]=\begin{pmatrix}1\\1\\3\end{pmatrix}+\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}+\beta\begin{pmatrix}2\\1\\-1\end{pmatrix}$ $\overrightarrow{OX}=2\begin{pmatrix}1\\1\\3\end{pmatrix}+2\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}+2\beta\begin{pmatrix}2\\1\\-1\end{pmatrix}-\begin{pmatrix}-1\\2\\0\end{pmatrix}$ $=\begin{pmatrix}3\\0\\6\end{pmatrix}+h\begin{pmatrix}0\\1\\1\end{pmatrix}+k\begin{pmatrix}2\\1\\-1\end{pmatrix}$ The equation of the locus of X is $\mathbf{r}=\begin{pmatrix}3\\0\\6\end{pmatrix}+h\begin{pmatrix}0\\1\\1\end{pmatrix}+k\begin{pmatrix}2\\1\\-1\end{pmatrix}, h,k\in\mathbb{R}$ The locus of X is a **plane** parallel to Π.

Qn.	Solution
12(a)	$y = 2ux^2$ ———————————(1) $\frac{\mathrm{d}y}{\mathrm{d}x} = 4ux + 2x^2 \frac{\mathrm{d}u}{\mathrm{d}x}$ —————(2) Substituting (1) and (2) into $2x^2 \frac{\mathrm{d}y}{\mathrm{d}x} - 4xy + y^2 = 0$, we have $2x^2\left(4ux + 2x^2 \frac{\mathrm{d}u}{\mathrm{d}x}\right) - 4x\left(2ux^2\right) + \left(2ux^2\right)^2 = 0$ $8ux^3 + 4x^4 \frac{\mathrm{d}u}{\mathrm{d}x} - 8ux^3 + 4u^2x^4 = 0$ $\frac{\mathrm{d}u}{\mathrm{d}x} = -u^2$ $\int -\frac{1}{u^2}\ \mathrm{d}u = \int\ \mathrm{d}x$ $\frac{1}{u} = x + c$ $u = \frac{1}{x+c}$ $\frac{y}{2x^2} = \frac{1}{x+c}$ $y = \frac{2x^2}{x+c}$
(b)	$\frac{\mathrm{d}\theta}{\mathrm{d}t} = k\left(32-\theta\right),\ k > 0$ $\int \frac{1}{32-\theta} \mathrm{d}\theta = \int k\ \mathrm{d}t$ $-\ln\left(32-\theta\right) = kt + A$ since $\theta < 32$ $\ln\left(32-\theta\right) = -kt - A$ $\left(32-\theta\right) = \mathrm{e}^{-kt-A}$ $32-\theta = \mathrm{e}^{-kt}\mathrm{e}^{-A} = C\mathrm{e}^{-kt}$ where $C = \mathrm{e}^{-A}$ $\theta = 32 - C\mathrm{e}^{-kt}$ Given that $\theta = 4, \frac{\mathrm{d}\theta}{\mathrm{d}t} = 2$ when $t = 0$ $\theta = 32 - C\mathrm{e}^{-kt} \Rightarrow 4 = 32 - C \Rightarrow C = 28$ $\frac{\mathrm{d}\theta}{\mathrm{d}t} = k\left(32-\theta\right) \Rightarrow 2 = k\left(32-4\right) \Rightarrow k = \frac{1}{14}$ $\therefore \theta = 32 - 28\mathrm{e}^{-\frac{1}{14}t}$ (shown)

<table>
<tr><td>(i)</td><td>$\theta = 32 - 28\mathrm{e}^{-\frac{1}{14}t}$
At $\theta = 20$,
$20 = 32 - 28\mathrm{e}^{-\frac{1}{14}t}$
$28\mathrm{e}^{-\frac{1}{14}t} = 12$
$-\frac{1}{14}t = \ln\frac{3}{7}$
$t = -14\ln\frac{3}{7} = 11.86 \approx 12$ min</td></tr>
<tr><td>(ii)</td><td>As $t \to \infty$, $\mathrm{e}^{-\frac{1}{14}t} \to 0$. $\therefore \theta \to 32$.
i.e. the temperature of the water increases and approaches the room temperature, i.e. $32^{\circ}\mathrm{C}$, for large values of t.</td></tr>
<tr><td>(iii)</td><td>
</td></tr>
</table>

Prelim Paper 2 Solutions

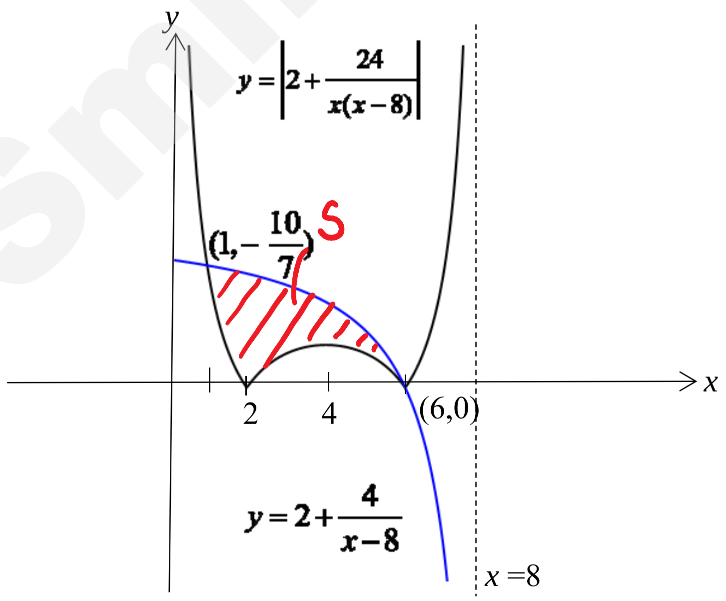
Pure Mathematics [40 Marks]

Qn.	Solution
1(a)	Let P_n be the statement $\sum_{r=1}^{n} 4r^3 = \left[n(n+1)\right]^2$ for $n \in \mathbb{Z}^+$. When $n=1$, LHS = 4 and RHS=$\left[1\times 2\right]^2 = 4$ $\therefore$ P_1 is true. Assume P_k is true for some $k \in \mathbb{Z}^+$. i.e. $\sum_{r=1}^{k} 4r^3 = \left[k(k+1)\right]^2$. To prove that P_{k+1} is true, i.e. $\sum_{r=1}^{k+1} 4r^3 = \left[(k+1)(k+2)\right]^2$. $\sum_{r=1}^{k+1} 4r^3 = \sum_{r=1}^{k} 4r^3 + 4(k+1)^3$ $= \left[k(k+1)\right]^2 + 4(k+1)^3$ $= (k+1)^2\left[k^2 + 4(k+1)\right]$ $= (k+1)^2\left(k^2+4k+4\right)$ $= (k+1)^2(k+2)^2$ $= \left[(k+1)(k+2)\right]^2$ $\therefore$ P_{k+1} is true. Hence P_k is true $\Rightarrow P_{k+1}$ is true, and since P_1 is true, by Mathematical Induction, P_n is true for **all** $n \in \mathbb{Z}^+$.
(b)	$f(r) - f(r-1) = \left(r^4 + 2r^3 + 2r^2 + r\right)$ $-\left[(r-1)^4 + 2(r-1)^3 + 2(r-1)^2 + (r-1)\right]$ $= r^4 + 2r^3 + 2r^2 + r$ $-r^4 + 4r^3 - 6r^2 + 4r - 1$ $-2r^3 + 6r^2 - 6r + 2$ $-2r^2 + 4r - 2$ $-r + 1$ $= 4r^3 + 2r$, $a = 4$ $\sum_{r=1}^{n} r\left(2r^2+1\right) = \frac{1}{2}\sum_{r=1}^{n}\left(4r^3+2r\right)$

	$=\frac{1}{2}\sum_{r=1}^{n}\left(\mathrm{f}(r)-\mathrm{f}(r-1)\right)$ $=\frac{1}{2}\begin{bmatrix} \mathrm{f}(1)-\mathrm{f}(0) \\ +\mathrm{f}(2)-\mathrm{f}(1) \\ +\mathrm{f}(3)-\mathrm{f}(2) \\ \vdots \\ \vdots \\ +\mathrm{f}(n-1)-\mathrm{f}(n-2) \\ +\mathrm{f}(n)-\mathrm{f}(n-1) \end{bmatrix}$ $=\frac{1}{2}\left(\mathrm{f}(n)-\mathrm{f}(0)\right)$ $=\frac{n^4+2n^3+2n^2+n}{2}$ *OR* $\frac{n(n+1)(n^2+n+1)}{2}$

Qn.	Solution
2(i)	Area of triangle $ABC=\frac{1}{2}\left\lvert\overrightarrow{CA}\times\overrightarrow{AB}\right\rvert=\frac{1}{2}\left\lvert\frac{2}{5}\mathbf{a}\times(\mathbf{b}-\mathbf{a})\right\rvert$ $=\frac{1}{5}\lvert\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{a}\rvert$ $=\frac{1}{5}\lvert\mathbf{a}\times\mathbf{b}\rvert$ Area of triangle $ABD=\frac{1}{2}\left\lvert\overrightarrow{DB}\times\overrightarrow{AB}\right\rvert=\frac{1}{2}\left\lvert\frac{\mu}{1+\mu}\mathbf{b}\times(\mathbf{b}-\mathbf{a})\right\rvert$ $=\frac{\mu}{2(1+\mu)}\lvert\mathbf{b}\times\mathbf{b}-\mathbf{b}\times\mathbf{a}\rvert$ since $\mu>0$ $=\frac{\mu}{2(1+\mu)}\lvert\mathbf{a}\times\mathbf{b}\rvert$ Area of triangle *ABD* = 2 (area of triangle *ABC*) $\frac{\mu}{2(1+\mu)}\lvert\mathbf{a}\times\mathbf{b}\rvert=\frac{2}{5}\lvert\mathbf{a}\times\mathbf{b}\rvert$ $5\mu=4(1+\mu)$ $\mu=4$
(ii)	Line BC: $\mathbf{r}=\overrightarrow{OB}+s\overrightarrow{BC}$, $s\in\mathbb{R}$ $=\mathbf{b}+s\left(\overrightarrow{OC}-\overrightarrow{OB}\right)$, $s\in\mathbb{R}$ $=\mathbf{b}+s\left(\frac{3}{5}\mathbf{a}-\mathbf{b}\right)$, $s\in\mathbb{R}$ $=\frac{3}{5}s\,\mathbf{a}+(1-s)\mathbf{b}$, $s\in\mathbb{R}$ Line AD: $\mathbf{r}=\overrightarrow{OA}+t\overrightarrow{AD}$, $t\in\mathbb{R}$

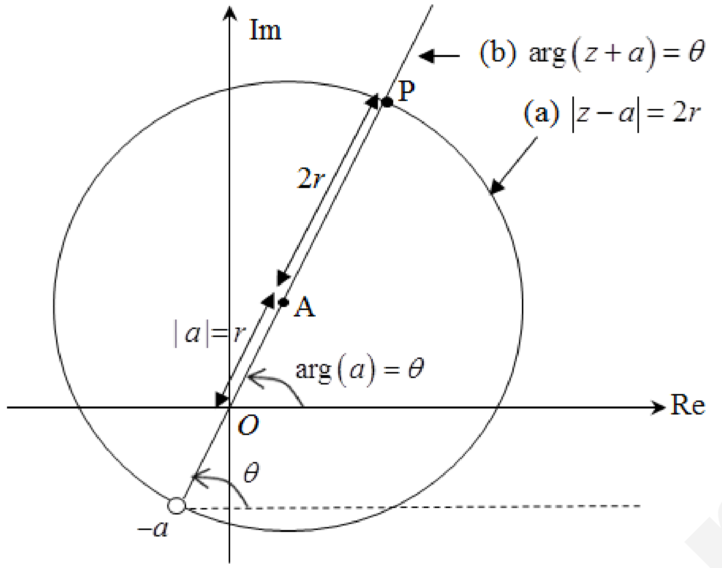
	$= \mathbf{a} + t\left(\overrightarrow{OD} - \overrightarrow{OA}\right)$, $t \in \mathbb{R}$ $= \mathbf{a} + t\left(\frac{1}{5}\mathbf{b} - \mathbf{a}\right)$, $t \in \mathbb{R}$ $= \frac{1}{5}t\ \mathbf{b} + (1-t)\mathbf{a}$, $t \in \mathbb{R}$ At point of intersection E, $\frac{3}{5}s\ \mathbf{a} + (1-s)\mathbf{b} = (1-t)\mathbf{a} + \frac{1}{5}t\ \mathbf{b}$ $\frac{3}{5}s = 1-t$ ...(1) $1-s = \frac{1}{5}t$ ...(2) Solving, $s = \frac{10}{11}$, $t = \frac{5}{11}$ and $\overrightarrow{OE} = \frac{3}{5}\left(\frac{10}{11}\right)\mathbf{a} + \left(1 - \frac{10}{11}\right)\mathbf{b} = \frac{6}{11}\mathbf{a} + \frac{1}{11}\mathbf{b}$
(iii)	Length of projection of $\mathbf{a}$ on $\mathbf{b}$ = $\left\lvert \mathbf{a} \cdot \hat{\mathbf{b}} \right\rvert$ $= \lvert\mathbf{a}\rvert \lvert\hat{\mathbf{b}}\rvert \cos 45°$ $= \lvert\mathbf{a}\rvert \cos 45°$ $= \frac{1}{\sqrt{2}}\lvert\mathbf{b}\rvert$ $\overrightarrow{OF} = \left(\frac{1}{\sqrt{2}}\lvert\mathbf{b}\rvert\right)\hat{\mathbf{b}} = \frac{1}{\sqrt{2}}\mathbf{b}$

Qn.	Solution
3(i) **[3]**	

(ii)	Area of S $=\int_1^2 \left(2+\frac{4}{x-8}\right)-\left[-\left(2+\frac{24}{x(x-8)}\right)\right]\ \mathrm{d}x$ $+\int_2^6 \left(2+\frac{4}{x-8}\right)-\left[2+\frac{24}{x(x-8)}\right]\ \mathrm{d}x$ $=\int_1^2 \left(2+\frac{4}{x-8}\right)-\left[-\left(2+\frac{3}{x-8}-\frac{3}{x}\right)\right]\ \mathrm{d}x$ $+\int_2^6 \left(2+\frac{4}{x-8}\right)-\left[2+\frac{3}{x-8}-\frac{3}{x}\right]\ \mathrm{d}x$ $=\int_1^2 \left(4+\frac{7}{x-8}-\frac{3}{x}\right)\ \mathrm{d}x \ +\int_2^6 \left[\frac{1}{x-8}+\frac{3}{x}\right]\ \mathrm{d}x$ $=\left[4x+7\ln\lvert x-8\rvert-3\ln\lvert x\rvert\right]_1^2+\left[\ln\lvert x-8\rvert+3\ln\lvert x\rvert\right]_2^6$ $=\left[4+7\ln 6-7\ln 7-3\ln 2\right]+\left[\ln 2-\ln 6+3\ln 6-3\ln 2\right]=4+4\ln 2+9\ln 3-7\ln 7$ $\therefore A=4,\ B=4,\ C=9,\ D=-7$
(iii)	Required Volume $=\pi\int_1^6 \left(2+\frac{4}{x-8}\right)^2-\left(2+\frac{24}{x(x-8)}\right)^2\ \mathrm{d}x$ $\approx 4.63989\pi=14.577\text{ units}^2\ (5\text{s.f})=14.6\text{ units}^2\ (3\text{s.f})$

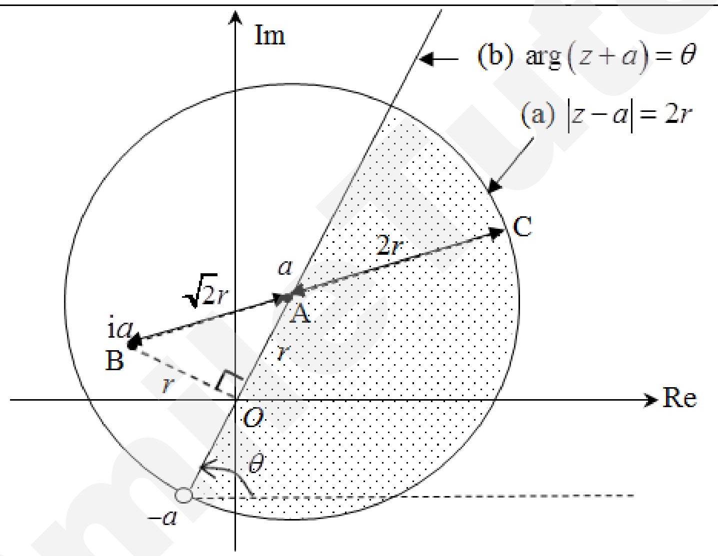
Qn.	Solution
4(a) **]**	$z^4=-4\mathrm{i}=4\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\times\mathrm{e}^{2k\pi\mathrm{i}}$ where $k\in\mathbb{Z}$ By De Moivre's Theorem, $z=\left[4\mathrm{e}^{\left(\frac{4k-1}{2}\right)\pi\mathrm{i}}\right]^{\frac{1}{4}}$ $=\sqrt{2}\mathrm{e}^{\left(\frac{4k-1}{8}\right)\pi\mathrm{i}},\ \ k=0,\pm1,2$ The quadrilateral is a square.

(b)(i)



(ii) At point of intersection of the loci, P, $z=3re^{i\theta}$ or $3r(\cos\theta+i\sin\theta)$

(iii)



Minimum $|z-\mathrm{i}a|=BO=r$

Maximum $|z-\mathrm{i}a|=BC=BA+AC=\sqrt{r^2+r^2}+2r=\left(\sqrt{2}+2\right)r$

Qn.	Solution
5(i)	Note: For the interval number, k

$$k=\frac{\text{total population}}{\text{Sample Size}}=\frac{\text{total population}}{5\%\text{ of total population}}=\frac{1}{5\%}=\frac{1}{5/100}=20$$

Use a computer to generate a random number from 1 to 20, say 5. Select the 5th employee to arrive at the office, thereafter every 20th employee that arrives, ie 5th, 25th, 45th, etc.

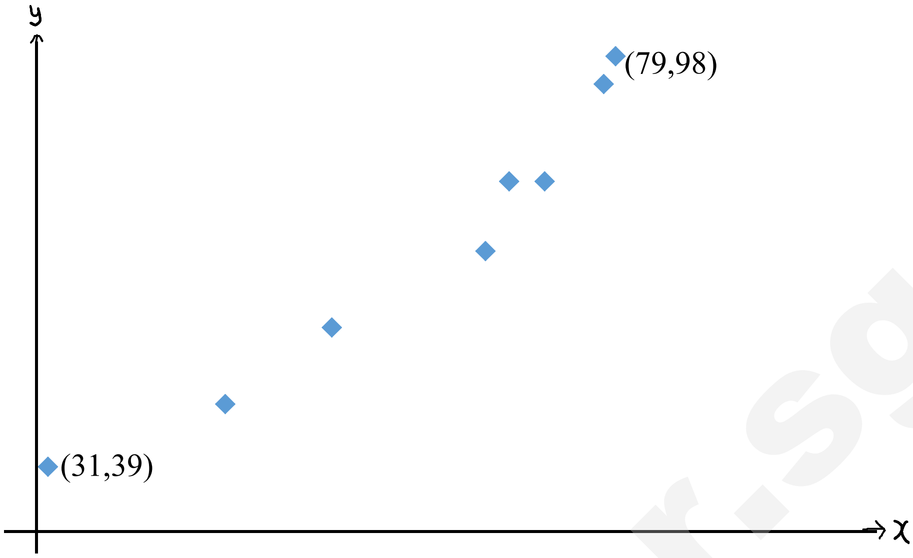
(ii) Any reasonable choice of strata: Age/Gender/Marital status etc.

Strata	1	2	3	total
Sample Size (n)	x	y	z	x+y+z = 5% of population

	Choose the individuals from each strata according to some convenient/non-random scheme such as the first *x*, *y*, *z* individuals committee encounters belong to strata 1,2,3 respectively such that sample size, *x*+*y*+*z* is 5% of the population. OR Choose the first "n" employee arrived at the company on a particular morning from each stratum as shown in the table.
Qn.	**Solution**
6 **[**	From GC, unbiased estimate of population mean, $\bar{x} = 4.89$ unbiased estimate of population variance, $s^2 = 0.2601281735^2 \left(or \ \frac{203}{3000} \right)$ $H_0: \mu = 5$ $H_1: \mu < 5$ Perform 1-tail t-test at $\lambda\%$ significance level Under H_0, Test statistic , $T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)$ where $\mu_0 = 5$ and $n = 10$ For the sample, $\bar{x} = 4.89$, $s = 0.2601281735$. Using a t-test, p-value $= 0.1069800566$ Since there is insufficient evidence for Jane to doubt the stall owner, we Do Not reject H_0, $p\text{-value} > \lambda\% = \frac{\lambda}{100} \Rightarrow \lambda < 10.7$ (3sf)
Qn.	**Solution**
7(i)	No. of selections = ${}^4C_1 \times {}^8C_2 = 112$
(ii)	No. of selections = ${}^{32}C_6 = 906192$
(iii)	No. of selections = $6 \times \left({}^8C_1\right)^2 \left({}^8C_2\right)^2 + 4 \times \left({}^8C_1\right)^3 \left({}^8C_3\right)$ $= 301056 + 114688$ $= 415744$ Alternative Solution **(I) By complement** No. of selections if All from 1 shelf = ${}^8C_4 \times 4 = 112$ All from 2 shelves = ${}^4C_2[\left({}^8C_3\right)^3 + ({}^8C_2 \times {}^8C_4 \times 2) + ({}^8C_1 \times {}^8C_5 \times 2)]$ All from 3 shelves = ${}^4C_3[\left({}^8C_2\right)^3 + (({}^8C_1)^2 \times {}^8C_4 \times 3) + ({}^8C_1 \times {}^8C_2 \times {}^8C_3 \times 3!)]$

	Answer = $906192-112-47712-442624=415744$ **(II) By Inclusion-Exclusion Principle** No. of selections if No cars are taken from 1 of the 4 shelves = ${}^{4}C_1 \times {}^{24}C_6 = 538384$ No cars are taken from 2 of the 4 shelves = ${}^{4}C_2 \times {}^{16}C_6 = 48048$ No cars are taken from 3 of the 4 shelves = 112 Answer = $906192-538384+48048-112=415744$

Qn.	Solution
8(a)	Since $\mathrm{P}(W<10)=\mathrm{P}(W>13)$, $\mu=\dfrac{10+13}{2}=11.5$. $2W-S \sim \mathrm{N}(3,\ 25+4\sigma^2)$ $\mathrm{P}(S>2W)=0.43$ $\mathrm{P}(2W-S<0)=0.43$ $\mathrm{P}(Z<\dfrac{0-3}{\sqrt{25+4\sigma^2}})=0.43$ From GC, $\dfrac{-3}{\sqrt{25+4\sigma^2}}=-0.176374$ Solving, $\sigma=8.13$.
8(b) (i)	Let X be the time taken by Joe to attend to a customer. $X \sim \mathrm{N}(10, 0.7^2)$ Required probability = ${}^{3}C_2 \times \left[\mathrm{P}(X<10)\right]^2 \mathrm{P}(X>10.5)=0.178$
(ii)	Let Y be the time taken by Joan to attend to a customer. $Y \sim \mathrm{N}(10.2, 0.75^2)$ $(X_1+X_2)-(Y_1+Y_2) \sim \mathrm{N}(-0.4, 2.105)$ $\mathrm{P}(\lvert (X_1+X_2)-(Y_1+Y_2)\rvert>3)$ $=1-\mathrm{P}(-3<(X_1+X_2)-(Y_1+Y_2)<3)$ = 0.0461 The following answers are accepted: (i) The time taken by Joe to attend to a customer is independent of the time taken by Joan to attend to a customer. (ii) The time taken by Joe to attend to one customer is independent of the time taken by Joe to attend to another customer.
Qn.	**Solution**
9	$\sum x=500,\ \sum y=509+m,\ n=8$ Since $(\bar{x},\bar{y})$ lies on the regression line of y on x, $\bar{y}=1.2\bar{x}-4$ $\dfrac{509+m}{8}=1.2\left(\dfrac{500}{8}\right)-4$ $m=59$

(i)	 From GC, $r = 0.96826 \approx 0.968$
(ii)	From GC, $\ln y = 3.0351 + 0.018945x$ $a = 3.0351 \approx 3.04$ $b = 0.018945 \approx 0.0189$ $r = 0.990499 \approx 0.990$
(iii)	The scatter plot of x and y shows a non-linear relationship as when x increases, y appears to be increasing at an increasing rate. Since the product moment correlation coefficient between x and $\ln y$ of 0.990 is closer to 1 than the product moment correlation coefficient between x and y of 0.968, $\ln y = a + bx$ is the better model.
	When $x = 75$, $\ln y = 3.0351 + 0.018945(75) = 4.455975$ $y = 86.140$ The student would have obtained 86.1 marks in the year-end examination.

Qn.	Solution
10(i)	$\mathrm{P}(A) + \mathrm{P}(B) - 1 \le \mathrm{P}(A \cap B) \le \min\{\mathrm{P}(A), \mathrm{P}(B)\}$ $\frac{7}{24} \le \mathrm{P}(A \cap B) \le \frac{5}{8}$ Greatest value of $\mathrm{P}(A \cap B)$ is $\frac{5}{8}$. Least value of $\mathrm{P}(A \cap B)$ is $\frac{7}{24}$.
(ii)	$\mathrm{P}(A' \cap B') = \mathrm{P}(A' \mid B')\mathrm{P}(B') = \frac{3}{8} \times \frac{1}{3} = \frac{1}{8}$
(iii)	$\mathrm{P}(A \cup B) = 1 - \mathrm{P}(A' \cap B') = \frac{7}{8}$

(iv)	$P(A\cup B)=P(A)+P(B)-P(A\cap B)$ $\frac{7}{8}=\frac{5}{8}+\frac{2}{3}-P(A\cap B)$ $P(A\cap B)=\frac{5}{12}$ $=\frac{5}{8}\times\frac{2}{3}=P(A)P(B)$ Hence, A and B are independent events. *OR* $\because P(A'\mid B')=\frac{3}{8}=1-\frac{5}{8}=P(A')$ A' and B' are independent events $\Rightarrow A$ and B are independent events
(v)	$P(A\cup B\cup C)=P(A)+P(B)+P(C)$ $-P(A\cap B)-P(A\cap C)-P(B\cap C)$ $+P(A\cap B\cap C)$ $\frac{11}{12}=\frac{5}{8}+\frac{2}{3}+\frac{3}{8}-\frac{5}{12}-\frac{1}{4}-\frac{1}{4}+P(A\cap B\cap C)$ $P(A\cap B\cap C)=\frac{1}{6}$

Qn. [	Solution
11 (i)	Let X denote the number of gift vouchers obtained (out of 8). $X \sim B(8, 0.07)$ $P(X = 2) = 0.0888$ (3 sf)
(ii)	Let Y denote the number of gift vouchers obtained (out of 31). $Y \sim B(31, 0.07)$ $P(Y \geq 3) = 1 - P(Y \leq 2) = 0.371$ (3 sf)
(iii)	Let V denote the number of gift vouchers obtained (out of 60). $V \sim B(60, 0.07)$ Since $n = 60$ is large, $p = 0.07$ is small such that $np = 4.2 < 5$, $V \sim Po(4.2)$ approximately $P(6 \leq V \leq 8) = P(V \leq 8) - P(V \leq 5) = 0.219$ (3 sf)
(iv)	Let W denote the number of gift vouchers obtained (out of n). $W \sim B(n, 0.07)$ For the condition of the question to be satisfied, the probability is $p = P(1 \leq W \leq 3)$ From GC, when $n = 25$, $p = 0.74343$ when $n = 26$, $p = 0.74365$ when $n = 27$, $p = 0.74250$ $\therefore$ Value of n for the probability p to be maximum = 26.

(v)	Let T denote the number of gifts exchanged in a day. $E(T)=10, \ \ Var(T)=25$ Let $T_1, T_2,...,T_{40}$ be a random sample from distribution X $n = 40$ large, by the Central Limit Theorem, $T_1+T_2+\cdots+T_{40} \sim N\left(40(10),40(25)\right)$ approximately $\mathrm{P}\left(T_1+T_2+\cdots+T_{40} > \mathrm{k}\right) \le 0.05$ From GC, $\mathrm{k} \ge 452.01$ Least $\mathrm{k} = 453$. The shop needs to stock at least 453 gifts.

River Valley High School 2016 H2 Maths Preliminary Examination - Paper 1

1. **(i)** The first three terms of a sequence are given by $u_1 = -4.4$, $u_2 = -4.1$ and $u_3 = -2.6$. Given that u_n is a quadratic polynomial in *n*, find u_n in terms of *n*. [3]

(ii) Find the set of values of *n* for which u_n is more than 15. [2]

2. Let $\mathrm{f}(x) = \dfrac{3x}{(x+4)(x+1)}$.

(i) Express f(*x*) in partial fractions. [1]

(ii) Find the series expansion, in ascending powers of *x*, of f(*x*) up to and including the term x^3 . State the values of *x* for which this expansion is valid. [3]

(iii) The first two terms of the expansion of f(*x*) are identical to that of the expansion of $ax(1-bx)^{-\frac{1}{2}}$. Find the exact values of *a* and *b*. [2]

3. **(a)** Differentiate e^{-x^2} with respect to *x*. Hence find $\int x^3 \mathrm{e}^{-x^2}\,\mathrm{d}x$. [4]

(b) Find the exact value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \left|\sin x\right| \mathrm{d}x$. [3]

4. On the first of January 2011, John opened a new special savings account and deposited \$*x* into his account. On the first day of each subsequent month, he would deposit another \$*x* into the account. A compound interest of 0.3% per month would be paid at the end of each month.

(i) Given that $x = 600$, find the total amount of money in the savings account on first of January 2013 just before John deposited money into his account. [3]

On first January 2013, instead of going to the bank to make a deposit into his savings account, John went to a yatch club and decided to purchase a yatch. For the yatch that he was interested to buy, he had to make a down payment of \$23000.

(ii) What should his minimum monthly deposit be (in multiples of \$10) so that his savings account would have sufficient amount for him to make the down payment? [2]

(iii) Assume that upon making the withdrawal for the down payment, there was \$190 left in his savings account. John transferred this amount from the savings account to a checking account which charges no interest. With the \$190 as the initial savings, he decided to make a monthly contribution of 90% of the amount he contributed in the previous month at the start of every subsequent month to the checking account. He claimed that he would be able to save \$2000 eventually. Do you agree with him? Explain. [3]

5. With reference to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively and $\mathbf{b}$ is a unit vector.

(i) Give a geometric description of $|\mathbf{b}\cdot\mathbf{a}|$. [1]

(ii) Given that S denotes the area of triangle OAB, show that $4S^2 = |\mathbf{a}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$. [3]

(iii) Given that C is a point on the line AB with position vector $\mathbf{c}$, explain why $(\mathbf{c}-\mathbf{a})\times(\mathbf{c}-\mathbf{b}) = \mathbf{0}$. [2]

(iv) Suppose $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and that the angle between $\mathbf{c}$ and $\mathbf{b}$ is $30°$, find in terms of $|\mathbf{a}|$ and $|\mathbf{c}|$, the ratio of the area of triangle OAB to that of triangle OCB. [3]

6. **(a)** The complex numbers z and w are such that $z = \dfrac{3a-5\mathrm{i}}{1+2\mathrm{i}}$ and $w = 1+13b\mathrm{i}$, where a and b are real numbers. Given that $z^* = w$, find the exact values of a and b. [4]

(b) Without using a graphic calculator, find the modulus and argument of the complex number $z = \dfrac{(1-\mathrm{i})^2}{(-1+\sqrt{3}\mathrm{i})^4}$, giving your answers in exact form. Hence evaluate z^6 exactly. [5]

7. A sequence u_1, u_2, u_3, … is such that $u_0 = 1$ and $u_{n+1} = u_n - \dfrac{2}{(2n+1)(2n+3)}$, for $n \geq 0$.

(i) Prove by induction that $u_n = \dfrac{1}{2n+1}$. [4]

(ii) Find $\displaystyle\sum_{n=1}^{N} \frac{1}{(2n+1)(2n+3)}$ in terms of N. [3]

(iii) Hence find $\displaystyle\sum_{n=0}^{N}\frac{1}{(2n+3)(2n+5)}$. [2]

(iv) State the value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}$. [1]

8. In a certain fish farm, the growth of the population of garoupa is studied. The population of garoupa at time t days is denoted by x (in thousands). It was found that the rate of birth per day is twice of x, and the rate of death per day is proportional to x^2.

(i) Given that there is no change in the population of garoupa when its population hits 10000, write down a differential equation relating $\frac{\mathrm{d}x}{\mathrm{d}t}$ and x. [2]

(ii) Its owner decides to sell away 1800 garoupa daily. Modify the differential equation in part **(i)** and show that the resulting differential equation can be written as $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{5}\left[(x-5)^2-a^2\right]$, where a is a constant to be determined. [2]

Given that the initial population of garoupa is 13000, solve this modified differential equation, expressing x in terms of t. [4]

Deduce the long term implication on the population of garoupa in the farm, and sketch the curve of x against t. [2]

9. The parametric equations of a curve C, are

$$x=4\cos\theta,\ y=\sin 2\theta, \text{ where } 0\le\theta\le\pi.$$

(i) Sketch C, stating the exact coordinates of any points of intersection with the axes. [2]

(ii) The point P on the curve has parameter p. Show that the equation of the tangent at P is $y=-\left(\frac{\cos 2p}{2\sin p}\right)x+2\cot p\cos 2p+\sin 2p$. [4]

(iii) The region S is the area bounded by C, the tangent at $\theta=\frac{\pi}{2}$ and the line $x=1$.

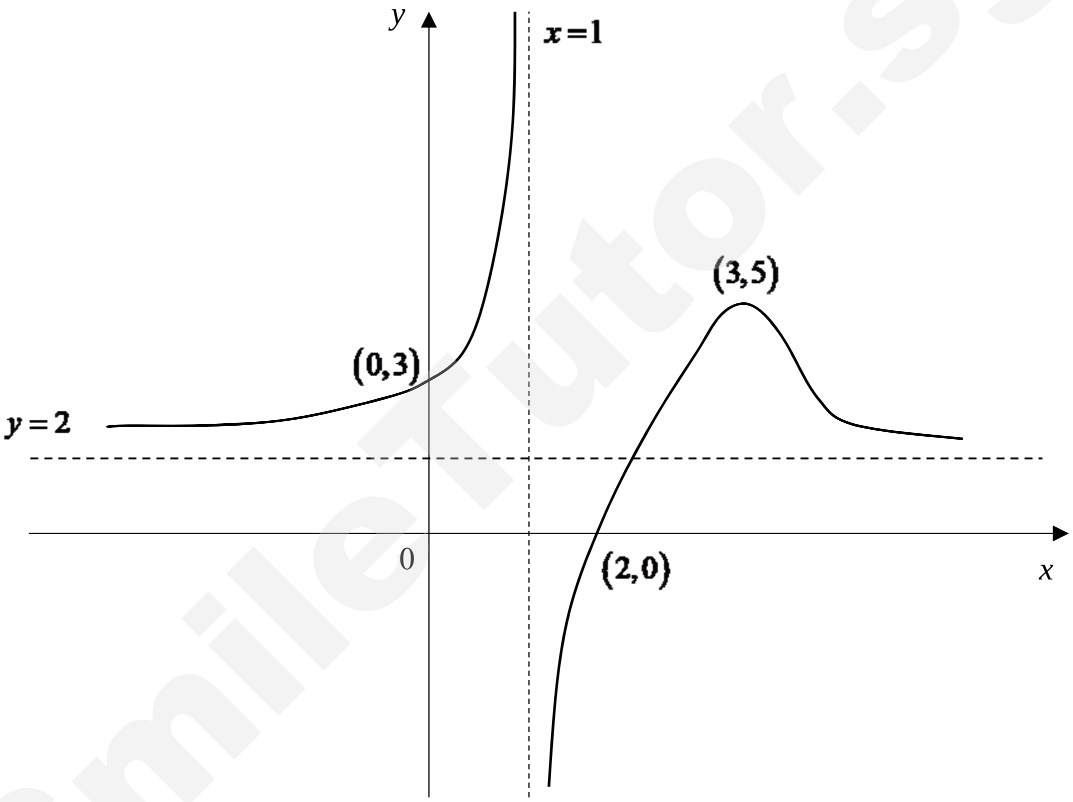
Show that the area of S can be expressed in the form $a+b\int_c^d \sin^2\theta\cos\theta\ \mathrm{d}\theta$, where a, b, c and d are exact constants to be determined. Hence evaluate the area of S numerically. [4]

(iv) C is transformed to give the curve C' given by

$$x = 2\cos\theta,\ y = \sin 2\theta - 1, \text{ where } 0 \le \theta \le \pi.$$

Describe the sequence of transformations from C to C'. [2]

10. The diagram shows the graph of $y = 2\mathrm{g}(x+1)$. The graph intersects the axes at $(0,3)$ and $(2,0)$ and has a turning point at $(3, 5)$. The asymptotes of the graph are at $x = 1$ and $y = 2$.



On separate diagrams, sketch the following graphs indicating the points corresponding to the axial intercepts, turning point and asymptotes where necessary.

(i) $y = \mathrm{g}(x)$ [3]

(ii) $y = \dfrac{1}{2\mathrm{g}(x+1)}$ [3]

(iii) $y = -\sqrt{2\mathrm{g}(x+1)}$ [3]

Suppose the above diagram shows the graph for $y = \mathrm{h}(x)$. Sketch the graph of $y = \mathrm{h}'(x)$. [3]

11. The function f is defined by

$$\mathrm{f}: x \mapsto -2x^2 + 12x - 19,\ x \in \mathbb{R}$$

(i) Sketch the graph of $y = \mathrm{f}(x)$, indicating clearly the coordinates of any turning point(s) and axial intercept(s). [1]

(ii) Give a reason why f does not have an inverse. [1]

(iii) If the domain of f is further restricted to $x \le k$, state the greatest value of k for which the function f^{-1} exists. [1]

(iv) Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. [3]

For the rest of the question, the domain of f is as found in part **(iii).**

(v) On the same diagram, sketch the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$, showing clearly the geometrical relationship among the graphs. [3]

The function g is defined by

$$\mathrm{g}: x \mapsto \mathrm{e}^{x^2-3},\ x \in \mathbb{R}$$

(vi) Explain why the composite function gf^{-1} exists. [1]

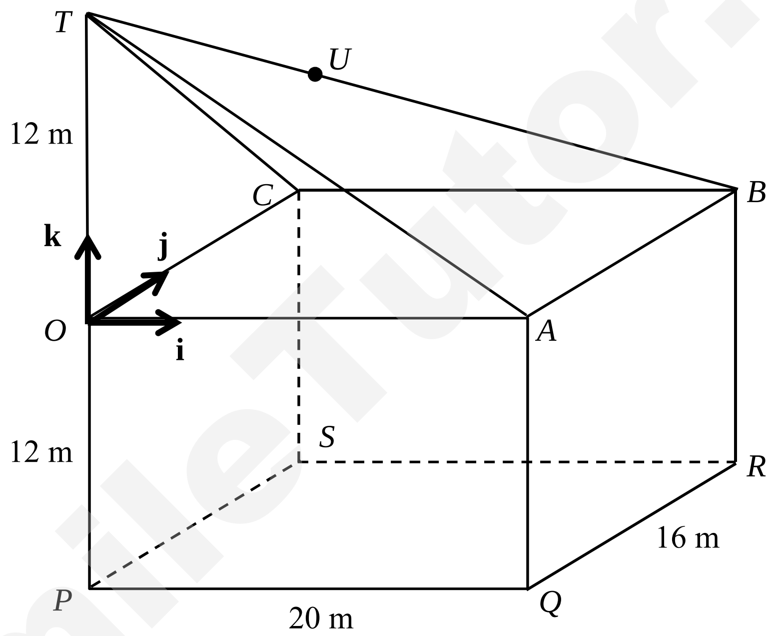
(vii) Find the range of gf^{-1}, giving your answer in the exact form. [2]

END OF PAPER

River Valley High School 2016 H2 Maths Preliminary Examination - Paper 2

Section A: Pure Mathematics [40 Marks]

1. The diagram below shows the structure of a military building which consists of two parts. For the first part, the rectangular plane *OABC* is on the ground level and the point *T* is 12 m vertically above *O*. The second part of the building is an underground store room in the form of a cuboid with sides $OP = AQ = BR = CS = 12$ m, $OA = CB = PQ = SR = 20$ m and $OC = AB = PS = QR = 16$ m.



To better study the structure using vector method, the point *O* is taken as the origin and vectors **i**, **j**, **k**, each of length 1 m, are taken along OA, OC and OT respectively.

(i) The military officials decide to install a surveillance camera *U* along the edge TB such that $TU : UB = 1 : 3$ for better coverage. Determine the position vector of *U* and hence the vector equation of the plane UCB in scalar product form. [3]

(ii) Another camera is to be placed at the point *P* for effective coverage. Find the acute angle between the planes UCB and PCB. [3]

(iii) As highly explosive items that need to be kept at low temperature are present in the underground store room, the military officials also place an infra-red sensor device at the point *T*. Determine the shortest distance of *T* from the plane PCB. [3]

2. **(i)** Solve the equation $z^5 - 16 - 16\sqrt{3}\mathrm{i} = 0$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [3]

(ii) The roots of the equation $z^5 - 16 - 16\sqrt{3}\mathrm{i} = 0$ are represented by $z_1,\ z_2,\ z_3,\ z_4$ and z_5, where $-\pi < \arg(z_1) < \arg(z_2) < \arg(z_3) < \arg(z_4) < \arg(z_5) \le \pi$. Show all these roots on an Argand diagram. [2]

(iii) Given that the complex number v satisfies the equation $|v - z_3| = |v - z_4|$, sketch the locus of the points which represent v on the same Argand diagram. Determine if this locus passes through the point which represent the complex number z_1. [2]

(iv) Another complex number w is such that $|w| \le 1$ and $|w - z_3| \le |w - z_4|$. Shade on the same diagram, the region representing w. Determine the range of value of $\arg(w - z_2)$. [3]

3. **(i)** It is given that $y\cos x = \mathrm{e}^x$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 + \tan x)$. [2]

(ii) By further differentiation, find the Maclaurin series for y, up to and including the term in x^2. [5]

(iii) Hence, find the set of values of x for which the value of y in part **(i)** is within $\pm\ 0.05$ of the value found by its Maclaurin series in part **(ii)**. [3]

4. The equation of a curve C is given by $y=\sqrt{\dfrac{3x}{4-x}}$.

(i) State the largest range of values of x for C to be defined. [1]

For the rest of this question, define C for the range of values found in part **(i)**.

(ii) Sketch C, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [2]

(iii) The region bounded by the curve, the line $y=3$ and the y-axis is denoted by R. Find the exact value of the area of R. [5]

(iv) The region R is rotated 2π radians about the line $y=3$ to form a solid. Write down the equation of the curve obtained when C is translated by 3 units in the negative y-direction. Hence find the volume of the solid formed. [3]

Section B: Statistics [60 Marks]

5. A school has 200 teachers of whom 5% are in the 21 – 30 age group, 60% are in the 31 – 40 age group and the rest are in the age group of 41 and above. During a meeting held in a Lecture Theatre, the principal intends to obtain a sample of 20 teachers for a survey. She decides to select 20 teachers from the last occupied row for the survey.

(i) Name the sampling method described. State a reason, in the context of the question, why this sampling method is not desirable. [2]

(ii) Suggest a method of obtaining a representative sample and describe how it may be carried out. [3]

6. For events A and B, it is given that $\mathrm{P}(A\,|\,B')=\dfrac{4}{7}$, $\mathrm{P}(B'\,|\,A')=\dfrac{2}{3}$ and $\mathrm{P}(A)=\dfrac{11}{20}$. Give a reason why events A and B are not independent. [1]
Find

(i) $\mathrm{P}(A\cup B)$, [3]

(ii) $\mathrm{P}(A\cap B')$. [2]

7. Find the number of ways to arrange the nine letters of the word PERMUTATE,

(i) in a circle; [2]

(ii) in a row with exactly one pair of identical letters together; [3]

(iii) in a row with identical letters separated (for example "PERMUATET", "PERTUTAEM" etc…). [3]

8. In a college Mathematics examination with a large candidature, the percentage mark X obtained by each male candidates was found to follow a Normal distribution with mean μ and standard deviation σ. It is further found that $\mathrm{P}(X > 45) = 0.85$ and $\mathrm{P}(X > 85) = 0.15$.

(i) Find the value of μ and σ. [3]

(ii) Find the least integral value of a such that the probability of a male candidate scoring less than a is at least 0.75. [2]

In the same examination, the percentage mark obtained by each female candidate was found to follow a Normal distribution with mean of 67 and standard deviation of 22.

(iii) Find the probability that the total percentage mark of two randomly selected female candidates is more than three times that of a randomly selected male candidate. [3]

One Mathematics teacher of the college claimed that the percentage mark of each candidate in the examination followed a Normal distribution of mean of 66 and standard deviation of 45. Comment on the validity of this claim. [1]

9. Company *A* packs and supplies salt in small packets. The mass of salt in one packet is denoted by *x* grams(g). The company claims that the mean mass of salt per packet is at least μ_0 g. A random sample of 12 packets of salt is taken and its mean and standard deviation are found to be 9.81g and 0.217g respectively.

Find the range of value of μ_0 for which Company *A*'s claim will not be rejected at the 5% significance level. State any assumption that you have made. [6]

Another company, Company *B*, claims that the mean mass of salt supplied by them per packet is 10 g. In a test against Company *B*'s claim at the α % significance level, the hypotheses are as follows:

Null hypothesis: $\mu_B = 10$,
Alternative hypothesis: $\mu_B \neq 10$,

where μ_B is the population mean mass of salt in a packet of salt from Company *B*.

The *p*-value is found to be 0.0438 (corrected to 3 significant figures). Explain the meaning of this *p*-value in the context of the question. [1]

State the range of values of α for which Company *B*'s claim is not rejected. For the range of values of α found, explain if Company *B*'s claim is still valid under a one-tailed test. [3]

10. During the Arts Festival period, JC students in a college are encouraged to view the arts exhibits put up by the Arts Talent Programme students in the canteen during lunch time from 12 noon to 2pm daily. Given that the JC students do not influence each other in their decision to view the exhibits, state a condition for the random variable X, which denotes the number of students who view the exhibits in independent 15-minute interval, to be well modelled by a Poisson distribution. [1]

Assuming that X has a Poisson distribution with mean 7.5, find

(i) the probability that less than 12 students view the arts exhibits from 12:30 pm to 1 pm on a particular day during the Arts Festival; [2]

(ii) the probability that there are three 30-minute intervals in which at least 12 students view the arts exhibits on a particular day during the Festival, [2]

(iii) the probability that the mean number of students who view the art exhibits daily from 12 noon to 2 pm on 65 school days is more than 60.5. [3]

The organising committee decides to study the students' support for the arts exhibits for the Arts Festival. By using a suitable approximation, determine the probability that for the first 50 days, there are more than 25 days whereby there is only one 30-minute interval among the four possible intervals daily from 12 noon to 2 pm, in which there are less than 12 students who view the arts exhibits. [3]

11. The table below gives the time t, in minutes, taken by Andy to complete a particular stage of a computer game x weeks after he has started playing the game.

x	1	2	3	4	5	6
t	48.2	32.5	22.7	18.0	16.4	14.6

(i) Draw a scatter diagram for the data, labelling the axes clearly. [2]

(ii) Calculate the product moment correlation coefficient and comment on why its value does not necessarily mean that the best model for the relationship between x and t is linear. [2]

(iii) Determine which of the following would be a better model for the set of data, justifying your choice clearly:

(A) $t = a(x-10)^2 + b$,

(B) $t = c\mathrm{e}^{-x} + d$,

where a, b, c and d are positive constants. [2]

(iv) For the better model identified in part **(iii)**, calculate the product moment correlation coefficient. [1]

(v) Andy would like to estimate the time taken by himself to complete the stage of the game 10 weeks after he has started playing the game. Find the equation of a suitable regression line and use it to obtain the estimate. Give two reasons why the estimate is not reliable. [4]

END OF PAPER

Solutions to 2016 RVHS Y6 H2 Maths Preliminary Exam II (Paper 1)

Question 1 [5 Marks]

(i)

Let $u_n = an^2 + bn + c$

$u_1 = -4.4$

$a + b + c = -4.4$ -------------- (1)

$u_2 = -4.1$

$4a + 2b + c = -4.1$ ----------- (2)

$u_3 = -2.6$

$9a + 3b + c = -2.6$ ----------- (3)

Solving (1), (2) and (3)

$a = 0.6$, $b = -1.5$ and $c = -3.5$

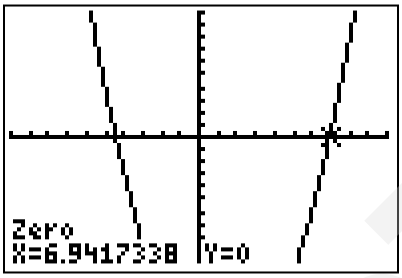
$u_n = 0.6n^2 - 1.5n - 3.5$

(ii)

$u_n > 15$

$0.6n^2 - 1.5n - 3.5 > 15$

$0.6n^2 - 1.5n - 18.5 > 0$



Since n is a positive integer, the set of values of n is $n \geq 7$.

Question 2 [6 Marks]

(i)

$$\mathrm{f}(x) = \frac{3x}{(x+4)(x+1)}$$

$$= \frac{A}{(x+4)} + \frac{B}{(x+1)}$$

$3x = A(x+1) + B(x+4)$

When $x = -1$, $-3 = 3B \quad \Rightarrow \quad B = -1$

When $x = -4$, $-12 = -3A \quad \Rightarrow \quad A = 4$

$$\mathrm{f}(x) = \frac{4}{(x+4)} - \frac{1}{(x+1)}$$

(ii)

$$\mathrm{f}(x) = 4(4+x)^{-1} - (1+x)^{-1}$$

$$= 4(4+x)^{-1} - (1+x)^{-1}$$

$$= 4\left[4^{-1}\left(1+\frac{x}{4}\right)^{-1}\right] - (1+x)^{-1}$$

$$= 1 + (-1)\left(\frac{x}{4}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{4}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{4}\right)^3 + \ldots$$

$$-\left[1 + (-1)(x) + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \ldots\right]$$

	$=1-\frac{x}{4}+\frac{x^2}{16}-\frac{x^3}{64}-1+x-x^2+x^3+...$ $=\frac{3}{4}x-\frac{15}{16}x^2+\frac{63}{64}x^3+...$ Range of validity for $\left(1+\frac{x}{4}\right)^{-1}$ is $-4<x<4$ Range of validity for $(1+x)^{-1}$ is $-1<x<1$ Therefore, overall range of validity is $-1<x<1$.	
(iii)	$ax(1-bx)^{-\frac{1}{2}}$ $=ax\left[1+\left(-\frac{1}{2}\right)(-bx)+...\right]$ $=ax+\frac{ab}{2}x^2+...$ Comparing the coefficient of x terms: $a=\frac{3}{4}$ Comparing the coefficients of x^2 terms: $\frac{ab}{2}=-\frac{15}{16} \quad \Rightarrow \quad b=-\frac{5}{2}$	

Question 3 [7 Marks]		
(a)	$\frac{d}{dx}e^{-x^2}=-2xe^{-x^2}$ $\int x^3e^{-x^2}dx$ $=\int\left(-\frac{1}{2}x^2\right)(-2xe^{-x^2})dx$ $=-\frac{1}{2}x^2e^{-x^2}+\int xe^{-x^2}dx$ $=-\frac{1}{2}x^2e^{-x^2}-\frac{1}{2}\int(-2x)e^{-x^2}dx$ $=-\frac{1}{2}x^2e^{-x^2}-\frac{1}{2}e^{-x^2}+c$	
(b)	$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}\sin x\lvert\sin x\rvert dx$ $=\int_{-\frac{\pi}{4}}^{0}\sin x(-\sin x)dx+\int_{0}^{\frac{\pi}{2}}\sin x(\sin x)dx$ $=-\int_{-\frac{\pi}{4}}^{0}\sin^2x\,dx+\int_{0}^{\frac{\pi}{2}}\sin^2x\,dx$ $=-\int_{-\frac{\pi}{4}}^{0}\frac{1-\cos 2x}{2}dx+\int_{0}^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}dx$	

$$= -\left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]_{-\frac{\pi}{4}}^{0} + \left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]_{0}^{\frac{\pi}{2}}$$

$$= -\left[(0) - \left(-\frac{\pi}{8} + \frac{1}{4}\right)\right] + \left[\left(\frac{\pi}{4}\right) - (0)\right]$$

$$= \frac{\pi}{8} + \frac{1}{4}$$

$$= \frac{\pi + 2}{8}$$

Question 4 [8 Marks]

(i)

Mth	Start of Month	End of Month
1	600	1.003×600
2	$1.003 \times 600 + 600$	$(1.003 \times 600 + 600) \times 1.003$ $= 1.003^2 \times 600 + 1.003 \times 600$
3	$1.003^2 \times 600 + 1.003 \times 600 + 600$	$1.003^3 \times 600 + 1.003^2 \times 600$ $+ 1.003 \times 600$
n		$1.003^n \times 600 + 1.003^{n-1} \times 600$ $+ \cdots + 1.003 \times 600$

At the end of December 2012, there are 24 months.

Amount in the account at the end of December 2012 after interest paid

$$= 1.003^{24} \times 600 + 1.003^{23} \times 600 + \cdots + 1.003 \times 600$$

$$= 600\left(1.003 + 1.003^2 + \cdots 1.003^{24}\right)$$

$$= 600\left[\frac{1.003\left(1.003^{24} - 1\right)}{1.003 - 1}\right]$$

$$= 14952.62754$$

$$\approx \$14952.63$$

(ii) Let the monthly contribution be $\$x$.

$$x\left[\frac{1.003\left(1.003^{24} - 1\right)}{1.003 - 1}\right] \geq 23000$$

$$x \geq 922.9147161$$

Since his monthly contributions are in multiple of $10, he should deposit $930 monthly into his savings account.

(iii)

Months	Amount contributed ($)
1	190
2	190×0.9
3	190×0.9^{2}
n	$190\times0.9^{n-1}$

Sum of the savings eventually

$$=190\left(1+0.9+0.9^{2}+0.9^{3}+\cdots+0.9^{n-1}+\cdots\right)$$

$$=190\left(\frac{1}{1-0.9}\right)$$

$=\$1900$, which is less than \$2000

Hence, John would not be able to save \$2000 in the long run.

Question 5 [9 Marks]

(i) $|\mathbf{b}\cdot\mathbf{a}|=|\mathbf{a}\cdot\mathbf{b}|$, where $\mathbf{b}$ is a unit vector, which gives the length of projection of the vector $\overrightarrow{\mathbf{OA}}$ onto the line OB (or on vector $\overrightarrow{\mathbf{OB}}$).

(ii)

$$S=\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

$$2S=|\mathbf{a}||\mathbf{b}|\sin\theta$$

$$4S^{2}=|\mathbf{a}|^{2}|\mathbf{b}|^{2}\sin^{2}\theta$$

$$4S^{2}=|\mathbf{a}|^{2}|\mathbf{b}|^{2}\left(1-\cos^{2}\theta\right)$$

$$=|\mathbf{a}|^{2}|\mathbf{b}|^{2}\left(1-\left(\frac{(\mathbf{a}\cdot\mathbf{b})}{|\mathbf{a}||\mathbf{b}|}\right)^{2}\right)$$

$$=|\mathbf{a}|^{2}|\mathbf{b}|^{2}-(\mathbf{a}\cdot\mathbf{b})^{2}$$

$$=|\mathbf{a}|^{2}-(\mathbf{a}\cdot\mathbf{b})^{2}\quad \textbf{since } |\mathbf{b}|=1$$

(iii) As C lies on AB, A, C and B are collinear.

$\Rightarrow$ $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are parallel

$$\therefore(\mathbf{c}-\mathbf{a})\times(\mathbf{c}-\mathbf{b})=\overrightarrow{AC}\times\overrightarrow{BC}$$

$$=\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{BC}\right|\sin 0\ \hat{\mathbf{n}}$$

$$=\mathbf{0},\quad \text{since } \sin 0=0$$

(iv)	$\mathbf{a}$ is perpendicular to $\mathbf{b}$ $\Rightarrow \mathbf{a}\cdot\mathbf{b}=0$ $\therefore S=\dfrac{\lvert\mathbf{a}\rvert}{2}$ Area of triangle OCB, S_1 $=\dfrac{1}{2}\lvert\mathbf{c}\times\mathbf{b}\rvert$ $=\dfrac{1}{2}\lvert\mathbf{c}\rvert\lvert\mathbf{b}\rvert\sin(30^\circ)$ $=\dfrac{1}{4}\lvert\mathbf{c}\rvert$, since $\lvert\mathbf{b}\rvert=1$ Hence, $S:S_1=\dfrac{\lvert\mathbf{a}\rvert}{2}:\dfrac{\lvert\mathbf{c}\rvert}{4}=2\lvert\mathbf{a}\rvert:\lvert\mathbf{c}\rvert$	

Question 6 [9 Marks]		
(a)	$z=\dfrac{3a-5\mathrm{i}}{1+2\mathrm{i}}$ $=\dfrac{3a-5\mathrm{i}}{1+2\mathrm{i}}\times\dfrac{1-2\mathrm{i}}{1-2\mathrm{i}}$ $=\dfrac{3a-10-(6a+5)\mathrm{i}}{5}$ Therefore, $z^*=\dfrac{3a-10+(6a+5)\mathrm{i}}{5}$ $z^*=w$ $\dfrac{3a-10+(6a+5)\mathrm{i}}{5}=1+13b\mathrm{i}$ Comparing real parts: $\dfrac{3a-10}{5}=1 \quad\Rightarrow\quad a=5$ Comparing imaginary parts: $\dfrac{6a+5}{5}=13b \quad\Rightarrow\quad b=\dfrac{7}{13}$	
(b)	$\lvert z\rvert=\dfrac{\lvert 1-\mathrm{i}\rvert^2}{\lvert -1+\sqrt{3}\mathrm{i}\rvert^4}$ $=\dfrac{\sqrt{2}^2}{2^4}=\dfrac{1}{8}$ $\arg(z)=\arg\left(\dfrac{(1-\mathrm{i})^2}{(-1+\sqrt{3}\mathrm{i})^4}\right)$ $=2\arg(1-\mathrm{i})-4\arg(-1+\sqrt{3}\mathrm{i})$ $=2\left(-\dfrac{\pi}{4}\right)-4\left(\dfrac{2\pi}{3}\right)$ $=-\dfrac{\pi}{2}-\dfrac{8\pi}{3} \quad =-\dfrac{19\pi}{6}=\dfrac{5\pi}{6}$ (principal range)	

	$\therefore z = \frac{1}{8}\left[\cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right]$ $\Rightarrow z^6 = \frac{1}{8^6}\left[\cos(5\pi) + \mathrm{i}\sin(5\pi)\right]$ $= \frac{1}{8^6}[-1+0]$ $= -\frac{1}{262144}$	

Question 7 [10 Marks]		
(i)	Let P_n be the statement $u_n = \frac{1}{2n+1}$ for $n \geq 0$. When $n = 0$, $LHS = u_0 = 1$ (Given) $RHS = \frac{1}{2(0)+1} = 1 = LHS$ $\therefore P_0$ is true. Assume P_k is true for some $k \in \mathbb{Z}$, $k \geq 0$, i.e. $u_k = \frac{1}{2k+1}$ Show that P_{k+1} is also true, i.e. $u_{k+1} = \frac{1}{2k+3}$ Using the recurrence relation, $u_{k+1} = u_k - \frac{2}{(2k+1)(2k+3)}$ $= \frac{1}{2k+1} - \frac{2}{(2k+1)(2k+3)}$ $= \frac{2k+3-2}{(2k+1)(2k+3)}$ $= \frac{2k+1}{(2k+1)(2k+3)} = \frac{1}{2k+3}$ P_{k+1} is true. Since P_0 is true and P_k is true, implies P_{k+1} is also true. By mathematical induction, P_n is true for $n \geq 0$.	

(ii)	$\sum_{n=1}^{N}\frac{1}{(2n+1)(2n+3)}$ $=\frac{1}{2}\sum_{n=1}^{N}(u_n-u_{n+1})$ $=\frac{1}{2}[u_1-\cancel{u_2}$ $+\cancel{u_2}-\cancel{u_3}$ $+\cancel{u_3}-\cancel{u_4}$ $\vdots$ $+\cancel{u_{N-1}}-\cancel{u_N}$ $+\cancel{u_N}-u_{N+1}]$ $=\frac{1}{2}(u_1-u_{N+1})$ $=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2N+3}\right)$	
(iii)	Let $n=r-1$, $\sum_{n=0}^{N}\frac{1}{(2n+3)(2n+5)}$ $=\sum_{r-1=0}^{r-1=N}\frac{1}{(2r-2+3)(2r-2+5)}$ $=\sum_{r=1}^{N+1}\frac{1}{(2r+1)(2r+3)}$ $=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2N+5}\right)$	
(iv)	As $N\to\infty$, $\frac{1}{2N+3}\to 0$, $\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}=\frac{1}{6}$	

Question 8 [10 marks]		
(i)	Let $\frac{dA}{dt}$ and $\frac{dB}{dt}$ be the rate of birth and rate of death respectively. $\frac{dA}{dt}=2x$, $\quad \frac{dB}{dt}\propto x^2 \quad \Rightarrow \quad \frac{dB}{dt}=kx^2$ $\frac{dx}{dt}=\frac{dA}{dt}-\frac{dB}{dt}=2x-kx^2$ When $x=10$, $\frac{dx}{dt}=0$: $0=2(10)-k(10)^2 \quad \Rightarrow \quad k=\frac{1}{5}$ $\therefore \frac{dx}{dt}=2x-\frac{x^2}{5}$	

(ii)

$$\therefore \frac{\mathrm{d}x}{\mathrm{d}t} = 2x - \frac{x^2}{5} - 1.8$$

$$= -\frac{1}{5}(x^2 - 10x + 9)$$

$$= -\frac{1}{5}\left[(x-5)^2 - 4^2\right], \text{ where } a = 4 \text{ (shown)}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{5}\left[(x-5)^2 - 4^2\right]$$

$$\int \frac{1}{(x-5)^2 - 4^2}\,\mathrm{d}x = \int -\frac{1}{5}\,\mathrm{d}t$$

$$\frac{1}{2(4)}\ln\left|\frac{(x-5)-4}{(x-5)+4}\right| = -\frac{1}{5}t + c$$

$$\ln\left|\frac{x-9}{x-1}\right| = -\frac{8}{5}t + 8c$$

$$\frac{x-9}{x-1} = \mathrm{e}^{-\frac{8}{5}t + 8c} = A\mathrm{e}^{-\frac{8}{5}t}$$

When $t = 0$, $x = 13$:

$$\frac{13-9}{13-1} = A \quad \Rightarrow \quad A = \frac{1}{3}$$

$$\frac{x-9}{x-1} = \frac{1}{3}\mathrm{e}^{-\frac{8}{5}t}$$

$$3(x-9) = (x-1)\mathrm{e}^{-\frac{8}{5}t}$$

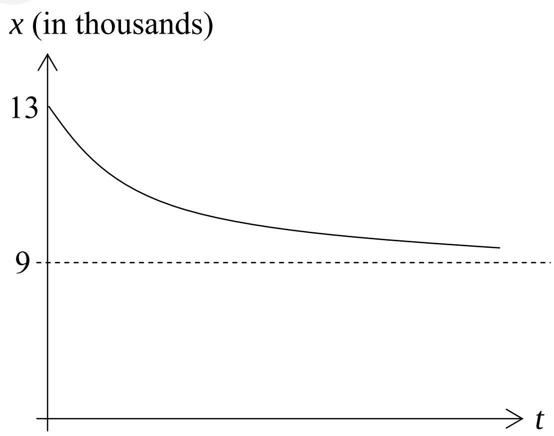
$$x\left(3 - \mathrm{e}^{-\frac{8}{5}t}\right) = 27 - \mathrm{e}^{-\frac{8}{5}t}$$

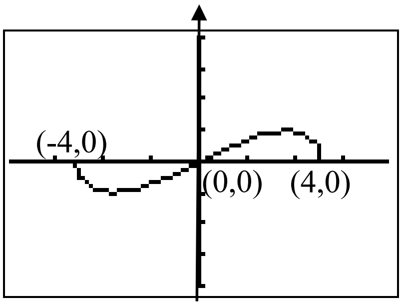
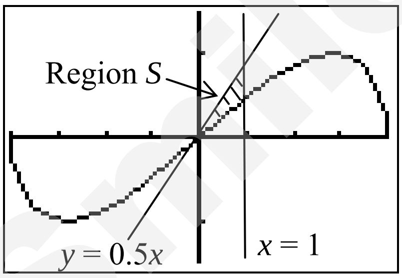
$$x = \frac{27 - \mathrm{e}^{-\frac{8}{5}t}}{3 - \mathrm{e}^{-\frac{8}{5}t}}$$

As $t \to \infty$, $\mathrm{e}^{-\frac{8}{5}t} \to 0$

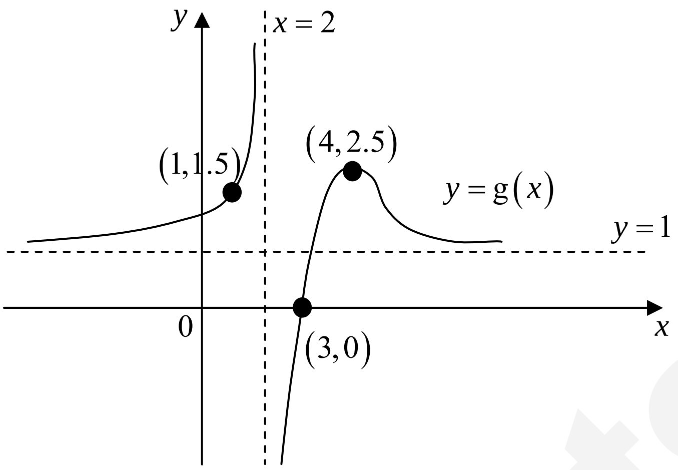
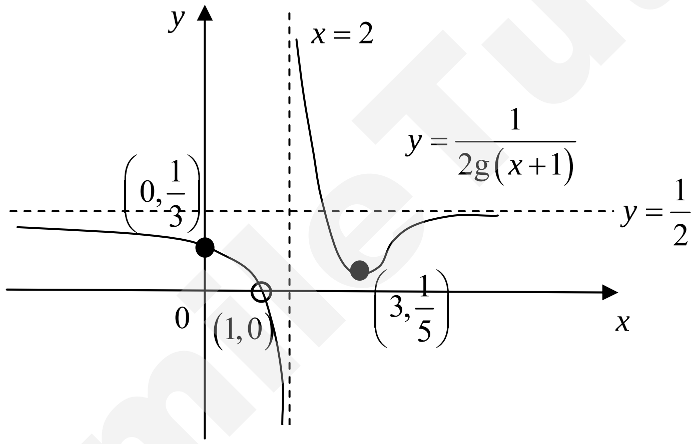
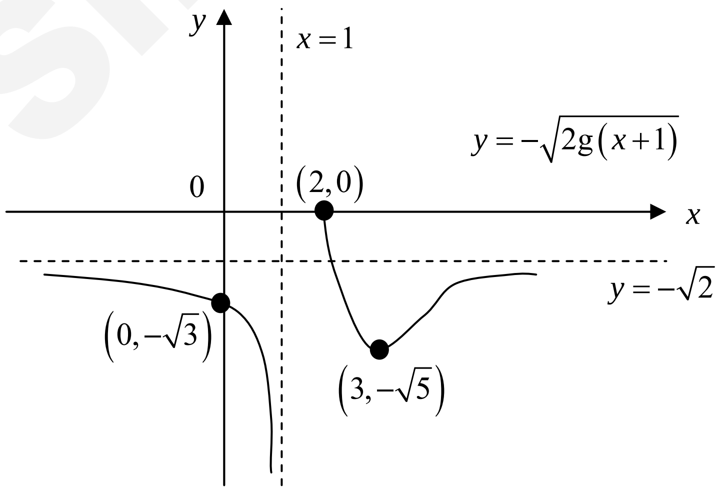
$$x \to \frac{27}{3} = 9$$

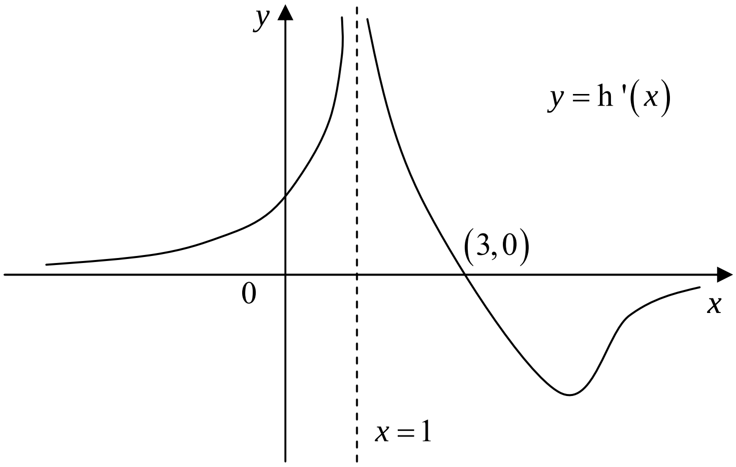
In the long term, the population of garoupa will stabilise at 9000.

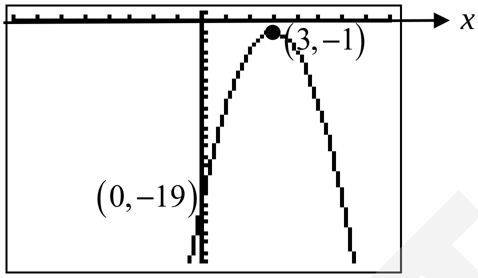


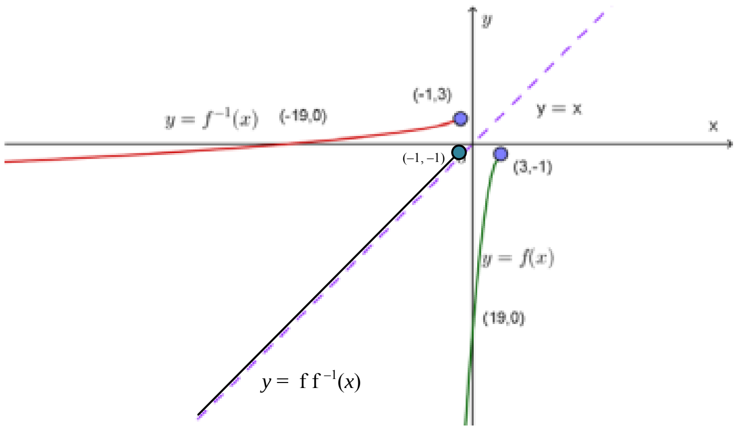
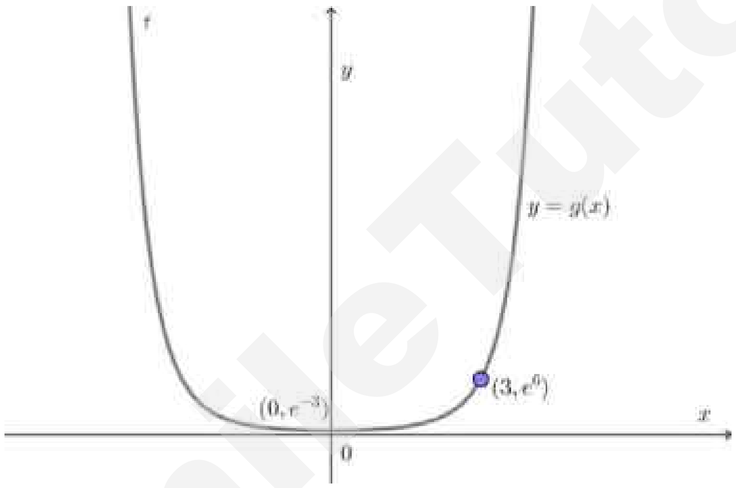
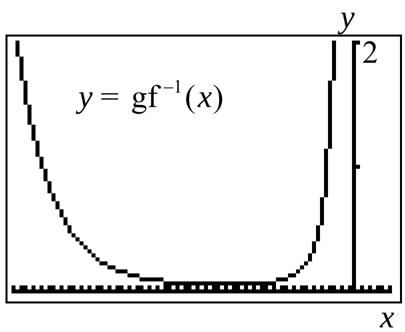
Question 9 [12 Marks]		
(i)		
(ii)	$x = 4\cos\theta$ $\quad$ $y = \sin 2\theta$ $\frac{dx}{d\theta} = -4\sin\theta$ $\quad$ $\frac{dy}{d\theta} = 2\cos 2\theta$ $\frac{dy}{dx} = \frac{2\cos 2\theta}{-4\sin\theta} = -\frac{\cos 2\theta}{2\sin\theta}$ Equation of tangent at P, $y - \sin 2p = -\frac{\cos 2p}{2\sin p}(x - 4\cos p)$ $y = -\left(\frac{\cos 2p}{2\sin p}\right)x + \frac{4\cos p\cos 2p}{2\sin p} + \sin 2p$ $= -\left(\frac{\cos 2p}{2\sin p}\right)x + 2\cot p\cos 2p + \sin 2p$	
(iii)	Equation of the tangent at $\theta = \frac{\pi}{2}$, $y = \frac{1}{2}x$  Area of S $= \int_0^1 \frac{x}{2}\,dx - \int_0^1 y\,dx$ $= \left[\frac{x^2}{4}\right]_0^1 - \int_{\pi/2}^{\cos^{-1}\left(\frac{1}{4}\right)} \sin 2\theta(-4\sin\theta\,d\theta)$ $= \frac{1}{4} + 8\int_{\pi/2}^{\cos^{-1}\left(\frac{1}{4}\right)} \sin^2\theta\cos\theta\,d\theta$ (shown) $= 0.00395$ units^2 (to 3 s.f.)	

(iv)	$C \to C' : \begin{cases} x \to 2x \\ y \to y-1 \end{cases}$ Sequence of transformations: - Scaling by factor 0.5 parallel to *x*-axis - Translation of 1 unit in the negative *y*-direction	

Question 10 [12 marks]		
(i)		
(ii)		
(iii)		

Question 11 [12 marks]		
(i)		
(ii)	The horizontal line, $y=-5$, cuts the graph more than once, hence f is not one-one $\Rightarrow$ f^{-1} does not exist.	
(iii)	Turning point of $y=f(x)$ occurs at $(3,-1)$. Hence greatest value of $k=3$	
(iv)	$y=-2x^2+12x-19$ $=-2(x-3)^2-1$ $(x-3)^2=\frac{y+1}{-2}$ $x-3=\pm\sqrt{\frac{y+1}{-2}}$ $x=3\pm\sqrt{\frac{y+1}{-2}}$ Since $x\le 3$, $x=3-\sqrt{\frac{y+1}{-2}}$ $f^{-1}(x)=3-\sqrt{\frac{x+1}{-2}}$ $D_{f^{-1}}=R_f=(-\infty,-1]$	

(v)		
(vi)	$R_{\mathrm{f}^{-1}} = (-\infty, 3]$ and $D_g = \mathbb{R}$ Since $R_{\mathrm{f}^{-1}} \subseteq D_g$, hence gf^{-1} exists.	
(vii)	$\underset{(-\infty,-1]}{D_{\mathrm{f}^{-1}}} \to \underset{(-\infty,3]}{R_{\mathrm{f}^{-1}}} \to \underset{[\mathrm{e}^{-3},\infty)}{R_{\mathrm{gf}^{-1}}}$  <u>ALTERNATIVELY</u>: $\mathrm{gf}^{-1}(x) = \mathrm{g}\left(3 - \sqrt{\frac{x+1}{-2}}\right)$ $= \mathrm{e}^{\left(3-\sqrt{\frac{x+1}{-2}}\right)^2 - 3}, \quad x \le -1$  From GC, minimum point is at $x = -19 \Rightarrow y = \mathrm{e}^{-3}$ $\therefore R_{\mathrm{gf}^{-1}} = [\mathrm{e}^{-3}, \infty)$	

Solutions to 2016 RVHS Y6 H2 Maths Preliminary Exam II (Paper 2)

Section A: Pure Mathematics [40 marks]

Question 1 [9 Marks]

(i)

We first have the following position vectors:

$$\overrightarrow{OT}=\begin{pmatrix}0\\0\\12\end{pmatrix},\ \overrightarrow{OA}=\begin{pmatrix}20\\0\\0\end{pmatrix},\ \overrightarrow{OB}=\begin{pmatrix}20\\16\\0\end{pmatrix}\text{ and }\overrightarrow{OC}=\begin{pmatrix}0\\16\\0\end{pmatrix}$$

Then by Ratio Theorem:
Since U divides TB in the ratio 1:3,

$$\overrightarrow{OU}=\frac{1}{4}\overrightarrow{OB}+\frac{3}{4}\overrightarrow{OT}$$

$$=\frac{1}{4}\begin{pmatrix}20\\16\\0\end{pmatrix}+\frac{3}{4}\begin{pmatrix}0\\0\\12\end{pmatrix}=\begin{pmatrix}5\\4\\9\end{pmatrix}$$

To find the vector equation of the plane UCB, we first have:

$$\overrightarrow{UB}=\overrightarrow{OB}-\overrightarrow{OU}=\begin{pmatrix}20\\16\\0\end{pmatrix}-\begin{pmatrix}5\\4\\9\end{pmatrix}=\begin{pmatrix}15\\12\\-9\end{pmatrix}$$

and

$$\overrightarrow{CB}=\overrightarrow{OB}-\overrightarrow{OC}=\begin{pmatrix}20\\0\\0\end{pmatrix}$$

We then apply cross product of the vectors to obtain normal vector to the plane UCB:

$$\overrightarrow{UB}\times\overrightarrow{CB}=\begin{pmatrix}15\\12\\-9\end{pmatrix}\times\begin{pmatrix}20\\0\\0\end{pmatrix}=\begin{pmatrix}0\\-180\\-240\end{pmatrix}=-60\begin{pmatrix}0\\3\\4\end{pmatrix}$$

Thus, the vector equation of the plan UCB in scalar product form is then $\mathbf{r}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}=\begin{pmatrix}0\\16\\0\end{pmatrix}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}=48$

(ii)

For the plane PCB, we first need to find the normal vector to the plane. We have:

$$\overrightarrow{PC}\times\overrightarrow{CB}=\left(\begin{pmatrix}0\\16\\0\end{pmatrix}-\begin{pmatrix}0\\0\\-12\end{pmatrix}\right)\times\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$=\begin{pmatrix}0\\16\\12\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\12\\-16\end{pmatrix}=4\begin{pmatrix}0\\3\\-4\end{pmatrix}$$

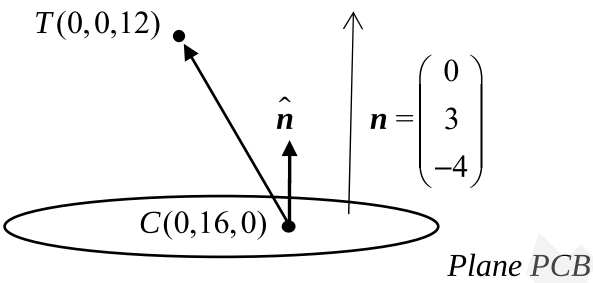
Therefore a vector normal to the plane PCB will be $\begin{pmatrix}0\\3\\-4\end{pmatrix}$.

Thus, the acute angle θ between plane UCB and PCB will be given by:

$$\cos\theta = \left| \frac{\begin{pmatrix}0\\3\\4\end{pmatrix} \bullet \begin{pmatrix}0\\3\\-4\end{pmatrix}}{\sqrt{3^2+4^2}\sqrt{3^2+(-4)^2}} \right| = \left| \frac{9-16}{\sqrt{3^2+4^2}\sqrt{3^2+(-4)^2}} \right| = \frac{7}{25}$$

That is, $\theta = 73.7^\circ$ (3 s.f.)

(iii) Method 1:



We note that $C = (0,16,0)$ is a point on the plane PCB and thus we projected the vector $\overrightarrow{CT}$ onto $\hat{\boldsymbol{n}}$.

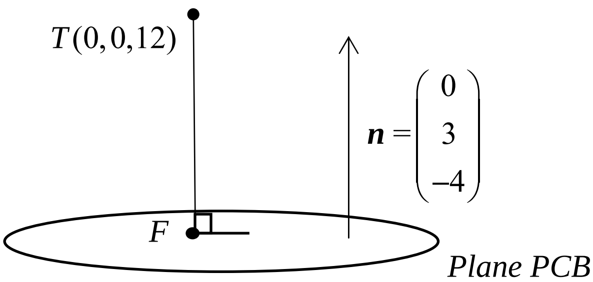
$$\overrightarrow{CT} = \overrightarrow{OT} - \overrightarrow{OC} = \begin{pmatrix}0\\-16\\12\end{pmatrix}$$

The shortest distance of T to the plane PCB is given by:

$$d = \left|\overrightarrow{CT} \bullet \hat{n}\right| = \frac{\left|\begin{pmatrix}0\\-16\\12\end{pmatrix} \bullet \begin{pmatrix}0\\3\\-4\end{pmatrix}\right|}{\sqrt{3^2+(-4)^2}}$$

$$= \frac{|-48-48|}{5} = \frac{96}{5} \text{ units}$$

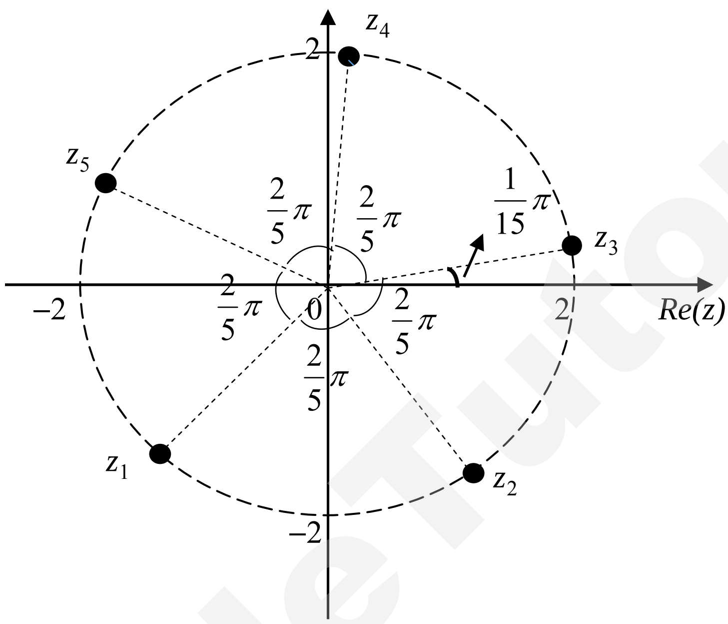
Method 2:



Let F be the foot of the perpendicular from point T to the plane PCB.

	Then $\overrightarrow{TF}=\lambda\mathbf{n}=\lambda\begin{pmatrix}0\\3\\-4\end{pmatrix}$ $\Rightarrow \overrightarrow{OF}-\overrightarrow{OT}=\lambda\begin{pmatrix}0\\3\\-4\end{pmatrix}\Rightarrow\overrightarrow{OF}=\overrightarrow{OT}+\lambda\begin{pmatrix}0\\3\\-4\end{pmatrix}=\begin{pmatrix}0\\3\lambda\\12-4\lambda\end{pmatrix}$ Since *F* is also a point on the plane *PCB*, $\overrightarrow{OF}$ will also satisfy the vector equation of plane *PCB.* We then have $\begin{pmatrix}0\\3\lambda\\12-4\lambda\end{pmatrix}\bullet\begin{pmatrix}0\\3\\-4\end{pmatrix}=\begin{pmatrix}0\\16\\0\end{pmatrix}\bullet\begin{pmatrix}0\\3\\-4\end{pmatrix}=48$ Thus, $9\lambda-4(12-4\lambda)=48$ $\Rightarrow 25\lambda=96$ $\Rightarrow \lambda=\frac{96}{25}$ Therefore, the perpendicular distance is $\left\|\overrightarrow{TF}\right\|=\frac{96}{25}\left\|\begin{pmatrix}0\\3\\-4\end{pmatrix}\right\|=\frac{96}{25}\sqrt{3^2+(-4)^2}=\frac{96}{5}$ units	

Question 2 [10 Marks]		
(i)	Given that $z^5-16-16\sqrt{3}i=0$, we have $z^5=16+16\sqrt{3}i=32e^{i\left(\frac{\pi}{3}\right)}$. Then by solving, the roots are given by: $z=32^{\frac{1}{5}}e^{i\,\frac{1}{5}\left(\frac{\pi}{3}+2k\pi\right)}$ $=2e^{i\left(\frac{1+6k}{15}\right)\pi}$ where $k=-2,-1,\ 0,\ 1,\ 2.$	

(ii)	Given that $-\pi < \arg(z_1) < \arg(z_2) < \arg(z_3) < \arg(z_4) < \arg(z_5) \le \pi$ $\Rightarrow$ we have the following roots: $z_1 = 2e^{i\left(-\frac{11}{15}\pi\right)}$, $z_2 = 2e^{i\left(-\frac{5}{15}\pi\right)}$, $z_3 = 2e^{i\left(\frac{1}{15}\pi\right)}$, $z_4 = 2e^{i\left(\frac{7}{15}\pi\right)}$ and $z_5 = 2e^{i\left(\frac{13}{15}\pi\right)}$. The roots z_1, z_2, z_3, z_4 and z_5 are then represented in the following Argand diagram:  Note that if the points Z_1, Z_2, Z_3, Z_4 and Z_5 represent the complex number z_1, z_2, z_3, z_4 and z_5 respectively on the Argand diagram, then $\angle Z_1OZ_2 = \angle Z_2OZ_3 = \angle Z_3OZ_4 = \angle Z_4OZ_5 = \angle Z_5OZ_1 = \dfrac{2\pi}{5}$	

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<td>(iii)</td>
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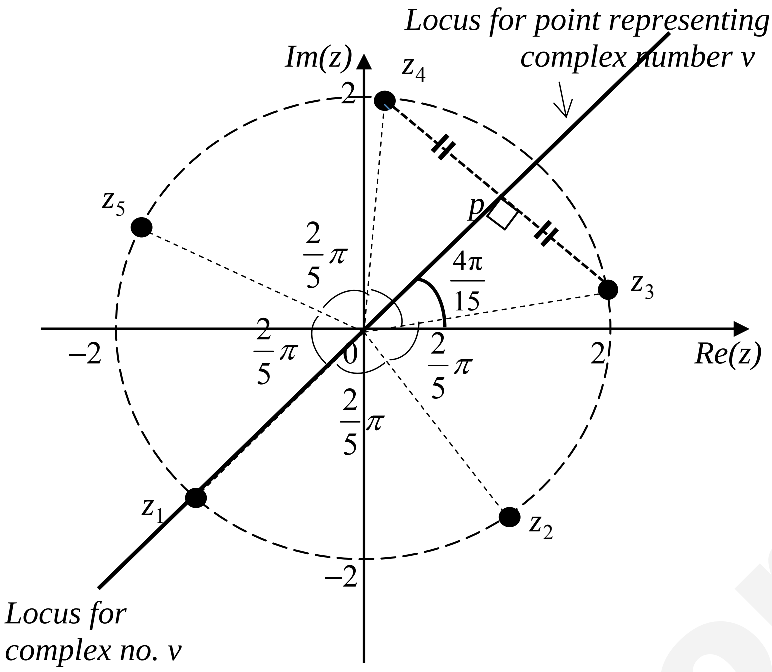

We note that if p is the complex number represented by the midpoint of the line joining Z_3 and Z_4, then

$$\arg(p)=\frac{\frac{1}{15}\pi+\frac{7}{15}\pi}{2}=\frac{4}{15}\pi .$$

Since $\arg(p)+\left|\arg(z_1)\right|=\frac{4}{15}\pi+\frac{11}{15}\pi=\pi$, we can conclude that the point representing z_1 lies on the locus of v.
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<td>(iv)</td>
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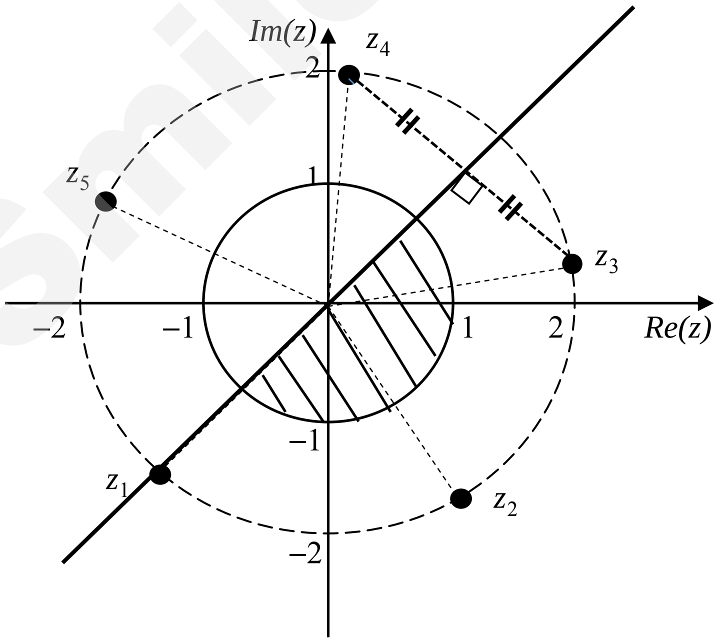

Shaded region represents possible region for complex number w.
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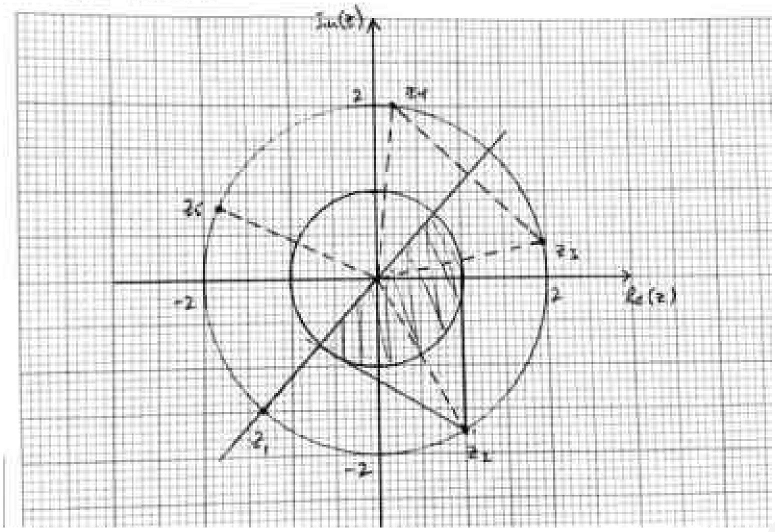
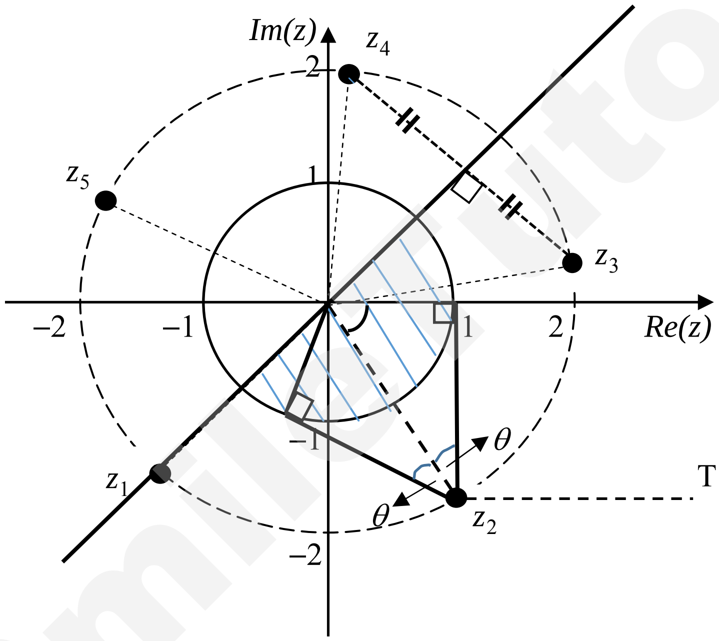
Q2. (ii) & (iv)





To find range of $\arg(w - z_2)$, we note that :

$$\angle OZ_2T = \pi - \frac{5}{15}\pi = \frac{2}{3}\pi$$

and $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$.

Thus the range of value of $\arg(w - z_2)$ is

$$\frac{2}{3}\pi - \frac{1}{6}\pi \le \arg(w - z_2) \le \frac{2}{3}\pi + \frac{1}{6}\pi$$

i.e. $\frac{1}{2}\pi \le \arg(w - z_2) \le \frac{5}{6}\pi$ or $1.57 \le \arg(w - z_2) \le 2.62$

Question 3 [10 Marks]

(i)

Let $y\cos x = e^x$ ------(1)

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cos x + y(-\sin x) = e^x$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cos x = y\sin x + y\cos x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\frac{\cos x + \sin x}{\cos x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(1+\tan x) \text{ ------(2)} \quad \text{(shown)}$$

(ii)

Differentiate with respect to x:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)(1+\tan x) + y(\sec^2 x) \text{ ------(3)}$$

At $x = 0$,

From (1): $y = \mathrm{f}(0) = \dfrac{e^0}{\cos 0} = 1$

From (2): $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (1)(1+\tan 0) \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(0) = 1$

From (3):

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (1)(1+\tan 0) + (1)\sec^2 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{f}''(0) = 2$$

$$\therefore y = \mathrm{f}(x) = \mathrm{f}(0) + \frac{\mathrm{f}'(0)}{1!}x + \frac{\mathrm{f}''(0)}{2!}x^2 + \ldots\ldots\ldots$$

$$= 1 + x + x^2 + \ldots\ldots\ldots\ldots \quad (\text{up to } x^2 \text{ term})$$

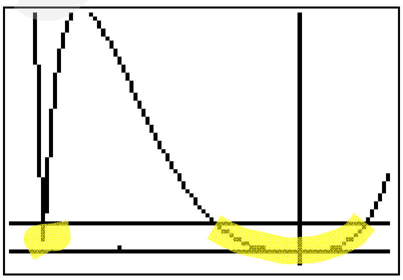
(iii)

Let $\mathrm{f}(x) = \dfrac{e^x}{\cos x}$, where $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$,

and $\mathrm{g}(x) = 1 + x + x^2$ (from (ii)).

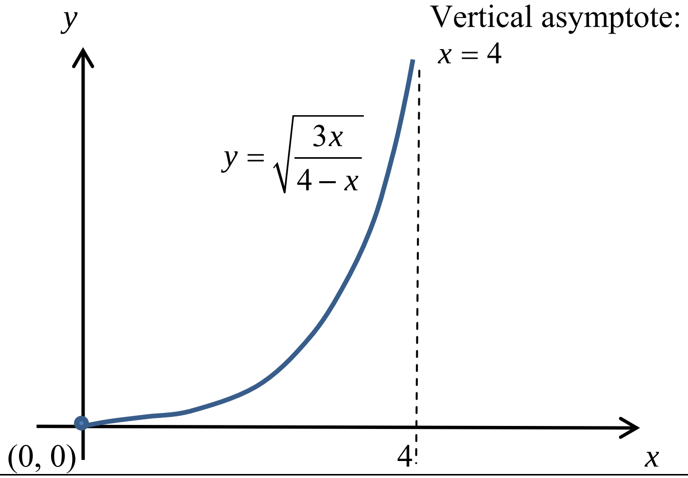
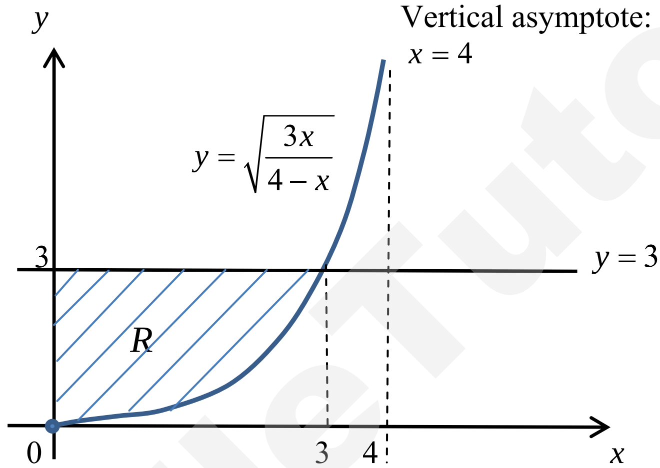
We thus solve $|\mathrm{f}(x) - \mathrm{g}(x)| < 0.05$.

Using GC, we plot $y = |\mathrm{f}(x) - \mathrm{g}(x)|$ and $y = 0.05$:



x-values of the intersection points are:
$-1.425019, -1.410526, -0.4704015$ and 0.3795259

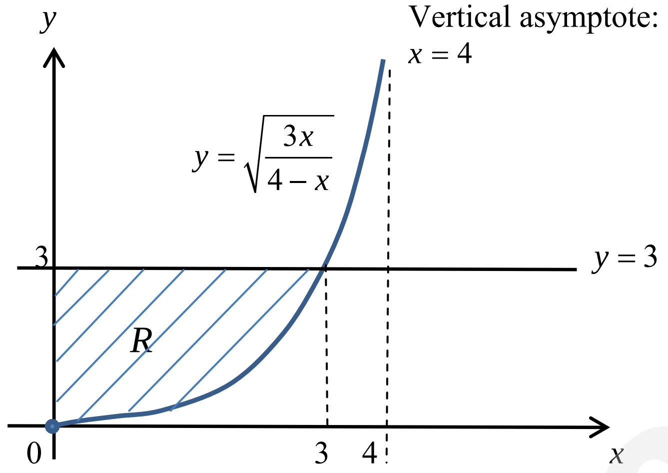
$\therefore$ Ans: $\{x : x \in \mathbb{R}, -1.43 < x < -1.41 \text{ or } -0.470 < x < 0.380\}$

Question 4 [11 Marks]		
(i)	The largest range of values of x for curve C to be defined is $0 \le x < 4$.	
(ii)		
(iii)	The area R is as shown in the shaded region below:  $y = \sqrt{\frac{3x}{4-x}}$ then $y^2 = \frac{3x}{4-x}$ $\Rightarrow 4y^2 - xy^2 = 3x$ $\Rightarrow x = \frac{4y^2}{3+y^2}$ Thus, the area of the region R $= \int_0^3 x \, \mathrm{d}y$ $= \int_0^3 \frac{4y^2}{3+y^2} \, \mathrm{d}y$ $= \int_0^3 4 - \frac{12}{3+y^2} \, \mathrm{d}y$ $= \left[4y - \frac{12}{\sqrt{3}}\tan^{-1}\left(\frac{y}{\sqrt{3}}\right)\right]_0^3$ $= \left(4\times 3 - \frac{12}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\right)\right) - 0$	

$$= 12 - \frac{12}{\sqrt{3}} \times \frac{\pi}{3}$$

$$= 12 - \frac{4\sqrt{3}}{3}\pi \text{ unit}^2$$

(iv) The area R is rotated 2π radian about the line $y = 3$:



The equation of curve when C is translated by -3 units in the y-direction is $y = \sqrt{\frac{3x}{4-x}} - 3$.

Next, let

$$y = 3 \Rightarrow \sqrt{\frac{3x}{4-x}} = 3 \Rightarrow \frac{3x}{4-x} = 9 \Rightarrow x = 3$$

The volume V generated when R is rotated completely about $y = 3$ is the same as the volume generated when the new curve is rotated about the x-axis from $x = 0$ to $x = 3$.

Therefore,

$$V = \int_0^3 \pi y^2 \, dx$$

$$= \pi \int_0^3 \left(\sqrt{\frac{3x}{4-x}} - 3 \right)^2 dx$$

$$= 28.6 \text{ unit}^3$$

Section B: Statistics [60 marks]

Question 5 [5 Marks]

(i) Quota Sampling.
This method is non-random as not every teacher has an equal chance of being selected. <u>OR</u>
This method does not give a representative sample.

(ii) Stratified Sampling.
Principal can draw **random** samples from each stratum as follows:

Age group	21 - 30	31 - 40	41 and above
No. of teachers selected	$0.05 \times 20 = 1$	$0.6 \times 20 = 12$	$20 - 1 - 12 = 7$

Question 6 [6 marks]		
	Since $\mathrm{P}(A) \neq \mathrm{P}(A \mid B')$, events A and B' are not independent. Thus A and B are not independent.	
(i)	$\mathrm{P}(B' \mid A') = \frac{2}{3} \Rightarrow \frac{\mathrm{P}(B' \cap A')}{\mathrm{P}(A')} = \frac{2}{3}$ $\Rightarrow \mathrm{P}(B' \cap A') = \frac{2}{3} \times \frac{9}{20} = \frac{3}{10}$ $\mathrm{P}(A \cup B) = 1 - \mathrm{P}(B' \cap A') = \frac{7}{10}$	
(ii)	$\mathrm{P}(A \mid B') = \frac{4}{7} \Rightarrow \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = \frac{4}{7}$ $\Rightarrow \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(A \cap B') + \mathrm{P}(A' \cap B')} = \frac{4}{7}$ $\Rightarrow \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(A \cap B') + \frac{3}{10}} = \frac{4}{7}$ $\Rightarrow \mathrm{P}(A \cap B') = \frac{2}{5}$	

Question 7 [8 Marks]		
(i)	$\frac{(9-1)!}{2!2!} = 10080$	
(ii)	For the case where <u>EE</u> are together and T, T separated: Arrange <u>EE</u>, P, R, M, U, A first, then slot in T, T to separate them. No. of ways = $6! \times \binom{7}{2} = 15120$ Hence, required no. of ways = $15120 \times 2 = 30240$	
(iii)	No. of ways without restrictions = $\frac{9!}{2!2!} = 90720$ No. of ways with 'E's together & 'T's together = 7! = 5040 Hence, required no. of ways = No. of ways without restrictions – n(each of the pair of identical letters together) – n(exactly one of the pairs is together) = 90720 – 5040 – 30240 = 55440	
	<u>ALTERNATIVELY</u> <u>Case 1</u>: E, E, T, T all separated No. of ways = $5! \times \binom{6}{4} \times \frac{4!}{2!2!} = 10800$	

Case 2: with exactly 1 E and 1 T together (i.e. ET, E, T separated)

No. of ways = $5! \times \binom{6}{3} \times 3! \times 2 = 28800$

Case 3: 2 pairs of ET, but the pairs separated

No. of ways = $5! \times \binom{6}{2} \times (2!)^2 = 7200$

Case 4: ETE or TET together

No. of ways = $\left[5! \times \binom{6}{2} \times 2!\right] \times 2 = 7200$

Case 5: ETET or TETE

No. of ways = $6! \times 2 = 1440$

Hence, required no. of ways

= 10800 + 28800 + 7200 + 7200 + 1440 = 55440

Question 8 [9 Marks]

(i) $\mathrm{P}(X > 45) = 0.85 \Rightarrow \mathrm{P}(X < 45) = 0.15$

Since $\mathrm{P}(X > 85) = 0.15$, by symmetry property of normal distribution curve, $\mu = \frac{45+85}{2} = 65$

Next,

$\mathrm{P}(X < 45) = 0.15$

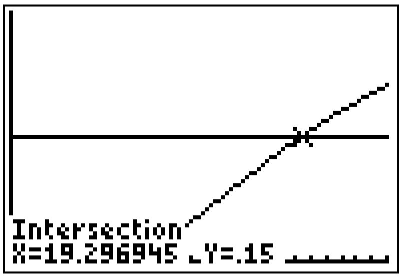
$\Rightarrow \mathrm{P}\left(Z < \frac{45-\mu}{\sigma}\right) = 0.15$

$\Rightarrow \mathrm{P}\left(Z < \frac{45-65}{\sigma}\right) = 0.15$

$\Rightarrow \frac{45-65}{\sigma} = -1.03643338$

$\Rightarrow \sigma = 19.296947 = 19.3$ (3 s.f.)

ALTERNATIVE METHOD



$y = \mathrm{P}(X < 45)$

$y = 0.15$

$\therefore \sigma = 19.296947 = 19.3$ (3 s.f.)

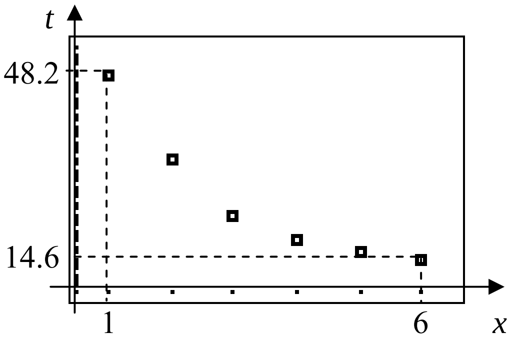
(ii)	If the probability of a male candidate scoring less than a is at least 0.75, then we have $\text{P}(X < a) \geq 0.75$. $X \sim N(65, 19.296947^2)$ $\text{P}(X < a) \geq 0.75$ $\Rightarrow a \geq 78.01559295$ Thus, the least integral value of a is 79.	
(iii)	Let Y denotes the random variable that represents the percentage marks obtained by a randomly selected female candidate in the Maths exam. Then $Y \sim N(67, 22^2)$. We first note that $\text{P}(Y_1 + Y_2 > 3X) = \text{P}(Y_1 + Y_2 - 3X > 0)$ Since $X \sim N(65, 19.296947^2)$, we have $\text{E}(Y_1 + Y_2 - 3X) = 67 + 67 - 3 \times 65 = -61$ $\text{Var}(Y_1 + Y_2 - 3X) = 22^2 + 22^2 + 3^2 \times 19.296947^2$ $= 4319.349472$ Then $Y_1 + Y_2 - 3X \sim N(-61, 4319.349472)$ Thus, $\text{P}(Y_1 + Y_2 > 3X)$ $= \text{P}(Y_1 + Y_2 - 3X > 0)$ $= 0.177$ (3 s.f.)	
	We note that for a random variable W with normal distribution of mean 66 and standard deviation 45, $\text{P}(W < 0) = 0.0712334139$ which is non negligible. As $W \geq 0$, we conclude that the teacher's claim is not valid. OR: If $W \sim N(66, 45)$, then 99.7% of the values would lie within $66 \pm 3 \times 45$, which contains a significant range of negative values. Thus, the teacher's claim is not valid.	

Question 9 [10 Marks]		
	Let μ be the mean mass of salt in a packet of salt from Company A. Test $\text{H}_0 : \mu = \mu_0$ against $\text{H}_1 : \mu < \mu_0$ at 5% significance level Test statistic: Under H_0, $T = \dfrac{\overline{X} - \mu_0}{s/\sqrt{12}} \sim t(11)$, where $s^2 = \dfrac{12}{11}\left(0.217^2\right) = 0.0513698182$ Critical region: Reject H_0 if $t_{\text{calc}} < -1.795884781$	

	Since H_0 is not rejected, $\dfrac{9.81-\mu_0}{\sqrt{\dfrac{0.0513698182}{12}}} \geq -1.795884781$ $\mu_0 \leq 9.927501081$ i.e. $\mu_0 \leq 9.93$ (to 3s.f.) (OR $\mu_0 \leq 9.92$) We assume that mass of salt follows a normal distribution.	
	p-value of 0.0438 is the lowest level of significance for which Company B's claim that the mean mass of salt per packet is 10 g is rejected.	
	To not reject Company B's claim (i.e. to not reject H_0), p-value $\geq \dfrac{\alpha}{100}$ $\Rightarrow \alpha \leq 0.0438 \times 100$ i.e. $\alpha \leq 4.38$ Supposed a one-tailed test is conducted instead, i.e. testing $H_0 : \mu_B = 10$ against $H_1 : \mu_B < 10$ p-value $= \dfrac{0.0438}{2} = 0.0219$. Then we would not be certain if p-value $< \alpha$ since $\alpha \leq 4.38$. (no conclusion)	

Question 10 [11 marks]		
	X can be well modelled by a Poisson distribution if (i) the mean number of JC students who viewed the arts exhibits in each 15-minute interval remains **<u>constant</u>**, *or* (ii) the mean number of JC students who viewed the arts exhibits in other time intervals is **<u>proportional</u>** to the mean number for each 15-minute interval.	
(i)	Let X' denotes the random variable that represents the number of students who view the arts exhibits in a 30-minute period. Then $X' \sim \text{Po}(15)$ since $X \sim \text{Po}(7.5)$ Then the required probability $= \text{P}(X' < 12) = \text{P}(X' \leq 11)$ $= 0.184751799$ $= 0.185$ (3 s.f.)	

(ii)	The required probability $= \binom{4}{3}(1-0.184751799)^3(0.184751799)^1$ $= = \frac{4!}{3!1!}(0.815248201)^3(0.184751799)^1$ $= 0.400422262$ $= 0.400$ (3 s.f.) OR: Let Y denotes the number of 30-minute intervals among the 4 possible ones from 12 noon to 2 pm in which there are at least 12 students who view the arts exhibits. Then $Y \sim \mathrm{B}(4,1-0.184751799) = \mathrm{B}(4, 0.815248201)$ Then required probability = $\mathrm{P}(Y = 3) = 0.400422262$ $= 0.400$ (3 s.f.)	
(iii)	Let W denotes the random variable that represents the number of students who view the arts exhibits from 12 noon to 2 pm on a randomly chosen day during the Arts Festival. Since $X \sim \mathrm{Po}(7.5)$, we have $W \sim \mathrm{Po}(7.5\times 8)$ i.e. Po(60) For the 65-day period, $n = 65$ (> 50) is large. By the Central Limit Theorem (CLT), $\overline{W} \sim \mathrm{N}\left(60, \frac{60}{65}\right)$ approximately. Thus, $\mathrm{P}(\overline{W} > 60.5) = 0.3013866388$ $= 0.301$ (3 s.f.)	
	Next, let V be the random variable that denotes the number of days among 50 whereby there is only one 30 minutes interval among the 4 possible intervals daily, in which there are less than 12 students who view the arts exhibits. Then $V \sim \mathrm{B}(n, p)$ where $n = 50$, $p = 0.400422262$ (part ii) Then upon checking, $n = 50$ is large, $np = 50 \times 0.40042262 = 20.0211131 > 5$, $nq = 50 \times (1 - 0.40042262) = 29.9788869 > 5$ We conclude that $V \sim N(np, npq)$ approximately. That is, $V \sim N(20.0211131, 12.00421727)$ approximately. Then, $\mathrm{P}(V > 25)$ $= \mathrm{P}(V \geq 26)$ $= \mathrm{P}(V \geq 25.5)$ (continuity correction) $= 0.0569000711$ $= 0.0569$ (3 s.f.)	

Question 11 [11 marks]		
(i)		
(ii)	$r=-0.9168956355 \approx -0.917$ Although $\lvert r \rvert$ is close to 1, suggesting a linear relationship between *x* and *t* (i.e $t=mx+k$), it does appear that the points follow a curvilinear trend from the scatter diagram.	
(iii)	(B) $t=ce^{-x}+d$ is a better model because as *x* increases, *t* decreases at a decreasing rate. As for the model $t=a(x-10)^2+b$, *t* might decrease at a decreasing rate initially as *x* increases (for $x < 10$) but it will increase eventually (for $x > 10$). This does make sense for the context of the question since more practice on a longer period ought to improve his proficiency thus taking less time to complete the stage of the computer game.	
(iv)	For $t=ce^{-x}+d$: $r=0.9852371391 \approx 0.985$ For $t=a(x-10)^2+b$: $r=0.954421565 \approx 0.954$	
(v)	A suitable regression line: $t=89.60094125\,e^{-x}+16.73059924$ i.e. $t=89.6\,e^{-x}+16.7$ When $x=10$, $t=89.60094125\,e^{-10}+16.73059924$ $=16.73466712 \approx 16.7$ mins The estimate is not reliable because: (1) $x=10$ is outside of data range (extrapolation) (2) Andy has already obtained timings less than 16.7 minutes for both week 5 and week 6. It is more likely that the timing will be shorter than 16.7 min by week 8. If the wrong line is chosen: $t=0.5024319662(x-10)^2+2.706822862$ When $x=10$. $t=0.5024319662(-2)^2+2.706822862$ $=4.716550727 \approx 4.72$ mins The estimate is not reliable because $x=10$ is outside of data range (extrapolation).	

SAINT ANDREW'S JUNIOR COLLEGE

Preliminary Examination

MATHEMATICS
Higher 2 **9740/01**

Paper 1

Monday **29 August 2016** **3 hours**

Additional materials : Answer paper
List of Formulae (MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks : **100**

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of 7 printed pages including this page.
[Turn over

1 The sum $\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)}$ is denoted by S_n.

(i) By using the method of differences, find an expression for S_n in terms of n. **[3]**

(ii) Hence find the value of S_n as n tends to infinity. **[1]**

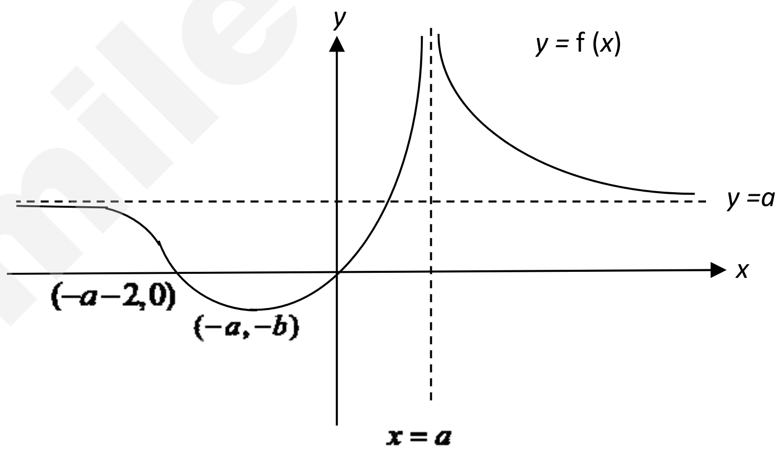
(iii) Find the smallest value of n for which S_n is within 2×10^{-4} of the sum to infinity. **[3]**

(iv) Using your answer in part **(i)**, find $\sum_{r=0}^{n} \frac{1}{(3r+1)(3r+4)}$ and deduce that

$\sum_{r=0}^{n} \frac{1}{(3r+4)^2} < \frac{1}{3}$. **[4]**

2 **(a)** By completing the square, or otherwise, state precisely a sequence of geometrical transformations which would transform the graph of $y = \ln(4x^2 - 16x + 15)$ onto the graph of $y = \frac{1}{2}\ln(4x^2 - 1)$. **[3]**

(b) The diagram shows the graph of $y = \mathrm{f}(x)$. The graph passes through the origin and the point $(-a-2, 0)$. It has a minimum point at $(-a, -b)$, $a > 1$, $b > 1$. The graph also has a vertical asymptote $x = a$ and a horizontal asymptote $y = a$.



Sketch, on separate diagrams, the graph of:

(i) $y = \frac{1}{\mathrm{f}(x)}$; **[2]**

(ii) $y = \mathrm{f}'(x)$, **[2]**

showing clearly all the asymptotes, turning points and axes intercepts.

3 **(a)** John deposits \$$x$ into a bank at the beginning of each year. The bank pays interest at a fixed rate of 5% of the amount at the end of each year. John then withdraws the interest as soon as it is added. Find, in terms of x and N, the total amount of interest he will collect at the beginning of $(N+1)$th year. **[3]**

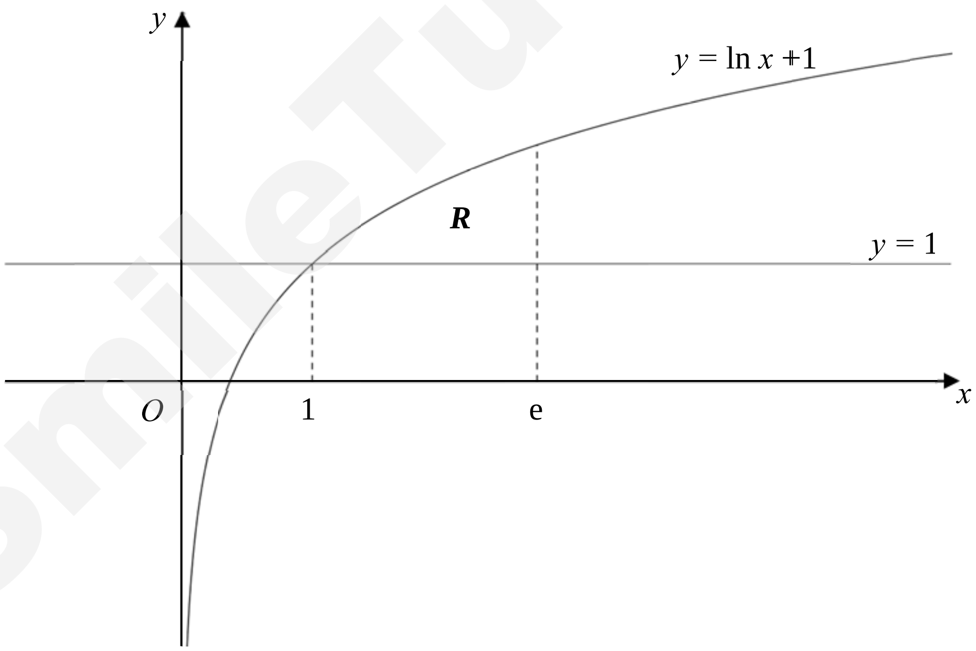
(b) An agricultural farm has 2000kg of vegetables. At the end of each week, the farm sells 10% of the vegetables and grows another 80kg on the farm.

(i) Find the amount of vegetables the farm has at the end of nth week, expressing your answer in the form $A(B^n)+C$, where A, B and C are constants to be determined. **[3]**

(ii) At which week will the amount of vegetables in the farm be first less than 835kg? **[2]**

4 **(i)** Find $\int u^2 e^u \, du$. **[3]**

(ii) The curve C has equation $y = \ln x + 1$ as shown below.



The region R is bounded by the curve C and the lines $x=1$, $x = \mathrm{e}$ and $y = 1$.

Write down the equation of the curve by translating C one unit in the negative y-direction.

Hence, using the substitution $u = \ln x$, evaluate the exact volume generated when R is rotated completely about the line $y=1$ by 2π radians. **[4]**

5 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \mathrm{e}^{2x} - 2\mathrm{e}^{x} + 3, \quad x \in \mathbb{R}$$

$$\mathrm{g}: x \mapsto \ln(2-x), \quad x \in \mathbb{R}, x < 2$$

(i) By sketching a graph, explain why the inverse function f^{-1} does not exist. **[2]**

(ii) Given that the domain of f is restricted to $(-\infty, a]$, state the maximum value of a for which f^{-1} exist. **[1]**

(iii) Using the value of a found in **(ii)** and by completing the square, find the inverse function f^{-1}. **[3]**

(iv) Find the exact range of gf^{-1}. **[2]**

6 Given that $y = \sqrt{4+\sin 2x}$, show that $y\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos 2x$. **[1]**

(i) By further differentiation of the above result, find the Maclaurin series for y in ascending powers of x up to and including the term in x^2 . **[3]**

(ii) Verify the correctness of the series found in **(i)** by using an appropriate standard series expansion. **[2]**

(iii) Deduce from part **(i)** the approximate value of $\int_0^{0.1} \sqrt{4-\sin 2x}\,\mathrm{d}x$, giving your answer to 5 significant figures. **[2]**

7 Relative to the origin O, the position vectors of two points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. The vector $\mathbf{a}$ is a unit vector which is perpendicular to $\alpha\mathbf{a}+\beta\mathbf{b}$, where $\alpha > 1$ and $\beta > 1$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{5\pi}{6}$.

(i) Show that $|\mathbf{b}| = \dfrac{2\sqrt{3}}{3}\left(\dfrac{\alpha}{\beta}\right)$. **[3]**

(ii) Give the geometrical interpretation of $|\mathbf{a} \bullet \mathbf{b}|$ and find its value in terms of α and β . **[3]**

(iii) The point M divides AB in the ratio $\lambda : 1-\lambda$ where $0 < \lambda < 1$. The point N is such that $OMNB$ is a parallelogram. Find $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and the area of the triangle OAN in terms of λ, α and β . **[5]**

8 The variables w, x and y are connected by the following differential equations:

$$\frac{dw}{dx} = -\frac{3}{2}w - 2 \quad \text{(A)}$$

$$\frac{dy}{dx} = w \quad \text{(B)}$$

(i) Solve equation (A) to find w in terms of x. **[3]**

(ii) Hence find y in terms of x. **[2]**

(iii) The result in part **(ii)** represents a family of curves. Some members of the family are straight lines. Write down the equation of one of these lines. On a single diagram, sketch your line together with a non-linear member of the family of curves that has your line as an asymptote, indicating clearly any axes intercepts. **[3]**

9 **(a)** **(i)** Solve $z^3 = 1 - i\sqrt{3}$, giving your answers in the form $re^{i\theta}$, where $r > 0$, and $-\pi < \theta \le \pi$. **[3]**

(ii) Show that

$$\left(z^n - 2e^{i\theta}\right)\left(z^n - 2e^{-i\theta}\right) = z^{2n} - 4z^n \cos\theta + 4,$$

Hence find the roots of the equation

$z^6 - 2z^3 + 4 = 0$ in the form of $re^{i\theta}$, where $r > 0$, and $-\pi < \theta \le \pi$. **[3]**

(b) Given that $z = \cos\theta + i\sin\theta$, show that $1 - z^2 = (-2i\sin\theta)z$. Given also that $0 < \theta < \pi$, find the modulus and argument of $1 - z^2$ in terms of θ. **[5]**

10

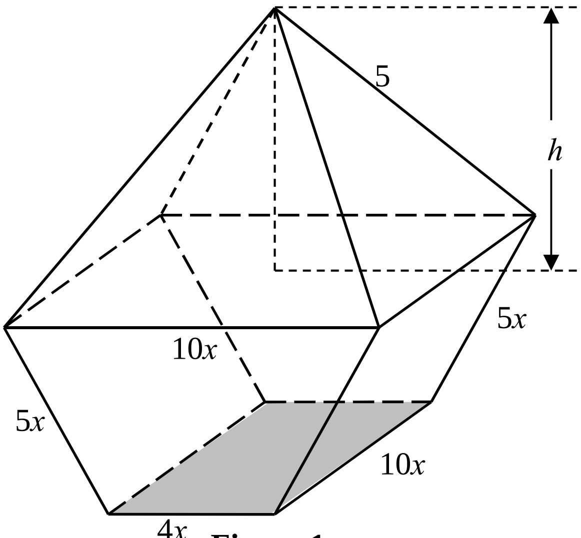


Figure 1

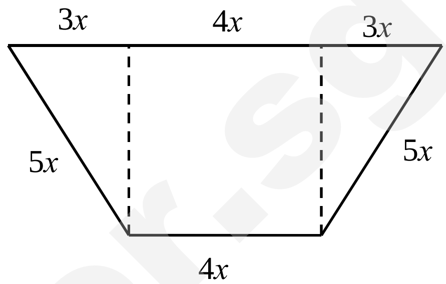


Figure 2

A designer decided to build a model as shown in **Figure 1** above, consisting of a base and a top. The base is made up of a prism with a cross-section of a trapezium where the length of the parallel sides are $4x$ cm and $10x$ cm (**Figure 2**). The top is a right pyramid with a square base of sides $10x$ cm, height h cm and a fixed slant height of 5 cm.

(i) Find an expression for the volume of the model, V, in terms of x. Given that $x = x_1$ is the value of x which gives the maximum value of V, show that x_1 satisfies the equation $13563x^4 - 7719x^2 + 625 = 0$. **[6]**

(ii) Find the two solutions to the equation in part **(i)** for which $x > 0$, giving your answers correct to 5 decimal places. **[2]**

(iii) Using both the solutions found in part **(ii)**, show that one of the values does not give a stationary value of V. Hence, write down the value of x_1. **[3]**

[Area of trapezium = $\frac{1}{2}(\text{sum of parallel sides}) \times \text{height}$;

Volume of pyramid = $\frac{1}{3}$base area $\times$ height]

11 The line l_1 and the planes p_1 and p_2 have equations as follows:

$$l_1 \;:\; x-5=-y-1, z=4;$$

$$p_1 \;:\; xa+z=5a+4,$$

$$p_2: \quad \mathbf{r}=\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\1\end{pmatrix}$$

where a is a positive constant and λ and μ are real numbers.

(i) Given that the acute angle between l_1 and p_1 is $\frac{\pi}{6}$, show that $a=1$. **[2]**

(ii) The planes p_1 and p_2 meet in the line l_2. Find a vector equation of l_2. **[2]**

(iii) Hence, find the values of α and β such that the system of equations

$$x+z=9$$
$$x+z=y$$
$$5x+4y+\alpha z=\beta$$

has

(a) more than one solution;

(b) exactly one solution.

If $\alpha=5$, $\beta=10$, give a geometrical interpretation of the relationship between the 3 equations. Explain your answer. **[6]**

End of Paper

SAINT ANDREW'S JUNIOR COLLEGE

Preliminary Examination

MATHEMATICS
Higher 2 **9740/02**

Paper 2

Thursday **15 September 2016** **3 hours**

Additional materials : Answer paper
List of Formulae (MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks : **100**

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
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Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
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You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of 7 printed pages including this page.
[Turn over

Section A: Pure Mathematics [40 marks]

1 Prove by the method of mathematical induction that $\sum_{r=2}^{n}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{n^{n-1}}{(n-1)!}\right]$. **[5]**

2 Solve the inequality $\frac{x^2+6x+8}{x-1}\geq 0$.

Hence, by completing the square, solve the inequality $\frac{y^2+2y+15}{|y+1|-1}\geq -6$. **[6]**

3 It is given that $\mathrm{f}(x)=\frac{5-ax^2}{1+x^2}$ where $a>1$, $a\in \mathbb{R}^+$

(i) Sketch $y=\mathrm{f}(x)$, showing clearly the coordinates of the turning point, any intersections with the axes and the equation(s) of any asymptote(s). **[3]**

(ii) By drawing a sketch of another suitable curve on the same diagram, find the number of real roots of the equation

$$x^4+(a+1)x^2-5=0 \ .$$ **[2]**

(iii) Let $\mathrm{g}(x)=x^4+(a+1)x^2-5$. Show that $\mathrm{g}(x)=\mathrm{g}(-x)$. What can be said about the four roots of the equation $\mathrm{g}(x)=0$? **[3]**

4 A curve C has parametric equations

$$x=a\sin 2t\ ,\ y=a\sin 3t$$

where $0\leq t\leq\frac{\pi}{2}$ and a is a positive constant.

(i) Find the gradient of C at the point $(a\sin 2\theta\ , a\sin 3\theta)$ where $0\leq\theta\leq\frac{\pi}{2}$. Hence, what can be said about the tangent to C as $\theta\to\frac{\pi}{4}$? **[3]**

(ii) Find the equation of the normal, in exact form, at the point where $t=\frac{\pi}{12}$. **[3]**

(iii) With the aid of a sketch, show that the area bounded by the curve C, the y-axis and the line $y = \frac{a\sqrt{2}}{2}$ can be written as

$$3a^2\int_0^{\frac{\pi}{12}}(\cos 3t \sin 2t)\,dt\ .$$

Hence, find the exact area of the region bounded by the curve C, the y-axis and the normal to the curve at $t = \frac{\pi}{12}$ in the form $ka^2\left[b\cos\frac{\pi}{12} + c\sin\frac{\pi}{12}\right] + da^2$, where k, b, c and d are constants to be determined. **[7]**

5 The complex number z is given by $z = re^{i\theta}$, where $1 \le r \le 2$ and $\frac{1}{6}\pi \le \theta \le \frac{3}{4}\pi$.

(i) State $|z|$ and arg(z) in terms of r and θ. Hence, draw an Argand diagram to show the locus of z as r and θ varies. You should identify the modulus and argument of the end-points of the locus. **[3]**

(ii) Find the exact minimum value of $|z+5-6\mathrm{i}|$ and the corresponding complex number z representing the point at which this minimum value occurs, giving your answer in the form x + iy, where x and y are real numbers. **[3]**

Another complex number w satisfies the equation $\arg(w - 2\sqrt{3}) = \frac{5\pi}{6}$.

(iii) On the same diagram as part **(i)**, sketch the locus of w and indicate the set of points that satisfies both the locus of z and w. **[2]**

Section B: Statistics [60 marks]

6 A survey is to be carried out to obtain feedback from the members of a new female-only fitness club regarding its various facilities and fitness classes. The membership of this fitness club comprises 5000 female members and the number of members belonging to the various age groups are given in the table below:

Age group	18 - 25	26-30	31 – 40	41 – 50	51 & above
Number of members	500	1000	1500	1500	500

It is proposed to carry out the survey by interviewing members who visit the club on a particular weekday in the morning.

(i) Explain why this proposed method is inappropriate. **[1]**

(ii) Suggest an appropriate method of carrying out the survey and describe how you intend to implement the sampling method to obtain a representative sample of 200 members. **[3]**

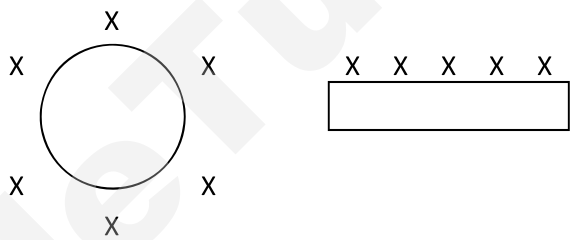
7 **(a)** For events X and Y, it is given that $P(X \cup Y) = \frac{5}{8}$, $P(X \cap Y') = \frac{7}{24}$ and $P(X'|Y) = \frac{9}{16}$.

(i) Find $P(X' \cap Y)$, **[3]**

(ii) Find $P(X)$ and determine if the events X and Y' are independent. **[3]**

(b) The Mathematics department consists of 5 female teachers and 6 male teachers. After a meeting, the department went to a nearby food court for lunch. Due to the lunch crowd, they only managed to find a circular table for 6 and a long table with a row of 5 seats as shown below. The seats at both tables are fixed and cannot be rearranged.

Ms Koh was among the Mathematics teachers who attended the lunch.



(i) Find the probability that Ms Koh is seated between 2 male teachers. **[4]**

(ii) Given that Ms Koh is seated between 2 male teachers, find the probability that the male and female teachers alternate at both tables. **[3]**

8 **(i)** Given that $X \sim N(\mu, \sigma^2)$ and $P(X < 28.4) = P(X > 77.6) = 0.012$, find the value of μ and σ. **[3]**

(ii) The mass, in grams, of a randomly chosen packet of sweets is normally distributed with mean μ and variance σ^2 obtained from part **(i)**. Every packet of sweets is priced at $1.20 per 100 grams. Find the probability that the sum of 4 packets of sweets cost at most $2.60. **[3]**

9 The hens on a farm lay either white or brown eggs. The eggs are randomly put into boxes of six. The farmer claims that the number of brown eggs in a box can be modelled by a binomial distribution $\mathrm{B}(6, p)$.

(i) State, in context, two assumptions to support the farmer's claim. **[2]**

(ii) Given that the probability a box contains at least 5 brown eggs is 0.04096, find the value of p. **[2]**

A supermarket orders 100 boxes of eggs daily.

(iii) By using a suitable approximation, find the probability that there are at least 90 boxes that contains at most 4 brown eggs in a particular day. **[3]**

(iv) The supermarket places a daily order of 100 boxes of eggs for 8 weeks. Estimate the probability that the mean number of boxes that contains at least 5 brown eggs in a day is between 4 and 7.
[You may assume that there are 7 days in a week.] **[3]**

10 A researcher wishes to investigate the length of time that patients spend with a doctor at a particular clinic. The time a patient spends with the doctor is denoted by X minutes. Based on past records, the clinic claims that the mean length of time for the doctor to see a patient is at most 10 minutes. To test this claim, the researcher recorded the actual times spent by the doctor to see a random sample of 12 patients.

$$\sum x = 147, \quad \sum x^2 = 1927.91$$

(i) Stating a necessary assumption, carry out an appropriate test, at the 5% significance level, to determine whether there is any evidence to doubt the clinic's claim. **[5]**

(ii) Suppose now that the population standard deviation of X is 15 and that the assumption made in part **(i)** is still valid. A new sample of n patients is obtained and the sample mean length of time is found to be unchanged. Using this sample, the researcher conducts another test and found that the null hypothesis is not rejected at the 5% significance level. Obtain an inequality involving n and find the set of values that n can take. **[3]**

11 An experiment is conducted to calibrate an anemometer*. In this calibration process, the wind speed X is fixed precisely and the resulting anemometer speed Y is recorded.

For a particular anemometer, this process produced the following set of measurements:

Wind speed (m/s), X	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0
Anemometer (revs/min), Y	24	28	47	92	164	236	312	360

(i) Calculate the product moment correlation coefficient between X and Y. **[1]**

(ii) Sketch a scatter diagram for the data. **[1]**
Explain why it is advisable to sketch the scatter diagram in addition to calculating the product moment correlation coefficient before interpreting this set of bivariate data. **[1]**

A proposed model for the above data is $Y = a + bX^2$.

(iii) Calculate the product moment correlation coefficient and the equation of the least squares regression line for the proposed model.
Explain whether the model for Y on X^2 or Y on X is a better model for the data set. **[3]**

(iv) Use an appropriate regression line to estimate the value of X when the value of Y is 120. Give a reason for the choice of your regression line. **[2]**

[* *An anemometer is a device commonly used in a weather station for measuring wind speed.*]

12 The managers of 2 branches of a travel agency were discussing whether the number of customers who bought the Luxury Cruise Package per week could be modelled by a Poisson distribution. One of the managers said, "It must be assumed that the number of customers who bought the package per week is a constant."

(i) Give a corrected version of the manager's statement, and explain why the correction is necessary. **[1]**

It is given that the number of customers who bought the Luxury Cruise Package per week can be modelled by a Poisson distribution. The average number of customers who bought the Luxury Cruise Package per week at Branch A and Branch B is 3.5 and 4.5 respectively. Assume that the number of customers who bought the package at Branch A and Branch B are independent.

(ii) Find the probability that, in a randomly chosen week, the total number of customers who bought the Luxury Package at both branches is between 5 and 10. **[2]**

(iii) Given that the probability that at most one customer bought the Luxury Cruise Package at Branch A in n weeks is less than 0.1, find the value of the least n. **[3]**

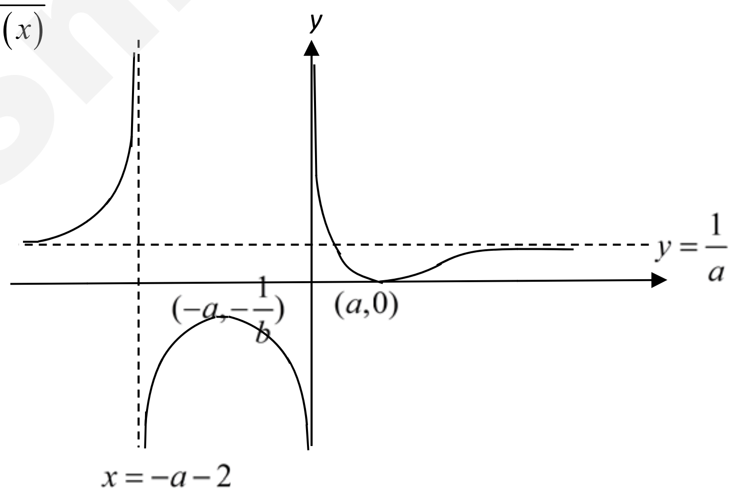
(iv) Using a suitable approximation, find the probability that, in one month, the number of customers who bought the Luxury Cruise Package at Branch B exceeds the number of customers at Branch A by not more than 5. **[4]**
[You may assume that there are 4 weeks in 1 month.]

Explain why the Poisson distribution may not be a good model for the number of customers who bought the Luxury Cruise Package in a year. **[1]**

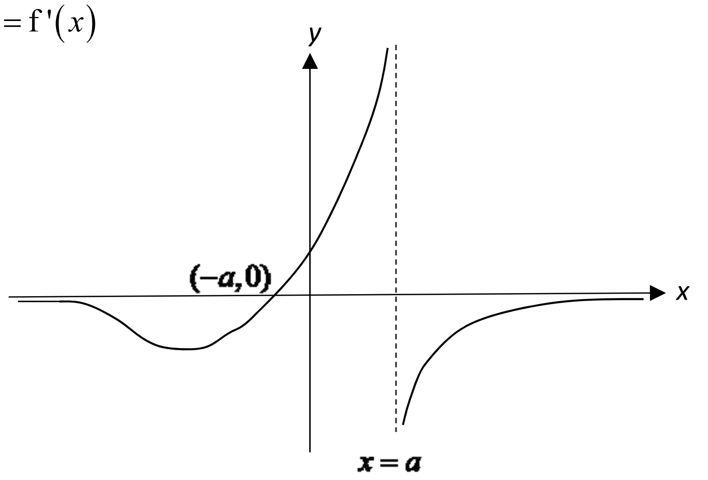
End of Paper

2016 Prelim Paper 1 Solution

1(i)	$\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)}$ $= \frac{1}{3}\sum_{r=1}^{n}\left(\frac{1}{3r-2} - \frac{1}{3r+1}\right)$ $= \frac{1}{3}[\ 1 - \cancel{\frac{1}{4}}$ $+\ \cancel{\frac{1}{4}} - \cancel{\frac{1}{7}}$ $+\ \cancel{\vdots}$ $+\ \cancel{\frac{1}{3n-5}} - \cancel{\frac{1}{3n-2}}$ $+\ \cancel{\frac{1}{3n-2}} - \frac{1}{3n+1}]$ $= \frac{1}{3}\left(1 - \frac{1}{3n+1}\right)$
1(ii)	As $n \to \infty, S_n \to \frac{1}{3}$ since $\frac{1}{3n+1} \to 0$
(iii)	$\lvert S_n - S \rvert < 2\times10^{-4}$ $\left\lvert \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) - \frac{1}{3} \right\rvert < 2\times10^{-4}$ $\left\lvert -\frac{1}{3}\left(\frac{1}{3n+1}\right) \right\rvert < 2\times10^{-4}$ $\frac{1}{3}\left(\frac{1}{3n+1}\right) < 2\times10^{-4}$ since $n \in \mathbb{Z}^+$ $\frac{1}{3n+1} < \frac{3}{5000}$ $3n+1 > \frac{5000}{3}$ $n > 555.2$ Hence, smallest $n = 556$.
(iv)	From (i) $\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)} = \frac{1}{(1)(4)} + \frac{1}{(4)(7)} + \frac{1}{(7)(10)} + \ldots + \frac{1}{(3n-2)(3n+1)} =$ $\frac{1}{3}\left(1 - \frac{1}{3n+1}\right)$

	$\sum_{r=0}^{n}\frac{1}{(3r+1)(3r+4)}=\frac{1}{(1)(4)}+\frac{1}{(4)(7)}+\frac{1}{(7)(10)}+\ldots+\frac{1}{(3n-2)(3n+1)}+\frac{1}{(3n+1)(3n+4)}$ $=\sum_{r=1}^{n+1}\frac{1}{(3r-2)(3r+1)}=\frac{1}{3}\left(1-\frac{1}{3n+4}\right)$ $(3r+4)^2>(3r+1)(3r+4)$ for $r\ge 0, r\in\mathbb{Z}$ $\frac{1}{(3r+4)^2}<\frac{1}{(3r+1)(3r+4)}$ $\sum_{r=0}^{n}\frac{1}{(3r+4)^2}<\sum_{r=0}^{n}\frac{1}{(3r+1)(3r+4)}<\frac{1}{3}$ since $1-\frac{1}{3n+4}<1$ for $n\in\mathbb{Z}\cup\{0\}$
2 (a)	$y=\ln(4x^2-16x+15)$ $=\ln(4(x^2-4x)+15)$ $=\ln(4(x-2)^2-16+15)$ $=\ln(4(x-2)^2-1)$ Let f(x) = $\ln(4(x-2)^2-1)$ and $y=\frac{1}{2}\ln(4x^2-1)=\frac{1}{2}$ f(x + 2) A translation to the –2 units in the direction of the x– axis followed by; A scaling of a scale factor of ½ parallel to y– axis .
(b)(i)	$y=\frac{1}{\mathrm{f}(x)}$ 

(ii) $y = \mathrm{f}'(x)$



3(a)

Year No	Amount withdrawn at the beginning of N years
1	0
2	$0.05x$
3	$0.05(2x)$
…	
$N+1$	$T_N = 0.05Nx$

$S_{N+1} = 0.05x(1+2+3...N)$

$= \dfrac{0.05xN(N+1)}{2}$

$= \dfrac{xN(N+1)}{40}$

(i)

Week No	Amount of vegetables on the farm
1	2000 (0.9) +80
2	$=\left((2000\times 0.9)+80\right)\times 0.9+80$ $=(2000\times 0.9^2)+(80\times 0.9)+80$

3	$=\left((2000\times0.9^2)+(80\times0.9)+80\right)\times0.9)+80$ $=(2000\times0.9^3)+(80\times0.9^2)+\left(80\times0.9\right)+80$
….	
n	$=(2000\times0.9^n)+(80\times0.9^{n-1})...+\left(80\times0.9\right)+80$ $=(2000\times0.9^n)+80\left[0.9^{n-1}+0.9^{n-2}+...1\right]$ $=(2000\times0.9^n)+80\left[\frac{(1-0.9^n)}{1-0.9}\right]$ $=(2000\times0.9^n)+800(1-0.9^n)$ $=(1200\times0.9^n)+800$

A =1200, $B = 0.9$, $C = 800$

(ii)

$(1200\times0.9^n)+800<835$

Solving,

$(1200\times0.9^n)<35$

$0.9^n<\frac{35}{1200}$

$n>33.549$

Least number is 34 weeks

Alternatively use GC

When n = 33, $(1200\times0.9^n)+800$=837.08

When n = 34, $(1200\times0.9^n)+800$=833.38

When n = 35, $(1200\times0.9^n)+800$= 830.04

4(i)

$U=u^2 \quad \frac{\mathrm{d}V}{\mathrm{d}u}=e^u$

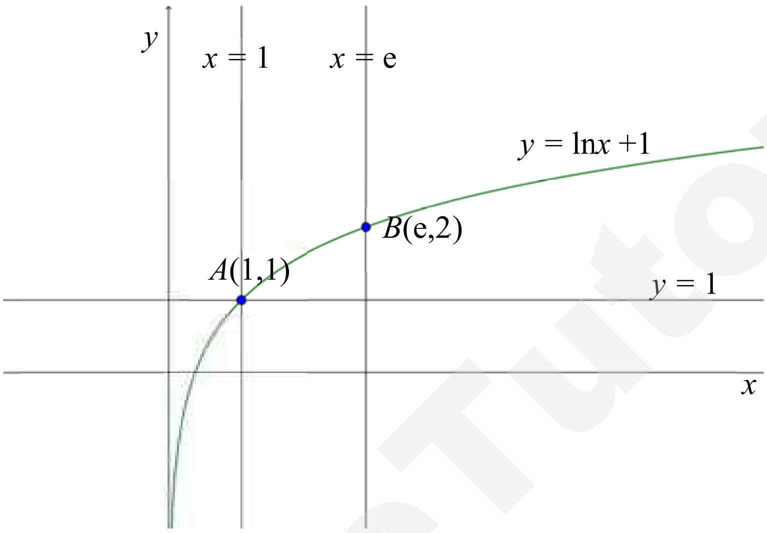
$\frac{\mathrm{d}U}{\mathrm{d}u}=2u \quad V=e^u$

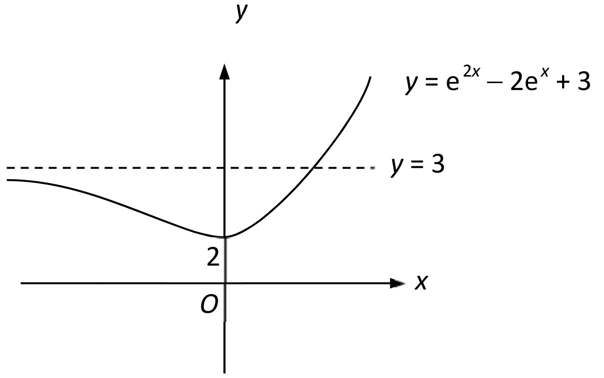
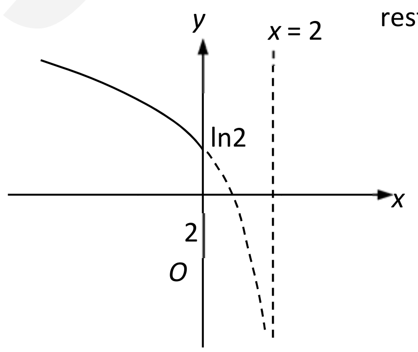
$\int u^2e^u\ \mathrm{d}u$

$=\left[u^2e^u\right]-2\int ue^u\ \mathrm{d}u$

$U=u \quad \frac{\mathrm{d}V}{\mathrm{d}u}=e^u$

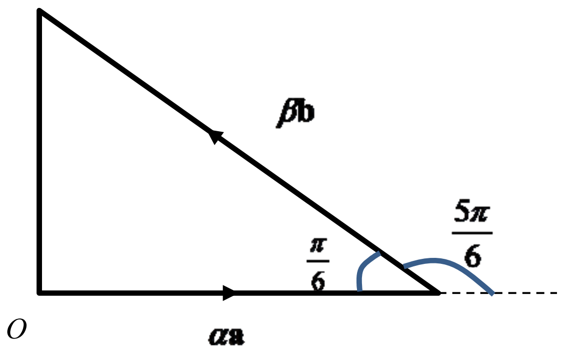
$\frac{\mathrm{d}U}{\mathrm{d}u}=1 \quad V=e^u$

	$u^2e^u - 2\int ue^u \, du$ $= u^2e^u - 2\left\{\left[ue^u\right] - \int e^u \, du\right\}$ $= u^2e^u - 2ue^u + 2e^u + C$, where C was an arbitrary constant
4	At $x = 1$, $y = 1$ At $x = \text{e}$, $y = 2$  After translations, the graph is: $y_1 = \ln x$, with A' (1 , 0) and B' (e , 1) $u = \ln x$ $x = e^u$ $\frac{dx}{du} = e^u$ $x = 1, u = 0$ $x = e, u = 1$ $V = \pi\int_1^e \left[(\ln x)^2\right] dx$ $= \pi\int_0^1 u^2 \frac{dx}{du} du$ $= \pi\int_0^1 u^2e^u \, du$ $= \pi\left[u^2e^u - 2ue^u + 2e^u\right]_0^1$ $= \pi\left[e - 2e + 2e - 2\right]$ $= \pi(e-2)$ units^3

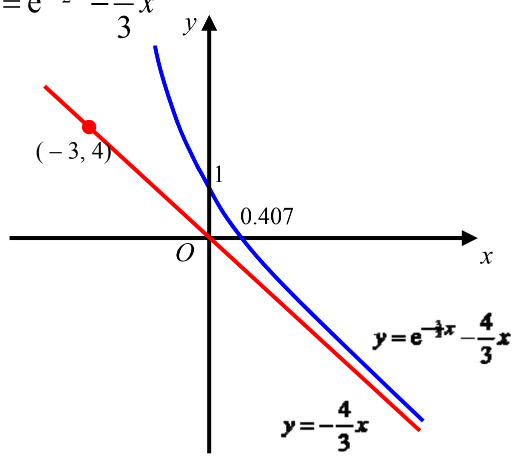
5 (i)	 As the horizontal line $y = k$, $k \in (2,3)$ cuts the graph $y = f(x)$ more than once, f is not one - one. Hence f^{-1} does not exist.
(ii)	Max $a = 0$.
(iii)	Let $y = e^{2x} - 2e^{x} + 3$ $y = (e^{x})^2 - 2e^{x} + 3$ $y = (e^{x} - 1)^2 + 2$ $x = \ln(1+\sqrt{y-2})$ (rejected, since $x \leq 0$) . or $\ln(1-\sqrt{y-2})$ $\therefore f^{-1} : x \mapsto \ln(1-\sqrt{x-2})$, $x \in \mathbb{R}$ $2 \leq x < 3$.
(iv)	$[2,3) \xrightarrow{f^{-1}} (-\infty,0] \xrightarrow{g} [\ln 2,\infty)$ $R_{gf^{-1}} = [\ln 2,\infty)$  Graph of g when D_g is restricted to $(-\infty,0]$

6	$y=\sqrt{4+\sin 2x}$ $y^2=4+\sin 2x$ Differentiating implicitly with respect to x, $2y\frac{\mathrm{d}y}{\mathrm{d}x}=2\cos 2x$ $y\frac{\mathrm{d}y}{\mathrm{d}x}=\cos 2x$ …. (1)
(i)	Differentiating (1) implicitly with respect to x, $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-2\sin 2x$ …. (2) When $x = 0$, $y=\sqrt{4+0}=2$ From (1) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}$ From (2) $2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{1}{2}\right)^2=0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{1}{8}$ The Maclaurin's Series of y is $y=2+\frac{1}{2}x-\frac{1}{8}\times\frac{x^2}{2!}+\cdots$ $y\approx 2+\frac{1}{2}x-\frac{x^2}{16}$, up to and including the term in x^2
(ii)	By using the standard series of $\sin x$, $\sin 2x \approx 2x$ $\sqrt{4+\sin 2x}\approx\sqrt{4+2x}$ $=(4+2x)^{\frac{1}{2}}$ $=2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}$

	$\sqrt{4+\sin 2x}=2\left[1+\frac{1}{2}\left(\frac{x}{2}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\cdots\right]$ $=2\left(1+\frac{x}{4}-\frac{x^2}{32}+\cdots\right)$ $=2+\frac{x}{2}-\frac{x^2}{16}+\cdots$
(iii)	$\int_0^{0.1}\sqrt{4-\sin 2x}\ \mathrm{d}x=\int_0^{0.1}\sqrt{4+\sin(-2x)}\ \mathrm{d}x$ $=\int_0^{0.1}\left(2-\frac{x}{2}-\frac{x^2}{16}\right)\mathrm{d}x$ (Replace x with $-x$) ≈ 0.19748 (to 5 s.f.)
7 (i)	$\mathbf{a}\bullet(\alpha\mathbf{a}+\beta\mathbf{b})=0$ $\alpha\lvert\mathbf{a}\rvert^2+\beta\mathbf{a}\bullet\mathbf{b}=0$ $\alpha+\beta\mathbf{a}\bullet\mathbf{b}=0$ $\mathbf{a}\bullet\mathbf{b}=-\frac{\alpha}{\beta}$ Since angle between **a** and **b** is $\frac{5\pi}{6}$, $\cos\left(\frac{5\pi}{6}\right)=\frac{\mathbf{a}\bullet\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ $-\frac{\sqrt{3}}{2}=\frac{-\frac{\alpha}{\beta}}{\lvert\mathbf{b}\rvert}$ $\lvert\mathbf{b}\rvert=\frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)$ (shown) Or

	 $\cos\left(\frac{\pi}{6}\right)=\frac{\alpha\lvert\mathbf{a}\rvert}{\beta\lvert\mathbf{b}\rvert}$ $\lvert\mathbf{b}\rvert=\frac{\alpha\lvert\mathbf{a}\rvert}{\beta\cos\left(\frac{\pi}{6}\right)}$ $\lvert\mathbf{b}\rvert=\frac{\alpha}{\beta\frac{\sqrt{3}}{2}}$ $=\frac{2\alpha}{\sqrt{3}\beta}$ $=\frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)$
(ii)	$\lvert\mathbf{a}\bullet\mathbf{b}\rvert$ is the length of projection of $\mathbf{b}$ onto $\mathbf{a}$ $\lvert\mathbf{a}\bullet\mathbf{b}\rvert=\left\lvert\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\left(\frac{5\pi}{6}\right)\right\rvert$ $=\lvert\mathbf{b}\rvert\frac{\sqrt{3}}{2}$ $=\frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)\frac{\sqrt{3}}{2}$ $=\left(\frac{\alpha}{\beta}\right)$
(iii)	By Ratio theorem, $\overrightarrow{OM}=\lambda\mathbf{b}+(1-\lambda)\mathbf{a}$

	$\overrightarrow{ON}=\overrightarrow{OM}+\overrightarrow{MN}$ $=\left[\lambda\mathbf{b}+(1-\lambda)\mathbf{a}\right]+\overrightarrow{OB}$ $=\left[\lambda\mathbf{b}+(1-\lambda)\mathbf{a}\right]+\mathbf{b}$ $=(\lambda+1)\mathbf{b}+(1-\lambda)\mathbf{a}$ Area of triangle OAN $=\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{ON}\right\|$ $=\frac{1}{2}\left\|\mathbf{a}\times\left[(\lambda+1)\mathbf{b}+(1-\lambda)\mathbf{a}\right]\right\|$ $=\frac{1}{2}\left\|(\lambda+1)\mathbf{a}\times\mathbf{b}+(1-\lambda)\mathbf{a}\times\mathbf{a}\right\|$ $=\frac{1}{2}(\lambda+1)\left\|\mathbf{a}\times\mathbf{b}\right\|$ since $\left\|\lambda+1\right\|=\lambda+1$ as $0<\lambda<1$ $=\frac{1}{2}(\lambda+1)\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|\left\|\sin\left(\frac{5\pi}{6}\right)\right\|$ $=\frac{(\lambda+1)}{2}\left(\frac{2\sqrt{3}}{3}\right)\left(\frac{\alpha}{\beta}\right)\left(\frac{1}{2}\right)$ $=\frac{(\lambda+1)\sqrt{3}}{6}\left(\frac{\alpha}{\beta}\right)$
8(i)	$\frac{\mathrm{d}w}{\mathrm{d}x}=-(\frac{3}{2}w+2)$ $\int\frac{1}{\frac{3}{2}w+2}\,\mathrm{d}w=\int-1\,\mathrm{d}x$ $\frac{2}{3}\int\frac{\frac{3}{2}}{\frac{3}{2}w+2}\,\mathrm{d}w=\int-1\,\mathrm{d}x$ $\frac{2}{3}\ln\left\|\frac{3}{2}w+2\right\|=-x+A$ where A is an arbitrary constant $\ln\left\|\frac{3}{2}w+2\right\|=-\frac{3}{2}x+\frac{3}{2}A$

	$\ln\left\lvert\frac{3}{2}w+2\right\rvert=-\frac{3}{2}x+\frac{3}{2}A$ $\left\lvert\frac{3}{2}w+2\right\rvert=\mathrm{e}^{-\frac{3}{2}x+\frac{3}{2}A}=\mathrm{e}^{\frac{3}{2}A}.\mathrm{e}^{-\frac{3}{2}x}$ $\frac{3}{2}w+2=\pm\mathrm{e}^{\frac{3}{2}A}.\mathrm{e}^{-\frac{3}{2}x}=B\mathrm{e}^{-\frac{3}{2}x}$ where $B=\pm\mathrm{e}^{\frac{3}{2}A}$ $w=\frac{2}{3}B\mathrm{e}^{-\frac{3}{2}x}-\frac{4}{3}$ $w=C\mathrm{e}^{-\frac{3}{2}x}-\frac{4}{3}$ where $C=\frac{2}{3}B$
8(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=C\mathrm{e}^{-\frac{3}{2}x}-\frac{4}{3}$ $\int\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x=\int\left(C\mathrm{e}^{-\frac{3}{2}x}-\frac{4}{3}\right)\mathrm{d}x$ $y=-\frac{2}{3}C\mathrm{e}^{-\frac{3}{2}x}-\frac{4}{3}x+E$ where E is an arbitrary constant $y=D\mathrm{e}^{-\frac{3}{2}x}-\frac{4}{3}x+E$ where $D=-\frac{2}{3}C$
8(iii)	When $D=0$ (or $C=0$) and $E=0$, $y=-\frac{4}{3}x$ When $D=1$ (or $C=-\frac{3}{2}$) [or any value of D or C] and $E=0$ [or any value of E corresponding to the choice of E above], $y=\mathrm{e}^{-\frac{3}{2}x}-\frac{4}{3}x$ 

9 (a)

(i)

$$z^3 = 1 - \mathrm{i}\sqrt{3}$$
$$= 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}+2k\pi\right)}, \ k \in \mathbb{Z}$$
$$z = 2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\,\frac{1}{3}\left(-\frac{\pi}{3}+2k\pi\right)}, \ k = 0, \pm 1$$
$$= 2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{9}+\frac{2k\pi}{3}\right)}, \ k = 0, \pm 1$$

$$z = 2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{5\pi}{9}}, 2^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{9}}, 2^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{7\pi}{9}}$$

(ii)

$$\left(z^n - 2\mathrm{e}^{\mathrm{i}\theta}\right)\left(z^n - 2\mathrm{e}^{-\mathrm{i}\theta}\right)$$
$$= z^{2n} - 2z^n\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right) + 4$$
$$= z^{2n} - 2z^n\left(\cos\theta + \mathrm{i}\sin\theta + \cos(-\theta) + \mathrm{i}\sin(-\theta)\right) + 4$$
$$= z^{2n} - 2z^n\left(\cos\theta + \mathrm{i}\sin\theta + \cos\theta - \mathrm{i}\sin\theta\right) + 4$$
$$= z^{2n} - 4z^n\cos\theta + 4 \quad \text{(shown)}$$

Hence,

$$z^6 - 2z^3 + 4 = 0$$

$$z^6 - 4\left(\frac{1}{2}\right)z^3 + 4 = 0$$

$$z^6 - 4\left(\cos\frac{\pi}{3}\right)z^3 + 4 = 0$$

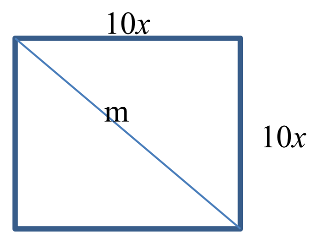
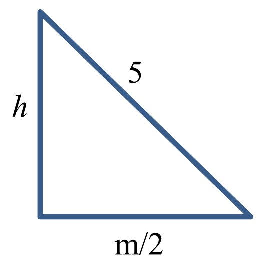
$$\left(z^3 - 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)\left(z^3 - 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right) = 0$$

The roots of are $z^3 = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$

$$z = 2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{5\pi}{9}}, 2^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{9}}, 2^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{7\pi}{9}} \quad \text{(from (i))}$$

Since the coefficients of the equation are all real, complex roots occur in conjugate

	pairs. Therefore, for $z^3 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$, the roots are $z = 2^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}, 2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}, 2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}}$
9(b)	$1 - z^2 = 1 - (\cos 2\theta + \mathrm{i}\sin 2\theta)$ $= 1 - \cos 2\theta - \mathrm{i}(2\sin\theta\cos\theta)$ $= 2\sin^2\theta - \mathrm{i}(2\sin\theta\cos\theta)$ $= (-2i\sin\theta)(\cos\theta + \mathrm{i}\sin\theta)$ $= (-2\mathrm{i}\sin\theta)z$ (shown) Alternatively : $1 - z^2 = 1 - (e^{i2\theta})$ $= e^{i\theta}(e^{-i\theta} - e^{i\theta})$ $= e^{i\theta}(\cos\theta - i\sin\theta - \cos\theta - i\sin\theta)$ $= z(-2i\sin\theta)$ (Shown) $\left\|1 - z^2\right\| = \left\|-2i\sin\theta\right\|\left\|z\right\| = 2\sin\theta$ $\arg(1 - z^2) = \arg(-2\mathrm{i}\sin\theta) + \arg(z)$ $= \arg(2\sin\theta) + \arg(-\mathrm{i}) + \arg(z)$ $= \theta - \frac{\pi}{2}$
10(i)	Let V be the total volume of the solid and h be the height of the pyramid. Height of trapezium, l $= \sqrt{25x^2 - 9x^2}$ $= \sqrt{16x^2}$ $= 4x$ $V_{\text{base}} = \frac{1}{2}(4x + 10x)(4x)(10x)$ $= 280x^3$

	 $m=\sqrt{2(10x)^2}$ $=10\sqrt{2}x$ $\frac{m}{2}=5\sqrt{2}x$  $h=\sqrt{25-\left(5\sqrt{2}x\right)^2}$ $=\sqrt{25-50x^2}$ $V_{pyramid}=\frac{1}{3}(10x)^2\left(\sqrt{25-50x^2}\right)$ $=\frac{100x^2}{3}\left(\sqrt{25-50x^2}\right)$ $V=280x^3+\frac{100x^2}{3}\sqrt{25-50x^2}$

	$\frac{\mathrm{d}V}{\mathrm{d}x}=840x^2+\frac{100}{3}\left\{x^2\left(\frac{1}{2\sqrt{25-50x^2}}\right)(-100x)+2x\sqrt{25-50x^2}\right\}$ $=840x^2+\frac{100}{3}\left\{2x\sqrt{25-50x^2}-\frac{50x^3}{\sqrt{25-50x^2}}\right\}$ $=840x^2+\frac{100}{3}(2x)\left\{\sqrt{25-50x^2}-\frac{25x^2}{\sqrt{25-50x^2}}\right\}$ $=\frac{20}{3}x\left\{126x+10\left[\sqrt{25-50x^2}-\frac{25x^2}{\sqrt{25-50x^2}}\right]\right\}$ For stationary values of V, $\frac{\mathrm{d}V}{\mathrm{d}x}=0$ Since $x>0$, $126x+10\left[\sqrt{25-50x^2}-\frac{25x^2}{\sqrt{25-50x^2}}\right]=0$ $126x=-10\left[\sqrt{25-50x^2}-\frac{25x^2}{\sqrt{25-50x^2}}\right]$ $126x\sqrt{25-50x^2}=-10\left[\left(25-50x^2\right)-25x^2\right]$ $126x\sqrt{25-50x^2}=-10\left[25-75x^2\right]$ $=750x^2-250$ Squaring both sides, $\left(126x\sqrt{25-50x^2}\right)^2=\left(750x^2-250\right)^2$ $15876x^2\left(25-50x^2\right)=562500x^4-375000x^2+62500$ $396900x^2-793800x^4=562500x^4-375000x^2+62500$ $3969x^2-7938x^4=5625x^4-3750x^2+625$ $13563x^4-7719x^2+625=0$
(ii)	Using G.C., since $x>0$, the two values of x are $0.6865562\approx0.68656$ or $0.3126699\approx0.31267$ (to 5 dp).
(iii)	$\frac{\mathrm{d}V}{\mathrm{d}x}=\frac{20}{3}x\left\{126x+10\left[\sqrt{25-50x^2}-\frac{25x^2}{\sqrt{25-50x^2}}\right]\right\}$

	For $x = 0.6865562$, $\frac{dV}{dx} = \frac{20}{3}(0.6865562)\{126(0.6865562)$ $+10\left[\sqrt{25-50(0.6865562)^2} - \frac{25(0.6865562)^2}{\sqrt{25-50(0.6865562)^2}}\right]\}$ $= 0.03789$ ≈ 0 For $x = 0.3126699$, $\frac{dV}{dx} = \frac{20}{3}(0.3126699)\{126(0.3126699)$ $+10\left[\sqrt{25-50(0.3126699)^2} - \frac{25(0.3126699)^2}{\sqrt{25-50(0.3126699)^2}}\right]\}$ $= 164.24$ $\neq 0$ Hence $x_1 = 0.68656$
11 (i)	$l_1 : \mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $p_1 : \mathbf{r} \bullet \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} = 5a + 4$ Acute $\angle$ between l_1 and $p_1 = \frac{\pi}{6}$ $\Rightarrow$ Acute $\angle$ between $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} = \frac{\pi}{3}$

	$\cos\left(\frac{\pi}{3}\right)=\frac{\left|\begin{pmatrix}a\\0\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\0\end{pmatrix}\right|}{\sqrt{a^2+1}\sqrt{2}}$ $\frac{1}{2}=\frac{a}{\sqrt{2\left(a^2+1\right)}}$ $2\left(a^2+1\right)=4a^2$ $2a^2=2$ $a=\pm 1$ Since $a>0$, $a=1$. (proven)
(ii)	p_2: $\mathbf{r}=\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\1\end{pmatrix}$ Normal vector of $p_2=\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $p_2:\mathbf{r}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$ Using GC, solving $x+z=9$ and $x-y+z=0$, $x=9-z$ $y=9$ $z=z$ Therefore equation of l_2, $\underset{\sim}{r}=\begin{pmatrix}9\\9\\0\end{pmatrix}+s\begin{pmatrix}-1\\0\\1\end{pmatrix}, s\in\mathbb{R}$
(iii)	$x+z=9\cdots(1)$ $x+z=y\cdots(2)$ $5x+4y+\alpha z=\beta\cdots(3)$ (1) and (2) are the cartesian equations of planes p_1 and p_2 respectively.

	Let $p_3 : \mathbf{r} \bullet \begin{pmatrix} 5 \\ 4 \\ \alpha \end{pmatrix} = \beta$ Since the system of equations is known to have more than one solution, p_1, p_2 and p_3 intersect at l_2. Therefore, l_2 lies in p_3. $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \\ \alpha \end{pmatrix} = 0$ $-5 + \alpha = 0$ $\alpha = 5$ $\begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \\ 5 \end{pmatrix} = \beta$ $\beta = 81$
	Since the system of equations is known to have exactly one solution, p_1, p_2 and p_3 intersect at a point. Therefore, l_2 intersects p_3 at a point. $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \\ \alpha \end{pmatrix} \neq 0$ $-5 + \alpha \neq 0$ $\alpha \neq 5$ $\beta \in \mathbb{R}$
	Since $\alpha = 5$ and $\beta \neq 81$, the planes do not intersect at a common point. Since the 3 planes are not parallel to each other, the 3 planes form a triangular prism.

1

Let P(n) be the statement, $\sum_{r=2}^{n}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{n^{n-1}}{(n-1)!}\right]$, , n = 2,3,4,... .

When n = 2,

LHS = $(2-1)\ln\left(\frac{2}{2-1}\right)=\ln 2$

RHS = $\ln\left[\frac{2^{2-1}}{(2-1)!}\right]=\ln 2$

$\therefore$ LHS = RHS

P(2) is true.

Assume that P(k) is true for some positive integer k, , k = 2,3,4,…

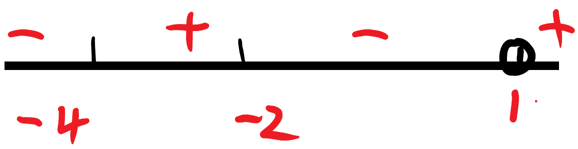
i.e. Assume $\sum_{r=2}^{k}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{k^{k-1}}{(k-1)!}\right]$.

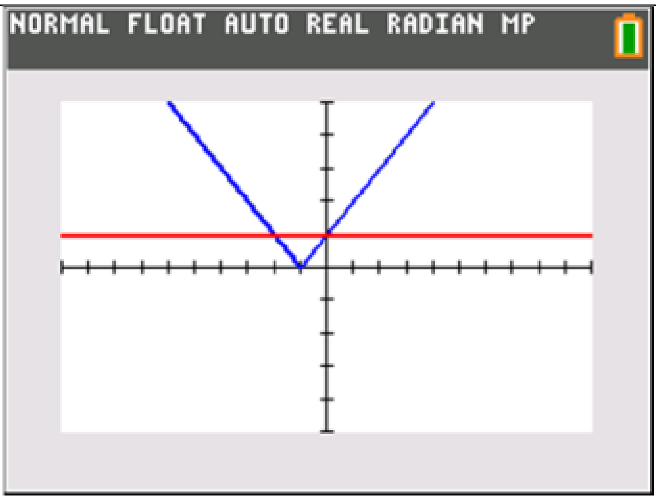
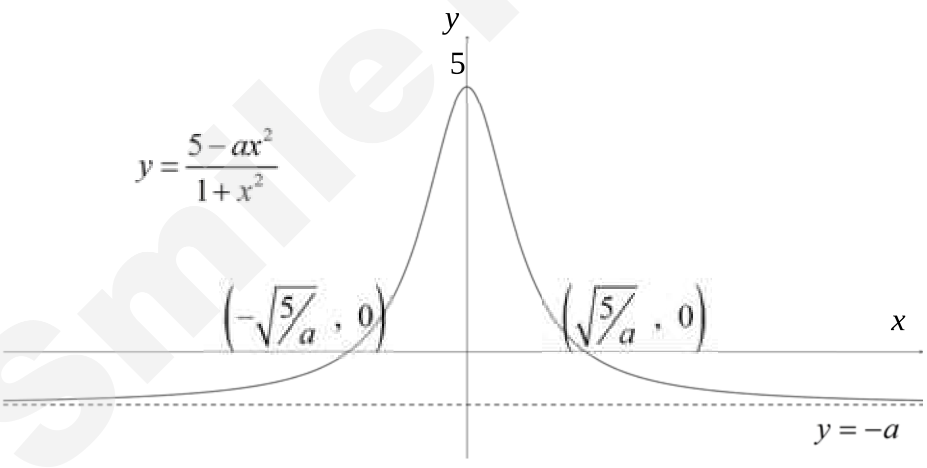
To prove P(k + 1) is true.

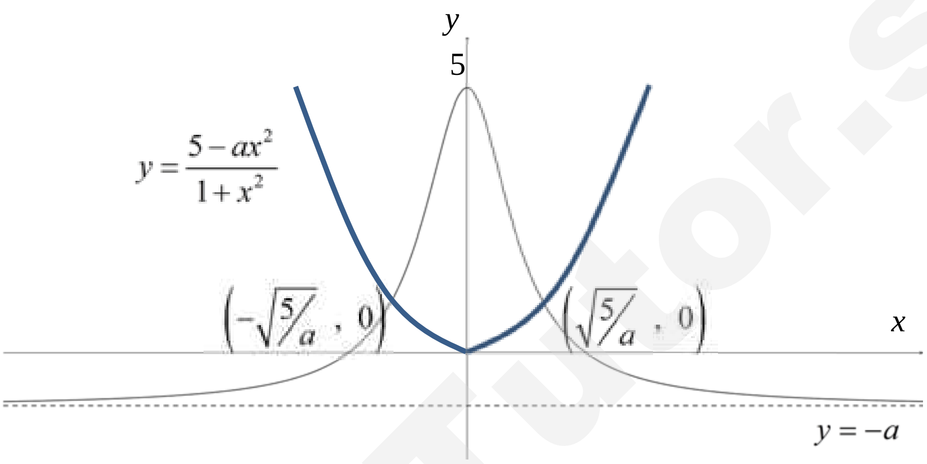
i.e. to prove $\sum_{r=2}^{k+1}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{(k+1)^{k}}{k!}\right]$

$$
\begin{aligned}
\text{LHS} &= \sum_{r=2}^{k+1}(r-1)\ln\left(\frac{r}{r-1}\right) \\
&= \sum_{r=2}^{k}(r-1)\ln\left(\frac{r}{r-1}\right)+k\ln\left(\frac{k+1}{k}\right) \\
&= \ln\left[\frac{k^{k-1}}{(k-1)!}\right]+k\ln\left(\frac{k+1}{k}\right) \\
&= \ln\left[\frac{k^{k-1}}{(k-1)!}\right]+\ln\left(\frac{k+1}{k}\right)^{k} \\
&= \ln\left[\frac{k^{k-1}}{(k-1)!}\times\frac{(k+1)^{k}}{k^{k}}\right] \\
&= \ln\left[\frac{k^{-1}(k+1)^{k}}{(k-1)!}\right] \\
&= \ln\left[\frac{(k+1)^{k}}{k(k-1)!}\right] \\
&= \ln\left[\frac{(k+1)^{k}}{k!}\right] \\
&= \text{RHS}
\end{aligned}
$$

Since P(2) is true and if P(k) is true, it implies that P(k+1) is true. By Mathematical Induction, P(n) is true for all positive integers $n \geq 2$.

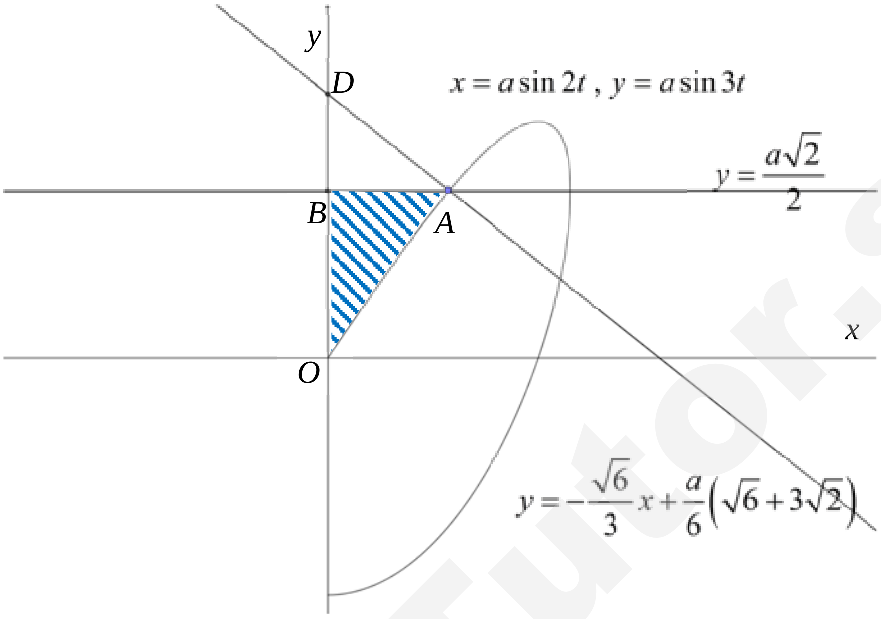
2	$\dfrac{x^2+6x+8}{x-1} \geq 0,\ x \neq 1$ $\dfrac{(x+2)(x+4)}{(x-1)} \geq 0$ Use sign test  $-4 \leq x \leq -2$ or $x > 1$. (Ans)
	$\dfrac{y^2+2y+15}{\lvert y+1\rvert-1} \geq -6$ $\dfrac{y^2+2y+15+6(\lvert y+1\rvert-1)}{\lvert y+1\rvert-1} \geq 0$ $\dfrac{(y+1)^2+14+6\lvert y+1\rvert-6}{\lvert y+1\rvert-1} \geq 0$ $\dfrac{(y+1)^2+6\lvert y+1\rvert+8}{\lvert y+1\rvert-1} \geq 0$ Since $(y+1)^2 = \lvert y+1\rvert^2$ Replace x by $\lvert y+1\rvert$ in (i) $-4 < \lvert y+1\rvert < -2$ (no solution since $\lvert y+1\rvert \geq 0$ for all real y) or $\lvert y+1\rvert > 1$ $\therefore\ y+1 > 1$ or $y+1 < -1$ (or use of the graphical method)

	 From GC, the intersection points are $(-2,1)$ and $(0,1)$ $y>0$ or $y<-2$ (Ans)
3(i)	$\mathrm{f}(x)=\frac{5-ax^2}{1+x^2}=\frac{5-a(1+x^2)+a}{1+x^2}=-a+\frac{5+a}{1+x^2}$, $a>1$ 

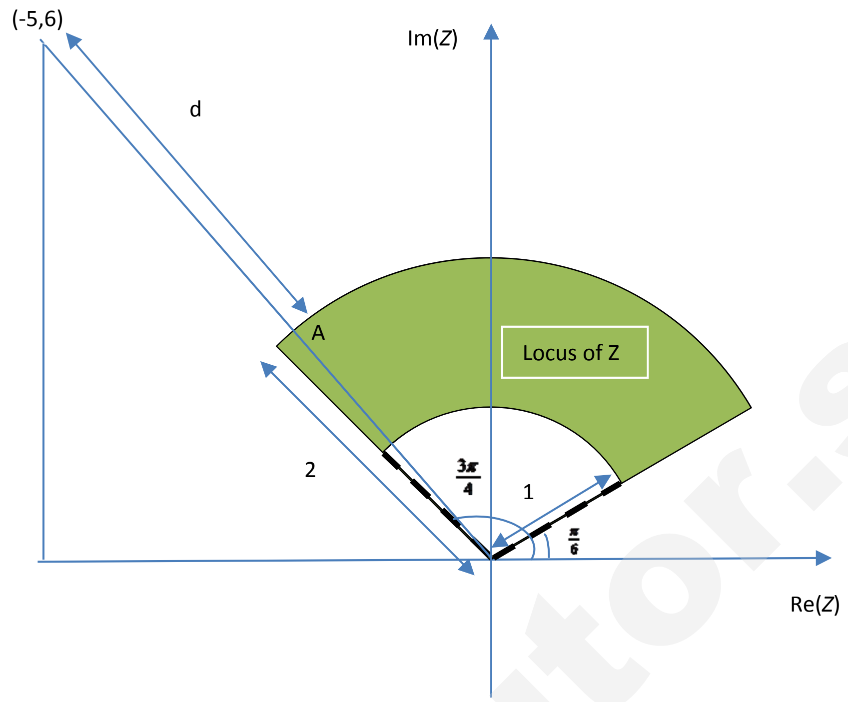
(ii)	$x^4+(a+1)x^2-5=0$ $x^4+x^2=5-ax^2$ $x^2(1+x^2)=5-ax^2$ $x^2=\dfrac{5-ax^2}{1+x^2}$ Hence we should sketch the curve $y=x^2$.  From the graph we can see that there are only two real roots. LHS = $\mathrm{g}(-x)=(-x)^4+(a+1)(-x)^2-5=x^4+(a+1)x^2-5=\mathrm{g}(x)$=RHS So g(–$x$) = g($x$) As there are only two real roots, the other two roots should be complex roots. As the coefficients of equation are all real, the remaining two roots must be a pair of complex conjugates. As there is only one pair the complex conjugates, and $\mathrm{g}(x)=\mathrm{g}(-x)=0$, then the complex conjugates must be purely imaginary. *Explanation*: If $z=x+\mathrm{i}y$ is a root where x, y are real, so $\mathrm{g}(x+\mathrm{i}y)=0$. Since $\mathrm{g}(z)=\mathrm{g}(-z)$ and $\mathrm{g}(z)=0$, then $\mathrm{g}(-z)=0$ and hence $z=-(x+\mathrm{i}y)=-x-\mathrm{i}y$ is also a root. As the complex roots need to be in conjugate pairs, then

	$\left(x+\mathrm{i}y\right)^{*}=-x-\mathrm{i}y$ $x-\mathrm{i}y=-x-\mathrm{i}y$ Comparing the Real Part, $x=-x$ $2x=0$ $x=0$ [Note that the imaginary part is not necessary, as it yields $-y=-y$ which is trivial.] Hence the complex roots must be purely imaginary. Alternative explanation: $x^4+(a+1)x^2-5=0$ $x^2=\dfrac{(a+1)\pm\sqrt{(a+1)^2-4(1)(-5)}}{2}$ $x^2=\dfrac{(a+1)\pm\sqrt{(a+1)^2+20}}{2}$ $x^2=\dfrac{(a+1)+\sqrt{(a+1)^2+20}}{2}$ or $x^2=\dfrac{(a+1)-\sqrt{(a+1)^2+20}}{2}$ $x=\pm\sqrt{\dfrac{(a+1)+\sqrt{(a+1)^2+20}}{2}}$ or $x=\pm\sqrt{\dfrac{(a+1)-\sqrt{(a+1)^2+20}}{2}}$ Since $\dfrac{(a+1)+\sqrt{(a+1)^2+20}}{2}>0$ and $\dfrac{(a+1)-\sqrt{(a+1)^2+20}}{2}<0$ then the complex roots must be purely imaginary.
4(i)	Given $x=a\sin 2t$, $y=a\sin 3t$, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=2a\cos 2t$, $\dfrac{\mathrm{d}y}{\mathrm{d}t}=3a\cos 3t$ Hence, At the point $\left(a\sin 2\theta, a\sin 3\theta\right)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3\cos 3\theta}{2\cos 2\theta}$ As $\theta\to\dfrac{\pi}{4}$, $\cos 2\theta\to\cos\dfrac{\pi}{2}=0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}\to-\infty$.

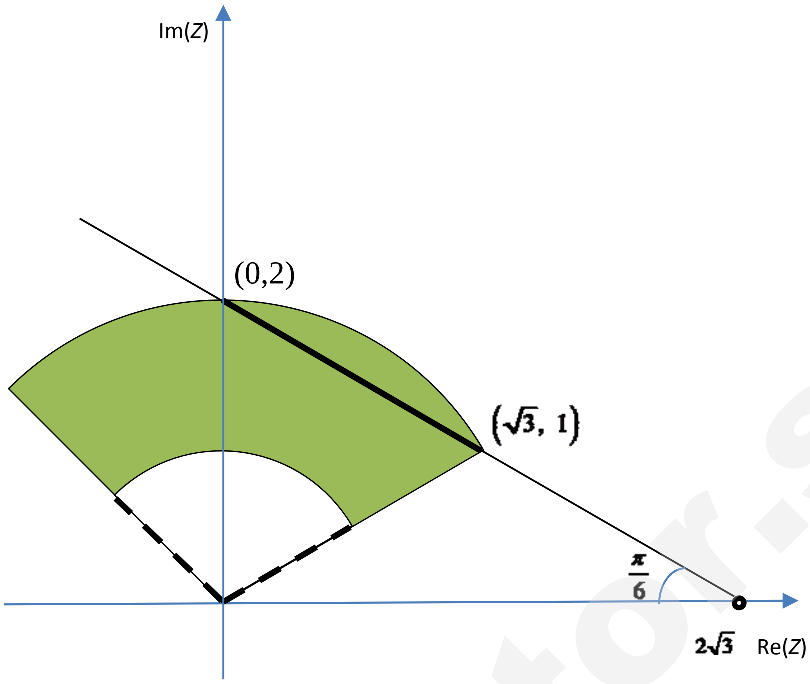
	i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}$ will be undefined. Hence, the tangent to C as $\theta \to \frac{\pi}{4}$ will become a vertical line, i.e. parallel to y-axis
(ii)	At $t = \frac{\pi}{12}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\cos\left(3\left(\frac{\pi}{12}\right)\right)}{2\cos\left(2\left(\frac{\pi}{12}\right)\right)}$ $= \frac{3\left(\frac{1}{\sqrt{2}}\right)}{2\left(\frac{\sqrt{3}}{2}\right)}$ $= \frac{\sqrt{6}}{2}$ Gradient of normal$= -\frac{2}{\sqrt{6}}$ Coordinates of point at $t = \frac{\pi}{12}$: $x = a\sin\left(2\left(\frac{\pi}{12}\right)\right)$ $= \frac{a}{2}$ $y = a\sin\left(3\left(\frac{\pi}{12}\right)\right)$ $= \frac{a\sqrt{2}}{2}$ Equation of normal: $y - \frac{a\sqrt{2}}{2} = -\frac{2}{\sqrt{6}}\left(x - \frac{a}{2}\right)$ $= -\frac{\sqrt{6}}{3}\left(x - \frac{a}{2}\right)$

	$y=-\frac{\sqrt{6}}{3}x+\frac{a\sqrt{6}}{6}+\frac{a\sqrt{2}}{2}$ $=-\frac{\sqrt{6}}{3}x+\frac{a}{6}\left(\sqrt{6}+3\sqrt{2}\right)$
(iii)	 $y=0, t=0$ $y=\frac{a\sqrt{2}}{2}, t=\frac{\pi}{12}$ (from (ii)) Area bounded by curve, y-axis and the line $y=\frac{\sqrt{2}a}{2}$, region OAB $=\int_0^{\frac{\sqrt{2}a}{2}} x\,\mathrm{d}y$ $=\int_0^{\frac{\pi}{12}}\left(a\sin 2t\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\mathrm{d}t$ $=\int_0^{\frac{\pi}{12}}\left(a\sin 2t\right)\left(3a\cos 3t\right)\mathrm{d}t$ $=3a^2\int_0^{\frac{\pi}{12}}\left(\cos 3t\sin 2t\right)\mathrm{d}t$ (Shown) y-intercept of normal is: $y=\frac{a}{6}\left(\sqrt{6}+3\sqrt{2}\right)$

	Hence, Required area, region *OADB* $=3a^2\int_0^{\frac{\pi}{12}}\left(\cos 3t\sin 2t\right)\mathrm{d}t+\frac{1}{2}\left(\frac{a}{2}\right)\left[\frac{a}{6}\left(\sqrt{6}+3\sqrt{2}\right)-\frac{a\sqrt{2}}{2}\right]$ $=\frac{3a^2}{2}\int_0^{\frac{\pi}{12}}\left(\sin 5t-\sin t\right)\mathrm{d}t+\frac{a^2}{4}\left[\frac{\left(\sqrt{6}+3\sqrt{2}\right)}{6}-\frac{\sqrt{2}}{2}\right]$ $=\frac{3a^2}{2}\left[-\frac{1}{5}\cos 5t+\cos t\right]_0^{\frac{\pi}{12}}+\frac{a^2}{4}\left(\frac{\sqrt{6}}{6}\right)$ $=\frac{3a^2}{2}\left[-\frac{1}{5}\cos\frac{5\pi}{12}+\cos\frac{\pi}{12}+\frac{1}{5}\cos 0-\cos 0\right]+\frac{a^2\sqrt{6}}{24}$ $=\frac{3a^2}{2}\left[-\frac{1}{5}\cos\frac{5\pi}{12}+\cos\frac{\pi}{12}-\frac{4}{5}\right]+\frac{a^2\sqrt{6}}{24}$ $=\frac{3a^2}{2}\left[\cos\frac{\pi}{12}-\frac{1}{5}\cos\left(\frac{\pi}{2}-\frac{\pi}{12}\right)\right]+a^2\left(\frac{\sqrt{6}}{24}-\frac{6}{5}\right)$ $=\frac{3a^2}{2}\left[\cos\frac{\pi}{12}-\frac{1}{5}\sin\frac{\pi}{12}\right]+a^2\left(\frac{\sqrt{6}}{24}-\frac{6}{5}\right)$ units^2 where $k=\frac{3}{2}, b=1, c=-\frac{1}{5}, d=\frac{\sqrt{6}}{24}-\frac{6}{5}$
5(i), (ii)	$\arg(z)=\theta$ $\|z\|=r$

	
(ii)	The minimum distance, d $= \sqrt{(-5)^2+6^2} - 2 = \sqrt{61} - 2$ Let θ be the basic angle of the point $(-5, 6)$. From the diagram, $\tan\theta = \dfrac{6}{5}$ so $\sin\theta = \dfrac{6}{\sqrt{61}}$ and $\cos\theta = \dfrac{5}{\sqrt{61}}$. Hence complex number that corresponds to point A, $z = 2(-\cos\theta + i\sin\theta) = -2\left(\dfrac{5}{\sqrt{61}}\right) + 2i\left(\dfrac{6}{\sqrt{61}}\right) = \dfrac{1}{\sqrt{61}}(-10+12i)$ $= -1.28 + 1.54i$

(iii)

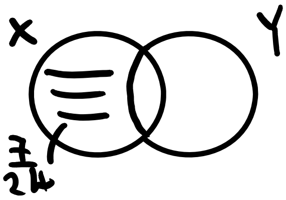


6(i) The people who visits the fitness centre on a particular weekday may not be representative of the members of the club as some members may be working or attending school and only go to club later in the day. Thus the sample collected would contain mainly women who are not working/do not attend school thereby under representing women of some age groups.

(ii) Stratified sampling method should be used to obtain a representative sample of 200 members.

1. Draw up a list of the members of the fitness centre and calculate the proportion for the different corresponding age group.

18 – 25	26-30	31 – 40	41 – 50	51 & above
500	1000	1500	1500	500
$\frac{500}{5000}\times 200 = 20$	$\frac{1000}{5000}\times 200 = 40$	$\frac{1500}{5000}\times 200 = 60$	$\frac{1500}{5000}\times 200 = 60$	$\frac{500}{5000}\times 200 = 20$

	2. Within each age group, use simple random sampling method to select the required sample size for survey.
7 (a)(i)	$\mathrm{P}(X \cup Y) = \mathrm{P}(Y) + \mathrm{P}(X \cap Y')$ $\frac{5}{8} = \mathrm{P}(Y) + \frac{7}{24}$ $\mathrm{P}(Y) = \frac{8}{24} = \frac{1}{3}$  Given $\mathrm{P}(X' \mid Y) = \frac{9}{16}$, $\frac{\mathrm{P}(X' \cap Y)}{\mathrm{P}(Y)} = \frac{9}{16}$ $\mathrm{P}(X' \cap Y) = \frac{9}{16}.\frac{1}{3} = \frac{3}{16}$
(ii)	$\mathrm{P}(X) = 1 - \left[\mathrm{P}(X \cup Y)' + \mathrm{P}(X' \cap Y)\right] = 1 - \frac{3}{8} - \frac{9}{48} = \frac{7}{16}$ $\mathrm{P}(X).\mathrm{P}(Y') = \frac{7}{16}.\frac{2}{3} = \frac{7}{24} = \mathrm{P}(X \cap Y')$ X and Y' are independent.
7(b) **(i)**	Without restriction $= \binom{11}{6}(6-1)!5! = 6652800$ Case 1: Ms Koh sits at round table with two male teachers No. of ways $= \binom{8}{3}\binom{6}{2}2!(4-1)!5! = 1209600$ Case 2: Ms Koh sits at long table with two male teachers No. of ways $= \binom{8}{2}\binom{6}{2}2!3!(6-1)! = 604800$ Required Probability

	$=\dfrac{\binom{8}{3}\binom{6}{2}2!(4-1)!5!+\binom{8}{2}\binom{6}{2}2!3!(6-1)!}{\binom{11}{6}(6-1)!5!}$ $=\dfrac{3}{11}$ or 0.273 (to 3.s.f)
(ii)	Required Probability $=\dfrac{P(\text{Ms koh sits between 2 male teachers and male and female teachers alternate})}{P(\text{Ms koh sits between 2 male teachers})}$ $=\dfrac{P(\text{male and female teachers alternate})}{P(\text{ms koh sits between 2 male teachers})}$ $=\dfrac{\dfrac{\binom{6}{3}\binom{5}{3}(3-1)!3!3!2!}{6652800}}{\dfrac{3}{11}}$ OR $\dfrac{\dfrac{6\times5\times4\times5\times4\times2\times3\times2}{6652800}}{\dfrac{3}{11}}$ $=\dfrac{\dfrac{1}{231}}{\dfrac{3}{11}}$ $=\dfrac{1}{63}$ or 0.159 (to 3.s.f)

Alternative Solution:

Required Probability

$$= \frac{P(\text{ms koh sits between 2 male teachers and male and female teachers alternate})}{P(\text{ms koh sits between 2 male teachers})}$$

Case 1: Ms Koh sits at the long table and the male and female teachers alternate

no. of ways = $\binom{6}{3}\binom{4}{1}3!2!3!(3-1)! = 11520$

Case 2: Ms Koh sits at the circular table and the male and female teachers alternate

no. of ways = $\binom{6}{3}\binom{4}{2}(3-1)!3!3!2! = 17280$

Total no. of ways ms koh sits between 2 male teachers and male and female teachers alternate

$= \binom{6}{3}\binom{4}{1}3!2!3!(3-1)! + \binom{6}{3}\binom{4}{2}(3-1)!3!3!2!$

$= 28800$

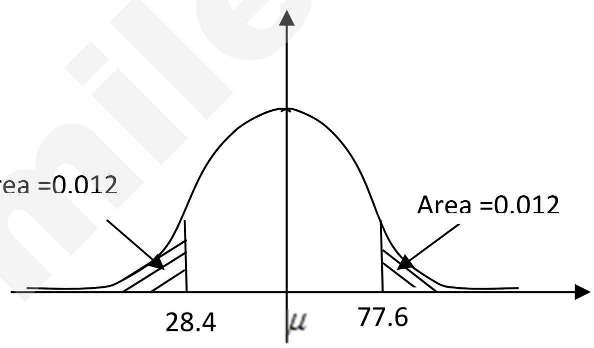
Required Probaility

$$= \frac{\frac{28800}{6652800}}{\frac{3}{11}}$$

$$= \frac{1}{63}$$

[Note that for the males and females to seat on alternate seats, on the round table there must be 3 males and 3 females and the long table there must be 3 males and 2 females]

8 (i)



Method 1 (recognise μ is the midpoint)

By symmetry, $\mu = \dfrac{28.4 + 77.6}{2}$ =53

$P(X < 28.4) = 0.012$

$P(Z < \dfrac{28.4 - 53}{\sigma}) = 0.012$

Using GC,

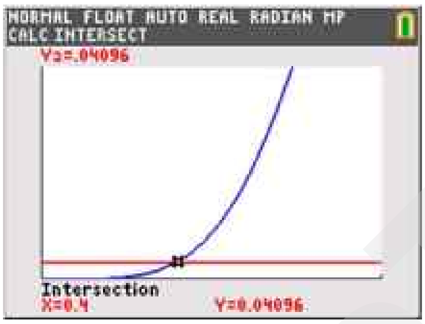
	$\frac{-24.6}{\sigma} = -2.25712924$ $\sigma = 10.8988$ $\sigma = 10.9$ (3 s.f.) <u>Method 2 (simultaneous equations- not recommended)</u> $P(X < 28.4) = 0.012$ $\quad$ $P(X < 77.6) = 0.988$ $P(Z < \frac{28.4-\mu}{\sigma}) = 0.012$ $\quad$ $P(Z < \frac{77.6-\mu}{\sigma}) = 0.012$ $\frac{28.4-\mu}{\sigma} = -2.25712924$ $\quad$ $\frac{77.6-\mu}{\sigma} = 2.25712924$ $28.4-\mu = -2.25712924\sigma$ $\quad$ $77.6-\mu = 2.25712924\sigma$ Solve simultaneously, $\mu = 53, \quad \sigma = 10.8988$
(ii)	Let X be the weight of a packet of sweets in grams. $X \sim N(53, 10.8988^2)$ <u>Method 1 (expression in terms of mass)</u> $X_1 + X_2 + ...X_4 \sim N(212, 475.13536)$ $\$1.20 \to 100g$ $\$2.60 \to \frac{2.6\times100}{1.2} = 216.667g$ $P(X_1 + X_2 + ...X_4 \le 216.667) = 0.58476$ $= 0.585$ (3 s.f.) $P(X_1 + X_2 + ...X_4 \le \frac{2.6\times100}{1.2}) = 0.58476$ $= 0.585$ (3 s.f.) <u>Method 2 (expression in terms of cost)</u> Let C denote the cost of 4 packets of sweets. $C = \frac{1.20}{100}(X_1 + X_2 + ...X_4) = 0.012(X_1 + X_2 + ...X_4)$ Then $E(C) = 0.012(4\times53) = 2.544$ $Var(C) = 0.012^2(4\times10.8988^2) = 0.0684195$

<table>
<tr><td></td><td>$P(C \le 2.60) = 0.5847618 = 0.585$ (3 s.f.)</td></tr>
<tr><td>9 (i)</td><td>The probability of picking a brown egg from a box is constant.

The colour of an egg is independent of other eggs.</td></tr>
<tr><td>(ii)</td><td>Let X be the r.v. “number of brown eggs in a box of 6 eggs”

$X \sim \mathrm{B}(6, p)$

$P(X \ge 5) = 0.04096$
$P(X = 5) + P(X = 6) = 0.04096$
$\binom{6}{5} p^5 (1-p) + \binom{6}{6} p^6 (1-p)^0 = 0.04096$



Using GC, $p = 0.4$</td></tr>
<tr><td>(iii)</td><td>Let A be the r.v. “number of boxes that contain at most 4 brown eggs in a box out of 100 boxes”

$A \sim \mathrm{B}(100,\ p_1)$, where $p_1 = 1 - 0.04096 = 0.95904$.

Let Y be the r.v. “number of boxes that contain at least 5 brown eggs in a box out of 100 boxes”

$Y \sim \mathrm{B}(100, 0.04096)$

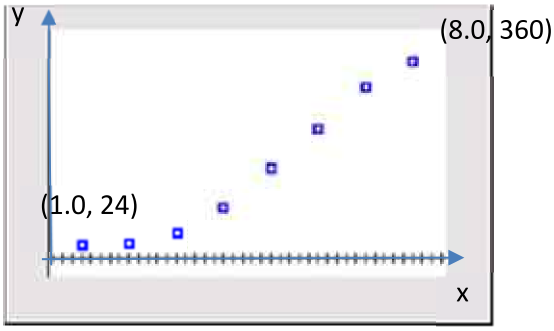
Note that $A + Y = 100$

Since $n = 100$ is large, $np = 100(0.04096) = 4.096 < 5$,

$Y \sim \mathrm{P_0}(4.096)$ approximately

$\mathrm{P}(A \ge 90) = \mathrm{P}(100 - Y \ge 90) = \mathrm{P}(Y \le 10) = 0.997$</td></tr>
<tr><td>(iv)</td><td>$Y \sim \mathrm{B}(100, 0.04096)$</td></tr>
</table>

	$E(Y)=100(0.04096)=4.096$ $Var(Y)=100(0.04096)(0.95904)=3.9282$ In 8 weeks, there are 56 days altogether. Mean number of boxes with at least 5 brown eggs is $\overline{Y}=\dfrac{Y_1+Y_2+Y_3+\cdots+Y_{56}}{56}$ Since sample size = 56 is large, by Central Limit Theorem, $\overline{Y} \sim \mathrm{N}\left(4.096, \dfrac{3.9282}{56}\right)$ approximately i.e. $\overline{Y} \sim \mathrm{N}(4.096, 0.0701469)$ $\mathrm{P}(4<\overline{Y}<7)= 0.641$ (correct to 3 sig fig)
10(i)	Let X be the random variable ' length of time a patient spent with the doctor' Given $\sum x = 147, \quad \sum x^2 = 1927.91$ The unbiased estimates of population mean μ and population variance σ^2 are $\overline{x}=12.25, \quad s^2=\dfrac{1}{n-1}[\sum x^2-\dfrac{(\sum x)^2}{n}]=11.56$ **Assumption:** The length of time, X, a patient spent with the doctor follows a normal distribution. Test H_0: $\mu=10$ H_1: $\mu>10$ Under H_0, $T=\dfrac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(11)$ Use a right tailed t-test at the 5% level of significance. From GC, p-value = 0.0213

	Since p-value = $0.0213 < 0.05$, we reject H_0. There is sufficient evidence to conclude at the 5% level of significance, the mean time spent with a patient is more than 10 minutes.
ii	Since population variance is given, z-test should be used. Test H_0: $\mu = 10$ H_1: $\mu > 10$ Under H_0, $Z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$ Since H_0 is not rejected, $z = \dfrac{12.25 - 10}{15/\sqrt{n}} < 1.64485$ $0.15\sqrt{n} < 1.64485$ $n < 120.25$ $\therefore$ Maximum n = 120. Set of values of n is $\{ n \in \mathbb{Z}^+ : 0 < n \le 120 \}$
11(i)	$r = 0.974$
(ii)	 Although the value of product moment correlation coefficient indicates a strong positive linear correlation however from the scatter diagram, X and Y follows a non-linear relationship, hence it is advisable to interpret the data using both the scatter diagram and the value of product moment correlation coefficient.

(iii)	Using GC, $Y = 9.67 + 5.81X^2$ $r = 0.993$ Since the value of product moment correlation coefficient for Y on X^2 is closer to 1, compared to the value of product moment correlation coefficient for Y on X, the new proposed model is a better model for the data set.
(iv)	$Y = 9.6714285 + 5.81190X^2$ When $y = 120$, $x = 4.36$ (3 s.f.) The line of Y on X^2 is used because the value of X is fixed precisely and hence X is the independent variable.
12(i)	It must be assumed that the <u>average</u> number of customers who bought the package per week is a constant. The number of customers who bought the package per week varies, and cannot be a constant.
(ii)	Let X and Y be the number of customers who bought the Luxury Cruise Package in a week at Branch A and Branch B respectively. X~Po(3.5) Y~Po(4.5) $X+Y$~Po(8) $\text{P}(5 < X+Y < 10) = \text{P}(X+Y \le 9) - \text{P}(X+Y \le 5) \approx 0.525$
(iii)	Let U be the number of customers who bought the Luxury Cruise Package in n weeks at Branch A. U~Po(3.5n) Given $\text{P}(U \le 1) < 0.1$ From GC, when $n = 1$, $\text{P}(U \le 1) = 0.136 > 0.1$ when $n = 2$, $\text{P}(U \le 1) = 0.0073 < 0.1$ Least $n = 2$
(iv)	Let S and T be the number of customers who bought the Luxury Cruise Package in one month at Branch A and Branch B respectively. S~Po(14)

	Since $\lambda > 10$, S~N(14, 14) approximately. T~Po(18) Since $\lambda > 10$, T~N(18, 18) approximately. T-S ~ N(4, 32) $\mathrm{P}(0 < T - S \leq 5) \xrightarrow{C.C.} \mathrm{P}(0.5 < T - S \leq 5.5) \approx 0.337$
	The mean number of customers who bought the Luxury Package might not be a constant from one week to another because of fluatuations such as sales, holidays, the economic climate etc. Hence the Poisson distribution may not be a good model for the number of customers who bought the package in a year.

SERANGOON JUNIOR COLLEGE

2016 JC2 PRELIMINARY EXAMINATION

MATHEMATICS

Higher 2 **9740/1**

15 Sept 2016

3 hours

Additional materials: Writing paper

List of Formulae (MF15)

**TIME : ** 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and class on the cover page and on all the work you hand in.
Write in blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets **[ ]** at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

Total marks for this paper is 100 marks

This question paper consists of 6 printed pages (inclusive of this page) and 2 blank pages.

Answer all questions [100 marks].

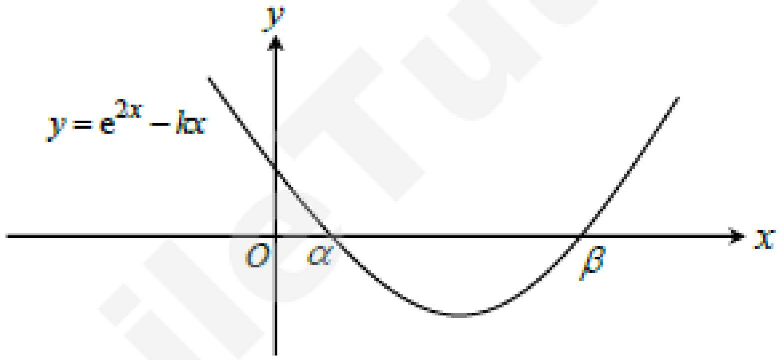
1 State a sequence of 3 transformations which transform the graph of $\mathrm{g}(x)=\mathrm{e}^{(6x+2)^2}+1$ to the graph of $\mathrm{f}(x)=\mathrm{e}^{(2x-2)^2}$. [3]

2 Using the standard series expansions, obtain the Maclaurin series of $\ln\left[(1+x)(1-2x)^3\right]$ in ascending powers of x, up to and including the term in x^3. [2]

(i) Find the set of values of x for which the above expansion is valid. [1]

(ii) Hence, find the range of values of x for which the expansion of $\left[\mathrm{e}^{\ln\left[\ln\frac{1}{(1+2x)^3}\right]}-\ln(1-x)\right](2+7x)^5$ is valid. [2]

3 (a)



The diagram shows the graph of $y=\mathrm{e}^{2x}-kx$, where k is a positive real number.
The two roots of the equation $\mathrm{e}^{2x}-kx=0$ are denoted by α and β, where $\alpha<\beta$.

It is given that there is a sequence of real numbers $x_1, x_2, x_3 \ldots$ that satisfies the recurrence relation, $x_{n+1}=\dfrac{1}{k}\mathrm{e}^{2x_n}$, for $n\geq 1$.

By considering $x_{n+1}-x_n$, prove that

$$x_{n+1}>x_n \text{ if } x_n<\alpha \text{ or } x_n>\beta.$$ [2]

(b) Prove by the method of mathematical induction that

$$\sum_{r=1}^{n}\cos r\theta=\frac{\sin(n+\frac{1}{2})\theta-\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta},$$ for all positive integers n. [5]

4 Andy and his fiancée signed up for a new 4-room flat in Boon Keng. They take up a housing loan of \$450,000 provided by BEST bank for the purchase. The couple pay a fixed monthly instalment of \$*A* on the first day of each month. A fixed interest rate of 1.6% is charged on the last day of each year, based on the remaining loan amount at the beginning of that year before the 1st monthly instalment is paid. If the first instalment is paid in January 2016,

(i) Show that the amount the couple owe the bank at the end of 2017 is $\$[464515.20 - 24.192A]$. [1]

(ii) Given that *A* is 1500, find the date and amount of the final repayment to the nearest cent. [5]

5 **(a)** It is given that $y = \dfrac{x^2 - x - 1}{x+1}$, $x \in \mathbb{R}$, $x \neq -1$. Without using a graphic calculator, find the set of values that *y* cannot take. [3]

(b) The curve *C* has equation $y = \dfrac{x^2 + b}{x - a}$, where $a > 0$, $b > a$ and $x \neq a$.

(i) Draw a sketch of the curve *C*, label clearly the equation(s) of its asymptote(s) and the coordinates of any intersection with the axes. [3]

(ii) By drawing an additional graph on the diagram drawn in **(i)**, state the number of real root(s) of the equation $x^2 + b = (x-a)(x^2 + a)$. [2]

6 **(a)** The equations of two planes p_1 and p_2 are

$$x + 4y + 2z = 7,$$
$$3x + \lambda y + 4z = \mu,$$

respectively, where λ and μ are constants.

(i) Given that the two planes intersect in a line *l*, with a vector equation given by

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \ s \in \mathbf{R},$$

show that the value of λ is 10 and find the value of μ. [3]

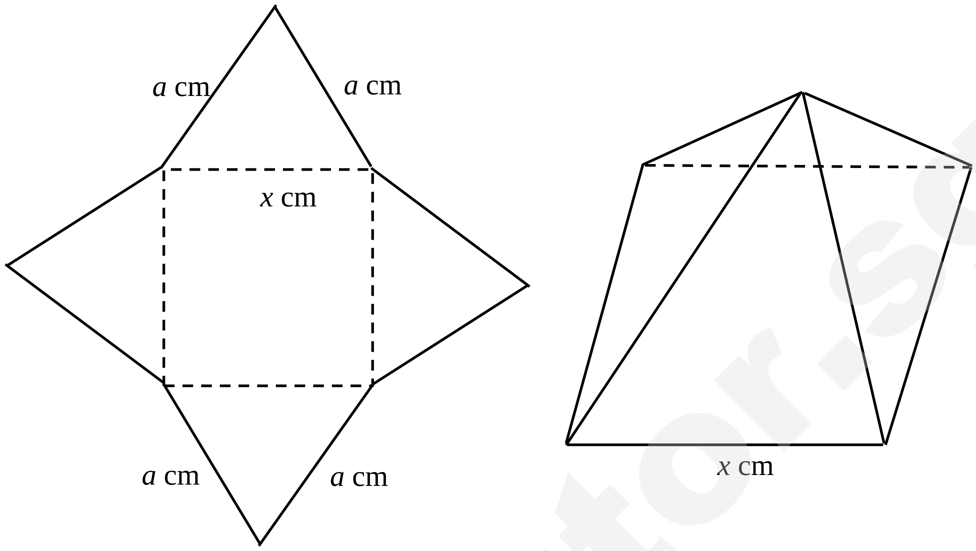
(ii) If plane p_3 is the reflection of p_1 in p_2, find the acute angle between p_1 and p_3. [2]

(b) Relative to the origin *O*, the points *A*, *B*, *C* and *D* have position vectors **a**, **b**, **c** and **d** respectively. It is given that λ and μ are non-zero numbers such that $\lambda\mathbf{a} + \mu\mathbf{b} - \mathbf{c} = \mathbf{0}$ and $\lambda + \mu = 1$,

(i) Show that *A*, *B* and *C* are collinear. [3]

(ii) If *O* is not on the line *AC* and $|\mathbf{c} \times \mathbf{a}|(\mathbf{b} - \mathbf{a}) = (\mathbf{c}\cdot\mathbf{d})\mathbf{d}$, determine the relationship between $\overrightarrow{AC}$ and $\overrightarrow{OD}$, explaining your answer clearly. [2]

7 A piece of metal with negligible thickness has been cut into a shape that is made up of four isosceles triangles each with base x cm and fixed sides a cm. Their bases frame to a form a square with sides of length x cm. A right pyramid is formed by folding along the dotted lines as shown in the diagram below.

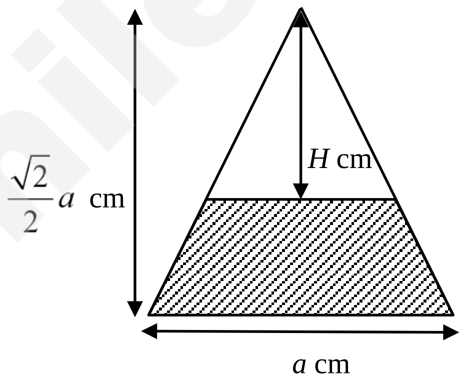


[Volume of a pyramid = $\frac{1}{3} \times$ base area $\times$ height]

(i) Show that the volume of the pyramid is $\frac{x^2}{3}\sqrt{a^2 - \frac{x^2}{2}}$ cm^3. [2]

(ii) Find the value of x, in terms of a, that will give maximum volume for the pyramid. [4]

(iii)



To make the pyramid into a paperweight with negligible thickness, a viscous fluid is pumped into the interior at a rate of 1 cm^3/s. Given that H cm is the perpendicular distance from the apex of the pyramid to the viscous fluid surface, $x = a$ and the height of the pyramid is $\frac{\sqrt{2}}{2}a$ cm, find the rate at which H is changing when $H = \frac{a}{2}$, giving your answer in terms of a. [3]

[The diagram above shows the cross sectional area of the pyramid.]

8 **(i)** Show that $\left(0, -\frac{1}{4}\right)$ lies on the locus $|z+2| = |z+1+2\text{i}|$. [1]

(ii) Sketch on a single Argand diagram the loci $|z+1+2\text{i}| = \sqrt{5}$ and $|z+2| = |z+1+2\text{i}|$. [4]

(iii) Hence indicate clearly on the Argand diagram the locus of z that satisfies the relations $|z+1+2\text{i}| \le \sqrt{5}$ and $|z+2| = |z+1+2\text{i}|$. [1]

(iv) Find the greatest and least possible values of $\arg(z+1+2\text{i})$, giving your answers in radians correct to 3 decimal places. [4]

9 The path travelled by an object measured with respect to the origin in the horizontal and vertical directions, at time t seconds, is denoted by the variables x and y respectively.

It is given that when $t = 0$, $x = 1$, $y = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$. The variables are related by the differential equations $\frac{\mathrm{d}y}{\mathrm{d}t} - y + \sqrt{\mathrm{e}^{2t} - 4y^2} = 0$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \cos^2 2t$.

(i) Using the substitution $y = w\mathrm{e}^t$, show that $\frac{\mathrm{d}w}{\mathrm{d}t} = -\sqrt{1-4w^2}$ and hence find y in terms of t. [6]

(ii) Find x in terms of t. [4]

10 Given that $\mathrm{f}(r) = \frac{3^r}{r+1}$, show that $\mathrm{f}(r+2) - \mathrm{f}(r) = \frac{(8r+6)3^r}{(r+1)(r+3)}$. [1]

(i) Find $\sum_{r=1}^{n} \frac{(4r+3)3^r}{(r+1)(r+3)}$ in terms of n. [2]

(ii) Hence find $\sum_{r=1}^{n} \frac{(4r+11)3^r}{(r+3)(r+5)}$ in terms of n. [4]

(iii) Using the result in **(ii)**, show that $\sum_{r=0}^{n} \left[\frac{r \cdot 3^r}{(r+5)^2}\right] - \frac{3^{n+1}}{4} < -\frac{51}{160}$. [3]

11 The functions f and g are defined as follows:

$$\mathrm{f} : x \mapsto -\left|x^2 + 2x\right|, \quad a < x \le 0$$

$$\mathrm{g} : x \mapsto -\sqrt{x+1}, \quad x > -1$$

(i) State the least value of a for the inverse function of f to exist. Hence, find f^{-1} in similar form. [4]

For the following parts, use the value of a found in part **(i)**.

(ii) Write down ff^{-1} in similar form. [1]

(iii) Find the rule for gf in the form $bx + c$, where $b, c \in$ □ . State its range. [3]

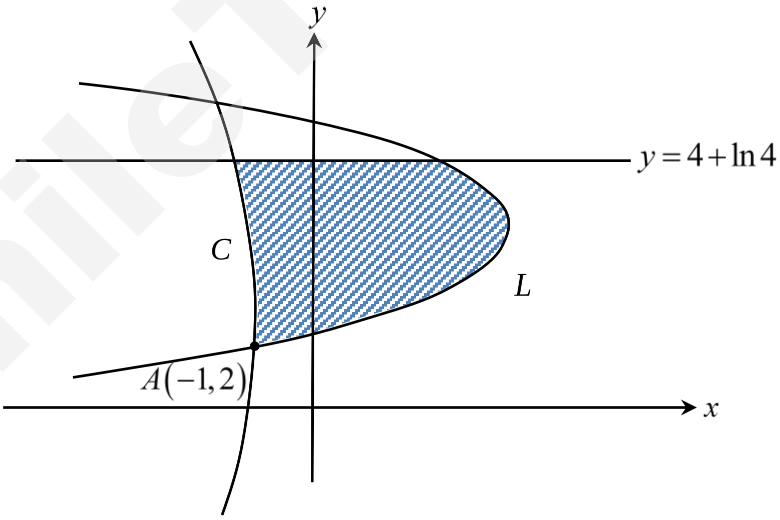
(iv) Find the exact range of x for which $\mathrm{f}\left(x - \frac{3}{2}\right) > \mathrm{g\,f}\left(x - \frac{3}{2}\right)$. [3]

12 **(a)(i)** Find $\frac{\mathrm{d}}{\mathrm{d}x}\left[(\ln x)^2\right]$. [1]

(ii) The curve C is defined by the parametric equations

$$x = \ln t - t, \quad y = 2t + \ln\left(t^2\right) \qquad \text{where } t > 0.$$

Another curve L is defined by the equation $(4 - y)^2 = 3 - x$. The graphs of C and L intersect at the point $A(-1, 2)$ as shown in the diagram below.



Find the exact area of the shaded region bounded by C, L and the line $A(-1,2)$. [6]

(b) The region R is the finite region enclosed by the curve $(y-1)^2 = 1 - x$ and the y-axis. The region S is the region in the 2nd quadrant enclosed by the curve $y = 2\tan\left(x + \frac{\pi}{4}\right)$ and the axes.

Find the total volume generated when region R and S is rotated through 2π radians about the x-axis, leaving your answers in exact form. [4]

END OF PAPER

SERANGOON JUNIOR COLLEGE

2016 JC2 PRELIMINARY EXAMINATION

MATHEMATICS

Higher 2 **9740/2**

21 Sept 2016

3 hours

Additional materials: Writing paper

List of Formulae (MF15)

**TIME : ** 3 hours

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Total marks for this paper is 100 marks

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Section A: Pure Mathematics [40 marks].

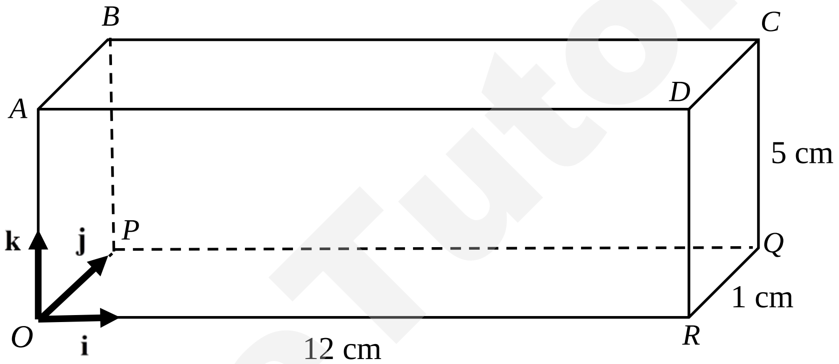
1 **(a)** If $0 < a < b$, solve $\int_0^b x|a-x|\,\mathrm{d}x$, leaving your answers in terms of a and b. [2]

(b)(i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3-x}{\sqrt{1-x}}\right)$. [1]

(ii) Find $\int \frac{3-x}{x^2-3x+2}\,\mathrm{d}x$. [2]

(iii) Hence find $\int \frac{1+x}{(1-x)^{\frac{3}{2}}}\tan^{-1}\sqrt{1-x}\,\mathrm{d}x$. [3]

2



The cuboid above is formed by the eight vertices *O, A, B, C, D, P, Q* and *R*. Perpendicular unit vectors **i**, **j**, **k** are parallel to *OR*, *OP* and *OA* respectively. The length of *OR*, *OP* and *OA* are 12 cm, 1 cm and 5 cm respectively.

(i) Find the Cartesian equation of line *AC*. [2]

(ii) Find the acute angle between *CA* and *CR*. Hence, find the shortest distance from *R* to *AC*. [4]

(iii) The point *T* is on *AC* produced such that $AT = \lambda AC$ and *M* is the midpoint of *OR*. The unit vector in the direction of *OT* is represented by the vector $\overrightarrow{OV}$. By considering the cross product of relevant vectors, find the ratio of the area of triangle *OVM* to the area of triangle *ORT* in terms of λ. [3]

3 **(a)** The complex number *w* is such that $w = a + \mathrm{i}b$, where *a* and *b* are non-zero [3] real numbers. The complex conjugate of w is denoted by w*. Given that $\frac{(w^*)^2}{w} = 3 - \mathrm{i}b$, solve for a and b and hence write down the possible values of w.

(b) **(i)** Without the use of a graphic calculator, find the roots of the equation $z^2 - 2z + 4 = 0$, leaving your answers in the form $re^{\mathrm{i}\theta}$, $r > 0$ and $-\pi < \theta \le \pi$. [2]

(ii) Let α and β be the roots found in **(b)(i)**. If $\arg(\alpha) > \arg(\beta)$, find $\left|\alpha^{10} - \beta^{10}\right|$ and $\arg\left(\alpha^{10} - \beta^{10}\right)$ in exact form. [3]

(c) **(i)** Show that the locus of z where $\arg(z + 2\sqrt{3} + \mathrm{i}) = -\frac{\pi}{6}$ passes through the point $(-\sqrt{3}, -2)$. [1]

(ii) Find the Cartesian equation of the locus of z in the form $y = mx + c$, stating its domain clearly. Leave your answer in exact form. [2]

4 A curve C has parametric equations

$$x = \sin t, \quad y = \cos t.$$

(i) Find the equations of tangent and normal to C at the point with parameter t. [3]

(ii) Points P and Q on C have parameters p and q respectively, where $0 < p < \frac{\pi}{2}$ and $0 < q < \frac{\pi}{2}$. The tangent at P meets the normal at Q at the point R. Show that the x–coordinate of R is $\frac{\sin q}{\cos(p-q)}$. Hence, find in similar form the y–coordinate of R in terms of p and q. [3]

The tangent at P meets the y-axis at the point A and the normal at Q meets the y-axis at the point B. Taking $q = \frac{\pi}{2} - p$,

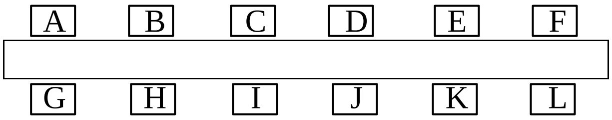
(iii) Show that the area of triangle ARB is $\frac{1}{2}\operatorname{cosec}(2p)$. [3]

(iv) Find the Cartesian equation of the locus of point R. [3]

Section B: Statistics [60 marks]

5 Nicole decides to celebrate her birthday with 9 boys and 2 girls whose names are Vanessa and Sally.

(a) **(i)** They have a dinner at a restaurant that can only offer them a rectangular table as shown in the following diagram, with seats labelled A to L as shown.



Find the number of ways in which at least one girl must be seated at the seats A, F, G and L. [2]

(ii) Find the number of ways in which they can sit if instead, the restaurant offers them 2 indistinguishable round tables of 6. [2]

(iii) After the dinner, they went for a movie together. They bought tickets for seats in a row. Find the number of ways where Nicole and Vanessa must be seated together but not Sally. [2]

(b) After the celebration, Nicole plays a card game with Vanessa. The pack of 20 cards are numbered 1 to 20. The two friends take turns to draw a card from the pack. If a prime number is drawn, the player wins the game. If a composite number (4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20) is drawn, the player loses the game and the other player wins. If the number '1' is drawn, the card is returned and the other player draws the next card. Nicole draws the first card. Find the probability of her winning the game. [3]

6 In a telephone enquiry service, 92% of calls to it are successfully connected. The probability of any call being successfully connected is constant. A random sample of 60 calls is taken each day.

(i) State, in context, an assumption needed for it to be well modelled by a binomial distribution. [1]

(ii) On a given day, it is found that at most 55 calls went through successfully. Find the probability that there are at least 50 successful calls in the sample of 60. [2]

(iii) Estimate the probability that the number of successful calls on any day is less than 55 in a sample of 60. [4]

(iv) The number of successful calls is recorded daily for 70 consecutive days. Find the approximate probability that the average number of successful calls in a day is not more than 55. [2]

7 **(a)** Tickets are sold for the closing ceremony of an international swimming competition. It is desired to sample 1% of the spectators to find their opinions of the goodie bags received during the closing ceremony.

(i) Give a reason why it would be difficult to use a stratified sample. [1]

(ii) Explain how a systematic sample could be carried out. [2]

(b) The random variable X has the distribution $\mathrm{N}(18, 3^2)$ and the random variable Y has the distribution $\mathrm{N}(\mu, \sigma^2)$. The random variable T is related to X and Y by the formula $T = \dfrac{X_1 + X_2 + 3Y}{4}$, where X_1 and X_2 are two independent observations of X. Given that $\mathrm{P}(T < 10) = \mathrm{P}(T > 30) = 0.0668$, find the value of σ and the exact value of μ. [5]

(c) A survey done on students in a particular college found that the amount of time a student spends on social media in a week is normally distributed with mean 7 hours and variance 4 hours^2.

Five students are randomly chosen. Find the probability that the fifth student is the second student who spends more than 10 hours a week on social media. [2]

8 An advertising display contains a large number of light bulbs which are continually being switched on and off every day in a week. The light bulbs fail independently at random times. Each day the display is inspected and any failed light bulbs are replaced. The number of light bulbs that fail in any one-day period has a Poisson distribution with mean 1.6.

(i) State, in the context of the question, one assumption that needs to be made for the number of light bulbs that fail per day to be well modelled by a Poisson distribution. [1]

(ii) Estimate the probability that there are fewer than 17 light bulbs that needs to be replaced in a period of 20 days. [2]

(iii) Using a suitable approximation, find the probability that there will be not fewer than 20 days with more than two light bulbs that will need to be replaced per day in a period of 8 weeks. [4]

(iv) The probability of at least three light bulbs having to be replaced over a period of n consecutive days exceeds 0.999. Write down an inequality in terms of n to express this information, and hence find the least value of n. [4]

9 **(a)** Observations of 10 pairs of values (x,y) are shown in the table below. [2]

x	1	2	3	4	5	6	7	8	9	10
y	0.5	0.6	0.8	0.95	a	1.21	1.36	1.55	1.87	2.11

It is known that the equation of the linear regression line of y on x is $y = 0.17321x + 0.24133$. Find a, correct to 2 decimal places.

(b) A student wanted to study the relationship between the number of commercial crimes (c) and the mean years of schooling (s) of the offenders. The following set of data was obtained.

Year	2009	2010	2011	2012	2013	2014	2015
Mean years of schooling (s)	9.7	10.1	10.2	10.3	10.5	10.6	10.7
No. of commercial crimes (c)	3359	3504	4080	3507	3947	5687	8329

(i) Draw a scatter diagram for these values. [2]

(ii) One of the values of c appears to be incorrect. Circle this point on your diagram and label it P. [1]

It is thought that the number of commercial crimes (c) can be modelled by one of the formulae after removing the point P.

(A) $c = a + b\left(100^s\right)$

(B) $c = a + bs$

(C) $c = a + b\ln s$

where a and b are non-zero constants.

(iii) With reference to the scatter diagram, explain clearly which model is the best model for this set of data. For the case identified, find the equation of a suitable regression line. [2]

(iv) Using the regression line found in **(iii)**, estimate the number of commercial crimes (to the nearest whole number) when the mean years of schooling reaches 11. [2]

(v) Comment on the reliability of your answer in part **(iv)**. [1]

10 In the latest Pokkinon Roll game, players go to a battle arena to use their Pokkinon character to battle against each other. Alvin and Billy are interested to know how long it takes before someone wins a battle. The time taken by a randomly chosen player to win a game follows a normal distribution.

(a) Alvin claims that on average, it will take at most 190.0 seconds to win a battle.

To verify his belief, he surveyed a randomly chosen sample of 45 Pokkinon Roll gamers and found out that the mean is 195.0 seconds with a variance of 206.0 seconds^2.

Carry out an appropriate test at 1% level of significance whether there is any evidence to doubt Alvin's claim. State an assumption needed for the calculation. [5]

(b) Billy also obtained his own data by recording the time taken, in seconds, by 5 randomly chosen gamers as shown below.

188.0 190.0 k 186.0 187.0

However, he believes that it will take 190.0 seconds on average to win a battle. When he conducted the test at 4.742% level of significance, his conclusion is one where the null hypothesis is not rejected. The sample mean time taken is denoted by $\bar{x}$.

Given that s^2 is the unbiased estimate of the population variance and that the maximum range of values of $\bar{x}$ is $188 \le \bar{x} \le a$, write down an equation involving s and a. [1]

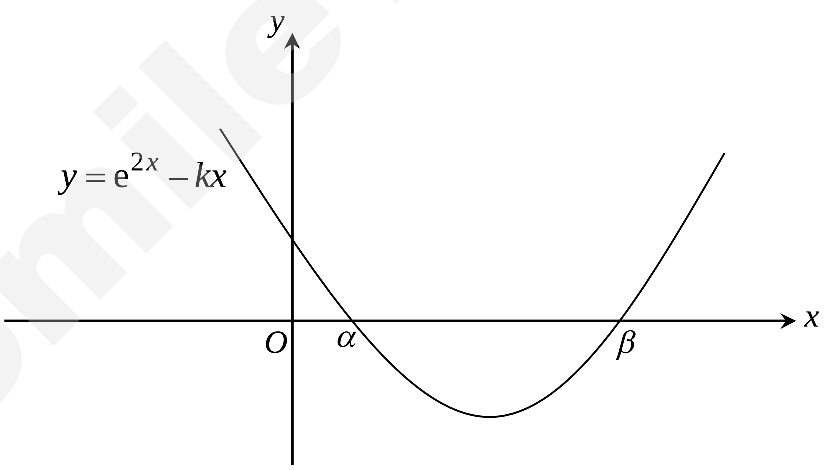
Hence or otherwise find the values of a and k, leaving your answers to the nearest integer. [5]

THE END

SRJC Paper 1 Solutions :

Answer all questions [100 marks].

1	State a sequence of 3 transformations which transform the graph of $\mathrm{g}(x)=\mathrm{e}^{(6x+2)^2}+1$ to the graph of $\mathrm{f}(x)=\mathrm{e}^{(2x-2)^2}$.	[3]
	Solution	
	I) Translation in the negative y-direction by 1 unit $y=\mathrm{e}^{(6x+2)^2}+1 \to y=\mathrm{e}^{(6x+2)^2}$	
	II) Stretch parallel to the x-axis by a scale factor 3. $y=\mathrm{e}^{(6x+2)^2} \to y=\mathrm{e}^{\left(6\left(\frac{x}{3}\right)+2\right)^2}=\mathrm{e}^{(2x+2)^2}$	
	III) Translation in the positive x-direction by 2 units y= $\mathrm{e}^{(2x+2)^2} \to y=\mathrm{e}^{(2(x-2)+2)^2}=\mathrm{e}^{(2x-2)^2}$	
	Alternatively	
	I) Translation in the negative y-direction by 1 unit $y=\mathrm{e}^{(6x+2)^2}+1 \to y=\mathrm{e}^{(6x+2)^2}$	
	II) Translation in the positive x-direction by 2/3 units $y=\mathrm{e}^{(6x+2)^2} \to y=\mathrm{e}^{\left(6\left(x-\frac{2}{3}\right)+2\right)^2}=\mathrm{e}^{(6x-2)^2}$	
	III) Stretch parallel to the x-axis by a scale factor 3. $y=\mathrm{e}^{(6x-2)^2} \to y=\mathrm{e}^{\left(6\left(\frac{x}{3}\right)-2\right)^2}=\mathrm{e}^{(2x-2)^2}$	
	Note: The translation can be step 1, 2 or 3.	
2	Using the standard series expansions, obtain the Maclaurin series of $\ln\left[(1+x)(1-2x)^3\right]$ in ascending powers of x, up to and including the term in x^3. **(i)** Find the set of values of x for which the above expansion is valid. **(ii)** Hence, find the range of values of x for which the expansion $\left[\mathrm{e}^{\ln\left[\ln\frac{1}{(1+2x)^3}\right]}-\ln(1-x)\right](2+7x)^5$ is valid.	[2] [1] [2]
	Solution	

	$\ln[(1+x)(1-2x)^3] = \ln(1+x) + 3\ln(1-2x)$ $= x - \frac{x^2}{2} + \frac{x^3}{3} + 3\left(-2x - 2x^2 - \frac{8x^3}{3}\right) + \cdots$	
	$= -5x - \frac{13x^2}{2} - \frac{23x^3}{3} + \cdots$	
	(i) So $-1 < -2x \le 1$ and $-1 < x \le 1$	
	$-\frac{1}{2} \le x < \frac{1}{2}$ and $-1 < x \le 1$	
	$\left\{x \in \mathbb{R} : -\frac{1}{2} \le x < \frac{1}{2}\right\}$	
	(ii) $\left[e^{\ln\left[\ln \frac{1}{(1+2x)^3}\right]} - \ln(1-x)\right](2+7x)^3 = \left[-3\ln(1+2x) - \ln(1-x)\right](2+7x)^3$	
	$= -\left[3\ln(1+2x) + \ln(1-x)\right](2+7x)^3$	
	From **(i)**, replace x by $-x$,	
	$-\frac{1}{2} < x \le \frac{1}{2}$ and $x \in \mathbb{R}$	
	$\therefore$ Range of values of x is $-\frac{1}{2} < x \le \frac{1}{2}$.	
3	**(a)** 	
	The diagram shows the graph of $y = e^{2x} - kx$, where k is a positive real number. The two roots of the equation $e^{2x} - kx = 0$ are denoted by α and β, where $\alpha < \beta$.	
	It is given that there is a sequence of real numbers $x_1, x_2, x_3 ...$ that satisfies the recurrence relation, $x_{n+1} = \frac{1}{k}e^{2x_n}$, for $n \ge 1$.	
	By considering $x_{n+1} - x_n$, prove that $x_{n+1} > x_n$ if $x_n < \alpha$ or $x_n > \beta$.	[2]

(b) Prove by the method of mathematical induction that

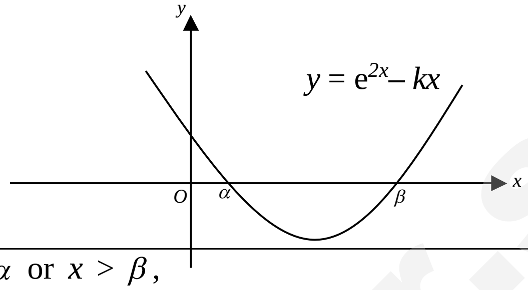
$$\sum_{r=1}^{n} \cos r\theta = \frac{\sin(n+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$$, for all positive integers n. [5]

Solution

(a) $x_{n+1} - x_n$

$= \dfrac{1}{k}e^{2x_n} - x_n$

$= \dfrac{e^{2x_n} - kx_n}{k}$



From given sketch, if $x < \alpha$ or $x > \beta$,

$e^{2x} - kx > 0$

So if $x_n < \alpha$ or $x_n > \beta$,

$x_{n+1} - x_n = \dfrac{e^{2x_n} - kx_n}{k} > 0$

Therefore, $x_{n+1} > x_n$ (Proven)

(b) Let P_n be the statement

$\cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta = \dfrac{\sin(n+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$ for $n \in \mathbb{Z}^+$.

To show P_1 is true,

$$\text{LHS} = \sum_{r=1}^{1} \cos r\theta = \cos\theta$$

$$\text{RHS} = \frac{\sin\frac{3}{2}\theta - \sin\frac{1}{2}\theta}{2\cos\frac{1}{2}\theta}$$

$$= \frac{2\cos\theta\sin(\frac{1}{2}\theta)}{2\sin\frac{1}{2}\theta}$$

$$= \cos\theta = \text{LHS}$$

$\therefore P_1$ is true.

Suppose P_k is true for some $k \in \mathbb{Z}^+$, i.e. $\displaystyle\sum_{r=1}^{k} \cos r\theta = \frac{\sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$

To show P_{k+1} is true, i.e. $\displaystyle\sum_{r=1}^{k+1} \cos r\theta = \frac{\sin(k+\frac{3}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$

$$\text{LHS} = \sum_{r=1}^{k+1} \cos r\theta$$

$$= \sum_{r=1}^{k} \cos r\theta + \cos(k+1)\theta$$

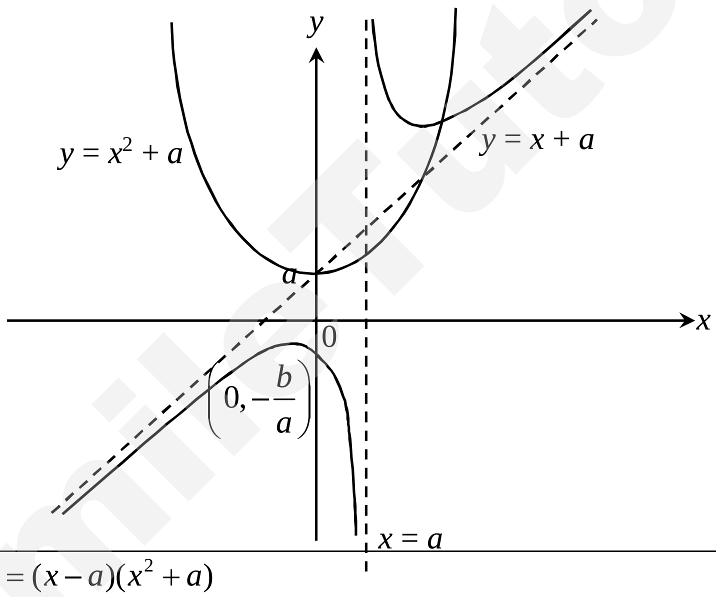
$$= \frac{\sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} + \cos(k+1)\theta$$

	$= \dfrac{\sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta + 2\cos(k+1)\theta\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$	
	$= \dfrac{\sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta + \sin(k+\frac{3}{2})\theta - \sin(k+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$ $= \dfrac{\sin(k+\frac{3}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} = \text{RHS}$	
	Hence, P_{k+1} is true if P_k is true. (Missing is "P")	
	Since P_1 is true and P_{k+1} is true when P_k is true, by Mathematical Induction, P_n is true for all positive integers n.	
4	Andy and his fiancée signed up for a new 4-room flat in Boon Keng. They take up a housing loan of \$450,000 provided by BEST bank for the purchase. The couple pay a fixed monthly instalment of \$$A$ on the first day of each month. Interest is charged on the last day of each year at a fixed rate of 1.6% of the remaining loan amount at the beginning of that year. If the first instalment is paid in January 2016,	
	(i) Show that the amount the couple owe the bank at the end of 2017 is $\$[464515.2 - 24.192A]$.	[1]
	(ii) Given that A is 1500, find the date and amount of the final repayment to the nearest cent.	[5]
	Solution	
	(i) Amt owe at the end of 2017 = (1.016)(1.016)(450000) – 1.016(12A) – 12A $= \$[464515.2 - 24.192A]$	
	(ii)	

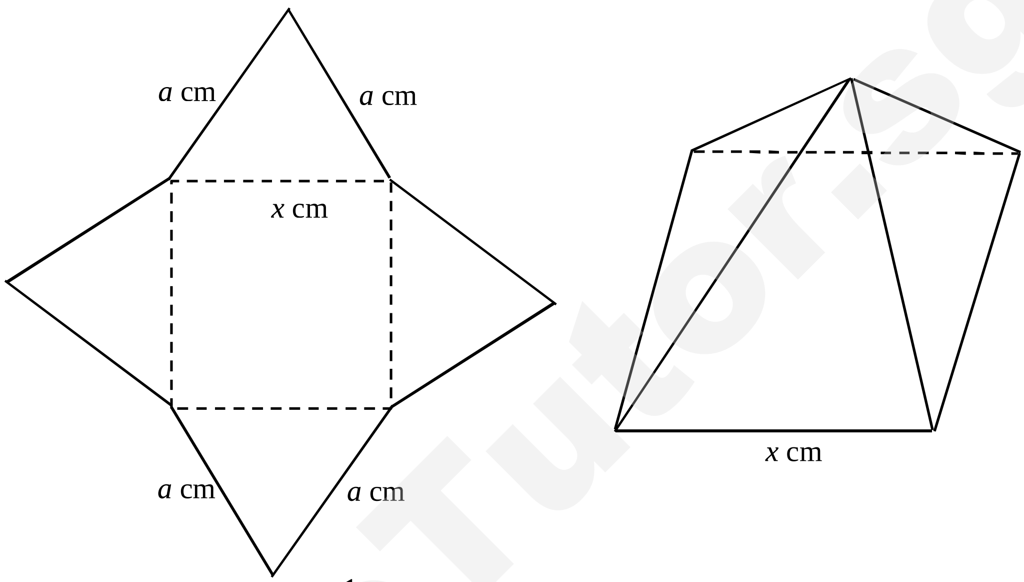
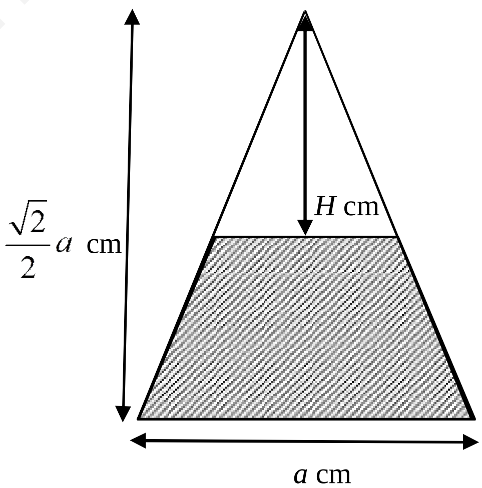
year	Amt owed at the beginning	Amt owed at the end of the year after paying 18000
1st	450000	1.016(450000) -18000
2nd	1.016(450000) -12A	(1.016)(1.016)(450000) – 1.016(18000) – 18000
3rd	(1.016^2)(450000) – 1.016(12A) – 12A	(1.016)(1.016^2)(450000) – (1.016)(1.016)(18000) – (1.016)(18000) – 18000

		…	…
	n^{th}		$(1.016^n)(450000)$ – $(1.016^{n-1})(18000)$ – $(1.016^{n-2})(18000)$ – …… – 18000

	Amount of money owe at the end of nth year $= 450000(1.016)^n - 18000(1+1.016+1.016^2+...+1.016^{n-1})$	
	Consider $450000(1.016)^n - 18000\left(\dfrac{1\left(1.016^n-1\right)}{1.016-1}\right) \le 0$	
	$450000(1.016)^n - 1125000\left(1.016^n - 1\right) \le 0$	
	Using G.C, $n \ge 32.2$	
	When n = 32,	
	Amount owe at the end of 32 years $= \$450000(1.016)^{32} - \$1125000\left(1.016^{32}-1\right) = \$3233.601$	
	Since they will be paying \$1500 each month, they will finished the payment on 1st March 2048. The last payment is \$233.60.	
5	**(a)** It is given that $y = \dfrac{x^2-x-1}{x+1}$, $x \in \mathbb{R}$, $x \ne -1$. Without using a graphic calculator, find the set of values that y cannot take.	[3]
	(b) The curve C has equation $y = \dfrac{x^2+b}{x-a}$, where $a > 0$, $b > a$ and $x \ne a$	
	(i) Draw a sketch of the curve C, label clearly the equation(s) of its asymptote(s) and the coordinates of any intersection with the axes.	[3]
	(ii) By drawing an additional graph on the diagram drawn in **(i)**, state the number of real root(s) of the equation $x^2+b=\left(x-a\right)\left(x^2+a\right)$.	[2]
	Solution	
	(a) Consider any horizontal line $y=k$, $k \in \mathbb{R}$. Consider the intersection of the graphs $y = \dfrac{x^2-x-1}{x+1}$ and $y=k$, i.e. $\dfrac{x^2-x-1}{x+1} = k$ $\Rightarrow x^2 - x - 1 = k(x+1)$ $\Rightarrow x^2 + x(-1-k) + (-1-k) = 0$	
	For the equation to have no real solutions,	

	Discriminant <0 $\Rightarrow (-1-k)^2-4(1)(-1-k)<0$	
	$\Rightarrow k^2+6k+5<0$ $(k+1)(k+5)<0$  $\therefore$The set of values that y cannot take is $\{y\in\mathbb{R}\ :-5<y<-1\}$	
	(bi) When $x = 0$, $y = -\dfrac{b}{a}$ $y=\dfrac{x^2+b}{x-a}=(x+a)+\dfrac{a^2+b}{x-a}$ $x = a$ and $y = x + a$, are equations of the asymptotes	
	(ii) 	
	$x^2+b=(x-a)(x^2+a)$ $\Rightarrow \dfrac{x^2+b}{x-a}=x^2+a$	
	By adding an additional graph in (i), i.e. $y=x^2+a$, no. of real root is 1.	
6	**(a)** The equations of two planes p_1 and p_2 are $x+4y+2z=7$, $3x+\lambda y+4z=\mu$, respectively, where λ and μ are constants.	
	(i) Given that the two planes intersect in a line l, with a vector equation given by $\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+s\begin{pmatrix}-2\\1\\-1\end{pmatrix},\ s\in\mathbb{R},$ show that the value of λ is 10 and find the value of μ.	[3]

	(ii) If plane p_3 is the reflection of p_1 in p_2, find the acute angle between p_1 and p_3.	[2]
	(b) Relative to the origin O, the points A, B, C and D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively. It is given that λ and μ are non-zero numbers such that $\lambda\mathbf{a} + \mu\mathbf{b} - \mathbf{c} = \mathbf{0}$ and $\lambda + \mu = 1$,	
	(i) Show that A, B and C are collinear.	[3]
	(ii) If O is not on the line AC and $\lvert\mathbf{c}\times\mathbf{a}\rvert(\mathbf{b}-\mathbf{a}) = (\mathbf{c}\bullet\mathbf{d})\mathbf{d}$, determine the relationship between $\overrightarrow{AC}$ and $\overrightarrow{OD}$, explaining your answer clearly.	[2]
	Solution	
	(ai) $\begin{pmatrix}-2\\1\\-1\end{pmatrix}\bullet\begin{pmatrix}3\\\lambda\\4\end{pmatrix}=0$	
	$\lambda = 10$	
	$\mu = \begin{pmatrix}1\\1\\1\end{pmatrix}\bullet\begin{pmatrix}3\\10\\4\end{pmatrix}=17$	
	(ii) Let θ be the angle between p_1 and p_2. $\cos\theta = \dfrac{\begin{pmatrix}1\\4\\2\end{pmatrix}\bullet\begin{pmatrix}3\\10\\4\end{pmatrix}}{\sqrt{1^2+4^2+2^2}\sqrt{3^2+10^2+4^2}}$	
	$\cos\theta = \dfrac{51}{\sqrt{21}\sqrt{125}}$	
	$\theta = 5.4869°$	
	Acute angle between p_1 and p_3 = $2(5.4869°) = 11.0°$	
	(bi) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ $= \lambda\mathbf{a} + \mu\mathbf{b} - \mathbf{a}$	
	$= (\lambda - 1)\mathbf{a} + \mu\mathbf{b}$	
	$= -\mu\mathbf{a} + \mu\mathbf{b}$	
	$= \mu(\mathbf{b} - \mathbf{a})$	
	Since $\overrightarrow{AC} = \mu\overrightarrow{AB}$ for some $\mu \in \mathbb{R}\backslash\{0\}$, A, B, C are collinear.	
	(ii) $\lvert\mathbf{c}\times\mathbf{a}\rvert(\mathbf{b}-\mathbf{a}) = (\mathbf{c}\bullet\mathbf{d})\mathbf{d} \Rightarrow \overrightarrow{AB} = k\overrightarrow{OD}$ for some $k \in \mathbb{R}$ as $\lvert\mathbf{c}\times\mathbf{a}\rvert \neq 0$ since O is not on AC	
	since $\overrightarrow{AB} = \mu\overrightarrow{AC}$ for some $\mu \in \mathbb{R}$	

	so $\overrightarrow{AC}$ is parallel to $\overrightarrow{OD}$.	
7	A piece of metal with negligible thickness has been cut into a shape that is made up of four isosceles triangles each with base x cm and fixed sides a cm. Their bases frame to a form a square with sides of length x cm. A right pyramid is formed by folding along the dotted lines as shown in the diagram below.  [Volume of a pyramid = $\frac{1}{3}\times$ base area $\times$ height]	
	(i) Show that the volume of the pyramid is $\frac{x^2}{3}\sqrt{a^2-\frac{x^2}{2}}$ cm^3.	[2]
	(ii) Find the value of x, in terms of a, that will give maximum volume for the pyramid.	[4]
	(iii)  To make the pyramid into a paperweight with negligible thickness, a viscous fluid is pumped into the interior at a rate of 1 cm^3/s. Given that H cm is the perpendicular distance from the apex of the pyramid to the viscous fluid surface,	

$x = a$ and the height of the pyramid is $\frac{\sqrt{2}}{2}a$ cm, find the rate at which H is changing when $H = \frac{a}{2}$, giving your answer in terms of a.

[The diagram above shows the cross sectional area of the pyramid.] [3]

Solution

(i) Let the height of the isosceles triangle be k cm.

$$k^2 + \frac{x^2}{4} = a^2$$

$$k^2 = a^2 - \frac{x^2}{4}$$

Therefore, height of the pyramid = $\sqrt{a^2 - \frac{x^2}{4} - \frac{x^2}{4}} = \sqrt{a^2 - \frac{x^2}{2}}$

Volume of pyramid, $V = \frac{x^2}{3}\sqrt{a^2 - \frac{x^2}{2}}$

(ii) $\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{4x}{3}\sqrt{a^2 - \frac{x^2}{2}} + \frac{x^2}{3}\left(\frac{1}{2}\right)\frac{-x}{\sqrt{a^2 - \frac{x^2}{2}}}$

$$= \frac{4x\left(a^2 - \frac{x^2}{2}\right) - x^3}{6\sqrt{a^2 - \frac{x^2}{2}}}$$

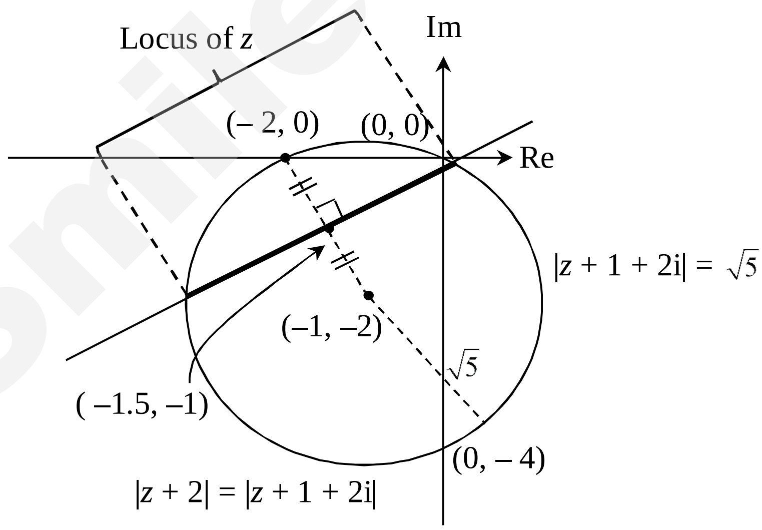
$$= \frac{4a^2x - 3x^2}{6\sqrt{a^2 - \frac{x^2}{2}}}$$

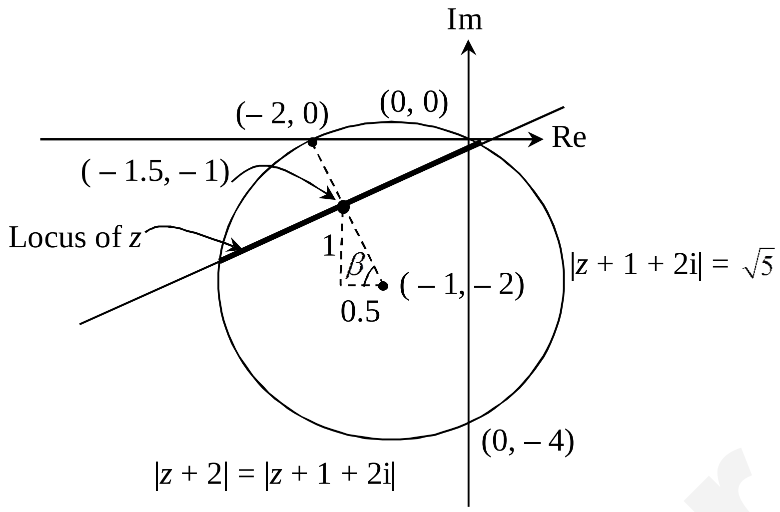
$$= \frac{2x\left(a - \frac{\sqrt{3}}{2}x\right)\left(a + \frac{\sqrt{3}}{2}x\right)}{3\sqrt{a^2 - \frac{x^2}{2}}}$$

When $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$,

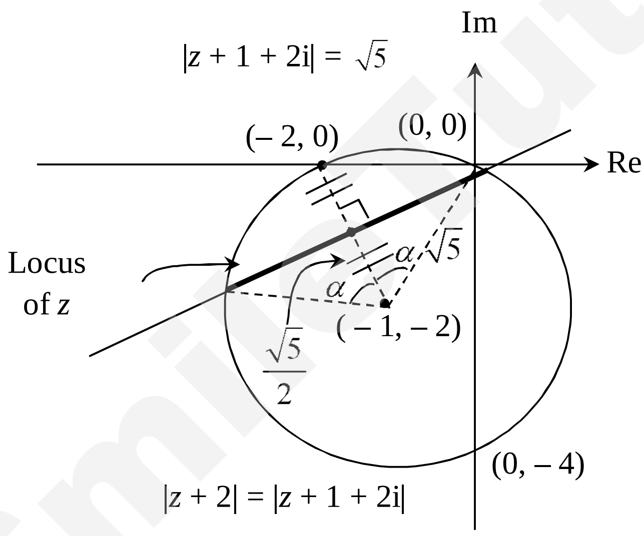
$$\frac{2x\left(a - \frac{\sqrt{3}}{2}x\right)\left(a + \frac{\sqrt{3}}{2}x\right)}{3\sqrt{a^2 - \frac{x^2}{2}}} = 0$$

	$\frac{2x}{3}\left(a^2-\frac{x^2}{2}\right)=\frac{x^3}{6}$	
	$x=\frac{2\sqrt{3}}{3}a$ or $-\frac{2\sqrt{3}}{3}a$ (rejected $\because x>0$)	
	When $x=\frac{2\sqrt{3}}{3}a^-$, $\frac{1}{\sqrt{a^2-\frac{x^2}{2}}}>0$, $\frac{2}{3}x>0$, $a-\frac{\sqrt{3}}{2}x>0$ and $a+\frac{\sqrt{3}}{2}x>0$, thus $\frac{\mathrm{d}V}{\mathrm{d}x}>0$. When $x=\frac{2\sqrt{3}}{3}a^+$, $\frac{1}{\sqrt{a^2-\frac{x^2}{2}}}>0$, $\frac{2}{3}x>0$, $a-\frac{\sqrt{3}}{2}x<0$ and $a+\frac{\sqrt{3}}{2}x>0$, thus $\frac{\mathrm{d}V}{\mathrm{d}x}<0$. Therefore $x=\frac{2\sqrt{3}}{3}a$ gives maximum volume.	
	(iii) Volume of pyramidal empty space, W cm^3, in the pyramid as it is being filled up $=\frac{1}{3}b^2H$, where b is the length of the square base $\frac{b}{H}=\sqrt{2}$ Therefore $W=\frac{2}{3}H^3 \Rightarrow \frac{\mathrm{d}W}{\mathrm{d}H}=2H^2$	
	When $H=\frac{a}{2}$	
	$\frac{\mathrm{d}W}{\mathrm{d}H}\times\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{\mathrm{d}W}{\mathrm{d}t}$ $2\left(\frac{a}{2}\right)^2\times\frac{\mathrm{d}H}{\mathrm{d}t}=-1$	
	$\frac{\mathrm{d}H}{\mathrm{d}t}=-\frac{2}{a^2}$	
	H is decreasing at a rate of $\frac{2}{a^2}$ cm/s.	
8	**(i)** Show that $\left(0,-\frac{1}{4}\right)$ lies on the locus $\lvert z+2\rvert=\lvert z+1+2\mathrm{i}\rvert$.	[1]

	(ii) Sketch on a single Argand diagram the loci $\lvert z+1+2\mathrm{i}\rvert=\sqrt{5}$ and $\lvert z+2\rvert=\lvert z+1+2\mathrm{i}\rvert$.	[4]
	(iii) Hence indicate clearly on the Argand diagram the locus of z that satisfies the relations $\lvert z+1+2\mathrm{i}\rvert\le\sqrt{5}$ and $\lvert z+2\rvert=\lvert z+1+2\mathrm{i}\rvert$.	[1]
	(iv) Find the greatest and least possible values of $\arg(z+1+2\mathrm{i})$, giving your answers in radians correct to 3 decimal places.	[4]
	Solution	
	(i) When $z=-\frac{1}{4}\mathrm{i}$ $\text{L.H.S}=\left\lvert-\frac{1}{4}\mathrm{i}+2\right\rvert=\sqrt{4\frac{1}{16}}$ $\text{R.H.S}=\left\lvert-\frac{1}{4}\mathrm{i}+1+2\mathrm{i}\right\rvert=\left\lvert1+\frac{7}{4}\mathrm{i}\right\rvert=\sqrt{4\frac{1}{16}}$ Hence $\left(0,-\frac{1}{4}\right)$ lies on $\lvert z+2\rvert=\lvert z+1+2\mathrm{i}\rvert$.	
	(ii) & (iii) 	
	(iv)	



$\beta = \tan^{-1}\left(\dfrac{1}{0.5}\right) = \tan^{-1}(2)$



$\alpha = \cos^{-1}\left[\dfrac{\left(\dfrac{\sqrt{5}}{2}\right)}{\sqrt{5}}\right] = \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$

Greatest value of arg (z +1+2i) = $\pi - (\beta - \alpha)$

$= 3.08164$

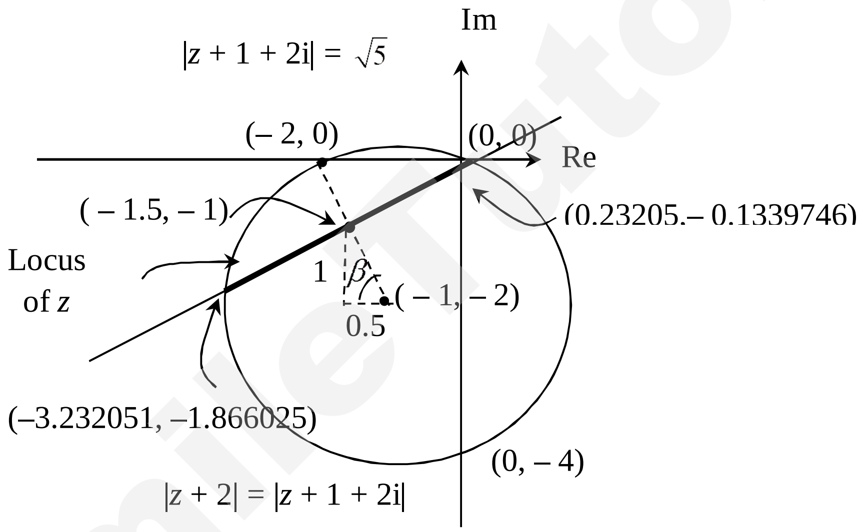
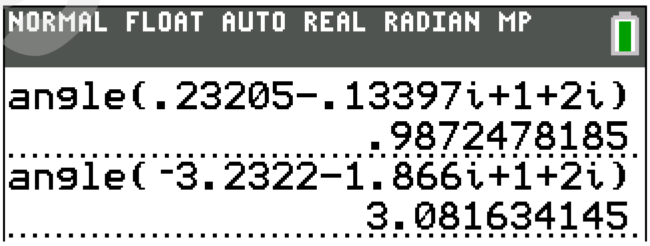
$= 3.082$

Least value of arg (z +1+2i) = $\pi - \beta - \alpha$

$= 0.987246 = 0.987$

(iv)

Alternative Method

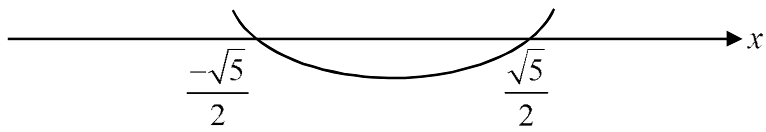
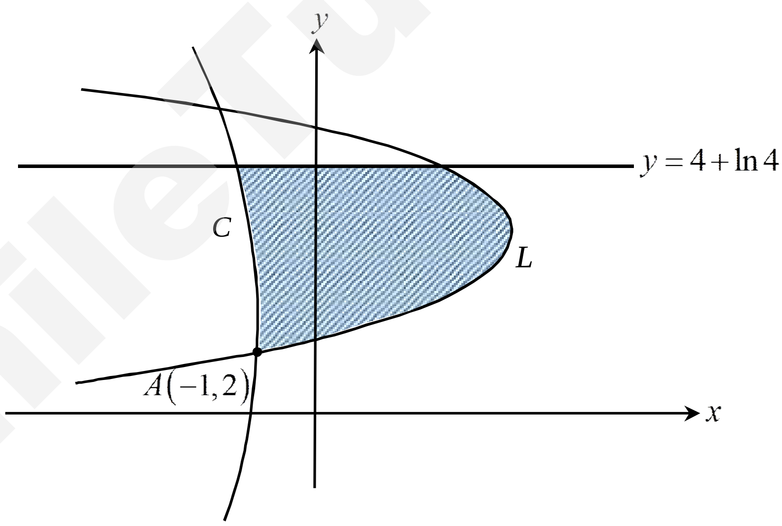
	Equation of circle: $(x + 1)^2 + (y + 2)^2 = 5$ $y = -2 \pm\sqrt{5-(x+1)^2}$ ------------------ (1) Gradient of the line passing through the $(-2, 0)$ and $(-1, -2)$ $= \dfrac{0-(-2)}{(-2)-(-1)} = -2$ Gradient of the perpendicular bisector $= -\dfrac{1}{-2} = \dfrac{1}{2}$ Equation of the perpendicular bisector: $y + 1 = \dfrac{1}{2}(x + 1.5)$ $y = \dfrac{1}{2}x - \dfrac{1}{4}$ ------------------ (2) Using GC, the points of intersection are $(-3.232051, -1.866025)$ and $(0.23205, -0.1339746)$.  Least value of arg $(z + 1 + 2i)$ when $z = 0.23205 - 0.1339746i$ is 0.987. Greatest value of arg $(z + 1 + 2i)$ when $z = -3.232051 - 1.866025i$ is 3.082  NORMAL FLOAT AUTO REAL RADIAN MP angle(.23205-.13397i+1+2i) .9872478185 angle(⁻3.2322-1.866i+1+2i) 3.081634145	
9	The path travelled by an object measured with respect to the origin in the horizontal and vertical directions, at time t seconds, is denoted by the variables x and y respectively.	

	It is given that when $t=0,\ x=1,\ y=0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=1$. The variables are related by the differential equations $\frac{\mathrm{d}y}{\mathrm{d}t}-y+\sqrt{e^{2t}-4y^2}=0$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\cos^2 2t$.	
	(i) Using the substitution $y=we^t$, show $\frac{\mathrm{d}w}{\mathrm{d}t}=-\sqrt{1-4w^2}$ and hence find y in terms of t.	[6]
	(ii) Find x in terms of t.	[4]
	Solution	
	(i) $\frac{\mathrm{d}y}{\mathrm{d}t}-y+\sqrt{e^{2t}-4y^2}=0$	
	$y=w\mathrm{e}^t$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}w}{\mathrm{d}t}\mathrm{e}^t+w\mathrm{e}^t$	
	$\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^t\left(\frac{\mathrm{d}w}{\mathrm{d}t}+w\right)$	
	$\mathrm{e}^t\left(\frac{\mathrm{d}w}{\mathrm{d}t}+w\right)-w\mathrm{e}^t+\sqrt{\mathrm{e}^{2t}-4\left(w\mathrm{e}^t\right)^2}=0$	
	$\mathrm{e}^t\left(\frac{\mathrm{d}w}{\mathrm{d}t}+w\right)-w\mathrm{e}^t+\mathrm{e}^t\sqrt{1-4w^2}=0$ $\frac{\mathrm{d}w}{\mathrm{d}t}+w-w+\sqrt{1-4w^2}=0$ $\frac{\mathrm{d}w}{\mathrm{d}t}=-\sqrt{1-4w^2}\ (\text{shown})$	
	$\frac{1}{\sqrt{1-4w^2}}\frac{\mathrm{d}w}{\mathrm{d}t}=-1$	
	$\frac{1}{2}\sin^{-1}(2w)=-t+C$	
	$w=\frac{1}{2}\sin(-2t+2C)$	
	$y=\frac{1}{2}\mathrm{e}^t\sin(-2t+C)$	
	When $t=0, y=0$ $\frac{1}{2}\sin(-2C)=0$ $C=0$ $\therefore y=\frac{1}{2}\mathrm{e}^t\sin(-2t)$ or $y=-\frac{1}{2}\mathrm{e}^t\sin(2t)$	
	(ii) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\cos^2 2t$	

	$=\dfrac{1+\cos 4t}{2}$	
	$\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{1}{2}\left(t+\dfrac{\sin 4t}{4}\right)+D_1$	
	$x=\dfrac{1}{2}\left(\dfrac{t^2}{2}-\dfrac{\cos 4t}{16}\right)+D_1t+D_2$	
	When $t=0, x=1, \dfrac{\mathrm{d}x}{\mathrm{d}t}=1$	
	$D_1=1, D_2=\dfrac{33}{32}$	
	$x=\dfrac{t^2}{4}-\dfrac{\cos 4t}{32}+t+\dfrac{33}{32}$	
10	Given that $\mathrm{f}(r)=\dfrac{3^r}{r+1}$, show that $\mathrm{f}(r+2)-\mathrm{f}(r)=\dfrac{(8r+6)3^r}{(r+1)(r+3)}$.	[1]
	(i) Find $\displaystyle\sum_{r=1}^{n}\frac{(4r+3)3^r}{(r+1)(r+3)}$ in terms of n.	[2]
	(ii) Hence find $\displaystyle\sum_{r=1}^{n}\frac{(4r+11)3^r}{(r+3)(r+5)}$ in terms of n.	[4]
	(iii) Using the result in **(ii)**, show that $\displaystyle\sum_{r=0}^{n}\left[\frac{r\cdot 3^r}{(r+5)^2}\right]-\frac{3^{n+1}}{4}<-\frac{51}{160}$.	[3]
	Solution	
	$\mathrm{f}(r+2)-\mathrm{f}(r)=\dfrac{3^{r+2}}{r+3}-\dfrac{3^r}{r+1}$	
	$=\dfrac{r\cdot 3^{r+2}+3^{r+2}-3^r\cdot r-3^{r+1}}{(r+1)(r+3)}$	
	$=\dfrac{8r\cdot 3^r+3^{r+1}(3-1)}{(r+1)(r+3)}$	
	$=\dfrac{(8r+6)3^r}{(r+1)(r+3)}$	
	(i) $\displaystyle\sum_{r=1}^{n}\frac{(4r+3)3^r}{(r+1)(r+3)}=\frac{1}{2}\sum_{r=1}^{n}\left[\mathrm{f}(r+2)-\mathrm{f}(r)\right]$	

	$=\frac{1}{2}\begin{bmatrix} \mathrm{f}(3)-\mathrm{f}(1) \\ +\mathrm{f}(4)-\mathrm{f}(2) \\ +\mathrm{f}(5)-\mathrm{f}(3) \\ +\ldots\ldots\ldots\ldots \\ +\mathrm{f}(n)-\mathrm{f}(n-2) \\ +\mathrm{f}(n+1)-\mathrm{f}(n-1) \\ +\mathrm{f}(n+2)-\mathrm{f}(n) \end{bmatrix} \quad =\frac{1}{2}\left[-\mathrm{f}(1)-\mathrm{f}(2)+\mathrm{f}(n+1)+\mathrm{f}(n+2)\right]$	]	
	$=-\frac{9}{4}+\frac{3^{n+1}}{2}\left[\frac{1}{n+2}+\frac{3}{n+3}\right]=-\frac{9}{4}+\frac{3^{n+1}}{2}\left[\frac{4n+9}{(n+2)(n+3)}\right]$	]	
	(ii) $\sum_{r=1}^{n}\frac{(4r+11)3^r}{(r+3)(r+5)}=\frac{1}{9}\sum_{r=1}^{n}\frac{(4r+11)3^{r+2}}{(r+3)(r+5)}$		
	$=\frac{1}{9}\sum_{r=3}^{n+2}\frac{(4r+3)3^r}{(r+1)(r+3)}$		
	$=\frac{1}{9}\left[\sum_{r=1}^{n+2}\frac{(4r+3)3^r}{(r+1)(r+3)}-\frac{21}{8}-\frac{99}{15}\right]$		
	$=\frac{1}{9}\left[-\frac{9}{4}+\frac{3^{n+3}(4n+17)}{2(n+4)(n+5)}-\frac{21}{8}-\frac{99}{15}\right]$		
	$=-\frac{51}{40}+\frac{3^{n+1}(4n+17)}{2(n+4)(n+5)}$		
	(iii) $\sum_{r=0}^{n}\frac{r\cdot 3^r}{(r+5)^2}=\frac{1}{4}\sum_{r=1}^{n}\frac{4r\cdot 3^r}{(r+5)^2}$		
	$<\frac{1}{4}\sum_{r=1}^{n}\frac{(4r+11)\cdot 3^r}{(r+5)^2}$		
	$<\frac{1}{4}\sum_{r=1}^{n}\frac{(4r+11)\cdot 3^r}{(r+3)(r+5)}$		
	So $\sum_{r=0}^{n}\frac{r\cdot 3^r}{(r+5)^2}<\frac{1}{4}\left[-\frac{51}{40}+\frac{3^{n+1}(4n+17)}{2(n+4)(n+5)}\right]$		
	$\Rightarrow \sum_{r=0}^{n}\frac{r\cdot 3^r}{(r+5)^2}-\frac{3^{n+1}}{4}<-\frac{51}{160}+\frac{3^{n+1}(4n+17)}{8(n+4)(n+5)}-\frac{3^{n+1}}{4}$		
	$=-\frac{51}{160}-\frac{3^{n+1}}{4}\left[1-\frac{4n+17}{2(n+4)(n+5)}\right]$		
	$<-\frac{51}{160}$ since $\frac{4n+17}{2(n+4)(n+5)}\le\frac{17}{18}$ for all $n\ge 0$		

11	The functions f and g are defined as follows: $\mathrm{f}: x \mapsto -\lvert x^2+2x \rvert, \quad a < x \le 0$ $\mathrm{g}: x \mapsto -\sqrt{x+1}, \quad x > -1$	
	(i) State the least value of a for the inverse function of f to exist. Hence, find f^{-1} in similar form.	[4]
	For the following parts, use the value of a found in part **(i)**.	
	(ii) Write down ff^{-1} in similar form.	[1]
	(iii) Find the rule for gf in the form $bx + c$, where $b, c \in \mathbb{R}$. State its range.	[3]
	(iv) Find the exact range of x for which $\mathrm{f}\left(x-\frac{3}{2}\right) > \mathrm{gf}\left(x-\frac{3}{2}\right)$.	[3]
	Solution	
	(i) Least $a=-1$	
	$\mathrm{f}(x) = -\left(-\left(x^2+2x\right)\right) = x^2+2x$	
	Let $y = \mathrm{f}(x) = x^2+2x$	
	$y = (x+1)^2 - 1$	
	$y+1 = (x+1)^2$	
	$x = -1 \pm \sqrt{y+1}$	
	$x = -1 + \sqrt{y+1} \; (\because -1 < x \le 0)$	
	$\mathrm{f}^{-1}: x \mapsto -1+\sqrt{x+1}, \quad -1 < x \le 0$	
	(ii) $\mathrm{ff}^{-1}: x \mapsto x, \quad -1 < x \le 0$	
	(iii) $\mathrm{gf}(x) = -\sqrt{x^2+2x+1}$	
	$= -\sqrt{(x+1)^2}$	
	$= -\lvert x+1 \rvert$	
	$= -x-1 \quad \left(\because D_{\mathrm{gf}} = (-1,0]\right)$	
	$R_{\mathrm{gf}} = [-1, 0)$	
	(iv) $\mathrm{f}\left(x-\frac{3}{2}\right) > \mathrm{gf}\left(x-\frac{3}{2}\right)$	
	$\left(x-\frac{3}{2}\right)^2 + 2\left(x-\frac{3}{2}\right) > -\left(x-\frac{3}{2}\right) - 1$	
	$x^2 - 3x + \frac{9}{4} + 3x - 3 + 1 - \frac{3}{2} > 0$	
	$x^2 - \frac{5}{4} > 0$	
	$\left(x-\frac{\sqrt{5}}{2}\right)\left(x+\frac{\sqrt{5}}{2}\right) > 0$	

		
	$x < -\frac{\sqrt{5}}{2}$ or $x > \frac{\sqrt{5}}{2}$	
	But after translation by 1.5 units in the positive x – direction, $D_f = D_{gf} = \left(\frac{1}{2}, \frac{3}{2}\right]$	
	$\therefore \frac{\sqrt{5}}{2} < x \le \frac{3}{2}$	
12	**(a)(i)** Find $\frac{d}{dx}\left[(\ln x)^2\right]$.	[1]
	(ii) The curve C is defined by the parametric equations $x = \ln t - t, \quad y = 2t + \ln(t^2)$ where $t > 0$.	
	Another curve L is defined by the equation $(4-y)^2 = 3-x$. The graphs of C and L intersect at the point $A(-1,2)$ as shown in the diagram below.	
		
	Find the exact area of the shaded region bounded by C, L and the line $y = 4 + \ln 4$.	[6]
	(b) The region R is the finite region enclosed by the curve $(y-1)^2 = 1-x$ and the y-axis. The region S is the region in the 2nd quadrant enclosed by the curve $y = 2\tan\left(x + \frac{\pi}{4}\right)$ and the axes.	
	Find the total volume generated when region R and S is rotated through 2π radians about the x-axis, leaving your answers in exact form.	[4]
	Solution	
	(ai) $\frac{d}{dx}\left[(\ln x)^2\right] = (2\ln x)\left(\frac{1}{x}\right) = \frac{2\ln x}{x}$	

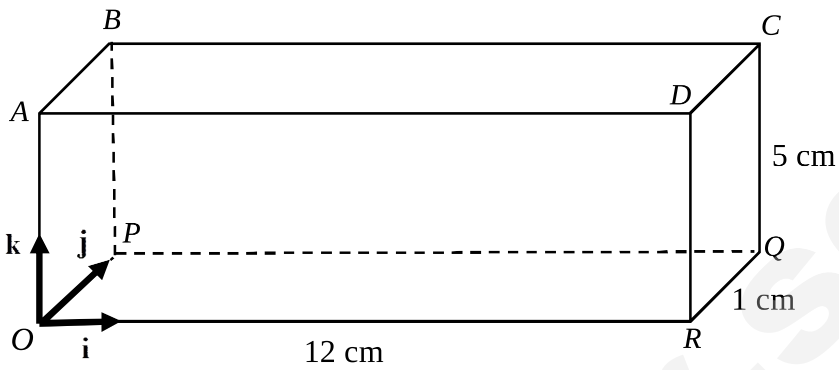
	(ii) Area = $\int_{2}^{4+\ln 4}\left[3-(4-y)^2\right]\mathrm{d}y-\int_{2}^{4+\ln 4}x\ \mathrm{d}y$	
	$=\left[3y-\frac{(4-y)^3}{(-3)}\right]_2^{4+\ln 4}-\int_1^2(\ln t-t)\left(2+\frac{2}{t}\right)\mathrm{d}t$	
	$=3(2+\ln 4)-\frac{1}{3}\left[(-\ln 4)^3-8\right]-\int_1^2\left(2\ln t+\frac{2\ln t}{t}-2t-2\right)\mathrm{d}t$	
	$=\frac{10}{3}+6\ln 2-\frac{(\ln 4)^3}{3}-2\int_1^2\ln t\ \mathrm{d}t-\left[(\ln t)^2\right]_1^2+\left[t^2\right]_1^2+\left[2t\right]_1^2$	
	$=\frac{10}{3}+6\ln 2-\frac{(\ln 4)^3}{3}-2\left[t\ln t\right]_1^2+\int_1^2 2\ \mathrm{d}t-(\ln 2)^2+3+2$	
	$=\frac{25}{3}+6\ln 2-\frac{(\ln 4)^3}{3}-4\ln 2+\left[2t\right]_1^2-(\ln 2)^2$	
	$=\frac{31}{3}+2\ln 2-\frac{(\ln 4)^3}{3}-(\ln 2)^2$	
	(b) Vol = $\pi\int_{-\frac{\pi}{4}}^{0}4\tan^2\left(x+\frac{\pi}{4}\right)\mathrm{d}x+\pi\int_0^1\left[\left(1+\sqrt{1-x}\right)^2-\left(1-\sqrt{1-x}\right)^2\right]\mathrm{d}x$	
	$=4\pi\int_{-\frac{\pi}{4}}^{0}\left(\sec^2\left(x+\frac{\pi}{4}\right)-1\right)\mathrm{d}x+\pi\int_0^1 4\sqrt{1-x}\ \mathrm{d}x$	
	$=4\pi\left[\tan\left(x+\frac{\pi}{4}\right)-x\right]_{-\frac{\pi}{4}}^{0}+4\pi\left[\frac{(1-x)^{\frac{3}{2}}}{-\frac{3}{2}}\right]_0^1$	
	$=4\pi\left(1-\frac{\pi}{4}\right)+\frac{8}{3}\pi=\frac{20}{3}\pi-\pi^2$	

END OF PAPER

SRJC Paper 2 Solutions

Section A: Pure Mathematics [40 marks].

1	**(a)** If $0<a<b$, solve $\int_0^b x\lvert a-x\rvert\,\mathrm{d}x$, leaving your answers in terms of a and b.	[2]
	(b)(i) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{3-x}{\sqrt{1-x}}\right)$.	[1]
	(ii) Find $\displaystyle\int \frac{3-x}{x^2-3x+2}\mathrm{d}x$.	[2]
	(iii) Hence find $\displaystyle\int \frac{1+x}{(1-x)^{\frac{3}{2}}}\tan^{-1}\sqrt{1-x}\ \mathrm{d}x$.	[3]
	Solution	
	(a) $\int_0^b x\lvert a-x\rvert\,\mathrm{d}x=\int_0^a x(a-x)\,\mathrm{d}x+\int_a^b x(x-a)\,\mathrm{d}x$	
	$=\left[\dfrac{ax^2}{2}-\dfrac{x^3}{3}\right]_0^a+\left[\dfrac{x^3}{3}-\dfrac{ax^2}{2}\right]_a^b$	
	$=\left(\dfrac{a^3}{2}-\dfrac{a^3}{3}\right)+\left(\dfrac{b^3}{3}-\dfrac{ab^2}{2}\right)-\left(\dfrac{a^3}{3}-\dfrac{a^3}{2}\right)$	
	$=\dfrac{a^3}{3}+\dfrac{b^3}{3}-\dfrac{ab^2}{2}$	
	(bi) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{3-x}{\sqrt{1-x}}\right)=\dfrac{-\sqrt{1-x}-(3-x)\left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{1-x}}\right)}{1-x}$	
	$=\dfrac{-2+2x+3-x}{2(1-x)^{3/2}}=\dfrac{x+1}{2(1-x)^{3/2}}$	
	(ii) $\displaystyle\int\frac{3-x}{x^2-3x+2}\mathrm{d}x=\int\frac{3-x}{(x-2)(x-1)}\mathrm{d}x$	
	$\displaystyle=\int\left[\frac{1}{x-2}-\frac{2}{x-1}\right]\mathrm{d}x$	
	$=\ln\lvert x-2\rvert-2\ln\lvert x-1\rvert+c$	
	(iii) $\displaystyle\int\frac{1+x}{(1-x)^{\frac{3}{2}}}\tan^{-1}\sqrt{1-x}\ \mathrm{d}x$ $\displaystyle=2\left(\frac{3-x}{\sqrt{1-x}}\right)\tan^{-1}\sqrt{1-x}-\int 2\left(\frac{3-x}{\sqrt{1-x}}\right)\left(\frac{1}{1+1-x}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{1-x}}\right)\mathrm{d}x$	
	$\displaystyle=2\left(\frac{3-x}{\sqrt{1-x}}\right)\tan^{-1}\sqrt{1-x}+\int\frac{3-x}{(2-x)(1-x)}\ \mathrm{d}x$	

	$= 2\left(\dfrac{3-x}{\sqrt{1-x}}\right)\tan^{-1}\sqrt{1-x} + \ln\lvert x-2\rvert - 2\ln\lvert x-1\rvert + c$	
2		
	The cuboid above is formed by the eight vertices *O*, *A*, *B*, *C*, *D*, *P*, *Q* and *R*. Perpendicular unit vectors **i**, **j**, **k** are parallel to *OR*, *OP* and *OA* respectively. The length of *OR*, *OP* and *OA* are 12 cm, 1 cm and 5 cm respectively.	
	(i) Find the Cartesian equation of line *AC*.	[2]
	(ii) Find the acute angle between *CA* and *CR*. Hence, find the shortest distance from *R* to *AC*.	[4]
	(iii) The point *T* is on *AC* produced such that $AT = \lambda AC$ and *M* is the midpoint of *OR*. The unit vector in the direction of *OT* is represented by the vector $\overrightarrow{OV}$. By considering the cross product of relevant vectors, find the ratio of the area of triangle *OVM* to the area of triangle *ORT* in terms of λ.	[3]
	Solution	
	(i) $\overrightarrow{AC} = 12\mathbf{i}+\mathbf{j}$	
	Equation of line *AC*: $\mathbf{r} = \begin{pmatrix}0\\0\\5\end{pmatrix} + \alpha\begin{pmatrix}12\\1\\0\end{pmatrix},\ \alpha \in \mathbb{R}$	
	Cartesian equation of line is $\dfrac{x}{12} = y, z = 5.$	
	(ii) Let the acute angle between *AC* and *RC* be *x*. $\cos x = \dfrac{\left\lvert\begin{pmatrix}12\\1\\0\end{pmatrix}\bullet\begin{pmatrix}0\\1\\5\end{pmatrix}\right\rvert}{\sqrt{145}\sqrt{26}}$	
	Therefore, $x = 89.1°$	
	Let the shortest distance required be *y*.	
	$\sin 89.07° = \dfrac{y}{\sqrt{26}}$	
	$y = 5.10$ cm	

(iii)

$$\vec{OC} = \frac{\vec{OT} + (\lambda - 1)\vec{OA}}{\lambda}$$

$$\vec{OT} = \lambda\vec{OC} - (\lambda - 1)\vec{OA}$$

$$= \lambda\begin{pmatrix}12\\1\\5\end{pmatrix} - (\lambda - 1)\begin{pmatrix}0\\0\\5\end{pmatrix}$$

$$= \begin{pmatrix}12\lambda\\\lambda\\5\end{pmatrix}$$

Area of triangle ORT = $\frac{1}{2}\left|\vec{OT} \times \vec{OR}\right|$

Area of triangle OVM

$$= \frac{1}{2}\left|\frac{\vec{OT}}{\left|\vec{OT}\right|} \times \frac{\vec{OR}}{2}\right|$$

$$= \frac{1}{2\sqrt{25+145\lambda^2}} \times \frac{1}{2}\left|\vec{OT} \times \vec{OR}\right|$$

Therefore the ratio of triangle OVM to area of triangle ORT is $1:2\sqrt{25+145\lambda^2}$.

3 **(a)** The complex number w is such that $w = a + \mathrm{i}b$, where a and b are non-zero real numbers. The complex conjugate of w is denoted by w^*. Given that $\frac{(w^*)^2}{w} = 3 - \mathrm{i}b$, solve for a and b and hence write down the possible values of w. [3]

(b) **(i)** Without the use of a graphic calculator, find the roots of the equation $z^2 - 2z + 4 = 0$, leaving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0$ and $-\pi < \theta \le \pi$. [2]

(ii) Let α and β be the roots found in **(b)(i)**. If $\arg(\alpha) > \arg(\beta)$, find $\left|\alpha^{10} - \beta^{10}\right|$ and $\arg\left(\alpha^{10} - \beta^{10}\right)$ in exact form. [3]

(c) **(i)** Show that the locus of z where $\arg(z + 2\sqrt{3} + \mathrm{i}) = -\frac{\pi}{6}$ passes through the point $(-\sqrt{3}, -2)$. [1]

(ii) Find the Cartesian equation of the locus of z in the form $y = mx + c$, stating its domain clearly. Leave your answer in exact form. [2]

Solution

(a) $\frac{(w^*)^2}{w} = 3 - \mathrm{i}b \Rightarrow \frac{(a - \mathrm{i}b)^2}{(a + \mathrm{i}b)} = 3 - \mathrm{i}b$

	$a^2-b^2-2\mathrm{i}ab=(3-\mathrm{i}b)(a+\mathrm{i}b)=3a+b^2+\mathrm{i}(-ab+3b)$		
	Equating the real and the imaginary parts:		
	$a^2-b^2=3a+b^2$ **.....**(1) and $-2ab=-ab+3b$ ...(2)		
	From (2) $a=-3$ since $b\neq 0$		
	From (1), $9-b^2=-9+b^2$ $b^2=9$ $b=\pm 3$ Possible values of w are $-3\pm 3\mathrm{i}$		
	(bi) $z^2-2z+4=0$ $z=\dfrac{2\pm\sqrt{4-16}}{2}=1\pm\sqrt{3}\mathrm{i}$ $\alpha=1+\sqrt{3}\mathrm{i}\ =2e^{\mathrm{i}\left(\frac{\pi}{3}\right)}$ and $\beta=1-\sqrt{3}\mathrm{i}\ =2e^{-\mathrm{i}\left(\frac{\pi}{3}\right)}$		
	(ii) $\alpha^{10}-\beta^{10}\ =2^{10}\left(e^{\mathrm{i}\left(\frac{10\pi}{3}\right)}-e^{-\mathrm{i}\left(\frac{10\pi}{3}\right)}\right)$		
	$=2^{10}\left(2\mathrm{i}\sin\dfrac{10\pi}{3}\right)$		
	$=2^{10}\left(2\mathrm{i}\sin\left(-\dfrac{2\pi}{3}\right)\right)$		
	$=2^{10}\left(2\left(-\dfrac{\sqrt{3}}{2}\right)\right)\mathrm{i}$ $=-1024\sqrt{3}\mathrm{i}$		
	$\left\|\alpha^{10}-\beta^{10}\right\|=1024\sqrt{3}$		
	So $\arg\left(\alpha^{10}-\beta^{10}\right)=-\dfrac{\pi}{2}$		
	(ci) When $z=-\sqrt{3}-2\mathrm{i}$, LHS = $\arg(-\sqrt{3}-2\mathrm{i}+2\sqrt{3}+\mathrm{i})=\arg(\sqrt{3}-\mathrm{i})$		
	$=-\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)=-\dfrac{\pi}{6}$ (Shown)		
	(ii) Gradient of the half line is $\tan(-\dfrac{\pi}{6})\ =-\dfrac{1}{\sqrt{3}}$		
	Equation : $y+2=-\dfrac{1}{\sqrt{3}}\left(x+\sqrt{3}\right),\ \ x>-2\sqrt{3}$ $y=-\dfrac{\sqrt{3}}{3}x-3$		

4	A curve C has parametric equations $$x=\sin t,\quad y=\cos t$$ **(i)** Find the equations of tangent and normal to C at the point with parameter t.	[3]
	(ii) Points P and Q on C have parameters p and q respectively, where $0<p<\frac{\pi}{2}$ and $0<q<\frac{\pi}{2}$. The tangent at P meets the normal at Q at the point R. Show that the x – coordinate of R is $\frac{\sin q}{\cos(p-q)}$. Hence, find in similar form the y – coordinate of R in terms of p and q.	[3]
	The tangent at P meets the y-axis at the point A and the normal at Q meets the y-axis at the point B. Taking $q=\frac{\pi}{2}-p$,	
	(iii) Show that the area of triangle ARB is $\frac{1}{2}\operatorname{cosec}(2p)$.	[3]
	(iv) Find the Cartesian equation of the locus of point R.	[3]
	Solution	
	(i) $x=\sin t$ $\qquad$ $y=\cos t$ $\frac{dx}{dt}=\cos t$ $\qquad$ $\frac{dy}{dt}=-\sin t$ $\frac{dy}{dx}=\frac{-\sin t}{\cos t}$ $=-\tan t$ Equation of tangent: $y-\cos t=(-\tan t)(x-\sin t)$ $y=(-\tan t)x+(\tan t)(\sin t)+\cos t$ $y=(-\tan t)x+\frac{\sin^2 t}{\cos t}+\frac{\cos^2 t}{\cos t}$ $y=(-\tan t)x+\sec t$ Equation of normal: $y-\cos t=(\cot t)(x-\sin t)$ $y=(\cot t)x-\cos t+\cos t$ $y=(\cot t)x$	
	(ii) Equation of tangent at P (with parameter p): $y=(-\tan p)x+\sec p$ Equation of normal at Q (with parameter q):	

$y = (\cot q)x$

Equating the equation of tangent at P and the equation of normal at Q, we have

$(\cot q)x = (-\tan p)x + \sec p$

$(\cot q + \tan p)x = \sec p$

$\left(\dfrac{\cos q}{\sin q} + \dfrac{\sin p}{\cos p}\right)x = \dfrac{1}{\cos p}$

$\left(\dfrac{\cos p \cos q + \sin p \sin q}{\cos p \sin q}\right)x = \dfrac{1}{\cos p}$

$\left(\dfrac{\cos(p-q)}{\cos p \sin q}\right)x = \dfrac{1}{\cos p}$

$x = \dfrac{\sin q}{\cos(p-q)}$ (Shown)

Substitute $x = \dfrac{\sin q}{\cos(p-q)}$ into the equation of normal at Q,

$y = (\cot q)\left(\dfrac{\sin q}{\cos(p-q)}\right)$

$y = \dfrac{\cos q}{\cos(p-q)}$

(iii)

Coordinates of A:

When $x = 0,\ y = \sec p$

A is $(0, \sec p)$ or $(0, \sin p \tan p + \cos p)$

Coordinates of B:

When $x = 0,\ y = 0$

B is $(0,0)$.

Since R is $\left(\dfrac{\sin q}{\cos(p-q)}, \dfrac{\cos q}{\cos(p-q)}\right)$,

Area of Triangle ARB

$= \dfrac{1}{2}(\sec p)\left(\dfrac{\sin q}{\cos(p-q)}\right)$

$= \dfrac{1}{2}\left(\dfrac{1}{\cos p}\right)\left(\dfrac{1}{2\sin p}\right)$

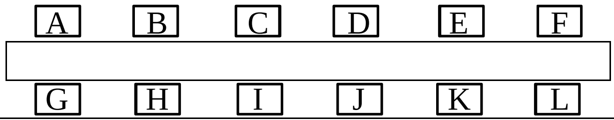
$= \dfrac{1}{2}\left(\dfrac{1}{2\sin p \cos p}\right)$

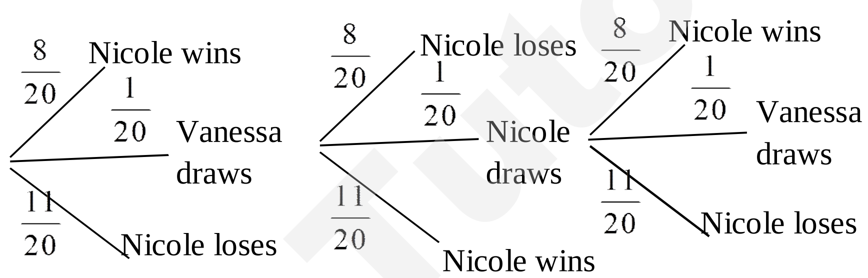
$= \dfrac{1}{2}\operatorname{cosec}(2p)$ (Shown)

(iv)

When $q=\frac{\pi}{2}-p$, $$x=\frac{\sin\left(\frac{\pi}{2}-p\right)}{\cos\left(2p-\frac{\pi}{2}\right)}=\frac{\cos p}{\sin 2p}=\frac{\cos p}{2\sin p\cos p}=\frac{1}{2\sin p}$$	**Alternatively,** Since $p=\frac{\pi}{2}-q$, $$x=\frac{\sin q}{\cos\left(\frac{\pi}{2}-2q\right)}=\frac{\sin q}{\sin 2q}=\frac{\sin q}{2\sin q\cos q}=\frac{1}{2\cos q}$$
$$y=\frac{\cos\left(\frac{\pi}{2}-p\right)}{\cos\left(2p-\frac{\pi}{2}\right)}=\frac{\sin p}{\sin 2p}=\frac{\sin p}{2\sin p\cos p}=\frac{1}{2\cos p}$$	**Alternatively,** $$y=\frac{\cos q}{\cos\left(\frac{\pi}{2}-2q\right)}=\frac{\cos q}{\sin 2q}=\frac{\cos q}{2\sin q\cos q}=\frac{1}{2\sin q}$$
Using $\sin^2 p+\cos^2 p=1$, $$\left(\frac{1}{2x}\right)^2+\left(\frac{1}{2y}\right)^2=1$$ $$\frac{1}{4x^2}+\frac{1}{4y^2}=1,\quad x>0, y>0$$	**Alternatively,** Using $\sin^2 q+\cos^2 q=1$, $$\left(\frac{1}{2y}\right)^2+\left(\frac{1}{2x}\right)^2=1$$ $$\frac{1}{4x^2}+\frac{1}{4y^2}=1,\quad x>0, y>0$$

Section B: Statistics [60 marks]

5	Nicole decides to celebrate her birthday with 9 boys and 2 girls whose names are Vanessa and Sally.	
	(a) (i) They have a dinner at a restaurant that can only offer them a rectangular table as shown in the following diagram, with seats labelled A to L as shown. 	
	Find the number of ways in which at least one girl must be seated at the seats A, F, G and L.	[2]
	(ii) Find the number of ways in which they can sit if instead, the restaurant offers them 2 indistinguishable round tables of 6.	[2]
	(iii) After the dinner, they went for a movie together. They bought tickets for seats in a row. Find the number of ways where Nicole and Vanessa must be seated together but not Sally.	[2]
	(b) After the celebration, Nicole plays a card game with Vanessa. The pack of 20 cards are numbered 1 to 20. The two friends take turns to draw a card from the pack. If a prime number is drawn, the player wins the game. If a composite number (4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20) is drawn, the player loses the game and the other player wins. If the number '1' is drawn, the card is returned and the other player draws the next card. Nicole draws the first card. Find the probability of her winning the game.	[3]
	Solution	
	(i) Number of ways $=12!-\binom{9}{4}4!8!$ $-\binom{9}{4}4!$ (for the selection and the arrangement of the 4 guys to be seated at A, F, G and L.)	
	$=357073920$	
	OR Number of ways $=12!-\binom{9}{5}8!4!$ $-\binom{9}{5}8!$ (for the selection of the 5 guys that is to be seated at B, C, D, E, H, I, J, K with the 3 girls)	
	$=357073920$	
	OR Number of ways $=12!-\binom{8}{3}9!3!$ $-\binom{8}{3}3!$ (for the selection of the 3 seat in the slot B, C, D, E, H, I, J, K)	
	$=357073920$	

	(ii) Number of ways $=\dfrac{\binom{12}{6}5!5!}{2!}$ – For $\binom{12}{6}5!5!$	
	$= 6652800$	
	(iii) Number of ways $=9!\binom{10}{2}2!2!$ **Alternative solution** $10!\times 9\times 2 = 65318400$ – For $\binom{10}{2}2!$ (the selection of the slots to separate Nicole and Vanessa with Sally)	
	$= 65318400$	
	(b)  	
	Probability of Nicole winning $=\frac{8}{20}\left[1+\left(\frac{1}{20}\right)^2+\left(\frac{1}{20}\right)^4+......\right]+\frac{11}{20}\left[\frac{1}{20}+\left(\frac{1}{20}\right)^3+\left(\frac{1}{20}\right)^5+...\right]$ First – 1st infinite series, Second – 2nd infinite series	
	$=\frac{8}{20}\cdot\frac{1}{1-\left(\frac{1}{20}\right)^2}+\frac{11}{20}\cdot\frac{\frac{1}{20}}{1-\left(\frac{1}{20}\right)^2}=\frac{3}{7}$	
6	In a telephone enquiry service, 92% of calls to it are successfully connected. The probability of any call being successfully connected is constant. A random sample of 60 calls is taken each day.	
	(i) State, in context, an assumption needed for it to be well modelled by a binomial distribution.	[1]
	(ii) On a given day, it is found that at most 55 calls went through successfully. Find the probability that there are at least 50 successful calls in the sample of 60.	[2]
	(iii) Estimate the probability that the number of successful calls on any day is less than 55 in a sample of 60.	[4]
	(iv) The number of successful calls is recorded daily for 70 consecutive days. Find the approximate probability that the average number of successful calls in a day is not more than 55.	[2]

Solution

(i) The event that a call is successfully connected is independent from the event of other calls being successfully connected.

(ii) Let X be the random variable denoting the number of successful calls, out of a sample of 60 calls.

$X \sim \mathrm{B}(60, 0.92)$

$\mathrm{P}(X \geq 50 \mid X \leq 55)$

$= \dfrac{\mathrm{P}(50 \leq X \leq 55)}{\mathrm{P}(X \leq 55)}$

$= \dfrac{\mathrm{P}(X \leq 55) - \mathrm{P}(X \leq 49)}{\mathrm{P}(X \leq 55)}$

$= \dfrac{0.52982 - 0.074926}{0.52982}$

= 0.986 (3s.f.)

(iii) Let Y be the random variable denoting the number of calls that are not successfully connected, out of a sample of 60 calls.

$Y \sim \mathrm{B}(60, 0.08)$

Since n = 60 is large, $np = 60(0.08) = 4.8 (< 5)$

$\therefore Y \sim \mathrm{Po}(4.8)$ approx

Number of successful calls	Number of calls that are not successfully connected
54	6
53	7
52	8

$\mathrm{P}(\text{less than 55 successful calls})$

$= \mathrm{P}(\text{at least 6 calls that are not successfully connected})$

$= \mathrm{P}(Y \geq 6)$

$= 1 - \mathrm{P}(Y \leq 5)$

= 0.349 (3s.f.)

(iv) $X \sim \mathrm{B}(60, 0.92)$

Since n = 70 is large, by Central Limit Theorem,

$\bar{X} \sim \mathrm{N}\left(55.2, \dfrac{4.416}{70}\right)$ approx

$\mathrm{P}(\bar{X} \leq 55) = 0.213 (\text{to 3 s.f})$

Alternative solution:

Let W be the random variable denoting the number of successful calls out of a sample of 4200.

$W \sim \mathrm{B}(4200, 0.92)$

Since n is large, np = 3864>5, nq = 336>5

$W \sim \mathrm{N}(3864, 309.12)$ approx

$\mathrm{P}(W \leq 55 \times 70) \xrightarrow{CC} \mathrm{P}(W < 3850.5) = 0.221 (\text{to 3 s.f})$

7	**(a)** Tickets are sold for the closing ceremony of an international swimming competition. It is desired to sample 1% of the spectators to find their opinions of the goodie bags received during the closing ceremony.	
	(i) Give a reason why it would be difficult to use a stratified sample.	[1]
	(ii) Explain how a systematic sample could be carried out.	[2]
	(b) The random variable X has the distribution $\mathrm{N}(18, 3^2)$ and the random variable Y has the distribution $\mathrm{N}(\mu, \sigma^2)$. The random variable T is related to X and Y by the formula $T = \dfrac{X_1 + X_2 + 3Y}{4}$, where X_1 and X_2 are two independent observations of X. Given that $\mathrm{P}(T < 10) = \mathrm{P}(T > 30) = 0.0668$, find the value of σ and the exact value of μ.	[5]
	(c) A survey done on students in a particular college found that the amount of time a student spends on social media in a week is normally distributed with mean 7 hours and variance 4 hours^2. Five students are randomly chosen. Find the probability that the fifth student is the second student who spends more than 10 hours a week on social media.	[2]
	Solution	
	(ai) Though the tickets issued might have a serial number indicated, but some people who have purchased the tickets, may not turn up for the closing ceremony and so it is difficult to obtain the actual sampling frame.	
	(ii) To have a sample consisting of 1% of the spectators present, the sampling interval will be 100. Randomly select a number between 1 to 100 say r. So at the entrance point, every rth person for each interval of 100 will be selected for the survey until the sample is obtained.	
	(b) $\mathrm{E}(T) = 20$	
	$\frac{1}{4}\left(2(18) + 3\mu\right) = 20$	
	$\mu = \frac{44}{3}$	
	$\mathrm{Var}(T) = \mathrm{Var}\left(\dfrac{X_1 + X_2 + 3Y}{4}\right)$	
	$= \frac{1}{4^2}\left(2\mathrm{Var}(X) + 3^2\mathrm{Var}(Y)\right)$	
	$= \frac{1}{4^2}\left(2(3^2) + 9\sigma^2\right) = \frac{9}{8} + \frac{9}{16}\sigma^2$	
	$\mathrm{P}(T < 10) = 0.0668$	
	$\mathrm{P}\left(Z < \dfrac{10-20}{\sqrt{\frac{9}{8} + \frac{9}{16}\sigma^2}}\right) = 0.0668$	

	$\dfrac{10-20}{\sqrt{\dfrac{9}{8}+\dfrac{9}{16}\sigma^2}}=-1.500$	
	$\left(\dfrac{10-20}{-1.500}\right)^2=\dfrac{9}{8}+\dfrac{9}{16}\sigma^2$	
	$\sigma=8.77533=8.78\,(3\text{sf})$	
	(c) Let X be the random variable "time taken by a randomly chosen student on social media".	
	$X \sim \mathrm{N}(7,2^2)$	
	Required probability $=4\left[\mathrm{P}\left(X>10\right)\right]^2\left[\mathrm{P}\left(X\le 10\right)\right]^3$	
	$=0.014508$	
	$=0.0145\,(3\text{ s.f})$	
8	An advertising display contains a large number of light bulbs which are continually being switched on and off every day in a week. The light bulbs fail independently at random times. Each day the display is inspected and any failed light bulbs are replaced. The number of light bulbs that fail in any one-day period has a Poisson distribution with mean 1.6.	
	(i) State, in the context of the question, one assumption that needs to be made for the number of light bulbs that fail per day to be well modelled by a Poisson distribution.	[1]
	(ii) Estimate the probability that there are fewer than 17 light bulbs that needs to be replaced in a period of 20 days.	[2]
	(iii) Using a suitable approximation, find the probability that there will be not fewer than 20 days with more than two light bulbs that will need to be replaced per day in a period of 8 weeks.	[4]
	(iv) The probability of at least three light bulbs having to be replaced over a period of n consecutive days exceeds 0.999. Write down an inequality in terms of n to express this information, and hence find the least value of n.	[4]
	Solution	
	(i) The average number of light bulbs that fail in a given time interval is proportional to the length of the time interval.	
	(ii) Let V be the random variable denoting "the number of light bulbs that needs to be replaced in 20 days." $V \sim \mathrm{Po}(20\times 1.6)$ $V \sim \mathrm{Po}(32)$ Since $\lambda = 32 > 10$, $V \sim \mathrm{N}(32, 32)$ approximately	

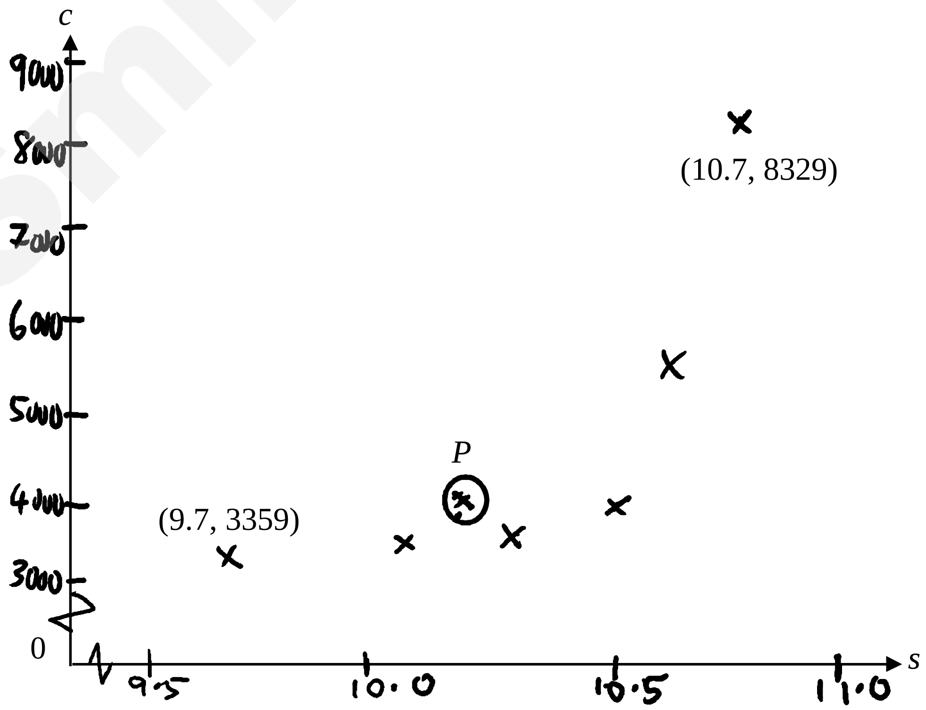
	$P(V < 17) \xrightarrow{\text{c.c}} P(V < 16.5)$ $= 0.003071651$ $= 0.00307$	
	(iii) $P(X > 2) = 1 - P(X \leq 2) = 0.21664$ Let Y be the random variable denoting "the number of days in which at least three light bulbs will need to be replaced out of 56 days" $Y \sim B(56, 0.21664)$ Since $n = 56$ is large, $np = 12.132$ (>5), $nq = 43.868$ $Y \sim N(12.132, 9.5036)$ approximately $P(Y \geq 20) \xrightarrow{\text{c.c}} P(Y > 19.5)$ $= 0.0084228$ $= 0.00842$	
	(iv) Let W be the random variable denoting "the number of light bulbs that need to be replaced in n consecutive days." $W \sim Po(1.6n)$ $P(W \geq 3) > 0.999$ $1 - P(W \leq 2) > 0.999$ $1 - 0.999 > P(W \leq 2)$ $0.001 > \frac{e^{-(1.6n)}(1.6n)^0}{0!} + \frac{e^{-(1.6n)}(1.6n)^1}{1!} + \frac{e^{-(1.6n)}(1.6n)^2}{2!}$ $e^{-(1.6n)} + e^{-(1.6n)}(1.6n) + e^{-(1.6n)}(1.28n^2) < 0.001$ $n = 7$, $P(W \leq 2) = 0.00102$ $n = 8$, $P(W \leq 2) = 0.000264$ $n = 9$, $P(W \leq 2) = 0.0000664$ Least value of n is 8.	
9	**(a)** Observations of 10 pairs of values (x,y) are shown in the table below.	

x	1	2	3	4	5	6	7	8	9	10
y	0.5	0.6	0.8	0.95	a	1.21	1.36	1.55	1.87	2.11

It is known that the equation of the linear regression line of y on x is $y = 0.17321x + 0.24133$. Find a, correct to 2 decimal places. [2]

(b) A student wanted to study the relationship between the number of commercial crimes (c) and the mean years of schooling (s) of the offenders. The following set of data was obtained.

Year	2009	2010	2011	2012	2013	2014	2015
Mean years of schooling (s)	9.7	10.1	10.2	10.3	10.5	10.6	10.7
No. of commercial crimes (c)	3359	3504	4080	3507	3947	5687	8329

	(i) Draw a scatter diagram for these values.	[2]
	(ii) One of the values of c appears to be incorrect. Circle this point on your diagram and label it P.	[1]
	It is thought that the number of commercial crimes (c) can be modelled by one of the formulae after removing the point P. (A) $c = a + b\left(100^s\right)$ (B) $c = a + bs$ (C) $c = a + b\ln s$ where a and b are non-zero constants.	
	(iii) With reference to the scatter diagram, explain clearly which model is the best model for this set of data. For the case identified, find the equation of a suitable regression line.	[2]
	(iv) Using the regression line found in **(iii)**, estimate the number of commercial crimes (to the nearest whole number) when the mean years of schooling reaches 11.	[2]
	(v) Comment on the reliability of your answer in part **(iv)**.	[1]
	Solution	
	(a) Using GC, $\overline{x} = 5.5$	
	$\overline{y} = \dfrac{10.95 + a}{10}$	
	Since $\left(\overline{x}, \overline{y}\right)$ lies on the regression line,	
	$\dfrac{10.95 + a}{10} = 0.17321(5.5) + 0.24133$	
	$a = 0.98985 \approx 0.99$ (correct to 2 decimal places)	
	(b) (i) and (ii)  	

	(iii) From the scatter diagram (after removing the outlier), as s increases, c increases at an increasing rate. Hence model (A) is the best model.	
	From GC, $c = 2862.048513 + \left(1.965434 \times 10^{-18}\right)\left(100^{s}\right)$	
	(iv) When s = 11,	
	$c = 2862.048513 + \left(1.965434 \times 10^{-18}\right)\left(100^{11}\right)$ ≈ 22516	
	(v) The estimate is unreliable because the data substituted is outside the data range and so the linear relationship between c and 100^{s} may not hold.	
10	In the latest Pokkinon Roll game, players go to a battle arena to use their Pokkinon character to battle against each other. Alvin and Billy are interested to know how long it takes before someone wins a battle. The time taken by a randomly chosen player to win a game follows a normal distribution.	
	(a) Alvin claims that on average, it will take at most 190.0 seconds to win a battle. To verify his belief, he surveyed a randomly chosen sample of 45 Pokkinon Roll gamers and found out that the mean is 195.0 seconds with a variance of 206.0 seconds^2.	
	Carry out an appropriate test at 1% level of significance whether there is any evidence to doubt Alvin's claim. State an assumption needed for the calculation.	[5]
	(b) Billy also obtained his own data by recording the time taken, in seconds, by 5 randomly chosen gamers as shown below.	
	188.0 190.0 k 186.0 187.0	
	However, he believes that it will take 190.0 seconds on average to win a battle. When he conducted the test at 4.742% level of significance, his conclusion is one where the null hypothesis is not rejected. The sample mean time taken is denoted by $\bar{x}$.	
	Given that s^2 is the unbiased estimate of the population variance and that the maximum range of values of $\bar{x}$ is $188 \le \bar{x} \le a$, write down an equation involving s and a.	[1]
	Hence or otherwise find the values of a and k, leaving your answers to the nearest integer.	[5]
	Solution	
	(a) Assume that the time taken to win any battle is independent of other battles.	
	Let Y denote the time taken to win a randomly chosen battle	
	$s^2 = \frac{45}{44}(206) = 210.68$	
	$H_0 : \mu = 190$ $H_1 : \mu > 190$	
	Under H_0 $\bar{Y} \sim N\left(190, \frac{210.68}{45}\right)$	
	$\mu = 190$, $\bar{y} = 195$, n = 45, s = $\sqrt{210.68}$	
	Using G.C, p-value is 0.0104	
	Since p value > 0.01, we do not reject H_0 and conclude that there is insufficient evidence to doubt Alvin's claim at 1% level of significance.	

	(b) $H_0 : \mu = 190$ $H_1 : \mu \neq 190$	
	2-tailed T-test at 4.742% level of significance	
	$T \sim t(4)$	
	$a = \dfrac{2.82844s}{\sqrt{5}} + 190$	
	$a = 1.26s + 190$	
	$188 \leq \bar{x} \leq a$	
	$\dfrac{-2\sqrt{5}}{s} \leq \dfrac{\bar{x} - 190}{\frac{s}{\sqrt{5}}} \leq \dfrac{\sqrt{5}(a-190)}{s}$	
	Since $\dfrac{-2\sqrt{5}}{s} = -2.82844$	
	$s = 1.5811$	
	$s^2 = 2.5$	
	So $a = \dfrac{2.82844(1.5811)}{\sqrt{5}} + 190 = 192$	
	OR by symmetry of curve, a = 192	
	$\sum x = 751 + k$	
	$\sum x^2 = 141009 + k^2$	
	$s^2 = \dfrac{1}{4}\left[141009 + k^2 - \dfrac{(751+k)^2}{5}\right]$	
	So $2.5 = \dfrac{1}{4}\left[141009 + k^2 - \dfrac{(751+k)^2}{5}\right]$	
	k =189 or k = 186.5 (rejected since $188 \leq \bar{x} \leq a$)	

THE END

TEMASEK JUNIOR COLLEGE, SINGAPORE

Preliminary Examination 2016
Higher 2

MATHEMATICS **9740/01**

Paper 1

30 August 2016

Additional Materials: Answer paper
List of Formulae (MF15)

3 hours

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **6** printed pages.

TEMASEK JUNIOR COLLEGE, SINGAPORE
PASSION PURPOSE DRIVE

[Turn over

1 A fitness assessment walk is conducted where participants walk briskly around a running path. The participants' walking time and heart rate are recorded at the end of the walk.

The formula for calculating the Fitness Index of a participant is as follows:

420 + (Age × 0.2) – (Walking Time × a) – (Body Mass Index × b) – (Heart Rate × c)

where a, b and c are real constants.

Data from 3 participants, Anand, Beng and Charlie are given in the table.

Name	Age	Walking Time	Body Mass Index	Heart Rate	Fitness Index
Anand	32	17.5	25	100	102.4
Beng	19	18.5	19	120	92.6
Charlie	43	17	23	90	121.2

Find the values of a, b and c. [3]

2 Solve the inequality $\dfrac{x^2-2a^2}{x}<a$, giving your answer in terms of a, where a is a positive real constant. [3]

Hence solve $\dfrac{x^2-2a^2}{|x|}<a$. [2]

3 **(i)** Use the substitution $u=x^2$ to find $\displaystyle\int \frac{x}{\sqrt{k^2-x^2}}\,\mathrm{d}x$ in terms of x and the constant k. [3]

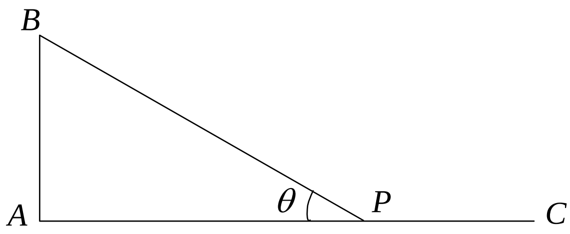
(ii) Find the exact value of $\displaystyle\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$, where

$$\mathrm{f}(x)=\begin{cases}\dfrac{2}{6-x^2}, & 0\le x<\sqrt{2},\\[2ex] \dfrac{x}{\sqrt{6-x^2}}, & \sqrt{2}\le x<2.\end{cases}$$ [3]

4 Relative to the origin O, the points A, B, M and N have position vectors **a**, **b**, **m** and **n** respectively, where **a** and **b** are non-parallel vectors. It is given that $\mathbf{m}=\lambda\mathbf{a}+(1-\lambda)\mathbf{b}$ and $\mathbf{n}=2(1-\lambda)\mathbf{a}-\lambda\mathbf{b}$ where λ is a real parameter.

Show that $\mathbf{m}\times\mathbf{n}=(3\lambda^2-4\lambda+2)\,(\mathbf{b}\times\mathbf{a})$. [2]

It is given that $|\mathbf{a}|=3$, $|\mathbf{b}|=4$ and the angle between **a** and **b** is $\dfrac{\pi}{6}$. Find the smallest area of the triangle MON as λ varies. [4]

5 

In the diagram, A and C are fixed points 500 m apart on horizontal ground. Initially, a drone is at point A and an observer is standing at point C. The drone starts to ascend vertically at a steady rate of $3\,\mathrm{m\,s^{-1}}$ as the observer starts to walk towards A with a steady speed of $4\,\mathrm{m\,s^{-1}}$. At time t, the drone is at point B and the observer is at point P.

Given that the angle APB is θ radians, show that $\theta = \tan^{-1}\left(\dfrac{3t}{500-4t}\right)$. [2]

(i) Find $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ in terms of t. [2]

(ii) Using differentiation, find the time t when the rate of change of θ is maximum. [4]

6 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \ln\left(x^2 - x + 1\right), \quad x \le 1,$$

$$\mathrm{g}: x \mapsto \mathrm{e}^x, \text{ for all real } x.$$

Sketch the graph of f and explain why f does not have an inverse. [2]

The function h is defined by

$$\mathrm{h}: x \mapsto \mathrm{f}(x), \ x \le k.$$

State the maximum value of k such that h^{-1} exists. [1]

Using this maximum value of k,

(i) show that the composite function gh exists, [1]

(ii) find $(\mathrm{gh})^{-1}(x)$, stating the domain of $(\mathrm{gh})^{-1}$. [4]

7 **(a)** The positive integers are grouped into sets as shown below, so that the number of integers in each set after the first set is three more than that in the previous set.

$$\{1\}, \{2, 3, 4, 5\}, \{6, 7, 8, 9, 10, 11, 12\}, \ldots$$

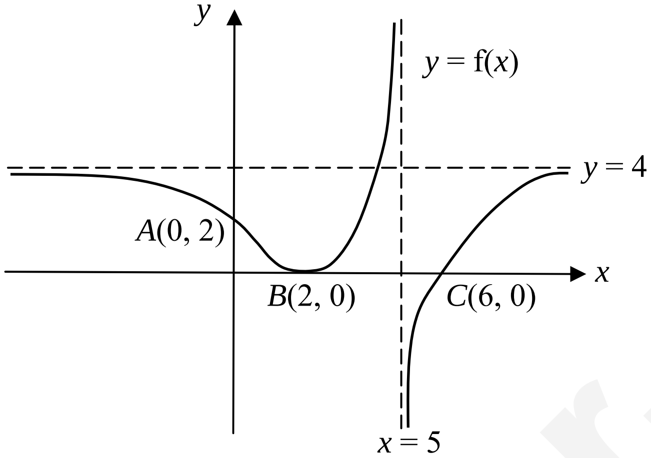
Find, in terms of r, the number of integers in the rth set. [1]

Show that the last integer in the rth set is $\dfrac{r}{2}(3r-1)$. [2]

Deduce, in terms of r, the first integer in the rth set. [2]

(b) Find $\displaystyle\sum_{r=1}^{n}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^r\right)$ in terms of n. [4]

8 The graph of $y = \mathrm{f}(x)$ intersects the axes at $A(0, 2)$, $B(2, 0)$ and $C(6, 0)$ as shown below. The lines $y = 4$ and $x = 5$ are asymptotes to the graph, and $B(2, 0)$ is a minimum point.



On separate diagrams, sketch the graphs of

(i) $y = \mathrm{f}(|x|)$, [2]

(ii) $y^2 = \mathrm{f}(x)$, [3]

(iii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

stating the equations of any asymptotes, coordinates of any stationary points and points of intersection with the axes.

9 **(a)** Given that x is small such that x^3 and higher powers of x can be neglected, show that

$$\frac{\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)}{\sqrt{2-\cos x}} \approx a + bx + cx^2 ,$$

for constants a, b and c to be determined. [4]

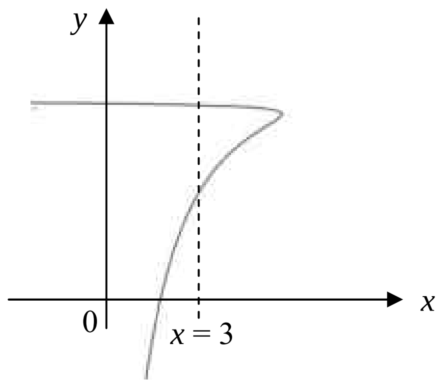
(b) The curve $y = \mathrm{f}(x)$ passes through the point $(0, -1)$ and satisfies the differential equation

$$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-y} .$$

(i) Find the Maclaurin series for y, up to and including the term in x^2. [3]

(ii) By using an appropriate expansion from the List of Formulae (MF15), obtain the Maclaurin series for $\ln(2+y)$, up to and including the term in x^2. [3]

10



The diagram shows the curve with parametric equations

$$x = 2t + t^2, \quad y = \frac{1}{(1-t)^2}, \quad \text{for } t < 1.$$

The curve has a vertical asymptote $x = 3$.

(i) Find the coordinates of the points where the curve cuts the y-axis. [2]

(ii) Find the equation of the tangent to the curve that is parallel to the y-axis. [4]

(iii) Express the area of the finite region bounded by the curve and the y-axis in the form $\int_a^b \mathrm{f}(t)\,\mathrm{d}t$, where a, b and f are to be determined. Use the substitution $u = 1 - t$ to find this area, leaving your answer in exact form. [5]

11 On the remote island of Squirro, ecologists introduced a non-native species of insects that can feed on weeds that are killing crops. Based on past studies, ecologists have observed that the birth rate of the insects is proportional to the number of insects, and the death rate is proportional to the square of the number of insects. Let x be the number of insects (in hundreds) on the island at time t months after the insects were first introduced.

Initially, 10 insects were released on the island. When the number of insects is 50, it is changing at a rate that is $\frac{3}{4}$ times of the rate when the number of insects is 100. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \beta x(2 - x)$$

where β is a positive real constant. [3]

Solve the differential equation and express x in the form $\frac{p}{1 + q\mathrm{e}^{-2\beta t}}$, where p and q are constants to be determined. [6]

Sketch the solution curve and state the number of insects on the island in the long run. [3]

[Turn over

12 **(a)** The complex numbers z_1 and z_2 satisfy the following simultaneous equations

$$2z_1 + \mathrm{i}\, z_2^* = 7 - 6\mathrm{i},$$

$$z_1 - \mathrm{i}\, z_2 = 6 - 6\mathrm{i}.$$

Find z_1 and z_2 in the form $x + y\mathrm{i}$, where x and y are real. [4]

(b) It is given that $w = \frac{1}{2} - \frac{1}{2}\mathrm{i}$. Find the modulus and argument of w, leaving your answers in exact form. [2]

It is also given that the modulus and argument of another complex number v is 2 and $\frac{\pi}{6}$ respectively.

(i) Find the exact values of the modulus and argument of $\frac{v}{w^*}$. [3]

(ii) By first expressing v in the form $\sqrt{c} + d\,\mathrm{i}$ where c and d are integers, find the real and imaginary parts of $\frac{v}{w^*}$ in surd form. [3]

(iii) Deduce that $\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$. [2]

End of Paper

H

TEMASEK JUNIOR COLLEGE, SINGAPORE

Preliminary Examination 2016
Higher 2

MATHEMATICS

9740/02

Paper 2

14 September 2016

Additional Materials: Answer Paper
List of Formulae (MF15)
Graph Paper

3 hours

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **6** printed pages.

Section A: Pure Mathematics [40 marks]

1 A sequence u_1, u_2, u_3, … is such that $u_1 = 2$ and

$$u_n = u_{n-1} - \frac{1}{n(n-1)}, \quad \text{for all integers } n \geq 2.$$

(i) By expressing $\frac{1}{r(r-1)}$ in partial fractions or otherwise, find $\sum_{r=2}^{n} \frac{1}{r(r-1)}$ in terms of n. [4]

(ii) By considering $\sum_{r=2}^{n} (u_r - u_{r-1})$ and using the result in part **(i)**, show that for all integers $n \geq 2$, u_n can be expressed in the form $a + \frac{b}{n}$, where a and b are constants to be determined. [3]

2 Solve the equation $w^4 + 2 - 2\sqrt{3}\,\mathrm{i} = 0$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leq \pi$ and $r > 0$. [4]

The roots represented by w_1 and w_2 are such that $-\frac{\pi}{2} < \arg(w_1) < 0$ and $\frac{\pi}{2} < \arg(w_2) < \pi$.

The complex number z satisfies the relations $|z - w_1| \geq |z - w_2|$ and $-\frac{\pi}{4} \leq \arg[z - (-1 + \mathrm{i})] \leq 0$.

On an Argand diagram, sketch the region R in which the points representing z can lie. [3]

Find the exact area of R. [3]

3 The line l has equation $\frac{x+1}{-1} = \frac{z+6}{2}$, $y = 4$ and the point A has coordinates $(-1, 3, -5)$.

(i) Find the position vector of the foot of the perpendicular from A to l. [3]

(ii) Plane p_1 contains l and A. Show that the equation of p_1 is $2x + y + z = -4$. [3]

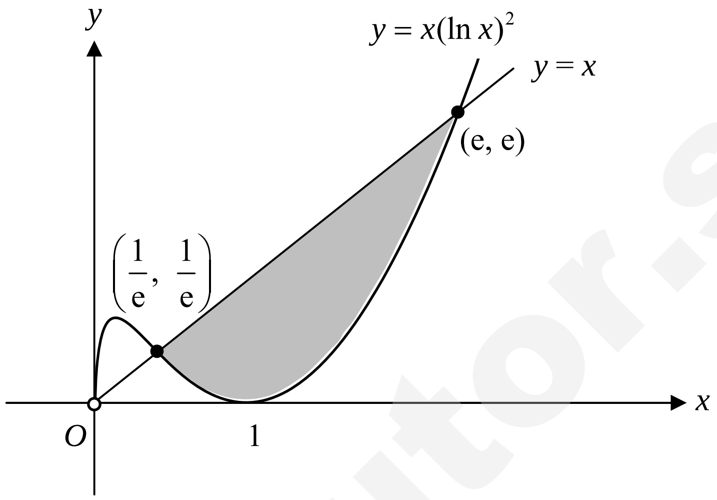
Given that the plane p_2 has equation $x + 2y + cz = -5$ where c is a negative constant, and that the acute angle between p_1 and p_2 is 60°, find the value of c. [3]

(iii) Find the equation of the line of intersection, m, between p_1 and p_2. [1]

(iv) The plane p_3 has equation $3x + \alpha y + 7z = \beta$ where α and β are constants. Given that the planes p_1, p_2 and p_3 have no common point, what can be said about the values of α and β? [2]

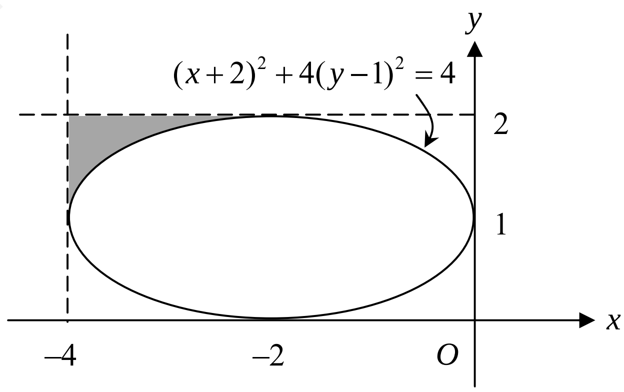
4 **(a)** The diagram below shows the graphs of $y = x(\ln x)^2$ and $y = x$. The two graphs intersect at the points $\left(\frac{1}{\mathrm{e}}, \frac{1}{\mathrm{e}}\right)$ and (e, e).

Find the exact area of the shaded region bounded by the graphs of $y = x(\ln x)^2$ and $y = x$. [5]



Hence, without integrating, find the exact area of the region bounded by the graphs of $y = x(\ln x)^2$ and the lines $y = \mathrm{e}$ and $x = \frac{1}{\mathrm{e}}$. [2]

(b) Find the volume of the solid formed when the shaded region bounded by the lines $x = -4$, $y = 2$ and the ellipse $(x+2)^2 + 4(y-1)^2 = 4$ is rotated through 2π radians about the y-axis. Give your answer correct to 1 decimal place. [4]



[Turn over

Section B: Statistics [60 marks]

5 An apartment block has 24 two-bedroom apartments and 88 four-bedroom apartments. A surveyor wishes to conduct interviews on 35 households in this block to learn about their household expenditure. She decides to use stratified sampling across apartment types in the block, assuming that only one household occupies each apartment.

(i) Give an advantage of this sampling method. [1]

(ii) Describe how a stratified sample can be obtained. [3]

6 A mathematician arranges all eight letters in the word PARALLEL to form different 8-letter code words. Find the number of different code words that can be formed if

(i) the code words start with an L and end with an A, [1]

(ii) the 2 A's are not adjacent to each other, [3]

(iii) there is exactly one letter between the first and second L, and exactly one letter between the second and third L. [3]

7 Anand, Beng and Charlie are selling cupcakes to raise funds for the charity, Boys And Girls Understand Singapore (BAGUS). Anand will bake 60% of the cupcakes, Beng will bake 40% of the cupcakes and Charlie will spread frosting on all the cupcakes.

20% of Anand's cupcakes will turn out flawed, while 12% of Beng's cupcakes will turn out flawed. Charlie has a 6% chance of spreading the frosting badly on any cupcake, regardless of whether it is flawed or otherwise.

If a cupcake is flawed or has frosting spread badly (or both), it is considered sub-standard. Otherwise, a cupcake is considered "good".

Find the probability that

(i) a randomly chosen cupcake is "good", [2]

(ii) a randomly chosen cupcake is either baked by Anand or is sub-standard but not both, [3]

(iii) a randomly chosen cupcake is baked by Anand given that it is sub-standard. [3]

8 The rate of growth, x units per hour, of a particular family of bacteria is believed to depend in some way on the controlled temperature T °C. Experiments were undertaken in the laboratory to investigate this and the results were tabulated as follows:

T	10	20	30	40	50	60	70	80
x	33	37	41	48	55	65	78	94

Draw a scatter diagram for these values, labelling the axes clearly. [1]

State, with a reason, which of the following model is most appropriate for the given data,

(A) $x = a + bT$ **(B)** $x = ae^{bT}$ **(C)** $x = a + b\ln T$

where a and b are constants, and $b > 0$. [1]

In addition, when $T = 90$, the rate of growth of bacteria was k units per hour.

Use the most appropriate model identified for the subsequent parts of this question.

(i) A suitable linear regression line was constructed based on all 9 pairs of transformed data, including the additional pair of data. If the values of a and b were determined to be 27.06 and 0.01497 respectively, find the value of k, correct to the nearest whole number. [4]

(ii) Find the product moment correlation coefficient for all 9 pairs of transformed data. [1]

(iii) Use the regression line in **(i)** to predict the growth rate of the bacteria when the temperature is $105\ °\mathrm{C}$, giving your answer to the nearest whole number. Comment on the reliability of this prediction. [2]

9 Small defects in a twill weave and satin weave occur randomly and independently at a constant mean rate of 1.2 defects and 0.8 defects per square metre respectively.

(i) Find the probability that there are exactly 9 defects in 7 square metres of twill weave. [2]

(ii) A box is made from 2 square metres of twill weave and 3 square metres of satin weave, chosen independently. A box is considered "faulty" if there are more than 10 defects. Show that the probability that a randomly chosen box is "faulty" is 0.0104, correct to 3 significant figures. [2]

(iii) A random batch of 50 boxes is delivered to a customer once every week. The customer can reject the entire batch if there are at least 2 "faulty" boxes in the batch. Using a suitable approximation, find the probability that the customer will reject the entire batch in a randomly selected week. [3]

Hence estimate the probability that the customer will reject 2 batches in a randomly selected period of 4 weeks. [2]

10 Newmob is a mobile phone service provider which sells several brands of mobile phones. uPhones and Samseng phones are sold at a subsidy to its subscribers. Each subscriber can either buy one uPhone or one Samseng phone or both one uPhone and one Samseng phone. The probability that a randomly chosen subscriber buys a uPhone is 0.3, and independently, the probability that the subscriber buys a Samseng phone is p.

(i) In a random sample of 50 subscribers, the probability that at most 20 subscribers buy a Samseng phone is twice the probability that exactly 15 subscribers buy a uPhone. Find the value of p. [3]

For the remainder of this question, you may take the value of p to be 0.4.

(ii) In a random sample of 50 subscribers, find the probability that the number of subscribers who buy a uPhone is greater than the expected number of subscribers who buy a Samseng phone. [2]

(iii) Each subsidy for the uPhone costs Newmob \$280 and each subsidy for a Samseng phone costs \$200. Newmob has 1000 subscribers. Using suitable approximations, find the probability that the total subsidy given by Newmob for uPhone purchases exceeds the total subsidy given for Samseng phone purchases. [5]

11 The volume of a packet of soya bean milk is denoted by V ml and the population mean of V is denoted by μ ml. A random sample of 80 packets of soya bean milk is taken and the results are summarised by

$$\sum(v-250) = -217, \quad \sum(v-250)^2 = 30738.$$

Test, at the 4% significance level, whether μ is less than 250 ml. [5]

Explain, in the context of the question, the meaning of 'at the 4% significance level'. [1]

In another test, using the same data, and also at the 4% significance level, the hypotheses are as follows.

Null hypothesis: $\mu = \mu_0$

Alternative hypothesis: $\mu \neq \mu_0$

Given that the null hypothesis is rejected in favour of the alternative hypothesis, find the set of possible values of μ_0. [4]

It is now given that $\mu = 250$ and the population variance is 385. A random sample of 50 packets of soya bean milk is taken and the total volume of the packets is denoted by T ml. By considering the approximate distribution of T and assuming that the volumes of all packets of soya bean milk are independent of one another, find $P(T > 12600)$. [3]

End of Paper

1

$420+6.4-17.5a-25b-100c=102.4 \qquad 17.5a+25b+100c=324 \quad ----- \quad (1)$

$420+3.8-18.5a-19b-120c=92.6 \quad \text{or} \quad 18.5a+19b+120c=331.2 \quad ---- \quad (2)$

$420+8.6-17a-23b-90c=121.2 \qquad 17a+23b+90c=307.4 \quad ----- \quad (3)$

Using GC, $a=\frac{58}{5},\ b=\frac{13}{5}, c=\frac{14}{25}$

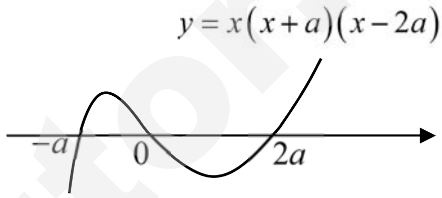
2

$\frac{x^2-2a^2}{x}<a\ , \quad x\neq 0$

$\frac{x^2-ax-2a^2}{x}<0$

$x(x+a)(x-2a)<0$

$x<-a \quad \text{or} \quad 0<x<2a$



Replace x by $|x|$,

$|x|<-a \qquad \text{or} \qquad 0<|x|<2a$

(no real solution) $\quad -2a<x<2a, \ \ x\neq 0$

3

(i) $\quad u=x^2 \ \Rightarrow\ \frac{\mathrm{d}u}{\mathrm{d}x}=2x \qquad \text{or} \qquad x=\sqrt{u} \ \Rightarrow\ \frac{\mathrm{d}x}{\mathrm{d}u}=\frac{1}{2\sqrt{u}}$

$$\int \frac{x}{\sqrt{k^2-x^2}}\,\mathrm{d}x=\frac{1}{2}\int\frac{1}{\sqrt{k^2-u}}\ \mathrm{d}u$$

$$=\frac{1}{2}\frac{\sqrt{k^2-u}}{\left(\frac{1}{2}\right)(-1)}+C$$

$$=-\sqrt{k^2-x^2}+C$$

(ii)

$$\int_0^2 \mathrm{f}(x)\ \mathrm{d}x=\int_0^{\sqrt{2}}\frac{2}{6-x^2}\ \mathrm{d}x+\int_{\sqrt{2}}^{2}\frac{x}{\sqrt{6-x^2}}\ \mathrm{d}x$$

$$=\left[\frac{2}{2\sqrt{6}}\ln\left(\frac{\sqrt{6}+x}{\sqrt{6}-x}\right)\right]_0^{\sqrt{2}}+\left[-\sqrt{6-x^2}\right]_{\sqrt{2}}^{2}$$

$$=\frac{1}{\sqrt{6}}\ln\left(\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}\right)-\sqrt{2}+2$$

4

$$\mathbf{m}\times\mathbf{n} = (\lambda\mathbf{a} + (1-\lambda)\mathbf{b}) \times (2(1-\lambda)\mathbf{a} - \lambda\mathbf{b})$$

$$= 2\lambda(1-\lambda)\,(\mathbf{a}\times\mathbf{a}) - \lambda^2\,(\mathbf{a}\times\mathbf{b}) \quad +2(1-\lambda)^2\,(\mathbf{b}\times\mathbf{a}) - \lambda(1-\lambda)\,(\mathbf{b}\times\mathbf{b})$$

$$= 2(1-\lambda)^2\,(\mathbf{b}\times\mathbf{a}) - \lambda^2\,(\mathbf{a}\times\mathbf{b}) \qquad \text{since } \mathbf{a}\times\mathbf{a} = \underset{\sim}{\mathbf{0}} = \mathbf{b}\times\mathbf{b}$$

$$= (2(1-\lambda)^2 + \lambda^2)\,(\mathbf{b}\times\mathbf{a}) \qquad \text{since } \mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$$

$$= (3\lambda^2 - 4\lambda + 2)\,(\mathbf{b}\times\mathbf{a})$$

Area of triangle MON $= \frac{1}{2}\left|\mathbf{m}\times\mathbf{n}\right| = \frac{1}{2}\left|(3\lambda^2 - 4\lambda + 2)(\mathbf{b}\times\mathbf{a})\right|$

$$= \frac{1}{2}\left|3\lambda^2 - 4\lambda + 2\right| \; \left|\mathbf{b}\times\mathbf{a}\right|$$

$$= \frac{1}{2}\left|3\left(\lambda - \frac{2}{3}\lambda\right)^2 + \frac{2}{3}\right| \; \left|\mathbf{b}\right|\left|\mathbf{a}\right|\sin\frac{\pi}{6}$$

$$= 3\left|3\left(\lambda - \frac{2}{3}\right)^2 + \frac{2}{3}\right|$$

$\therefore$ smallest area is $3\times\frac{2}{3} = 2$ units^2

<u>Alternative solution</u>

Using GC, the minimum value of $3\lambda^2 - 4\lambda + 2$ occurs when $\lambda = \frac{2}{3}$

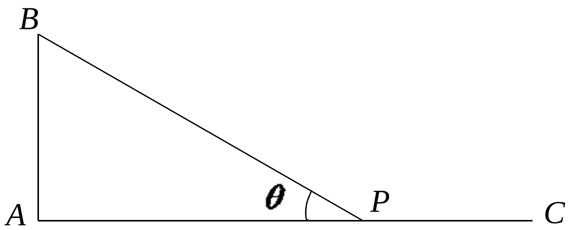
$\therefore$ smallest area $= \frac{1}{2}\left[3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 2\right]\left|\mathbf{b}\right|\left|\mathbf{a}\right|\sin\frac{\pi}{6} \qquad = \frac{1}{2}\left[\frac{2}{3}\right]\times 6 = 2$ units^2

5

At time t, $AB = 3t$, $AP = 500 - 4t$

$$\tan\theta = \frac{AB}{AP} = \frac{3t}{500-4t}$$

$$\theta = \tan^{-1}\left(\frac{3t}{500-4t}\right) \qquad \text{(shown)}$$



(i) $$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{1+\left(\frac{3t}{500-4t}\right)^2} \times \frac{(500-4t)(3) - 3t(-4)}{(500-4t)^2}$$

$$= \frac{(500-4t)^2}{(500-4t)^2 + (3t)^2} \times \frac{1500}{(500-4t)^2} = \frac{1500}{9t^2 + (500-4t)^2}$$

$$\left(= \frac{1500}{25t^2 - 4000t + 250000} = \frac{60}{t^2 - 160t + 10000} \right)$$

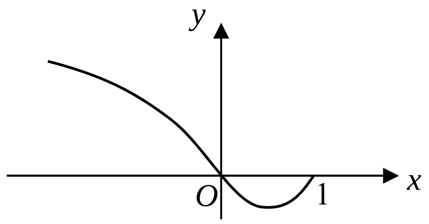
(ii) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\right)=\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=\dfrac{-1500\left(18t+2\left(500-4t\right)\left(-4\right)\right)}{\left(9t^2+\left(500-4t\right)^2\right)^2}=\dfrac{-1500\left(50t-4000\right)}{\left(9t^2+\left(500-4t\right)^2\right)^2}$

$\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=0 \Rightarrow -1500(50t-4000)=0 \Rightarrow t=80$

t	80^-	80	80^+
$\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$	+ve	0	−ve
slope	/	—	\

Using first derivative test, rate of change of θ is maximum at $t = 80$

6



The line $y = 0$ cuts the graph of f twice, thus f is not one-one and so f does not have an inverse.

Using GC, minimum value of f occurs when $x=\dfrac{1}{2}$

<u>OR</u> $x^2-x+1=\left(x-\dfrac{1}{2}\right)^2+\dfrac{3}{4} \Rightarrow$ minimum point: $\left(\dfrac{1}{2},\dfrac{3}{4}\right)$

Hence maximum value of k is $\dfrac{1}{2}$

(i) Since $R_{\mathrm{h}}=\left[\ln\dfrac{3}{4},\infty\right)\subseteq\left(-\infty,\infty\right)=D_{\mathrm{g}}$, the function gh exists.

(ii) $\mathrm{gh}\left(x\right)=\mathrm{g}\left(\ln\left(x^2-x+1\right)\right)=\mathrm{e}^{\ln(x^2-x+1)}=x^2-x+1,\quad x\le\dfrac{1}{2}$

Let $y=\mathrm{gh}\left(x\right)$

$y=x^2-x+1=\left(x-\dfrac{1}{2}\right)^2+\dfrac{3}{4}$

$\Rightarrow x=\dfrac{1}{2}-\sqrt{y-\dfrac{3}{4}} \quad \left(\text{reject } x=\dfrac{1}{2}+\sqrt{y-\dfrac{3}{4}} \quad \because x\le\dfrac{1}{2}\right)$

$\therefore \left(\mathrm{gh}\right)^{-1}\left(x\right)=\dfrac{1}{2}-\sqrt{x-\dfrac{3}{4}}$

$\mathrm{D}_{(\mathrm{gh})^{-1}}=\mathrm{R}_{\mathrm{gh}}=\left[\dfrac{3}{4},\ \infty\right)$

7

(a)

	{1},	{2, 3, 4, 5},	{6, 7, 8, 9, 10, 11, 12},	…
Set	1st	2nd	3rd	, …
No. of terms	1	4	7	, …

No. of integers in rth set $= 1 + (r-1)3$
$= 3r - 2$

Last integer in rth set $=$ Sum of no. of terms from 1st to rth set

$$= 1 + 4 + 7 + ... + (3r-2)$$

$$= \frac{r}{2}\left[2(1) + (r-1)(3)\right]$$

$$\text{or } \frac{r}{2}\left[1 + (3r-2)\right]$$

$$= \frac{r}{2}(3r-1)$$

Hence first integer in rth set $= \frac{r}{2}(3r-1) - (3r-2) + 1$

$$\text{or } \frac{1}{2}(r-1)\left[3(r-1)-1\right] + 1$$

$$= \frac{3r^2 - 7r + 6}{2}$$

(b)

$$\sum_{r=1}^{n}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + ... + \left(\frac{1}{2}\right)^r\right)$$

$$= \sum_{r=1}^{n}\left(\frac{1\left(1 - \left(\frac{1}{2}\right)^{r+1}\right)}{1 - \frac{1}{2}}\right)$$

$$= 2\sum_{r=1}^{n}\left(1 - \tfrac{1}{2}\left(\tfrac{1}{2}\right)^r\right)$$

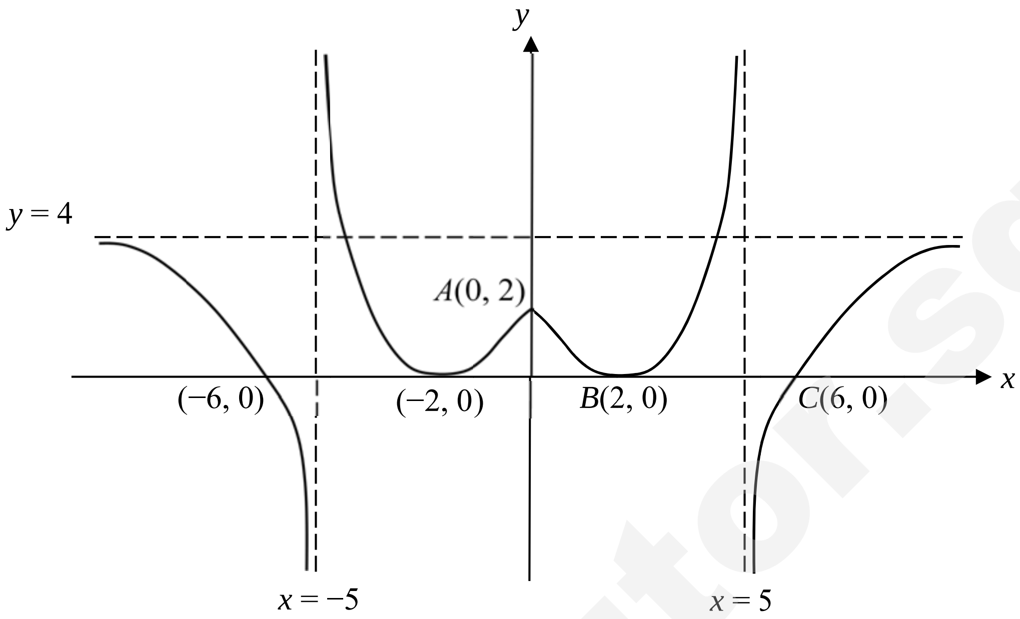
$$= 2n - \sum_{r=1}^{n}\left(\tfrac{1}{2}\right)^r$$

$$= 2n - \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}}$$

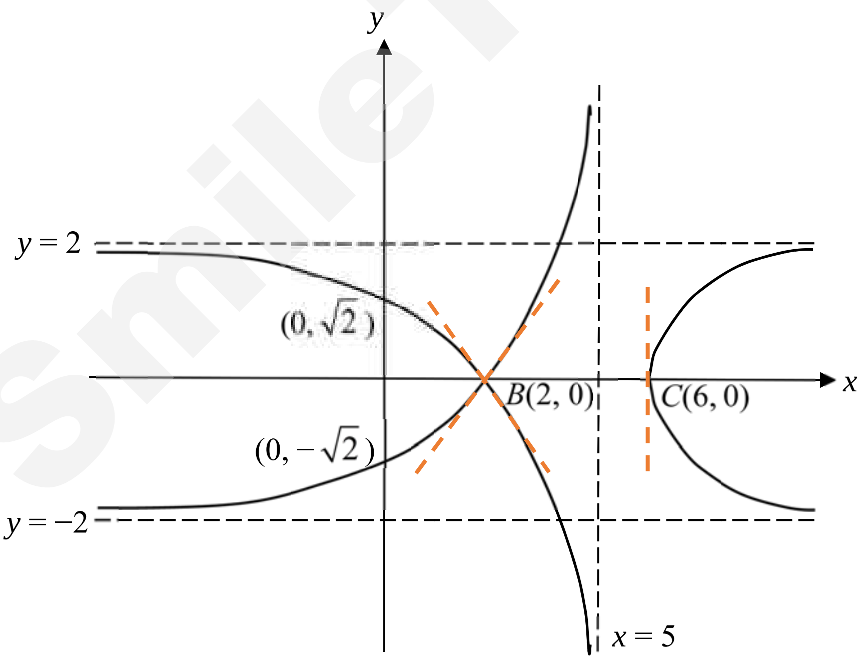
$$= 2n - 1 + \left(\tfrac{1}{2}\right)^n$$

8

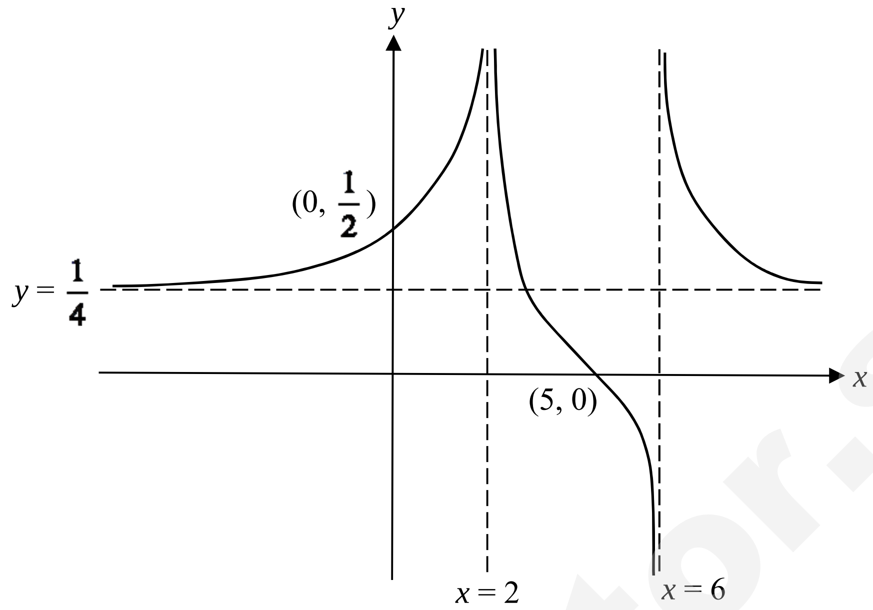
(i) $y = \mathrm{f}(|x|)$



(ii) $y^2 = \mathrm{f}(x)$



(iii) $y = \dfrac{1}{\mathrm{f}(x)}$



9

(a) $$\frac{\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)}{\sqrt{2-\cos x}} = \frac{\sqrt{2}\left(\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x\right)}{\sqrt{2-\cos x}}$$

$$= \frac{\sin x + \cos x}{\sqrt{2-\cos x}}$$

$$\approx \frac{x + \left(1 - \frac{x^2}{2}\right)}{\sqrt{2 - \left(1 - \frac{x^2}{2}\right)}}$$

$$= \frac{1 + x - \frac{x^2}{2}}{\sqrt{1 + \frac{x^2}{2}}}$$

$$= \left(1 + x - \frac{x^2}{2}\right)\left(1 + \frac{1}{2}x^2\right)^{-\frac{1}{2}}$$

$$= \left(1 + x - \frac{x^2}{2}\right)\left(1 - \frac{1}{4}x^2 + \ldots\right)$$

$$\approx 1 + x - \frac{3}{4}x^2 + \ldots$$

(b)(i) Given $(1+x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-y}$

Implicit differentiate w.r.t. x, $(1+x^2)\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}=-e^{-y}\dfrac{dy}{dx}$

When $x=0$, $y=-1$, $\dfrac{dy}{dx}=e$ and $\dfrac{d^2y}{dx^2}=-e^2$

So $y=-1+ex-\dfrac{e^2}{2}x^2+\ldots$

(ii) $\ln(2+y)=\ln\left(1+ex-\dfrac{e^2}{2}x^2+\ldots\right)$ use result in (i)

$=\left(ex-\dfrac{e^2}{2}x^2+\ldots\right)-\dfrac{1}{2}\left(ex-\dfrac{e^2}{2}x^2+\ldots\right)^2$ use standard series for ln

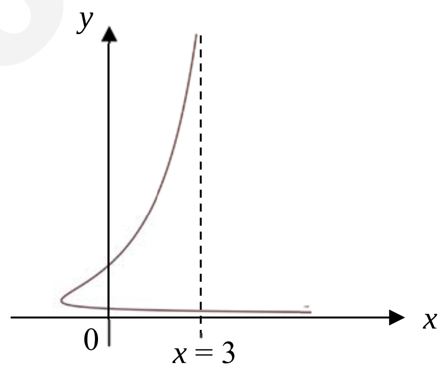
$\approx ex-\dfrac{e^2}{2}x^2-\dfrac{1}{2}(ex)^2$

$=ex-e^2x^2$

10

(i) When $x=0$, $t(2+t)=0 \Rightarrow t=0$ or $t=-2$

Coordinates are (0, 1) and (0, $\frac{1}{9}$)



(ii) $\dfrac{dx}{dt}=2+2t$, $\dfrac{dy}{dt}=\dfrac{2}{(1-t)^3}$

$\therefore \dfrac{dy}{dx}=\dfrac{1}{(1+t)(1-t)^3}$

When tangent is parallel to y-axis,

$(1+t)(1-t)^3=0 \Rightarrow t=-1$ or $t=1$ (vertical asymptote)

Equation of tangent is $x=-1$

(iii) Area $=-\int_{\frac{1}{9}}^{1} x \; dy$

$=-\int_{-2}^{0}(2t+t^2)\cdot\dfrac{2}{(1-t)^3}\,dt$

$=\int_{3}^{1}(2(1-u)+(1-u)^2)\cdot\dfrac{2}{u^3}\,du$

$=-2\int_{1}^{3}\dfrac{u^2-4u+3}{u^3}\,du$

$=-2\int_{1}^{3}\left(\dfrac{1}{u}-\dfrac{4}{u^2}+\dfrac{3}{u^3}\right)du$

$=-2\left[\ln u+\dfrac{4}{u}-\dfrac{3}{2u^2}\right]_1^3$

$=-2\left[\left(\ln 3+\dfrac{4}{3}-\dfrac{3}{18}\right)-\left(4-\dfrac{3}{2}\right)\right]=\dfrac{8}{3}-2\ln 3$

Let $u = 1-t$ $\dfrac{du}{dt}=-1$ When $t=0$, $u=1$ When $t=-2$, $u=3$

11

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{birth rate} - \text{death rate}$$

$$= \lambda x - \beta x^2 \qquad \text{where } \lambda \text{ and } \beta \text{ are positive real constants}$$

Given $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=\frac{1}{2}} = \frac{3}{4}\times\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=1}$

$$\lambda\left(\frac{1}{2}\right) - \beta\left(\frac{1}{2}\right)^2 = \frac{3}{4}(\lambda - \beta) \quad \Rightarrow \lambda = 2\beta$$

Hence $\frac{\mathrm{d}x}{\mathrm{d}t} = \beta x(2-x)$

$$\int \frac{1}{2x-x^2}\,\mathrm{d}x = \beta\int \mathrm{d}t$$

$$\frac{1}{2}\int\left(\frac{1}{x} + \frac{1}{2-x}\right)\mathrm{d}x = \beta\int \mathrm{d}t$$

$$\frac{1}{2}\left[\ln|x| - \ln|2-x|\right] = \beta t + c$$

$$\frac{1}{2}\left[\ln\left|\frac{x}{2-x}\right|\right] = \beta t + c$$

$$\frac{x}{2-x} = A\mathrm{e}^{2\beta t} \text{ where } A = \pm\mathrm{e}^{2c}$$

Subst $t=0,\ x=0.1 \Rightarrow \frac{0.1}{1.9} = A \Rightarrow A = \frac{1}{19}$

$$x = \frac{2}{19}\mathrm{e}^{2\beta t} - \frac{1}{19}x\,\mathrm{e}^{2\beta t}$$

$$x = \frac{\frac{2}{19}\mathrm{e}^{2\beta t}}{1+\frac{1}{19}\mathrm{e}^{2\beta t}}$$

$$= \frac{2\mathrm{e}^{2\beta t}}{19+\mathrm{e}^{2\beta t}} = \frac{2}{1+19\mathrm{e}^{-2\beta t}}$$

Alternative solution:

$$\int \frac{1}{2x-x^2}\,\mathrm{d}x = \beta\int \mathrm{d}t$$

$$-\int \frac{1}{(x-1)^2-1}\,\mathrm{d}x = \beta\int \mathrm{d}t$$

$$-\frac{1}{2}\left[\ln\left|\left(\frac{x-1-1}{x-1+1}\right)\right|\right] = \beta t + c$$

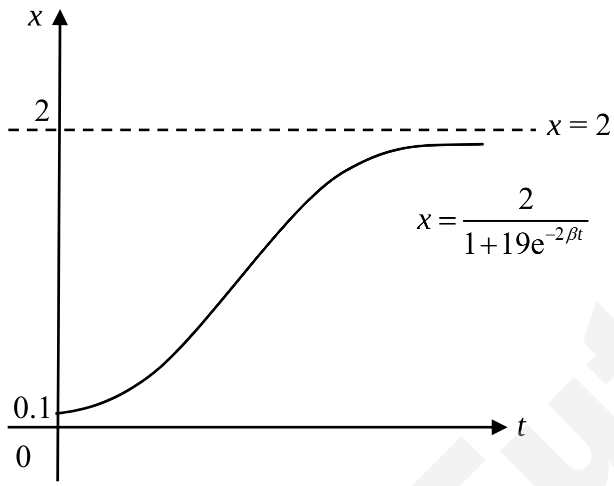
$$-\frac{1}{2}\left[\ln\left|\left(\frac{x-2}{x}\right)\right|\right] = \beta t + c$$

$$\frac{x-2}{x} = Ae^{-2\beta t} \quad \text{where } A = \pm e^{-2c}$$

Subst $t = 0,\ x = 0.1 \Rightarrow \dfrac{-1.9}{0.1} = A \Rightarrow A = -19$

$$x\left(1+19e^{-2\beta t}\right) = 2$$

$$x = \frac{2}{1+19e^{-2\beta t}}$$



The number of insects will approach 200 in the long run.

12

(a) $2z_1 + \mathrm{i}\,z_2^* = 7-6\mathrm{i}$ --- (1)

$z_1 - \mathrm{i}\,z_2 = 6-6\mathrm{i}$ --- (2)

(1) – (2) x 2: $\mathrm{i}\,z_2^* + 2\mathrm{i}\,z_2 = 7-6\mathrm{i}-2(6-6\mathrm{i}) = -5+6\mathrm{i}$

$z_2^* + 2z_2 = 6+5\mathrm{i}$

Since $z_2^* + 2z_2 = 3\mathrm{Re}(z_2) + \mathrm{Im}(z_2)\mathrm{i} = 6+5\mathrm{i}$, $z_2 = 2+5\mathrm{i}$

Sub $z_2 = 2+5\mathrm{i}$ into (2): $z_1 = 6-6\mathrm{i}+\mathrm{i}(2+5\mathrm{i}) = 1-4\mathrm{i}$

(b) $|w| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \dfrac{1}{\sqrt{2}}$

$\arg(w) = \tan^{-1}(-1) = -\dfrac{\pi}{4}$

(i) $\left|\dfrac{v}{w^*}\right| = \dfrac{|v|}{|w^*|} = \dfrac{|v|}{|w|} = \dfrac{2}{\left(\frac{1}{\sqrt{2}}\right)} = 2\sqrt{2}$

$$\arg\left(\frac{v}{w^*}\right) = \arg(v) - \arg(w^*) = \arg(v) + \arg(w) = \frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}$$

(ii) $v = 2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right) = \sqrt{3} + \mathrm{i}$

$$\frac{v}{w^*} = \frac{\sqrt{3}+\mathrm{i}}{\frac{1}{2}+\frac{1}{2}\mathrm{i}} = \frac{2\left(\sqrt{3}+\mathrm{i}\right)}{1+\mathrm{i}} \times \frac{1-\mathrm{i}}{1-\mathrm{i}}$$

$$= \left(\sqrt{3}+1\right) + \left(1-\sqrt{3}\right)\mathrm{i}$$

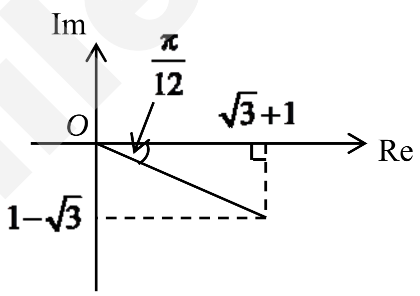
$\therefore \; \mathrm{Re}\left(\frac{v}{w^*}\right) = \sqrt{3}+1$ and $\mathrm{Im}\left(\frac{v}{w^*}\right) = 1-\sqrt{3}$

<u>Alternative solution</u>

$$\frac{1}{w^*} = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right] = 1-\mathrm{i}$$

$$\frac{v}{w^*} = \left(\sqrt{3}+\mathrm{i}\right)\left(1-\mathrm{i}\right) = \sqrt{3} - \sqrt{3}\mathrm{i} + \mathrm{i} + 1 = \left(\sqrt{3}+1\right) + \left(1-\sqrt{3}\right)\mathrm{i}$$

(iii) Using results in **(i)** and **(ii)**,



From the Argand diagram, $\tan\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = 2-\sqrt{3}$

1i	Let $\frac{1}{r(r-1)}=\frac{A}{r}+\frac{B}{r-1}$ $A(r-1)+Br=1$ Sub $r=0 \Rightarrow A=-1$ Sub $r=1 \Rightarrow B=1$ So $\frac{1}{r(r-1)}=\frac{1}{r-1}-\frac{1}{r}$ $\sum_{r=2}^{n}\frac{1}{r(r-1)}=\sum_{r=2}^{n}\left(\frac{1}{r-1}-\frac{1}{r}\right)$ $=\frac{1}{1}-\cancel{\frac{1}{2}}$ $+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}$ $\vdots$ $\vdots$ $+\cancel{\frac{1}{n-2}}-\cancel{\frac{1}{n-1}}$ $+\cancel{\frac{1}{n-1}}-\frac{1}{n}$ $=1-\frac{1}{n}$
1ii	$\sum_{r=2}^{n}(u_r-u_{r-1}) = -\sum_{r=2}^{n}\frac{1}{r(r-1)}$ $\cancel{u_2}\quad -u_1$ $+\cancel{u_3}\quad -\cancel{u_2}$ $\vdots \qquad = -\left(1-\frac{1}{n}\right)$ $+\cancel{u_{n-1}}-\cancel{u_{n-2}}$ $+u_n\quad -\cancel{u_{n-1}}$ $u_n=u_1-1+\frac{1}{n}=1+\frac{1}{n}$

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$w^4 + 2 - 2\sqrt{3}\,\mathrm{i} = 0 \;\Rightarrow\; w^4 = -2 + 2\sqrt{3}\,\mathrm{i}$

$\qquad\qquad\qquad\qquad = 4\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$

$\qquad\qquad\qquad\qquad = 4\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3} + 2k\pi\right)}$

$\Rightarrow\; w = (4)^{\frac{1}{4}}\left[\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}+2k\pi\right)}\right]^{\frac{1}{4}} = \sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}+\frac{k\pi}{2}\right)}, \qquad$ where $k = 0, \pm 1, -2$

$\therefore w = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{6}},\; \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}},\; \sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)},\; \sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$

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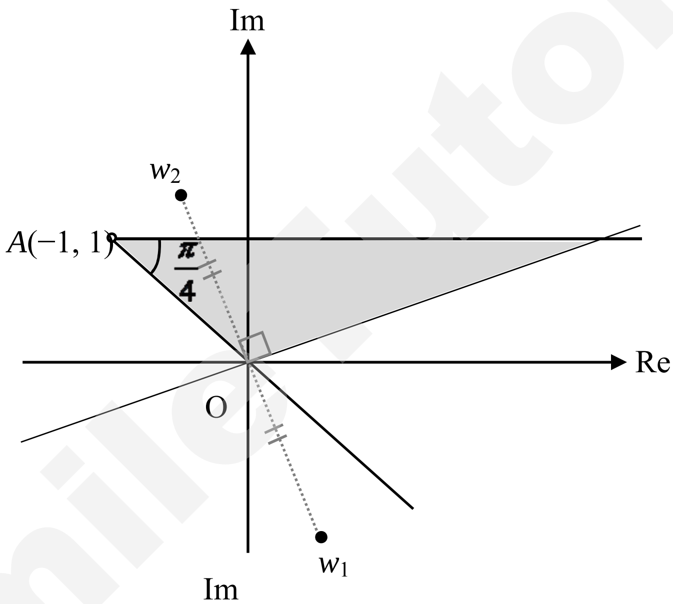
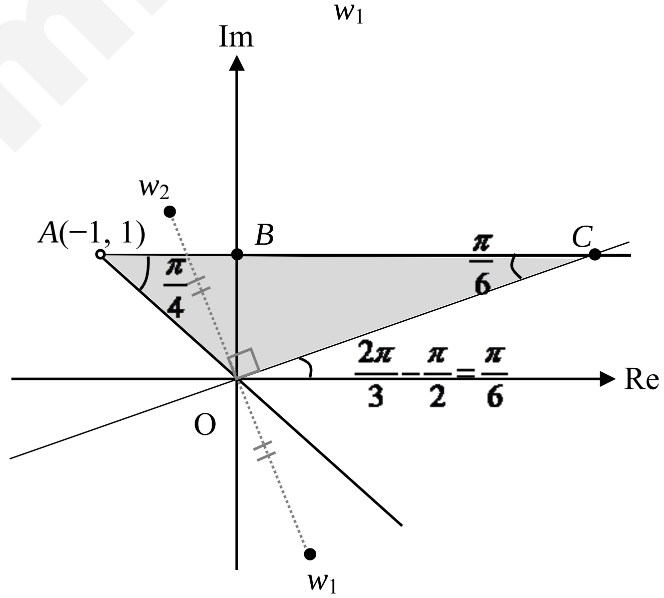
$\therefore w_1 = \sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}, w_2 = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$





$\tan\frac{\pi}{6} = \frac{1}{BC}$

$BC = \sqrt{3}$

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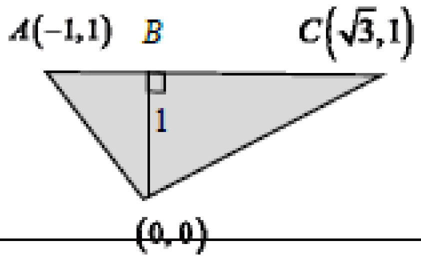
Area of shaded region

$$= \frac{1}{2}\left(1+\sqrt{3}\right)(1)$$

$$= \frac{1+\sqrt{3}}{2}$$



3i

Let $\lambda = \dfrac{x+1}{-1} = \dfrac{z+6}{2}$, $\quad y = 4$

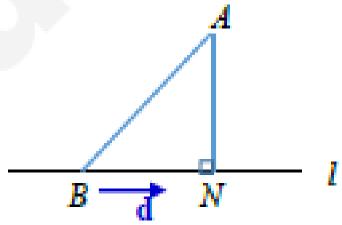
$x = -1-\lambda, \quad z = -6+2\lambda, \quad y = 4$

$$l: \mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

Let N be the foot of the perpendicular from A to l.

$$\overrightarrow{ON} = \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1+\lambda \\ 4 \\ -6+2\lambda \end{pmatrix} \quad \text{for some } \lambda \in \mathbb{R}$$

$$\Rightarrow \overrightarrow{AN} = \begin{pmatrix} -\lambda \\ 1 \\ -1+2\lambda \end{pmatrix}$$



$$\overrightarrow{AN} \perp l \quad \text{i.e.} \quad \begin{pmatrix} -\lambda \\ 1 \\ -1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 0 \quad \Rightarrow \lambda = \frac{2}{5}$$

$$\text{Thus } \overrightarrow{ON} = \frac{1}{5}\begin{pmatrix} -7 \\ 20 \\ -26 \end{pmatrix}$$

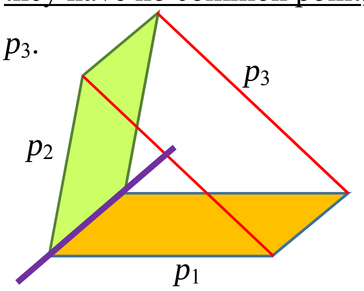
3ii

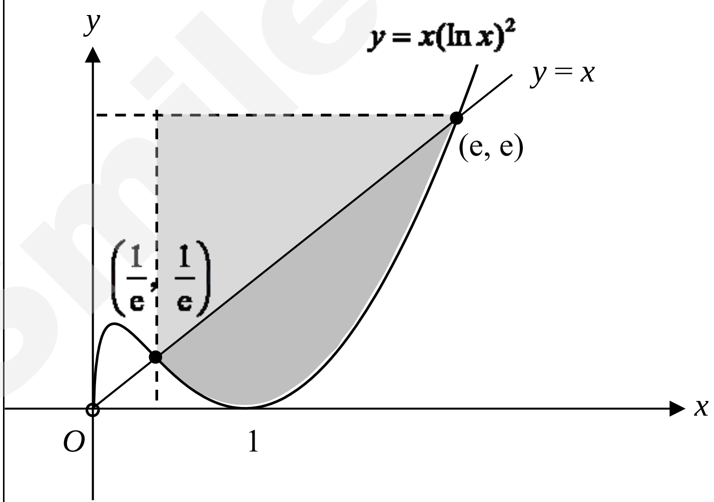
Let B be the point on l with coordinates $(-1, 4, 6)$

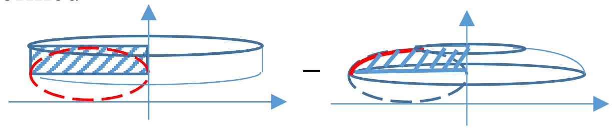
$$\overrightarrow{BA} = \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \text{or} \quad \overrightarrow{AN} = -\frac{1}{5}\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$$

A normal to p_1 is $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Equation of p_1 is $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = -4$ i.e $2x + y + z = -4$

	Equation of p_2 is $\mathbf{r}\cdot\begin{pmatrix}1\\2\\a\end{pmatrix}=-5$ $\cos 60^\circ=\dfrac{\left\|\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\c\end{pmatrix}\right\|}{\sqrt{6}\sqrt{5+c^2}}$ $\dfrac{1}{2}=\dfrac{\|4+c\|}{\sqrt{6}\sqrt{5+c^2}}$ $30+6c^2=4\left(c^2+8c+16\right)$ $c^2-16c-17=0$ $(c-17)(c+1)=0$ $c=17$ (rejected since $c<0$) or $c=-1$
3iii	p_1: $2x+y+z=-4$ p_2: $x+2y+cz=-5$ Using GC, $x=-1-\boldsymbol{t}$ $y=-2+\boldsymbol{t}$ $z=\boldsymbol{t}$ Equation of m is $\mathbf{r}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}+t\begin{pmatrix}-1\\1\\1\end{pmatrix}$, $t\in\mathbb{R}$
3iv	Since the <u>3 planes are not parallel</u> and <u>they have no common point</u>, m is parallel to p_3 but not contained in p_3.  m is perpendicular to $\mathbf{n}_3$: $\begin{pmatrix}3\\\alpha\\7\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=0\Rightarrow\alpha=-4$ Also $(-1, -2, 0)$ on m does not lie in p_3: $3x+\alpha y+7z=\beta$ Thus $3(-1)+(-4)(-2)+7(0)\neq\beta\ \Rightarrow\beta\neq 5$

4a	Area of shaded region $=\frac{1}{2}\left(\mathrm{e}+\frac{1}{\mathrm{e}}\right)\left(\mathrm{e}-\frac{1}{\mathrm{e}}\right) - \int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}} x(\ln x)^2 \, \mathrm{d}x$ $=\frac{1}{2}\left(\mathrm{e}^2-\frac{1}{\mathrm{e}^2}\right)-\left(\left[\frac{x^2}{2}(\ln x)^2\right]_{\frac{1}{\mathrm{e}}}^{\mathrm{e}}-\int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}}\frac{x^2}{2}\frac{2\ln x}{x}\,\mathrm{d}x\right)$ $=\frac{1}{2}\left(\mathrm{e}^2-\frac{1}{\mathrm{e}^2}\right)-\left(\left[\frac{\mathrm{e}^2}{2}-\frac{1}{2\mathrm{e}^2}\right]-\int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}} x\ln x\,\mathrm{d}x\right)$ $=\frac{1}{2}\left(\mathrm{e}^2-\frac{1}{\mathrm{e}^2}\right)-\frac{1}{2}\left(\mathrm{e}^2-\frac{1}{\mathrm{e}^2}\right)+\left(\left[\frac{x^2}{2}\ln x\right]_{\frac{1}{\mathrm{e}}}^{\mathrm{e}}-\int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}}\frac{x^2}{2}\frac{1}{x}\,\mathrm{d}x\right)$ $=\frac{1}{2}\left(\mathrm{e}^2+\frac{1}{\mathrm{e}^2}\right)-\frac{1}{2}\left[\frac{x^2}{2}\right]_{\frac{1}{\mathrm{e}}}^{\mathrm{e}}$ $=\frac{1}{2}\left(\mathrm{e}^2+\frac{1}{\mathrm{e}^2}\right)-\frac{1}{2}\left(\frac{\mathrm{e}^2}{2}-\frac{1}{2\mathrm{e}^2}\right)$ $=\frac{1}{4}\left(\mathrm{e}^2+\frac{3}{\mathrm{e}^2}\right)$  Area = $\frac{1}{4}\left(\mathrm{e}^2+\frac{3}{\mathrm{e}^2}\right)+\frac{1}{2}\left(\mathrm{e}-\frac{1}{\mathrm{e}}\right)^2$
4b	Given $(x+2)^2+4(y-1)^2=4$ $(x+2)^2=4[1-(y-1)^2] \Rightarrow x+2=\pm2\sqrt{1-(y-1)^2}$ The shaded region is bounded by the section of the ellipse where $x \le -2$. Hence $x=-2-2\sqrt{1-(y-1)^2}$.

	Volume of solid formed  $= \pi 4^2(2-1) - \pi\int_1^2 x^2 \text{ dy}$ $= 16\pi - \pi\int_1^2 \left(-2-2\sqrt{1-(y-1)^2}\right)^2 \text{ dy}$ $= 9.6 \text{ units}^3$

5i	Stratified sampling ensures that households occupying two-bedroom apartments and households occupying four-bedroom apartments are **proportionately represented** in the sample.
5ii	Total number of units = 24 + 88 = 112 Number of 2-bedroom households needed = $\dfrac{24}{112}\times 35 = 7.5$ Number of 4-bedroom households needed = $\dfrac{88}{112}\times 35 = 27.5$ She should choose 8 two-bedroom apartments and 27 four-bedroom apartments (or 7 two-bedroom apartments and 28 four-bedroom apartments) Using the list of apartment numbers as sampling frames for the two types of apartments, she would use simple random sampling (or systematic sampling) to select the 2-bedroom apartments and 4-bedroom apartments to be interviewed.

6i	Number of ways $= \dfrac{6!}{2!} = 360$ P AA R LLL E
6ii	(By Slotting) Number of ways $= \binom{7}{2}\dfrac{6!}{3!} = 2520$ **Method 2** (Complement) Number of ways to arrange letters without restriction $= \dfrac{8!}{2!3!} = 3360$ Number of ways to arrange letters with both A's together $= \dfrac{7!}{3!} = 840$ Total number of ways $= \dfrac{8!}{2!3!} - \dfrac{7!}{3!} = 2520$

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<tr><td>6iii</td><td>
Number of ways $= 4\times\frac{5!}{2!} = 240$ L _ L _ L _ _ _

Method 2
Number of ways $=\binom{5}{2}\times 2!\times\frac{4!}{2!} = 240$
Number of ways $= \frac{{}^5C_1 \times {}^4C_1 \times 4!}{2!} = 240$
Method 3:
Case 1: LALAL_ _ _
Number of ways $= 4! = 24$
Case 2: LXLAL_ _ _ or LALXL_ _ _
Number of ways $= 2\times\binom{3}{1}\times 4! = 144$
Case 3: LXLYL_ _ _ or LYLXL_ _ _
Number of ways $= 2\times\binom{3}{2}\times\frac{4!}{2!} = 72$
Total = 240
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<tr><td>7i</td><td>
P(a randomly chosen cupcake is “good”)

= 0.6 × 0.8 × 0.94 + 0.4 × 0.88 × 0.94

= 0.78208 = 0.782 (3sf)
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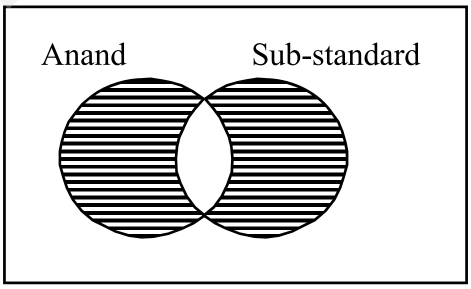
P(cupcake is either baked by Anand, is sub-standard but not both)



= P(cupcake is baked by Anand and is good) +
P(cupcake is baked by Beng and is sub-standard)

= 0.6 × 0.8 × 0.94 + 0.4 × (0.12 + 0.88 × 0.06)
= 0.520 (3sf)

Method 2
= 0.6 × 0.8 × 0.94 + 0.4 × (1 – 0.88 × 0.94)
= 0.520 (3sf)
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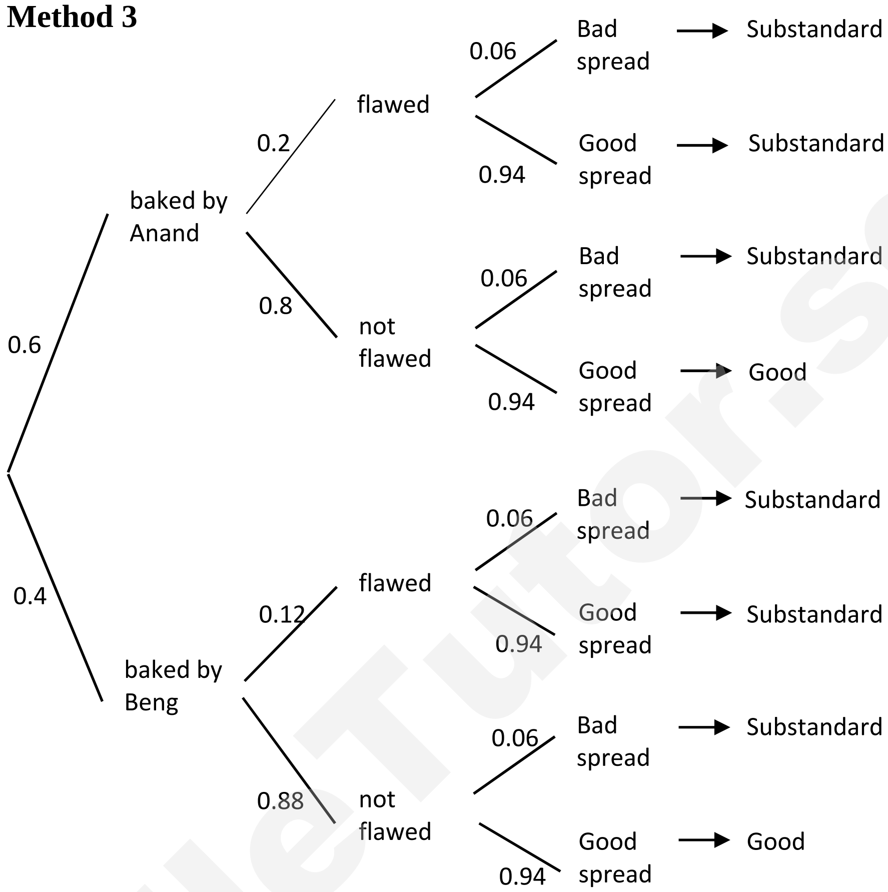
Method 3



=P(baked by Anand) + P(sub-standard) – 2 P(baked by Anand and sub-standard)
$= 0.6 + (1 - 0.78208) - 2\,(\,0.6 \times (1 - 0.8 \times 0.94)\,)$
$= 0.520$ (3sf)
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<tr>
<td>7iii</td>
<td>
P(cupcake baked by Anand | cupcake is sub-standard)

$$= \frac{\text{P(cupcake is sub-standard and baked by Anand)}}{\text{P(cupcake is sub-standard)}}$$

$$= \frac{0.6\times(0.2+0.8\times0.06)}{1-0.78208} \quad \text{or} \quad \frac{0.6\times(1-0.8\times0.94)}{1-0.78208}$$

$= 0.683$ (3sf)
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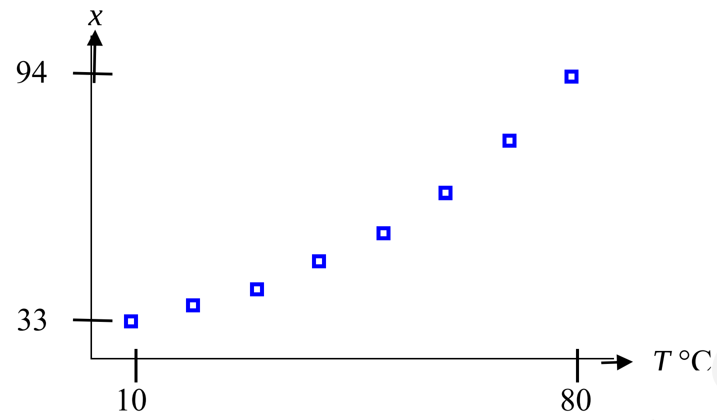
1

8	
8	**Model B** is the most appropriate as the curve is concave upwards and x increases with T, which is the same as the shape shown in the scatter diagram. OR **Model B** is the most appropriate since x increases at an increasing rate as T increases, which is the same as the shape shown in the scatter diagram.
8i	**Regression line for Model B:** $\ln x = \ln a + bT$ $\ln x = \ln(27.06) + 0.01497T$ $\overline{T} = \dfrac{10+20+30+40+50+60+70+80+90}{9} = 50$ and $\dfrac{\sum \ln x}{9}$ $= \dfrac{\ln 33 + \ln 37 + \ln 41 + \ln 48 + \ln 55 + \ln 65 + \ln 78 + \ln 94 + \ln k}{9}$ $= \dfrac{31.77392262 + \ln k}{9}$ Since ($\dfrac{\sum \ln x}{9}$, $\overline{T}$) lies on the regression line, $\dfrac{31.77392262 + \ln k}{9} = \ln(27.06) + 0.01497(50)$ $\Rightarrow k = 104$ (nearest whole number)
8ii	From GC, $r = 0.997$
8iii	When $T = 65$, $\ln x = \ln(27.06) + 0.01497(105)$

	$x = 130$ (nearest whole number) The estimated growth rate cannot be taken as reliable as the temperature 105^0C, from which the value of $x = 130$ is computed from, lies outside the data range of *T* i.e. [10, 90].

9i	Let *T* be the number of defects in 7 m^2 of twill weave. $T \sim P_o(1.2\times 7)$, i.e. $T \sim P_o(8.4)$ $P(T = 9) = 0.129025899 = 0.129$
9ii	Let *W* be the number of defects in a box. $W \sim P_o(2\times 1.2 + 3\times 0.8)$, i.e. $W \sim P_o(4.8)$ $P(\text{box is faulty}) = P(W > 10)$ $= 1 - P(W \le 10)$ $= 0.0104$ (shown)
9iii	Let *F* be the number of "faulty" boxes in a random batch of 50 boxes. $F \sim \text{B}(50,\ 0.0104)$ Since *n* is large and *np* = 0.52 < 5, $F \sim \text{P}_\text{o}(0.52)$ approximately. P(customer rejects the entire batch) $= \text{P}(F \ge 2)$ $= 1 - \text{P}(F \le 1)$ ≈ 0.096329 $= 0.0963$ or 0.0966 (3 sig fig) Required probability $\approx \binom{4}{2}(0.096329)^2(1-0.096329)^2$ $= 0.0455$ or 0.0457 (3 sig fig) Alternative Solution: Let *X* be the number of batches out of 4 that are rejected. $X \sim \text{B}(4,\ 0.096329)$ $\text{P}(X = 2) = 0.0455$ or 0.0457

10i

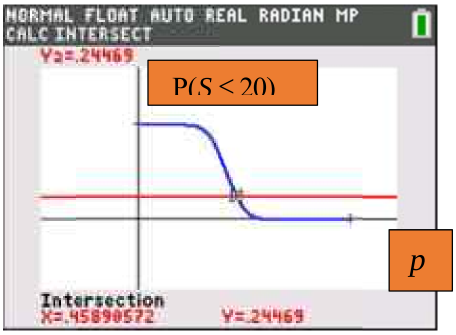
Let U and S be the number of subscribers who purchased a uPhone and a Samseng phone respectively in a sample of size n

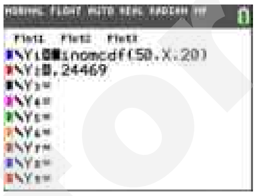
$U \sim \mathrm{B}(n, 0.3) \qquad S \sim \mathrm{B}(n, p)$

When $n = 50$,

$$\mathrm{P}(S \le 20) = 2\mathrm{P}(U = 15)$$
$$\mathrm{P}(S \le 20) = 0.24469$$

From GC,





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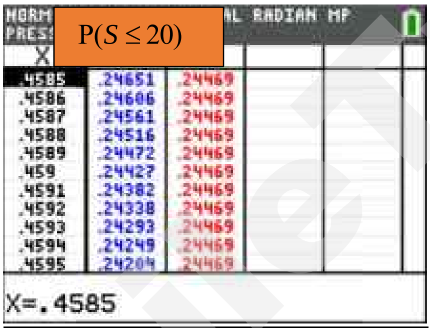


X	Y_1	Y_2
.4585	.24651	.24469
.4586	.24606	.24469
.4587	.24561	.24469
.4588	.24516	.24469
.4589	.24472	.24469
.459	.24427	.24469
.4591	.24382	.24469
.4592	.24338	.24469
.4593	.24293	.24469
.4594	.24249	.24469
.4595	.24204	.24469

X=.4585

$p = 0.459$ (3 sig fig)

10ii

Given $p = 0.4$, $n = 50$, $S \sim \mathrm{B}(50, 0.4)$

$\mathrm{E}(S) = 50\,(0.4) = 20$

$\mathrm{P}(U > 20) = 1 - \mathrm{P}(U \le 20)$

$\quad = 0.0478$ (3 sig fig)

10iii

When $n = 1000$

$U \sim \mathrm{B}(1000, 0.3)$

Since n is large, $np = 300 > 5$, $n\,(1 - p) = 700 > 5$

$U \sim \mathrm{N}(300, 210)$ approximately

$S \sim \mathrm{B}(1000, 0.4)$

Since n is large, $np = 400 > 5$, $n\,(1 - p) = 600 > 5$

$S \sim \mathrm{N}(400, 240)$ approximately

	$280U - 200S \sim N\,(\,280(300) - 200(400),\ 280^2(210) + 200^2(240)\,)$ $280U - 200S \sim N\,(\,4000,\ 26064000\,)$ $P\,(\,280U > 200S\,)$ $= P(\,280U - 200S > 0\,)$ $= P(\,280U - 200S > 0.5\,)$ (by continuity correction) $= 0.783$ (3 sig fig)

11i	H_0: $\mu = 250$ H_1: $\mu < 250$ Level of significance: 4% $\bar{v} = \frac{-217}{80} + 250 = 247.2875$ $s^2 = \frac{1}{79}\left(30738 - \frac{(-217)^2}{80}\right) = 381.6378165$ Since $n = 80$ is large, by Central Limit Theorem, $\bar{V}$ follows a normal distribution approximately. Test statistic: $\frac{\bar{V} - \mu}{\left(\frac{s}{\sqrt{n}}\right)} \sim N(0, 1)$ approximately If H_0 is true, p-value $= 0.107 > 0.04$, we do not reject H_0. (Or $z_{calc} = -1.24 < -1.751$) There is insufficient evidence at 4% level significance to conclude that $\mu < 250$. 'at the 4% significance level' means there is a 4% chance that we wrongly concluded that $\mu < 250$. (Or wrongly rejected $\mu = 250$ when it is true) H_0: $\mu = \mu_0$ H_1: $\mu \neq \mu_0$ If H_0 is true, $z_{cal} = \frac{\bar{v} - \mu_0}{\left(\frac{s}{\sqrt{n}}\right)}$ In order to reject H_0, z_{cal} must lie in the critical region.

	$\dfrac{247.2875-\mu_0}{\sqrt{\dfrac{381.6378165}{80}}}<-2.053749$ or $\dfrac{247.2875-\mu_0}{\sqrt{\dfrac{381.6378165}{80}}}>2.053749$ $\mu_0>251.77$ or $\mu_0<242.80$ Set of possible values of $\mu_0=\{\mu_0:\ \mu_0<243$ or $\mu_0>252\}$ (Also accept 242 other than 243) $T=V_1+V_2+V_3+....+V_{50}$ Given E(V) =250 and Var(V) = 385 Since n = 50 is large, by Central Limit Theorem, $T\sim$ N(50×250, 50× 385) = N(12500, 19250) approximately. P(T > 12600) = 0.236 (3 sig fig)

H2

TAMPINES JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

MATHEMATICS 9740/01

Paper 1

Tuesday, 13 Sep 2016

3 hours

Additional Materials: Answer paper

List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in, including the Cover Page.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 A curve C has equation $\qquad e^{x+y} + e = (3y+1)^2$.

(i) By considering $\dfrac{dy}{dx}$, show that C has no stationary points. [5]

(ii) Write down an equation relating x and y at which the tangent is parallel to the y-axis. [1]

2 Referred to the origin O, the points A and B have position vectors given by $\mathbf{a} = \begin{pmatrix} \cos t \\ -\sin t \\ 0.5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \cos 2t \\ \sin 2t \\ -1 \end{pmatrix}$ respectively, where t is a real parameter such that $0 \le t < \pi$.

(i) Show that $\mathbf{a} \cdot \mathbf{b} = p + \cos(qt)$, where p and q are constants to be determined. [2]

(ii) Hence find the exact value of t for which $\angle AOB$ is a maximum. [3]

3 **(i)** Describe a sequence of transformations that will transform the curve with equation $y = \dfrac{1}{x^2}$ on to the curve with equation $y = \dfrac{4}{(x-1)^2}$. [2]

(ii) It is given that

$$f(x) = \begin{cases} x+2 & \text{for } 0 < x \le 2, \\ \dfrac{4}{(x-1)^2} & \text{for } 2 < x \le 3, \end{cases}$$

and that $f(x) = f(x+3)$ for all real values of x.

Sketch the graph of $y = f(x)$ for $-2 \le x \le 6$. [3]

4 **Do not use a calculator in answering this question.**

One root of the equation $z^3 + az^2 + bz + 15 = 0$, where a and b are real, is $z = 1 + 2i$.

(i) Write down the other complex root. [1]

(ii) Explain why the cubic equation must have one real root. [1]

(iii) Find the value of the real root and the values of a and b. [5]

5 Functions f and g are defined by

$$\text{f} : x \mapsto \frac{3x-1}{3x-3}, \quad x \in \mathbb{R},\ x < 1,$$
$$\text{g} : x \mapsto \sqrt{x-2}, \quad x \in \mathbb{R},\ 2 \le x < 3.$$

(i) Find $\text{f}^{-1}(x)$. [2]

(ii) Show that $\text{f}^2(x) = x$. Hence find the exact value of $\text{f}^{2017}(0)$. [2]

(iii) Show that the composite function fg exists. Find an expression for $\text{fg}(x)$ and state the domain and range of fg. [5]

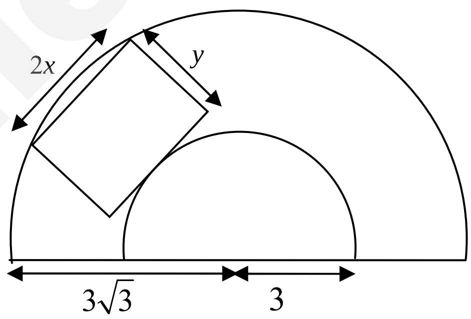
6 It is given that $\text{f}(x) = \dfrac{x+3}{(1-x)^n}$, where $-1 < x < 1$ and n is a positive integer.

(i) Find the binomial expansion of $\text{f}(x)$, in ascending powers of x, up to and including the term in x^2. [4]

(ii) Given that the coefficient of x^2 in the above expansion is 21, find the value of n. [3]

(iii) Given now that $n = 2$, by substituting a suitable value of x into the expansion in part **(i)**, find the exact value of $\displaystyle\sum_{r=1}^{\infty} \frac{4r-1}{4^{r-1}}$. [3]

7



The figure shows a rectangular sheet of length $2x$ metres and breadth y metres to be placed in a horizontal position along a garden walkway bounded by low vertical fence of which a horizontal cross-section is two concentric semicircles of radii 3 metres and $3\sqrt{3}$ metres. One side of the sheet of length $2x$ metres must be tangential to the inner fence, and the two ends of the opposite side must touch the outer fence, as shown in the figure. The rectangular sheet is assumed to have negligible thickness.

(i) By finding x^2 in terms of y, show that the area A square metres of the sheet, is given by $A = 2\sqrt{18y^2 - 6y^3 - y^4}$. [3]

(ii) Use differentiation to find, the maximum value of A, proving that it is a maximum. [7]

8 The plane p_1 passes through the points $A(4,1,1)$ and $B(2,1,0)$ and is parallel to the vector $4\mathbf{i}-\mathbf{j}+2\mathbf{k}$. A line l has equation $\frac{x-2}{-2}=y+1=z+4$.

(i) Show that a vector perpendicular to the plane p_1 is parallel to $\mathbf{i}-2\mathbf{k}$. Find the equation of p_1 in scalar product form. [3]

(ii) Find the coordinates of the point C at which l intersects p_1. [3]

(iii) The point D with coordinates (2, −1, −4) lies on l. Find the position vector of the foot of the perpendicular from D to p_1. Find the coordinates of the point E which is the mirror image of D in p_1. [4]

The plane p_2 contains the line l and the point A.

(iv) The planes p_1 and p_2 meet in a line m. Find a vector equation for m. [2]

9 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = 1 \quad \text{and} \quad 3u_{n+1} = 2u_n - 1 \text{ for } n \geq 1 .$$

(i) Use the method of mathematical induction to prove that

$$u_n = 3\left(\frac{2}{3}\right)^n - 1. \qquad [5]$$

(ii) Find $\sum_{r=1}^{n} u_r$. [3]

(iii) Give a reason why the series $\sum_{r=1}^{n}\left(u_r+1\right)$ converges, and write down the value of the sum to infinity. [2]

(iv) Explain, with the aid of a sketch, whether the value of $\sum_{r=1}^{\infty}\left(u_r+1\right)$ is an overestimation or underestimation of the value of $\int_0^{\infty} 3\left(\frac{2}{3}\right)^x \mathrm{d}x$. [2]

10 The mass, x grams, of a certain substance present in a chemical reaction at time t minutes satisfies the differential equation

$$\frac{dx}{dt} = k(4 + 2x - x^2),$$

where $0 \le x \le 1$ and k is a constant. It is given that $x = 1$ and $\frac{dx}{dt} = -\frac{1}{2}$ when $t = 0$.

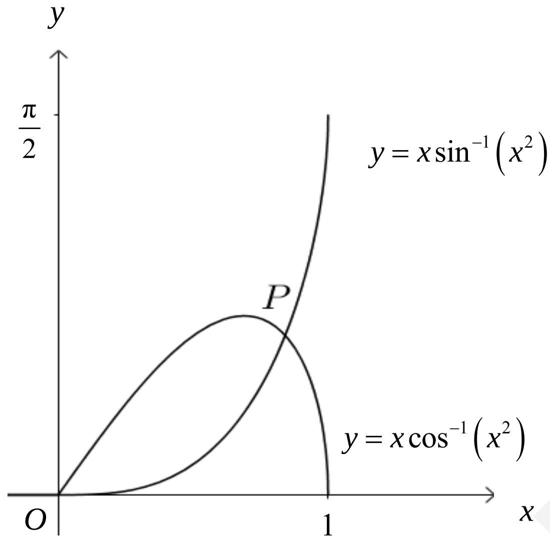
(i) Show that $k = -\frac{1}{10}$. [1]

(ii) By first expressing $4 + 2x - x^2$ in completed square form, find t in terms of x. [5]

(iii) Hence find the time taken for there to be none of the substance present in the chemical reaction, giving your answer correct to 3 decimal places. [1]

(iv) Express the solution of the differential equation in the form $x = f(t)$ and sketch the part of the curve with this equation which is relevant in this context. [5]

11



The diagram shows the curves with equations $y = x\sin^{-1}(x^2)$ and $y = x\cos^{-1}(x^2)$, where $0 \le x \le 1$. The curves meet at the point P with coordinates $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{4}\left(\frac{1}{\sqrt[4]{2}}\right)\right)$.

(i) Find the derivative of $\sqrt{1-x^4}$. [1]

(ii) Find the exact value of the area of the region bounded by the two curves. [5]

(iii) The region bounded by the curve $y = x\sin^{-1}(x^2)$, the line $y = \frac{\pi}{4}\left(\frac{1}{\sqrt[4]{2}}\right)$ and the y-axis is rotated about the y-axis through $360°$. By considering the parametric equations

$$x = t \quad \text{and} \quad y = t\sin^{-1}(t^2),$$

show that the volume of the solid formed is given by

$$\pi\int_0^{\frac{1}{\sqrt[4]{2}}} \left[\frac{2t^4}{\sqrt{1-t^4}} + t^2\sin^{-1}(t^2)\right] \mathrm{d}t.$$ [3]

(iv) Hence find the volume of the solid formed when the region bounded by the curve $y = x\sin^{-1}(x^2)$ and the line $y = \frac{\pi}{4}x$ is rotated through $360°$ about the y-axis. Give your answer correct to 5 significant figures. [3]

End of Paper

H2

TAMPINES JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

MATHEMATICS 9740/02

Paper 2

Thursday, 15 Sep 2016

3 hours

Additional Materials: Answer paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in, including the Cover Page.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 The curve C has parametric equations

$$x = 2\cos t \ , \qquad y = 3\sin t \ .$$

(i) Find the equation of the normal to C at the point P with parameter θ, leaving your answer in terms of θ. [3]

(ii) This normal to C at the point P meets the x- and y-axes at points A and B respectively. Find the cartesian equation of the locus of the midpoint of AB as θ varies. [4]

2 Using partial fractions, find the exact value of

$$\int_0^1 \frac{15x^2 - x + 17}{(2x+1)(x^2+4)}\,\mathrm{d}x \ . \qquad [8]$$

3 Mrs X wants to sew handmade gifts for the guests attending her daughter's wedding. On the first day of gift preparation, she spends 270 minutes of her time. Subsequently, she will decrease the amount of time spent each day on gift preparation by a certain amount. The total time taken to complete her gift preparation is 9000 minutes.

(i) Mrs X spends, on each subsequent day, 2.5% less time on gift preparation than on the previous day. Find, to the nearest minute, the time Mrs X spends on the 10th day, and find the minimum number of days Mrs X requires to complete her gift preparation. [6]

(ii) It takes 20 minutes for Mrs X to complete one gift. If Mrs X has the opportunity to make more gifts using this model, in theory, find the maximum number of gifts she can complete. [2]

(iii) After doing some calculations, Mrs X realises that she has to decrease the number of days spent on gift preparation. She still spends 270 minutes on the first day, but on each subsequent day, the amount of time spent is 4 minutes less than on the previous day. Find the minimum number of days Mrs X requires to complete her gift preparation.

[3]

4 (a) The complex number w is given by $3+3(\sqrt{3})\mathrm{i}$.

(i) Find the modulus and argument of w, giving your answers in exact form. [2]

(ii) Without using a calculator, find the smallest positive integer value of n for which $\left(\frac{w^3}{w^*}\right)^n$ is a real number. [4]

(b) The complex number z is such that $z^5=-4\sqrt{2}$.

(i) Find the values of z in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [4]

(ii) Show the roots on an Argand diagram. [2]

(iii) The roots represented by z_1 and z_2 are such that $0<\arg(z_1)<\arg(z_2)<\pi$. The locus of all points z such that $|z-z_1|=|z-z_2|$ intersects the line segment joining points representing z_1 and z_2 at the point P. P represents the complex number p. Find, in exact form, the modulus and argument of p. [2]

Section B: Statistics [60 marks]

5 It is desired to conduct a survey among university students regarding the use of the university's facilities. A random sample of 100 students is obtained.

(i) Explain what is meant in this context by the term 'a random sample'. [2]

(ii) Now it is necessary to obtain a representative range by faculties. Name a more appropriate sampling method and explain how it can be carried out. [2]

6 75% of the employees in a factory own a cell phone.

(i) A random sample of 8 employees is taken. Find the probability that the number of employees who own a cell phone is between 4 and 6 inclusive. [2]

In an industrial park, every factory has 160 employees.

(ii) Use a suitable approximation to find the probability that in a randomly selected factory, at least 115 employees own a cell phone. [3]

(iii) A random sample of 15 factories in the industrial park is taken. Find the probability that at most 11 of these factories each have at least 115 employees who own a cell phone. [2]

7 The average number of parking tickets that a traffic warden issues per day is being investigated.

(i) State, in context, two assumptions that need to be made for the number of parking tickets issued per day to be well modelled by a Poisson distribution. [2]

Assume that the number of parking tickets issued per day has the distribution Po (3.6).

(ii) Find the probability that, in seven days, the traffic warden issues more than 22 parking tickets altogether. [3]

(iii) The probability that the traffic warden issues more than N parking tickets altogether in 10 days is less than 0.05. Using a suitable approximation, find the least possible value of N. [2]

8 A fruit stall sells papayas. The mass of papayas is denoted by X kg. The stall owner claims that the mean mass of the papayas is at least 1.2 kg. The masses of a random sample of 8 papayas are summarised by

$$\sum x = 8.84, \quad \sum x^2 = 9.95.$$

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Stating a necessary assumption, test at the 5% level of significance whether there is any evidence to doubt the stall owner's claim. [5]

9 Two players A and B decide to play two consecutive card games. A fair coin is tossed to decide which player has the first move in the first game. The loser of the first game has the first move in the second game. A player must win both games to be declared the overall winner. If each player wins a game, the result is a draw.

When A has the first move in a game, the probability that A wins that game is $\frac{2}{3}$. When B has the first move in a game, the probability that B wins that game is $\frac{3}{5}$. Every game ends with either A or B as the winner.

(i) Show that the probability that two consecutive card games end in a draw is $\frac{142}{225}$. [2]

(ii) Given that A wins the second game, find the probability that two consecutive card games end in a draw. [3]

(iii) To make their games more enjoyable, A and B agree to change the procedure for deciding who has the first move in the first game. As a result of their new procedure, the probability of A having the first move in the first game is p. Find the exact value of p which gives A and B equal chances of winning both games. [3]

10(a) Sketch a scatter diagram that might be expected when x and y are related approximately by $y = px^2 + q$ in each of the cases **(i)** and **(ii)** below. In each case your diagram should include 5 points, approximately equally spaced with respect to x, and with all x- and y-values positive.

(i) p is positive and q is positive,

(ii) p is negative and q is positive. [2]

(b) A car website gives the following information on the ages in months (m) and resale price in dollars (P) of used passenger cars of a particular model.

m	10	20	28	35	40	45	56	62	70	74
P	110600	79900	78700	69200	66100	60200	53800	50600	46700	43800

It is thought that the price after m months can be modelled by one of the formulae

$$P = am + b\ , \quad P = c\ln m + d\ ,$$

where a, b, c and d are constants.

(i) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(A) m and P,

(B) $\ln m$ and P. [2]

(ii) Explain which of $P = am + b$ and $P = c\ln m + d$ is the better model and find the equation of a suitable line for this model. [3]

(iii) Use the equation of your regression line to estimate the price of a used passenger car that is 80 months old. Comment on the reliability of your estimate. [2]

11 A group of 5 girls and 7 boys play ice breaker games to get to know each other.

(i) The group stands in a line.

(a) Find the number of different possible arrangements. [1]

(b) The girls have names that start with different letters. Find the number of different possible arrangements in which all the girls are separated, with the girls' names in alphabetical order. [2]

(ii) The group forms two circles of 6, with one circle inside the other, such that each person in the inner circle stands facing a person in the outer circle. Find the number of different possible arrangements. [2]

(iii) The group has to split into a group of 3, a group of 4, and a group of 5. Find the number of possible ways in which the groups can be chosen if there is no girl in at least one of the groups. [4]

12 A supermarket sells two types of guava, A and B. The masses, in grams, of the guava of each type have independent normal distributions. The means and standard deviations of these distributions are shown in the following table.

	Mean (g)	Standard deviation (g)
Type A	200	12
Type B	175	12

Find the probability that

(i) the total mass of 4 randomly chosen guava of type A is more than 810 g, [2]

(ii) the mean mass of 4 randomly chosen guava of type A differs from the mean mass of 3 randomly chosen guava of type B by at least 30 g. [3]

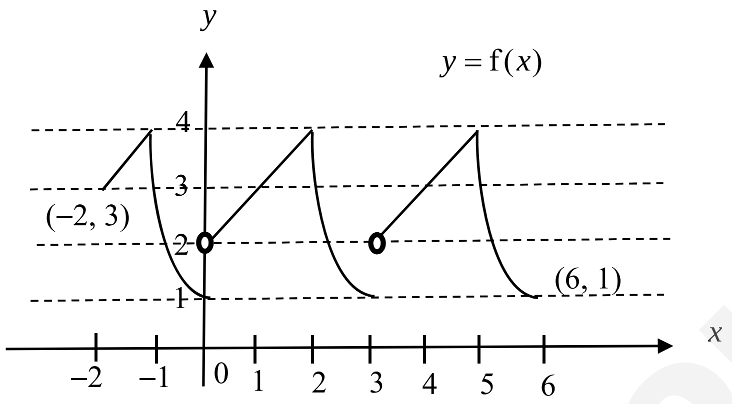
Mr Tan buys 20 guavas, m of them are guava of type A and the rest are guava of type B.

(iii) Find the least value of m such that the probability that the total mass of these 20 guavas exceeding 3500 g is more than 0.95. (Answers obtained by trial and improvement from a calculator will obtain no marks.) [4]

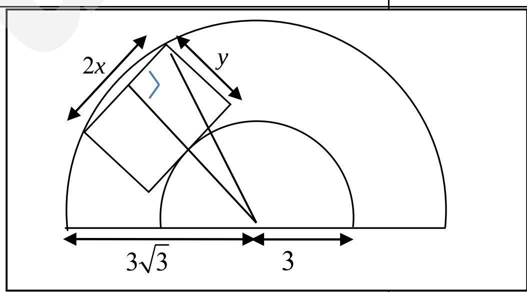
End of Paper

Tampines Junior College
2016 JC2 Preliminary Examination
H2 Mathematics Paper 1
Solution

1(i)	$\mathrm{e}^{x+y}+\mathrm{e}=(3y+1)^2$ Differentiate w.r.t x: $\mathrm{e}^{x+y}\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)=2(3y+1)\left(3\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $\mathrm{e}^{x+y}+\mathrm{e}^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x}=(18y+6)\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}\left(18y+6-\mathrm{e}^{x+y}\right)=\mathrm{e}^{x+y}$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^{x+y}}{18y+6-\mathrm{e}^{x+y}}$ Since $\mathrm{e}^{x+y}>0 \quad \forall x\in\mathbb{R}, \forall y\in\mathbb{R}$, $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}\neq 0$ $\therefore$ Curve C has no stationary points.	
(ii)	Since tangent is parallel to the y-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined $\therefore 18y+6-\mathrm{e}^{x+y}=0$ $6(3y+1)=\mathrm{e}^{x+y}$	
2(i)	$\mathbf{a}\bullet\mathbf{b}=\begin{pmatrix}\cos t\\ -\sin t\\ 0.5\end{pmatrix}\cdot\begin{pmatrix}\cos 2t\\ \sin 2t\\ -1\end{pmatrix}=\cos t\cos 2t-\sin t\sin 2t-\frac{1}{2}$ $=\cos(t+2t)-\frac{1}{2}=\cos 3t-\frac{1}{2}$ $p=-\frac{1}{2}, q=3$	
(ii)	$\lvert \underset{\sim}{a}\rvert\lvert \underset{\sim}{b}\rvert\cos\angle AOB=\cos(3t)-\frac{1}{2}$ For maximum $\angle \boldsymbol{AOB}$, since $0\le\angle \boldsymbol{AOB}\le\pi$ and $\cos\theta$ is a decreasing function over $[0,\pi]$, we aim to minimize $\cos\angle \boldsymbol{AOB}$, ie. To minimize $\cos 3\boldsymbol{t}-\frac{1}{2}$. Thus, $\cos 3t=-1\Rightarrow 3t=\pi$ (since $0\le t<\pi$) ie. $t=\frac{\pi}{3}$.	

3(i)	$f(x) = \dfrac{1}{x^2}$ and $4\,f(x-1) = \dfrac{4}{(x-1)^2}$ Translate $y = \dfrac{1}{x^2}$ by 1 unit in the direction of x-axis followed by a scaling of a factor of 4 parallel to y-axis (order can be reversed)	
(ii)		Note that the end-points of two functions do not meet. Remember to draw the given curve first, then repeat.
4(i)	The other roots is $1 - 2i$	
(ii)	Since the polynomial has <u>all real coefficients</u>, complex roots must occur in <u>conjugate pair</u>. The polynomial is of <u>degree 3</u> so there must be a pair of complex conjugate roots and <u>one real root</u>.	
(iii)	$z^3 + az^2 + bz + 15 = 0$ $(1+2i)^3 + a(1+2i)^2 + b(1+2i) + 15 = 0$ $-11 - 2i + a(-3+4i) + b(1+2i) + 15 = 0$ $(4 - 3a + b) + i(-2 + 4a + 2b) = 0$ Equating real and imaginary parts, $4 - 3a + b = 0$ or $-2 + 4a + 2b = 0$ Solving the equations, $a = 1,\ b = -1$ $z^3 + z^2 - z + 15 = (z^2 - 2z + 5)(z+3) = 0$ $\alpha = -3$ <u>Alternative method:</u> Let the real root be α. The factors are $(z - 1 - 2i)(z - 1 + 2i)(z - \alpha) = [(z-1)^2 + 4](z - \alpha)$ $= (z^2 - 2z + 5)(z - \alpha) = z^3 + az^2 + bz + 15 = 0$ Compare constant term: $5\alpha = -15$, $\alpha = -3$ Compare coefficient of z^2: $a = 3 - 2 = 1$ Compare coefficient of z: $b = 5 - 6 = -1$	

5(i)	Let $y=\dfrac{3x-1}{3x-3}$ $y(3x-3)=3x-1$ $3x(y-1)=3y-1$ $x=\dfrac{3y-1}{3y-3}$ $f^{-1}(x)=\dfrac{3x-1}{3x-3},\ x<1$	
(ii)	$f^2(x)=ff^{-1}(x)=x \ \Rightarrow f^3(x)=f(x)$, so odd power gives f($x$) $f^{2017}(0)=f(0)$ $=\dfrac{3(0)-1}{3(0)-3}=\dfrac{1}{3}$	
(iii)	Since $R_g=[0,1)\subseteq(-\infty,1)=D_f$, fg exists. $fg(x)=\dfrac{(3\sqrt{x-2})-1}{(3\sqrt{x-2})-3}$ $D_{fg}=D_g=[2,3)$ $R_{fg}=\left(-\infty,\dfrac{1}{3}\right]$	
6(i)	$f(x)=\dfrac{x+3}{(1-x)^n}$ $=(x+3)(1-x)^{-n}$ $=(x+3)\left(1+(-n)(-x)+\dfrac{(-n)(-n-1)}{2!}(-x)^2+\dots\right)$ $=(x+3)\left(1+nx+\left(\dfrac{n(n+1)}{2}\right)x^2+\dots\right)$ $=x+nx^2+3+3nx+\left(\dfrac{3n(n+1)}{2}\right)x^2+\dots$ $=3+(3n+1)x+\left(n+\dfrac{3n(n+1)}{2}\right)x^2+\dots$	

(ii)	$n+\dfrac{3n(n+1)}{2}=21$ $3n^2+5n-42=0$ $n=-\dfrac{14}{3}\left(\text{reject as } n\in\mathbb{Z}^+\right)$ or $n=3$ $\therefore n=3$	
(iii)	When $n=2$, $\dfrac{x+3}{(1-x)^2}=3+7x+11x^2+\ldots$ $\displaystyle\sum_{r=1}^{\infty}\frac{4r-1}{4^{r-1}}=3+\frac{7}{4}+\frac{11}{4^2}+\ldots=3+7\left(\frac{1}{4}\right)+11\left(\frac{1}{4}\right)^2+\ldots$ By substituting $x=\dfrac{1}{4}$ into $\dfrac{x+3}{(1-x)^2}$ in part **(i)**, $\displaystyle\sum_{r=1}^{\infty}\frac{4r-1}{4^{r-1}}=\frac{\frac{1}{4}+3}{\left(1-\frac{1}{4}\right)^2}=\frac{52}{9}$	
7(i)	$(3+y)^2+x^2=\left(3\sqrt{3}\right)^2$ $9+6y+y^2+x^2=27$ $x^2=18-6y-y^2$ Area of the sheet, $A=2xy$ $=2y\sqrt{18-6y-y^2}$ $=2\sqrt{18y^2-6y^3-y^4}$  	
(ii)	$\dfrac{dA}{dy}=2\dfrac{1}{2\sqrt{18y^2-6y^3-y^4}}\left(36y-18y^2-4y^3\right)$ $=\dfrac{2y\left(18-9y-2y^2\right)}{\sqrt{18y^2-6y^3-y^4}}$ [OR $A^2=4\left(18y^2-6y^3-y^4\right)$ $2A\dfrac{dA}{dy}=4\left(36y-18y^2-4y^3\right)$ $A\dfrac{dA}{dy}=4y\left(18-9y-2y^2\right)$]	

For maximum A, $\quad \dfrac{\mathrm{d}A}{\mathrm{d}y}=0$

$4y\left(18-9y-2y^2\right)=0$

$2y^2+9y-18=0$ or $y=0$ (reject as $y>0$)

$(2y-3)(y+6)=0$

$y=\dfrac{3}{2}$ or $y=-6$ (reject as $y>0$)

y	$\left(\frac{3}{2}\right)^-$	$\left(\frac{3}{2}\right)$	$\left(\frac{3}{2}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}y}$	>0	0	<0

A is maximum when $y=\dfrac{3}{2}$

When $y=\dfrac{3}{2}$,

Maximum $A == 2\sqrt{18\left(\frac{3}{2}\right)^2-6\left(\frac{3}{2}\right)^3-\left(\frac{3}{2}\right)^4} = \dfrac{9\sqrt{3}}{2}$ =7.79 m^2

8(i)

$$\left[\begin{pmatrix}4\\1\\1\end{pmatrix}-\begin{pmatrix}2\\1\\0\end{pmatrix}\right]\times\begin{pmatrix}4\\-1\\2\end{pmatrix}=\begin{pmatrix}2\\0\\1\end{pmatrix}\times\begin{pmatrix}4\\-1\\2\end{pmatrix}=\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

Hence a vector perpendicular to the plane p_1 is parallel to $\mathbf{i}-2\mathbf{k}$.

Equation of the plane p_1 : $\mathbf{r}\bullet\begin{pmatrix}1\\0\\-2\end{pmatrix}=\begin{pmatrix}4\\1\\1\end{pmatrix}\bullet\begin{pmatrix}1\\0\\-2\end{pmatrix}=2$

(ii)

Equation of l : $\mathbf{r}=\begin{pmatrix}2\\-1\\-4\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$

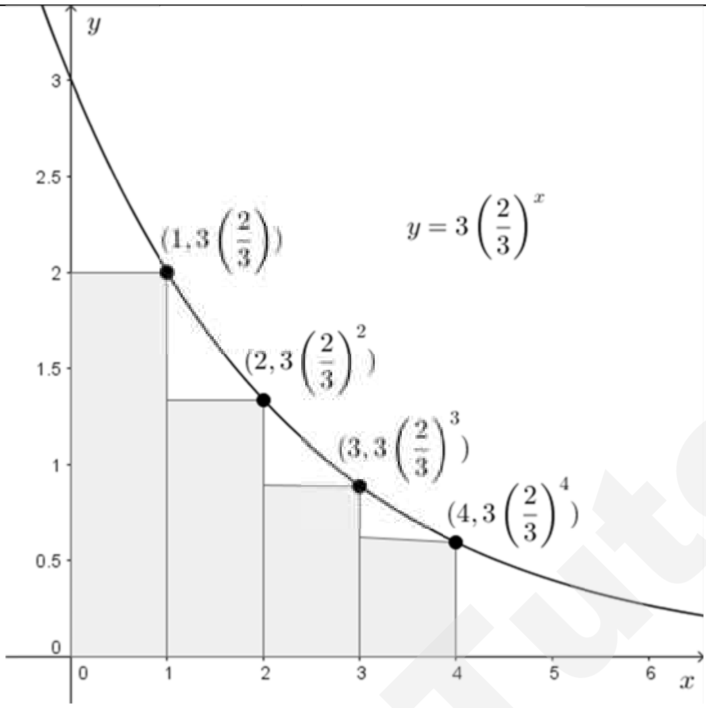
At C, $\begin{pmatrix}2-2\lambda\\-1+\lambda\\-4+\lambda\end{pmatrix}\bullet\begin{pmatrix}1\\0\\-2\end{pmatrix}=2$

$2-2\lambda+8-2\lambda=2$

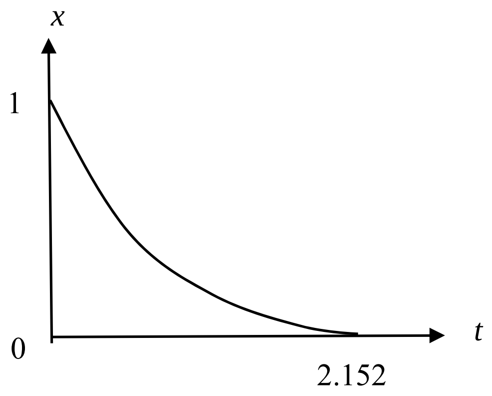
$\Rightarrow \lambda=2$

	$\therefore \overrightarrow{OC} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$ Coordinates $C(-2,1,-2)$	
(iii)	Let F be the foot of perpendicular from D to the plane p_1 . Equation of FD: $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $\beta \in \mathbb{R}$ At F , $\begin{pmatrix} 2+\beta \\ -1 \\ -4-2\beta \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 2$ $2+\beta+8+4\beta = 2$ $\Rightarrow \beta = -\frac{8}{5}$ $\therefore \overrightarrow{OF} = \begin{pmatrix} 2-\frac{8}{5} \\ -1 \\ -4-2(-\frac{8}{5}) \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ -1 \\ -\frac{4}{5} \end{pmatrix}$ Let E be the image of D under a reflection in the plane p_1 . $\therefore \overrightarrow{OE} = 2\begin{pmatrix} \frac{2}{5} \\ -1 \\ -\frac{4}{5} \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ -1 \\ \frac{12}{5} \end{pmatrix}$ Coordinates $E\left(-\frac{6}{5},-1,\frac{12}{5}\right)$	
(iv)	The line of intersection of the planes p_1 and p_2 is AC, $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -3 \end{pmatrix} = -3\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ Equation of line m : $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$	

9(i)	Let P_n be the statement $u_n = 3\left(\frac{2}{3}\right)^n - 1$ for all $n \in \mathbb{Z}^+$. When $n = 1$, LHS = $u_1 = 1$ RHS = $3\left(\frac{2}{3}\right)^1 - 1 = 1$ LHS = RHS $\therefore P_1$ is true. Assume P_k is true for some $k \in \mathbb{Z}^+$. i.e. $u_k = 3\left(\frac{2}{3}\right)^k - 1$ Prove that P_{k+1} is true. i.e. $u_{k+1} = 3\left(\frac{2}{3}\right)^{k+1} - 1$ $u_{k+1} = \frac{2u_k - 1}{3}$ $= \frac{2\left(3\left(\frac{2}{3}\right)^k - 1\right) - 1}{3}$ $= 2\left(\frac{2}{3}\right)^k - 1$ $= 3\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^k - 1$ $= 3\left(\frac{2}{3}\right)^{k+1} - 1$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true. Since P_1 is true, P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.	
(ii)	$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}\left[3\left(\frac{2}{3}\right)^r - 1\right]$ $= 3\sum_{r=1}^{n}\left(\frac{2}{3}\right)^r - n$ $= 3\left(\frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}}\right) - n$ $= 6\left(1-\left(\frac{2}{3}\right)^n\right) - n$	

(iii)	As $n \to \infty, \left(\frac{2}{3}\right)^n \to 0. \quad \therefore \sum_{r=1}^{n}\left(u_r+1\right) \to 6.$ The sum to infinity is 6 OR $S_\infty = 6$.	
(iv)	 $\sum_{r=1}^{\infty}\left(u_r+1\right) = \sum_{r=1}^{\infty} 3\left(\frac{2}{3}\right)^r = 3\left(\frac{2}{3}\right)^1 + 3\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + \ldots$ The value of $\sum_{r=1}^{\infty}\left(u_r+1\right)$ can be obtained by adding up the areas of the rectangles of width 1 as seen in the sketch. Since the total area of all the rectangles is lesser than the total area under the curve for $x \geq 0$, $\sum_{r=1}^{\infty}\left(u_r+1\right)$ is an underestimation of $\int_0^\infty 3\left(\frac{2}{3}\right)^x \mathrm{d}x$.	
10(i)	$\frac{\mathrm{d}x}{\mathrm{d}t} = k(4+2x-x^2)$ when $t=0$, $x=1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{2}$, $-\frac{1}{2} = k(4+2-1)$ $k = -\frac{1}{10}$	
(ii)	$4+2x-x^2 = 4-(x^2-2x)$ $= 4-[(x-1)^2-1]$ $= 5-(x-1)^2$	

	$\frac{dx}{dt} = k(4+2x-x^2)$ $\int \frac{1}{4+2x-x^2}\,dx = \int k\,dt$ $\int \frac{1}{5-(x-1)^2}\,dx = \int -\frac{1}{10}\,dt$ $\frac{1}{2(\sqrt{5})}\ln\left\|\frac{\sqrt{5}+(x-1)}{\sqrt{5}-(x-1)}\right\| = -\frac{1}{10}t+c$ When $t=0$, $x=1$ $c=\ln 1=0$ $t=\frac{-10}{2(\sqrt{5})}\ln\left\|\frac{\sqrt{5}+(x-1)}{\sqrt{5}-(x-1)}\right\|$ $t=\frac{5}{\sqrt{5}}\ln\left\|\frac{\sqrt{5}+(x-1)}{\sqrt{5}-(x-1)}\right\|^{-1}$ $t=\sqrt{5}\ln\left\|\frac{\sqrt{5}-(x-1)}{\sqrt{5}+(x-1)}\right\|$ $t=\sqrt{5}\ln\left\|\frac{\sqrt{5}-x+1}{\sqrt{5}+x-1}\right\|$	
(iii)	When $x=0$, $t=\sqrt{5}\ln\left\|\frac{\sqrt{5}+1}{\sqrt{5}-1}\right\|$ Time taken is 2.152 minutes	
(iv)	$t=\sqrt{5}\ln\left\|\frac{\sqrt{5}-x+1}{\sqrt{5}+x-1}\right\|$ $\ln\left\|\frac{\sqrt{5}-x+1}{\sqrt{5}+x-1}\right\| = \frac{t}{\sqrt{5}}$ $\frac{\sqrt{5}-x+1}{\sqrt{5}+x-1} = e^{\frac{t}{\sqrt{5}}}$ $(\sqrt{5}+x-1) = e^{-\frac{t}{\sqrt{5}}}(\sqrt{5}-x+1)$ $e^{-\frac{t}{\sqrt{5}}}x + x = e^{-\frac{t}{\sqrt{5}}}(\sqrt{5}+1)-\sqrt{5}+1$	

	$x=\dfrac{\mathrm{e}^{-\frac{t}{\sqrt{5}}}(\sqrt{5}+1)-\sqrt{5}+1}{1+\mathrm{e}^{-\frac{t}{\sqrt{5}}}}$ 	
11 (i)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1-x^4}\right)=\dfrac{-2x^3}{\sqrt{1-x^4}}$	
(ii)	Area of region $=\int_0^{\frac{1}{\sqrt[4]{2}}} x\left(\cos^{-1}\left(x^2\right)-\sin^{-1}\left(x^2\right)\right)\mathrm{d}x$ $=\left[\left(\dfrac{x^2}{2}\right)\left(\cos^{-1}\left(x^2\right)-\sin^{-1}\left(x^2\right)\right)\right]_0^{\frac{1}{\sqrt[4]{2}}}-\int_0^{\frac{1}{\sqrt[4]{2}}}\left(\dfrac{x^2}{2}\right)\left(\dfrac{-4x}{\sqrt{1-x^4}}\right)\mathrm{d}x$ $=\dfrac{1}{2\sqrt{2}}\left(\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right)-\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)\right)-\int_0^{\frac{1}{\sqrt[4]{2}}}\dfrac{-2x^3}{\sqrt{1-x^4}}\mathrm{d}x$ $=\dfrac{1}{2\sqrt{2}}\left(\dfrac{\pi}{4}-\dfrac{\pi}{4}\right)-\left[\sqrt{1-x^4}\right]_0^{\frac{1}{\sqrt[4]{2}}}$ $=1-\dfrac{\sqrt{2}}{2}$	
(iii)	Volume of solid formed $=\pi\int_0^{\frac{\pi}{4}\left(\frac{1}{\sqrt[4]{2}}\right)} x^2\,\mathrm{d}y$ $=\pi\int_0^{\frac{1}{\sqrt[4]{2}}} t^2\left[\dfrac{2t^2}{\sqrt{1-t^4}}+\sin^{-1}\left(t^2\right)\right]\mathrm{d}t$ $=\pi\int_0^{\frac{1}{\sqrt[4]{2}}}\left[\dfrac{2t^4}{\sqrt{1-t^4}}+t^2\sin^{-1}\left(t^2\right)\right]\mathrm{d}t$	

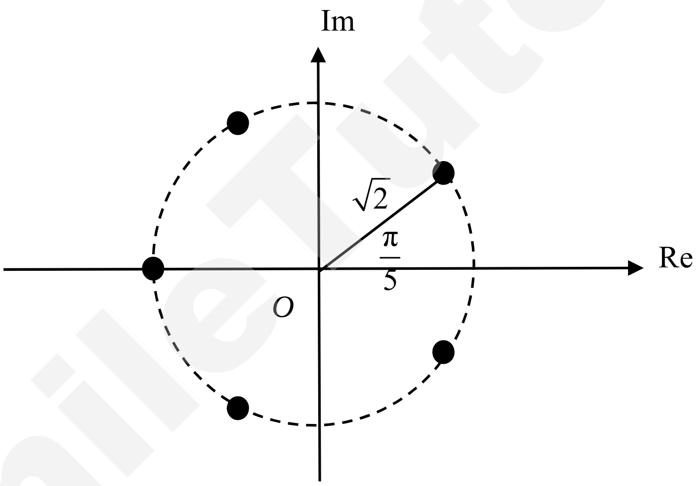
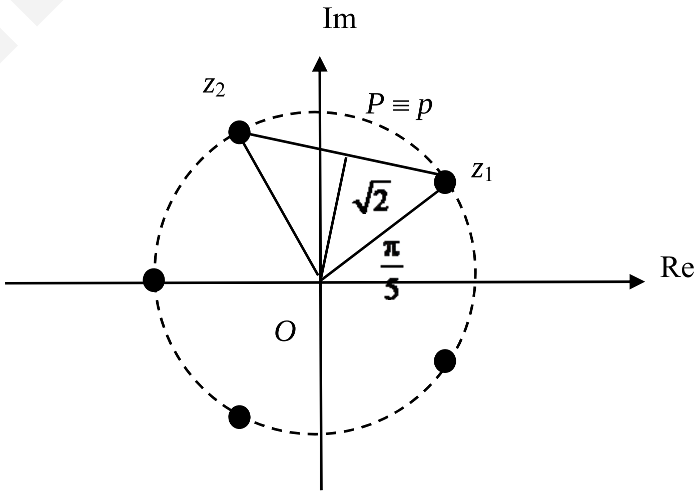
(iv)	Required volume $= \pi \int_0^{\frac{1}{\sqrt[4]{2}}} \left[\frac{2t^4}{\sqrt{1-t^4}} + t^2 \sin^{-1}\left(t^2\right) \right] \mathrm{d}t - \frac{1}{3}\pi \left(\frac{1}{\sqrt[4]{2}} \right)^2 \left(\frac{\pi}{4} \left(\frac{1}{\sqrt[4]{2}} \right) \right)$ $= 0.909285 - 0.489042$ $= 0.42024 \ (5 \text{ sig. fig.})$	

Tampines Junior College
2016 JC2 Preliminary Examination
H2 Mathematics Paper 2
Solution

1(i)	$x = 2\cos t, \quad y = 3\sin t$ $\frac{dx}{dt} = -2\sin t \quad \frac{dy}{dt} = 3\cos t$ $\frac{dy}{dx} = \frac{3\cos t}{-2\sin t}$ $= -\frac{3}{2}\cot t$ Gradient of normal to C at $P(2\cos\theta, 3\sin\theta)$ is $\frac{2}{3}\tan\theta$. Equation of normal to C at P is $y - 3\sin\theta = \left(\frac{2}{3}\tan\theta\right)(x - 2\cos\theta)$ $y = \left(\frac{2}{3}\tan\theta\right)(x - 2\cos\theta) + 3\sin\theta$ $y = \left(\frac{2}{3}\tan\theta\right)x + \frac{5}{3}\sin\theta$	
(ii)	When $y = 0$, $-3\sin\theta = \left(\frac{2}{3}\frac{\sin\theta}{\cos\theta}\right)(x - 2\cos\theta)$ $x = -\frac{9}{2}\cos\theta + 2\cos\theta$ $= -\frac{5}{2}\cos\theta$ $A\left(-\frac{5}{2}\cos\theta, 0\right)$ When $x = 0$, $y = \left(\frac{2}{3}\frac{\sin\theta}{\cos\theta}\right)(-2\cos\theta) + 3\sin\theta$ $= \frac{5}{3}\sin\theta$ $B\left(0, \frac{5}{3}\sin\theta\right)$ Midpoint of $AB = \left(-\frac{5}{4}\cos\theta, \frac{5}{6}\sin\theta\right)$. $x = -\frac{5}{4}\cos\theta, \quad y = \frac{5}{6}\sin\theta$ $\sin^2\theta + \cos^2\theta = 1$	

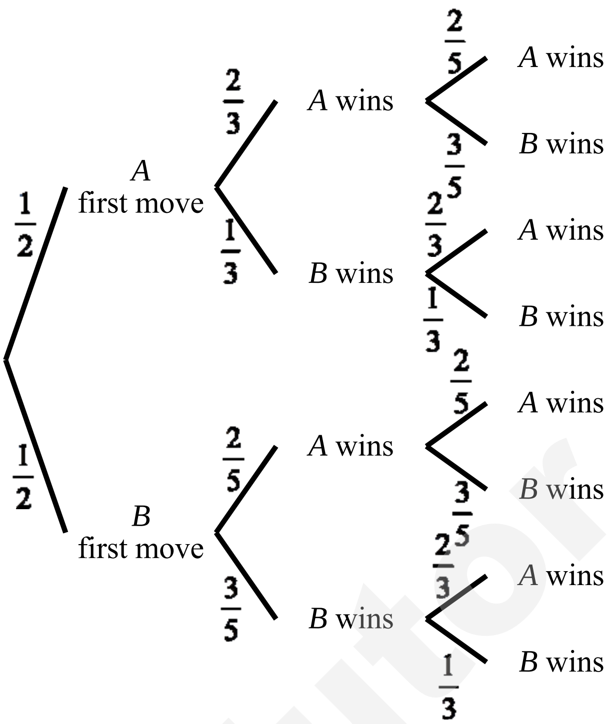
	$\left(\frac{6}{5}y\right)^2 + \left(\frac{4}{5}x\right)^2 = 1$ $16x^2 + 36y^2 = 25$	Note that no restriction of θ given in question
2	Let $\frac{15x^2 - x + 17}{(2x+1)(x^2+4)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+4}$ $15x^2 - x + 17 = A(x^2+4) + (Bx+C)(2x+1)$ Sub $x = -\frac{1}{2}$, $A = 5$ Sub $x = 0$, $17 = 4A + C$ $C = -3$ Sub $x = 1$, $31 = 25 + 3(B-3)$ $B = 5$ $\frac{15x^2 - x + 17}{(2x+1)(x^2+4)} = \frac{5}{2x+1} + \frac{5x-3}{x^2+4}$ $\int_0^1 \frac{15x^2 - x + 17}{(2x+1)(x^2+4)}\,\mathrm{d}x$ $= \int_0^1 \frac{5}{2x+1}\,\mathrm{d}x + 5\int_0^1 \frac{x}{x^2+4}\,\mathrm{d}x - 3\int_0^1 \frac{1}{x^2+4}\,\mathrm{d}x$ $= \frac{5}{2}\int_0^1 \frac{2}{2x+1}\,\mathrm{d}x + \frac{5}{2}\int_0^1 \frac{2x}{x^2+4}\,\mathrm{d}x - 3\int_0^1 \frac{1}{x^2+2^2}\,\mathrm{d}x$ $= \left[\frac{5}{2}\ln\lvert 2x+1\rvert + \frac{5}{2}\ln\lvert x^2+4\rvert - \frac{3}{2}\tan^{-1}(\frac{x}{2})\right]_0^1$ $= \frac{5}{2}\ln 3 + \frac{5}{2}\ln 5 - \frac{3}{2}\tan^{-1}\left(\frac{1}{2}\right) - \frac{5}{2}\ln 1 - \frac{5}{2}\ln 4$ $= \frac{5}{2}\ln\left(\frac{15}{4}\right) - \frac{3}{2}\tan^{-1}\left(\frac{1}{2}\right)$	
3(i)	Let r be the common ratio. Let n be the number of days required. $r = 0.975$ Time spent on 10th day $= (270)(0.975)^{10-1}$ $= 214.98$ ≈ 215 minutes	

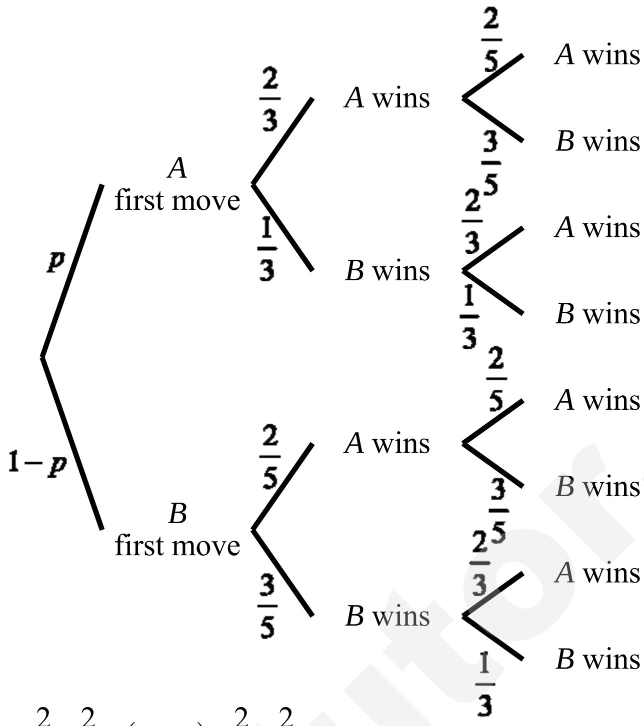
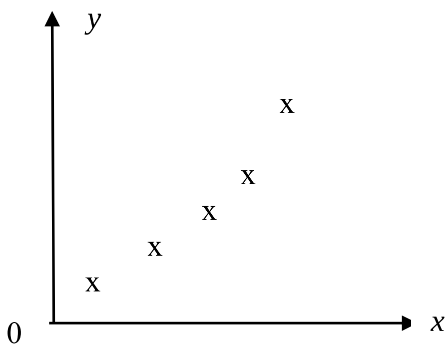
	$\dfrac{270\left(1-0.975^n\right)}{1-0.975} \geq 9000$ $\Rightarrow 10800\left(1-0.975^n\right) \geq 9000$ $\Rightarrow 1-0.975^n \geq \dfrac{5}{6}$ $\Rightarrow 0.975^n \leq \dfrac{1}{6}$ $\Rightarrow n \geq \dfrac{\ln\frac{1}{6}}{\ln 0.975} = 70.771$ Hence, $n = 71$	
(ii)	Theoretical maximum total time $= \dfrac{270}{1-0.975} = 10800$ Maximum number of gifts $= \dfrac{10800}{20} = 540$	
(iii)	Total number of minutes $= \dfrac{n}{2}\left(2(270)+(n-1)(-4)\right) \geq 9000$ $\Rightarrow 2n^2 - 272n + 9000 \leq 0$ From GC, $56.864 \leq n \leq 79.136$. Hence, $n = 57$ (time spent each day on task must be positive)	
4(a) (i)	$\lvert w \rvert = \sqrt{3^2 + (3\sqrt{3})^2} = 6$ $\arg(w) = \tan^{-1}\left(\dfrac{3\sqrt{3}}{3}\right) = \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$	
(ii)	$\left(\dfrac{w^3}{w^*}\right)^n = \left(\dfrac{\left(6e^{i\frac{\pi}{3}}\right)^3}{6e^{-i\frac{\pi}{3}}}\right)^n = \left(\dfrac{216e^{i\pi}}{6e^{-i\frac{\pi}{3}}}\right)^n = 36^n e^{i\frac{4\pi n}{3}}$ Since $\left(\dfrac{w^3}{w^*}\right)^n$ is real, $\sin\left(\dfrac{4\pi n}{3}\right) = 0$. Hence, $\dfrac{4\pi n}{3} = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$ $n = 0, \dfrac{3}{4}, \dfrac{3}{2}, \dfrac{9}{4}, 3, \ldots$ Smallest positive integer value of n is 3.	Alternatively, $\left(\dfrac{w^3}{w^*}\right)^n$ is real, means lying on real axis, so $\dfrac{4\pi n}{3} = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$

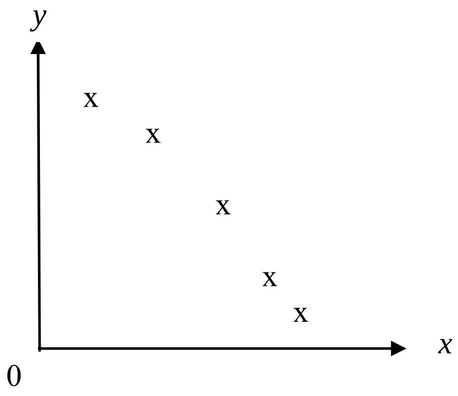
(b) (i)	$z^5 = -4\sqrt{2}$ $= 4\sqrt{2}e^{i\pi}$ $z = \left(4\sqrt{2}e^{i(\pi+2k\pi)}\right)^{1/5}, \quad k = -2, -1, 0, 1, 2$ $= \sqrt{2}e^{i\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right)}$ When $k = 0, \quad z_1 = \sqrt{2}e^{i\left(\frac{\pi}{5}\right)}$ When $k = 1, \quad z_2 = \sqrt{2}e^{i\left(\frac{3\pi}{5}\right)}$ When $k = 2, \quad z_3 = \sqrt{2}e^{i\pi}$ When $k = -2, \quad z_4 = \sqrt{2}e^{i\left(-\frac{3\pi}{5}\right)}$ When $k = -1, \quad z_5 = \sqrt{2}e^{i\left(-\frac{\pi}{5}\right)}$	
(ii)		
(iii)		

5(i)	A random sample is obtained by selecting 100 students from the population of university students such that every student has an <u>equal chance of being selected</u> and the selection of students is made <u>independently</u>.	
(ii)	A more appropriate sampling method is stratified sampling. Divide the students into <u>mutually exclusive strata (groups) such as the faculties the students belong to</u> and then <u>randomly select the students with the sample size proportional to the relative size of each stratum</u> to form the sample of size 100. The students to be included in the sample can then be selected from each stratum using random sampling or systematic sampling.	
6(i)	Let X be the number of employees who own a cell phone out of 8 employees. Then $X \sim B(8, 0.75)$ $P(4 \le X \le 6) = P(X \le 6) - P(X \le 3)$ $= 0.60562$ ≈ 0.606 (3 s.f.)	
(ii)	Let Y be the number of employees who own a cell phone out of 160 employees Then $Y \sim B(160, 0.75)$ Since $n = 160 > 50$ is large, $np = 160(0.75) = 120 > 5$, $n(1-p) = 160(0.25) = 40 > 5$, $np(1-p) = 30$, $Y \sim N(120, 30)$ approximately $P(Y \ge 115) = P(Y > 114.5)$ (c.c.) $= 0.84235$ ≈ 0.842 (3 s.f.)	
(iii)	Let W be the number of factories with at least 115 employees who own a cell phone Then $W \sim B(15, 0.84235)$ $P(W \le 11) = 0.202$ (3 s.f.)	
7(i)	The possible assumptions are: (a) The average / mean number of parking tickets issued per day is constant. (b) The parking tickets are issued independently / randomly.	
(ii)	Let Y be the number of parking tickets issued in seven days. Then $Y \sim Po(3.6 \times 7)$ i.e. $Y \sim Po(25.2)$ $P(Y > 22) = 1 - P(Y \le 22)$ $= 0.69634 \approx 0.696$ (3 s.f.)	

(iii)	Let W be the number of parking tickets issued in 10 days. Then W ~ Po (36) Since $\lambda = 36 > 10$ is large , W ~ N(36, 36) approximately $P(W > N) < 0.05 \Rightarrow P(W > N + 0.5) < 0.05$ (using c.c) Using G.C., When $N = 45$, $P(W > N + 0.5) = 0.05667 > 0.05$ When $N = 46$, $P(W > N + 0.5) = 0.04006 < 0.05$ Hence, the least possible value of N is 46 <u>Alternative solution</u> $P(W < N + 0.5) > 0.95$ $\frac{N+0.5-36}{6} > 1.64485$ $N > 45.369$ Hence, the least possible value of N is 46	
8(i)	Unbiased estimate of population mean: $\bar{x} = \frac{\sum x}{n} = \frac{8.84}{8} = 1.105$ Unbiased estimate of population variance: $s^2 = \frac{1}{7}\left(\sum x^2 - \frac{(\sum x)^2}{8}\right) = \frac{1}{7}\left(9.95 - \frac{8.84^2}{8}\right) = 0.02597$	
(ii)	Let X be the mass of papayas To test $H_0: \mu = 1.2$ against $H_1: \mu < 1.2$ at 5% level of significance Since $n = 8$ is small, population variance is unknown, Use t-test. Assume that mass of papayas follows a normal distribution. Test Statistics: $T = \frac{\bar{X} - 1.2}{\sqrt{\frac{s^2}{8}}} \sim t(7)$ under H_0 $t_{\text{test}} = \frac{1.105 - 1.2}{\sqrt{\frac{0.02597}{8}}} = -1.67$ Using t-test, by GC, p-value = 0.0697 Since p-value = 0.0697 > 0.05, we do not reject H_0 and conclude that there is insufficient evidence, at the 5% significance level, to reject the stall owner's claim.	

9(i)	 $\mathrm{P}(\text{games end in a draw}) = \frac{1}{2}\left(\frac{2}{3}\times\frac{3}{5}+\frac{1}{3}\times\frac{2}{3}+\frac{2}{5}\times\frac{3}{5}+\frac{3}{5}\times\frac{2}{3}\right)$ $= \frac{142}{225}$	
(ii)	$\mathrm{P}(A \text{ wins the second game and games end in a draw})$ $= \frac{1}{2}\left(\frac{1}{3}\times\frac{2}{3}+\frac{3}{5}\times\frac{2}{3}\right)$ $= \frac{14}{45}$ $\mathrm{P}(A \text{ wins the second game}) = \frac{1}{2}\left(\frac{2}{3}\times\frac{2}{5}+\frac{1}{3}\times\frac{2}{3}+\frac{2}{5}\times\frac{2}{5}+\frac{3}{5}\times\frac{2}{3}\right)$ $= \frac{118}{225}$ $\mathrm{P}(\text{games end in a draw} \mid A \text{ wins the second game})$ $= \frac{\mathrm{P}(A \text{ wins the second game and games end in a draw})}{\mathrm{P}(A \text{ wins the second game})}$ $= \frac{\frac{14}{45}}{\frac{118}{225}}$ $= \frac{35}{59}$	

(iii)	 $\mathrm{P}(A \text{ wins}) = p\times\frac{2}{3}\times\frac{2}{5}+(1-p)\times\frac{2}{5}\times\frac{2}{5}$ $= \frac{8}{75}p+\frac{4}{25}$ $\mathrm{P}(B \text{ wins}) = p\times\frac{1}{3}\times\frac{1}{3}+(1-p)\times\frac{3}{5}\times\frac{1}{3}$ $= -\frac{4}{45}p+\frac{1}{5}$ $\mathrm{P}(A \text{ wins}) = \mathrm{P}(B \text{ wins})$ $\Rightarrow \frac{8}{75}p+\frac{4}{25} = -\frac{4}{45}p+\frac{1}{5}$ $\Rightarrow p = \frac{9}{44}$	
10(a) (i)	 	

(ii)		
(b)(i)	(A) r = – 0.9492 (B) r = – 0.9916	
(ii)	$P = c\ln m + d$ is a better model since r is closer to −1. As m increases, $P = c\ln m + d$ decreases but cannot be less than zero. / P decreases at a decreasing rate. Equation of regression line is $P = -31800\ln m + 182000$	
(iii)	When $m = 80$, $P = -31825.69267\ln 80 + 181762.4239$ $P = 42300$ Estimate is not reliable. $m = 80$ is outside data range. Extrapolation is not a good practice.	
11(i) (a)	Number of different possible orders $= 12! = 479001600$	
(b)	Number of different possible orders $= 7! \times {}^{8}C_{5}$ $= 282240$	
(ii)	Number of different possible arrangements $= {}^{12}C_{6} \times \dfrac{6!}{6} \times 6!$ $= 79833600$	
(iii)	<u>Case 1: No girls in group of 3</u> Number of ways $= {}^{7}C_{3} \times {}^{9}C_{4} \times {}^{5}C_{5} = 4410$ <u>Case 2: No girls in group of 4</u> Number of ways $= {}^{7}C_{4} \times {}^{8}C_{3} \times {}^{5}C_{5} = 1960$ <u>Case 3: No girls in group of 5</u> Number of ways $= {}^{7}C_{5} \times {}^{7}C_{3} \times {}^{4}C_{4} = 735$ <u>Case 4: No girls in group of 3 and group of 4</u> Number of ways $= {}^{7}C_{3} \times {}^{4}C_{4} \times {}^{5}C_{5} = 35$ Total number of ways $= 4410 + 1960 + 735 - 35 = 7070$	

	Alternatively, we can use the complement method (not recommended) (There are 6 cases for every group to have at least one girl.) Number of ways $= {}^{12}C_3 \times {}^{9}C_4 \times {}^{5}C_5 - {}^{5}C_2 \times {}^{7}C_1 \times {}^{3}C_2 \times {}^{6}C_2 \times {}^{1}C_1 \times {}^{4}C_4$ $- {}^{5}C_2 \times {}^{7}C_1 \times {}^{3}C_1 \times {}^{6}C_3 \times {}^{2}C_2 \times {}^{3}C_3 - {}^{5}C_1 \times {}^{7}C_2 \times {}^{4}C_2 \times {}^{5}C_2 \times {}^{2}C_2 \times {}^{4}C_4$ $- {}^{5}C_3 \times {}^{2}C_1 \times {}^{7}C_3 \times {}^{1}C_1 \times {}^{4}C_4 - {}^{5}C_1 \times {}^{7}C_2 \times {}^{4}C_1 \times {}^{5}C_3 \times {}^{3}C_3 \times {}^{2}C_2$ $- {}^{5}C_1 \times {}^{7}C_2 \times {}^{4}C_3 \times {}^{5}C_1 \times {}^{1}C_1 \times {}^{4}C_4$ $= 7070$	
12(i)	Let X be the mass of guava of type A Let Y be the mass of guava of type B $X \sim \text{N}(200,\ 12^2)$ and $Y \sim \text{N}(175,\ 12^2)$ Let T be the total mass of 4 guava of type A. $T \sim \text{N}(4\times 200, 4\times 12^2)$ $T \sim \text{N}(800, 576)$ $\text{P}(T > 810) = 0.33846 \approx 0.338$	
(ii)	$\overline{X} \sim \text{N}(200,36)$, $\overline{Y} \sim \text{N}(175,48)$, $\overline{X} - \overline{Y} \sim \text{N}(25,84)$ $\text{P}\left(\left\lvert\overline{X}-\overline{Y}\right\rvert \geq 30\right) = 1 - \text{P}\left(\left\lvert\overline{X}-\overline{Y}\right\rvert \leq 30\right)$ $= 0.29269$ $= 0.293$	
(iii)	Let V be the total mass of m guava of type A Let W be the total mass of $(20-m)$ guava of type B $V \sim \text{N}(200m, 144m)$, $W \sim \text{N}(175(20-m), 144(20-m))$ $V + W \sim \text{N}(25m + 3500, 2880)$ $\text{P}(V+W > 3500) > 0.95$ $\Rightarrow \text{P}\left(Z > \frac{-25m}{\sqrt{2880}}\right) > 0.95$ $\Rightarrow \text{P}\left(Z < \frac{25m}{\sqrt{2880}}\right) > 0.95$ $\Rightarrow \text{P}\left(Z < \frac{25m}{\sqrt{2880}}\right) > \text{P}(Z < 1.64485)$ $\frac{25m}{\sqrt{2880}} > 1.64485$ $m > 3.53$ Least value of $m = 4$	

VICTORIA JUNIOR COLLEGE
Preliminary Examination
Higher 2

MATHEMATICS **9740/01**
Paper 1
Friday **8am – 11am**

16 September 2016 **3 hours**

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

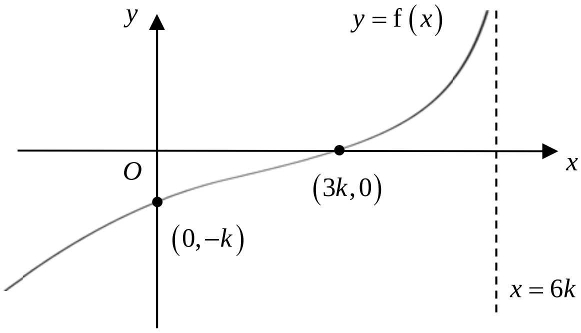
Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 6 printed pages

 VICTORIA JUNIOR COLLEGE **[Turn over**

1



The diagram shows the curve with equation $y = \mathrm{f}(x)$, $x < 6k$, $k > \frac{1}{3}$. The curve crosses the x-axis and y-axis at the points $(3k, 0)$ and $(0, -k)$ respectively. Sketch $y = \mathrm{f}(|x|+1)$. [3]

2 Indicate on a single Argand diagram, the set of points whose complex numbers satisfy the following inequalities

$$\left|\frac{z-6-5\mathrm{i}}{2}\right| \leq 4 \quad \text{and} \quad |2\mathrm{i}-4-z| \geq |z+4-10\mathrm{i}|.$$

Hence, find the least value and greatest value of $\arg(z-6+4\mathrm{i})$. [7]

3 **(a)** Without using a calculator, solve the inequality

$$\frac{4-7x}{x-3} \geq x.$$ [4]

(b) In 2016, Edwin, his father and his grandfather have an average age of 53. In the same year, the sum of one-half of his grandfather's age, one-third of his father's age and one-fourth of Edwin's age is 65. Twenty-two years ago, his grandfather's age was twice the sum of his father's age and his age. What are their respective ages in 2016? [You can assume that Edwin's age in 2016 is more than 22.] [3]

4 The function f is defined by $\mathrm{f}: x \mapsto (x-2)^2 + k, \quad x \leq 2$. It is given that f^{-1} exists.

(i) When $k = 1$,

(a) define f^{-1} in a similar form, [3]

(b) sketch, on a single diagram, the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$. [3]

(ii) State the set of values of k, such that the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ has no real solutions. [1]

5 A sequence $u_1,\ u_2,\ u_3, \ldots$ is such that $u_1 = 0$ and

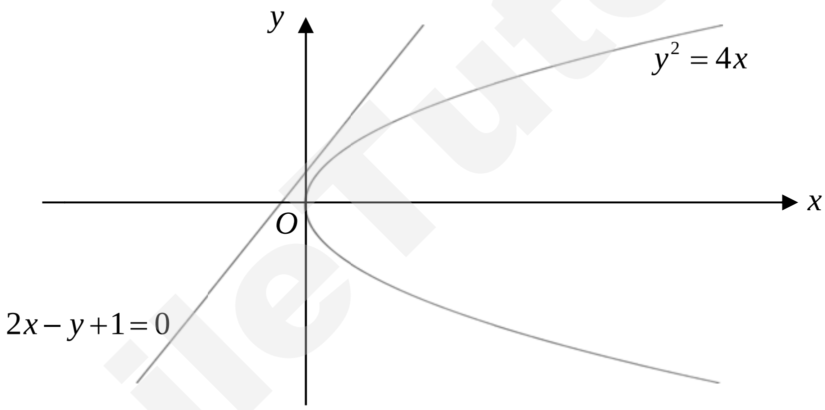
$$u_{n+1} = u_n + \frac{2-n^2}{(n+2)!}, \text{ for all } n \geq 1.$$

(i) Show that $u_2 = \frac{1}{6}$, and find the values of u_3 and u_4. [2]

(ii) Hence, give a conjecture for u_n in the form $\frac{n-1}{[\mathrm{f}(n)]!}$, where $\mathrm{f}(n)$ is a function of n to be determined. [1]

(iii) Use the method of mathematical induction to prove your conjecture in part **(ii)** for all positive integers n. [4]

6



A curve C has equation $y^2 = 4x$ and a line l has equation $2x - y + 1 = 0$. The diagram above shows the graphs of C and l.

$B(b, 2\sqrt{b})$ is a fixed point on C and A is an arbitrary point on l. State the geometrical relationship between the line segment AB and l if the distance from B to A is the least. [1]

Taking the coordinates of A as $(a, 2a+1)$, find an equation relating a and b for which AB is the least. [2]

Deduce that when AB is the least, $(AB)^2 = m\left(2b - 2\sqrt{b} + 1\right)^2$ where m is a constant to be found. Hence or otherwise, find the coordinates of the point on C that is nearest to l, as b varies. [5]

7 **(a)** Differentiate xe^{x^3} with respect to x. Hence, find $\int x^2\left(1+3x^3\right)e^{x^3}\,dx$. [4]

(b) The variables x and y are related by the differential equation

$$\frac{dy}{dx}=\frac{\sec^2 x}{2\sec^2 x+4\tan x+7}.$$

Using the substitution $u=\tan x$, find the general solution of the differential equation. [5]

8 **(i)** Use the method of differences to find, in terms of n,

$$\sum_{r=2}^{n}\ln\left[\frac{r(r+2)}{(r+1)^2}\right].$$ [4]

(ii) Give a reason why the series is convergent and state the sum to infinity. [2]

(iii) Given $\sum_{r=2}^{13}\ln\left[\frac{(2r)(2r+4)}{(r+1)^2}\right]=\ln\left(\frac{p}{q}\right)$, where p and q are integers and $\frac{p}{q}$ is in the simplest form, find the values of p and q. [3]

9 **(i)** Sketch the graph with equation $x^2+(y-r)^2=r^2$, where $r>0$ and $y\le r$. [2]

A hemispherical bowl of fixed radius r cm is filled with water. Water drains out from a hole at the bottom of the bowl at a constant rate.

Use your graph in part **(i)** to show that when the depth of water is h cm (where $h\le r$), the volume of water in the bowl is given by

$$V=\frac{\pi h^2}{3}(3r-h),$$ [3]

(ii) Given that a full bowl of water would become empty in 24 s, find the rate of decrease, in terms of r and h, of the depth of water in the bowl at the instant when the depth of water is h cm. [3]

(iii) Without any differentiation, determine, in terms of r, the slowest rate at which the depth of water is decreasing. [1]

10 The equations of planes p_1 and p_2 are

$$x-5y+2z=13,$$
$$-2x+y+5z=1,$$

respectively.

(i) Find the acute angle between p_1 and p_2. [2]

The planes p_1 and p_2 intersect in a line l.

(ii) Find a vector equation of l. [2]

The plane p_3 is perpendicular to both p_1 and p_2. The three planes p_1, p_2 and p_3 intersect at the point $(a,0,b)$, where a and b are constants.

(iii) Show that $a=7$ and $b=3$. [2]

The plane $\prod$ is parallel to p_3 and the distance between $\prod$ and p_3 is $4\sqrt{11}$ units.

(iv) Find the two possible cartesian equations of $\prod$. [4]

11 **(a)** An arithmetic progression which consists of $2n$ terms has first term a and common difference d. The third, fifth and twelfth terms of the arithmetic progression are also three distinct consecutive terms of a geometric progression. Find the sum of the even-numbered terms, i.e. the 2^{nd}, 4^{th}, … , $(2n)^{\text{th}}$ terms, of the arithmetic progression in terms of a and n. [5]

(b) To renovate his new HDB flat, Douglas is considering taking up a bank loan of $\$40,000$ from Citybank on 1^{st} July 2016. The bank charges a monthly interest of 0.5% on the outstanding amount owed at the end of each month.

Douglas will pay a fixed amount, $\$x$, to the bank at the beginning of each month, starting from September 2016.

(i) Taking July 2016 as the 1^{st} month, show that the amount of money owed at the beginning of the 5^{th} month is

$$1.005^4(40000)-200x\left(1.005^3-1\right).$$ [3]

(ii) If Douglas wishes to pay up his loan within 5 years, find the minimum amount of each monthly repayment. [2]

(iii) Using the value found in part **(ii)**, calculate the interest (to the nearest dollar) that Citybank has earned in total from Douglas's loan at the end of his last repayment. [2]

12 The curve C has equation $y = \frac{\mathrm{f}(x)}{x+a}$, where $\mathrm{f}(x)$ is a quadratic expression, a is a constant and $a \neq \pm 3$. It is given that the coordinates of the points of intersection of C with the x-axis are $(3,0)$ and $(-3,0)$, and the equation of the oblique asymptote is $y = \frac{1}{2}x + 1$.

(i) Find $\mathrm{f}(x)$, and show that $a = -2$. [5]

(ii) Sketch C, indicating clearly the equations of the asymptotes, and the coordinates of the points of intersection of C with the x- and y-axes. [2]

A tangent to C is parallel to the line $y = x + 2$. Find the possible equations of this tangent, leaving your answer in an exact form. [5]

[End of Paper]

VICTORIA JUNIOR COLLEGE
Preliminary Examination
Higher 2

MATHEMATICS **9740/02**
Paper 1
Wednesday **8am – 11am**

21 September 2016 **3 hours**

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
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Answer **all** the questions.
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This document consists of 5 printed pages

Section A: Pure Mathematics [40 marks]

1 When an object moves through a fluid, it experiences a force that slows it down. This force is called the drag force. At low speeds, it is known that the drag force causes the rate of change in the speed of the object to be proportional to its speed. You may assume that the experiment described below is carried out at low speeds and the only factor that affects the speed is the drag force.

An experiment is conducted to find out how the speed of an object changes as it moves through a certain fluid. When the speed of the object slows down to a speed of D m s^{-1}, a sensor is triggered and the subsequent speeds of the object are recorded.

(i) Show that the speed of the object, v m s^{-1}, at t s after the sensor is triggered, is given by

$v = De^{-pt}$, where p is a positive constant. [4]

(ii) On a single diagram, sketch the curves, C_1 and C_2, of v against t corresponding to $p = \mathrm{e}$ and $p = \frac{1}{\mathrm{e}}$.

State a single transformation the maps C_1 onto C_2. [3]

2 Given that $y = \sqrt{\left(\mathrm{e}^x \cos^2 x\right)}$, show that $2y\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - \mathrm{e}^x \sin 2x$. [2]

(i) Find the series expansion of y in ascending powers of x up to and including the term in x^2. [3]

(ii) Hence, or otherwise, find the series expansion of $\dfrac{1}{\sqrt{\left(\mathrm{e}^x \cos^2 x\right)}}$ in ascending powers of x up to and including the term in x^2. [3]

3 **(a)** The points A, B, C and D represent the complex numbers $-2+5\mathrm{i}$, z_1, $4+\mathrm{i}$ and z_2 respectively. Given that $ABCD$ is a square, labelled in an anti-clockwise direction, show that $z_1 = -1$. Find z_2. [4]

(b) Show that the equations $z^5 = z^*$ and $|z| = 1$ can be reduced to $z^n = 1$, where n is a positive integer to be determined. Find all possible values of z in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. [4]

Given further that $0 < \arg(z) < \frac{\pi}{2}$, find the smallest positive real number k for $\dfrac{(1+\mathrm{i})}{z^k}$ to be purely imaginary. [4]

4 **(a)** Referred to the origin *O*, the points *A* and *B* have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The points *C* on *AB* and *D* on *OB* are such that $2AC = CB$ and $2OD = 3DB$. Show that a vector equation of the line *m* passing through *C* and *D* can be written as

$$\mathbf{r} = \frac{3}{5}\mathbf{b} + \lambda(5\mathbf{a} - 2\mathbf{b}),\ \lambda \in \mathbb{R}\ .$$ [4]

It is given that $|\mathbf{a}| = 2, |\mathbf{b}| = 5$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$. The point *F* on *m* is such that *F* is nearest to *O*. Show that the position vector of *F* can be written as $k(5\mathbf{a} + 2\mathbf{b})$, where *k* is a constant to be found. [4]

(b) Plane π has equation $3x + 2y + 5z = 45$.

Obtain a vector equation of π in the form

$$\mathbf{r} = \mathbf{t} + \lambda\mathbf{u} + \mu\mathbf{v},\ \lambda, \mu \text{ are real parameters},$$

given that $\mathbf{t}$ and $\mathbf{u}$ are of the form $p\mathbf{i} + p\mathbf{j}$ and $2\mathbf{i} + q\mathbf{j}$ respectively, where p and q are constants to be determined, and $\mathbf{u}$ is perpendicular to $\mathbf{v}$. [5]

Section B: Statistics [60 marks]

5 The head of the Physical Education department of a school wants to gather students' views about the school's efforts in promoting student participation in physical activities. On a particular afternoon, he surveys the first 30 students who turn up at the school gymnasium.

(i) Explain why the above method may not be suitable for the purpose of his survey. [1]

(ii) Describe another sampling method that would yield a sample that is more appropriate in this context. [2]

6 Numbers in this question are formed using only the digits 1, 2, 6, 7 and 9.

(i) How many 4-digit numbers can be formed if repetition of digits is allowed? [1]

(ii) How many even numbers between 10,000 and 30,000 can be formed, if each digit can only be used once? [2]

(iii) A "trick" number is a 6-digit number formed using exactly 3 different digits, and that each digit is smaller than or equal to the following digit. How many "trick" numbers can be formed? [e.g. 127777 and 667799 are "trick" numbers, 111122 and 192992 are not "trick" numbers.] [5]

7 Box A contains 10 red, 8 blue and 7 green balls. Box B contains 2 white and 3 black balls. All the balls are indistinguishable except for their colours. Three balls are taken from Box A and two balls are taken from Box B, at random and without replacement.

Mr Wong guesses that there are at least 1 red ball and exactly 2 black balls taken, while Mr Tan guesses that all the balls taken are of different colours.

(i) Show that the probability that Mr Wong is correct is 0.241, correct to 3 significant figures. [3]

(ii) Find the probability that Mr Tan is correct. [2]

(iii) Find the probability that Mr Wong is correct, given that Mr Tan is wrong. [3]

8 A shop sells two brands of refrigerators which are in the same price range. The number of Tahichi refrigerators sold per week is a random variable with the distribution Po(1.3) and the number of Sungsam refrigerators sold per week is a random variable with the distribution Po(1.1).

(i) Show that the probability of a total of at least 10 refrigerators being sold in a randomly chosen 4-week period is 0.491, correct to 3 significant figures. [3]

(ii) A 4-week period is called a "good" period if at least 10 refrigerators are sold. Find, using a suitable approximation, the probability that, in 52 randomly chosen 4-week periods, there are more than 25 but at most 32 "good" periods. [4]

(iii) State, in the context of this question, two assumptions needed for your calculations in part **(i)** to be valid. Explain why one of these assumptions may not hold in this context. [3]

9 The masses of grade A durians from a plantation are normally distributed with mean 1.96 kg and standard deviation 0.24 kg and the masses of grade B durians from the same plantation are normally distributed with mean 1.00 kg and standard deviation σ kg.

The probability that a randomly chosen grade B durian has a mass of more than 0.8 kg is 0.95. Show that $\sigma = 0.122$, correct to 3 significant figures. [3]

(i) 50 grade A and 1 grade B durians are randomly picked from this plantation. Find the probability that the average mass of the 50 grade A durians is more than twice the mass of the grade B durian. Explain whether there is a need to use Central Limit Theorem in your working. [4]

(ii) A wholesaler buys 50 grade B durians. Using a suitable approximation, find the probability that more than 47 of the durians will have a mass of more than 0.8 kg. [4]

10 An ice-cream shop owner in Singapore wishes to find out how the daily sales of ice-cream depend on the daily average temperature. The following data are collected over 10 days.

Day	1	2	3	4	5	6	7	8	9	10
Daily average temperature, $t\ ^{\circ}C$	24.0	25.1	26.2	31.0	28.4	34.0	27.2	32.9	33.5	29.5
No. of cups of ice creams sold in one day, x	100	130	140	171	158	179	150	176	178	163

(i) Without calculating the equation of the regression line of x on t, find the coordinates of a point that will lie on this line. [2]

(ii) Draw a scatter diagram to illustrate the data and find the product moment correlation coefficient between x and t. [2]

(iii) **Without any calculations**, explain whether a quadratic model is more appropriate than a linear model to fit the data. [1]

(iv) The model $x = a(34.2 - t)^2 + b$ is used to fit the data. Calculate the least squares estimates of a and b. [2]

(v) By using the values found in part **(iv)**, estimate the expected number of cups of ice creams sold in 1 day if the daily average temperature is 31.0°C. [1]

11 The mass X g, of one loaf of "Gardener" wholemeal bread is a random variable with mean μ g, which is claimed to be 400g. A random sample of 5 loaves of wholemeal bread has masses in g as follows,

371.3, 399.4, 402.3, 388.3, 400.4.

Carry out a test at the 5% significance level to determine whether this claim is valid, stating clearly any assumption made. [4]

Another random sample of 50 loaves of wholemeal bread is taken, with results summarised below,

$$\sum(x-400) = -102.4\ ,\ \sum(x-400)^2 = 8030.2\,.$$

Using the second sample, another test was carried out at the k% significance level to determine the validity of the claim. Find the set of possible values of k for which the test concludes that the claim is incorrect.
Explain, in the context of the question, the meaning of "k% significance level". [5]

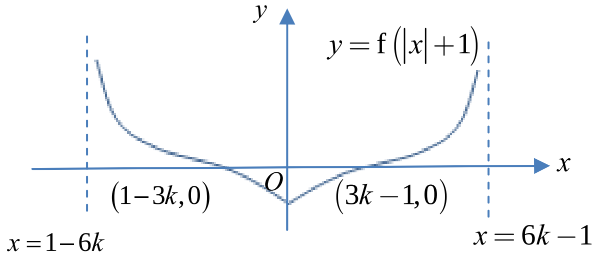
n hypothesis tests are carried out at 4% level of significance to test the validity of the claim.

Given that μ is indeed 400g, find the least value of n such that the probability of at most 1 test making a wrong conclusion is less than 0.05. [3]

[End of Paper]

2016 VJC JC2 Prelim Paper 1 Solutions/Comments

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>1</td><td>



</td></tr>
<tr><td>2</td><td>

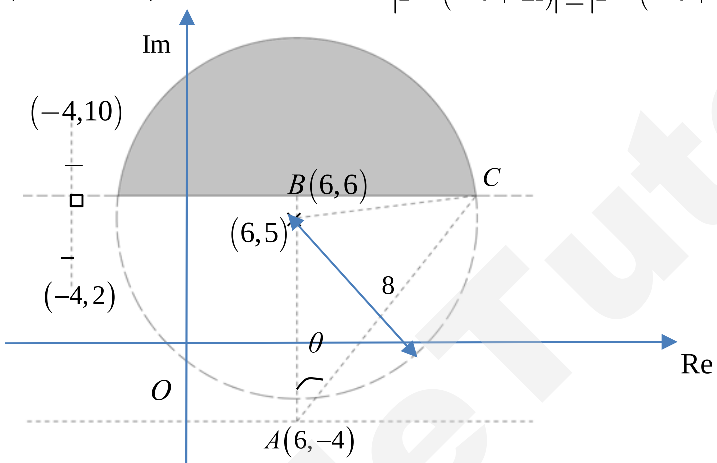
$\left|\dfrac{z-6-5\mathrm{i}}{2}\right| \le 4$

$\left|z-(6+5\mathrm{i})\right| \le 8$

$\left|2\mathrm{i}-4-z\right| \ge \left|z+4-10\mathrm{i}\right|$

$\left|z+4-2\mathrm{i}\right| \ge \left|z+4-10\mathrm{i}\right|$

$\left|z-(-4+2\mathrm{i})\right| \ge \left|z-(-4+10\mathrm{i})\right|$



By Pythagoras theorem, $BC^2 + 1^2 = 8^2 \Rightarrow BC = \sqrt{63}$

$\theta = \tan^{-1}\dfrac{\sqrt{63}}{10}$

Min $\arg(z-6+4\mathrm{i}) = \dfrac{\pi}{2} - \tan^{-1}\dfrac{\sqrt{63}}{10} = 0.900$

Max $\arg(z-6+4\mathrm{i}) = \dfrac{\pi}{2} + \tan^{-1}\dfrac{\sqrt{63}}{10} = 2.24$

</td></tr>
<tr><td></td><td>

Alternative : Equation of circle is $(x-6)^2 + (y-5)^2 = 64 --(1)$

Equation of perpendicular bisector is $y = 6 \quad ---(2)$

Substituting (2) into (1)

$(x-6)^2 + (6-5)^2 = 64 \Rightarrow x = 6 \pm \sqrt{63}$

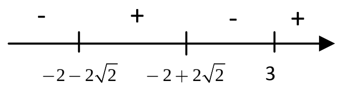
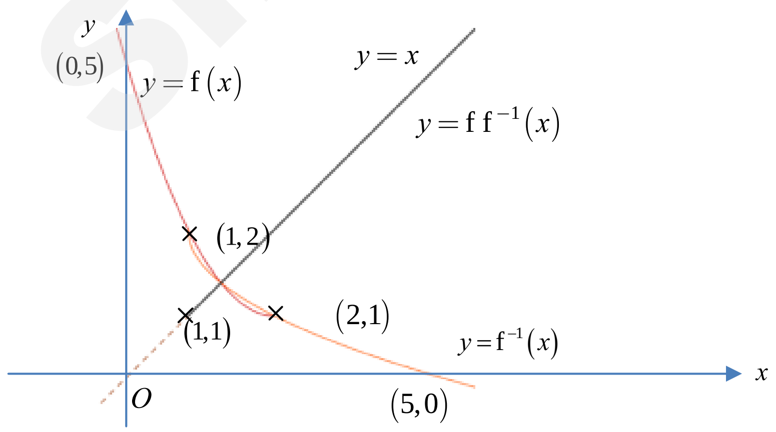
Min $\arg(z-6+4\mathrm{i}) = \arg\left(6+\sqrt{63}+6\mathrm{i}-6+4\mathrm{i}\right)$

$= \arg\left(\sqrt{63}+10\mathrm{i}\right) = \tan^{-1}\dfrac{10}{\sqrt{63}} = 0.900$

Max $\arg(z-6+4\mathrm{i}) = \arg\left(6-\sqrt{63}+6\mathrm{i}-6+4\mathrm{i}\right)$

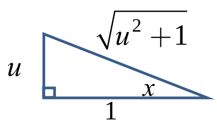
$= \arg\left(-\sqrt{63}+10\mathrm{i}\right) = \pi - \tan^{-1}\dfrac{10}{\sqrt{63}} = 2.24$

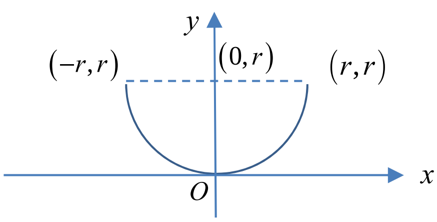
</td></tr>
</table>

Qn	Solution
3a	$\dfrac{4-7x}{x-3} \geq x$ $\dfrac{4-7x-x(x-3)}{x-3} \geq 0$ $\dfrac{x^2+4x-4}{x-3} \leq 0$ $\dfrac{(x+2)^2-8}{x-3} \leq 0$ $\dfrac{(x+2+2\sqrt{2})(x+2-2\sqrt{2})}{x-3} \leq 0$  $\therefore x \leq -2-2\sqrt{2}$ or $-2+2\sqrt{2} \leq x < 3$
b	Let e, f and g be the ages of Edwin, his father and his grandfather respectively. $e+f+g=53\times 3=159 \quad ----\ (1)$ $\frac{1}{4}e+\frac{1}{3}f+\frac{1}{2}g=65 \quad ----\ (2)$ $g-22=2(f-22+e-22)$ $2e+2f-g=66 \quad ----\ (3)$ From GC, $e=24, f=51, g=84$. The ages of Edwin, his father and his grandfather are 24, 51 and 84 respectively.
4ia	When $k=1$, $\mathrm{f}: x \mapsto (x-2)^2+1,\ x\in\mathbb{R},\ x\leq 2$ Let $y=(x-2)^2+1$ $(x-2)^2=y-1$ $x-2=\pm\sqrt{y-1}$ $x=2\pm\sqrt{y-1}$ Since $x\leq 2$, $x=2-\sqrt{y-1}$, $\mathrm{f}^{-1}: x\mapsto 2-\sqrt{x-1},\ x\geq 1$
b	
ii	$\{k\in\mathbb{R}: k>2\}$

Qn	Solution
5i	$u_2 = u_1 + \frac{2-1^2}{(1+2)!} = 0 + \frac{1}{6} = \frac{1}{6}$ $u_3 = u_2 + \frac{2-2^2}{(2+2)!} = \frac{1}{6} - \frac{2}{24} = \frac{1}{12}$ $u_4 = u_3 + \frac{2-3^2}{(3+2)!} = \frac{1}{12} - \frac{7}{120} = \frac{1}{40}$
ii	$u_2 = \frac{1}{6} = \frac{2-1}{(2+1)!}$, $u_3 = \frac{1}{12} = \frac{2}{24} = \frac{3-1}{(3+1)!}$, $u_4 = \frac{1}{40} = \frac{3}{120} = \frac{4-1}{(4+1)!}$ By observation, a conjecture is that $u_n = \frac{n-1}{(n+1)!}$
iii	Let P_n be the statement $u_n = \frac{n-1}{(n+1)!}$, for all $n \in \mathbb{Z}^+$. Check P_1: LHS = u_1 = 0 RHS = $\frac{1-1}{(1+1)!} = 0$ $\therefore P_1$ is true Assume that P_k is true for some positive integer k i.e. $u_k = \frac{k-1}{(k+1)!}$ We want to show that P_{k+1} is true. i.e. $u_{k+1} = \frac{k}{(k+2)!}$ $\text{LHS} = u_{k+1}$ $= u_k + \frac{2-k^2}{(k+2)!}$ $= \frac{k-1}{(k+1)!} + \frac{2-k^2}{(k+2)!}$ $= \frac{(k-1)(k+2)+2-k^2}{(k+2)!}$ $= \frac{k^2+k-2+2-k^2}{(k+2)!}$ $= \frac{k}{(k+2)!} = \text{RHS}$ Since P_1 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
6	If the distance *AB* is the least, the line segment *AB* is perpendicular to *l*.
	$B(b, 2\sqrt{b})$ and $A(a, 2a+1)$ Gradient of $BA = \frac{2\sqrt{b}-2a-1}{b-a}$

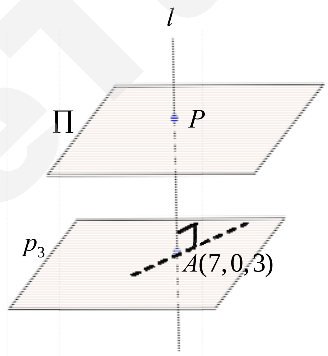
Qn	Solution
	Since gradient of l is 2, $\frac{2\sqrt{b}-2a-1}{b-a}=-\frac{1}{2}$ $\Rightarrow 4\sqrt{b}-4a-2=a-b$ $\Rightarrow a=\frac{1}{5}\left(b+4\sqrt{b}-2\right)$
	$\left(AB\right)^2=\left(2\sqrt{b}-2a-1\right)^2+\left(b-a\right)^2$ $=\left(2\sqrt{b}-2a-1\right)^2+\left(4\sqrt{b}-4a-2\right)^2$ $=5\left(2\sqrt{b}-2a-1\right)^2$ $=5\left(2\sqrt{b}-\frac{2}{5}\left(b+4\sqrt{b}-2\right)-1\right)^2$ $=5\left(\frac{2}{5}\sqrt{b}-\frac{2}{5}b-\frac{1}{5}\right)^2$ $=\frac{1}{5}\left(2\sqrt{b}-2b-1\right)^2$ $=\frac{1}{5}\left(2b-2\sqrt{b}+1\right)^2$ $2AB\frac{\mathrm{d}AB}{\mathrm{d}b}=\frac{2}{5}\left(2b-2\sqrt{b}+1\right)\left(2-\frac{1}{\sqrt{b}}\right)$ When $\frac{\mathrm{d}AB}{\mathrm{d}b}=0$, $\frac{2}{5}\left(2b-2\sqrt{b}+1\right)\left(2-\frac{1}{\sqrt{b}}\right)=0$ Consider $\left(2b-2\sqrt{b}+1\right)=0$ Since $\left(-2\right)^2-4\left(2\right)\left(1\right)<0$, $\left(2b-2\sqrt{b}+1\right)=0$ has no real solution. $2-\frac{1}{\sqrt{b}}=0\Rightarrow b=\frac{1}{4}$ the point on C nearest to l is $\left(b,2\sqrt{b}\right)=\left(\frac{1}{4},1\right)$.
	Alternative : $\left(AB\right)^2=\frac{1}{5}\left(2b-2\sqrt{b}+1\right)^2$ $=\frac{4}{5}\left(b-\sqrt{b}+\frac{1}{2}\right)^2$ $=\frac{4}{5}\left(\left(\sqrt{b}-\frac{1}{2}\right)^2-\frac{1}{4}+\frac{1}{2}\right)^2$ $=\frac{4}{5}\left(\left(\sqrt{b}-\frac{1}{2}\right)^2+\frac{1}{4}\right)^2$

Qn	Solution
	Since $\left(\sqrt{b}-\frac{1}{2}\right)^2 \geq 0$ for all real b, $(AB)^2$ is the least when $\sqrt{b}=\frac{1}{2}$, that is, $b=\frac{1}{4}$. Hence the point on C nearest to l is $\left(b, 2\sqrt{b}\right)=\left(\frac{1}{4},1\right)$
	Alternative : When $(AB)^2$ is the least, tangent to C at B is parallel to l. i.e. gradient of tangent to C = 2 $y^2=4x$ $2y\frac{\mathrm{d}y}{\mathrm{d}x}=4$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{y}$ At $\left(b,2\sqrt{b}\right)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{2\sqrt{b}}=\frac{1}{\sqrt{b}}=2$ $b=\frac{1}{4}$ $\therefore$ coordinates on C nearest to l is $\left(\frac{1}{4},1\right)$.
7a	$\frac{\mathrm{d}}{\mathrm{d}x}xe^{x^3}=e^{x^3}+x3x^2e^{x^3}$ $=e^{x^3}(1+3x^3)$ $\int x^2\left(1+3x^3\right)e^{x^3}\,\mathrm{d}x=xe^{x^3}x^2-\int xe^{x^3}\,2x\,\mathrm{d}x$ $=xe^{x^3}x^2-\frac{2}{3}\int 3x^2e^{x^3}\,\mathrm{d}x$ $=x^3e^{x^3}-\frac{2}{3}e^{x^3}+C$
b	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\sec^2 x}{2\sec^2 x+4\tan x+7}$ $y=\int\frac{\sec^2 x}{2\sec^2 x+4\tan x+7}\,\mathrm{d}x$ $=\int\frac{1}{2\left(u^2+1\right)+4u+7}\,\mathrm{d}u$ $=\int\frac{1}{2u^2+4u+9}\,\mathrm{d}u$ $=\frac{1}{2}\int\frac{1}{u^2+2u+\frac{9}{2}}\,\mathrm{d}u$ $=\frac{1}{2}\int\frac{1}{(u+1)^2+\frac{7}{2}}\,\mathrm{d}u$ $u=\tan x$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\sec^2 x$  $\sec x=\sqrt{u^2+1}$ $\sec^2 x=u^2+1$ Or $\sec^2 x=\tan^2 x+1$ $=u^2+1$

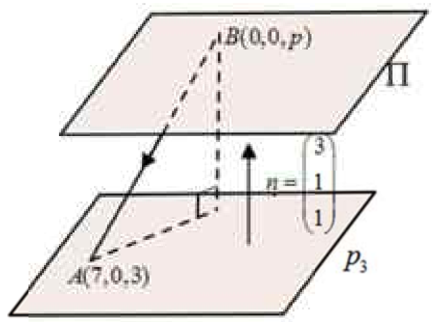
Qn	Solution
	$=\frac{1}{2}\left(\frac{\sqrt{2}}{\sqrt{7}}\right)\tan^{-1}\left(\frac{\sqrt{2}(u+1)}{\sqrt{7}}\right)+C$ $=\frac{1}{\sqrt{14}}\tan^{-1}\left(\frac{\sqrt{2}(\tan x+1)}{\sqrt{7}}\right)+C$
8i	$\sum_{r=2}^{n}\ln\left[\frac{r(r+2)}{(r+1)^2}\right]$ $=\sum_{r=2}^{n}\left(\ln r-2\ln(r+1)+\ln(r+2)\right)$ $\begin{array}{rcccccc} = & & \ln 2 & - & 2\ln 3 & + & \cancel{\ln 4} \\ & + & \ln 3 & - & \cancel{2\ln 4} & + & \cancel{\ln 5} \\ & + & \cancel{\ln 4} & - & \cancel{2\ln 5} & + & \cancel{\ln 6} \\ & & & & \vdots & & \\ & + & \cancel{\ln(n-2)} & - & \cancel{2\ln(n-1)} & + & \cancel{\ln n} \\ & + & \cancel{\ln(n-1)} & - & \cancel{2\ln n} & + & \ln(n+1) \\ & + & \cancel{\ln n} & - & 2\ln(n+1) & + & \ln(n+2) \end{array}$ $=\ln 2-\ln 3-\ln(n+1)+\ln(n+2)$ $=\ln\frac{2}{3}+\ln\frac{n+2}{n+1}$
ii	As $n\to\infty$, $\ln\frac{n+2}{n+1}\to\ln 1=0$, $\ln\frac{2}{3}+\ln\frac{n+2}{n+1}\to\ln\frac{2}{3}$. Since the series tends to a constant, it converges. The sum to infinity is $\ln\frac{2}{3}$.
iii	$\sum_{r=2}^{13}\ln\left[\frac{(2r)(2r+4)}{(r+1)^2}\right]=\sum_{r=2}^{13}\left(\ln 4+\ln\left[\frac{r(r+2)}{(r+1)^2}\right]\right)$ $=12\ln 4+\ln\frac{2}{3}+\ln\frac{15}{14}$ $=\ln\frac{83886080}{7}$
9i	

Qn	Solution	
ii	$\begin{aligned} V &= \pi\int_0^h x^2 \mathrm{d}y \\ &= \pi\int_0^h \left(r^2-(y-r)^2\right)\mathrm{d}y \\ &= \pi\left[r^2y-\frac{(y-r)^3}{3}\right]_0^h \\ &= \pi\left(r^2h-\frac{(h-r)^3}{3}-\frac{r^3}{3}\right) \\ &= \pi\begin{pmatrix} r^2h-\frac{h^3}{3}+h^2r \\ -hr^2+\frac{r^3}{3}-\frac{r^3}{3}\end{pmatrix} \\ &= \pi\left(h^2r-\frac{h^3}{3}\right) \\ &= \frac{\pi h^2}{3}(3r-h) \end{aligned}$	Alternative: $= \pi\int_0^h \left(r^2-(y^2-2ry+r^2)\right)\mathrm{d}y$ $= \pi\int_0^h \left(2ry-y^2\right)\mathrm{d}y$ $= \pi\left[ry^2-\frac{1}{3}y^3\right]_0^h$ $= \vdots$
iii	$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\frac{2}{3}\pi r^3}{24} = -\frac{\pi r^3}{36}$ $\frac{\mathrm{d}V}{\mathrm{d}h} = \pi\left(2hr-h^2\right)$ $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{\pi\left(2hr-h^2\right)}\left(-\frac{\pi r^3}{36}\right)$ $= -\frac{r^3}{36\left(2hr-h^2\right)}$ Rate of decrease is $\frac{r^3}{36\left(2hr-h^2\right)}$ cm^3 s$^{-1.}$	
iv	The rate of decrease of the depth is the least when the bowl is full, i.e. $h = r$. $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{r^3}{36r(2r-r)} = -\frac{r}{36}$ The slowest rate at which the depth of water is decreasing is $\frac{r}{36}$ cm s^{-1}.	
10i	Let θ be the angle between p_1 and p_2. $\cos\theta = \frac{\begin{pmatrix}1\\-5\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\5\end{pmatrix}}{\sqrt{30}\sqrt{30}} = \frac{3}{30} \Rightarrow \theta = 84.3^\circ$	
ii	$x-5y+2z=13,$ $-2x+y+5z=1,$ From GC, $x=-2+3\lambda, y=-3+\lambda, z=\lambda$	

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td></td><td>$\therefore$ equation of l is $\underset{\sim}{r} = \begin{pmatrix}-2\\-3\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\1\end{pmatrix}, \lambda \in \mathbb{R}$</td></tr>
<tr><td>iii</td><td>The point of intersection of p_1, p_2 and p_3 is the point of intersection of l and p_3. $(a,0,b)$ is a point on l.
$$\begin{pmatrix}a\\0\\b\end{pmatrix} = \begin{pmatrix}-2\\-3\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\1\end{pmatrix} \Rightarrow \begin{matrix}a = -2+3\lambda\\ \lambda = 3\\ b = \lambda\end{matrix}$$
$\therefore a = -2+3\times3 = 7$ and $b = 3$

Alternatively, subst $x = a, y = 0, z = b$ into equation of planes
$a + 2b = 13 --(1)$
$-2a + 5b = 1---(2)$
$(1)\times 2 \quad 2a + 4b = 26--(3)$
$(2)+(3) \qquad 9b = 27 \Rightarrow b = 3$
$a = 7$</td></tr>
<tr><td>iv</td><td>l is perpendicular to p_3 and intersect p_3 at $A(7,0,3)$. Let P be a point on l such that $AP = 4\sqrt{11}$, then P lies in $\prod$.
$$\overrightarrow{AP} = \pm 4\sqrt{11}\cdot\frac{1}{\sqrt{11}}\begin{pmatrix}3\\1\\1\end{pmatrix} = \pm 4\begin{pmatrix}3\\1\\1\end{pmatrix}$$
$$\overrightarrow{OP} = \overrightarrow{OP} + \overrightarrow{AP}$$
$$= \begin{pmatrix}7\\0\\3\end{pmatrix} \pm 4\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}-5\\-4\\-1\end{pmatrix} \text{ or } \begin{pmatrix}19\\4\\7\end{pmatrix}$$


$$\underset{\sim}{r}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}-5\\-4\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix} = -20 \text{ or } \underset{\sim}{r}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}19\\4\\7\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix} = 68$$
two possible cartesian equations of $\prod$ are
$3x + y + z = -20$ and $3x + y + z = 68$.

Alternatively,
The cartesian equation of $\prod$ is of the form $3x + y + z = p$.
$x = 0$, $y = 0$ and $z = p$ satisfy $3x + y + z = p$, $B(0,0,p)$ is a point in Π.


Distance between $\prod$ and p_3 is $4\sqrt{11}$.</td></tr>
</table>

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td></td><td>

$\left|\overrightarrow{BA}\cdot\hat{\underset{\sim}{n}}\right| = 4\sqrt{11}$

$\left|\begin{pmatrix}7\\0\\3-p\end{pmatrix}\cdot\frac{1}{\sqrt{11}}\begin{pmatrix}3\\1\\1\end{pmatrix}\right| = 4\sqrt{11}$

$\frac{1}{\sqrt{11}}\left|24-p\right| = 4\sqrt{11}$

$\left|p-24\right| = 44$

$p-24 = -44$ or 44

$p = -20$ or 68

two possible cartesian equations of $\prod$ are

$3x+y+z=-20$ and $3x+y+z=68$.
</td></tr>
<tr><td>11a</td><td>

$\frac{a+4d}{a+2d} = \frac{a+11d}{a+4d}$

$a^2+8ad+16d^2 = a^2+13ad+22d^2$

$5ad = -6d^2$

Since the terms are distinct, $d \neq 0$, $d = -\frac{5}{6}a$

Required sum $= \frac{n}{2}\left(\left(a+d\right)+\left(a+\left(2n-1\right)d\right)\right)$

$= \frac{n}{2}\left(2a+2nd\right) = n\left(a+nd\right)$
</td></tr>
<tr><td>bi</td><td>
<table>
<tr><th>n</th><th>Beginning</th><th>End</th></tr>
<tr><td>1</td><td>40000</td><td>40000(1.005)</td></tr>
<tr><td>2</td><td>40000(1.005)</td><td>$40000(1.005)^2$</td></tr>
<tr><td>3</td><td>$40000(1.005)^2 - x$</td><td>$40000(1.005)^3 - 1.005x$</td></tr>
<tr><td>4</td><td>$40000(1.005)^3 - 1.005x - x$</td><td>$40000(1.005)^3 - 1.005^2x - 1.005x$</td></tr>
<tr><td>5</td><td>$40000(1.005)^4 - 1.005^2x - 1.005x - x$</td><td></td></tr>
</table>

Amount at the beginning of 5th month

$= 40000(1.005)^4 - 1.005^2x - 1.005x - x$

$= 40000\left(1.005\right)^4 - \frac{x\left(1.005^3-1\right)}{1.005-1}$

$= 40000\left(1.005\right)^4 - 200x\left(1.005^3-1\right)$
</td></tr>
<tr><td>ii</td><td>

He wishes to repay his in 5 years, $n = 60$

$40000\left(1.005\right)^{59} - 200x\left(1.005^{58}-1\right) \leq 0$

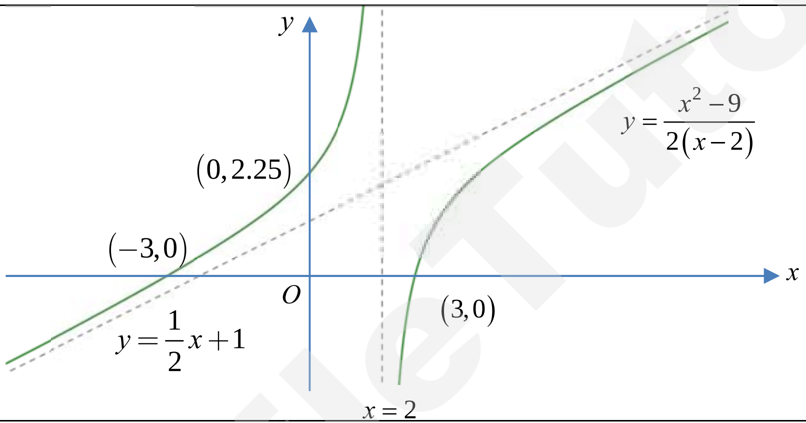
$x \geq \frac{40000\left(1.005\right)^{59}}{200\left(1.005^{58}-1\right)}$

$x \geq 800.17$

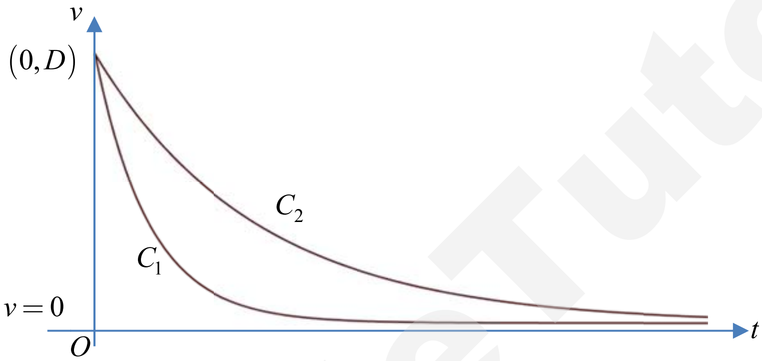
His minimum repayment is \$800.17
</td></tr>
<tr><td>iii</td><td>

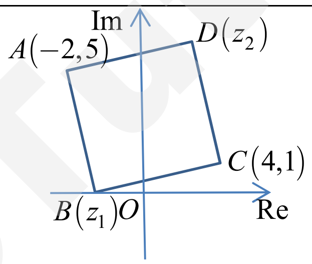
Amount interest bank earned = \$(800.17(58) – 40000)

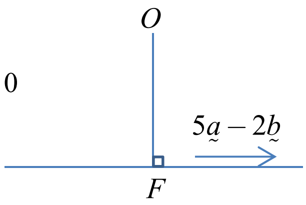
= \$6410.06 = \$6410 (nearest dollar)
</td></tr>
</table>

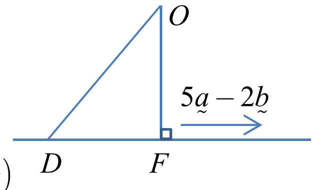
Qn	Solution
12	Since f(x) is a quadratic expression and $\mathrm{f}(3)=\mathrm{f}(-3)=0$, $\mathrm{f}(x)=k(x^2-9)$. $\dfrac{k(x^2-9)}{x+a}=\dfrac{1}{2}x+1+\dfrac{b}{x+a}$ $\dfrac{kx^2-9k}{x+a}=\dfrac{\left(\frac{1}{2}x+1\right)(x+a)+b}{x+a}$ $=\dfrac{\frac{1}{2}x^2+\left(1+\frac{1}{2}a\right)x+a+b}{x+1}$ Comparing coefficients, $k=\dfrac{1}{2}\Rightarrow \mathrm{f}(x)=\dfrac{1}{2}(x^2-9)$ $\therefore 1+\dfrac{1}{2}a=0\Rightarrow a=-2$ (shown) $a+b=-\dfrac{9}{2}\Rightarrow b=-\dfrac{5}{2}$
	
	$y=\dfrac{x^2-9}{2(x-2)}=\dfrac{1}{2}x+1-\dfrac{5}{2(x-2)}$ $\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2}+\dfrac{5}{2(x-2)^2}$ When $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1$, $\dfrac{1}{2}+\dfrac{5}{2(x-2)^2}=1\Rightarrow \dfrac{5}{2(x-2)^2}=\dfrac{1}{2}$ $(x-2)^2=5$ $x=2\pm\sqrt{5}$
	When $x=2+\sqrt{5}$, $y=\dfrac{1}{2}(2+\sqrt{5})+1-\dfrac{5}{2\sqrt{5}}=2$ When $x=2-\sqrt{5}$, $y=\dfrac{1}{2}(2-\sqrt{5})+1-\dfrac{5}{2(-\sqrt{5})}=2$ The equations of tangent are $y-2=x-(2\pm\sqrt{5})$ $y=x-\sqrt{5}$ or $y=x+\sqrt{5}$

2016 VJC JC2 Prelim Paper 2 Solutions/Comments

Qn	Solution
1i	Since speed is decreasing and v is positive, $\frac{\mathrm{d}v}{\mathrm{d}t}=-kv, \quad$ where k is a positive constant $\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}t}=-k$ $\int \frac{1}{v}\,\mathrm{d}v=\int -k\,\mathrm{d}t$ $\ln v=-kt+C \quad \because v>0$ $v=B\mathrm{e}^{-kt}$ When $t = 0$s, $v = D$ m s^{-1} $B = D$ Let $k = p$, hence $v=D\mathrm{e}^{-pt}$, where p is a positive constant.
ii	 Stretch C_1 parallel to the t-axis, factor e^2, v-axis is invariant.
2	$y=\sqrt{\mathrm{e}^x\cos^2 x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^x\cos^2 x-2\mathrm{e}^x\sin x\cos x}{2\sqrt{\mathrm{e}^x\cos^2 x}}$ $=\frac{y^2-\mathrm{e}^x\sin 2x}{2y}$ $2y\frac{\mathrm{d}y}{\mathrm{d}x}=y^2-\mathrm{e}^x\sin 2x$ **Alternative Solution** $y=\sqrt{\mathrm{e}^x\cos^2 x}$ $y^2=\mathrm{e}^x\cos^2 x$ $2y\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x\cos^2 x-2\mathrm{e}^x\sin x\cos x$ $=y^2-\mathrm{e}^x\sin 2x$
i	$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2y\frac{\mathrm{d}y}{\mathrm{d}x}-\mathrm{e}^x\sin 2x-2\mathrm{e}^x\cos 2x$ When $x = 0$, $y=\sqrt{\mathrm{e}^0\cos^2 0}=1$

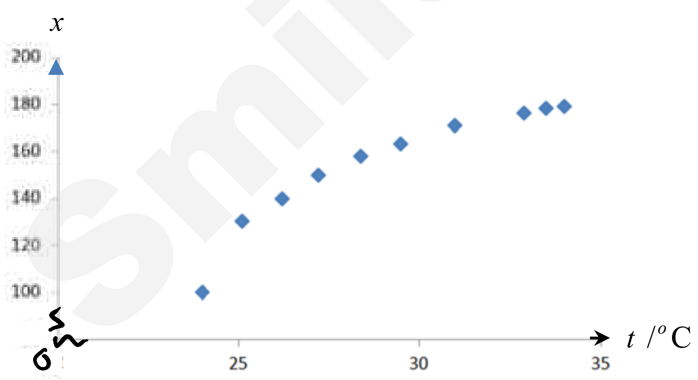
Qn	Solution
	$2\dfrac{dy}{dx}=1-0 \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{2}$ $2\left(\dfrac{1}{2}\right)^2+2\dfrac{d^2y}{dx^2}=2\left(\dfrac{1}{2}\right)-0-2 \Rightarrow \dfrac{d^2y}{dx^2}=-\dfrac{3}{4}$ $y=1+\dfrac{1}{2}x-\dfrac{3}{4}\left(\dfrac{x^2}{2!}\right)+\ldots$ $\quad=1+\dfrac{1}{2}x-\dfrac{3}{8}x^2+\ldots$
ii	$\dfrac{1}{\sqrt{e^x\cos^2 x}}=\left(1+\dfrac{1}{2}x-\dfrac{3}{8}x^2+\ldots\right)^{-1}$ $=1+(-1)\left(\dfrac{1}{2}x-\dfrac{3}{8}x^2+\ldots\right)+\dfrac{(-1)(-2)}{2!}\left(\dfrac{1}{2}x+\ldots\right)^2+\ldots$ $=1-\dfrac{1}{2}x+\dfrac{3}{8}x^2+\dfrac{1}{4}x^2+\ldots$ $=1-\dfrac{1}{2}x+\dfrac{5}{8}x^2+\ldots$
3a	From the diagram, $\arg(4+i-z_1)+\dfrac{\pi}{2}=\arg(-2+5i-z_1)$ $i(4+i-z_1)=(-2+5i-z_1)$ $4i-1-iz_1=-2+5i-z_1$ $(1-i)z_1=-1+i$ $z_1=-1$ 
	Midpoint of AC is $\left(\dfrac{-2+4}{2},\dfrac{5+1}{2}\right)=(1,3)$ Let $z_2=x+iy$ Since the diagonals of a square bisect other, Midpoint of BD is $(1,3)$ $\left(\dfrac{x-1}{2},\dfrac{y+0}{2}\right)=(1,3)$ $\therefore x=3,\ y=6$ $z_2=3+6i$
bi	Let $z=e^{i\theta} \Rightarrow z^*=e^{-i\theta}=\dfrac{1}{z}$ $z^5=z^{-1}$ $z^6=1$ **Alternatively** $z^5=z^*$ $z^6=zz^*=\lvert z\rvert^2$ $z^6=1$
	$z^6=e^{2k\pi i},\ k\in\mathbb{Z}$ $z=e^{\frac{k\pi i}{3}},\ k=0,1,2,3,4,5$ $=1, e^{\frac{\pi}{3}i}, e^{\frac{2\pi}{3}i}, -1, e^{\frac{4\pi}{3}i}, e^{\frac{5\pi}{3}i}$

Qn	Solution
ii	Since $0 < \arg(z) < \frac{\pi}{2}$, $z = e^{\frac{\pi}{3}i} \Rightarrow z^k = e^{\frac{k\pi}{3}i}$ $\frac{(1+i)}{z^k} = \sqrt{2}e^{i\frac{\pi}{4}}.e^{-\frac{k\pi}{3}i} = \sqrt{2}e^{\left(\frac{\pi}{4}-\frac{k\pi}{3}\right)i}$ If $\frac{(1+i)}{z^k}$ is purely imaginary, $\frac{\pi}{4}-\frac{k\pi}{3} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$ Since k is positive, $\frac{\pi}{4}-\frac{k\pi}{3} = -\frac{\pi}{2}, -\frac{3\pi}{2}, \dots$ $\frac{k\pi}{3} = \frac{3\pi}{4}, \frac{7\pi}{4}, \dots$ Smallest positive k when $\frac{k\pi}{3} = \frac{3\pi}{4}$ Smallest positive $k = \frac{9}{4}$
4a	$\overrightarrow{OC} = \frac{1}{3}(2\underset{\sim}{a}+\underset{\sim}{b}), \overrightarrow{OD} = \frac{3}{5}\underset{\sim}{b}$ $\overrightarrow{CD} = \frac{3}{5}\underset{\sim}{b} - \frac{1}{3}(2\underset{\sim}{a}+\underset{\sim}{b}) = -\frac{2}{15}(5\underset{\sim}{a}-2\underset{\sim}{b})$ Since line m passes through D and is parallel to CD, $\underset{\sim}{r} = \overrightarrow{OD} + \mu\overrightarrow{CD}$ $= \frac{3}{5}\underset{\sim}{b} + \frac{2}{15}\mu(2\underset{\sim}{b}-5\underset{\sim}{a})$ $= \frac{3}{5}\underset{\sim}{b} - \frac{2}{15}\mu(5\underset{\sim}{a}-2\underset{\sim}{b})$ $\underset{\sim}{r} = \frac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a}-2\underset{\sim}{b}), \lambda \in \mathbb{R}$ Equation of m is $\underset{\sim}{r} = \frac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a}-2\underset{\sim}{b}), \lambda \in \mathbb{R}$.
	F is a point on m $\therefore \overrightarrow{OF} = \frac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a}-2\underset{\sim}{b})$ for a value of λ  $\overrightarrow{OF}$ is perpendicular to $l \Rightarrow \overrightarrow{OF}\cdot(5\underset{\sim}{a}-2\underset{\sim}{b}) = 0$ $\Rightarrow \left[\frac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a}-2\underset{\sim}{b})\right]\cdot(5\underset{\sim}{a}-2\underset{\sim}{b}) = 0$ $\Rightarrow 3(\underset{\sim}{a}\cdot\underset{\sim}{b}) - \frac{6}{5}(\underset{\sim}{b}\cdot\underset{\sim}{b}) + \lambda\left[25(\underset{\sim}{a}\cdot\underset{\sim}{a}) - 20(\underset{\sim}{a}\cdot\underset{\sim}{b}) + 4(\underset{\sim}{b}\cdot\underset{\sim}{b})\right] = 0$ $\Rightarrow 3(\underset{\sim}{a}\cdot\underset{\sim}{b}) - \frac{6}{5}\lvert\underset{\sim}{b}\rvert^2 + \lambda\left[25\lvert\underset{\sim}{a}\rvert^2 - 20(\underset{\sim}{a}\cdot\underset{\sim}{b}) + 4\lvert\underset{\sim}{b}\rvert^2\right] = 0$ Since $\underset{\sim}{a}\cdot\underset{\sim}{b} = \lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos 60^\circ = 2\times5\times\frac{1}{2} = 5$ $\therefore 3(5) - \frac{6}{5}(5)^2 + \lambda\left[25(2)^2 - 20(5) + 4(5)^2\right] = 0$ $\Rightarrow \lambda = \frac{3}{20}$

Qn	Solution
	$\therefore \overrightarrow{OF}=\frac{3}{5}\underset{\sim}{b}+\frac{3}{20}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)=\frac{3}{20}\left(5\underset{\sim}{a}+2\underset{\sim}{b}\right)$
	Alternative Method  $\overrightarrow{DF}=\left(\overrightarrow{DO}\cdot\frac{5\underset{\sim}{a}-2\underset{\sim}{b}}{\left\|5\underset{\sim}{a}-2\underset{\sim}{b}\right\|}\right)\frac{5\underset{\sim}{a}-2\underset{\sim}{b}}{\left\|5\underset{\sim}{a}-2\underset{\sim}{b}\right\|}$ $=\frac{1}{\left\|5\underset{\sim}{a}-2\underset{\sim}{b}\right\|^2}\left(-\frac{3}{5}\underset{\sim}{b}\cdot\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)\right)\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)$ $=\frac{-3\underset{\sim}{a}\cdot\underset{\sim}{b}+\frac{6}{5}\left\|\underset{\sim}{b}\right\|^2}{\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)\cdot\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)$ $=\frac{-3(5)+\frac{6}{5}(5)^2}{25\left\|\underset{\sim}{a}\right\|^2-20\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)+4\left\|\underset{\sim}{b}\right\|^2}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)$ $=\frac{15}{25(2)^2-20(5)+4(5)^2}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)$ $=\frac{3}{20}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)$ $\therefore \overrightarrow{OF}=\frac{3}{5}\underset{\sim}{b}+\frac{3}{20}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)=\frac{3}{20}\left(5\underset{\sim}{a}+2\underset{\sim}{b}\right)$
b	The equation of the plane π is $3x+2y+5z=45$. $(p,p,0)$ lies in $\pi \Rightarrow 3p+2p+0=45 \Rightarrow p=9$ $\begin{pmatrix}2\\q\\0\end{pmatrix}$ is perpendicular to $\begin{pmatrix}3\\2\\5\end{pmatrix} \Rightarrow \begin{pmatrix}2\\q\\0\end{pmatrix}\cdot\begin{pmatrix}3\\2\\5\end{pmatrix}=0$ $6+2q=0 \Rightarrow q=-3$ Since $\underset{\sim}{v}$ is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{n}$, $\underset{\sim}{u}\times\underset{\sim}{n}=\begin{pmatrix}3\\2\\5\end{pmatrix}\times\begin{pmatrix}2\\-3\\0\end{pmatrix}=\begin{pmatrix}15\\10\\-13\end{pmatrix}$ $\underset{\sim}{r}=\begin{pmatrix}9\\9\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\0\end{pmatrix}+\mu\begin{pmatrix}15\\10\\-13\end{pmatrix}, \lambda,\mu\in\mathbb{R}$ **Alternative method to find $\underset{\sim}{v}$** Let $\underset{\sim}{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}3\\2\\5\end{pmatrix}=0$ and $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\0\end{pmatrix}=0$

Qn	Solution
	$3x + 2y + 5z = 0$ and $2x - 3y = 0$ $x = -\frac{15}{13}z,\ y = -\frac{10}{13}z,\ z = z$ Let $z = 13$ (any non-zero number will work) $\mathbf{v} = \begin{pmatrix} -15 \\ -10 \\ 13 \end{pmatrix}$
5i	He will not get to survey the students who do not go to the school gymnasium. Hence, the sample obtained is biased.
ii	He can obtain a numbered list of all the students (labelled 1 to N) in the school. Using a random number generator, he generates 30 **distinct** numbers. He will survey the students corresponding the numbers generated. **Alternatively.** Let the total number of students be N Sampling interval = $\frac{N}{30}$ He can obtain a numbered list of all the students (labelled 1 to N) in the school. Using a random number generator, select a starting number k where $1 \le k \le \frac{N}{30}$. He can interview the students corresponding to the numbers $k,\ k+\frac{N}{30}, k+\frac{N}{15}, \ldots, k+\frac{29N}{30}$.
6i	Number of 4-digit numbers = $5^4 = 625$
ii	Case1: Starts with 1 1 2 No. of ways = 2(3!) = 12 1 __ __ __ 6 Case 2: starts with 2 No. of ways = 3! = 6 2 __ __ __ 6 Total number of ways = 18
iii	Case 1: XXXXYZ No. of ways = ${}^5C_3({}^3C_1) = 30$ Case 2: XXXYYZ No. of ways = ${}^5C_3({}^3C_1)({}^2C_1) = 60$ Case 3: XXYYZZ No. of ways = ${}^5C_3 = 10$ Total number of ways = 100
7i	P(Mr Wong is correct) = $\left(1 - \frac{{}^{15}C_3}{{}^{25}C_3}\right) \times \frac{{}^3C_2}{{}^5C_2} = 0.24065 = 0.241$
ii	P(Mr Tan is correct) = $\frac{{}^{10}C_1 \times {}^8C_1 \times {}^7C_1}{{}^{25}C_3} \times \frac{{}^3C_1 \times {}^2C_1}{{}^5C_2} = \frac{84}{575} = 0.146$ <u>Alternative method:</u> P(Mr Tan is correct) $= \frac{10 \times 8 \times 7}{25 \times 24 \times 23} \times 3! \times \frac{3 \times 2}{5 \times 4} \times 2!$

Qn	Solution
	$=\dfrac{84}{575}=0.146$
iii	P(Mr Wong's guess is right, given that Mr Tan's guess is wrong) $=\dfrac{\text{P(Mr Wong is correct and Mr Tan is wrong)}}{\text{P(Mr Tan is wrong)}}$ $=\dfrac{0.24065}{1-0.14609}$ $=0.282$
8i	Let T be the total number of refrigerators sold in a 4-week period. $T \sim \text{Po}((1.3 + 1.1) \times 4)$ $T \sim \text{Po}(9.6)$ $\text{P}(T \geq 10) = 1 - \text{P}(T \leq 9) = 0.49114 = 0.491$ (3sf)
ii	Let X be number of good periods out of 52. $X \sim \text{B}(52, 0.491)$ or $X \sim B(52,\ 0.49114)$ Since $np = 25.532 > 5$ and $np(1-p) = 26.468 > 5$ $X \sim \text{N}(25.532, 12.996)$ approx. or $X \sim N(25.539, 12.996)$ approx. $\text{P}(25 < X \leq 32) = \text{P}(25.5 < X \leq 32.5) = 0.477$ (or 0.478)
iii	We need to assume that the sales of **all** the refrigerators are independent of one another. We also need to assume that the **average rate** of refrigerators being sold is constant. The first assumption may not hold as the two brands of refrigerator are in the same price range and they can be competing in terms of sales. OR The average rate of refrigerators sold is unlikely to be a constant due to sale, festive seasons, economic conditions etc.
9	Let A kg and B kg be masses of a randomly chosen grade A and grade B durian respectively. $A \sim \text{N}(1.96, 0.24^2)$ and $B \sim \text{N}(1.00, \sigma^2)$ $\text{P}(B > 0.8) = 0.95$ $\text{P}\left(Z > \dfrac{0.8-1.00}{\sigma}\right)=0.95$ $\text{P}\left(Z \leq \dfrac{0.8-1.00}{\sigma}\right)=0.05$ $\dfrac{-0.2}{\sigma}=-1.64485 \Rightarrow \sigma = 0.12159 \approx 0.122$
i	$A \sim \text{N}(1.96, 0.24^2)$ and $B \sim \text{N}(1.00, \sigma^2)$ $\overline{A} \sim N\left(1.96, \dfrac{0.24^2}{50}\right)$ and

Qn	Solution
	$2B \sim N(2.00, 2^2(0.122^2))$ or $2B \sim N(2.00, 2^2(0.12159)^2)$ $\overline{A} - 2B \sim N(-0.04, 0.060688)$ or $\overline{A} - 2B \sim N(-0.04, 0.060290)$ $P(\overline{A} - 2B > 0) = 0.436$ (or 0.435) Central limit theorem is not needed because the masses of grade *A* durians follow a normal distribution.
ii	Let *Y* be the number of grade *B* durians with a mass of more than 0.8 kg out of 50 durians. $Y \sim B(50, 0.95)$ $np = 50 \times 0.95 = 47.5 > 5$ and $n(1-p) = 50 \times 0.05 = 2.5 < 5$ Let *Y'* be the number of grade *B* durians with a mass $\leq$ 0.8 kg out of 50 durians. $Y' \sim Po(2.5)$ approx. $P(Y > 47) = P(50 - Y' > 47)$ $= P(Y' \leq 2)$ $= 0.544$
10i	$\bar{t}$ and $\bar{x}$ $\bar{t} = 29.18, \bar{x} = 154.5$ Hence, (29.18, 154.5) lies on the regression line *x* on *t*.
ii	 $r = 0.934$ (3s.f.)
iii	From the scatter diagram, *x* increases by decreasing amounts as *t* increases. Hence, a quadratic model might be more appropriate.
iv	By GC, $a = -0.673$ (3sf), $b = 179$ (3sf)
v	Substituting $t = 31.0$, $x = -0.67342(34.2 - 31.0)^2 + 179.28$ $= 172.388$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td></td><td>Expected number of cups of ice cream sold is 172.</td></tr>
<tr><td>11</td><td>$H_0: \mu = 400$
$H_1: \mu \neq 400$
Level of significance: 5%
Test Statistic: When H_0 is true, $T = \dfrac{\overline{X} - 400}{S/\sqrt{5}}$
Computation: $\nu = 5 - 1 = 4$.
By GC, $\overline{x} = 392.34, s = 12.971$, p-value = 0.257 (3sf)

Conclusion: Since p-value = 0.257 > 0.05, H_0 is not rejected at 5% level of significance. So there is insufficient evidence to conclude that the claim is invalid.

It is assumed that the masses of loaves of “Gardener” wholemeal bread follow a normal distribution.</td></tr>
<tr><td></td><td>$\overline{x} = 400 - \dfrac{102.4}{50} = 397.952$
$s^2 = \dfrac{1}{49}\left(8030.2 - \dfrac{(-102.4)^2}{50}\right) = 159.60$

$H_0: \mu = 400$
$H_1: \mu \neq 400$
Level of significance: k%
Test Statistic: When H_0 is true, $Z = \dfrac{\overline{X} - 400}{\sqrt{159.6017306}/\sqrt{55}}$

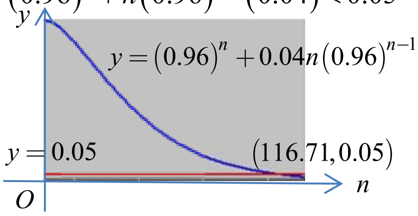
Computation:
By GC, $\overline{x} = 397.952$, p-value = 0.252 (3sf)

For H_0 to be rejected at k% level of significance,
$p\text{-value} \le \dfrac{k}{100} \Rightarrow k \ge 25.2$
Set of values = $\{k \in \mathbb{R} : k \ge 25.2\}$

“k% significance level” in this context means there is a probability of $\dfrac{k}{100}$ (or k%) that the test will conclude that the mean mass of “Gardener” wholemeal bread is not 400g, when it is actually 400g.</td></tr>
<tr><td></td><td>Let Y be the number of wrong conclusions out of n hypothesis tests
$Y \sim B(n, 0.04)$
$P(Y \le 1) < 0.05$

By GC,
<table><tr><th>n</th><th>$P(Y \le 1)$</th></tr><tr><td>116</td><td>0.05121</td></tr><tr><td>117</td><td>0.04952</td></tr></table>Least n = 117

Alternatively</td></tr>
</table>

Qn	Solution
	$P(Y \le 1) < 0.05$ $(0.96)^n + n(0.96)^{n-1}(0.04) < 0.05$  Least $n = 117$

YISHUN JUNIOR COLLEGE
2016 JC2 PRELIMINARY EXAMINATION

MATHEMATICS **9740/01**
Higher 2

Paper 1 **18 AUGUST 2016**
THURSDAY 0800h – 1100h

Additional materials :
Answer paper
Graph paper
List of Formulae (MF15)

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TIME 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and CTG in the spaces provided on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, write down the question number of the questions attempted, model of calculator used on the spaces provided on the cover page. Tie your cover page on top of the answer scripts before submission.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 A bakery sells strawberry, blueberry and walnut muffins. During a promotion, a customer purchased all 3 types of muffins and twice as many strawberry muffins as walnut muffins. The promotion price of each strawberry, blueberry and walnut muffin is $1.60, $1.75 and $2.20 respectively. Given that the customer paid $53.40 for 30 muffins, find the number of each type of muffins purchased. [4]

2 Solve the inequality $\dfrac{2}{x+2} \geq \dfrac{x+1}{3}$. [3]

Hence, find the range of values of x for which $\dfrac{2}{x+3} \geq \dfrac{x+2}{3}$. [2]

3 **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x\mathrm{e}^{-x}\right)$. [1]

(ii) Hence, find $\displaystyle\int \frac{x-x^2}{\mathrm{e}^x}\,\mathrm{d}x$. [4]

4 It is given that $\mathrm{h}(x) = x\cos x$ for $0 \leq x \leq \dfrac{\pi}{2}$. It is also known that $\mathrm{h}(-x) = \mathrm{h}(x)$ and $\mathrm{h}(\pi + x) = -\mathrm{h}(x)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{h}(x)$ for $-2\pi \leq x \leq 2\pi$. [3]

(ii) On a separate diagram, sketch the graph of $y^2 = \mathrm{h}(x)$ for $-2\pi \leq x \leq 2\pi$. [2]

5 The planes p_1, p_2 and p_3 have equations $3x+4y-7z=2$, $x-2y=4$ and $5x-4y+a\,z=3$ respectively, where a is a constant. The point C has position vector $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

(i) Given that $a=2$, find the coordinates of the point of intersection of p_1, p_2 and p_3. [2]

(ii) Find the coordinates of the foot of perpendicular from C to p_2. [3]

(iii) Find the value of a such that p_1, p_2 and p_3 have no common points. [3]

6 **Do not use a calculator in answering this question.**

The complex number z is given by $z = \dfrac{3+\mathrm{i}}{2-\mathrm{i}}$.

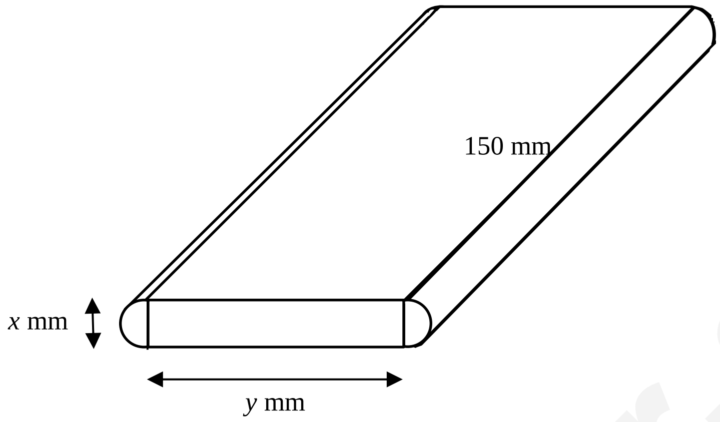
(i) Find $|z|$ and arg z in exact form. [2]

(ii) Hence, find the exact values of x and y, where $-\pi < y \leq \pi$, such that

$$\mathrm{e}^{x+\mathrm{i}2y} = \frac{3+\mathrm{i}}{2-\mathrm{i}}.$$ [3]

(iii) Find the smallest positive integer n such that $\left(\dfrac{z^2}{z^*}\right)^n$ is purely imaginary. [3]

7 A company manufactures a container of length 150 mm. The container has a uniform cross section made up of a rectangle y mm by x mm and 2 semi-circles of diameter x mm (see diagram).



Given that the container has a volume of 7200 mm^3, find the exact value of x which gives a container of minimum external surface area. [8]

8 **(a)** A bowl of hot soup is placed in a room where the temperature is a constant 20 °C. As the soup cools down, the rate of decrease of its temperature θ °C after time t minutes is proportional to the difference in temperature between the soup and its surroundings. Initially, the temperature of the soup is 80 °C and the rate of decrease of the temperature is 4 °C per minute. By writing down and solving a differential equation, show that $\theta = 20 + 60\,\mathrm{e}^{-\frac{1}{15}t}$. [6]

Find the time it takes the soup to cool to half of its initial temperature. [2]

(b) The gradient of a curve C is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^2.$$

Use the substitution $u = x + y$ to show that the above equation reduces to

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 + u^2.$$ [2]

Hence find y in terms of x given that C passes through the origin. [2]

9 **(a)** Given that $y = \ln(\cos x)$, show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -1.$$ [2]

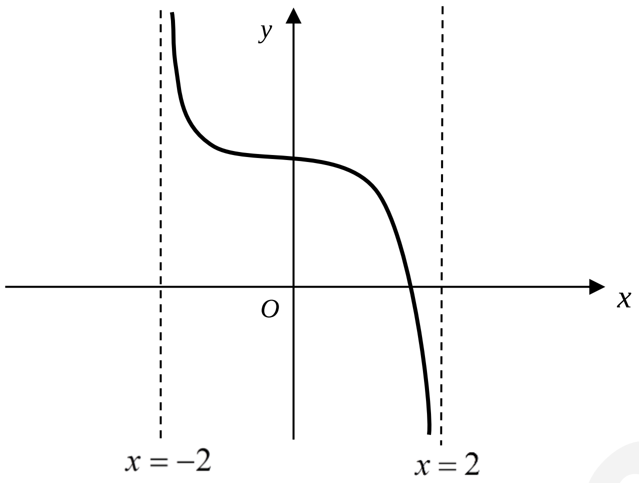
(i) By further differentiation of this result, find the Maclaurin series for y, up to and including the term in x^4. [5]

(ii) Hence, find the Maclaurin series for $\tan x$, up to and including the term in x^3. [2]

(b) Using an appropriate expansion from MF15, find the first three terms of the Maclaurin series for $\ln(k+x)^n$, where n and k are positive constants. [3]

10 **(i)** Find $\displaystyle\int \frac{1}{1+(3-y)^2}\,\mathrm{d}y$. [1]

(ii)



The diagram shows the curve with equation $y=3-\dfrac{x}{\sqrt{4-x^2}}$. Find the exact volume of revolution when the region bounded by the curve, the line $y=1$ and the y-axis is rotated completely about the y-axis. [4]

By using the substitution $x=2\sin\theta$, find the exact area of the region bounded by the curve, the line $x=1$ and the axes. [4]

11 In a training session, athletes run from a starting point S towards their coach in a straight line. When they reach the coach, they run back to S along the same straight line. A lap is completed when athletes return to S. At the beginning of the training session, the coach stands at A_1 which is 25 m away from S. After the first lap, the coach moves from A_1 to A_2 and after the second lap, he moves from A_2 to A_3 and so on. The points A_1, A_2, A_3, …, are increasingly further away from S in a straight line where A_iA_{i+1} = 1 m, $i \in \mathbb{Z}^+$. The training session will stop only when the athletes have run more than 1500 m.

An athlete completes his first lap in 20 seconds but the time for each subsequent lap is 15% more than the time for the preceding lap. Given that the athlete must complete each lap he runs and there is no resting time between laps, find the least amount of time to complete the training session, giving your answer correct to the nearest minute. [5]

Assuming that the athlete runs at a constant speed for each lap, find the number of complete laps when he has run for 15 minutes. [2]
Hence, find the distance from S and the direction of travel of the athlete after he has run for exactly 15 minutes. [3]

12 Planes p and q are perpendicular to each other. Plane p has equation $\mathbf{r} \cdot \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = -4$ and plane q contains the line l with equation $x = 0,\ y+1 = \dfrac{z-2}{4}$.

(i) Find a cartesian equation of q. [4]

(ii) Find a vector equation of the line m where p and q meet. [2]

(iii) Find the coordinates of the point C at which l intersects m. [3]

(iv) The points A and B have coordinates $(0, -1, 2)$ and $(x, 0, 0)$. If the area of triangle ABC is increasing at a rate of 17 units^2 per second, find the rate of change of x when $x = \sqrt{5}$. [5]

~ End of Paper ~

YISHUN JUNIOR COLLEGE
2016 JC2 PRELIMINARY EXAMINATION

MATHEMATICS **9740/02**
Higher 2

Paper 2 **24 August 2016**
WEDNESDAY 0800h – 1100h

Additional materials :
Answer paper
Graph paper
List of Formulae (MF15)

TIME 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and CTG in the spaces provided on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, write down the question number of the questions attempted, model of calculator used on the spaces provided on the cover page. Tie your cover page on top of the answer scripts before submission.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = 1 \quad \text{and} \quad u_{n+1} = nu_n + 1 \quad \text{for } n \geq 1.$$

Use the method of induction to prove that

$$u_n = (n-1)!\sum_{r=0}^{n-1}\frac{1}{r!}.$$ [4]

Hence, find the exact value of $\lim_{n\to\infty}\frac{u_n}{(n-1)!}$. [1]

2 Show that $\mathrm{f}(x) = \frac{5x-2}{x(x-1)(x+2)}$ can be expressed as $\frac{A}{x-1}+\frac{B}{x}+\frac{C}{x+2}$, where A, B and C are constants to be determined. [1]

Hence, find $\sum_{r=2}^{n}\mathrm{f}(r)$. (There is no need to express your answer as a single algebraic fraction.) [3]

Explain, with the aid of a sketch of $y = \mathrm{f}(x)$, $x > 1$, why $\sum_{r=2}^{n}\mathrm{f}(r) > \int_2^{n+1}\mathrm{f}(x)\,\mathrm{d}x$ for $n \geq 2$. [2]

3 The parametric equations of a curve C are $x = t - a\sin t$, $y = t\cos t$, where $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ and a is a constant. It is given that the normal to C at $x = 0$ is parallel to the x-axis.

(i) Show that $a = 1$. [3]

(ii) Sketch C, giving the coordinates of any points of intersection with the axes. [2]

(iii) Find the area of the region enclosed by C and the x-axis. [3]

4 The functions f and g are defined as follows.

$$\mathrm{f} : x \mapsto \frac{x^2+1}{2x}, \quad x > 0,$$

$$\mathrm{g} : x \mapsto \frac{1}{x}, \quad x > 0.$$

(i) Determine whether the composite function gf exists. If it exists, define gf in a similar form and find the range of gf. [4]

(ii) Give a reason why f does not have an inverse function. [1]

(iii) If the domain of f is further restricted to $x \geq k$, state the least value of k for which the function f^{-1} exists. Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. [4]

5 **(i)** Solve the equation

$$z^6 + 64 = 0,$$

giving the roots in the form $re^{i\alpha}$, where $r > 0$ and $-\pi < \alpha \le \pi$. [3]

(ii) The roots in part **(i)** represented by z_n, where $1 \le n \le 6$, are such that $-\pi < \arg(z_1) < \arg(z_2) < \cdots < \arg(z_6) \le \pi$. Show the roots on an Argand diagram and describe geometrically the relationship between the roots. [4]

The complex number w satisfies the equation $\left|iw + 4 + 4\sqrt{3}i\right| = 2$.

(iii) On the same Argand diagram, sketch the locus $\left|iw + 4 + 4\sqrt{3}i\right| = 2$. [3]

(iv) Hence, find the maximum possible value of $|w - z_n|$. [2]

Section B: Statistics [60 marks]

6 The CEO of a company with 40 000 employees wishes to investigate employees' opinions about the food stalls in the staff canteen. 2% of the employees will be chosen to take part in the survey. Explain briefly how the CEO could carry out a survey using

(i) random sampling, [2]

(ii) quota sampling. [2]

7 In a certain town, every car license plate number is a 4-digit number where the digits are chosen from 1 to 9 and cannot be repeated.

Find the number of different car license plate numbers if

(i) there are no restrictions, [1]

(ii) the digits of the car license plate number must not be in ascending order from left to right, [2]

(iii) exactly one of the digits is an even number. [2]

Due to an increasing population, it is decided that the digits used in the car license plate number can be repeated.

(iv) Find the number of different car license plate numbers where no digits can be larger than the third digit. [2]

8 **(a)** Given that events X and Y are independent, prove that events X and Y' are independent. [2]

(b) For events A and B, it is given that $P(A) = 0.5$, $P(B) = 0.6$ and $P(A' | B') = 0.3$. Find

(i) $P(A \cap B)$, [3]

(ii) $P(B' | A)$. [2]

Stating your reason, determine if events A and B are

(iii) mutually exclusive, [1]

(iv) independent. [1]

9 Rickie takes the train home after work on weekdays.

(i) The number of days in a week where Rickie finds a seat on the train is denoted by A. State, in context, two assumptions needed for A to be well modelled by a binomial distribution. [2]

Assume now that A has the distribution B(5, 0.65).

(ii) Rickie is contented if he finds a seat on two or three days in a week. Using a suitable approximation, find the probability that in a year (52 weeks), Rickie is contented in no more than 30 weeks. [5]

10 In a certain country, it is to be assumed that the number of drug trafficking cases per week can be modelled by the distribution Po(0.2) and the number of cigarette trafficking cases per week can be modelled by the independent distribution Po(0.7).

(i) Find the probability that, in a randomly chosen period of 8 weeks,

(a) the country has more than 6 drug trafficking cases, [2]

(b) the total number of drug and cigarette trafficking cases is fewer than 5. [2]

(ii) The probability that the country sees fewer than 2 drug trafficking cases in a period of n weeks is less than 0.01. Express this information as an inequality in n, and hence find the smallest possible integer value of n. [3]

(iii) Give two reasons in context why the assumptions made at the start of the question may not be valid. [2]

11 In this question, you should state clearly the values of the parameters of any normal distribution you use.

The masses, in grams, of towels manufactured by companies Alpha and Bravo are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean	Standard Deviation
Alpha	μ	20
Bravo	275	15

(i) Given that 6.68% of the towels from Alpha have mass more than 380 grams, show that the value of μ is 350 grams, correct to 3 significant figures. [2]

Towels from Alpha and Bravo are soaked in water to investigate their absorbency. A soaked towel from Alpha is 60% heavier than its dry towel, while a soaked towel from Bravo is 50% heavier than its dry towel.

(ii) Find the probability that the total mass of 4 soaked towels from Alpha and 2 soaked towels from Bravo exceeds 3 kilograms. [4]

12 A new Burger Chain, Burger Queen, claims that the mean waiting time for a burger is at most 4 minutes. The CEO of Burger Jack decides to record the waiting time, x minutes, for a burger at Burger Queen at 80 different locations. The results are summarised by

$$\sum(x-4)=25\ ,\ \sum(x-4)^2=140\,.$$

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Test, at the 1% level of significance, whether there is any evidence to doubt Burger Queen's claim. [4]

(iii) It is assumed that the standard deviation of the waiting time for a Burger Queen burger is 1.5 minutes. Given that the mean waiting time at another 80 locations is $\bar{x}$, use an algebraic method to find the set of values of $\bar{x}$ for which Burger Queen's claim would not be rejected at the 10% level of significance. [3]

13 **(i)** Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A), (B), (C) below. In each case your diagram should include 6 points, approximately equally spaced with respect to x, and with all x- and y- values positive. The letters a, b, c, d, e and f represent constants.

(A) $y=a+bx^2$, where a is positive and b is negative,

(B) $y=c+\dfrac{d}{x}$, where both c and d are positive,

(C) $y=e+fx$, where e is positive and f is negative. [3]

An archaeologist found an unknown substance on an excavation trip. Research is being carried out to investigate how the mass of the substance varies with time, measured from when it is placed in a cooled chamber. Observations at successive times give the data shown in the following table.

Time (x hours)	100	800	1500	3000	6000	8000
Mass (y grams)	25	8	5	4	3.5	3.3

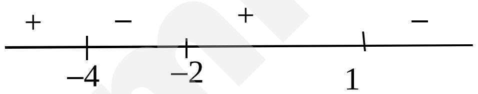
(ii) Draw the scatter diagram for these values, labelling the axes. [1]

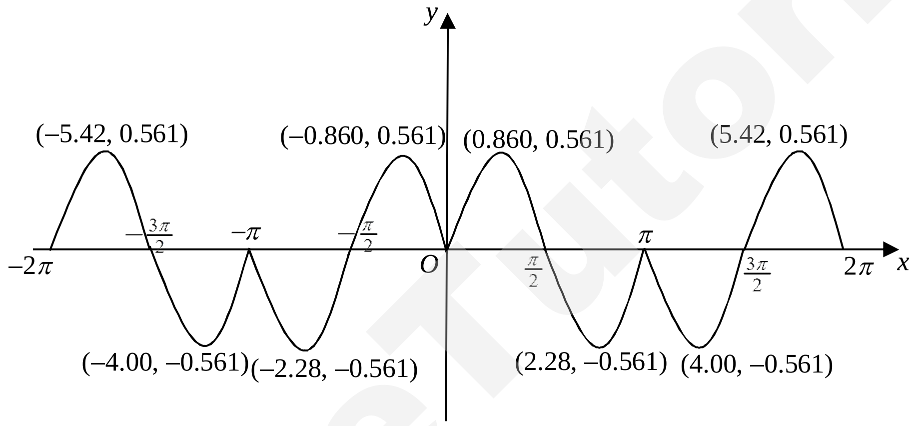
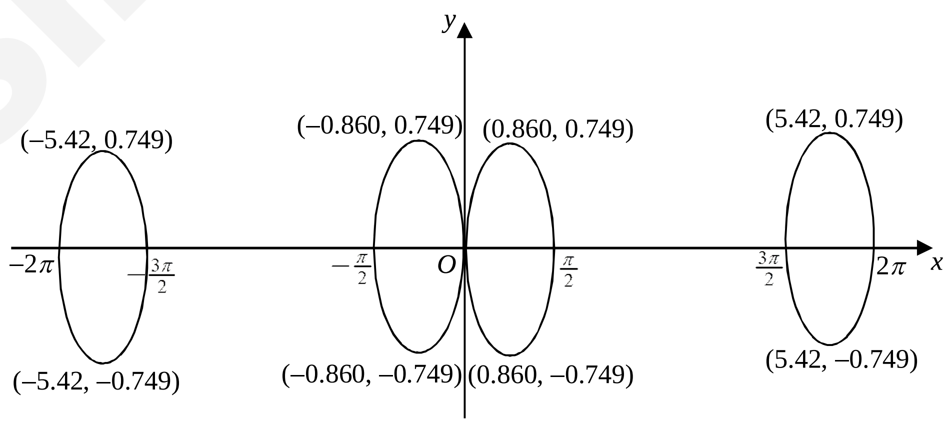
(iii) Explain which of the three cases in part **(i)** is the most appropriate for modelling these values, and calculate the product moment correlation coefficient for this case. [2]

(iv) Use the case that you identified in part **(iii)** to find the equation of a suitable regression line and estimate the time when the mass of the substance is 10 grams. Comment on the reliability of the estimate. [3]

~ End of Paper ~

YISHUN JUNIOR COLLEGE
2016 JC2 PRELIMINARY EXAM PAPER 1
H2 MATHEMATICS
SOLUTIONS

Qn	Solution
1	Let S, B and W be the number of strawberry, blueberry and walnut muffins purchased respectively. $S+B+W=30$ $1.6S+1.75B+2.2W=53.40$ $S=2W \Rightarrow S-2W=0$ From GC, $S=12,\ B=12,\ W=6$
2	$\dfrac{2}{x+2} \geq \dfrac{x+1}{3}$ $\dfrac{2}{x+2}-\dfrac{x+1}{3} \geq 0$ $\dfrac{6-(x+1)(x+2)}{3(x+2)} \geq 0$ $\dfrac{-x^2-3x+4}{3(x+2)} \geq 0$ $\dfrac{-(x+4)(x-1)}{3(x+2)} \geq 0$  Hence, $x \leq -4$ or $-2 < x \leq 1$ For $\dfrac{2}{x+3} \geq \dfrac{x+2}{3}$ From above, $x+1 \leq -4$ or $-2 < x+1 \leq 1$ $x \leq -5$ or $-3 < x \leq 0$

3(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathrm{e}^{-x}\right) = -x\mathrm{e}^{-x} + \mathrm{e}^{-x}$ $= \mathrm{e}^{-x}(1-x)$
(ii)	$\int \frac{x - x^2}{\mathrm{e}^x}\,\mathrm{d}x = \int x\frac{(1-x)}{\mathrm{e}^x}\,\mathrm{d}x$ $= x^2\mathrm{e}^{-x} - \int x\mathrm{e}^{-x}\mathrm{d}x$ $= x^2\mathrm{e}^{-x} - \left[-x\mathrm{e}^{-x} + \int \mathrm{e}^{-x}\mathrm{d}x\right]$ $= x^2\mathrm{e}^{-x} + x\mathrm{e}^{-x} + \mathrm{e}^{-x} + C$
4(i)	
4(ii)	

5(i)	$3x+4y-7z=2$ $x-2y=4$ $5x-4y+2z=3$ From GC, $x=-\frac{15}{31}, y=-\frac{139}{62}, z=-\frac{55}{31}$ $\therefore$ the point of intersection is $\left(-\frac{15}{31},-\frac{139}{62},-\frac{55}{31}\right)$
(ii)	Let F be the foot of perpendicular from C to plane. Equation of CF: $\mathbf{r}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix}, \lambda\in\mathbb{R}$ $\overrightarrow{OF}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix}$, for some $\lambda\in\mathbb{R}$ F also lies on plane p_2 $\overrightarrow{OF}\bullet\begin{pmatrix}1\\-2\\0\end{pmatrix}=4$ $\begin{pmatrix}-1+\lambda\\2-2\lambda\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\0\end{pmatrix}=4$ $-1+\lambda-4+4\lambda=4$ $5\lambda=9$ $\lambda=\frac{9}{5}$ $\therefore\overrightarrow{OF}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\frac{9}{5}\begin{pmatrix}1\\-2\\0\end{pmatrix}=\begin{pmatrix}4/5\\-8/5\\1\end{pmatrix}$ the coordinates is $\left(\frac{4}{5},-\frac{8}{5},1\right)$
(iii)	$3x+4y-7z=2$ $x-2y=4$ From GC, p_1 and p_2 intersect at the line $\mathbf{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\mu\begin{pmatrix}14\\7\\10\end{pmatrix}, \mu\in\mathbb{R}$ No common points $\Rightarrow$ p_3 must be parallel to the line.

	$\begin{pmatrix}5\\-4\\a\end{pmatrix}\cdot\begin{pmatrix}14\\7\\10\end{pmatrix}=0$ $42+10a=0$ $a=-4.2$
6(i)	$z=\dfrac{3+\mathrm{i}}{2-\mathrm{i}}=\dfrac{(3+\mathrm{i})(2+\mathrm{i})}{2^2+1}=\dfrac{1}{5}(5+5\mathrm{i})=1+\mathrm{i}$ Therefore, $\lvert z\rvert=\sqrt{2}$ [Or $\lvert z\rvert=\left\lvert\dfrac{3+\mathrm{i}}{2-\mathrm{i}}\right\rvert=\dfrac{\sqrt{10}}{\sqrt{5}}=\sqrt{2}$] arg $z=\dfrac{\pi}{4}$
(ii)	$\mathrm{e}^{x+\mathrm{i}2y}=z$ $\mathrm{e}^{x}\mathrm{e}^{\mathrm{i}2y}=\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ $\Rightarrow \mathrm{e}^{x}=\sqrt{2}$ or $\mathrm{e}^{\mathrm{i}2y}=\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ or $\mathrm{e}^{\mathrm{i}\left(-\frac{7\pi}{4}\right)}$ $\Rightarrow x=\ln\sqrt{2}=\dfrac{1}{2}\ln 2$ or $y=\dfrac{\pi}{8}$ or $-\dfrac{7\pi}{8}$
(iii)	For $\left(\dfrac{z^2}{z^*}\right)^n$ to be purely imaginary, $\arg\left(\dfrac{z^2}{z^*}\right)^n=(2k+1)\dfrac{\pi}{2},\ k\in\mathbb{Z}$ $n\left[2\arg z-\arg z^*\right]=(2k+1)\dfrac{\pi}{2}$ $n\left[\dfrac{\pi}{2}+\dfrac{\pi}{4}\right]=(2k+1)\dfrac{\pi}{2}$ $n=\dfrac{2}{3}(2k+1)$ Hence, the smallest positive integer n = 2

7	Let A and V be the external surface area and volume of the container respectively. $A = 300y + 150\pi x + 2xy + 2\pi\left(\frac{x}{2}\right)^2$ $V = 150xy + 150\pi\left(\frac{x}{2}\right)^2$ $7200 = 150xy + \frac{75}{2}\pi x^2$ $y = \frac{48}{x} - \frac{\pi x}{4}$ Substitute $y = \frac{48}{x} - \frac{\pi x}{4}$ into surface area equation: $A = 300\left(\frac{48}{x} - \frac{\pi x}{4}\right) + 150\pi x + 2x\left(\frac{48}{x} - \frac{\pi x}{4}\right) + \frac{\pi x^2}{2}$ $= \frac{14400}{x} + 96 + 75\pi x$ $\frac{dA}{dx} = -14400x^{-2} + 75\pi$ To find least surface area, $\frac{dA}{dx} = 0$ $-\frac{14400}{x^2} + 75\pi = 0$ $x = \sqrt{\frac{192}{\pi}}$ or $x = -\sqrt{\frac{192}{\pi}}$ (rejected $\because x > 0$) By 2nd Derivative Test $\frac{d^2A}{dx^2} = \frac{28800}{x^3}$ When $x = \sqrt{\frac{192}{\pi}}$, $\frac{d^2A}{dx^2} > 0$, A is minimum.
8(a)	Note that $\theta \geq 20$ or $\theta - 20 \geq 0$ $\frac{d\theta}{dt} = -k(\theta - 20)$, $k > 0$ Given when $t = 0$, $\theta = 80$, $\frac{d\theta}{dt} = -4$ $-4 = -k(80 - 20) \Rightarrow k = \frac{1}{15}$ $\frac{d\theta}{dt} = -\frac{1}{15}(\theta - 20)$

	$\int \frac{1}{\theta - 20}\, d\theta = -\frac{1}{15}\int 1\, dt$ $\ln(\theta - 20) = -\frac{1}{15}t + C$ $(\because \theta - 20 > 0)$ $\theta - 20 = e^{-\frac{1}{15}t+C}$ $\theta - 20 = Ae^{-\frac{1}{15}t}$, where $A = e^c$ $\Rightarrow \theta = 20 + Ae^{-\frac{1}{15}t}$ Alternatively: $\ln\lvert\theta - 20\rvert = -\frac{1}{15}t + C$ $\lvert\theta - 20\rvert = e^{-\frac{1}{15}t+C}$ $\theta - 20 = Ae^{-\frac{1}{15}t}$, where $A = \pm e^c$ $\Rightarrow \theta = 20 + Ae^{-\frac{1}{15}t}$ When $t = 0$, $\theta = 80$, $\Rightarrow 80 = 20 + A$ i.e. $A = 60$ $\therefore \theta = 20 + 60e^{-\frac{1}{15}t}$ $40 = 20 + 60e^{-\frac{1}{15}t}$ $e^{-\frac{1}{15}t} = \frac{1}{3}$ $-\frac{1}{15}t = \ln\frac{1}{3} = -\ln 3$ $t = 15\ln 3$
(b)	$u = x + y$ $\Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$ $\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$ $\frac{dy}{dx} = (x+y)^2$ $\Rightarrow \frac{du}{dx} - 1 = u^2$ $\Rightarrow \frac{du}{dx} = 1 + u^2$ $\int \frac{1}{1+u^2}\, du = \int 1\, dx$ $\tan^{-1}(u) = x + C$ $\tan^{-1}(x+y) = x + C$ When $x = 0$, $y = 0$, $C = 0$ $\therefore x + y = \tan x$ Hence, $y = \tan x - x$

9(a)	$y=\ln(\cos x)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sin x}{\cos x}=-\tan x$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\sec^2 x=-\left(1+\tan^2 x\right)$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-1$
(i)	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ When $x = 0$, $y = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0$, $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=-2$ $y=\ln(\cos x)=\frac{x^2}{2!}(-1)+\frac{x^4}{4!}(-2)+\ldots$ $= -\frac{1}{2}x^2-\frac{1}{12}x^4+\ldots$
(ii)	$\tan x=-\frac{\mathrm{d}y}{\mathrm{d}x}$ $=-\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{2}x^2-\frac{1}{12}x^4+\ldots\right]$ $=-\left(-x-\frac{1}{3}x^3+\ldots\right)$ $=x+\frac{1}{3}x^3+\ldots$

(b)	$\mathrm{f}(x) = \ln(k+x)^n$ $= n\ln(k+x)$ $= n\ln\left(k\left(1+\frac{x}{k}\right)\right)$ $= n\left[\ln k + \ln\left(1+\frac{x}{k}\right)\right]$ $= n\ln k + n\ln\left(1+\frac{x}{k}\right)$ $= n\ln k + n\left(\frac{x}{k} - \frac{1}{2}\left(\frac{x}{k}\right)^2 + \cdots\right)$ $= n\ln k + \frac{nx}{k} - \frac{nx^2}{2k^2} + \cdots$
10 (i)	$\int \frac{1}{1+(3-y)^2}\,\mathrm{d}y = (-1)\tan^{-1}\left(\frac{3-y}{1}\right) + C$ $= -\tan^{-1}(3-y) + C$
(ii)	$y = 3 - \frac{x}{\sqrt{4-x^2}}$ $\frac{x}{\sqrt{4-x^2}} = 3-y$ $\frac{x^2}{4-x^2} = (3-y)^2$ $x^2 = (3-y)^2(4-x^2)$ $x^2 = 4(3-y)^2 - x^2(3-y)^2$ $x^2 + x^2(3-y)^2 = 4(3-y)^2$ $x^2 = \frac{4(3-y)^2}{1+(3-y)^2}$ $= 4 - \frac{4}{1+(3-y)^2}$

	Volume of revolution about the y-axis $=\pi\int_1^3 x^2\ \mathrm{d}y$ $=\pi\int_1^3 4-\frac{4}{1+(3-y)^2}\ \mathrm{d}y$ $=\pi\int_1^3 4\ \mathrm{d}y-4\pi\int_1^3\frac{1}{1+(3-y)^2}\ \mathrm{d}y$ $=\pi[4y]_1^3-4\pi\left[-\tan^{-1}(3-y)\right]_1^3$ $=8\pi-4\pi\tan^{-1}(2)$
	Using the substitution $x=2\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=2\cos\theta$ When $x=1, \sin\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6}$ When $x=0, \sin\theta=0\Rightarrow\theta=0$ Area under the curve $=\int_0^1 3-\frac{x}{\sqrt{4-x^2}}\ \mathrm{d}x$ $=\int_0^{\frac{\pi}{6}}\left(3-\frac{2\sin\theta}{\sqrt{4-4\sin^2\theta}}\right)(2\cos\theta)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{6}} 6\cos\theta-\frac{4\sin\theta\cos\theta}{2\cos\theta}\ \mathrm{d}\theta$ $=[6\sin\theta+2\cos\theta]_0^{\frac{\pi}{6}}$ $=1+\sqrt{3}$

11	Distance travelled per lap is in AP: $a_1 = 50$, $d = 2 \times 1 = 2$. Given total distance travelled > 1500 $\frac{n}{2}[\,2(50) + (n-1)2] > 1500$ $n^2 + 49n - 1500 > 0$ $(n + 70.33)(n - 21.33) > 0$ $n < -70.33$ or $n > 21.33$ Since $n \in \mathbb{Z}^+$, least $n = 22$ Time taken per lap is in GP: $a_1 = 20$, $r = 1.15$ Required least time taken $= \frac{20\left((1.15)^{22} - 1\right)}{1.15 - 1}$ ≈ 2752.6 s ≈ 46 min $\frac{20\left((1.15)^{n} - 1\right)}{1.15 - 1} \geq 900$ $(1.15)^n \geq 7.75$ $n \geq \frac{\ln 7.75}{\ln 1.15}$ $n \geq 14.65$ Number of complete laps = 14 $S_{14} = \frac{20\left((1.15)^{14} - 1\right)}{1.15 - 1}$ $= 810.094$ s He needs to run for another $900 - 810.094 = 89.906$ s Distance $T_{15} = 50 + (15 - 1)2 = 78$ $\frac{89.906}{20(1.15)^{14}} \times 78 = 49.555$ He is running <u>towards</u> S and at a distance $78 - 49.555 \approx$ <u>28.4 m</u> away from S.

12 (i)	$l:\mathbf{r}=\begin{pmatrix}0\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\4\end{pmatrix},\lambda\in\mathbb{R}.$ $\mathbf{n}_2=\begin{pmatrix}-2\\3\\1\end{pmatrix}\times\begin{pmatrix}0\\1\\4\end{pmatrix}$ $=\begin{pmatrix}11\\8\\-2\end{pmatrix}$ $q:\mathbf{r}\bullet\begin{pmatrix}11\\8\\-2\end{pmatrix}=\begin{pmatrix}0\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}11\\8\\-2\end{pmatrix}$ $=-12$ $\therefore$ the Cartesian equation is $11x+8y-2z=-12$
(ii)	$-2x+3y+z=-4$ $11x+8y-2z=-12$ From GC, $x=-\frac{4}{49}+\frac{2}{7}\mu, y=-\frac{68}{49}-\frac{1}{7}\mu, z=\mu$ $\therefore\mathbf{r}=\begin{pmatrix}-\frac{4}{49}\\-\frac{68}{49}\\0\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\7\end{pmatrix},\mu\in\mathbb{R}$
(iii)	$\begin{pmatrix}0\\-1+\lambda\\2+4\lambda\end{pmatrix}=\begin{pmatrix}-4/49+2\mu\\-68/49-\mu\\7\mu\end{pmatrix}$ From GC, $\lambda=-\frac{3}{7},\mu=\frac{2}{49}$ Substitute $\lambda=-\frac{3}{7}$ into l: $\mathbf{r}=\begin{pmatrix}0\\-1\\2\end{pmatrix}-\frac{3}{7}\begin{pmatrix}0\\1\\4\end{pmatrix}=\begin{pmatrix}0\\-\frac{10}{7}\\\frac{2}{7}\end{pmatrix}$ $\therefore$ the coordinates of C is $\left(0,-\frac{10}{7},\frac{2}{7}\right)$.

(iv)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$ $=\begin{pmatrix}x\\0\\0\end{pmatrix}-\begin{pmatrix}0\\-1\\2\end{pmatrix}$ $=\begin{pmatrix}x\\1\\-2\end{pmatrix}$ $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$ $=\begin{pmatrix}0\\-\frac{10}{7}\\\frac{2}{7}\end{pmatrix}-\begin{pmatrix}0\\-1\\2\end{pmatrix}$ $=\begin{pmatrix}0\\-\frac{3}{7}\\-\frac{12}{7}\end{pmatrix}$ Area of $ABC,\ R=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}x\\1\\-2\end{pmatrix}\times\begin{pmatrix}0\\-3/7\\-12/7\end{pmatrix}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}-18/7\\12x/7\\-3x/7\end{pmatrix}\right\|=\frac{3}{14}\left\|\begin{pmatrix}-6\\4x\\-x\end{pmatrix}\right\|$ $=\frac{3}{14}\sqrt{(-6)^2+(4x)^2+(-x)^2}$ $=\frac{3}{14}\sqrt{36+16x^2+x^2}$ $=\frac{3}{14}\sqrt{36+17x^2}$

	$\frac{dR}{dx} = \frac{3}{14} \times \frac{1}{2}\left(36+17x^2\right)^{-\frac{1}{2}}\left(34x\right)$ $= \frac{102x}{28\sqrt{36+17x^2}}$ $= \frac{51x}{14\sqrt{36+17x^2}}$ when $x=\sqrt{5}, \frac{dx}{dt} = \frac{dx}{dR} \times \frac{dR}{dt}$ $= \frac{14(11)}{51\sqrt{5}} \times 17$ $= \frac{154}{3\sqrt{5}}$ units per second

YISHUN JUNIOR COLLEGE
2016 JC2 PRELIMINARY EXAM PAPER 2
H2 MATHEMATICS
SOLUTIONS

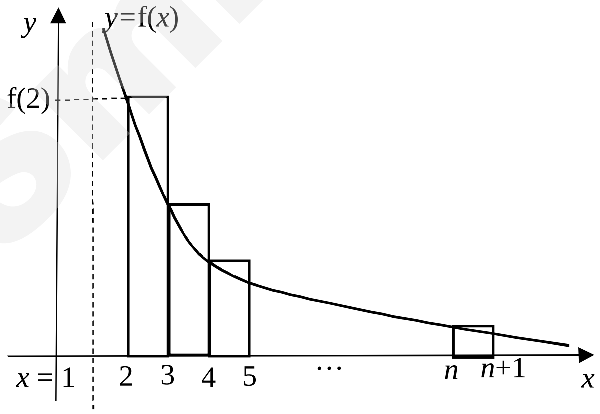
Qn	Solution
1	Let P_n be the statement "$u_n = (n-1)!\sum_{r=0}^{n-1}\frac{1}{r!}$" When $n=1$, L.H.S. = u_1 = 1 (Given) R.H.S = $(1-1)!\left(\frac{1}{0!}\right) = 1$ Hence, P_1 is true. Assume that P_k is true for some $k \in \mathbb{Z}^+$, ie, $u_k = (k-1)!\sum_{r=0}^{k-1}\frac{1}{r!}$. To show P_{k+1} is true, ie, $u_{k+1} = (k)!\sum_{r=0}^{k}\frac{1}{r!}$. $u_{k+1} = ku_k + 1$ $= k(k-1)!\sum_{r=0}^{k-1}\frac{1}{r!} + 1$ $= k!\sum_{r=0}^{k-1}\frac{1}{r!} + \frac{k!}{k!}$ $= k!\left(\sum_{r=0}^{k-1}\frac{1}{r!} + \frac{1}{k!}\right)$ $= k!\sum_{r=0}^{k}\frac{1}{r!}$ Therefore, P_k is true $\Rightarrow$ P_{k+1} is true By Mathematical Induction, P_n is true for all $n \geq 1$. $\therefore \frac{u_n}{(n-1)!} = \sum_{r=0}^{n-1}\frac{1}{r!}$ $\lim_{n\to\infty}\frac{u_n}{(n-1)!} = \sum_{r=0}^{\infty}\frac{1}{r!} = e^x\vert_{x=1} = e$

2

$$f(x)=\frac{5x-2}{x(x-1)(x+2)}=\frac{1}{x-1}+\frac{1}{x}-\frac{2}{x+2}$$

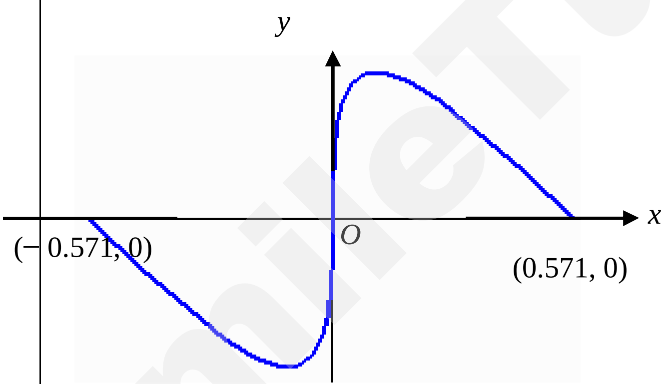
$$\sum_{r=2}^{n} f(r)=\sum_{r=2}^{n}\left(\frac{1}{r-1}+\frac{1}{r}-\frac{2}{r+2}\right)$$

$$\begin{aligned}
&=\frac{1}{1}+\frac{1}{2}-\cancel{\frac{2}{4}}\\
&+\frac{1}{2}+\frac{1}{3}-\cancel{\frac{2}{5}}\\
&+\frac{1}{3}+\cancel{\frac{1}{4}}-\cancel{\frac{2}{6}}\\
&+\cancel{\frac{1}{4}}+\cancel{\frac{1}{5}}-\cancel{\frac{2}{7}}\\
&+\cancel{\frac{1}{5}}+\cancel{\frac{1}{6}}-\cancel{\frac{2}{8}}\\
&+\cdots\\
&+\cancel{\frac{1}{n-4}}+\cancel{\frac{1}{n-3}}-\cancel{\frac{2}{n-1}}\\
&+\cancel{\frac{1}{n-3}}+\cancel{\frac{1}{n-2}}-\cancel{\frac{2}{n}}\\
&+\cancel{\frac{1}{n-2}}+\cancel{\frac{1}{n-1}}-\frac{2}{n+1}\\
&+\cancel{\frac{1}{n-1}}+\frac{1}{n}-\frac{2}{n+2}\\
&=\frac{8}{3}-\left(\frac{1}{n}+\frac{2}{n+1}+\frac{2}{n+2}\right)
\end{aligned}$$



From the above diagram,
Sum of areas of (n–1) rectangles
> Area under graph of y = f(x) from x = 2 to n+1

$$(1)f(2)+(1)f(3)+(1)f(4)+\ldots+(1)f(n) > \int_2^{n+1} f(x)\,dx \;\therefore\; \sum_{r=2}^{n} f(r) > \int_2^{n+1} f(x)\,dx$$

3(i)	$x = t - a\sin t \quad y = t\cos t$ $\frac{dx}{dt} = 1 - a\cos t \quad \frac{dy}{dt} = -t\sin t + \cos t$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ $= \frac{-t\sin t + \cos t}{1 - a\cos t}$ normal parallel to *x*-axis $\Rightarrow 1 - a\cos t = 0$ $\Rightarrow a = \frac{1}{\cos t}$ --- (1) When $x = 0$, $t - a\sin t = 0$ ----- (2) Sub (1) into (2): $t - \tan t = 0$ From GC, $t = 0$ (since $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$) Sub into (1): $a = \frac{1}{\cos 0} = 1$ (shown)
(ii)	
(iii)	Area of region $= \left\lvert \int_{1-\frac{\pi}{2}}^{0} y \, dx \right\rvert + \int_{0}^{\frac{\pi}{2}-1} y \, dx$ $= \left\lvert \int_{-\frac{\pi}{2}}^{0} t\cos t\left(1-\cos t\right) dt \right\rvert + \int_{0}^{\frac{\pi}{2}} t\cos t\left(1-\cos t\right) dt$ $= 0.408$

4(i)

gf exists when $R_f \subseteq D_g$.

Since $R_f = [1,\infty) \subseteq (0,\infty) = D_g$, therefore gf exists.

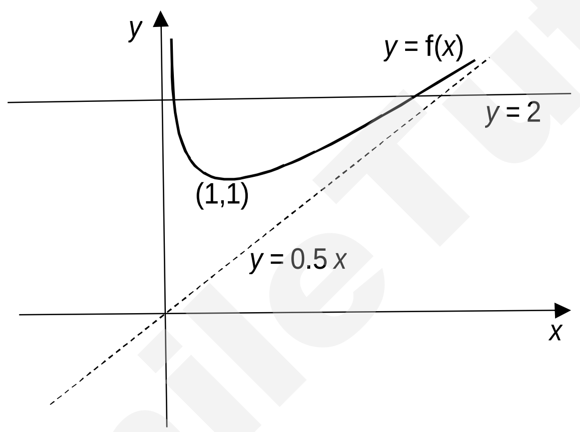
$$\text{gf}(x) = \text{g}\left(\frac{x^2+1}{2x}\right)$$
$$= \frac{2x}{x^2+1}$$

$$\text{gf} : x \mapsto \frac{2x}{x^2+1}, \quad x > 0$$

$$(0,\infty) \xrightarrow{\text{f}} [1,\infty) \xrightarrow{\text{g}} (0,1]$$

$$\therefore R_{gf} = (0,1]$$

4(ii)



The line y = 2 cuts the graph of f more than once. Hence, f is not a one–one function. Therefore, f does not have an inverse.

4(iii)

From the graph, least value of k = 1

$x^2 - 2yx + 1 = 0$

$$x = \frac{2y \pm \sqrt{4y^2-4}}{2} = y \pm \sqrt{y^2-1}$$

Since $x \geq 1$, $x = y + \sqrt{y^2-1}$

$$\text{f}^{-1}(x) = x + \sqrt{x^2-1}, \; x \geq 1$$

5(i)

$z^6 = -64 = 64\,e^{i\pi}$

$\quad = 64\,e^{i(2k+1)\pi},\ k \in \mathbb{Z}$

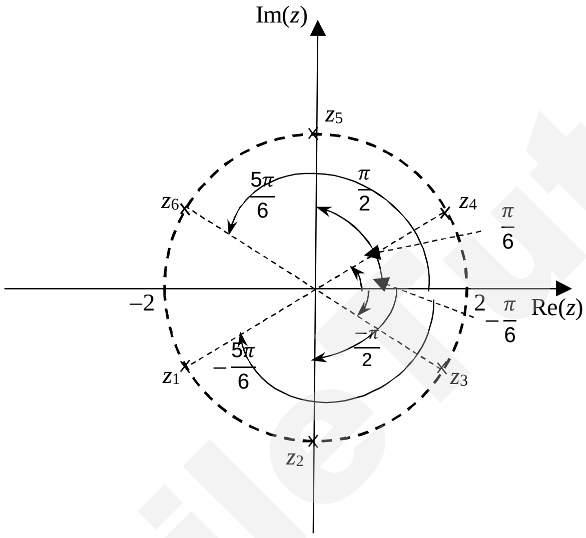
$z = 2\,e^{i\left(\frac{2k+1}{6}\right)\pi},\ k = 0, \pm 1, \pm 2, -3$

$z = 2e^{i\left(\frac{\pi}{6}\right)}, 2e^{i\left(\frac{\pi}{2}\right)}, 2e^{i\left(-\frac{\pi}{6}\right)}, 2e^{i\left(\frac{5\pi}{6}\right)}, 2e^{i\left(-\frac{\pi}{2}\right)}, 2e^{i\left(-\frac{5\pi}{6}\right)}$

(ii)

$z_1 = 2e^{i\left(-\frac{5\pi}{6}\right)}, z_2 = 2e^{i\left(-\frac{\pi}{2}\right)}, z_3 = 2e^{i\left(-\frac{\pi}{6}\right)}$

$z_4 = 2e^{i\left(\frac{\pi}{6}\right)}, z_5 = 2e^{i\left(\frac{\pi}{2}\right)}, z_6 = 2e^{i\left(\frac{5\pi}{6}\right)}$



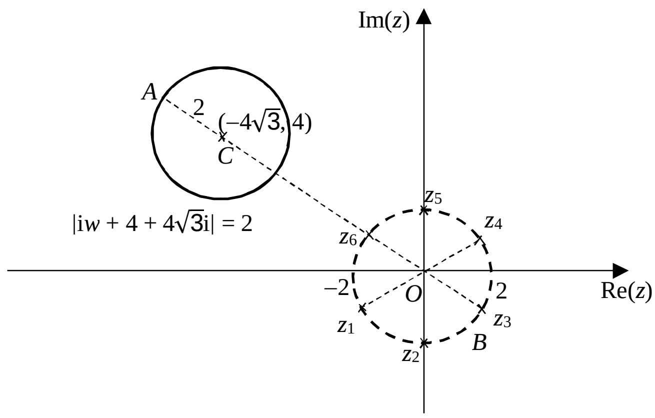
The 6 points representing the 6 roots are the vertices of a regular hexagon.

(iii)

$|iw + 4 + 4\sqrt{3}\,i| = 2$

$|\,i\,|\,|w - i(4 + 4\sqrt{3}\,i)| = 2$

$|w - (-4\sqrt{3} + 4i)| = 2$



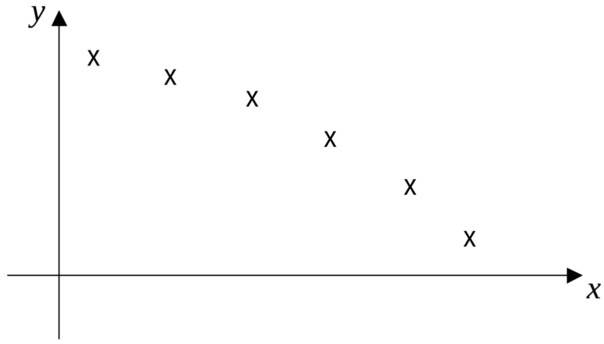
(iv)	As arg $(-4\sqrt{3}+4\mathrm{i}) = \dfrac{5\pi}{6}$, O, C and B are collinear. Hence, maximum $\lvert w - z_n\rvert = \lvert w - z_3\rvert = AB$ $= OC + CA + OB$ $= \sqrt{\left(4\sqrt{3}\right)^2 + 4^2} + 2 + 2$ $= 12$
6(i) **(ii)**	Number the employees from 1 to 40 000. Randomly select 800 numbers using a random number generator. The employees corresponding to these 800 numbers are selected for the survey. The manager can survey the first 400 male employees and first 400 female employees who step into the canteen on a particular day.
7(i) **(ii)** **(iii)** **(iv)**	No of ways = $9\times8\times7\times6$ $= 3024$ No of ways = $3024 - C_4^9$ $= 2898$ No of ways = $C_1^4 \times C_3^5 \times 4!$ $= 960$ No of ways= $\sum_{r=1}^{9} r^3$ $= 2025$
8(a) **(b)** **(i)**	$\mathrm{P}(X \cap Y') = \mathrm{P}(X) - \mathrm{P}(X \cap Y)$ $= \mathrm{P}(X) - \mathrm{P}(X)\times\mathrm{P}(Y)$ since X and Y are ind. $= P(X)[1 - P(Y)]$ $= \mathrm{P}(X)\times\mathrm{P}(Y')$ Since $\mathrm{P}(X \cap Y') = \mathrm{P}(X)\times\mathrm{P}(Y')$, events X and Y' are independent. P(A'|B')=0.3 $\dfrac{\mathrm{P}(A'\cap B')}{\mathrm{P}(B')} = 0.3$ $\mathrm{P}(A'\cap B') = 0.3\times(1-0.6)$ $=0.12$ $\mathrm{P}(A\cup B) = 1 - 0.12 = 0.88$ $P(A\cup B) = P(A) + P(B) - P(A\cap B)$ $\mathrm{P}(A\cap B) = 0.5+0.6-0.88$ $=0.22$

(ii)	$P(B' \cap A) = P(A) - P(A \cap B)$ $= 0.5 - 0.22 = 0.28$ $P(B' \mid A) = \dfrac{P(B' \cap A)}{P(A)}$ $= \dfrac{0.28}{0.5} = 0.56$
(iii)	Since $P(A \cap B) = 0.22 \neq 0$, events A and B are not mutually exclusive
(iv)	$P(A \cap B) = 0.22$ $P(A) \times P(B) = 0.5 \times 0.6 = 0.3$ Since $P(A \cap B) \neq P(A) \times P(B)$, events A and B are not independent.
9(i) **(ii)**	(1) The probability of Rickie finding a seat on the train is constant every weekday. (2) The event of Rickie finding a seat on the train on one weekday is independent of the other weekdays. A~B(5, 0.65) $P(A = 2 \text{ or } 3) = P(A \leq 3) - P(A \leq 1)$ $= 0.51756$ Let X be the number of weeks, out of 52, that Rickie is contented. Then $X \sim B(52,\ 0.51756)$ Since n = 52 is large, np = 52(0.51756)=26.91312 > 5, nq = 52(1−0.51756)=25.08688 > 5, $X \sim N(26.91312,\ 52 \times 0.51756 \times (1 - 0.51756))$ approx. i.e. $X \sim N(26.91312, 12.98397)$ approx. $P(X \leq 30) \rightarrow P(X < 30.5)$ using continuity correction ≈ 0.840 (3 s.f)

10(i) (a)	Let X and Y be the r.v. "number of drug trafficking and cigarette trafficking cases in 8 weeks respectively". Then $X \sim \text{Po}(1.6)$, $Y \sim \text{Po}(5.6)$ $\text{P}(X > 6) = 1 - \text{P}(X \le 6)$ ≈ 0.00134 (3 s.f)
(i) (b)	$X + Y \sim \text{Po}(7.2)$ $\text{P}(X + Y < 5) = \text{P}(X + Y \le 4)$ ≈ 0.156 (3 s.f)
(ii)	Let W be the number of drug trafficking cases in a period of n weeks. Then $W \sim \text{Po}(0.2n)$ $\text{P}(W < 2) < 0.01$ $\text{P}(W \le 1) < 0.01$ $\text{e}^{-0.2n} + 0.2n(\text{e}^{-0.2n}) < 0.01$ Using GC, If n = 33, $\text{P}(W \le 1) = 0.0103 > 0.01$ If n = 34, $\text{P}(W \le 1) = 0.00869 < 0.01$ $\therefore$ smallest possible integer value of n is 34. Alternative method: Plot graph Using GC, $n > 33.19176$ $\therefore$ smallest possible integer value of n is 34.
(iii)	The number of drug and cigarette trafficking cases per week may decrease after police's raids so the trafficking cases may not occur with a constant average rate. A person can be both a drug and cigarette trafficker so the drug trafficking cases and cigarette trafficking cases may not be independent.
11(i)	Let A and B be the mass of a randomly chosen towel from Alpha and Bravo respectively. $\text{P}(A > 380) = 0.0668$ $\text{P}(A \le 380) = 0.9332$ $\text{P}(Z \le \frac{380 - \mu}{20}) = 0.9332$ Using GC, $\frac{380 - \mu}{20} = 1.5000556$

	$380-\mu=30.001112$ $\mu=349.998888$ $=350$ (3 s.f)
(ii)	$T=1.6(A_1+A_2+A_3+A_4)+1.5(B_1+B_2)$ $\sim \mathrm{N}(1.6\times4\times350+1.5\times2\times275,\ 1.6^2\times4\times20^2+1.5^2\times2\times15^2)$ i.e. $T\sim \mathrm{N}(3065,\ 5108.5)$ $\mathrm{P}(T>3000)=0.81843...$ ≈ 0.818 (3 s.f)
12(i)	Unbiased estimate of population mean, $\bar{x}=\frac{\sum(x-4)}{80}+4$ $=\frac{25}{80}+4=4.3125$ Unbiased estimate of population variance, $s^2=\frac{1}{80-1}\left(140-\frac{25^2}{80}\right)$ $=1.673259\ldots$ ≈ 1.67 (3 s.f)
(ii)	$H_0:\mu=4$ $H_1:\mu>4$ Under H_0, the test statistic is $Z=\frac{\bar{X}-\mu}{S/\sqrt{n}}\sim N(0,1)$ approx. (by CLT), where $\mu=4,\ \bar{x}=4.3125,\ s^2=1.6733,\ n=80$ Level of significance=1% Using GC, p-value=0.015357 (5 s.f) Since p-value=0.015357>0.01, we do not reject H_0 and conclude that at the 1% level, there is no sufficient evidence to doubt Burger Queen's claim.
(iii)	Critical value=1.28155 In order not to reject H_0, $\frac{\bar{x}-4}{1.5/\sqrt{80}}<1.28155$ $\bar{x}<4.2149...$ Required set = $\{\bar{x}\in\mathbb{R}^+:\bar{x}<4.21\}$

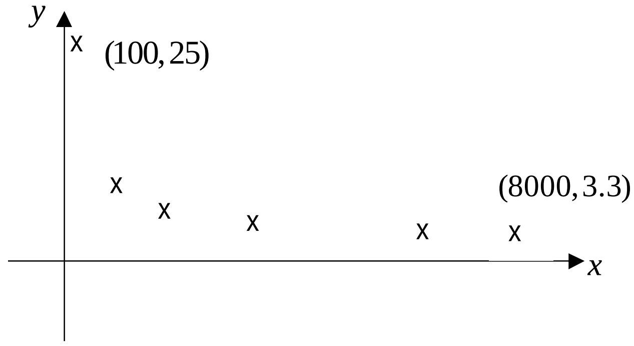
13(i) (A) $y = a + bx^2$, where a is positive and b is negative



(B) $y = c + \frac{d}{x}$, where both c and d are positive

(C) $y = e + fx$, where e is positive and f is negative

(ii)



(iii) Case (B) is the most appropriate because as x increases, y decreases at a decreasing rate.

From GC, $r = 0.994981\ldots$

≈ 0.995 (3s.f)

(iv) From GC, $y = 3.62896 + \dfrac{2154.915}{x}$

$y = 3.63 + \dfrac{2150}{x}$ (3 sf)

When y =10, $10 = 3.62896 + \dfrac{2154.915}{x}$

$x \approx 338$ (3s.f)

The estimate is likely to be reliable as y =10 is within the range of given y values and r value is close to 1.