2015 Prelim Topical-Complex Numbers

1

ACJC/2015/II/2

The complex number z satisfies the equation $\left|\frac{z+\mathrm{i}-3}{2+\mathrm{i}z}\right|=1$.

(i) Using an algebraic method, find the purely imaginary number that satisfies the given relation. [2]

(ii) Sketch the locus of the points representing z, labelling the coordinates of the y-intercept. [2]

(iii) Describe the locus of the points representing w such that $|w-4\mathrm{i}|=a$, where a is a non-zero constant. Hence find the exact value of a such that there is exactly one value of z that satisfies $\left|\frac{z+\mathrm{i}-3}{2+\mathrm{i}z}\right|=1$ and $|z-4\mathrm{i}|=a$. [3]

For this value of a, find the exact value of z that satisfies the above conditions, giving your answer in the form $x+\mathrm{i}y$ where $x,\ y\in\mathbb{R}$. [2]

2

AJC/2015/I/3

The complex number z satisfies

$$|z-4-3\mathrm{i}|\le 2 \qquad \text{and} \qquad \pi<\arg\left[(z-4)^2\right]<2\pi$$

(i) Sketch clearly the locus of z on an Argand diagram. [3]

(ii) Find the range of values of $|z-8|$. [2]

(iii) Find maximum value of $\arg(z-8)$. [3]

3

AJC/2015/II/2

Solve the equation $z^5+32=0$, expressing your answers in the form $re^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [2]

z_1, z_2 and z_3 are three of the roots of $z^5+32=0$ such that $0<\arg z_1<\arg z_2<\arg z_3\le\pi$.

(i) Find the smallest positive integer n such that $\left(\frac{z_1}{z_2^*}\right)^n$ is real and positive. [3]

(ii) The points A and B represent the roots z_1 and z_3 respectively in the Argand diagram. The line segment BA' is obtained by rotating the line segment BA through $\frac{\pi}{2}$ clockwise about the point B. Find the real part of the complex number represented by point A', giving your answer in exact trigonometric form. [4]

4	**CJC/2015/I/1** A calculator is **not** to be used in answering this question. By considering $w = z^2 - z$ or otherwise, solve $$z^4 - 2z^3 - 2z^2 + 3z - 10 = 0 \text{ where } z \in \mathbb{C},$$ leaving the roots in exact form. [4]
5	**CJC/2015/II/3** A calculator is **not** to be used in answering this question. The complex numbers a and b are given by $\frac{1+\mathrm{i}}{1-\mathrm{i}}$ and $\frac{\sqrt{2}}{1-\mathrm{i}}$ respectively. **(i)** Find the moduli and arguments of a and b. [3] **(ii)** In an Argand diagram, the points A, B and C represent the complex numbers a, b and $a+b$ respectively. The origin is denoted by O. By considering the quadrilateral $OACB$ and the argument of $a+b$, show that $\tan\left(\frac{3\pi}{8}\right) = 1+\sqrt{2}$. [3] **(iii)** Using a single Argand diagram, sketch the loci **(a)** $\lvert z-a \rvert = 2$, **(b)** $\arg(z-b) = \frac{\pi}{2}$. Find the exact complex number z, in the form $x+\mathrm{i}y$, that satisfies parts **(a)** and **(b)**. [5]
6	**DHS/2015/I/1** A graphic calculator is **not** to be used in answering this question. **(i)** Find the value of $(1+4\mathrm{i})^2$, showing clearly how you obtain your answer. [1] **(ii)** Given that $1+2\mathrm{i}$ is a root of the equation $$z^2 - z + (a+b\mathrm{i}) = 0,$$ find the values of the real numbers a and b. [2] **(iii)** For these values of a and b, solve the equation in part **(ii)**. [2]
7	**DHS/2015/I/4** The complex number z is given by $(\sqrt{3})\mathrm{e}^{\mathrm{i}\left(\frac{1}{6}\pi\right)}$. **(i)** Given that $w = (1+\mathrm{i})z$, find $\lvert w \rvert$ and $\arg w$ in exact form. [2] **(ii)** Without using a calculator, find the smallest positive integer n such that w^n is purely imaginary. State the modulus of w^n when n takes this value. [4]

8	**DHS/2015/II/3**
	(i) Given that the complex number z satisfies the equation $z=\frac{4}{z^*}$, show that $\|z\|=2$. [1]
	(ii) Express $1-\mathrm{i}\sqrt{3}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$. [1]
	(iii) On a single Argand diagram, sketch the loci
	(a) $z=\frac{4}{z^*}$,
	(b) $\|z+2\|=\|z-1+\mathrm{i}\sqrt{3}\|$. [3]
	(iv) The complex numbers z_1 and z_2 satisfy the equations $z=\frac{4}{z^*}$ and $\|z+2\|=\|z-1+\mathrm{i}\sqrt{3}\|$, where $\arg(z_1)>\arg(z_2)$. Find the exact values of z_1 and z_2, giving your answers in the form $x+\mathrm{i}y$. [4]
	(v) Another complex number w satisfies $w=\frac{4}{w^*}$ and $\frac{1}{2}\pi<\arg w<\pi$.
	Explain why $\arg(z_1-w)-\arg(z_2-w)=\frac{1}{2}\pi$. [1]
9	**HCI/2015/I/10**
	(a) The equation $z^3-az^2+2az-4\mathrm{i}=0$, where a is a constant, has a root i.
	(i) Briefly explain why i^* may not necessarily be a root of the equation. [1]
	(ii) Show that $a=2+\mathrm{i}$. [2]
	(iii) Hence, find the remaining roots of the equation in exact form. [5]
	(b) The complex number z satisfies the equations $\|z^*-1+\mathrm{i}\|=2$ and $\arg(z-2\mathrm{i})=\frac{\pi}{4}$. By considering $z=x+\mathrm{i}y$, find z. [5]
10	**HCI/2015/II/3**
	The complex number z satisfies the inequalities
	$\|z\|\le\|z+2-2\mathrm{i}\|$ and $\frac{3\pi}{4}<\arg(-2+2\mathrm{i}-z)\le\pi$.
	(i) Sketch the locus of z on an Argand diagram. [5]
	(ii) Find the exact range of $\arg(z+3)$. [3]
	(iii) Given that $\mathrm{Re}(z)\le 2$, find the area of the region where z can lie in. [2]
11	**IJC/2015/I/5**
	A graphic calculator is **not** to be used in answering this question.
	The complex number z is given by $2+\mathrm{i}\sqrt{3}$.
	(i) Find z^4 in the form $x+\mathrm{i}y$, showing your working. [2]
	(ii) Given that z is a root of the equation $2w^4+aw^2+bw+49=0$, find the values of the real numbers a and b. [3]
	(iii) Using these values of a and b, find all the roots of this equation in exact form. [4]

12	**IJC/2015/II/4** **(a)** The complex number z satisfies the relations $\lvert z-3\rvert \le 3$ and $\lvert z-3-3\mathrm{i}\rvert = \lvert z\rvert$. **(i)** Illustrate both of these relations on a single Argand diagram. [3] **(ii)** Find exactly the maximum and minimum possible values of $\lvert z\rvert^2$. [4] **(b)** The complex number w is given by $\left(\dfrac{-\sqrt{3}+\mathrm{i}}{\sqrt{2}-\mathrm{i}\sqrt{2}}\right)^2$. Without using a calculator, find **(i)** $\lvert w\rvert$ and the exact value of $\arg w$, [4] **(ii)** the set of values of n, where n is a positive integer, for which $w^n w^*$ is a real number. [4]
13	**JJC/2015/I/3** The complex number z satisfies the following relations $z = 3+\mathrm{i}k$, where k is a positive real variable, and $\lvert z-1\rvert \le 3$. Illustrate both of these relations on a single Argand diagram. Hence or otherwise find the maximum value of arg z. [6]
14	**JJC/2015/I/5** **Do not use a graphic calculator in answering this question.** **(i)** Solve the equation $z^3+8=0$, giving the roots in the form $x + y$i. [3] **(ii)** Show the roots on an Argand diagram. [2] **(iii)** Given that the roots with $x>0$ satisfy the equation $\lvert z-a\rvert = 2$, where a is a positive real constant, find the value of a. [2]
15	**JJC/2015/II/1** **(a)** Given that $\arg(a+\mathrm{i}b)=\theta$, where a and b are real and positive, find in terms of θ and/or π, the value of **(i)** $\arg(a-\mathrm{i}b)$, **(ii)** $\arg(b+\mathrm{i}a)$. [3] **(b)** Solve the simultaneous equations $$\mathrm{i}z+2w=0 \text{ and } z-w^*=3,$$ giving z and w in the form a + ib, where a and b are real. [5]

16	**MI/2015/I/11**

(a) **(i)** A student wrote the statement "if $z = x + y\text{i}$, where x and y are real, is a root of the equation $\text{P}(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 = 0$, $n \in \mathbb{Z}^+$, then $z = x - iy$ is also a root" in his notes. Explain whether the statement is always true. **[1]**

(ii) Given that $2 + \text{i}$ is a root of the equation $4z^3 - 11z^2 + 25 = 0$, without the use of a calculator, find the other roots of the equation. **[4]**

(b) **(i)** Solve the equation

$$z^4 + 1 + \sqrt{3}\text{i} = 0,$$

giving the roots in the form $re^{\text{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. **[4]**

(ii) Show the roots on an Argand diagram. **[2]**

(c) Given that $|z^2| = 2$, $|wz| = 2\sqrt{2}$, $\arg(-\text{i}z) = \dfrac{\pi}{4}$ and $\arg\left(\dfrac{z^2}{w}\right) = -\dfrac{5\pi}{6}$, find w in the form $a + b\text{i}$, where a and b are real coefficients to be determined. **[4]**

17	**MI/2015/II/4**

In an Argand diagram, the points A and B represent the complex numbers 3i and $\sqrt{3}$ + 2i respectively.

(i) Sketch, on an Argand diagram, the locus of the points representing the complex number z such that $|z - 3\text{i}| = |z - \sqrt{3} - 2\text{i}|$. [2]

(ii) Show that the locus in part **(i)** passes through the point (0, 1). [1]

(iii) A circle with centre (0, h) and radius k passes through A and B. Deduce the value of h and sketch this circle on the same diagram as the locus in part **(i)**. [3]

(iv) Indicate on the Argand diagram, the region which represents z such that $|z - \text{i}h| \le k$ and $|z - 3\text{i}| \ge |z - \sqrt{3} - 2\text{i}|$. Find the range of values of $\arg(z + 2 - \text{i})$. [5]

18	**MJC/2015/I/10**

(a) On the same diagram, sketch the loci

(i) $|z - 3 - 6\text{i}| = 4$,

(ii) $\arg(z - 3 - 2\text{i}) = \dfrac{\pi}{6}$. [4]

The complex number w is represented by the point of intersection of the loci in parts **(i)** and **(ii)**. Find w in the form x + iy, leaving the values of x and y in the exact non-trigonometrical form. [3]

	(b) Sketch the locus of z such that $2 \le \|z\| < 4$ and the argument of z follows an arithmetic progression where the first term is $-\frac{3\pi}{4}$ radians with common difference $\frac{\pi}{2}$ radians. Find the exact minimum value of $\|z+3\mathrm{i}\|$. [4]
19	**MJC/2015/II/3** **(a)** It is given that $1+2\mathrm{i}$ is a root of $x^3+ax^2+bx-5=0$, where a and b are real. Find the values of a and b and the other roots. [4] **(b)** **(i)** Find the possible values of z such that $\|z^4\|=2$ and z^4 is a negative real number. [3] Let z_1, z_2, z_3 and z_4 be the values of z found in part **(i)** such that $-\pi < \arg(z_1) < \arg(z_2) < \arg(z_3) < \arg(z_4) \le \pi$. **(ii)** Write down a complex number w such that $z_2 = wz_1$. [1] **(iii)** Hence, or otherwise, find the exact value of $\|wz_3 - w^* z_3\|^2$, where w^* is the conjugate of w. [2]
20	**NJC/2015/I/12** **Do not use a graphing calculator for this question.** **(a)** Let $z_1 = \sqrt{2} - (\sqrt{2})\mathrm{i}$ and $z_2 = 1 + (\sqrt{3})\mathrm{i}$. **(i)** Find $z_3 = -\frac{(z_2)^2}{z_1^*}$ exactly in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [4] **(ii)** On a single Argand diagram, sketch the points representing the complex numbers z_1, z_2 and z_3. Label each point and show clearly any geometrical relationships. [3] **(iii)** Explain whether there is a fixed complex number k such that z_1, z_2 and z_3 are all roots of the equation $z^3 = k$. [2] **(b)** Given that $w = \frac{1}{\mathrm{e}^{\mathrm{i}4\theta} - 1}$, show that $\mathrm{Re}(w) = -\frac{1}{2}$ and find $\mathrm{Im}(w)$ in terms of θ. [4]

21 **NJC/2015/II/4**

The complex number z satisfies both the relations

$$|z+3-3\text{i}| \le 5\sqrt{2} \text{ and } 0 \le \arg(z-6) \le \frac{3}{4}\pi .$$

(i) On an Argand diagram, shade the region in which the points representing z can lie. [3]

(ii) Label the point(s) that correspond to the maximum value of $|z-4-10\text{i}|$ on your diagram with the letter P. (You do not have to find the coordinates of the point(s).) [2]

(iii) Express the smallest value of $|z-4-10\text{i}|$ in the form $m\sqrt{2}$, where m is an integer-valued constant to be determined. Show your working clearly. [2]

It is given that the complex number w satisfies the relation $|w+3-3\text{i}| \le 5\sqrt{2}$ only.

(iv) Find the minimum value of $|\arg(w-6)|$, giving your answer in radians, correct to 3 decimal places. [4]

22 **NYJC/2015/I/11**

A sequence of complex numbers $w_1, w_2, w_3, \ldots$ satisfies the recurrence relation

$$w_{n+1} = \sqrt{w_n}\ ,\ n \ge 1,$$

with $w_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\text{i}$ and $0 \le \arg(w_n) \le \frac{\pi}{2}$.

(i) Express w_1, w_2, w_3, w_4 in exponential form. **[2]**

(ii) If $\arg(w_n) = \theta_n$, show that $\theta_{n+1} = \frac{1}{2}\theta_n$ where $n \ge 1$. Deduce that θ_n is a geometric sequence and hence find the value of $\sum_{n=1}^{\infty}\theta_n$. **[4]**

(iii) Explain why the locus of all points z such that $|z-w_3| = |z-w_4|$ passes through the origin. Hence find the exact cartesian equation of this locus. **[4]**

23 **NYJC/2015/II/3**

The complex number z satisfies $|z-2+\text{i}| \le \sqrt{5}$ and $|z-3-\text{i}| \le \sqrt{10}$.

(i) On an Argand diagram, sketch the region in which the point representing z can lie. **[4]**

(ii) Show that the complex number $z_1 = 4-2\text{i}$ satisfies the equation $|z-2+\text{i}| = \sqrt{5}$ and $|z-3-\text{i}| = \sqrt{10}$. **[2]**

(iii) Find the area of the region that z lies. **[3]**

(iv) Find the range of values of $\arg(z-2-4\text{i})$. **[3]**

24	**PJC/2015/I/7**

The polynomial P(z) has real coefficients. The equation P(z) = 0 has a root $re^{i\theta}$, where $r > 0$ and $0 < \theta \le \pi$.

(i) Write down a second root in terms of r and θ, and hence show that a quadratic factor of P(z) is $z^2 - 2rz\cos\theta + r^2$. [3]

(ii) Solve the equation $z^4 = -625$, expressing the solutions in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [3]

(iii) Use your answers in parts **(i)** and **(ii)** to express $z^4 + 625$ as the product of two quadratic factors with real coefficients, giving each factor in non-trigonometrical form. [3]

25	**PJC/2015/II/3**

Sketch, on a single Argand diagram, the set of points representing all complex numbers satisfying both of the following inequalities:

$$\left|z - \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)\right| \le \frac{\sqrt{3}}{2} \quad \text{and} \quad -\frac{\pi}{3} \le \arg(z - 2\mathrm{i}) \le 0. \qquad [4]$$

Hence find

(i) the minimum value of $|\mathrm{i} - 2z|$, [3]

(ii) the exact value of the complex number z such that $\arg(z)$ is minimum. [3]

26	**RI/2015/I/3**

A calculator is **not** to be used in answering this question.

The complex number z satisfies $|z| \le 1$, where $-\frac{3\pi}{4} \le \arg(z) \le -\frac{\pi}{2}$.

(i) On an Argand diagram, sketch the region in which the point representing z can lie. **[2]**

Given that $w = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$, find in exact form,

(ii) the range of values of arg(wz), **[2]**

(iii) the greatest value of $|z - w|$. **[2]**

27	**RI/2015/I/12** A calculator is **not** to be used in answering this question. **(a)** Find the 6 roots of the equation $z^6 = -\mathrm{i}$, giving your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. **[3]** **(b)** The equation $z^3 + az^2 + 6z + 2 = 0$, where a is a real constant, has 2 roots which are purely imaginary. Find the value of a and solve the equation. **[4]** **(c)** **(i)** Prove that $\dfrac{1+\sin\frac{3\pi}{8}+\mathrm{i}\cos\frac{3\pi}{8}}{1+\sin\frac{3\pi}{8}-\mathrm{i}\cos\frac{3\pi}{8}} = \cos\frac{\pi}{8}+\mathrm{i}\sin\frac{\pi}{8}$. **[3]** **(ii)** Hence find the two smallest positive integer values of n for which $\left(\dfrac{1+\sin\frac{3\pi}{8}+\mathrm{i}\cos\frac{3\pi}{8}}{1+\sin\frac{3\pi}{8}-\mathrm{i}\cos\frac{3\pi}{8}}\right)^n - \mathrm{i} = 0.$ **[2]**
28	**RVHS/2015/I/7** A graphic calculator is **not** to be used in answering this question. (i) Find the roots of the equation $z^2 = -8\mathrm{i}$ in the form $a + b\mathrm{i}$. [4] (ii) Hence, sketch on a single Argand diagram, the roots of $w^4 = -64$. [3] (iii) Find the roots of the equation $z^2 + (2+2\mathrm{i})z + 4\mathrm{i} = 0$. [3]
29	**RVHS/2015/II/3** The complex numbers z and w are such that $\lvert z - 2\mathrm{i} \rvert = 1$ $\arg(w) = k$ (i) Given $k = \frac{\pi}{4}$, sketch the loci on an Argand diagram. Give the geometrical description of the loci. [4] (ii) Find the values of k such that the loci intersect at exactly one point. [3] (iii) Given that $k = \frac{5\pi}{6}$, find the values of w and z in the form $x + \mathrm{i}y$ that minimize $\lvert w - z\rvert$. [4]
30	**SAJC/2015/I/13** **(a)** A complex number $z = x + iy$, where $x, y \in \mathbb{R}$, is represented by the point P in an Argand diagram. The complex number $w = \dfrac{z-2i}{z+4}$, where $z \neq -4$, has its real part zero. By using $z = x + iy$, or otherwise, show that the locus of P in the Argand diagram is a circle. Hence find the equation of the circle, stating clearly its centre and radius. **[4]**

(b) The complex number z satisfies the relations $|z+2-i| \le \sqrt{5}$ and $\arg(z-1+2i) = \dfrac{3\pi}{4}$.

(i) Illustrate both of these relations on a single Argand diagram. **[4]**

(ii) Find the exact minimum and maximum value of $|z-1+2i|$. **[2]**

(iii) Find the minimum and maximum values of $\arg(w-1+2i)$, where w satisfies $|w+2-i| = \sqrt{5}$. **[3]**

31 SAJC/2015/II/4

(a) Solve the equation $w^4 + 1 - \sqrt{3}i = 0$, expressing the roots in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. Show the roots on an Argand diagram, showing the relationship between them clearly. **[5]**

(b) A complex number $z = x + iy$ has modulus r and argument θ, where $0 < \theta < \dfrac{\pi}{2}$. The complex numbers v and w are defined by $v = -y + xi$ and $w = x^2 - y^2 + 2xyi$.

(i) Express v and w in terms of z. **[2]**

(ii) Hence, or otherwise, express vw in exponential form in terms of r and θ. **[2]**

[2]

(iii) If $vw = -4 - 4\sqrt{3}i$, solve for z in exponential form, giving your answer in exact form. **[4]**

32 SRJC/2015/I/11

(a) The complex number $-1+i$ is a root of $3z^3 + 13z^2 + az + b = 0$, where a and b are real constants. Find the values of a and b. [3]

(b) The complex number z is such that $z^4 = z^*$.

(i) Write down all possible value(s) of $|z|$. [2]

(ii) Find all possible values of z in exponential form. [3]

(iii) Given that $0 < \arg(z) < \dfrac{\pi}{2}$, find the smallest positive real number k for z^k to be purely imaginary. [3]

33 SRJC/2015/II/4

(a) The complex number z is given by $z = re^{i\theta}$, where $r > 0$ and $0 \le \theta \le \dfrac{1}{2}\pi$.

(i) Given that $w = (\sqrt{3} - i)z$, find $|w|$ in terms of r and $\arg w$ in terms of θ. [2]

(ii) For a fixed value of r, draw on the same Argand diagram the the locus of z and w as θ varies. [2]

(iii) If $r = 1.5$ units, calculate (in terms of π) the length of the locus of w for $\mathrm{Im}(w) \ge 0$ as θ varies. [2]

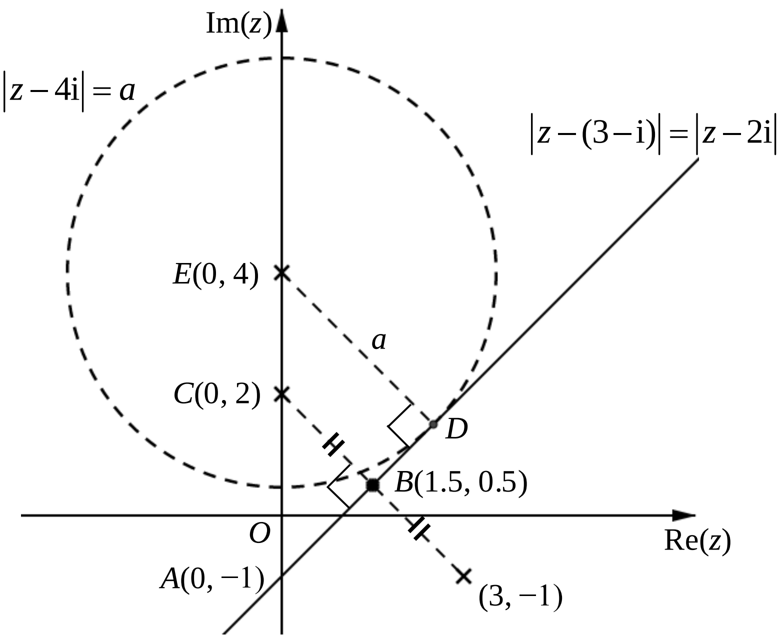
(b) Sketch on a single Argand diagram the set of points representing all complex numbers v satisfying both of the following inequalities:

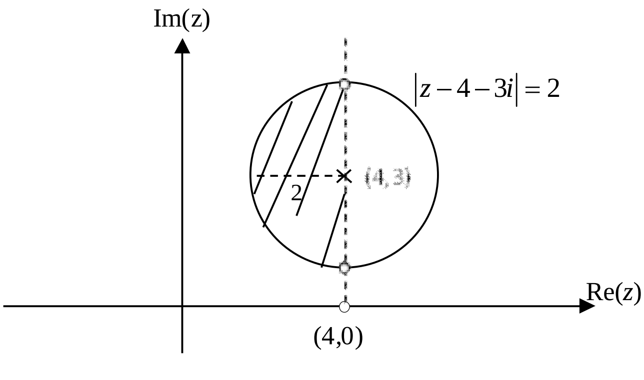
$$|v - 5 - 8i| \le 5 \quad \text{and} \quad |v - 12 - 8i| \ge |v - 12 - 10i|.$$

Hence find (in radians) the least value of $\arg(v - 5 + 3i)$. [6]

34	**TJC/2015/I/5** The complex number z satisfies the equation $\dfrac{1+z^3}{1-z^3}=\sqrt{3}\,\mathrm{i}$. Without the use of a graphing calculator, express z^3 in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r \geq 0$ and $-\pi < \theta \leq \pi$. Hence find the roots of the equation. [6]
35	**TJC/2015/II/1** Sketch, on an Argand diagram, the locus of the point representing the complex number z such that $\arg(z-\sqrt{3}+\mathrm{i})=\dfrac{5\pi}{6}$. [2] Give a geometrical description of the locus of the point representing the complex number w such that $\lvert w+\mathrm{i}\rvert = k$, where k is real. [1] **(i)** Given that the two loci intersect at exactly one point, show that $k = a$ or $k \geq b$ where a and b are real constants to be determined. [3] **(ii)** In the case when k takes the value of a, find the complex number representing the point of intersection, in the form x + iy, where x and y are exact. [3]
36	**TPJC/2015/I/3** The complex number z satisfies both $\lvert z-3+4\mathrm{i}\rvert \leq 5$ and $\lvert z\rvert \leq \lvert z-6+8\mathrm{i}\rvert$. **(i)** On an Argand diagram, sketch the region in which the point representing z can lie. [4] It is given that $-\pi < \arg(z-3-6\mathrm{i}) \leq \pi$ and that $\lvert\arg(z-3-6\mathrm{i})\rvert$ is as small as possible. **(ii)** Find z in the form $x+\mathrm{i}y$. [3] **(iii)** Find the value of $\arg(z-3-6\mathrm{i})$ in radians, correct to 4 significant figures. [2]
37	**TPJC/2015/II/3** **(i)** Solve the equation $z^3 = 4\sqrt{2}(-1-\mathrm{i})$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [3] **(ii)** Hence, find in exact form, the roots of the equation $w^3 = 4\sqrt{2}(-1+\mathrm{i})$. [2] It is given that $\alpha = z^3 + \dfrac{1}{(z^*)^3}$. **(iii)** Without using a calculator, find an exact expression for α. Give your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [2] **(iv)** Without using a calculator, find the three smallest positive whole number values of n for which $\dfrac{\alpha^n}{\alpha^*}$ is a real number. [4]

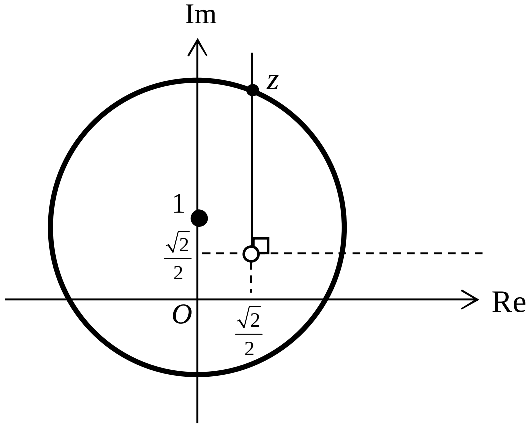
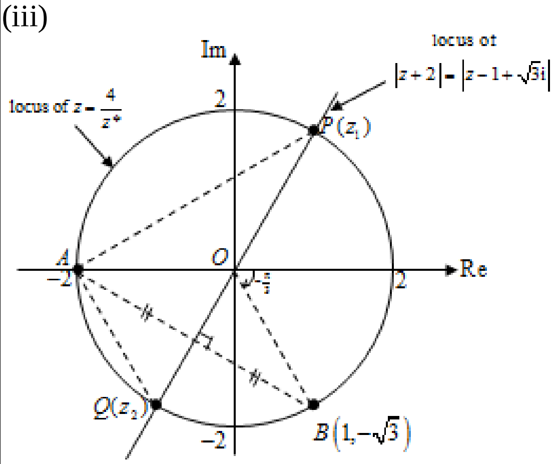
38	**VJC/2015/I/8** **(i)** If $z = x + \mathrm{i}\,y$, where $x, y \in \mathbb{R}$, prove that $\left(z^2\right)^* = \left(z^*\right)^2$. [2] **(ii)** Solve the equation $z^2 = 1 - 4\sqrt{3}\mathrm{i}$, giving your answers exactly in the form $x + \mathrm{i}y$. [4] **(iii)** Use your answers in part **(ii)** to solve the equation $w^2 = 4 + 16\sqrt{3}\mathrm{i}$. [2] **(iv)** The roots in part **(ii)** are represented by z_1 and z_2. Given that $\arg\left(z^2\right) = \theta$, find $\arg\left(z_1 z_2\right)$, giving your answer in terms of θ. [2]
39	**VJC/2015/II/3** The complex number z satisfies the equation $\lvert z - 5 + 7\mathrm{i} \rvert = 6$. **(i)** Show the locus of z on an Argand diagram. [2 **(ii)** If the locus in part **(i)** intersects the locus $\arg\left(\mathrm{z} - a - 2\mathrm{i}\right) = -\dfrac{\pi}{2}$ at two distinct points, where a is a real number, find the set of values that a can take. [2 **(iii)** Given that the locus in part **(i)** intersects the locus $\lvert z + 2 + 4\mathrm{i} \rvert = k$ at exactly one point, find two possible exact values of k. [3 Using one value of k found, find exactly the value of z represented by the point of intersection, giving your answer in the form $x + \mathrm{i}y$. [3
40	**YJC/2015/I/11** **(a)** Sketch on an Argand diagram, the set of points representing all complex numbers z satisfying $$\lvert z \rvert \le 2,\ \ z + z^* \ge 2 \ \text{ and }\ 0 \le \arg(z) \le \frac{\pi}{4}.$$ Hence, find the greatest exact values of $\lvert z - 4 - 4\mathrm{i} \rvert$ and $\arg(z-4)$. [6] **(b)** Solve $w^3 + 1 = 0$ and hence deduce the solutions of $\left(\dfrac{z+1}{z}\right)^3 = -1$, giving each answer in the form $a + \mathrm{i}b$, where a and b are real. [6]
41	**YJC/2015/II/1** **(a)** Using an Argand diagram, show that any complex number $a + \mathrm{i}b$, where a and b are real numbers, may be expressed in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$. State the geometrical meaning of r and θ. [3] Write down the cartesian equation of the locus of the point representing $3(\cos\theta + \mathrm{i}\sin\theta)$ as θ varies. [1] **(b)** Given that $z = \cos\alpha + \mathrm{i}\sin\alpha$, show that $z + \dfrac{1}{z} = 2\cos\alpha$ and $z - \dfrac{1}{z} = 2\mathrm{i}\sin\alpha$. Hence, find $\dfrac{z^2+1}{z^2-1}$ in terms of α. [5]

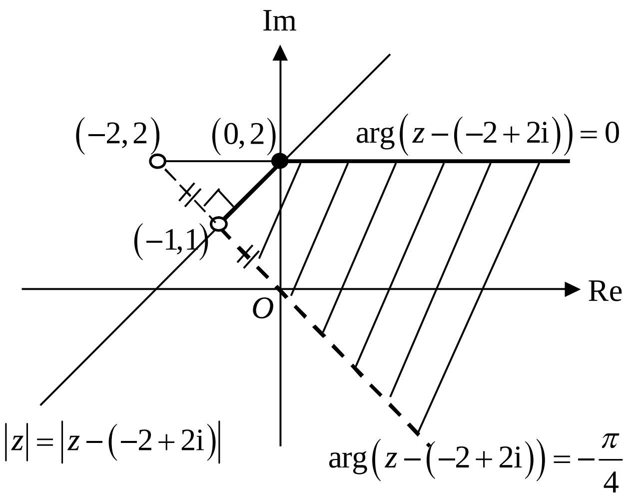
<table>
<tr><th>S/No</th><th>Answers</th></tr>
<tr><td>1</td><td>(i) $-i$
(ii)


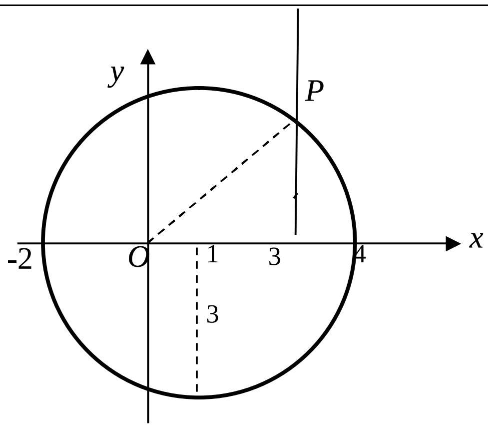
(iii) The locus of points representing w such that $\left|w-4\text{i}\right|=a$ is a circle with centre (0, 4) and radius a units.
$a=\dfrac{5\sqrt{2}}{2}$, $z=\dfrac{5}{2}+\dfrac{3}{2}\text{i}$</td></tr>
<tr><td>2</td><td>

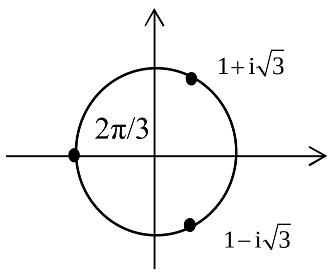
(ii) $\sqrt{17}<\left|z-8\right|\le 7$ (iii) 2.91</td></tr>
<tr><td>3</td><td>$2e^{-\frac{3\pi}{5}i},2e^{-\frac{\pi}{5}i},2e^{\frac{\pi}{5}i},2e^{\frac{3\pi}{5}i},2e^{\pi i}$

(i) smallest n = 5
(ii) $-2+2\sin\dfrac{\pi}{5}$</td></tr>
<tr><td>4</td><td>$z=\dfrac{1\pm\sqrt{21}}{2},\dfrac{1\pm\sqrt{7}\text{i}}{2}$</td></tr>
<tr><td>5</td><td>(i) $\left|a\right|=1$; $\left|b\right|=1$; $\arg\left(a\right)=\arg\left(\dfrac{1+\text{i}}{1-\text{i}}\right)=\dfrac{\pi}{2}$; $\arg\left(b\right)=\dfrac{\pi}{4}$
(iii)</td></tr>
</table>

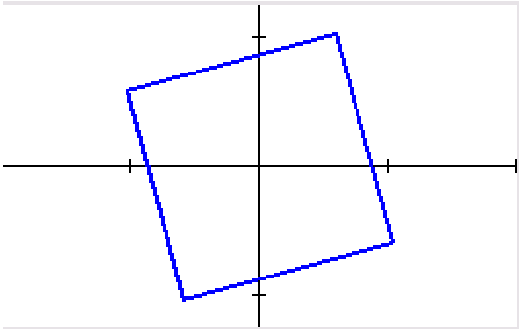
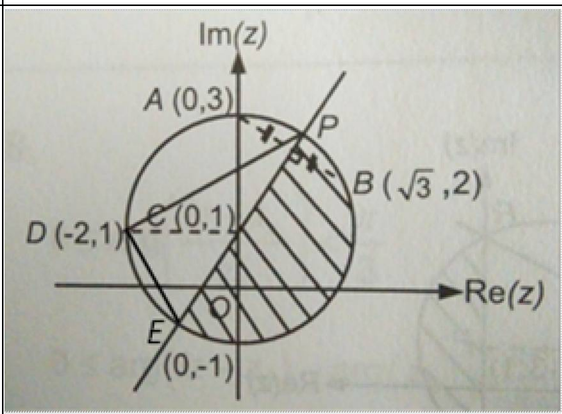
	 $\therefore z=\frac{\sqrt{2}}{2}+\left(1+\sqrt{\frac{7}{2}}\right)\mathrm{i}$
6	(ii) $a=4,\ b=-2$; (iii) $1+2\mathrm{i}$ or $-2\mathrm{i}$
7	(i) $\lvert w\rvert=\sqrt{6},\quad \arg(w)=\frac{5\pi}{12}$; (ii) $n=6,\ \lvert w^6\rvert=216$
8	(ii) $2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$; (iii)  (iv) $z_1=1+\sqrt{3}\mathrm{i}$ and $z_2=-1-\sqrt{3}\mathrm{i}$
9	(a)(i) It is not necessarily true because to conclude that i^* is a root, the coefficients of the equation must be real. (iii) $z=1+\sqrt{3}\mathrm{i}$ or $z=1-\sqrt{3}\mathrm{i}$ (b) $1+3\mathrm{i}$

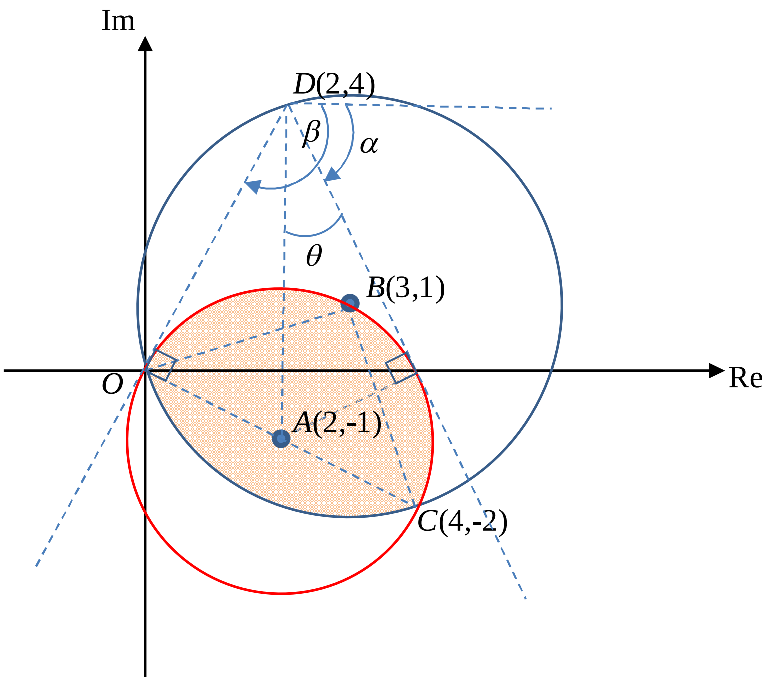
<table>
<tr><td>10</td><td>

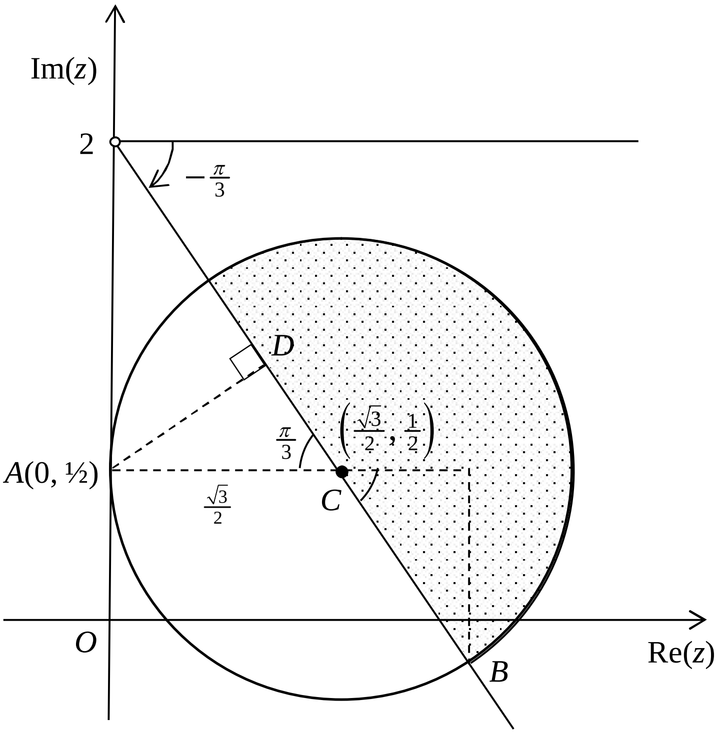
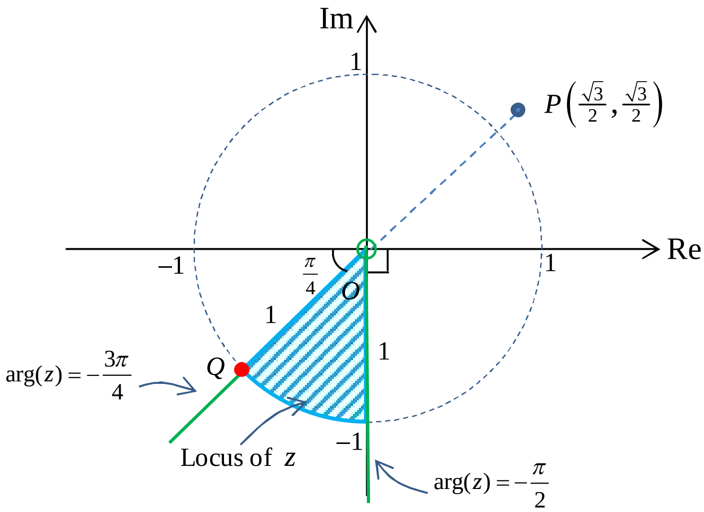
(ii) $-\frac{\pi}{4}<\arg(z+3)\le\tan^{-1}\left(\frac{2}{3}\right)$; (iii) 7</td></tr>
<tr><td>11</td><td>(i) $-47+8\sqrt{3}i$
(ii) $a=-11,\ b=28$
(iii) $2-\sqrt{3}i, -2\pm\frac{\sqrt{2}}{2},\ 2+\sqrt{3}i$</td></tr>
<tr><td>12</td><td>(a)(ii)minimum = $\frac{9}{2}$; maximum = $18+9\sqrt{2}$
(b)(i) $|w|=1,\ \arg w=\frac{\pi}{6}$
(b)(ii) $\{n: n\in\mathbb{Z}^+, n=6k-5,\quad k\in\mathbb{Z}^+\}$</td></tr>
<tr><td>13</td><td>

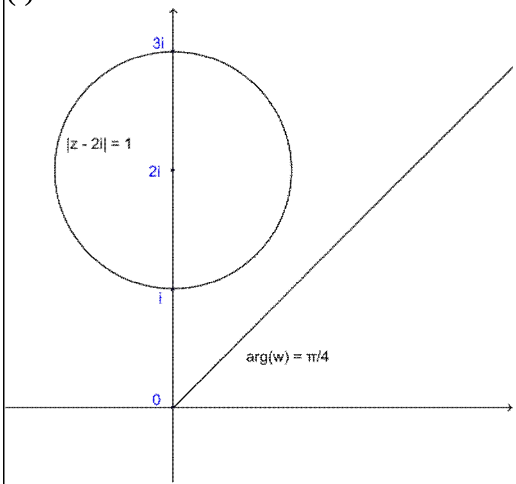
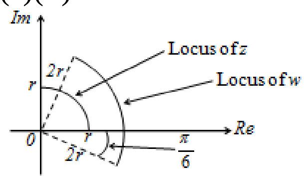
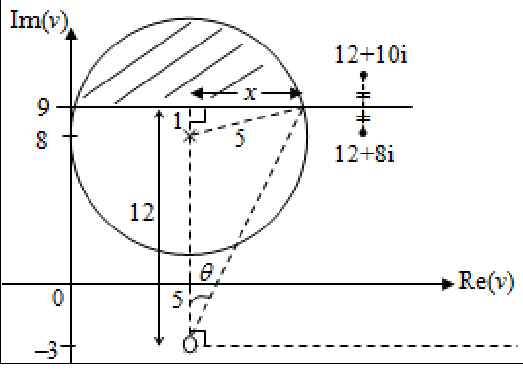
Max arg z = 0.641</td></tr>
<tr><td>14</td><td>(i) $z=-2, 1+i\sqrt{3}, 1-i\sqrt{3}$
(ii)


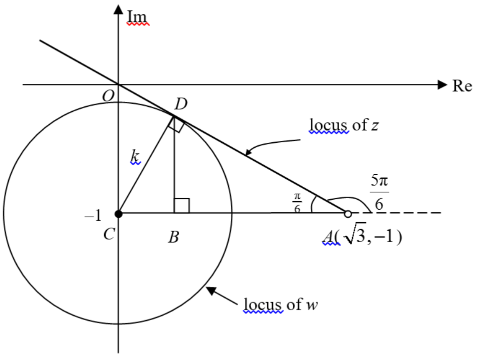
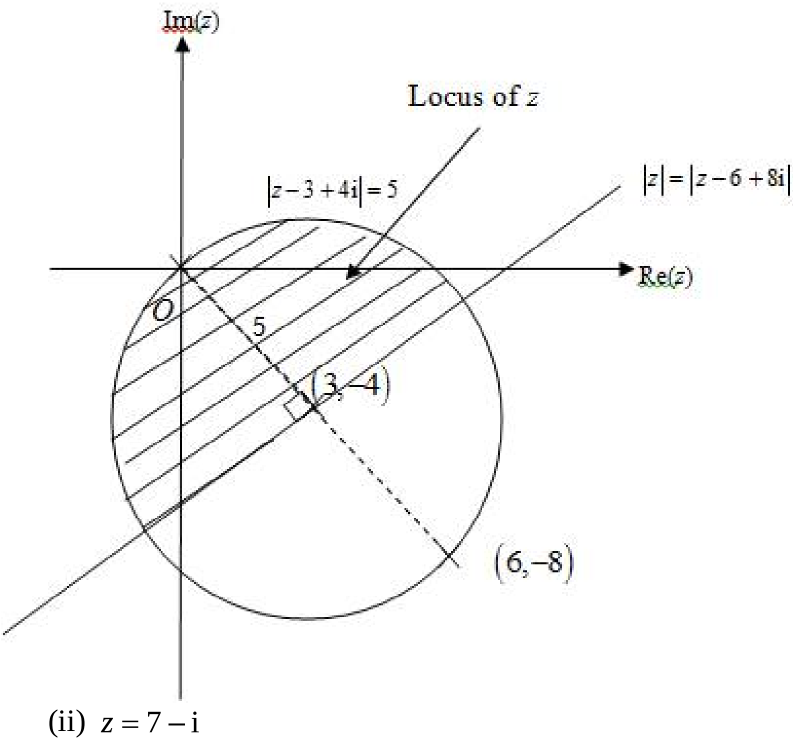
(iii) a = 2</td></tr>
</table>

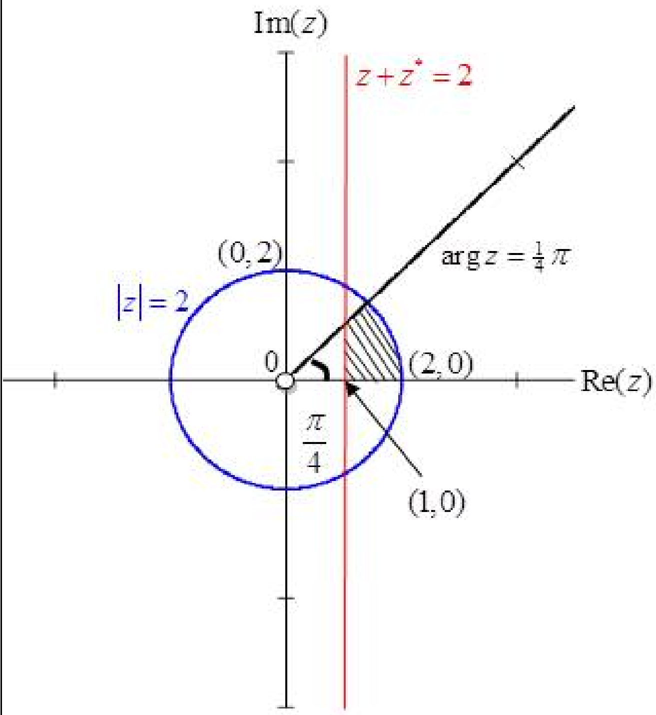
15	(a)(i) $-\theta$; (ii) $\frac{\pi}{2}-\theta$ (b) $w=1-2\mathrm{i}$; $z=4+2\mathrm{i}$
16	(ai) The statement is true when the coefficients of the equation, a_i for $i=0,\ 1,\ \ldots,\ n$ are real. (aii) 2 - i and $-\frac{5}{4}$ (bi) $z=2^{\frac{1}{4}}e^{-i\frac{\pi}{6}},\ 2^{\frac{1}{4}}e^{i\frac{\pi}{3}},\ 2^{\frac{1}{4}}e^{i\frac{5\pi}{6}}$ or $2^{\frac{1}{4}}e^{-i\frac{2\pi}{3}}$ (bii)  (c) $\lvert w\rvert=2,\ \arg w=\frac{\pi}{3},\ w=1+\mathrm{i}\sqrt{3}$
17	 h = 1 $-\frac{\pi}{3}\le \arg(z)\le\frac{\pi}{6}$
18	(a) $w=3+2\sqrt{3}+4\mathrm{i}$ (b) $\frac{3}{\sqrt{2}}$
19	(a)(i) $a=-3, b=7$ The other roots are $1-2\mathrm{i}$ and 1 (b)(i) $2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{3}{4}\right)\pi}, 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{1}{4}\right)\pi}, 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{1}{4}\right)\pi}, 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{3}{4}\right)\pi}$ (b)(ii) $w=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ or $w=\mathrm{i}$ (b)(iii) $4\sqrt{2}$
20	(a)(i) $z_3=2\mathrm{e}^{-\mathrm{i}\frac{7\pi}{12}}$ (a)(iii) No, because the difference in argument between any pair of adjacent complex numbers is not constant, **OR** the difference in argument between z_1 and z_2 is not $2\pi/3$ (or any other pair of the two complex numbers) **OR**

	$(z_1)^3 = \left(2e^{-i\frac{\pi}{4}}\right)^3 = 8e^{-i\frac{3\pi}{4}}$ but $(z_2)^3 = \left(2e^{i\frac{\pi}{3}}\right)^3 = 8e^{i\pi} \neq (z_1)^3$ (b) $\mathrm{Im}(w) = -\frac{1}{2}\cot 2\theta$
21	(iii) $2\sqrt{2}$ (iv) 1.979 rad (to 3 d.p.)
22	**(i)** $w_1 = e^{\frac{\pi i}{4}},\ w_2 = e^{\frac{\pi i}{8}},\ w_3 = e^{\frac{\pi i}{16}},\ w_4 = e^{\frac{\pi i}{32}}$ **(ii)** $\sum_{n=1}^{\infty}\theta_n = \frac{\pi}{2}$ **(iii)** $y = x\tan\left(\frac{3\pi}{64}\right)$
23	(i)  **(iii)** $5(\pi - 1)$ **(iv)** $-2.03 \le \arg(z - 2 - 4i) \le -1.11$
24	(i) $z = re^{-i\theta}$ (ii) $z = 5e^{i\left(-\frac{3\pi}{4}\right)}, 5e^{i\left(-\frac{\pi}{4}\right)}, 5e^{i\left(\frac{\pi}{4}\right)}, 5e^{i\left(\frac{3\pi}{4}\right)}$ (iii) $(z^2 - 5\sqrt{2}z + 25)(z^2 + 5\sqrt{2}z + 25)$

25	 (i) $\frac{3}{2}$ (ii) $\frac{3\sqrt{3}}{4}-\frac{1}{4}\mathrm{i}$
26	(i)  (ii) $-\frac{\pi}{2}\le\arg(wz)\le-\frac{\pi}{4}$. (iii) $\frac{\sqrt{6}}{2}+1$
27	(a) $z=\mathrm{e}^{\mathrm{i}\frac{(4k-1)\pi}{12}}$, $k=-2,-1,0,1,2,3$ (b) $\sqrt{6}\,\mathrm{i}$, $-\sqrt{6}\,\mathrm{i}$ and $-\frac{1}{3}$ (c)(ii) 4 and 20
28	(i) $z=2-2\mathrm{i}$ or $-2+2\mathrm{i}$ (iii) $-2\mathrm{i}$ or -2

29	(i)  $\lvert z-2i\rvert=1$ is a circle centred on (0, 2) with radius 1. $\arg(w)=\dfrac{\pi}{4}$ is the half line from the origin (excluding the origin), which makes an angle of $\dfrac{\pi}{4}$ with the positive real-axis. (ii) $k=\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$ (iii) $z=\dfrac{-1}{2}+i\dfrac{4-\sqrt{3}}{2}$, $w=\dfrac{-\sqrt{3}}{2}+\dfrac{i}{2}$
30	(ii) $\sqrt{18}-\sqrt{5},\sqrt{18}+\sqrt{5}$ (iii) 1.80 rad , 2.91 rad
31	$w=2^{1/4}e^{i(-\frac{5\pi}{6})},2^{1/4}e^{i(-\frac{\pi}{3})},2^{1/4}e^{i(\frac{\pi}{6})},2^{1/4}e^{i(\frac{2\pi}{3})}.$ $vw=r^3e^{i(\frac{\pi}{2}+3\theta)}$, $z=2e^{i\left(\frac{5\pi}{18}\right)}$
32	(a) a = 20 and b = 14 (b)(i) |z| = 0 or 1 (b)(ii) z = 0, 1, $e^{\frac{2\pi}{5}i}$, $e^{-\frac{2\pi}{5}i}$, $e^{\frac{4\pi}{5}i}$, $e^{-\frac{4\pi}{5}i}$ (b)(iii) $\dfrac{5}{4}$
33	(a)(i) $\theta-\dfrac{\pi}{6}$ (a)(ii) (a)(iii) π units  (b) 1.18 rad 

34	$z^3 = e^{i\frac{\pi}{3}}$; $z = e^{i\frac{\pi}{9}}, e^{-i\frac{5\pi}{9}}, e^{i\frac{7\pi}{9}}$
35	 The locus of *w* is a <u>circle centered at (0, -1) and radius *k* units</u>. (i) $a = \frac{\sqrt{3}}{2}$, $b = \sqrt{3}$ (ii) $\frac{\sqrt{3}}{4} - \frac{1}{4}i$
36	**(i)**  (ii) $z = 7 - i$ (iii) – 1.052
37	(i) $z = 2e^{-i\frac{\pi}{4}}$, $2e^{i\frac{5\pi}{12}}$, $2e^{-i\frac{11\pi}{12}}$ (ii) $w = 2e^{i\frac{\pi}{4}}$, $2e^{-i\frac{5\pi}{12}}$, $2e^{i\frac{11\pi}{12}}$ (iii) $\frac{65}{8}e^{-i\frac{3\pi}{4}}$ (iv) 3, 7 and 11
38	(ii) $z = 2 - \sqrt{3}i$ or $-2 + \sqrt{3}i$ (iii) $w = 4 + 2\sqrt{3}i$ or $-4 - 2\sqrt{3}i$ (iv) $\theta + \pi$

39	(ii) $\{a \in \mathbb{R} : -1 < a < 11\}$ (iii) $\sqrt{58}-6$ and $\sqrt{58}+6$ For $k=\sqrt{58}+6$, $z=\left(5+\dfrac{42}{\sqrt{58}}\right)-\left(\dfrac{18}{\sqrt{58}}+7\right)\mathrm{i}$
40	**(a)**  Shaded region represents the set of required points. Greatest value of $\lvert z-4-4\mathrm{i}\rvert=\sqrt{(4-1)^2+4^2}=5$ Greatest value of $\arg(z-4)=\pi$ (b) $w=\mathrm{e}^{\frac{1}{3}(2k+1)\pi\mathrm{i}}$, where $k=0,\pm1$ $k=0$, $w=\cos\frac{1}{3}\pi+\mathrm{i}\sin\frac{1}{3}\pi=\dfrac{1}{2}+\mathrm{i}\dfrac{\sqrt{3}}{2}$ $k=-1$, $w=\cos\frac{1}{3}\pi-\mathrm{i}\sin\frac{1}{3}\pi=\dfrac{1}{2}-\mathrm{i}\dfrac{\sqrt{3}}{2}$ $k=1$, $w=\cos\pi+\mathrm{i}\sin\pi=-1$
41	**(a)** $a=r\cos\theta$ and $b=r\sin\theta$ Therefore $a+b\mathrm{i}=r\cos\theta+(r\sin\theta)\mathrm{i}=r(\cos\theta+\mathrm{i}\sin\theta)$ $x^2+y^2=9$ **(b)** $-\mathrm{i}\cot\alpha$

2015 Prelim Topical-Differential Equations

1	**AJC/II/1** **(a)** The variables w, x and y are connected by the following differential equations: $$e^{2x}\frac{dw}{dx}=w^2 \qquad \text{and} \qquad \frac{dy}{dx}=w.$$ **(i)** Verify that $w=\dfrac{2}{e^{-2x}+A}$ is the general solution of $e^{2x}\dfrac{dw}{dx}=w^2$, where A is an arbitrary constant. [2] **(ii)** Hence find y in terms of x. [3] **(b)** A tank initially contains 50 grams of salt dissolved in 100 litres of water. Brine that contains 2 grams of salt per litre of brine flows into the tank at a rate of 5 litres per minute. The solution is kept thoroughly mixed and flows out from the tank at a rate of 5 litres per minute. Given that the amount of salt in the tank at time t minutes is given by S, show that $$\frac{dS}{dt}=\frac{200-S}{20}.$$ Hence find the time, in minutes, at which the concentration of salt in the tank reaches 1 gram per litre. [7]
2	**ACJC/I/8** In order to model a particular predator-prey relationship, a biology student came up with the following differential equations: $$\frac{dx}{dt}=1-\frac{x}{100} \qquad \text{(A)}$$ $$\frac{dy}{dt}=x-100 \qquad \text{(B)},$$ where the variables x and y denote the number (in thousands) of predator and prey respectively, t days after the start of the observation. There were 50 000 predators at the start of the observation. **(i)** By solving equation (A), show that $x=100-ke^{-0.01t}$, where k is a constant to be determined. [4] **(ii)** What can you say about the population of the predator after several years? [1] **(iii)** In the long run, the model shows that number of prey approaches 5 million. Using your answer in **(i)**, find y in terms of t. [3]

3	**CJC/II/4** In a research project, the population is modelled by the following *logistic* differential equation, $$\frac{\mathrm{d}P}{\mathrm{d}t}=0.64P\left(1-\frac{P}{10}\right),$$ where P is the population function of time t. **(i)** Solve the differential equation by expressing P in terms of t, given that $P=1$ when $t=0$. **[5]** Sketch the solution curve for $t\geqslant 0$. Comment on the population in the long run. **[2]** An alternative model for the population is the *Gompertz* function, which is the solution to the following differential equation, $$\frac{\mathrm{d}P}{\mathrm{d}t}=0.4P\left(\ln 10-\ln P\right).$$ **(ii)** By solving the differential equation, show that the general solution is $P=10\mathrm{e}^{-A\mathrm{e}^{-0.4t}}$, where A is a constant. **[3]** Given the same initial condition that $P=1$ when $t=0$, sketch the solution curve of the particular solution for $t\geqslant 0$ on the same diagram in part **(i)**. Comment on the similarity and difference between the two models. **[4]**
4	**DHS/I/10** **(a)**Show that the substitution $w=xy^2$ reduces the differential equation $$2xy\frac{\mathrm{d}y}{\mathrm{d}x}=4x^2y^4-y^2+1$$ to the form $$\frac{\mathrm{d}w}{\mathrm{d}x}=aw^2+b,$$ where a and b are to be determined. Hence obtain the general solution in the form $y^2=\mathrm{f}(x)$. [5] **(b)**A certain species of bird with a population of size n thousand at time t months satisfies the differential equation $$\frac{\mathrm{d}^2n}{\mathrm{d}t^2}=\mathrm{e}^{-\frac{1}{4}t}.$$ Find the general solution of this differential equation. [2] Sketch three members of the family of solution curves, given that $n=30$ when $t=0$. [4]

5	**HCI/I/8** Mac has a $400\,000\ \text{m}^2$ farm and on his farm, an area of $60\,000\ \text{m}^2$ is covered in weeds in June, and in September, the area increases to $69\,500\ \text{m}^2$. The growth of weeds is such that the area covered in weeds increases at a monthly rate directly proportional to its area. At the same time, Mac does weeding at a constant rate of $4\,000\ \text{m}^2$ per month. Let the area of the farm covered in weeds at time t (in months) be $A\ \text{m}^2$. **(i)** By considering a differential equation, show that $A = \alpha e^{kt} + \lambda$, where α, k and λ are constants to be determined. [5] **(ii)** The region covered in weeds is in the shape of a circle. Find the monthly rate at which the radius of the region changes when the radius is 200 m. [2] **(iii)** Mac understands that having some weeds on the farm can be beneficial. Find the monthly rate at which Mac needs to do weeding if $\frac{dA}{dt} = 0$ in September. [2] **(iv)** Comment on the significance of $\frac{dA}{dt} = 0$ in the context of this question. [1]
6	**IJC/II/2** **(i)** Find the general solution of the differential equation $x^3 \frac{d^2 y}{dx^2} = 2 - x$. [3] **(ii)** It is given that $y = 1$ when $x = 1$. On a single diagram, sketch three members of the family of solution curves for $x > 0$. [5]
7	**JJC/II/2** A forensic expert with the Criminal Investigation Department arrived at a crime scene at 12 midnight, finding a body on the floor. It was noted that the air-conditioner thermostat was set at $25\ ^\circ\text{C}$, and at the time he arrived, the temperature of the body was $32\ ^\circ\text{C}$. A second measurement was taken at 1 am, and the temperature of the body was found to be $30\ ^\circ\text{C}$. It is known that the rate of decrease of the temperature of the body, $\theta\ ^\circ\text{C}$, is proportional to the temperature difference, $(\theta - 25)\ ^\circ\text{C}$ between the body and the room. By setting up and solving a differential equation, show that $$\theta = 7\left(\frac{5}{7}\right)^t + 25,$$ where t is the number of hours after midnight. [6] Assuming that at the time of death, the body temperature was $37\ ^\circ\text{C}$, estimate this time of death. [2]

8	**MI/PU3/I/5** (i) Find the general solution of the differential equation $\frac{dy}{dx} = 1 - y^2$. **[3]** (ii) Find the particular solution of the differential equation for which $y = \frac{1}{3}$ when $x = 0$. **[1]** (iii) What can you say about the gradient of every solution curve as $x \to \pm\infty$? **[1]** (iv) Sketch, on a **single** diagram, the graph of the solution found in part (ii), together with 2 other members of the family of solution curves. **[3]**
9	**MJC/I/8** A cup of hot liquid is placed in a room where the temperature is a constant $25°\text{C}$. As the liquid cools down, the rate of decrease of its temperature $\theta°\text{C}$ after time t minutes is proportional to the temperature difference $(\theta - 25)°\text{C}$. Initially the temperature of the liquid is $75°\text{C}$. **(i)** Find θ in terms of t and sketch this solution curve. [7] **(ii)** After 10 minutes, the temperature of the liquid was recorded to be $35°\text{C}$. Find the time it takes for the liquid to cool from $75°\text{C}$ to $30°\text{C}$, giving your answer to the nearest minute. [3]
10	**NYJC/I/2** The result of an experiment is modelled by the following differential equation $$\frac{d^2x}{dt^2} = \frac{4}{(t+1)^3}$$ where x is the distance of an object measured from a fixed point at any time t. **(i)** Find x in terms of t, given that the initial position is at $x = 3$. **[4]** **(ii)** Sketch **three** members of the family of solution curves for the solution in **(i)**, stating the equation of any asymptotes. Your sketch should show the different characteristics of the family. **[3]**

11 **NJC/I/10**

The population (in thousands) of fish present in a lake at time t years is denoted by x. It is found that the growth rate of x is proportional to $(200 - 2t - x)$.

It is given that the initial population of the fish in the lake is 8000 and the population grows at a rate of 16000 per year initially. Show that the growth rate of x at time t years can be modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{200 - 2t - x}{12}.$$ [2]

Find x in terms of t by using the substitution $u = 2t + x$. Deduce, to the nearest number of years, the time taken for the fish to die out in this lake. [6]

It is given that the solution curve that describes that population size of the fish at time t years intersects the graph of $x = 200 - 2t$ at the point (t_1, x_1). Describe, in context, what t_1 and x_1 represent. [2]

12 **PJC/I/1**

The volume of water V in a filtration tank at time t satisfies the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 5 - kV,$$

where k is a positive constant. Find V in terms of k and t, given that the tank is initially empty. [5]

State what happens to V for large values of t. [1]

13 **RI/II/4**

(a) The variables u and t are related by the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (a-u)(b-u),$$

where a and b are positive constants such that $u < a$ and $u < b$.
It is given that $u = 0$ when $t = 0$.
Find, simplifying your answer,

(i) u, in terms of t and a, when $a = b$, **[3]**

(ii) t, in terms of u, a and b, when $a \neq b$. **[5]**

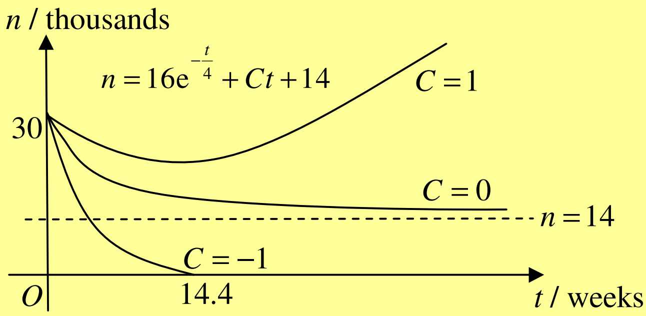
(b) A differential equation is of the form $y = px + qx\frac{\mathrm{d}y}{\mathrm{d}x}$, where p and q are constants. Its general solution is $y = x + \frac{C}{x}$, where C is an arbitrary constant.

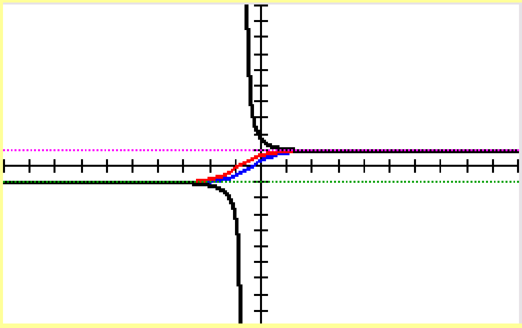
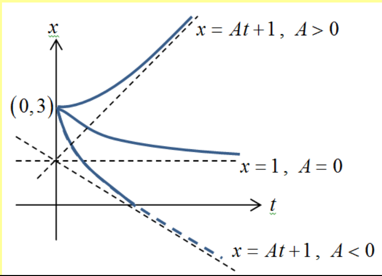
(i) Find the values of p and q. **[2]**

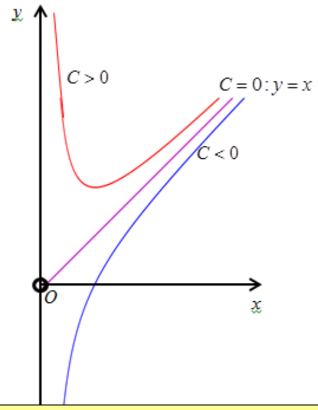
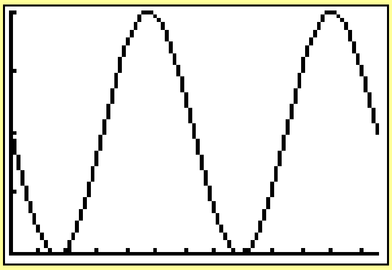
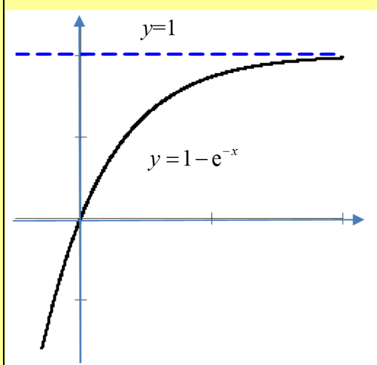
(ii) Sketch, on a single diagram, for $x > 0$, a member of the family of solution curves for each of the following cases: $C = 0$, $C > 0$ and $C < 0$. **[3]**

14	**RVHS/II/4** (a) Show that the differential equation $$x\frac{\mathrm{d}u}{\mathrm{d}x}+u-\sqrt{4-(ux)^2}=0$$ may be reduced by means of the substitution $y=ux+2$ to $$\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{4y-y^2}\ .$$ [2] Hence, find the general solution of the differential equation, leaving your answer in exact form. [4] (b) The displacement s (metres) of an object moving in a straight line from a fixed point O is related to time t (seconds) by the differential equation $$\frac{\mathrm{d}s}{\mathrm{d}t}=\sqrt{4s-s^2}\ .$$ (i) Sketch the solution curve of the particular solution for $0\le t\le 4\pi$ given that $s=1$ when $t=\frac{5\pi}{6}$. [4] (ii) Describe the motion of the object and comment on whether the differential equation in s and t is an appropriate model in the real-life context. [2]
15	**SRJC/II/2** A tank contains water which is heated by an electric water heater working under the action of a thermostat. When the water heater is first switched on, the temperature of the water is $35^{\circ}\mathrm{C}$. The heater causes the temperature to increase at a rate $r^{\circ}\mathrm{C}$ per minute, where r is a constant, until the water temperature hits $75\ ^{\circ}\mathrm{C}$. The heater then switches off. **(i)** Write down, in terms of r, the time taken for the temperature to increase from $35^{\circ}\mathrm{C}$ to $75\ ^{\circ}\mathrm{C}$. [1] The temperature of the water then immediately starts to decrease. The temperature of the water at time t minutes after the heater is switched off is $\theta\ ^{\circ}\mathrm{C}$. It is known that the temperature of the water decreases at a variable rate $k(\theta-25)\ ^{\circ}\mathrm{C}$ per minute, where k is a positive constant, until $\theta=35$. **(ii)** Write down a differential equation involving θ and t, to represent the situation as the temperature is decreasing. [1] **(iii)** Given that when $\theta=55$, the temperature is decreasing at a rate of $5\ ^{\circ}\mathrm{C}$ per minute, find the total length of time for the temperature to increase from $35^{\circ}\mathrm{C}$ to $75\ ^{\circ}\mathrm{C}$ and then decrease to $35^{\circ}\mathrm{C}$, leaving your answer in exact form, in terms of r. [7]
16	**TPJC/I/1** A parachutist leaves the aircraft with zero speed and falls vertically downward. At t seconds later, his speed, v metres per second, and the distance fallen, y metres, satisfy the following differential equations respectively. $$\frac{\mathrm{d}v}{\mathrm{d}t}=10-0.02v \qquad \text{(A)}$$

	$$\frac{\mathrm{d}y}{\mathrm{d}t}=v \qquad \text{(B)}$$ **(i)** Solve equation (A) to show that $v=500\left(1-\mathrm{e}^{-0.02t}\right)$. [4] **(ii)** Hence find y in terms of t. [2] **(iii)** Find the distance fallen by the parachutist when his speed is 50 metres per second. [2]
17	**TJC/II/3** **(a)** By considering a standard series expansion, find the general solution of the differential equation $x=1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\frac{1}{2!}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\frac{1}{3!}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+\cdots+\frac{1}{r!}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^r+\cdots$. [4] **(b)** An empty rectangular tank has vertical sides of depth H metres and a horizontal base of unit area. Water is pumped into the tank at a constant rate such that if no water flows out, the tank can be filled up in time T seconds. Water flows out at a rate which is proportional to the depth of water in the tank. At time t seconds, the depth of water in the tank is x metres. When the depth of water is 1 metre, it remains at this constant value. Show that $\frac{\mathrm{d}x}{\mathrm{d}t}=k(1-x)$, where k is a constant in terms of H and T. [2] Find x in terms of t, H and T. [4]
18	**VJC/I/7** A tank initially contains 400 litres of solution with 100 kg of salt dissolved in it. A solution containing 0.125 kg of salt per litre flows into the tank at a rate of 12 litres per minute and the solution flows out at the same rate. You should assume that the inflow is instantaneously and thoroughly mixed with the contents of the tank. If the amount of salt in the tank is q kg at the end of t minutes, show that $$\frac{\mathrm{d}q}{\mathrm{d}t}=1.5-0.03q. \qquad [2]$$ Find the time taken for the concentration of salt in the tank to reach 0.16 kg per litre. [5] (Concentration of salt = the amount of salt per unit volume of solution in the tank.) State what happens to q for large values of t. Sketch a graph of q against t. [3]
19	**YJC/I/4** Find the solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{\mathrm{d}y}{\mathrm{d}x}$ in the form $y=\mathrm{f}(x)$, given that $y=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=1$ when $x=0$. [4] Sketch the solution curve, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [2]

S/No	Topic	Answers(includes comments and graph)
1	**AJC**	(a)(ii) $\frac{1}{A}\ln\left\|1+Ae^{2x}\right\|+B,$ (b) 8.11 min (3 s.f.)
2	**ACJC**	(ii) The population of the predators approaches 100 000 after several years. (iii) $y=5000e^{-0.01t}+5000.$
3	**CJC**	(i) $\therefore P=\frac{10}{1+9e^{-0.64t}}$ (ii) $P=10e^{-Ae^{-0.4t}}$ (<u>Similarity</u>) The population approaches 10 asymptotically in the long run in both models. (<u>Difference</u>) The population in the second model would increase faster first then slower compared to the first model. OR The Gompertz function will take a longer time to approach 10 as compared to the Logistic function.
4	**DH**	(i) $y^2=\frac{\tan(2x+D)}{2x}$; (ii) $n=16e^{-\frac{t}{4}}+Ct+D, D=14$ 
5	**HCI**	(i) $A=24000e^{0.111t}+36000$ with $\alpha=24000$ (3 s.f.) and $\lambda=36000$ (3 s.f.); (ii) 7.93 (3 s.f.); (iii) $n=7720$ (3 s.f.); (iv) $\frac{dA}{dt}=0$ means that the rate which Mac needs to cut the weeds is equal to the rate the weeds grow. Thus, the area covered in weeds is unchanged.
6	**IJC**	(i) $y=\frac{1}{x}+\ln\left\|x\right\|+Cx+D$

7	**JJC**	$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-k(\theta-25)$; 10.24pm
8	**MI**	(i) $y=\frac{Ae^{2x}-1}{Ae^{2x}+1}$ (ii) $y=\frac{2e^{2x}-1}{2e^{2x}+1}$ (iii) $\frac{\mathrm{d}y}{\mathrm{d}x}\to 0$ (iv) 
9	**MJC**	(i) $\theta=25+50\mathrm{e}^{-kt}$ (ii) $t\approx 14$ (to nearest min)
10	**NYJC**	**(i)** $x=\frac{2}{t+1}+At+1$ **(ii)** 
11	**NJC**	$x=224-2t-216\mathrm{e}^{-\frac{t}{12}}$; $t\approx 112$ (years); x_1 is the maximum population size of the fish. t_1 is the number of years for the population to reach its maximum.
12	**PJC**	$V=\frac{5}{k}\left(1-\mathrm{e}^{-kt}\right)$ $V\to\frac{5}{k}$
13	**RJC**	(a)(i) $u=\frac{a^2t}{at+1}$ (a)(ii) $t=\frac{1}{b-a}\ln\frac{a(b-u)}{b(a-u)}$ (b)(i) $p=2$ and $q=-1$ (b)(ii)

		
14	**RVH**	(a) $u=\dfrac{2\sin\left(x+c\right)}{x}$ (b)(i)  (ii) The object oscillates about the starting point which is 2m from O, with an amplitude of 2m. The motion assumes the absence of resistance whereby the amplitude remains constant which is unrealistic in real-life.
15	**SRJC**	(i) $\dfrac{40}{r}$ mins (ii) $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=-k\left(\theta-25\right)$ (iii) $\left(\dfrac{40}{r}+6\ln 5\right)$ mins
16	**TPJC**	(ii) $y=500(t+50\mathrm{e}^{-0.02t})-25000$ (iii) $y=134$ (to 3 s.f.)
17	**TJC**	(a) $y=x\ln x-x+C$ (b) $x=1-\mathrm{e}^{-\frac{H}{T}t}$
18	**VJC**	42.4 mins The amount of salt in the tank <u>decreases</u> to 50kg.
19	**YJC**	$y=\mathrm{f}(x)=1-\mathrm{e}^{-x}$ 

2015 Prelim Topical-Mathematical Induction

1 **AJC/2015/I/Q8**

(i) Prove by the method of mathematical induction that

$$\sum_{r=2}^{n} \frac{2}{(r+3)(r+5)} = \frac{11}{30} - \frac{2n+9}{(n+4)(n+5)}.$$ [5]

(ii) Hence find $\sum_{r=4}^{n+4} \frac{2}{r(r+2)}$. [3]

(iii) Deduce that $\sum_{r=4}^{n+4} \frac{1}{(r+1)^2} < \frac{9}{40}$. [2]

2 **ACJC/2015/I/Q7**

A sequence u_1 ,u_2, u_3, … is given by

$$u_1 = -2 \quad \text{and} \quad u_n = u_{n-1} + \ln\left(1 + \frac{2n-1}{(n-1)^2}\right) - 2 \qquad \text{for } n \geq 2.$$

(i) Use the method of mathematical induction to prove that for all positive integers n,

$$u_n = 2(\ln n - n)$$ [4]

(ii) Hence find $\sum_{r=15}^{n} \frac{e^{u_r}}{r^2}$ in terms of n. [3]

(iii) Give a reason why the series $\sum_{r=1}^{\infty} \frac{e^{u_r}}{r^2}$ converges, and write down its exact value. [2]

3 **CJC/2015/I/Q4**

(i) Show that $r(r+1) = \frac{1}{3}\left[r(r+1)(r+2) - (r-1)r(r+1)\right]$. [1]

(ii) Hence, using the method of difference, find the sum $1\times2 + 2\times3 + 3\times4 + \cdots + n(n+1)$. [3]

(iii) Prove by mathematical induction that

$$1\times2\times3 + 2\times3\times4 + 3\times4\times5 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3).$$ [4]

(iv) Based on the results in parts **(ii)** and **(iii)**, write a reasonable conjecture for the sum of the series $\sum_{r=1}^{n} r(r+1)(r+2)(r+3)$. [1]

4 DHS/2015/II/Q1

A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = \frac{1}{4}$ and

$$u_{n+1} = u_n - \frac{4}{n^2(n+1)(n+2)^2}, \quad \text{for all } n \geq 1.$$

(i) Use the method of mathematical induction to prove that $u_n = \dfrac{1}{n^2(n+1)^2}$. [4]

(ii) Hence find $\displaystyle\sum_{n=1}^{N} \frac{1}{n^2(n+1)(n+2)^2}$. [2]

(iii) Give a reason why the series in part **(ii)** is convergent and state the sum to infinity. [2]

(iv) Use your answer to part **(ii)** to find $\displaystyle\sum_{n=2}^{N} \frac{1}{n(n^2-1)^2}$. [2]

5 HCI/2015/II/Q2

It is given that $\mathrm{f}(r) = (3r-2)(3r+1)$, where $r \in \mathbb{Z}^+$.

(i) Use the result $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ to show that $\displaystyle\sum_{r=1}^{n} \mathrm{f}(r) = n\left(an^2 + bn + c\right)$, where a, b and c are constants to be determined. [3]

Let $S_n = \displaystyle\sum_{r=1}^{n} \frac{1}{\mathrm{f}(r)}$.

(ii) Show that $S_2 = \frac{2}{7}$, and evaluate S_3 and S_4. [2]

(iii) State a conjecture for S_n in the form $\dfrac{A}{B+1}$, where A and B are in terms of n. Prove the conjecture by mathematical induction. [5]

6 IJC/2015/I/Q4

A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = 1 \text{ and } u_{n+1} = \frac{4u_n - 1}{8} \text{ for } n \geq 1.$$

(i) Find the values of u_2 and u_3. [1]

(ii) It is given that $u_n \to l$ as $n \to \infty$. Showing your working, find the exact value of l. [1]

(iii) For this value of l, use the method of mathematical induction to prove that

$$u_n = 5\left(\frac{1}{2}\right)^{n+1} + l$$ [4]

(iv) Hence find $\sum_{n=1}^{N} u_n$. [3]

7 **JJC/2015/I/Q2**

A sequence u_0, u_1, u_2, …is defined by $u_0 = 3$ and $u_{n+1} = 1 - 2u_n$, where n □□□□

(i) Prove by induction that $u_n = \frac{1}{3}\left[1+8(-2)^n\right]$, for all $n \ge 0$. [4]

(ii) State, briefly giving a reason for your answer, whether the sequence is convergent. [1]

8 **MI/2015/I/Q10**

(i) Show that $\dfrac{2(3^r)}{3(1+3^{r-1})(1+3^r)} = \dfrac{1}{1+3^{r-1}} - \dfrac{1}{1+3^r}$. [1]

(ii) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{3^r}{(1+3^{r-1})(1+3^r)} = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{1+3^n}\right).$$ [4]

(iii) Use the method of mathematical induction to prove the result in part (ii). [5]

(iv) Hence find $\sum_{r=1}^{\infty} \dfrac{3^{r+1}}{(1+3^{r-1})(1+3^r)}$. [3]

9 **MJC/2015/II/Q1**

The sequence $u_1,\ u_2,\ u_3,...$ is such that $u_1 = 3$ and

$$u_{n+1} = 3 - \frac{2}{u_n}, \quad \text{for all } n \ge 1.$$

(i) By writing the first 3 terms of this sequence, show that a possible conjecture for u_n is $\dfrac{a^{n+1}-1}{a^n-1}$, where a is a positive integer to be determined. [2]

(ii) Using the value of a found in part **(i)**, prove by induction that $u_n = \dfrac{a^{n+1}-1}{a^n-1}$ for all $n \ge 1$. [4]

(iii) Hence determine if the limit of $u_1 u_2 u_3 ... u_n$ exists as $n \to \infty$. [2]

10 NYJC/2015/II/Q1

Prove by the method of mathematical induction that

$$\sum_{r=1}^{n} \cos\left(2r\theta\right) = \frac{\sin\left(2n+1\right)\theta - \sin\theta}{2\sin\theta}$$

for all positive integers n. [5]

Hence, evaluate $\sum_{r=1}^{n} \cos\left(r\pi\right)$. [3]

11 NJC/2015/I/Q1

Prove by the method of mathematical induction that

$$\sum_{n=0}^{N} \frac{3n+2}{(n+1)!3^{n+1}} = 1 - \frac{1}{(N+1)!3^{N+1}}$$

for $N \geq 0$. [5]

12 PJC/2015/I/Q10

Prove by mathematical induction that $\sum_{r=1}^{n} \sin(2r\theta) = \frac{\cos\theta - \cos(2n+1)\theta}{2\sin\theta}$ for all positive integers n. [6]

Hence, find an expression for $\sin\theta\cos\theta + \sin 2\theta \cos 2\theta + ... + \sin n\theta \cos n\theta$ in terms of θ and n. [2]

13 RI/2015/I/Q4

Prove by induction that $\sum_{r=2}^{n} \frac{2^r\left(r-1\right)}{r\left(r+1\right)} = \frac{2^{n+1}}{n+1} - 2$, for $n \geq 2$. [5]

Hence find $\sum_{r=3}^{2N} \frac{2^r\left(r-1\right)}{r\left(r+1\right)}$. [2]

14 RVHS/2015/I/Q5

(a) Given that $u_{n+1} = u_n - \frac{n^2+n-1}{\left(n+1\right)!}$ and $u_0 = 1$, prove by induction that

$$u_n = \frac{n+1}{n!} \text{ for } n \geq 0$$

[5]

(b) Using the recurrence relation in part (a), find

$$\sum_{r=1}^{N} \frac{(r+1)^2 + r}{(r+2)!}.$$

[4]

15 SAJC/2015/IQ2

Prove by mathematical induction that,

$$\sin x+\sin 11x+\sin 21x+...+\sin(10n+1)x=\frac{\cos 4x-\cos\left(10n+6\right)x}{2\sin 5x}$$

where $\sin 5x\neq 0$, for non-negative integers n. [5]

16 SRJC/2015/IQ4

A sequence u_n is given by $u_1=1$ and $u_{n+1}=u_n+(n+1)\left(\frac{1}{2}\right)^n$, for $n\in\mathbb{Z}^+$.

Prove using mathematical induction that $u_n = 4-(n+2)\left(\frac{1}{2}\right)^{n-1}$. [5]

17 TPJC/2015/I/Q2

The sum, S_n, of the first n terms of a sequence $u_1,\ u_2,\ u_3,\ \dots$ is given by

$$S_n=\sum_{r=1}^{n}r(3^{r-1})=\frac{1}{4}+\frac{3^n}{4}\left(2n-1\right).$$

(i) Use the method of mathematical induction to prove that $S_n=\frac{1}{4}+\frac{3^n}{4}\left(2n-1\right)$. [5]

(ii) Find a recurrence relation for u_{n+1} in terms of u_n and n. [2]

(iii) Determine the behaviour of the sequence. [1]

18 TJC/2015/I/Q4

The sequence of real numbers $u_1,u_2,u_3,\ ...$ is defined by

$$u_{n+1}=\frac{n+2}{n+4}u_n \text{ and } u_1=a, \text{ where } n\geq 1 \text{ and } a\in\mathbb{R}.$$

(i) Prove by mathematical induction that $u_n=\frac{12a}{(n+2)(n+3)}$ for $n\geq 1$. [4]

(ii) Determine the limit of $n(n+2)\frac{u_n}{u_1}$ as $n\to\infty$. [2]

19 VJC/2015/I/Q1

A sequence $w_1,\ w_2,\ w_3,\ ...$ is such that

$$w_{n+1}=\frac{1}{n}[(n+1)w_n+1],\ \text{where } n\in\mathbb{Z}^+,\ \text{and}$$

$$w_1=a,\ \text{where } a \text{ is a constant.}$$

Use the method of mathematical induction to prove that $w_n=an+(n-1)$. [5]

20 YJC/2015/I/Q7b

Prove by mathematical induction that for all positive integers n,

$$\sum_{r=1}^{n} \frac{r(2^r)}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}.$$ [4]

S/No	Answers
1	(ii) $\frac{9}{20} - \frac{2n+11}{(n+5)(n+6)}$
2	(ii) $\frac{e^{-28} - e^{-2n}}{e^2 - 1}$ (iii) $\sum_{r=1}^{\infty} \frac{e^{u_r}}{r^2} = \sum_{r=1}^{\infty} e^{-2r}$ converges as $\left\|e^{-2}\right\| < 1$. $\sum_{r=1}^{\infty} e^{-2r} = \frac{e^{-2}}{1-e^{-2}} = \frac{1}{e^2 - 1}$ (Alternatively, students may use part (ii) by taking n to infinity, and then adding on the sum of the first 14 terms.)
3	(ii) $\frac{1}{3}n(n+1)(n+2)$ (iv) $\sum_{r=1}^{n} r(r+1)(r+2)(r+3) = \frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$
4	(ii) $\frac{1}{16} - \frac{1}{4(N+1)^2(N+2)^2}$ (iii) $\frac{1}{16}$ (iv) $\frac{1}{16} - \frac{1}{4N^2(N+1)^2}$
5	(i) $a = 3$, $b = 3$ and $c = -2$ (ii) $S_3 = \frac{3}{10}$, $S_4 = \frac{4}{13}$ (iii) $S_n = \frac{n}{3n+1}$
6	(i) $u_2 = \frac{3}{8}$, $u_3 = \frac{1}{16}$ (ii) $l = -\frac{1}{4}$ (iv) $\frac{5}{2}\left(1 - \frac{1}{2^N}\right) - \frac{1}{4}N$

7	(ii) The sequence is **divergent** as $n \to \infty$, $(-2)^n$ does not converge to a finite number.
8	(iv) $\frac{9}{4}$
9	(i) $a = 2$ (iii) The limit does not exist
10	$\frac{(-1)^n - 1}{2}$
12	$\frac{\cos\theta - \cos(2n+1)\theta}{4\sin\theta}$
13	$\frac{2^{2N+1}}{2N+1} - \frac{8}{3}$
14	(b) $\frac{3}{2} - \frac{N+3}{(N+2)!}$
17	(ii) $u_{n+1} = 3u_n + 3^n$ (iii) The sequence increases and diverges.
18	(ii) 12

2015 Prelim Topical- Sequences and Series

1 ACJC/I/13

The owner of a newly opened café decided to rent a painting from an artist as part of the decoration of the café. They set about drafting up a contract for the terms of the rental.
The artist proposed a rental contract (Version 1) stating that the owner will pay the artist \$15 for the 1st day of rental and for each subsequent day, the daily rental cost will increase by \$0.50.

(i) On which day of the rental will the owner first have to pay the artist more than \$39 as the daily rental rate? [2]

The owner proposed an alternative contract (Version 2), where the daily rental rate is such that on the nth day of the rental, the amount of money, in dollars, the owner has to pay to the artist is given by the function

$$\mathrm{f}(n)=\frac{12000}{4n^2+4n-3}.$$

(ii) Express $\mathrm{f}(n)$ in the form $\dfrac{A}{2n-1}+\dfrac{B}{2n+3}$, where A and B are constants to be determined. [1]

(iii) Hence show that with Version 2 of the contract, the total amount of money the artist will receive at the end of m days of rental is

$$4000-\frac{12000(m+1)}{(2m+1)(2m+3)}.$$ [3]

(iv) The artist accepted Version 2 of the contract, and terminated the contract at the end of k days. Given that the artist received more money in total from Version 2 than if he had chosen Version 1, find the largest possible value of k. [3]

2 AJC/II/4

Betty needs to decorate a wall of length 5 metres for a party. She attaches hooks, starting from the extreme left end of the wall, and numbers each hook “1”, “2”, “3” and so on. The spacing between the 1^{st} and 2^{nd} hook is 50 cm and each subsequent spacing is 2 cm shorter than the previous spacing. This will continue till she is unable to place the next hook due to insufficient space.

(i) How many hooks can she attach in total? [4]

Betty also cuts a 6 metre roll of ribbon into pieces of varying lengths. The first piece cut off is of length 80 cm and each subsequent piece cut off is 10 % shorter than the previous piece.

(ii) If the length of the remaining roll of ribbon is less than 1 metre after n cuts, find the smallest value of n. [3]

Betty wants to hang the ribbons between every 2 hooks such that the ribbons are of decreasing lengths starting from that between the 1^{st} and 2^{nd} hook. She starts by using the ribbon of length 80 cm (Assume negligible length of ribbon is required to hang them on the hooks).

(iii) By considering the length of the ribbon to be hung between the n^{th} and $(n+1)^{th}$ hook, write down an inequality to be satisfied by n. [1]

(iv) Hence find the number of hooks which will have ribbons hung on them. [2]

3 CJC/I/3

John decided to embark on a 100-day skipping exercise challenge. The duration of his skipping exercise each day, in seconds, follows an arithmetic progression for the odd days, and another arithmetic progression for the even days.

Day	**Duration (s)**
Day 1	20
Day 2	20
Day 3	29
Day 4	30
Day 5	38
Day 6	40
Day 7	47
Day 8	50
Day 9	56
Day 10	60
$\vdots$	$\vdots$

(i) Find the duration of the skipping exercise that he does on the 75^{th} day. [2]

(ii) Find the total duration of the skipping exercise that he does for the first 75 days. [3]

(iii) After the 75^{th} day, John thinks that the workout is too strenuous so he decides to modify the workout such that he does 80% of the previous day's duration. Find the duration of the skipping exercise that he does on the 100^{th} day. [2]

(iv) Comment on the practicality of this modification. [1]

4 CJC/I/5

Sequence U is defined by the following recurrence relation,

$$u_1 = 1, \quad u_{n+1} = \frac{1}{2}u_n + 1 \text{ for all } n \in \mathbb{Z}^+ .$$

Sequence V is defined by $v_n = u_n - 2$ for all $n \in \mathbb{Z}^+$.

(i) Find the recurrence relation between v_{n+1} and v_n. Hence show that the sequence V is a geometric progression with common ratio $\frac{1}{2}$. [3]

(ii) Find the limit of the sequence V and that of the sequence U when $n \to \infty$. [3]

(iii) Find the sum to infinity of the sequence V. [1]

(iv) Find the sum of the first n terms of the sequence U.

Hence show that its sum to infinity does not exist. [3]

5 DHS/II/4

A farmer owns a parcel of land that is partially covered with weed. The area that is covered with weed increases by 80 m^2 each week. The farmer decides to start weeding. At the start of the first week of weeding, 500 m^2 of the land is covered with weed.

(i) In option 1, the farmer removes weed from 10% of the area covered with weed at the end of each week.

(a) Find the area covered with weed at the end of the second week. [2]

(b) Show that the area covered with weed at the end of the *n*th week is given by $(0.9^n(500)+k(1-0.9^n))\ \text{m}^2$, where k is a constant to be determined. [3]

(c) Find the area covered with weed at the end of a week in the long run. [1]

(ii) In option 2, the farmer removes weed from an area of 50 m^2 at the end of the first week. He removes weed from an additional area of 10 m^2 at the end of each subsequent week. Thus he removes weed from an area of 60 m^2 at the end of the second week, and 70 m^2 at the end of the third week, and so on.

(a) Show that the change in the area covered with weed in the *n*th week is given by $(40-10n)\ \text{m}^2$. [1]

(b) Hence, or otherwise, find the area covered with weed at the end of the *n*th week in terms of n. [3]

6 HCI/I/9

A researcher conducted a study on the radioisotope, Iodine-131 (I-131) which has a half-life of 8 days (i.e., the amount of I-131 is halved every 8 days). He first introduced $1000\,\text{mg}$ of I-131 in an empty Petri dish on Day 1 and tracked the amount of I-131 in the dish.

(i) State the amount of I-131 in the dish at the end of 16 days. [1]

After every 16 days, i.e., on Day 17, Day 33, Day 49 etc., the researcher added $1000\,\text{mg}$ of I-131 to the dish.

(ii) Find the amount of I-131, to the nearest mg, in the dish immediately after $1000\,\text{mg}$ of I-131 was added on Day 49. [3]

(iii) Show that the amount of I-131 in the dish will never exceed $1334\,\text{mg}$. [2]

The researcher discovered that he accidentally used a different radioisotope, Iodine-125 (I-125) on Day 1, which has a half-life of 60 days instead. He checked that he had indeed used the correct I-131 on other occasions.

(iv) Find the total amount of radioisotopes I-125 and I-131 in the dish on Day 121, giving your answer correct to the nearest mg. [4]

7 IJC/I/9

At a robotic exhibition, a miniature robotic mouse is programmed to walk 2 different simulation paths from a starting point O to and from a series of points, $P_1, P_2, P_3, \ldots,$ increasingly far away in a straight line. The robotic mouse starts at O and walks stage 1 from O to P_1 and back to O, then stage 2 from O to P_2 and back to O, and so on. The miniature robotic mouse can walk a maximum distance of 100 m before its battery runs out.

(i)

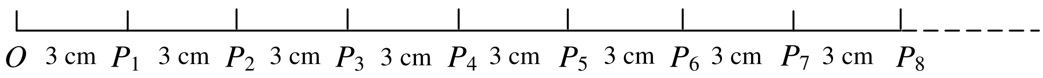


Fig. 1

In simulation A, the distances between adjacent points are all 3 cm (see Fig. 1).

(a) Find the distance travelled by the robotic mouse that completes the first 12 stages of simulation A. [2]

(b) Write down an expression for the distance travelled by the robotic mouse that completes n stages of simulation A. Hence find the greatest number of stages that the robotic mouse can complete before its battery runs out. [4]

(ii)

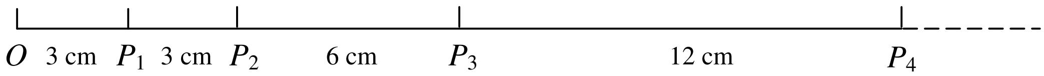


Fig. 2

In simulation B, the distances between the points are such that OP_1 = 3 cm, P_1P_2 = 3 cm, P_2P_3 = 6 cm and $P_nP_{n+1} = 2P_{n-1}P_n$ (see Fig. 2). Write down an expression for the distance travelled by the robotic mouse that completes N stages of simulation B. Hence find the distance from O, and the direction of travel, of the robotic mouse at the instance when the battery runs out in simulation B. [5]

8 JJC/I/7

An athlete hopes to represent Singapore at the SEA Games in 2015 and he embarks on a rigorous training programme.

For his first training session, he ran a distance of 7.5 km. For his subsequent training sessions, he ran a distance of 800 m more than the previous training session.

(i) Express, in terms of n, the distance (in km) he ran on his n th training session. [1]

(ii) Find the minimum number of training sessions required for him to run a total distance of at least 475 km. [3]

After a month, he realised that his progress was unsatisfactory and he decided to modify the training. For the modified training programme, he ran a distance of x km for the first session, and on each subsequent training session, the distance covered is $\frac{6}{5}$ times of the previous session.

(iii) Find x, to the nearest integer, if he covered a distance of 14.93 km on the 6th training session. [2]

(iv) Using the answer in **(iii)**, and denoting the total distance after n training sessions by G_n, write down an expression for G_n in terms of n.

Hence show that $\sum_{n=1}^{N} G_n$ may be expressed in the form $aG_N + bN$, where a and b are integers to be determined. [4]

9 MI/I/4

(i) The n^{th} term of a sequence is $\mathrm{T}_n = \ln 3x^{n-1}$ where x is a constant. Show that the sequence is an arithmetic progression for all positive integers n. [2]

(ii) When $\ln 3$ is subtracted from the 19^{th}, 7^{th} and 3^{rd} terms of the arithmetic progression in part **(i)**, these terms become the first three terms of a geometric progression.

(a) Find the common ratio of the geometric progression. [2]

(b) Find the range of values of x for which the sum of the first 20 terms of the arithmetic progression exceeds the sum to infinity of the geometric progression. [3]

10 MJC/I/6

Matthew embarks on a skipping regime, 6 days a week to lose weight. In week 1, he skips 50 times per day. In week 2, he skips 60 times per day. On each subsequent week, the number of skips per day is 10 more than on the previous week.

(i) Show that the number of skips completed by Matthew in week 8 is 720. [1]

(ii) After n weeks, Matthew found that he exceeded 5000 skips in total. Express this information as an inequality in n and hence find the least value of n. [4]

As a result of the skipping, Matthew starts to lose weight. He measures his initial weight and records his weight at the end of each week and notices that his weights follow a geometric progression. At the end of week 32, Matthew's weight is 83 kg.

(iii) Given that he lost 10% of his initial weight at the end of week 25, find Matthew's initial weight. [3]

11 NJC/II/2

(a) Joanne begins a monthly savings plan. In the first month, she puts \$1000 into her savings, and for each subsequent month she puts 5% more than in the previous month.

How many months would it take for Joanne's total savings to first exceed \$20000? [3]

Jim starts a monthly savings plan two months later than Joanne with an initial savings of \$2000, and for each subsequent month he puts \$100 more than in the previous month. The table below shows the person whose total savings exceeds the other person's total savings in the *N*th month since Joanne started saving.

N	Person with more total savings in Nth month
1	Joanne
2	Joanne
3	Joanne
4	Joanne
5	Jim
6	Jim
$\vdots$	$\vdots$
$n-1$	Jim
n	Joanne
$\vdots$	$\vdots$

Find the value of n. [4]

(b) Suppose instead that Joanne puts a fixed amount of \$1000 into her bank account on the first day of every month. The interest rate is r% per month, so that on the last day of each month the amount in the account on that day is increased by r%.

At the end of 2 years, the total amount in her account will be at least \$30000. Find the smallest value of r, correct to 1 decimal place. [3]

12 NYJC/I/9

(a) A geometric sequence $x_1, x_2, x_3, \ldots$ has first term a and common ratio r, where $a > 0$, $r > 0$.

The sequence of numbers $y_1, y_2, y_3, \ldots$ satisfies the relation $y_n = \log_4(x_n)$ for $n \in \mathbb{Z}^+$.

(i) If the product of x_5 and x_{21} is 4096, find the value of $\sum_{k=1}^{25} \log_4 x_k$. **[4]**

(ii) Show that $y_1, y_2, y_3, \ldots$ is an arithmetic sequence. **[2]**

(b) The output of an oil field in any year is 6% less than in the preceding year. Prove that the total output of the oil mined from the oil field cannot exceed 17 times the output in the first year. **[2]**

It is decided to close the mine when the total output exceeds 16 times the output in the first year. Suppose the oil field was first mined in the year 1997, determine the earliest year in which the oil field will be closed. **[2]**

13 PJC/I/6

An art exhibition features sculptures of Singa and Nila, which are the mascots for the SEA Games held in Singapore in 1993 and 2015 respectively. The number of Singa and Nila sculptures corresponds to the number of sports contested at the two editions of the games respectively. The sculptures are of varying heights.

The Singa sculptures are displayed in a line such that the tallest Singa sculpture is in the middle. Starting from both ends of the line, the height of each subsequent Singa sculpture is 10 cm more than the preceding Singa sculpture, up to the middle Singa sculpture.

(i) Given that 29 sports were contested at the 1993 SEA Games and the total height of all the Singa sculptures is 3120 cm, find the height of the tallest and shortest sculptures. [3]

The Nila sculptures are displayed in order of descending height. The height of the tallest Nila sculpture is 210 cm. The height of each subsequent Nila sculpture is 5% shorter than the height of the preceding Nila sculpture.

(ii) Given that the shortest Nila sculpture is the only Nila sculpture to have a height of less than 35 cm, find the number of sports contested at the 2015 SEA Games. [2]

(iii) Find, to 2 decimal places, the height of the shortest Nila sculpture and the total height of all the Nila sculptures. [3]

14 RI/I/2

The first term of an arithmetic series is 5 and the 25th term of the series is $\frac{31}{5}$. Find the least value of n such that the sum of the first n terms of the series exceeds 1000. [4]

15 RI/II/2(a)

Given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(2n+1)(n+1)$, find $\sum_{r=N+1}^{2N} \left(7^{r+1} + 3r^2\right)$ in terms of N, simplifying your answer.

[3]

16 RVHS/I/3

(a) The sum of the first n terms of a sequence is given by $S_n = 2n^2 + 3n$. Prove that the given sequence is an arithmetic progression. Find the least value of n such that the sum of the first n terms exceeds 2015. [4]

(b) In an extreme cake-making competition, Kimberley is tasked to make a multi-layered cake in which each layer is in the shape of a cylinder, starting with the base layer of height 10 cm. For each subsequent layer above, it must be smaller than the previous layer. In particular, the height of any layer above the base must be $(5k)$ % of the height of the previous layer, where k is a positive integer. The layers are joined together using whipped cream.

(i) Show that the height of an n-layer cake is given by

$$\frac{200}{20-k}\left[1-\left(\frac{k}{20}\right)^n\right] \text{cm.} \qquad [1]$$

(ii) It is found that a cake with height exceeding 1.2 m will become structurally unstable. Using k = 19, find the maximum number of layers that Kimberley's cake can have before it becomes structurally unstable. [2]

(iii) State a possible assumption used in part (ii). [1]

17 SAJC/II/2

(a) Adrian has signed up at a driving centre to learn how to drive. His first lesson is 40 minutes long. Each subsequent lesson is 5 minutes longer than the previous lesson, so that the second lesson is 45 minutes long, the third lesson is 50 minutes long, and so on. The centre requires a student to have attended at least 60 hours of lessons before he is qualified to take the driving test. Find the minimum number of lessons that Adrian has to attend before he can take the test. **[4]**

(b) A sequence of real numbers $u_1, u_2, u_3, \ldots$, where $u_1 \neq 0$, is defined such that the $(n+1)$th term of the sequence is equal to the sum of the first n terms, where $n \in \mathbb{Z}^+$. Prove that the sequence $u_2, u_3, u_4, \ldots$ follows a geometric progression. **[3]**

Hence find $\sum_{r=1}^{N+1} u_r$ in terms of u_1 and N. **[2]**

18 SRJC/I/10

(a) An arithmetic progression, A, has first term, a, and a non-zero common difference, d. The first, fifth and fourteenth term of A are equal to the third, second and first term of a geometric progression G respectively.

(i) Show that the common ratio of G is $\frac{4}{9}$. [3]

(ii) Determine the ratio of the sum to infinity of G to the sum to infinity of the odd-numbered terms of G. [3]

(b) In January 2001, John borrowed \$29 000 from a bank. Interest was charged at 4.5% per year and was calculated at the end of each year starting from 2001. John planned to pay back a fixed amount of \$$x$ on the first day of each month, starting from the first month after his graduation. John graduated from his study in at the end of 2004 and started payment in January 2005.

(i) Show that the amount owed at the end of 2005 was

$$1.045^5(29000) - 1.045(12x). \qquad [1]$$

Taking 2005 as the first year, find an exact expression for the amount John owed at the end of the n^{th} year, simplifying your answer in terms of n and x. [3]

(ii) If the loan must be repaid within 8 years from John's graduation, find the minimum amount, correct to the nearest dollar, John must pay each month. [2]

19 TJC/II/2

(a) Given that the sequence $5,\ 11,\ 17,\ \cdots,\ x$ is arithmetic, solve the equation

$$5+11+17+\cdots+x=2760.$$ [4]

(b) Mr Tan set aside \$80,000 for his two sons. On the first day of the year that his sons turned 7 and 17 years old, he deposited \$$x$ into the younger son's bank account and the remaining sum of money into the elder son's bank account. Mr Tan adds a further \$1000 into the younger son's account on the first day of each subsequent year. The bank pays a compound interest at a rate of 2% per annum on the last day of each year.

Each son will withdraw the full sum of money from his account (after interest had been added) on the last day of the year that he turns 21 years old.

(i) Find the amount of money the elder son will withdraw in terms of x. [1]

(ii) Show that the younger son will withdraw $\$\left(1.02^{15}x+51000\left(1.02^{14}-1\right)\right)$. [3]

Find the value of x if Mr Tan wanted both sons to receive the same withdrawal amount, giving your answer to the nearest integer. [2]

20 TPJC/I/8

Trainer A runs 100m for stage 1 of an exercise and increases the training distance by 50m for each subsequent stage.

(i) Find the distance run by Trainer A on the 15th stage of the exercise. [2]

(ii) Write down an expression for S_A, the distance run by Trainer A who completes n stages of the exercise. Hence find the least number of stages that Trainer A needs to complete to run at least 10km. [3]

In the same exercise, Trainer B runs 50m for stage 1, and increases the training distances by 20% for each subsequent stage.

(iii) Find the distance run by Trainer B on the 15th stage of the exercise. [2]

(iv) Write down an expression for S_B, the distance run by Trainer B who completes n stages of the exercise. Hence find the least number of stages, n, such that $S_B > S_A$. [3]

(v) For this value of n found in part **(iv)**, find the difference in distance between Trainer B and Trainer A after n stages of the exercise, giving your answer to the nearest whole number. [1]

21 VJC/I/6

On 1 January 2015, Mrs Koh put \$1000 into an investment fund which pays compound interest at a rate of 8% per annum on the last day of each year. She puts a further \$1000 into the fund on the first day of each subsequent year until she retires.

(i) If she retires on 31 December 2040, show that the total value of her investment on her retirement day is \$86351, correct to the nearest dollar. [4]

On 1 January 2015, Mr Woo put \$1000 into a savings plan that pays no interest. On the first day of each subsequent year, he saves \$80 more than the previous year. Thus, he saves \$1080 on 1 January 2016, \$1160 on 1 January 2017, and so on.

(ii) By forming a suitable inequality, find the year in which Mr Woo will first have saved over \$86351 in total. [4]

22 YJC/I/8

(a) Mary had 1922 marbles and she decided to pack them into different bags. She placed 6 marbles in the first bag. Each subsequent bag that she packed contained 6 marbles more than the previous bag. She continued to fill the bags until there was not enough marbles to fill the next bag. Calculate the number of marbles that were left unpacked. [4]

(b) On the first day of February 2015, a bank loans a man \$10,000 at an interest rate of 1.5% per month. This interest is added on the last day of each month and is calculated based on the amount due on the first day of the month. The man agrees to make repayments on the first day of each subsequent month. Each repayment is \$1200 except for the final repayment which is less than \$1200.
The amount that he owes at the start of each month is taken to be the amount he still owes just after the monthly repayment has been made. Find the date and amount of the final repayment to the nearest cent. [4]

S/No	Schools	Answers(includes comments and graph)
1	**ACJC**	(i) 50th day of rental (ii) $\frac{3000}{2n-1}-\frac{3000}{2n+3}$ (iv) 99
2	**AJC**	(i) 14 (ii) 10 (iii) $80(0.9)^{n-1} \geq 50-2(n-1)$ (iv) 10
3	**CJC**	(i) 353 (ii) 14487 (iii) 1.33
4	**CJC**	(i) $v_{n+1}=\frac{1}{2}v_n$ (ii) limit of sequence V is 0; limit of sequence U is 2 (iii) -2 (iv) $2\left(\frac{1}{2}\right)^n-2+2n$
5	**DHS**	(i)(a) $541.8\ \ \mathrm{m}^2$; (b) $0.9^n(500)+720(1-0.9^n)\ \mathrm{m}^2$, $k=720$; (c) 720; (ii)(b) $500+35n-5n^2\ \mathrm{m}^2$
6	**HCI**	(i) 250 mg (ii) 1328.125 mg = 1328 mg (nearest mg) (iv) 917 mg (nearest mg)
7	**IJC**	(i)(a) 468 (i)(b) $3n(n+1)$; Greatest n = 57 (ii) $6(2^N-1)$; 2282 cm; direction of travel is towards O
8	**JJC**	(i) $6.7 + 0.8n$ (ii) 27 (iii) 6 (iv) $30\left[\left(\frac{6}{5}\right)^n-1\right]$; $6G_N-30N$
9	**MI**	(ii)(a) $\frac{1}{3}$ (ii)(b) $x>e^{-\frac{20}{163}\ln 3}=0.874$
10	**MJC**	(ii) Least value of n is 10 (iii) 95.0 kg
11	**NJC**	(a) 15 months; n = 50 (b) least value of r = 1.8 (1 d.p.)
12	**NYJC**	(a)(i) 75 (b) 2049
13	**PJC**	(i) Height of the shortest Singa sculpture is 40 cm Height of the tallest Singa sculpture is 180 cm (ii)The number of sports contested at the 2015 SEA Games is 36. (iii)The height of the shortest Nila sculpture is 34.88 cm.

		The total height of all the Nila sculptures is 3537.33 cm.
14	**RI**	124
15	**RI**	(a) $\frac{7^{N+2}(7^N-1)}{6}+\frac{N(2N+1)(7N+1)}{2}$
16	**RVHS**	(a) $n = 32$ (b)(i) 17 (b)(ii) thickness of cream negligible
17	**SAJC**	(a) 32 (b) $u_1 2^N$
18	**SRJC**	(a)(ii) $\frac{13}{9}$ (b)(i) $1.045^{n+4}(29000)-\frac{836x}{3}(1.045^n-1)$ (b)(ii) \$419 (to 3s.f.)
19	**TJC**	(a) 179 (b)(i) $1.02^5(80000-x)$ \$29402 (correct to nearest integer)
20	**TPJC**	(i) 800 m (ii) $S_A = 25n(3+n)$; Least number of stages = 19 (iii) 642 m (iv) $S_B = 250(1.2^n-1)$; Least number of stages = 23 (v) 1362 m
21	**VJC**	(ii) 2050
22	**YJC**	(a) 122 marbles (b) Final payment of \$1162.70 (nearest cents) is paid on 1st November 2015

1. **2015/ACJC/II/2**

i) $\left|\dfrac{z+i-3}{2+iz}\right|=1$

Let $z=ki$ where $k\in\mathbb{R}$

$|ki+i-3|=|2+i(ki)|$

$|-3+(k+1)i|^2=|2-k|^2$

$(-3)^2+(k+1)^2=(2-k)^2$

$9+k^2+2k+1=4-4k+k^2$

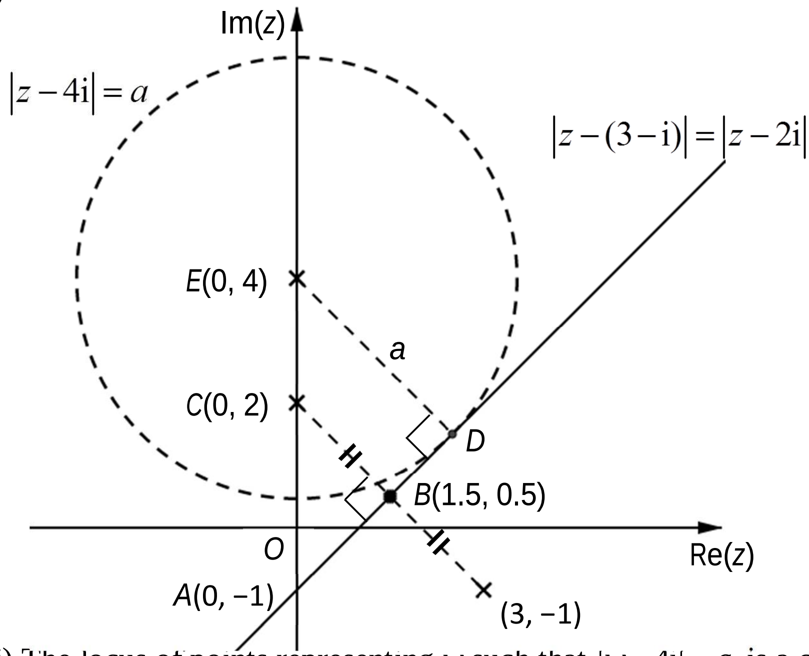
$6k=-6$

$k=-1$

$|z-3+i|=|-i||2+iz|$

$|z-(3-i)|=|z-2i|$

ii)



iii) The locus of points representing w such that $|w-4i|=a$ is a circle with centre (0, 4) and radius a units.

To have exactly one value of z satisfying the 2 conditions, the perpendicular bisector should be tangent to the circle (see sketch in **2(ii)**)

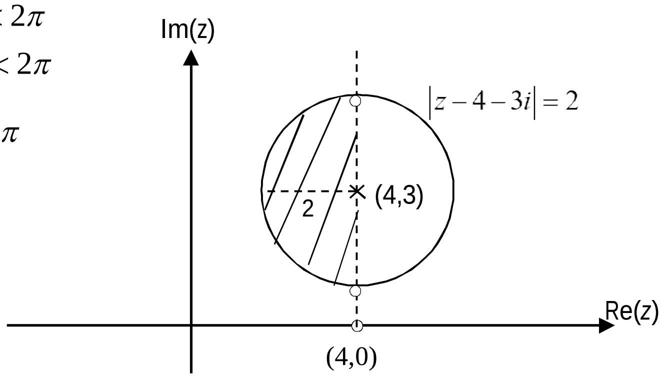
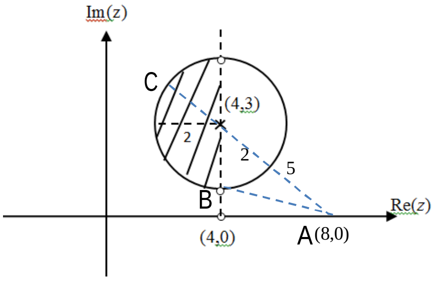
Gradient of $\perp$ bisector $=-1\div\dfrac{-1-2}{3-0}$

$=1$

$\Rightarrow$ $\perp$ bisector makes an angle of $\frac{\pi}{4}$ rad. with the horizontal axis

Consider right-angled triangle *ADE*:

$\sin\dfrac{\pi}{4}=\dfrac{a}{4-(-1)}$

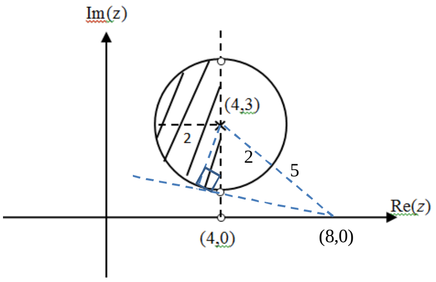
	$a=\dfrac{5}{\sqrt{2}}$ or $\dfrac{5\sqrt{2}}{2}$ $z=\dfrac{5}{\sqrt{2}}\cos\dfrac{\pi}{4}+\mathrm{i}\left(4-\dfrac{5}{\sqrt{2}}\sin\dfrac{\pi}{4}\right)$ $=\dfrac{5}{2}+\dfrac{3}{2}\mathrm{i}$ $\therefore\ x=\dfrac{5}{2},\ y=\dfrac{3}{2}$ **Alternative method for finding *a*:** Triangles *ABC* and *ADE* are similar right-angled triangles with common angle $\angle BAC$ $\sin\angle BAC=\dfrac{BC}{AC}=\dfrac{DE}{AE}$ $\dfrac{\sqrt{(0-1.5)^2+(2-0.5)^2}}{3}=\dfrac{a}{5}$ $a=\dfrac{5}{3}\times\sqrt{\dfrac{9}{2}}$ $=\dfrac{5}{\sqrt{2}}$ or $\dfrac{5\sqrt{2}}{2}$
2.	**<u>2015/AJC/I/3</u>** i) $\pi<\arg(z-4)^2<2\pi$ $\pi<2\arg(z-4)<2\pi$ $\dfrac{\pi}{2}<\arg(z-4)<\pi$  ii)  From diagram,

$AB < |z-8| \le AC$

$\sqrt{1^2+4^2} < |z-8| \le \sqrt{3^2+4^2}+2$

$\sqrt{17} < |z-8| \le 7$

iii)



maximum $\arg(z-8)$

$= \pi - \tan^{-1}\dfrac{3}{4} + \sin^{-1}\dfrac{2}{5}$

$= 2.9096$

$= 2.91\text{rad} \ \ (3\text{sf})$

3. **<u>2015/AJC/II/2</u>**

$z^5 + 32 = 0$

$z^5 = -32 = 32e^{(\pi+2k\pi)i}$

$z = 2e^{(\frac{\pi}{5}+\frac{2k\pi}{5})i}, \qquad k = 0, \pm1, \pm2$

$= 2e^{-\frac{3\pi}{5}i}, 2e^{-\frac{\pi}{5}i}, 2e^{\frac{\pi}{5}i}, 2e^{\frac{3\pi}{5}i}, 2e^{\pi i}$

i)

<u>Method 1</u>

$$\left(\frac{z_1}{z_2*}\right)^n = \frac{2^n e^{\left(\frac{n\pi}{5}\right)i}}{2^n e^{\left(-\frac{3n\pi}{5}\right)i}} = e^{\left(\frac{4n\pi}{5}\right)i}$$

For $\left(\dfrac{z_1}{z_2*}\right)^n$ to be real and positive, smallest $n = 5$

<u>Method 2</u>

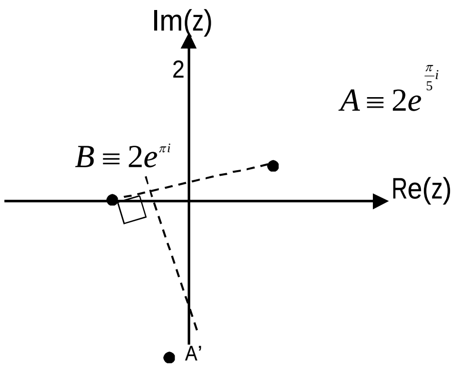
$$\begin{aligned}\arg\left(\frac{z_1}{z_2*}\right)^n &= n\left[\arg(z_1) - \arg(z_2*)\right] \\ &= n\left[\arg(z_1) - (-\arg(z_2))\right] \\ &= n\left[\frac{\pi}{5} + \frac{3\pi}{5}\right] \\ &= \frac{4n\pi}{5}\end{aligned}$$

For $\left(\dfrac{z_1}{z_2*}\right)^n$ to be real and positive,

$$\frac{4n\pi}{5} = 2k\pi, \quad k \in \mathbb{Z}$$

$n = \dfrac{5}{2}k, \quad k \in \mathbb{Z}$, so smallest $n = 5$

ii)



Let the complex number represented by A' be x+iy

BA rotates 90° about B to get BA':

$$(x+iy) - 2e^{i\pi} = (-i)\left(2e^{i\frac{\pi}{5}} - 2e^{i\pi}\right)$$

$$x+iy = (-2) - i\left[2e^{\frac{1}{5}\pi i} - (-2)\right] \quad \text{since } e^{i\pi} = -1$$

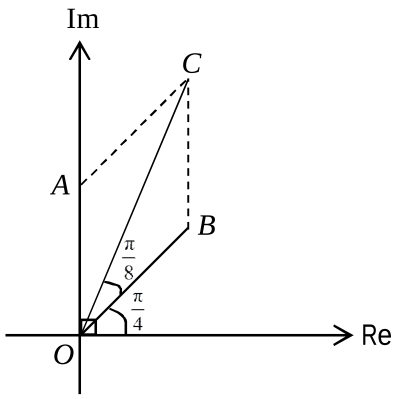
$$x+iy = -2 - i\left[2\cos\frac{\pi}{5} + 2i\sin\frac{\pi}{5} + 2)\right]$$

Real part = $-2 - 2i^2\sin\dfrac{\pi}{5} = -2 + 2\sin\dfrac{\pi}{5}$

4. **2015/CJC/I/1**

$$w^2 = \left(z^2 - z\right)^2$$
$$= z^4 - 2z^3 + z^2$$
$$z^4 - 2z^3 - 2z^2 + 3z - 10 = 0$$
$$\left(z^4 - 2z^3 + z^2\right) - 3z^2 + 3z - 10 = 0$$
$$\left(z^4 - 2z^3 + z^2\right) - 3\left(z^2 - z\right) - 10 = 0$$
$$w^2 - 3w - 10 = 0$$
$$(w-5)(w+2) = 0$$

$w = 5$	or	$w = -2$
$z^2 - z = 5$		$z^2 - z = -2$
$z^2 - z - 5 = 0$		$z^2 - z + 2 = 0$
$z = \dfrac{1 \pm \sqrt{1 - 4(-5)}}{2}$		$z = \dfrac{1 \pm \sqrt{1 - 4(2)}}{2}$

	$=\dfrac{1\pm\sqrt{21}}{2}$ $\qquad$ $=\dfrac{1\pm\sqrt{-7}}{2}$ $=\dfrac{1\pm\sqrt{7}\text{i}}{2}$
5.	**<u>2015/CJC/II/3</u>** (i) $\lvert a\rvert=\left\lvert\dfrac{1+\text{i}}{1-\text{i}}\right\rvert=\dfrac{\lvert 1+\text{i}\rvert}{\lvert 1-\text{i}\rvert}=\dfrac{\sqrt{2}}{\sqrt{2}}=1$ $\lvert b\rvert=\left\lvert\dfrac{\sqrt{2}}{1-\text{i}}\right\rvert=\dfrac{\sqrt{2}}{\lvert 1-\text{i}\rvert}=\dfrac{\sqrt{2}}{\sqrt{2}}=1$ $\arg(a)=\arg\left(\dfrac{1+\text{i}}{1-\text{i}}\right)$ $=\arg(1+\text{i})-\arg(1-\text{i})$ $=\dfrac{\pi}{4}-\left(-\dfrac{\pi}{4}\right)$ $=\dfrac{\pi}{2}$ $\arg(b)=\arg\left(\dfrac{\sqrt{2}}{1-\text{i}}\right)$ $=\arg\left(\sqrt{2}\right)-\arg(1-\text{i})$ $=0-\left(-\dfrac{\pi}{4}\right)$ $=\dfrac{\pi}{4}$ (ii)  $\angle BOC=\dfrac{\frac{\pi}{2}-\frac{\pi}{4}}{2}=\dfrac{\pi}{8}$ $\arg(a+b)=\dfrac{\pi}{8}+\dfrac{\pi}{4}=\dfrac{3\pi}{8}$ $a=\dfrac{1+\text{i}}{1-\text{i}}\cdot\dfrac{1+\text{i}}{1+\text{i}}=\dfrac{1+2\text{i}+\text{i}^2}{1-\text{i}^2}=\dfrac{2\text{i}}{2}=\text{i}$ $b=\dfrac{\sqrt{2}}{1-\text{i}}\cdot\dfrac{1+\text{i}}{1+\text{i}}=\dfrac{\sqrt{2}(1+\text{i})}{1-\text{i}^2}=\dfrac{\sqrt{2}}{2}(1+\text{i})$

$\therefore a+b = \mathrm{i} + \frac{\sqrt{2}}{2}(1+\mathrm{i}) = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}+1\right)\mathrm{i}$

$\tan\left(\frac{3\pi}{8}\right) = \frac{\frac{\sqrt{2}}{2}+1}{\frac{\sqrt{2}}{2}} = 1 + \frac{2}{\sqrt{2}} = 1+\sqrt{2}$ (shown)

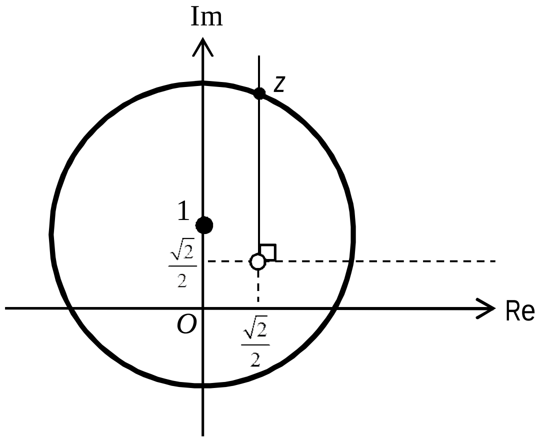
(iii)

(a) $|z-\mathrm{i}| = 2 \Rightarrow |z-(0+\mathrm{i})| = 2$

Locus is a circle, centre (0, 1) and radius 2

(b) $\arg(z-b) = \frac{\pi}{2} \Rightarrow \arg\left(z - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}\right)\right) = \frac{\pi}{2}$

Locus is a half-line from the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ (excluding the angle itself) making an angle of $\frac{\pi}{2}$ to the positive Re-axis direction.



Equation of circle: $x^2 + (y-1)^2 = 2^2$

When $x = \frac{\sqrt{2}}{2}$,

$$\left(\frac{\sqrt{2}}{2}\right)^2 + (y-1)^2 = 2^2$$

$$\frac{1}{2} + (y-1)^2 = 4$$

$$(y-1)^2 = \frac{7}{2}$$

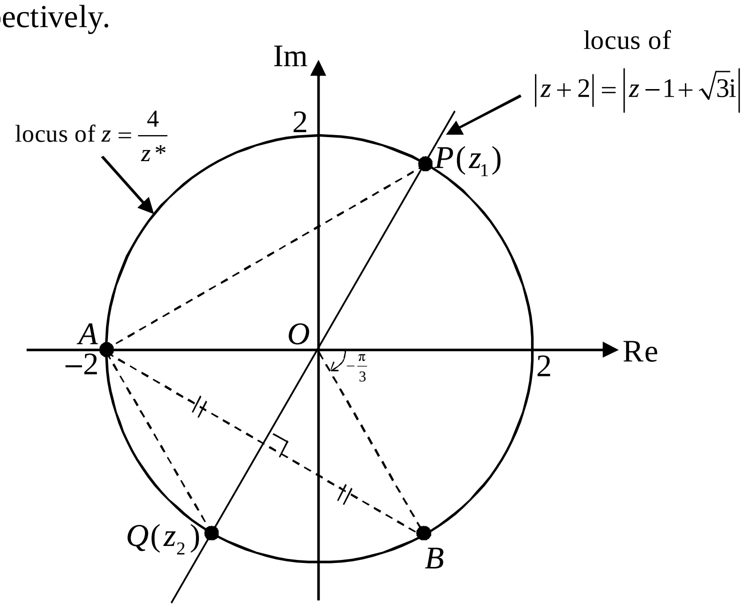
$$y-1 = \pm\sqrt{\frac{7}{2}}$$

$$y = 1+\sqrt{\frac{7}{2}} \text{ or } 1-\sqrt{\frac{7}{2}} \text{ (rej. } \because y \text{ is positive)}$$

$\therefore z = \frac{\sqrt{2}}{2} + \left(1+\sqrt{\frac{7}{2}}\right)\mathrm{i}$

	Alternative for finding *y* From diagram, using Pythagoras' theorem, $y=1+\sqrt{2^2-\left(\frac{\sqrt{2}}{2}\right)^2}=1+\sqrt{\frac{7}{2}}$
6.	**2015/DHS/I/1** (i) $(1+4\mathrm{i})^2=1+8\mathrm{i}+(4\mathrm{i})^2=-15+8\mathrm{i}$ (ii) $(1+2\mathrm{i})^2-(1+2\mathrm{i})+(a+b\mathrm{i})=0$ $(1+4\mathrm{i}-4)-1-2\mathrm{i}+a+b\mathrm{i}=0$ $(a-4)+(2+b)\mathrm{i}=0$ Compare Re and Im parts, $a=4,\ b=-2$ (iii) $z^2-z+(4-2\mathrm{i})=0$ $z=\frac{1\pm\sqrt{(-1)^2-4(1)(4-2\mathrm{i})}}{2(1)}$ $=\frac{1\pm\sqrt{-15+8\mathrm{i}}}{2}$ $=\frac{1\pm(1+4\mathrm{i})}{2}$ $=1+2\mathrm{i}$ or $-2\mathrm{i}$
7.	**2015/DHS/I/4** (i) **Method 1** $\lvert w\rvert=\lvert(1+\mathrm{i})z\rvert$ $=\lvert 1+\mathrm{i}\rvert\lvert z\rvert$ $=\sqrt{2}\sqrt{3}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}$ $=\sqrt{6}$ $\arg(w)=\arg((1+\mathrm{i})z)$ $=\arg(1+\mathrm{i})+\arg(z)$ $=\frac{\pi}{4}+\frac{\pi}{6}=\frac{5\pi}{12}$ **Method 2**

	$w=(1+\mathrm{i})z$ $=\left(\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)\left(\sqrt{3}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}\right)$ $=\left(\sqrt{6}\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{12}\right)}\right)$ $\therefore \lvert w\rvert=\sqrt{6},\quad \arg(w)=\dfrac{5\pi}{12}$ (ii) $w^n=\left(\sqrt{6}\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{12}\right)}\right)^n$ $=\left(\sqrt{6}\right)^n\mathrm{e}^{\mathrm{i}\left(\frac{5n\pi}{12}\right)}$ $=\left(\sqrt{6}\right)^n\left(\cos\left(\dfrac{5n\pi}{12}\right)+\mathrm{i}\sin\left(\dfrac{5n\pi}{12}\right)\right)$ For w^n to be purely imaginary, $\cos\left(\dfrac{5n\pi}{12}\right)=0$ $\dfrac{5n\pi}{12}=\dfrac{\pi}{2},\dfrac{3\pi}{2},\dfrac{5\pi}{2},\ldots$ $n=\dfrac{6}{5},\dfrac{18}{5},6,\ldots$ smallest positive integer $n=6$ When $n=6$, $\lvert w^6\rvert=\left(\sqrt{6}\right)^6=216$
8	**<u>2015/DHS/II/3</u>** (i) $z=\dfrac{4}{z^*}$ $zz^*=4$ $\lvert z\rvert^2=4$ $\lvert z\rvert=2 \because \lvert z\rvert>0$ (shown) (ii) $1-\sqrt{3}\mathrm{i}=2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$

	(iii) Let A, B, P, Q be points that represent the complex numbers -2, $1-\sqrt{3}\mathrm{i}$, z_1 and z_2 respectively. 
	(iv) **Method 1** Since $P(z_1)$ and $Q(z_2)$ lie on circle (centred at origin) of radius 2, $\|z_1\|=\|z_2\|=2$ $\angle AOB=\frac{2\pi}{3}$ $\angle AOQ=\angle QOB=\frac{\pi}{3}$ (angle bisector) $\therefore \arg(z_2)=-\frac{2\pi}{3}$ $\Rightarrow z_2=2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}=-1-\sqrt{3}\mathrm{i}$ $\arg(z_1)=\frac{\pi}{3}$ (angle on straight line) $\Rightarrow z_1=2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}\right)}=1+\sqrt{3}\mathrm{i}$ **Method 2** Cartesian equation of circle: $x^2+y^2=2^2$ ---(1)

Gradient of line *AB*: $\dfrac{-\sqrt{3}-0}{1-(-2)}=-\dfrac{\sqrt{3}}{3}=-\dfrac{1}{\sqrt{3}}$

Gradient of perpendicular bisector $=-\dfrac{1}{-\frac{1}{\sqrt{3}}}=\sqrt{3}$

Cartesian equation of perpendicular bisector:

$y-0=\sqrt{3}(x-0)$

$y=\sqrt{3}x$ ---(2)

Solving (1) and (2),

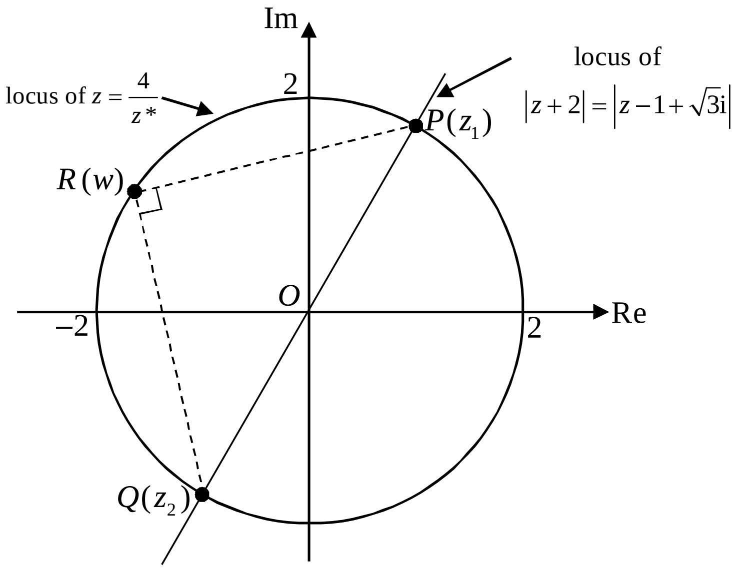
$x=1$ or $x=-1$

$y=\sqrt{3}$ $\quad y=-\sqrt{3}$

$\therefore z_1=1+\sqrt{3}\mathrm{i}$ and $z_2=-1-\sqrt{3}\mathrm{i}$

(v)

Let *R* be the point that represent the complex numbers *w*.



Note that *PQ* forms the diameter of the circle centred at origin with radius 2 units.

$\arg(z_1-w)-\arg(z_2-w)$

$=\angle PRQ=\dfrac{\pi}{2}$ (right angle in semicircle)

9. **<u>2015/HCI/I/10</u>**

(a)(i) It is not necessarily true because to conclude that i^* is a root, the coefficients of the equation must be real.

	(a)(ii) Sub $w=\mathrm{i}$ into $z^3-az^2+2az-4\mathrm{i}=0$ $\mathrm{i}^3-a\mathrm{i}^2+2a\mathrm{i}-4\mathrm{i}=0 \Rightarrow -\mathrm{i}+a+2a\mathrm{i}-4\mathrm{i}=0 \Rightarrow a(1+2\mathrm{i})=5\mathrm{i} \Rightarrow a=2+\mathrm{i}$ (a)(iii) Let $\left(bz^2+cz+d\right)\left(z-\mathrm{i}\right)=0$ By inspection, $b=1$ $d=4$, $\left(z^2+cz+4\right)\left(z-\mathrm{i}\right)=0$ Compare z terms: $-\mathrm{i}c+4=2a \Rightarrow c=\dfrac{4+2\mathrm{i}-4}{-\mathrm{i}}=-2$ Thus $z^2+cz+4=0$ $\Rightarrow z=\dfrac{2\pm\sqrt{4-4(1)(4)}}{2}=1\pm\dfrac{\sqrt{-12}}{2}=1\pm\sqrt{3}\mathrm{i}$ $\Rightarrow z=1+\sqrt{3}\mathrm{i}$ or $z=1-\sqrt{3}\mathrm{i}$ (b) $\arg\left(z-2\mathrm{i}\right)=\dfrac{\pi}{4}$ $\Rightarrow \tan\dfrac{\pi}{4}=\dfrac{y-2}{x} \Rightarrow y=x+2$ where $y>2$, $x>0$ --------- (1) From $\left\|z^*-1+\mathrm{i}\right\|=2$ $\Rightarrow \left\|x-\mathrm{i}y-1+\mathrm{i}\right\|=2 \Rightarrow \left\|\left(x-1\right)-\mathrm{i}\left(y-1\right)\right\|=2$ $\Rightarrow \left(x-1\right)^2+\left(y-1\right)^2=4$ --------- (2) Sub (1) into (2): $x^2-2x+1+x^2+2x+1=4 \Rightarrow 2x^2=2 \Rightarrow x=\pm1$ Since $x>0$, therefore $x=1 \Rightarrow z=x+\mathrm{i}y=1+3\mathrm{i}$
11.	**<u>2015/IJC/I/5</u>** (i) $z^4=\left(2+\sqrt{3}\mathrm{i}\right)^4$ $=2^4+{}^4C_1\left(2\right)^3\left(\sqrt{3}\mathrm{i}\right)+{}^4C_2\left(2\right)^2\left(\sqrt{3}\mathrm{i}\right)^2$ $+{}^4C_3\left(2\right)\left(\sqrt{3}\mathrm{i}\right)^3+\left(\sqrt{3}\mathrm{i}\right)^4$ $=16+32\sqrt{3}\mathrm{i}-72-24\sqrt{3}\mathrm{i}+9$ $=-47+8\sqrt{3}\mathrm{i}$ (ii) <u>Method 1</u> $2w^4+aw^2+bw+49=0$

	$2\left(-47+8\sqrt{3}\mathrm{i}\right)+a\left(2+\sqrt{3}\mathrm{i}\right)^2+b\left(2+\sqrt{3}\mathrm{i}\right)+49=0$ $-94+16\sqrt{3}\mathrm{i}+4a+a\left(4\sqrt{3}\mathrm{i}\right)-3a+2b+b\sqrt{3}\mathrm{i}+49=0$ $-45+a+2b+\left(4a+b+16\right)\sqrt{3}\mathrm{i}=0$ Comparing the coefficient of real part $a+2b=45------(1)$ Comparing the coefficient of imaginary part $4a+b=-16-------(2)$ $a=-11,\ b=28$ <u>Method 2</u> Since the coefficients of the equation are real, $z^*=2-\sqrt{3}\mathrm{i}$ is also a root $\left(w-\left(2+\sqrt{3}\mathrm{i}\right)\right)\left(w-\left(2-\sqrt{3}\mathrm{i}\right)\right)$ $\left(\left(w-2\right)-\sqrt{3}\mathrm{i}\right)\left(\left(w-2\right)+\sqrt{3}\mathrm{i}\right)$ $=\left(w-2\right)^2-\left(\sqrt{3}\mathrm{i}\right)^2$ $=w^2-4w+7$ $2w^4+aw^2+bw+49=\left(w^2-4w+7\right)\left(2w^2+pw+7\right)$ By comparing coeff of w^3 , (or inspection mtd) $p-8=0$ $p=8$ comparing coeff of w^2 : $a=7-32+14=-11$ comparing coeff of w: $b=56-28=28$ (iii) **<u>Method 1</u>** Since the coefficients of the equation are real, $z^*=2-\sqrt{3}\mathrm{i}$ is also a root. $\left(w-\left(2+\sqrt{3}\mathrm{i}\right)\right)\left(w-\left(2-\sqrt{3}\mathrm{i}\right)\right)$ $\left(\left(w-2\right)-\sqrt{3}\mathrm{i}\right)\left(\left(w-2\right)+\sqrt{3}\mathrm{i}\right)$ $=\left(w-2\right)^2-\left(\sqrt{3}\mathrm{i}\right)^2$ $=w^2-4w+7$ By long division or inspection,

$$2w^4 - 11w^2 + 28w + 49 = \left(w^2 - 4w + 7\right)\left(2w^2 + 8w + 7\right)$$

$$2w^2 + 8w + 7 = 0$$

$$w = \frac{-8 \pm \sqrt{64 - 4(2)(7)}}{2(2)}$$

The other roots are $2 - \sqrt{3}\mathrm{i}, -2 \pm \frac{\sqrt{2}}{2}$

Method 2

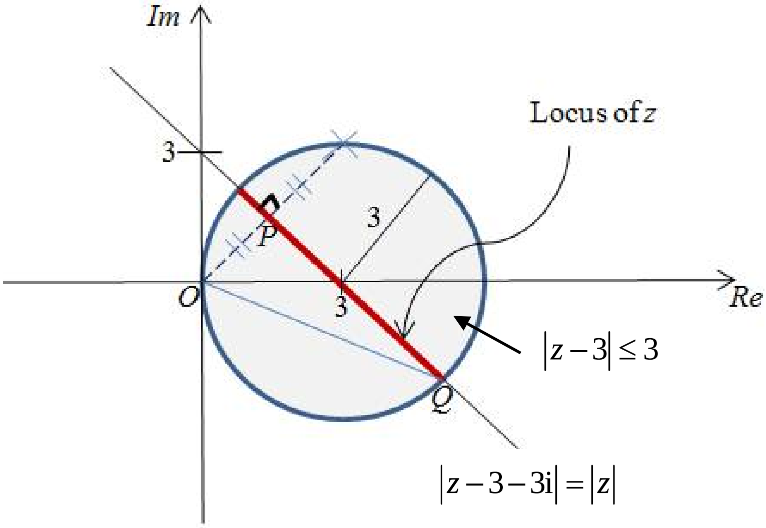
$$2w^2 + 8w + 7 = 0$$

$$w = \frac{-8 \pm \sqrt{64 - 4(2)(7)}}{2(2)} = -2 \pm \frac{\sqrt{2}}{2}$$

The other roots are $2 - \sqrt{3}\mathrm{i}, -2 \pm \frac{\sqrt{2}}{2}$

12. **2015/IJC/II/4**

(a)(i)



(a)(ii)

Min value of $|z|^2 = OP^2 = \left(3\cos\frac{\pi}{4}\right)^2 = \frac{9}{2}$

Max value of $|z|^2$

$= OQ^2$

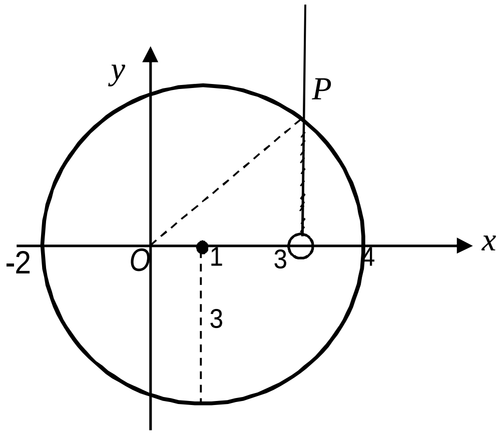
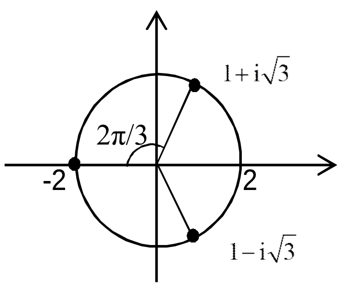
$= 3^2 + 3^2 - 2(3)(3)\cos\frac{3\pi}{4}$

$= 18 + 9\sqrt{2}$

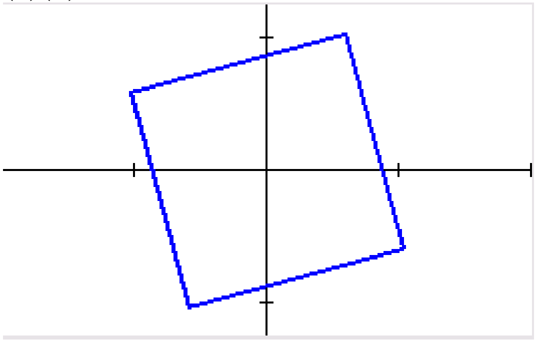
(b)(i)

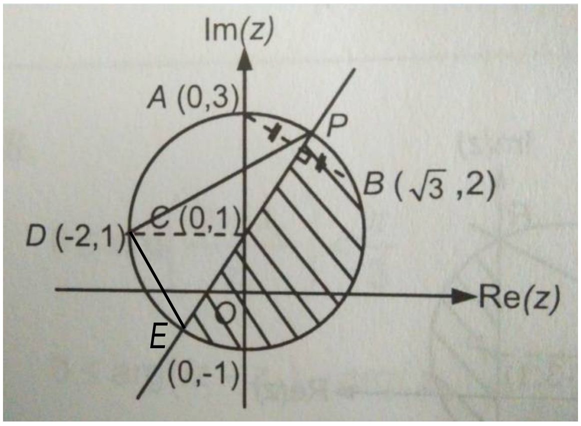
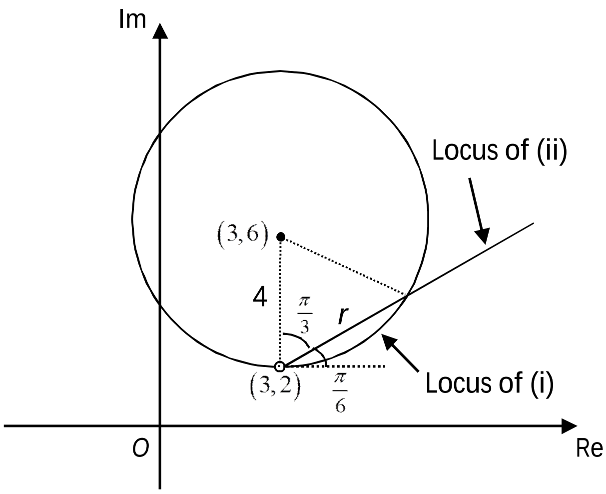
Method 1:

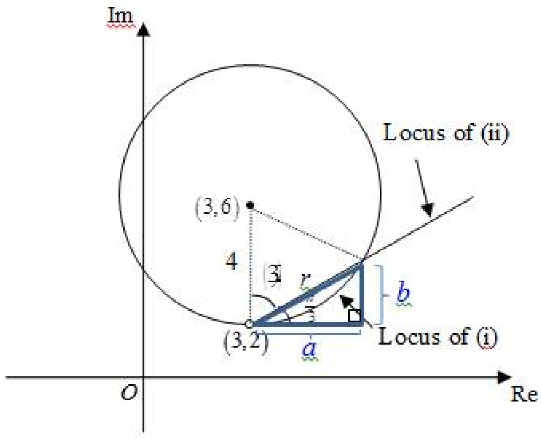
	$w=\left(\dfrac{-\sqrt{3}+\mathrm{i}}{\sqrt{2}-\sqrt{2}\mathrm{i}}\right)^2=\left(\dfrac{2\mathrm{e}^{\frac{5\pi}{6}\mathrm{i}}}{2\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}}\right)^2=\mathrm{e}^{\frac{13\pi}{6}\mathrm{i}}=\mathrm{e}^{\frac{\pi}{6}\mathrm{i}}$ $\therefore\ \lvert w\rvert=1$ and $\arg w=\dfrac{\pi}{6}$ **<u>Method 2:</u>** $\lvert w\rvert=\left\lvert\dfrac{-\sqrt{3}+\mathrm{i}}{\sqrt{2}-\sqrt{2}\mathrm{i}}\right\rvert^2=\dfrac{\lvert-\sqrt{3}+\mathrm{i}\rvert^2}{\lvert\sqrt{2}-\sqrt{2}\mathrm{i}\rvert^2}=\dfrac{4}{4}=1$ $\arg w=\arg\left(\dfrac{-\sqrt{3}+\mathrm{i}}{\sqrt{2}-\sqrt{2}\mathrm{i}}\right)^2$ $=2\left[\arg\left(-\sqrt{3}+\mathrm{i}\right)-\arg\left(\sqrt{2}-\mathrm{i}\sqrt{2}\right)\right]$ $=2\left[\dfrac{5\pi}{6}-\left(-\dfrac{\pi}{4}\right)\right]-2\pi$ $=\dfrac{\pi}{6}$ **<u>Method 3:</u>** $w=\left(\dfrac{-\sqrt{3}+\mathrm{i}}{\sqrt{2}-\sqrt{2}\mathrm{i}}\right)^2$ $=\dfrac{3-2\sqrt{3}\mathrm{i}-1}{2-4\mathrm{i}-2}$ $=\dfrac{2-2\sqrt{3}\mathrm{i}}{4\mathrm{i}}$ $=\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}$ $\therefore\ \lvert w\rvert=1$ and $\arg w=\dfrac{\pi}{6}$ (b)(ii) $\arg w^n w^*=n\arg w-\arg w=(n-1)\dfrac{\pi}{6}$ Since $w^n w^*$ is a real number, and n is a positive integer, $(n-1)\dfrac{\pi}{6}=k\pi,\quad k\in\mathbb{Z}^+\cup\{0\}$ $n=6k+1,\quad k\in\mathbb{Z}^+\cup\{0\}$ Or $n=6k-5,\quad k\in\mathbb{Z}^+$ $\left\{n:n\in\mathbb{Z}^+,n=6k-5,\quad k\in\mathbb{Z}^+\right\}$

13.	**2015/JJC/I/3** At P, $\lvert 3+\mathrm{i}k-1\rvert=3$ $4+k^2=9$ $k=\sqrt{5}, k>0$ $\max \arg z=\arg\left(3+\mathrm{i}\sqrt{5}\right)=0.641 \text{ rad}$ 
14.	**2015/JJC/I/5** (i) $z^3=-8=8\mathrm{e}^{\mathrm{i}\pi}=8\mathrm{e}^{\mathrm{i}(\pi+2k\pi)}, k=0, \pm 1$ $z=2\mathrm{e}^{\mathrm{i}\left(\frac{\pi+2k\pi}{3}\right)}$ $=2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}, 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}, 2\mathrm{e}^{\mathrm{i}\pi}$ $=-2, 2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right), 2\left(\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ $z=-2, 1+\mathrm{i}\sqrt{3}, 1-\mathrm{i}\sqrt{3}$  By Symmetry a = 2 or $\lvert 1-\mathrm{i}\sqrt{3}-a\rvert=2$ $\quad$ $\lvert 1+\mathrm{i}\sqrt{3}-a\rvert=2$ $(a-1)^2+3=2^2$ or $(1-a)^2+3=2^2$ $a=0(rej)$ or 2 $\quad$ $a=0(rej)$ or 2

15.	**2015/JJC/II/1** (a)(i)(ii) $\arg(a-\mathrm{i}b)=-\theta$ $\arg(b+\mathrm{i}a)=\arg(\mathrm{i})(a-\mathrm{i}b)=\frac{\pi}{2}-\theta$ (b) $\mathrm{i}z+2w=0------(1)$ $z-w^*=3 \Rightarrow z=3+w^*------(2)$ Sub (2) into (1) $\mathrm{i}(3+w^*)+2w=0$ $3\mathrm{i}+\mathrm{i}w^*+2w=0$ Let $w=x+\mathrm{i}y$ $3\mathrm{i}+\mathrm{i}(x-\mathrm{i}y)+2(x+\mathrm{i}y)=0$ $(y+2x)+\mathrm{i}(3+x+2y)=0$ Comparing real and imaginary parts, $y+2x=0------(3)$ $3+x+2y=0----(4)$ Sub (3) into (4) $3+x-4x=0$ $x=1$ $y=-2$ $\therefore w=1-2\mathrm{i}$ $z=3+1+2\mathrm{i}=4+2\mathrm{i}$
16.	**2015/MI/I/11** (a)(i) The statement is true when the coefficients of the equation, a_i for $i=0,\ 1,\ \ldots,\ n$ are real. (a)(ii) Since the coefficients of the equation are real and $2+i$ is a root, $2-\mathrm{i}$ is also a root. $4z^3-11z^2+25=0$ $4\left(z-(2+\mathrm{i})\right)\left(z-(2-\mathrm{i})\right)(z+a)=0$ Comparing the coefficient of z^0 : $4(2+\mathrm{i})(2-\mathrm{i})b=25$ $b=\frac{5}{4}$ $\left(z-(2+\mathrm{i})\right)\left(z-(2-\mathrm{i})\right)\left(z+\frac{5}{4}\right)=0$ $z=2\pm\mathrm{i}$ or $-\frac{5}{4}$ The other roots are 2 - i and $-\frac{5}{4}$.

	(b)(i) $z^4+1+\mathrm{i}\sqrt{3}=0$ $z^4=2e^{i\left(-\frac{2\pi}{3}+2k\pi\right)},\ k=0,\ \pm1,\ 2$ $z=2^{\frac{1}{4}}e^{i\left(-\frac{\pi}{6}+\frac{k\pi}{2}\right)},\ k=0,\ \pm1,\ 2$ $z=2^{\frac{1}{4}}e^{-i\frac{\pi}{6}},\ 2^{\frac{1}{4}}e^{i\frac{\pi}{3}},\ 2^{\frac{1}{4}}e^{i\frac{5\pi}{6}}$ or $2^{\frac{1}{4}}e^{-i\frac{2\pi}{3}}$ (b)(ii)  (c) $\lvert z^2\rvert=2$ $\lvert z\rvert=\sqrt{2}$ $\lvert wz\rvert=2\sqrt{2}$ $\lvert w\rvert\lvert z\rvert=2\sqrt{2}$ $\lvert w\rvert=2$ $\arg(-\mathrm{i}z)=\frac{\pi}{4}$ $\arg z=\frac{\pi}{4}-\arg(-\mathrm{i})$ $=\frac{3\pi}{4}$ $\arg\left(\frac{z^2}{w}\right)=-\frac{5\pi}{6}$ $2\arg z-\arg w=-\frac{5\pi}{6}$ $\arg w=2\left(\frac{3\pi}{4}\right)+\frac{5\pi}{6}-2\pi$ $=\frac{\pi}{3}$ $w=2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$ $=1+\mathrm{i}\sqrt{3}$
17.	<u>**2015/MI/II/4**</u>

	(i)(ii)  Midpoint of A and B = $\frac{\sqrt{3}}{2}+\frac{5}{2}i$ (iii) $h = 1$ (iv) From diagram, $\angle PCD = \pi - (\tan^{-1}\sqrt{3}) = \frac{2\pi}{3}$ As ΔPDC is isos. Δ, greatest arg = $\frac{1}{2}\left(\pi - \frac{2\pi}{3}\right) = \frac{\pi}{6}$ and smallest arg = $-\left[\frac{1}{2}\left(\pi - \frac{\pi}{3}\right)\right] = -\frac{\pi}{3}$ Range is $-\frac{\pi}{3} \le z \le \frac{\pi}{6}$.
18	**<u>2015/MJC/I/10</u>** (a)  $r = 4$ (equilateral triangle)



$$a = r\cos\frac{\pi}{6} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

$$b = r\sin\frac{\pi}{6} = 4\left(\frac{1}{2}\right) = 2$$

$$w = 3 + 2\sqrt{3} + (2+2)\mathrm{i}$$
$$= 3 + 2\sqrt{3} + 4\mathrm{i}$$

Alternative:

$$w = 3 + 2\mathrm{i} + 4\mathrm{e}^{\frac{\pi}{6}\mathrm{i}}$$
$$= 3 + 2\mathrm{i} + 4\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$$
$$= 3 + 2\mathrm{i} + 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$$
$$= 3 + 2\sqrt{3} + 4\mathrm{i}$$

Alternative:

$(x-3)^2 + (y-2)^2 = 16 ---(1)$

$(x-3)^2 + (y-6)^2 = 16 ---(2)$

$(1)-(2)$

$(y-2)^2 - (y-6)^2 = 0$

$(4)(2y-8) = 0$

$y = 4$

Sub $y = 4$ into (1)

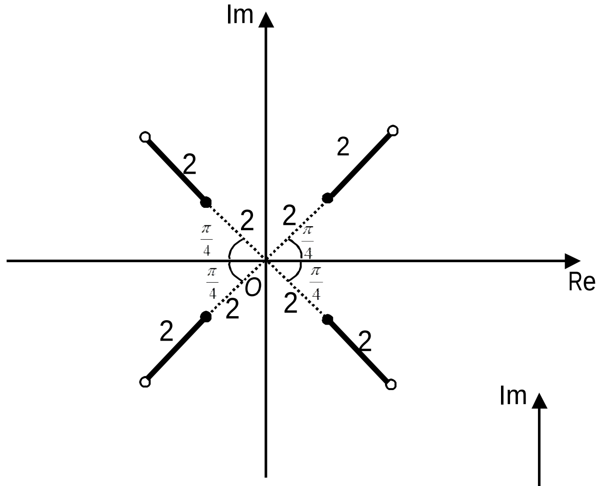
$(x-3)^2 + 4 = 16$

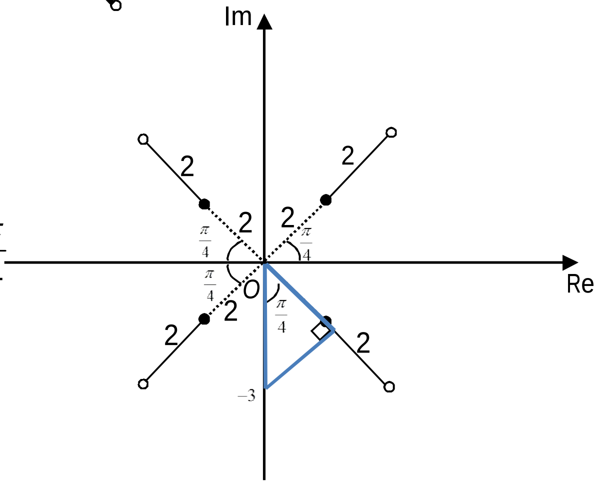
$x = 3 \pm \sqrt{12}$

Since $x > 0,\ x = 3 + \sqrt{12} = 3 + 2\sqrt{3}$

$\therefore w = 3 + 2\sqrt{3} + 4\mathrm{i}$

(b)





Minimum $|z+3\text{i}| = 3\sin\frac{\pi}{4}$

$$= \frac{3}{\sqrt{2}}$$

19. **<u>2015/MJC/II/3</u>**

(a)(i)

Method 1:

Since $1+2\text{i}$ is a root to $x^3+ax^2+bx-5=0$,

$$(1+2\text{i})^3 + a(1+2\text{i})^2 + b(1+2\text{i}) - 5 = 0$$

$$(-11-2\text{i}) + a(-3+4\text{i}) + b(1+2\text{i}) - 5 = 0$$

Comparing real part,

$$-11-3a+b-5=0$$

$$-3a+b=16---(1)$$

Comparing imaginary part,

$$-2+4a+2b=0$$

$$4a+2b=2---(2)$$

Solving (1) and (2), $a=-3, b=7$.

$$x^3-3x^2+7x-5=0$$

Using GC, the other roots are $1-2\text{i}$ and 1.

Method 2:

Since the coefficients of the polynomial are all real,

$1+2\text{i}$ is a root implies $1-2\text{i}$ is also a root.

	$\left(x-(1+2\mathrm{i})\right)\left(x-(1-2\mathrm{i})\right)=\left((x-1)^2-(2\mathrm{i})^2\right)$ $=x^2-2x+5$ By comparing coefficients, $x^3+ax^2+bx-5=\left(x^2-2x+5\right)(x-1)$ By comparing coefficient of x^2, $a=-1-2=-3$ By comparing coefficient of x, $b=2+5=7$ Therefore, the other roots are $1-2\mathrm{i}$ and 1. (b)(i) $z^4=2\mathrm{e}^{\mathrm{i}\pi}$ $z=2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{\pi+2k\pi}{4}\right)}$ $=2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{1}{4}+\frac{k}{2}\right)\pi}\qquad k=-2,-1,0,1$ $=2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{3}{4}\right)\pi},2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{1}{4}\right)\pi},2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{1}{4}\right)\pi},2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{3}{4}\right)\pi}$ (b)(ii) $w=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ or $w=\mathrm{i}$ (b)(iii) **Hence method:** $\left\|wz_3-w^*z_3\right\|^2=\left\|z_4-z_2\right\|^2$ $=\left(2\left(2^{\frac{1}{4}}\right)\right)^2$ $=4\sqrt{2}$ **Otherwise method:** $\left\|wz_3-w^*z_3\right\|^2=\left\|z_3\left(w-w^*\right)\right\|^2$ $=\left\|z_3\right\|^2\left\|2\mathrm{i}\right\|^2$ $=4\left(2^{\frac{1}{4}}\right)^2$ $=4\left(2^{\frac{1}{2}}\right)=4\sqrt{2}$
20.	**<u>2015/NJC/I/12</u>** (a)(i) **<u>Method 1: Using Exponential Forms</u>** $z_1=\sqrt{2}-\sqrt{2}\mathrm{i}=2\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$; $z_2=1+\sqrt{3}\mathrm{i}=2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$

$$z_3 = -\frac{z_2^{\ 2}}{z_1^{\ *}}$$

$$= e^{i\pi}\frac{\left(2e^{i\frac{\pi}{3}}\right)^2}{\left(2e^{-i\frac{\pi}{4}}\right)^*}$$

$$= e^{i\pi}\left(\frac{4e^{i\frac{2\pi}{3}}}{2e^{i\frac{\pi}{4}}}\right)$$

$$= \frac{2e^{i\frac{5\pi}{3}}}{e^{i\frac{\pi}{4}}}$$

$$= 2e^{i\left(\frac{5\pi}{3}-\frac{\pi}{4}\right)}$$

$$= 2e^{i\frac{17\pi}{12}} = 2e^{i\left(-\frac{7\pi}{12}\right)}$$

Method 2: Applying Laws of Modulus & Argument

$$|z_1| = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = 2, \qquad |z_2| = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = 2,$$

$$\arg(z_1) = -\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = -\frac{\pi}{4} \qquad \arg(z_2) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

Hence

$$\left|-\frac{z_2^{\ 2}}{z_1^{\ *}}\right| = \frac{|z_2|^2}{|z_1|} = \frac{2^2}{2} = 2$$

$$\arg\left(-\frac{z_2^{\ 2}}{z_1^{\ *}}\right) = \arg(-1) + \arg\left(z_2^{\ 2}\right) - \arg\left(z_1^{\ *}\right)$$

$$= \pi + 2\arg(z_2) + \arg(z_1)$$

$$= \pi + 2\times\frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{17\pi}{12} \equiv -\frac{7\pi}{12}$$

Therefore $z_3 = 2e^{i\left(-\frac{7\pi}{12}\right)}$.

Method 3: Using Cartesian Form

$$z_3 = -\frac{{z_2}^2}{{z_1}^*}$$

$$= -\frac{\left(1+\sqrt{3}\mathrm{i}\right)^2}{\sqrt{2}+\sqrt{2}\mathrm{i}}$$

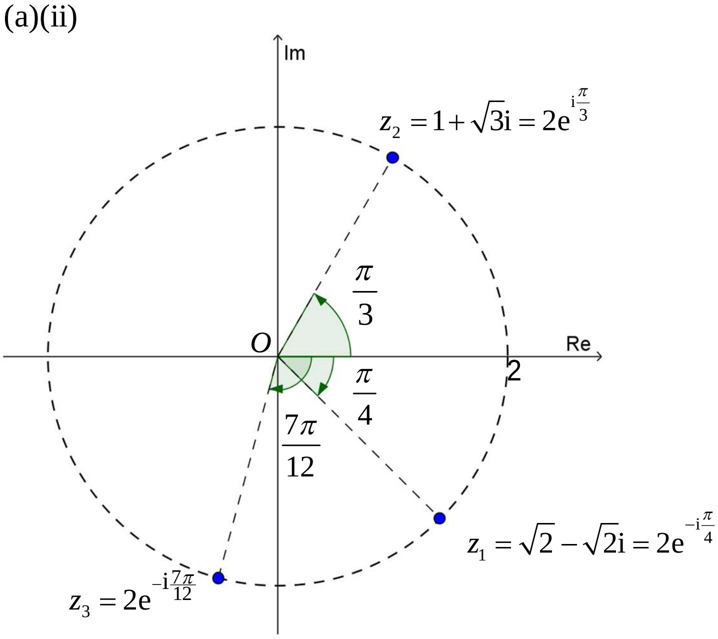
$$= -\frac{1+2\sqrt{3}\mathrm{i}-3}{\sqrt{2}+\sqrt{2}\mathrm{i}}$$

$$= \frac{2-2\sqrt{3}\mathrm{i}}{\sqrt{2}+\sqrt{2}\mathrm{i}}$$

$$= \frac{4\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}}{2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}$$

$$= 2\mathrm{e}^{-\mathrm{i}\frac{7\pi}{12}}$$

(a)(ii)



(a)(iii) No, because…

…the difference in argument between any pair of adjacent complex numbers is not constant, **OR**

…the difference in argument between z_1 and z_2 is not $2\pi/3$ (or any other pair of the two complex numbers) **OR**

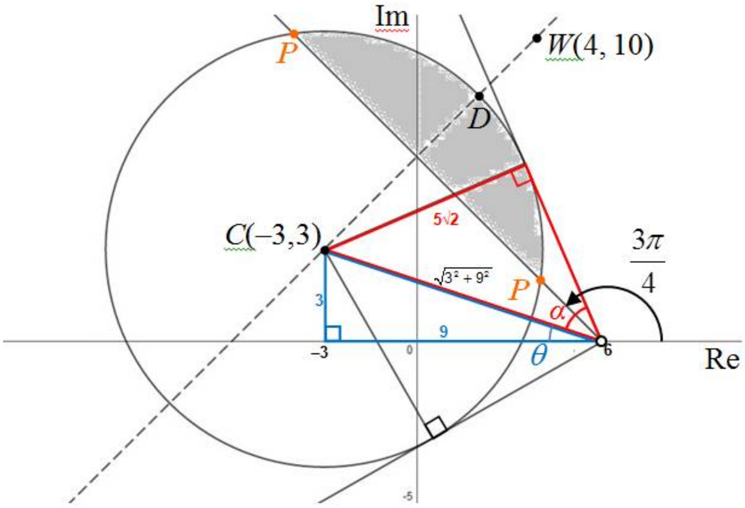
$$\left(z_1\right)^3 = \left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\right)^3 = 8\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}} \text{ but}$$

$$\left(z_2\right)^3 = \left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)^3 = 8\mathrm{e}^{\mathrm{i}\pi} \neq \left(z_1\right)^3$$

(b)

<u>Method 1: Using exponential form</u>

	$$\frac{1}{e^{i4\theta}-1}=\frac{1}{e^{i2\theta}\left(e^{i2\theta}-e^{-i2\theta}\right)}$$ $$=\left(\frac{1}{e^{i2\theta}-e^{-i2\theta}}\right)e^{-i2\theta}$$ $$=\frac{1}{2i\sin 2\theta}\left(\cos 2\theta-i\sin 2\theta\right)$$ $$=-\left(\frac{1}{2}\cot 2\theta\right)i-\frac{1}{2}$$ $$=-\frac{1}{2}-\left(\frac{1}{2}\cot 2\theta\right)i$$ Hence $\mathrm{Re}(w)=-\frac{1}{2}$ (shown) and $\mathrm{Im}(w)=-\frac{1}{2}\cot 2\theta$ **<u>Method 2: Using polar form</u>** $$\frac{1}{e^{i4\theta}-1}=\frac{1}{\cos 4\theta+i\sin 4\theta-1}\left(\frac{\cos 4\theta-1-i\sin 4\theta}{\cos 4\theta-1-i\sin 4\theta}\right)$$ $$=\frac{\cos 4\theta-1-i\sin 4\theta}{\left(\cos 4\theta-1\right)^2+\left(\sin 4\theta\right)^2}$$ $$=\frac{\cos 4\theta-1-i\sin 4\theta}{\cos^2 4\theta-2\cos 4\theta+1+\sin^2 4\theta}$$ $$=\frac{-\left(1-\cos 4\theta\right)-i\sin 4\theta}{2\left(1-\cos 4\theta\right)}$$ $$=-\frac{1}{2}-\frac{2\sin 2\theta\cos 2\theta}{2\left(1-\left(1-2\sin^2 2\theta\right)\right)}i$$ $$=-\frac{1}{2}-\frac{1}{2}i\cot 2\theta$$ Hence $\mathrm{Re}(w)=-\frac{1}{2}$ (shown) and $\mathrm{Im}(w)=-\frac{1}{2}\cot 2\theta$
21.	**<u>2015/NJC/II/4</u>** (i), (ii)



(iii) Let W be the point representing $4+10\mathrm{i}$. Then

$$\begin{aligned}\min\left|z-4-10\mathrm{i}\right| &= WD\\ &= WC-CD\\ &= \sqrt{(4-(-3))^2+(10-3)^2}-5\sqrt{2}\\ &= 7\sqrt{2}-5\sqrt{2}\\ &= 2\sqrt{2} \text{ i.e. } m=2 \text{ (shown)}\end{aligned}$$

(iv) Minimum value of $\left|\arg(w-6)\right|$

$=\pi-\alpha-\theta$

$$=\pi-\sin^{-1}\left(\frac{5\sqrt{2}}{\sqrt{3^2+9^2}}\right)-\tan^{-1}\left(\frac{3}{9}\right)$$

$$=\pi-\sin^{-1}\left(\frac{\sqrt{50}}{\sqrt{90}}\right)-\tan^{-1}\left(\frac{3}{9}\right)$$

$$=\pi-\sin^{-1}\left(\sqrt{\frac{5}{9}}\right)-\tan^{-1}\left(\frac{1}{3}\right)$$

$=1.978773429$ rad

$=1.979$ rad (to 3 d.p.)

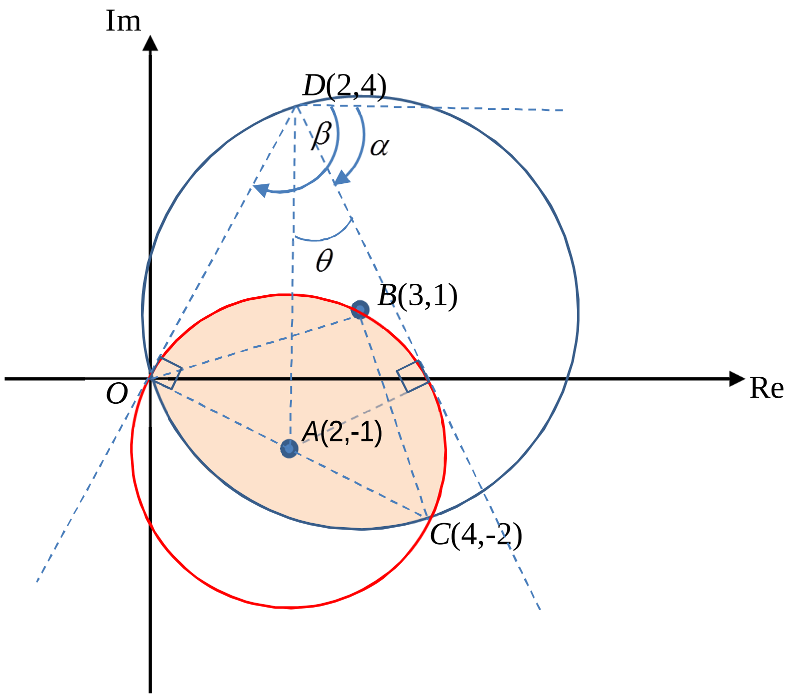
22. **2015/NYJC/I/11**

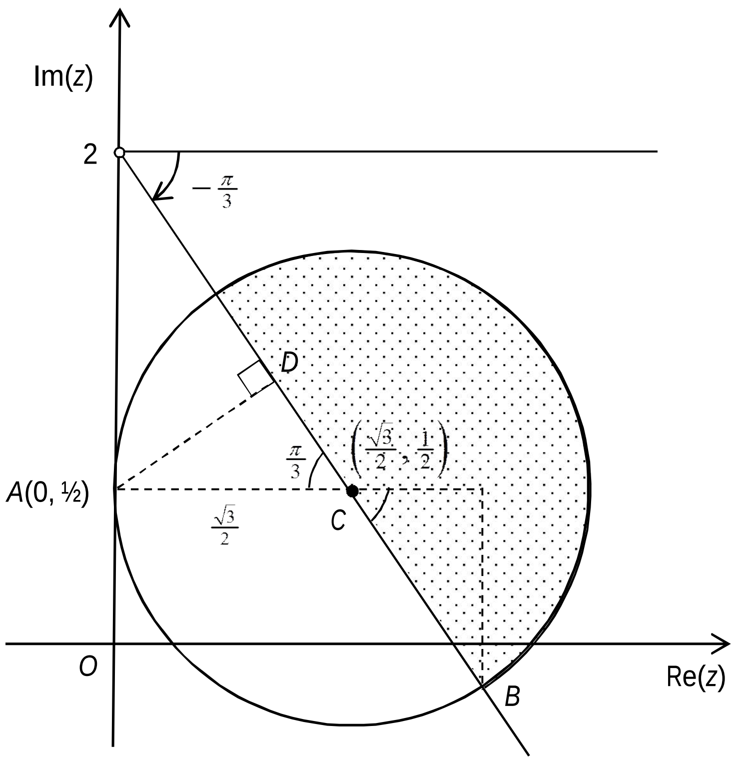
(i) $w_1=e^{\frac{\pi i}{4}}$, $w_2=e^{\frac{\pi i}{8}}$, $w_3=e^{\frac{\pi i}{16}}$, $w_4=e^{\frac{\pi i}{32}}$

(ii) Note that $\theta_1=\dfrac{\pi}{4}$ and $\arg(w_{n+1})=\dfrac{1}{2}\arg(w_n)$. Thus $\theta_{n+1}=\dfrac{1}{2}\theta_n$.

Since $\dfrac{\theta_{n+1}}{\theta_n}=\dfrac{1}{2}$ for all $n\geq 1$, thus θ_n is a geometric sequence with common ratio $\dfrac{1}{2}$.

$$\sum_{n=1}^{\infty}\theta_n=\frac{\theta_1}{1-\frac{1}{2}}=\frac{\pi}{2}$$

	(iii) By (i), $\|w_3\|=\|w_4\|=1$. Thus the origin satisfies the equation $\|z-w_3\|=\|z-w_4\|$. Thus the locus of points satisfying the equation $\|z-w_3\|=\|z-w_4\|$ passes through the origin. Let α be the angle between the line and the positive real axis. Since the perpendicular bisector is also the angle bisector, $\alpha=\frac{1}{2}\left(\frac{\pi}{16}+\frac{\pi}{32}\right)=\frac{3\pi}{64}$. Thus the exact Cartesian equation is $y=x\tan\left(\frac{3\pi}{64}\right)$.
23.	**<u>2015/NYJC/II/3</u>** (i)  (ii) Since $\|z_1-2+i\|=\sqrt{5}$ and $\|z_1-3-i\|=\sqrt{10}$, thus $z_1=4-2i$ satisfies the equation $\|z-2+i\|=\sqrt{5}$ and $\|z-3-i\|=\sqrt{10}$. (iii) Note that triangle *OBC* is a right angle triangle. Further *B* lies on the locus of $\|z-2+i\|=\sqrt{5}$. Thus $area=\frac{1}{2}\pi\left(\sqrt{5}\right)^2+\left[\frac{1}{4}\pi\left(\sqrt{10}\right)^2-\frac{1}{2}\left(\sqrt{10}\right)^2\right]$ $=\frac{5\pi}{2}+\frac{5\pi}{2}-5$ $=5(\pi-1)$ (iv) $\theta=\sin^{-1}\frac{\sqrt{5}}{5}$ Thus $\alpha=-\frac{\pi}{2}+\theta$ and $\beta=-\frac{\pi}{2}-\theta$.

	Hence the required range is $-2.03 \le \arg(z-2-4\mathrm{i}) \le -1.11$.
24.	**<u>2015/PJC/I/7</u>** (i) $z = r\mathrm{e}^{\mathrm{i}\theta}$ is a root $\Rightarrow$ $z = r\mathrm{e}^{-\mathrm{i}\theta}$ is another root since P(z) has real coefficients. A quadratic factor of $\mathrm{P}(z)$ $=\left(z-r\mathrm{e}^{\mathrm{i}\theta}\right)\left(z-r\mathrm{e}^{-\mathrm{i}\theta}\right)$ $=z^2 - zr\mathrm{e}^{-\mathrm{i}\theta} - zr\mathrm{e}^{\mathrm{i}\theta} + r^2$ $=z^2 - zr\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right) + r^2$ $=z^2 - zr\left(\cos\theta + \mathrm{i}\sin\theta + \cos(-\theta) + \mathrm{i}\sin(-\theta)\right) + r^2$ $=z^2 - zr\left(\cos\theta + \mathrm{i}\sin\theta + \cos\theta - \mathrm{i}\sin\theta\right) + r^2$ $=z^2 - 2rz\cos\theta + r^2$ (shown) (ii) $z^4 = -625 = 5^4\mathrm{e}^{\mathrm{i}(\pi+2n\pi)}$ $z = 5\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}+\frac{n\pi}{2}\right)}, n = 0, \pm1, -2$ so $z = 5\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}, 5\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}, 5\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}\right)}, 5\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$ (iii) $z^4 + 625 = (z-5\mathrm{e}^{\mathrm{i}(\frac{\pi}{4})})(z-5\mathrm{e}^{\mathrm{i}(-\frac{\pi}{4})})(z-5\mathrm{e}^{\mathrm{i}(\frac{3\pi}{4})})(z-5\mathrm{e}^{\mathrm{i}(-\frac{3\pi}{4})})$ $\qquad = [z^2-(2)(5)z\cos\left(\frac{\pi}{4}\right)+5^2][z^2-(2)(5)z\cos\left(\frac{3\pi}{4}\right)+5^2]$ $=(z^2-5\sqrt{2}z+25)(z^2+5\sqrt{2}z+25)$
25.	**<u>2015/PJC/II/3</u>** 

(i) $|\mathrm{i}-2z| = \left|(-2)(z-\frac{1}{2}\mathrm{i})\right| = 2\left|z-\frac{1}{2}\mathrm{i}\right|$

Min $\left|z-\frac{1}{2}\mathrm{i}\right| = AD = \frac{\sqrt{3}}{2}\sin\frac{\pi}{3} = \frac{3}{4}$

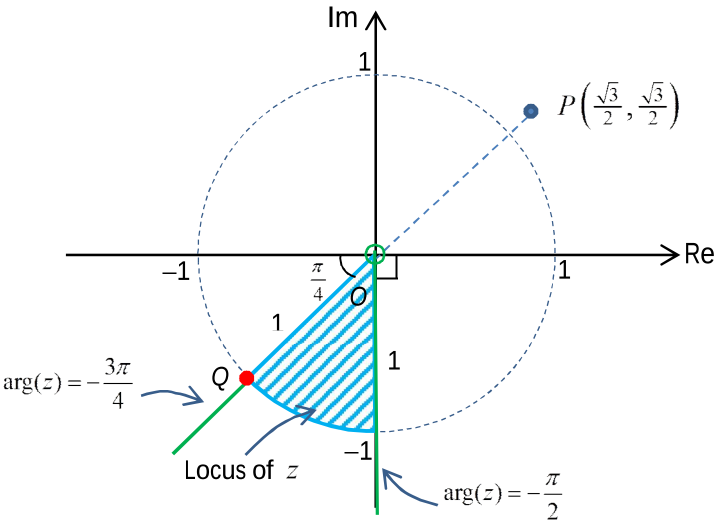
Hence min value of $|\mathrm{i}-2z|$ is $\frac{3}{2}$

(ii) Let z_B be the complex number such that arg (z) is least.

$$z_B = \left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\cos\frac{\pi}{3}\right)+\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\sin\frac{\pi}{3}\right)\mathrm{i} = \frac{3\sqrt{3}}{4}-\frac{1}{4}\mathrm{i}$$

26. **2015/RI/I/3**

(i)



(ii) Given $w = \frac{\sqrt{3}}{2}+\mathrm{i}\frac{\sqrt{3}}{2}$, $\arg(w) = \frac{\pi}{4}$.

Now $\arg(wz) = \arg(w)+\arg(z)$.

Since $-\frac{3\pi}{4} \le \arg(z) \le -\frac{\pi}{2}$,

$$-\frac{\pi}{2} \le \arg(wz) \le -\frac{\pi}{4}.$$

(iii) Greatest value of $|z-w| = PQ$

$$= |w|+1$$

$$= \left|\frac{\sqrt{3}}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right|+1$$

$$= \frac{\sqrt{6}}{2}+1$$

27. **2015/RI/I/12**

(a) $z^6 = -\mathrm{i} = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$

$$= \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} \times \mathrm{e}^{\mathrm{i}2k\pi} \quad \text{where } k \in \mathbb{Z}$$

$$= \mathrm{e}^{\mathrm{i}\frac{(4k-1)\pi}{2}}$$

By De Moivre's Theorem,

$$z = \left[\mathrm{e}^{\mathrm{i}\frac{(4k-1)\pi}{2}}\right]^{\frac{1}{6}}$$

$$= \mathrm{e}^{\mathrm{i}\frac{(4k-1)\pi}{12}}, \quad k = -2, -1, 0, 1, 2, 3$$

(b)

Method 1:

Since the coefficients of all the terms in $z^3 + az^2 + 6z + 2 = 0$ are real, and the equation has 2 roots which are purely imaginary, let the roots be $s\,\mathrm{i}$, $-s\,\mathrm{i}$ and t, where s and t are real.

Then, $z^3 + az^2 + 6z + 2 = (z + s\,\mathrm{i})(z - s\,\mathrm{i})(z - t)$

$$= (z^2 + s^2)(z - t)$$

Compare coefficients of

z^2: $\quad a = -t \Rightarrow a = \frac{1}{3}$

z: $\quad 6 = s^2$

z^0: $\quad 2 = -s^2 t \Rightarrow t = -\frac{1}{3}$

$\therefore$ The roots are $\sqrt{6}\,\mathrm{i}$, $-\sqrt{6}\,\mathrm{i}$ and $-\frac{1}{3}$.

Method 2:

Let one of the imaginary root of $z^3 + az^2 + 6z + 2 = 0$ be $s\mathrm{i}$, where s is real.

Then, $(s\mathrm{i})^3 + a(s\mathrm{i})^2 + 6(s\mathrm{i}) + 2 = 0$

$$-s^3\mathrm{i} - as^2 + 6s\mathrm{i} + 2 = 0$$

Comparing real and imaginary parts,

Real part: $\quad -as^2 + 2 = 0 \quad \Rightarrow \quad as^2 = 2$

Imaginary part: $\quad -s^3 + 6s = 0 \quad \Rightarrow \quad s = 0$ or $\sqrt{6}$ or $-\sqrt{6}$

Clearly, $z = 0$ is not a solution of $z^3 + az^2 + 6z + 2 = 0$,
so $s \neq 0$.

$\therefore$ a is $\frac{2}{s^2} = \frac{1}{3}$ and the 2 imaginary roots are $\sqrt{6}\,\mathrm{i}$ and $-\sqrt{6}\,\mathrm{i}$.

Now, $(z + \sqrt{6}\,\mathrm{i})(z - \sqrt{6}\,\mathrm{i}) = z^2 + 6$

So $z^3 + \frac{1}{3}z^2 + 6z + 2 = (z^2 + 6)\left(z + \frac{1}{3}\right)$

$\therefore$ The roots are $\sqrt{6}\,\mathrm{i}$, $-\sqrt{6}\,\mathrm{i}$ and $-\frac{1}{3}$.

NOTE:
A calculator is not allowed as stated in the instruction for this question. So, students cannot use PolySmlt2 to find the 3rd root.

(c)(i)
<u>Method 1</u> :

$$\frac{1+\sin\frac{3\pi}{8}+\mathrm{i}\cos\frac{3\pi}{8}}{1+\sin\frac{3\pi}{8}-\mathrm{i}\cos\frac{3\pi}{8}}$$

$$=\frac{1+\cos\left(\frac{\pi}{2}-\frac{3\pi}{8}\right)+\mathrm{i}\sin\left(\frac{\pi}{2}-\frac{3\pi}{8}\right)}{1+\cos\left(\frac{\pi}{2}-\frac{3\pi}{8}\right)-\mathrm{i}\sin\left(\frac{\pi}{2}-\frac{3\pi}{8}\right)}$$

$$=\frac{1+\cos\frac{\pi}{8}+\mathrm{i}\sin\frac{\pi}{8}}{1+\cos\frac{\pi}{8}-\mathrm{i}\sin\frac{\pi}{8}}$$

$$=\frac{1+\cos\frac{\pi}{8}+\mathrm{i}\sin\frac{\pi}{8}}{1+\cos\left(-\frac{\pi}{8}\right)+\mathrm{i}\sin\left(-\frac{\pi}{8}\right)}$$

$$=\frac{1+\mathrm{e}^{\mathrm{i}\frac{\pi}{8}}}{1+\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}}}$$

$$=\frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{8}}\left(\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}}+1\right)}{1+\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}}}$$

$$=\mathrm{e}^{\mathrm{i}\frac{\pi}{8}}$$

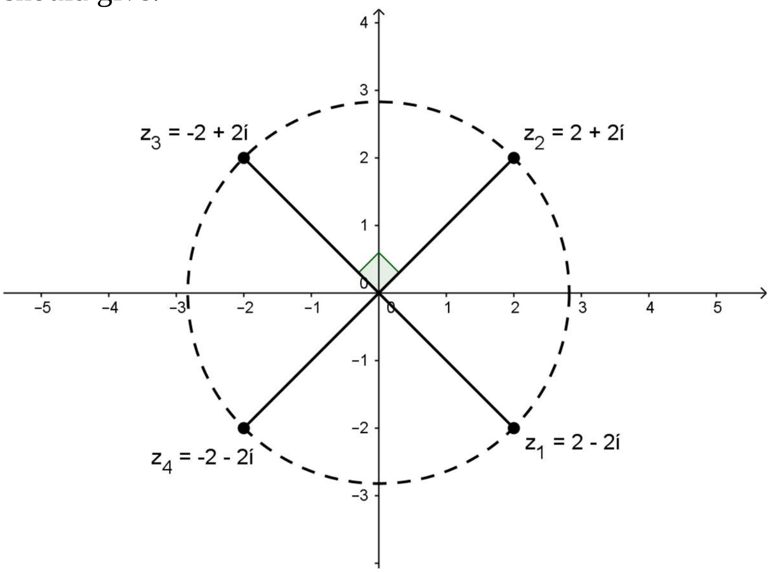
$$=\cos\frac{\pi}{8}+\mathrm{i}\sin\frac{\pi}{8} \quad \text{(proven)}$$

<u>Method 2</u>:

$$\frac{1+\sin\frac{3\pi}{8}+\mathrm{i}\cos\frac{3\pi}{8}}{1+\sin\frac{3\pi}{8}-\mathrm{i}\cos\frac{3\pi}{8}}$$

$$=\frac{1+\sin\frac{3\pi}{8}+\mathrm{i}\cos\frac{3\pi}{8}}{1+\sin\frac{3\pi}{8}-\mathrm{i}\cos\frac{3\pi}{8}}\times\frac{1+\sin\frac{3\pi}{8}+\mathrm{i}\cos\frac{3\pi}{8}}{1+\sin\frac{3\pi}{8}+\mathrm{i}\cos\frac{3\pi}{8}}$$

	$=\dfrac{\left(1+\sin\dfrac{3\pi}{8}\right)^2-\cos^2\dfrac{3\pi}{8}+2\cos\dfrac{3\pi}{8}\left(1+\sin\dfrac{3\pi}{8}\right)\mathrm{i}}{\left(1+\sin\dfrac{3\pi}{8}\right)^2+\cos^2\dfrac{3\pi}{8}}$ $=\dfrac{1+2\sin\dfrac{3\pi}{8}+\sin^2\dfrac{3\pi}{8}-\cos^2\dfrac{3\pi}{8}+2\cos\dfrac{3\pi}{8}\left(1+\sin\dfrac{3\pi}{8}\right)\mathrm{i}}{1+2\sin\dfrac{3\pi}{8}+\sin^2\dfrac{3\pi}{8}+\cos^2\dfrac{3\pi}{8}}$ $=\dfrac{2\sin\dfrac{3\pi}{8}+2\sin^2\dfrac{3\pi}{8}+2\cos\dfrac{3\pi}{8}\left(1+\sin\dfrac{3\pi}{8}\right)\mathrm{i}}{2+2\sin\dfrac{3\pi}{8}}$ $=\dfrac{2\sin\dfrac{3\pi}{8}\left(1+\sin\dfrac{3\pi}{8}\right)+2\cos\dfrac{3\pi}{8}\left(1+\sin\dfrac{3\pi}{8}\right)\mathrm{i}}{2\left(1+\sin\dfrac{3\pi}{8}\right)}$ $=\sin\dfrac{3\pi}{8}+\mathrm{i}\cos\dfrac{3\pi}{8}$ $=\cos\left(\dfrac{\pi}{2}-\dfrac{3\pi}{8}\right)+\mathrm{i}\sin\left(\dfrac{\pi}{2}-\dfrac{3\pi}{8}\right)$ $=\cos\dfrac{\pi}{8}+\mathrm{i}\sin\dfrac{\pi}{8}$ (proven) (c)(ii) To find *n* such that $\left(\dfrac{1+\sin\dfrac{3\pi}{8}+\mathrm{i}\cos\dfrac{3\pi}{8}}{1+\sin\dfrac{3\pi}{8}-\mathrm{i}\cos\dfrac{3\pi}{8}}\right)^n-\mathrm{i}=0$ $\left[\cos\left(\dfrac{\pi}{8}\right)+\mathrm{i}\sin\left(\dfrac{\pi}{8}\right)\right]^n-\mathrm{i}=0$ [from (i)] $\cos\left(\dfrac{n\pi}{8}\right)+\mathrm{i}\sin\left(\dfrac{n\pi}{8}\right)=\mathrm{i}$ <u>Method 1</u>: $\arg\left[\cos\left(\dfrac{n\pi}{8}\right)+\mathrm{i}\sin\left(\dfrac{n\pi}{8}\right)\right]=\dfrac{\pi}{2}$ By observation, the 2 smallest positive integer values of *n* are 4 and 20. <u>Method 2</u>: $\dfrac{n\pi}{8}=2k\pi+\dfrac{\pi}{2},\quad k\in\mathbb{Z}_0^+$ $n=16k+4$

	$\therefore$ The 2 smallest positive integer values of n are 4 and 20.
28.	**<u>2015/RVHS/I/7</u>** (i) Method 1: Let $z = x + \mathrm{i}y$ where $x, y \in \mathbb{R}$ $z^2 = -8\mathrm{i}$ $x^2 - y^2 + 2xy\mathrm{i} = -8\mathrm{i}$ Comparing real and imaginary parts, $x^2 - y^2 = 0$ and $2xy = -8$ $x = \pm y$ $\quad xy = -4$ When $x = y, y^2 = -4 \Rightarrow$ no solution When $x = -y, -y^2 = -4 \Rightarrow y = \pm 2, x = \mp 2$ $\therefore z = 2 - 2\mathrm{i}$ or $-2 + 2\mathrm{i}$ Method 2: $z^2 = -8\mathrm{i}$ $z^2 = 8\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}+2k\pi\right)}$ where $k \in \mathbb{Z}$ $z = 2\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}+k\pi\right)}$ for $k = 0, 1$ $z = 2\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}$ or $2\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}\right)}$ $z = 2 - 2\mathrm{i}$ or $-2 + 2\mathrm{i}$ (ii) $w^4 = -64$ $w^2 = \pm 8\mathrm{i} \Rightarrow w^2 = 8\mathrm{i}$ or $w^2 = -8\mathrm{i}$ The roots of the second equation are the conjugates of those of the first. Since two of the roots are those found in (i), representing them on argand diagram should give:  (iii)

$$z^2+(2+2\mathrm{i})z+4\mathrm{i}=0$$

$$z=\frac{-(2+2\mathrm{i})\pm\sqrt{(2+2\mathrm{i})^2-4(1)(4\mathrm{i})}}{2}$$

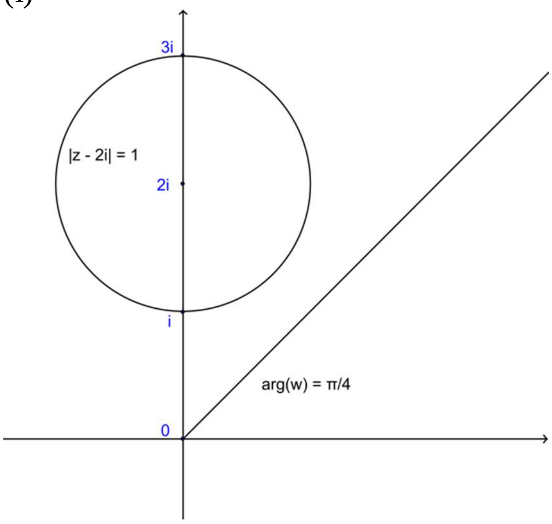
$$=-1-\mathrm{i}\pm\frac{1}{2}\sqrt{-8\mathrm{i}}$$

$$=-1-\mathrm{i}\pm\frac{1}{2}(2-2\mathrm{i})$$

$$=-2\mathrm{i}\quad\text{or}\quad-2$$

29. **<u>2015/RVHS/2/3</u>**

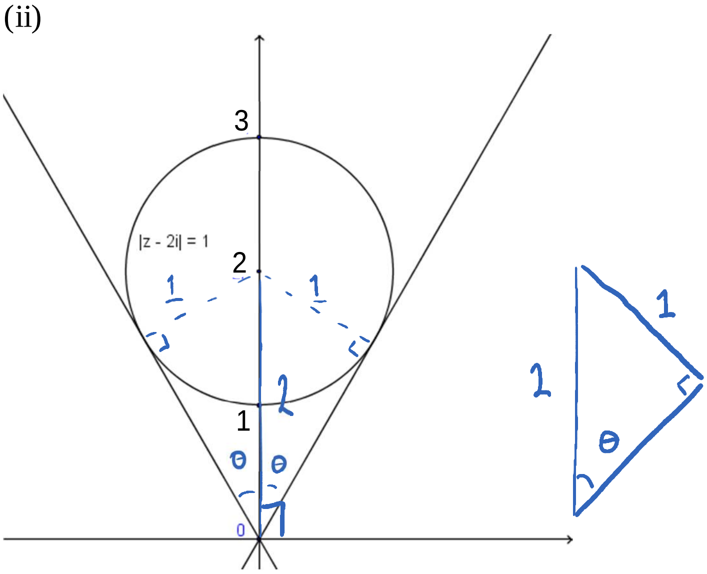
(i)



$|z-2i|=1$ is a circle centred on (0, 2) with radius 1.

$\arg(w)=\dfrac{\pi}{4}$ is the half line from the origin (excluding the origin), which makes an angle of $\dfrac{\pi}{4}$ with the positive real-axis.

(ii)

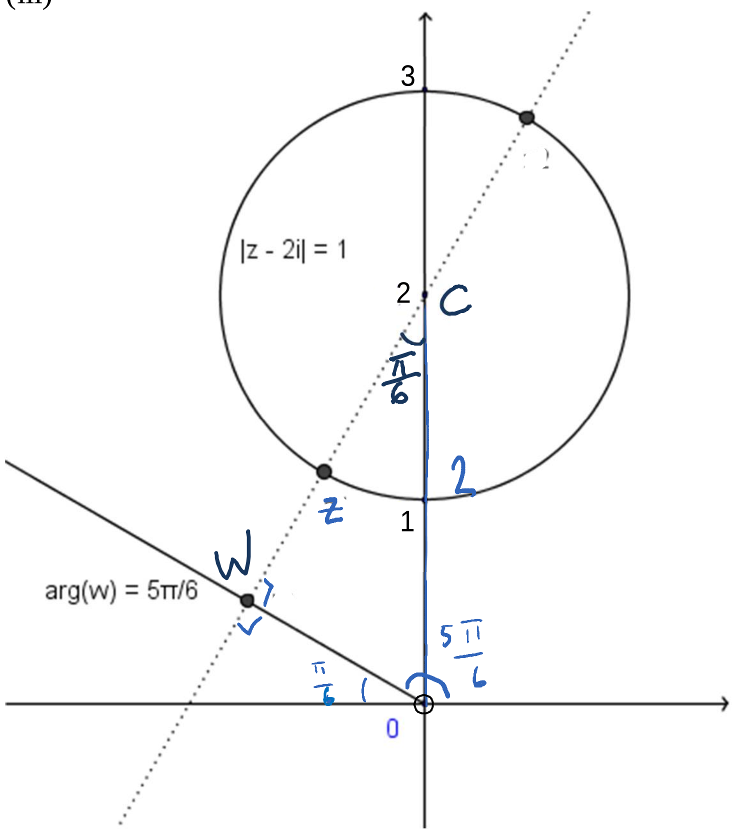


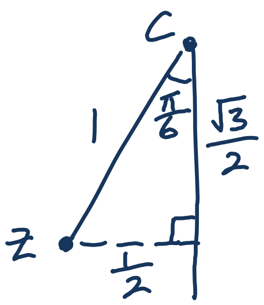
Intersect at one point implies the half line is a tangent to the circle.

$\sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$

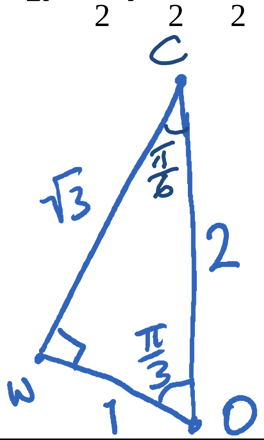
Therefore, $k = \dfrac{\pi}{2} - \dfrac{\pi}{6} = \dfrac{\pi}{3}$ or $k = \dfrac{\pi}{2} + \dfrac{\pi}{6} = \dfrac{2\pi}{3}$

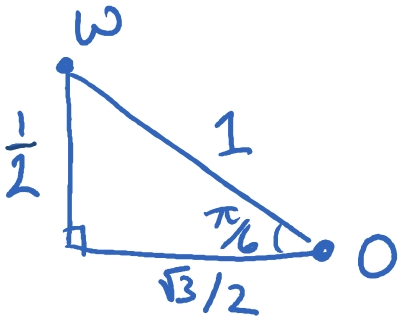
(iii)





$z = 2i - \dfrac{\sqrt{3}}{2}i - \dfrac{1}{2} = \dfrac{-1}{2} + i\dfrac{4-\sqrt{3}}{2}$





	$w = \frac{-\sqrt{3}}{2} + \frac{i}{2}$
30.	**<u>2015/SAJC/I/13</u>**

(a)

$w = \frac{z-2i}{z+4}$, where $z \neq -4$,

Let $z = x+iy$,

$$w = \frac{(x+iy)-2i}{(x+iy)+4} \cdot \frac{(x+4)-iy}{(x+4)-iy}$$

$$= \frac{(x^2+4x+y(y-2))+i(-xy+x(y-2)+4(y-2)}{(x+4)^2+y^2}$$

If Re(w) = 0, then

$$\frac{x^2+4x+y^2-2y}{(x+4)^2+y^2} = 0,$$

$\Rightarrow x^2+4x+y^2-2y=0$

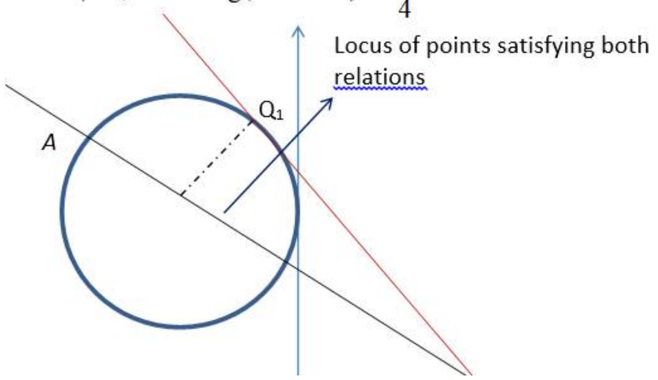
$\Rightarrow (x+2)^2 - 4 + (y-1)^2 - 1 = 0$

$\Rightarrow (x+2)^2 + (y-1)^2 = (\sqrt{5})^2$

$\therefore$ The locus of P is a circle with centre at (-2,1) and radius $\sqrt{5}$ units (Shown)

(b)(i)

$|z+2-i| \leq \sqrt{5}$ and $\arg(z-1+2i) = \frac{3\pi}{4}$



(ii)

Minimum $|z-1+2i| = PB = BC - CP = \sqrt{18} - \sqrt{5}$ units

Maximum $|z-1+2i| = AB = BC + AC = \sqrt{18} + \sqrt{5}$ units

(iii)

$$\sin\theta = \frac{\sqrt{5}}{\sqrt{18}}$$

Minimum arg(z -1+2i)

$$= \frac{3\pi}{4} - \theta$$

$$= \frac{3\pi}{4} - \sin^{-1}\left(\frac{\sqrt{5}}{\sqrt{18}}\right)$$

$$= \frac{3\pi}{4} - 0.55512$$

$$= 1.80 \text{ rad}$$

Maximum arg(z -1+2i)

$$= \frac{3\pi}{4} + \theta$$

$$= \frac{3\pi}{4} + \sin^{-1}\left(\frac{\sqrt{5}}{\sqrt{18}}\right)$$

$$= 2.91 \text{ rad}$$

31. **<u>2015/SAJC/II/4</u>**

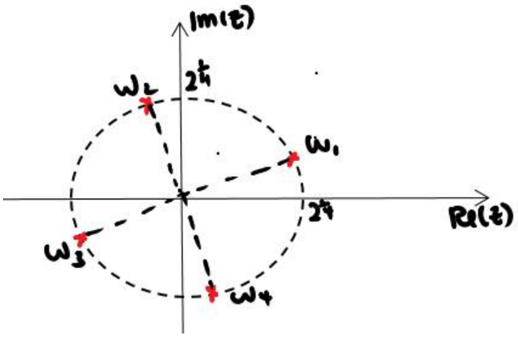
(a) $w^4 + 1 - \sqrt{3}i = 0,$

$$w^4 = -1 + \sqrt{3}i$$

$$= 2e^{i\left(\frac{2\pi}{3} + 2k\pi\right)}, \quad k = -2, -1, 0, 1$$

$$w = 2^{1/4} e^{i\left(\frac{\pi}{6} + \frac{k\pi}{2}\right)}, \quad k = -2, -1, 0, 1$$

$$\therefore w = 2^{1/4} e^{i\left(-\frac{5\pi}{6}\right)},\ 2^{1/4} e^{i\left(-\frac{\pi}{3}\right)},\ 2^{1/4} e^{i\left(\frac{\pi}{6}\right)},\ 2^{1/4} e^{i\left(\frac{2\pi}{3}\right)}.$$



(b)(i) $-y + xi = iz$ (Ans)

$wx^2 - y^2 + 2xyi = z^2$ (Ans)

(ii)

$$\therefore\ vw = iz^3$$

$$|vw| = |iz^3| = |z|^3 = r^3$$

arg (vw) = arg (v) + arg(w)

= arg(iz) + arg(z^2)

$$= \arg(i) + \arg(z) + 2\arg(z)$$

$$= \frac{\pi}{2} + 3\theta$$

	$\therefore vw = r^3 e^{i(\frac{\pi}{2}+3\theta)}$ (Ans) (b)(iii) $-4-4\sqrt{3}\mathrm{i} = 8\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$ Thus, $r^3 e^{i(\frac{\pi}{2}+3\theta)} = 8\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$ Since $\frac{\pi}{2} < 3\theta + \frac{\pi}{2} < 2\pi$, we need to compare $r^3 e^{i(\frac{\pi}{2}+3\theta)} = 8\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$ Thus, comparing modulus and argument, $r = 2$ and $\frac{\pi}{2} + 3\theta = \frac{4\pi}{3}$ $\theta = \frac{5\pi}{18}$ Thus, $z = 2e^{\mathrm{i}\left(\frac{5\pi}{18}\right)}$ **<u>Alternative Method</u>** $iz^3 = -4-4\sqrt{3}\mathrm{i} = 8\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$ $z^3 = \frac{8e^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}}{\mathrm{i}} = \frac{8e^{\mathrm{i}\left(\frac{-2\pi}{3}\right)}}{e^{\mathrm{i}\left(\frac{\pi}{2}\right)}} = 8e^{\mathrm{i}\left(-\frac{7\pi}{6}\right)} = 8e^{\mathrm{i}\left(\frac{5\pi}{6}\right)}$ $z^3 = 8e^{\mathrm{i}\left(\frac{5\pi}{6}+2k\pi\right)}$ for $k \in \mathbb{Z}$ $z = 2e^{\mathrm{i}\left(\frac{5\pi}{18}+\frac{2k\pi}{3}\right)}$ for $k = -1, 0, 1$ $z = 2e^{\mathrm{i}\left(-\frac{7\pi}{18}\right)}, 2e^{\mathrm{i}\left(\frac{5\pi}{18}\right)}, 2e^{\mathrm{i}\left(\frac{17\pi}{18}\right)}$ Since it's given that $0 < \theta < \frac{\pi}{2}$, $z = 2e^{\mathrm{i}\left(\frac{5\pi}{18}\right)}$
32.	**<u>2015/SRJC/I/11</u>** (a) Since $-1 + \mathrm{i}$ is a root of $3z^3 + 13z^2 + az + b = 0$, then we have $3(-1+\mathrm{i})^3 + 13(-1+\mathrm{i})^2 + a(-1+\mathrm{i}) + b = 0$ $(6+6\mathrm{i}) - 26\mathrm{i} - a + a\mathrm{i} + b = 0$ $(6 - a + b) + \mathrm{i}(-20 + a) = 0$ By comparing real and imaginary parts, we have $6 - a + b = 0$ and $-20 + a = 0$ $\Rightarrow$ $a = 20$ and $b = 14$ OR: Since $-1 + \mathrm{i}$ is a root of $3z^3 + 13z^2 + az + b = 0$, then $-1 - \mathrm{i}$ is also a root by the Conjugate Root Theorem. Let $3z^3 + 13z^2 + az + b = (z + 1 - \mathrm{i})(z + 1 + \mathrm{i})(3z + A)$ $= (z^2 + 2z + 2)(3z + A)$ By comparing coefficients of z^2, $13 = A + 6$ $\Rightarrow A = 7$ By comparing constants, $b = 2A = 14$ By comparing coefficients of z, $a = 2A + 6 = 20$

(b) (i) $|z^4| = |z^*| = |z|$

$\Rightarrow |z|^4 - |z| = 0$

$\Rightarrow |z|(|z|^3 - 1) = 0$

$\Rightarrow |z| = 0 \text{ or } 1$

(ii) If $|z| = 1$, then let $z = e^{i\theta}$ for $-\pi < \theta \le \pi$

$(e^{i\theta})^4 = e^{-i\theta}$

$e^{5\theta i} = 1 = e^{i(0 + 2k\pi)}$

$e^{i\theta} = e^{\frac{2k\pi}{5}i}$ for $k = 0, \pm 1, \pm 2$

If $|z| = 0$, then $z = 0$.

$\therefore z = 0,\ 1,\ e^{\frac{2\pi}{5}i},\ e^{-\frac{2\pi}{5}i},\ e^{\frac{4\pi}{5}i}, e^{-\frac{4\pi}{5}i}$

(iii) Since $0 < \arg(z) < \dfrac{\pi}{2}$, $z^k = \left(e^{\frac{2\pi}{5}i}\right)^k = e^{\frac{2k\pi}{5}i}$

If z^k is purely imaginary, then $\arg(z^k) = m\pi + \dfrac{\pi}{2}$ for $m \in \mathbb{Z}$.

$$\Rightarrow \frac{2k\pi}{5} = m\pi + \frac{\pi}{2}$$

$$\Rightarrow k = \frac{5}{2}\left(m + \frac{1}{2}\right)$$

Hence, smallest positive real number k is $\dfrac{5}{4}$

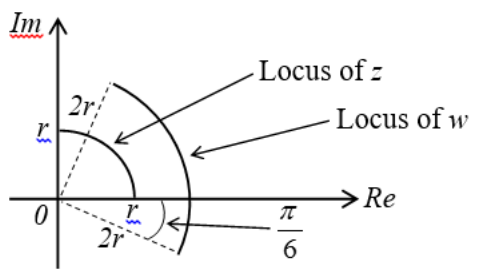
33. **<u>2015/SRJC/II/4</u>**

(a)

(i) $|w| = \left|\sqrt{3} - i\right||z| = \sqrt{(\sqrt{3})^2 + 1^2}\,r = 2r$

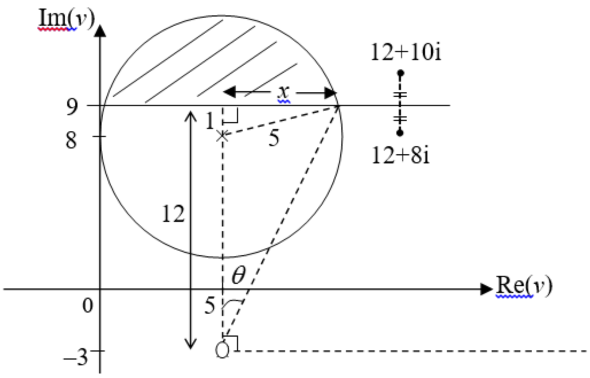
$\arg(w) = \arg\left(\sqrt{3} - i\right) + \arg(z) = \theta - \dfrac{\pi}{6}$

(ii)



(iii) Length of the locus of $w = \dfrac{1}{6}(2)(\pi)(3) = \pi$ units

(b)



From the diagram, $x^2+1^2=5^2 \Rightarrow x=\sqrt{24}$

$$\tan\theta=\frac{\sqrt{24}}{12} \Rightarrow \theta=0.3876 \text{ rad}$$

Therefore, the least value of $\arg(v-5+3\mathrm{i})=\frac{\pi}{2}-0.3876=1.18$ rad

34. **<u>2015/TJC/I/5</u>**

$$\frac{1+z^3}{1-z^3}=\sqrt{3}\mathrm{i}$$

$$1+z^3=\sqrt{3}\mathrm{i}-\sqrt{3}z^3\mathrm{i}$$

$$z^3(1+\sqrt{3}\mathrm{i})=\sqrt{3}\mathrm{i}-1$$

$$z^3=\frac{\sqrt{3}\mathrm{i}-1}{1+\sqrt{3}\mathrm{i}}$$

$$=\frac{2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}}{2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}}=\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$$

<u>Alternative</u>

$$z^3=\frac{\sqrt{3}\mathrm{i}-1}{\sqrt{3}\mathrm{i}+1}$$

$$=\frac{\sqrt{3}\mathrm{i}-1}{\sqrt{3}\mathrm{i}+1}\times\frac{\sqrt{3}\mathrm{i}-1}{\sqrt{3}\mathrm{i}-1}$$

$$=\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$$

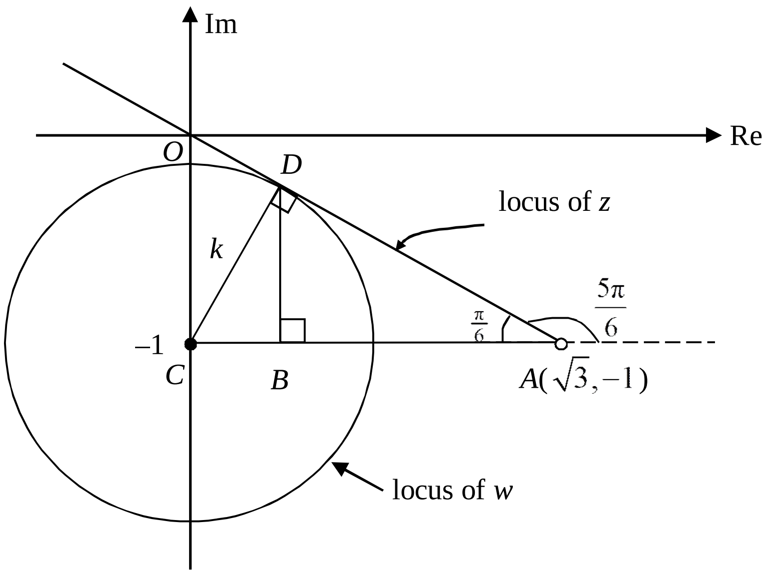
$$=\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}$$

$$z^3=\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}+2n\pi\right)}, \quad n\in\mathbb{Z}$$

$$z=\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{9}+\frac{2n\pi}{3}\right)}, \quad n=0,\pm1$$

$$z=\mathrm{e}^{\mathrm{i}\frac{\pi}{9}},\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}}$$

35.	**2015/TJC/II/1**



The locus of w is a <u>circle centered at (0, -1) and radius k units</u>

(i) Two loci intersect exactly at one point: $k = CD$ or $k \geq AC$

In triangle ACD: $\sin\frac{\pi}{6} = \frac{CD}{\sqrt{3}} \Rightarrow CD = \frac{\sqrt{3}}{2}$

and $AC = \sqrt{3}$, $\therefore k = \frac{\sqrt{3}}{2}$ or $k \geq \sqrt{3}$

i.e. $a = \frac{\sqrt{3}}{2}$, $b = \sqrt{3}$

(ii) Consider triangle BCD: $\angle BCD = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

$$BC = k\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{4} ,$$

$$BD = k\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{4}$$

At point D : $x = \frac{\sqrt{3}}{4}, y = -(1 - \frac{3}{4}) = -\frac{1}{4}$,

$\therefore$ complex number representing point D = $\frac{\sqrt{3}}{4} - \frac{1}{4}\mathrm{i}$

Alternatively,

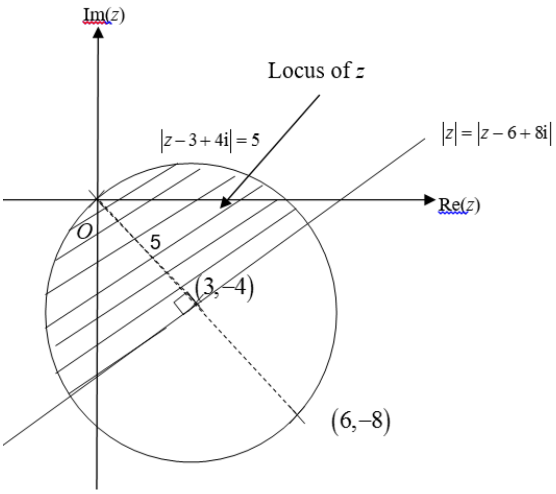
$OD = 1\cos\frac{\pi}{3} = \frac{1}{2}$

$\therefore$ complex number representing point D = $\frac{1}{2}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right)$

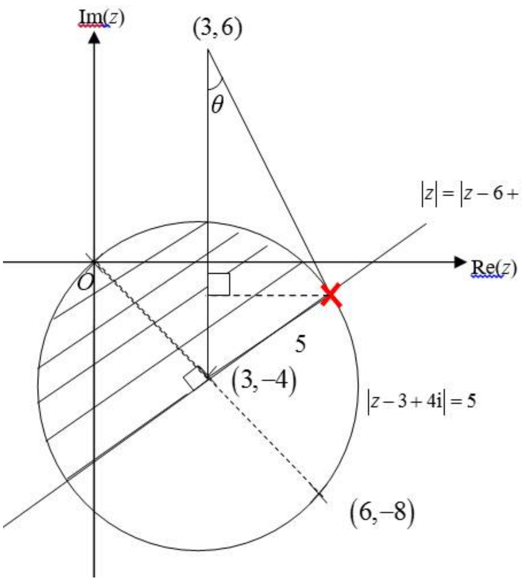
$$= \frac{\sqrt{3}}{4} - \frac{1}{4}\mathrm{i}$$

36.	**2015/TPJC/I/3**

(i)



(ii)



Method 1:

Circle: $(x-3)^2+(y+4)^2=25$

Perpendicular Bisector: $y=\dfrac{3}{4}x-\dfrac{25}{4}$

At intersection point:

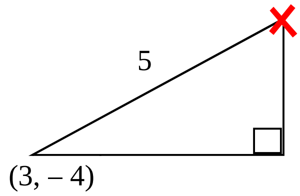
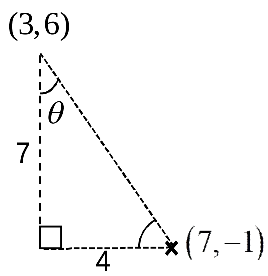
$16x^2-96x+144+9x^2+54x+81=400$

$25x^2-150x-175=0$

$x^2-6x-7=0$

$x=-1$ (rejected $\because x>0$) or 7

$z=7-\mathrm{i}$

	Method 2: Gradient of perpendicular bisector $=\frac{3}{4}$  Thus, $\tan\alpha=\frac{3}{4}$ $x=3+5\cos\alpha=3+5\left(\frac{4}{5}\right)=7$ $y=-4+5\sin\alpha=-4+5\left(\frac{3}{5}\right)=-1$ $z=7-\text{i}$ (iii) **Method 1:** $\tan\theta=\frac{4}{7}\quad\Rightarrow\quad\theta=\tan^{-1}\left(\frac{4}{7}\right)$ $\arg\left(z-3-6\text{i}\right)=-\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{4}{7}\right)\right]$ $=\tan^{-1}\left(\frac{4}{7}\right)-\frac{\pi}{2}$ $=-1.052\quad(4\text{ s.f.})$  **Method 2:** Using GC: $\arg\left(z-3-6\text{i}\right)=\arg(7-\text{i}-3-6\text{i})$ $=\arg(4-7\text{i})$ $=-1.052\text{ (4 s.f.)}$
37.	**2015/TPJC/II/3** (i) $z^3=4\sqrt{2}\left(-1-\text{i}\right)$ $=8\text{e}^{-\text{i}\frac{3\pi}{4}}$ $z=2\text{e}^{\text{i}\left(-\frac{1}{4}\pi+\frac{2}{3}k\pi\right)},\qquad k=-1,0,1$ $z=2\text{e}^{-\text{i}\frac{\pi}{4}},\ 2\text{e}^{\text{i}\frac{5\pi}{12}},\ 2\text{e}^{-\text{i}\frac{11\pi}{12}}$ (ii) $w^3=4\sqrt{2}\left(-1+\text{i}\right)=\left(z^3\right)^*=\left(z^*\right)^3$ $w=z^*$ $w=2\text{e}^{\text{i}\frac{\pi}{4}},\ 2\text{e}^{-\text{i}\frac{5\pi}{12}},\ 2\text{e}^{\text{i}\frac{11\pi}{12}}$ (iii) $\alpha=z^3+\frac{1}{\left(z^*\right)^3}$

$$\alpha = z^3 + \frac{1}{\left(z^3\right)^*}$$

$$= 2^3 \mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}} + \frac{1}{2^3 \mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}}$$

$$= \left(8 + \frac{1}{8}\right)\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}$$

$$= \frac{65}{8}\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}$$

(iv) $$\frac{\alpha^n}{\alpha^*} = \frac{\left(\frac{65}{8}\right)^n \mathrm{e}^{-\mathrm{i}\frac{3n\pi}{4}}}{\frac{65}{8}\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}}$$

$$= \left(\frac{65}{8}\right)^{n-1} \mathrm{e}^{-\mathrm{i}\left(\frac{3n\pi}{4}+\frac{3\pi}{4}\right)}$$

$$= \left(\frac{65}{8}\right)^{n-1} \mathrm{e}^{-(n+1)\left(\frac{3\pi}{4}\right)\mathrm{i}}$$

$$= \left(\frac{65}{8}\right)^{n-1} \left[\cos\left(-(n+1)\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-(n+1)\frac{3\pi}{4}\right)\right]$$

For $\frac{\alpha^n}{\alpha^*}$ to be real, $\sin\left(-(n+1)\frac{3\pi}{4}\right) = 0$

$$\sin\left(-(n+1)\frac{3\pi}{4}\right) = 0$$

$$(n+1)\frac{3\pi}{4} = k\pi, \ k \in \mathbb{Z}$$

$$n = \frac{4k}{3} - 1, \ \ k \in \mathbb{Z}$$

3 smallest positive whole number n = 3, 7 and 11.

38. **<u>2015/VJC/I/8</u>**

(i)

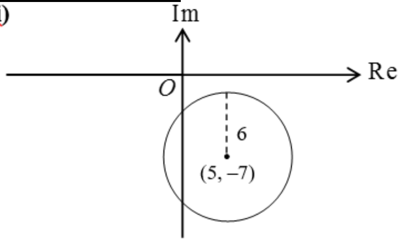
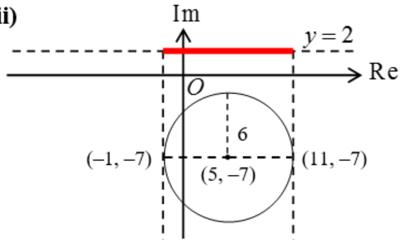
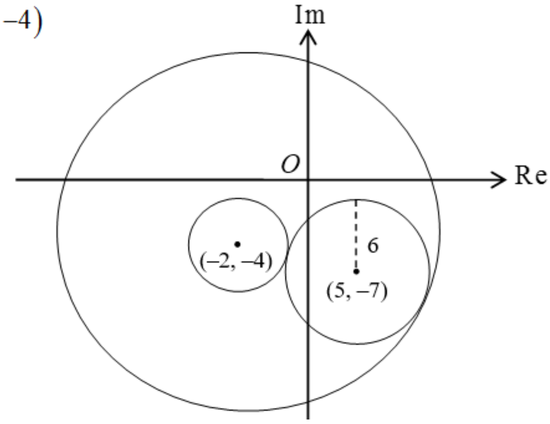
$$\text{LHS} = \left((x + \mathrm{i}\,y)^2\right)^*$$

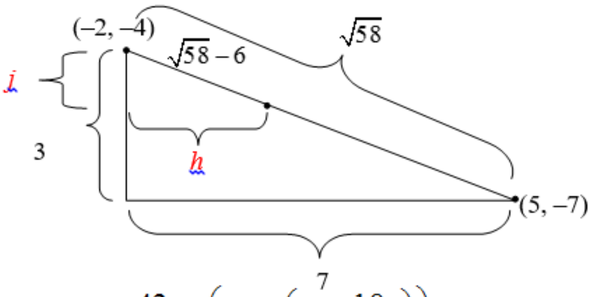
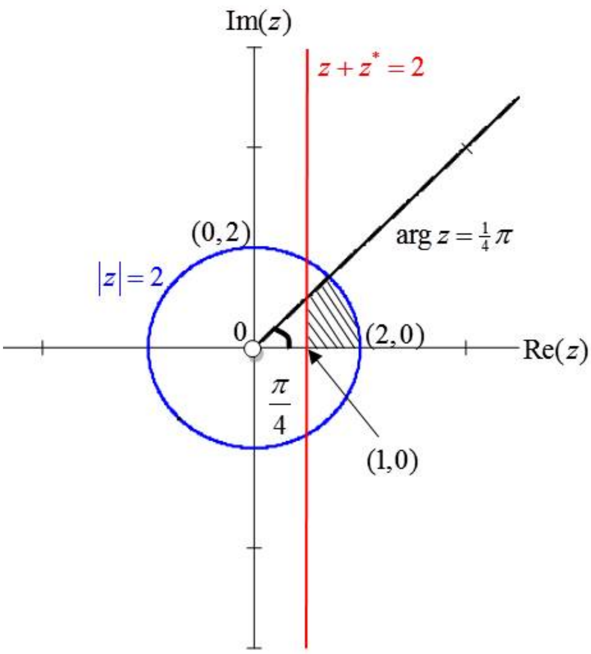
$$= \left(x^2 + (\mathrm{i}\,y)^2 + 2xy\,\mathrm{i}\right)^*$$

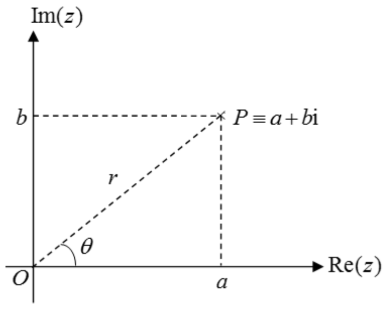
$$= \left(x^2 - y^2 + 2xy\,\mathrm{i}\right)^*$$

$$= x^2 - y^2 - 2xy\,\mathrm{i}$$

	$\text{RHS} = \left((x+\mathrm{i}\,y)^*\right)^2$ $= \left(x-\mathrm{i}\,y\right)^2$ $= x^2 + \left(\mathrm{i}\,y\right)^2 - 2xy\,\mathrm{i}$ $= x^2 - y^2 - 2xy\,\mathrm{i}$ **(ii)** Let $z = x+\mathrm{i}\,y$, where $x, y \in \mathbb{R}$ $(x+\mathrm{i}\,y)^2 = 1-4\sqrt{3}\,\mathrm{i}$ $\Rightarrow \begin{cases} x^2-y^2=1 & -(1) \\ 2xy=-4\sqrt{3} & -(2) \end{cases}$ $(2) \Rightarrow y = \dfrac{-2\sqrt{3}}{x}$ $(1) \Rightarrow x^2 - \dfrac{12}{x^2} = 1 \Rightarrow x^4 - x^2 - 12 = 0 \Rightarrow (x^2-4)(x^2+3)=0$ $x \in \mathbb{R} \Rightarrow x^2 \geqslant 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2, \quad y = \mp\sqrt{3}$ $\therefore z = 2-\sqrt{3}\,\mathrm{i}$ or $-2+\sqrt{3}\,\mathrm{i}$ **(iii)** $w^2 = 4+16\sqrt{3}\,\mathrm{i} = 4(1+4\sqrt{3}\,\mathrm{i})$ * both sides: $\left(w^2\right)^* = 4(1-4\sqrt{3}\,\mathrm{i})$ using (i) : $\left(w^*\right)^2 = 4(1-4\sqrt{3}\,\mathrm{i})$ using (ii) : $w^* = \sqrt{4}\,\left(2-\sqrt{3}\,\mathrm{i}\right)$ or $\sqrt{4}\,\left(-2+\sqrt{3}\,\mathrm{i}\right)$ $= 4-2\sqrt{3}\,\mathrm{i}$ or $-4+2\sqrt{3}\,\mathrm{i}$ $\therefore w = 4+2\sqrt{3}\,\mathrm{i}$ or $-4-2\sqrt{3}\,\mathrm{i}$ **(iv)** $z_1 = 2-\sqrt{3}\,\mathrm{i}$ and $z_2 = -2+\sqrt{3}\,\mathrm{i}$ Given: $\arg\left(z^2\right) = \theta$. $\arg\left(z_1 z_2\right) = \arg\left[\left(-2+\sqrt{3}\mathrm{i}\right)\left(2-\sqrt{3}\mathrm{i}\right)\right]$ $= \arg\left(-4+4\sqrt{3}\mathrm{i}+3\right)$ $= \arg\left(-1+4\sqrt{3}\mathrm{i}\right)$ $= \arg\left(-z^2\right)$ $= \arg\left(-1\right) + \arg\left(z^2\right)$ $= \theta + \pi$ **<u>Alternative</u>** Given: $\arg\left(z^2\right) = \theta$ (where $\theta < 0$)

	$\therefore \arg(z) = \dfrac{\theta}{2} + n\pi$ $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ $= \dfrac{\theta}{2} + \left(\dfrac{\theta}{2} + \pi\right)$ $= \theta + \pi$
39.	**<u>2015/VJC/II/3</u>** **(i)**  **(ii)**  $\{a \in \mathbb{R} : -1 < a < 11\}$ **Q3(iii)** $\lvert z + 2 + 4\mathrm{i} \rvert = k$ is a circle with radius k and centre $(-2, -4)$  Distance between the two centres $= \sqrt{7^2 + 3^2} = \sqrt{58}$ Possible exact values of $k = \sqrt{58} - 6$ and $\sqrt{58} + 6$

	For $k=\sqrt{58}-6$ $\frac{h}{7}=\frac{\sqrt{58}-6}{\sqrt{58}} \Rightarrow h=7-\frac{42}{\sqrt{58}}$ $\frac{j}{3}=\frac{\sqrt{58}-6}{\sqrt{58}} \Rightarrow j=3-\frac{18}{\sqrt{58}}$  $\therefore z=-2+7-\frac{42}{\sqrt{58}}+\left(-4-\left(3-\frac{18}{\sqrt{58}}\right)\right)\mathrm{i}$ $=\left(5-\frac{42}{\sqrt{58}}\right)+\left(\frac{18}{\sqrt{58}}-7\right)\mathrm{i}$ Similarly for $k=\sqrt{58}+6$, $z=\left(5+\frac{42}{\sqrt{58}}\right)-\left(\frac{18}{\sqrt{58}}+7\right)\mathrm{i}$
40.	**<u>2015/YJC/I/11</u>** (a)  Shaded region represents the set of required points. Greatest value of $\|z-4-4\mathrm{i}\|=\sqrt{(4-1)^2+4^2}=5$ Greatest value of $\arg(z-4)=\pi$

	(b) $w^3+1=0 \Rightarrow w^3=-1=\mathrm{e}^{(2k\pi+\pi)\mathrm{i}}$, where $k\in\mathbb{Z}$ $w=\mathrm{e}^{\frac{1}{3}(2k+1)\pi\mathrm{i}}$, where $k=0,\pm1$ $k=0$, $w=\cos\frac{1}{3}\pi+\mathrm{i}\sin\frac{1}{3}\pi=\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}$ $k=-1$, $w=\cos\frac{1}{3}\pi-\mathrm{i}\sin\frac{1}{3}\pi=\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}$ $k=1$, $w=\cos\pi+\mathrm{i}\sin\pi=-1$ <u>Alternative</u> $w^3+1=(w+1)(w^2-w+1)=0$ $w=-1,\ \frac{1\pm\sqrt{1-4}}{2}$ i.e. $w=-1,\ \frac{1\pm\mathrm{i}\sqrt{3}}{2}$ For $\left(\frac{z+1}{z}\right)^3=-1$, let $\frac{z+1}{z}=w$, Then $z+1=wz$ and $z=\frac{1}{w-1}$ i.e. $z=\frac{1}{\mathrm{e}^{\frac{1}{3}(2k+1)\pi\mathrm{i}}-1}$ where $k=0,\pm1$ $z=\frac{1}{\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}-1}=-\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}$ $z=\frac{1}{\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}-1}=-\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}$ $z=\frac{1}{-1-1}=-\frac{1}{2}$
41.	<u>**2015/YJC/II/1**</u> (a)  $a=r\cos\theta$ and $b=r\sin\theta$ Therefore $a+b\mathrm{i}=r\cos\theta+(r\sin\theta)\mathrm{i}=r(\cos\theta+\mathrm{i}\sin\theta)$

Let P be the point on the Argand diagram representing the complex number $a+b\mathrm{i}$.
r is the distance of the point P to the origin.

θ is the directed angle made by the line segment joining origin to point P, with the positive real axis.

Cartesian equation of locus: $x^2+y^2=9$

(b)
Given $z=\cos\alpha+\mathrm{i}\sin\alpha$
By De Moivre's Theorem

$$\frac{1}{z}=z^{-1}=\left(\cos\alpha+\mathrm{i}\sin\alpha\right)^{-1}=\cos\alpha-\mathrm{i}\sin\alpha$$

OR

$$\frac{1}{z}=z^{-1}=\mathrm{e}^{-\mathrm{i}\alpha}=\cos\alpha-\mathrm{i}\sin\alpha$$

OR

$$\frac{1}{z}=\frac{1}{\cos\alpha+\mathrm{i}\sin\alpha}\times\frac{\cos\alpha-\mathrm{i}\sin\alpha}{\cos\alpha-\mathrm{i}\sin\alpha}$$
$$=\frac{\cos\alpha-\mathrm{i}\sin\alpha}{\cos^2\alpha-\mathrm{i}^2\sin^2\alpha}$$
$$=\frac{\cos\alpha-\mathrm{i}\sin\alpha}{\cos^2\alpha+\sin^2\alpha}$$
$$=\cos\alpha-\mathrm{i}\sin\alpha$$

OR

$$zz^*=|z|^2=1\Rightarrow\frac{1}{z}=z^*=\cos\alpha-\mathrm{i}\sin\alpha$$

$$z+\frac{1}{z}=\cos\alpha+\mathrm{i}\sin\alpha+\cos\alpha-\mathrm{i}\sin\alpha$$
$$=2\cos\alpha \text{ (Shown)}$$
$$z-\frac{1}{z}=\cos\alpha+\mathrm{i}\sin\alpha-(\cos\alpha-\mathrm{i}\sin\alpha)$$
$$=2\mathrm{i}\sin\alpha \text{ (Shown)}$$

$$\left(z+\frac{1}{z}\right)\div\left(z-\frac{1}{z}\right)=\frac{z^2+1}{z}\times\frac{z}{z^2-1}$$

Hence $\dfrac{z^2+1}{z^2-1}=\dfrac{2\cos\alpha}{2\mathrm{i}\sin\alpha}=-\dfrac{\mathrm{i}\cos\alpha}{\sin\alpha}$ or $-\mathrm{i}\cot\alpha$

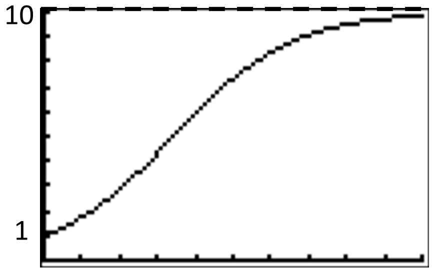
Topical Practice Questions: Differential Equations Solutions (2015 Prelim)

1	$\frac{\mathrm{d}w}{\mathrm{d}x}=-2(\mathrm{e}^{-2x}+A)^{-2}(-2\mathrm{e}^{-2x})=\frac{4\mathrm{e}^{-2x}}{(\mathrm{e}^{-2x}+A)^2}$ $\therefore$ LHS = $\mathrm{e}^{2x}\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{4}{(\mathrm{e}^{-2x}+A)^2}=w^2$ = RHS (verified) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{\mathrm{e}^{-2x}+A}$ $y=\int\frac{2}{\mathrm{e}^{-2x}+A}\,\mathrm{d}x$ $=\frac{1}{A}\int\frac{2A\mathrm{e}^{2x}}{1+A\mathrm{e}^{2x}}\,\mathrm{d}x$ $=\frac{1}{A}\ln\left\|1+A\mathrm{e}^{2x}\right\|+B,$ where B is an arbitrary constant. Let S be the amount of salt (in grams) at time t minutes. $\frac{\mathrm{d}S}{\mathrm{d}t}$ = rate of salt entering tank − rate of salt leaving tank $=2(5)-\frac{S}{100}(5)$ $=\frac{200-S}{20}$ g/min $\frac{1}{200-S}\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{1}{20}$ $\int\frac{1}{200-S}\,\mathrm{d}S=\int\frac{1}{20}\,\mathrm{d}t$ $-\ln\left\|200-S\right\|=\frac{1}{20}t+c$ $200-S=A\mathrm{e}^{-\frac{1}{20}t}$ $S=200-A\mathrm{e}^{-\frac{1}{20}t}$ When $t=0$, $S=50$ $\Rightarrow$ $50=200-A$ $\Rightarrow$ $A=150$ $\therefore$ $S=200-150\mathrm{e}^{-\frac{1}{20}t}$ When concentration is 1 g/litre, $S=100$. $\Rightarrow 100=200-150\mathrm{e}^{-\frac{1}{20}t}$ $\Rightarrow$ $\mathrm{e}^{-\frac{1}{20}t}=2/3$ $\therefore$ $t=-20\ln(2/3)=8.11$ min (3 s.f.)
2	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{100-x}{100}$

	$\int \frac{1}{100-x}\ \mathrm{d}x = \int 0.01\ \mathrm{d}t$ $\ln\lvert 100-x\rvert = -0.01t + k$ $100 - x = A\mathrm{e}^{-0.01t}, \quad A = \pm \mathrm{e}^{k}$ $x = 100 - A\mathrm{e}^{-0.01t}.$ When $t = 0, \quad x = 50 \Rightarrow A = 50.$ $\therefore x = 100 - 50\mathrm{e}^{-0.01t}.$ As $t \to \infty, \quad x \to 100.$ The population of the predators approaches 100 000 after several years. $\frac{\mathrm{d}y}{\mathrm{d}t} = -50\mathrm{e}^{-0.01t}.$ $\int \frac{\mathrm{d}y}{\mathrm{d}x}\ \mathrm{d}x = \int -50\mathrm{e}^{-0.01t}\ \mathrm{d}t$ $y = 5000\mathrm{e}^{-0.01t} + c.$ As $t \to \infty, \quad y \to 5000 \Rightarrow c = 5000.$ $\therefore y = 5000\mathrm{e}^{-0.01t} + 5000.$
3	$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.64P\left(1 - \frac{P}{10}\right)$ $\frac{\mathrm{d}P}{\mathrm{d}t} = 0.064P(10-P)$ $\frac{1}{P(10-P)}\frac{\mathrm{d}P}{\mathrm{d}t} = 0.064$ $\int \frac{1}{P(10-P)}\mathrm{d}P = \int 0.064\mathrm{d}t$ $\frac{1}{10}\int \frac{1}{P} + \frac{1}{10-P}\mathrm{d}P = \int 0.064\mathrm{d}t$ $\int \frac{1}{P} + \frac{1}{10-P}\mathrm{d}P = \int 0.64\mathrm{d}t$ $\ln\lvert P\rvert - \ln\lvert 10-P\rvert = 0.64t + C$ $\ln\left\lvert \frac{P}{10-P}\right\rvert = 0.64t + C$ $\frac{P}{10-P} = \pm\mathrm{e}^{0.64t+C}$ $\frac{P}{10-P} = A\mathrm{e}^{0.64t}$, where $A = \pm\mathrm{e}^{c}$ $\frac{10-P}{P} = B\mathrm{e}^{-0.64t}$, where $B = \frac{1}{A}$ $\frac{10}{P} = 1 + B\mathrm{e}^{-0.64t}$ $P = \frac{10}{1 + B\mathrm{e}^{-0.64t}}$

When $t=0$, $P=1$,

$$1=\frac{10}{1+Be^{0}}$$
$$B=9$$
$$\therefore P=\frac{10}{1+9e^{-0.64t}}$$



The population will approach 10 asymptotically in the long run.

$$\frac{dP}{dt}=0.4P\ln\left(\frac{10}{P}\right)$$
$$\frac{1}{P(\ln 10-\ln P)}\frac{dP}{dt}=0.4$$
$$\int\frac{1}{P(\ln 10-\ln P)}dP=\int 0.4\,dt$$
$$\ln\left|\ln 10-\ln P\right|=-0.4t+C$$
$$\ln 10-\ln P=\pm e^{-0.4t+C}$$
$$\ln 10-\ln P=Ae^{-0.4t},\text{ where } A=\pm e^{C}$$
$$\ln P=\ln 10-Ae^{-0.4t}$$
$$P=10e^{-Ae^{-0.4t}}$$

When $t=0$, $P=1$,

$$1=10e^{-Ae^{0}}$$
$$A=\ln 10$$
$$\therefore P=10e^{-(\ln 10)e^{-0.4t}}$$

(Similarity)
The population approaches 10 asymptotically in the long run in both models.

(Difference)
- The population in the second model would increase faster first then slower compared to the first model.

The Gompertz function will take a longer time to approach 10 as compared to the Logistic function.

4

$$2xy\frac{dy}{dx}=4x^{2}y^{4}-y^{2}+1$$

Given $w = xy^2 \Rightarrow \dfrac{dw}{dx} = 2xy\dfrac{dy}{dx} + y^2$

$$2xy\frac{dy}{dx} = \frac{dw}{dx} - y^2$$

$$\therefore \ \frac{dw}{dx} - y^2 = 4w^2 - y^2 + 1$$

$$\frac{dw}{dx} = 4w^2 + 1, \quad \text{ie} \quad a = 4, \quad b = 1$$

$$\frac{1}{w^2 + \frac{1}{4}}\frac{dw}{dx} = 4$$

$$\int \frac{1}{w^2 + \frac{1}{4}}\,dw = 4\int 1\,dx$$

$$2\tan^{-1}(2w) = 4x + C$$

$$\tan^{-1}(2w) = 2x + D \quad (\text{where } D = C/2)$$

$$2w = \tan(2x + D)$$

$$y^2 = \frac{\tan(2x + D)}{2x}$$

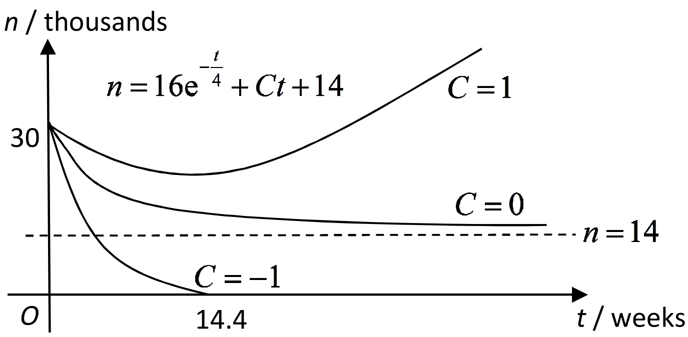
$$\frac{d^2n}{dt^2} = e^{-\frac{t}{4}}$$

$$\frac{dn}{dt} = -4e^{-\frac{t}{4}} + C$$

$$n = 16e^{-\frac{t}{4}} + Ct + D$$

Given $n = 30$, $t = 0$,

$30 = 16 + D \Rightarrow D = 14$



SR: If all 3 marks awarded, deduct 1 mark if

– Curve(s) drawn for $n < 0$ or $t < 0$ **or**

– value of C not clearly indicated for each graph **or**

axes not labelled (condone missing units)

5

$\frac{\mathrm{d}A}{\mathrm{d}t} = kA - 4000 \Rightarrow \int \frac{1}{kA-4000}\,\mathrm{d}A = \int 1\,\mathrm{d}t$

$\Rightarrow \frac{1}{k}\int \frac{1}{A-\frac{4000}{k}}\,\mathrm{d}A = \int 1\,\mathrm{d}t$

$\Rightarrow \frac{1}{k}\ln\left|A-\frac{4000}{k}\right| = t + C \Rightarrow \left|A-\frac{4000}{k}\right| = \mathrm{e}^{kt+kC}$

$\Rightarrow A-\frac{4000}{k} = \alpha\mathrm{e}^{kt}$ $\left(\text{where } \alpha = \pm\mathrm{e}^{kC}\right)$

$\Rightarrow A = \alpha\mathrm{e}^{kt} + \frac{4000}{k}$

Sub $(t, A) = (0, 60000)$:

$60000 = \alpha + \frac{4000}{k} \Rightarrow \alpha = 60000 - \frac{4000}{k}$ ---- (1)

Sub $(t, A) = (3, 69500)$:

$69500 = \alpha\mathrm{e}^{3k} + \frac{4000}{k}$ ---- (2)

Sub (1) into (2):

$69500 = \left(60000 - \frac{4000}{k}\right)\mathrm{e}^{3k} + \frac{4000}{k}$

Using G.C., $k \approx 0.1111343 = 0.111$ (3 s.f.)

Thus $A = \left(60000 - \frac{4000}{k}\right)\mathrm{e}^{kt} + \frac{4000}{k}$

$\Rightarrow A = 24000\mathrm{e}^{0.111t} + 36000$ with $\alpha = 24000$ (3 s.f.) and $\lambda = 36000$ (3 s.f.)

Since $A = \pi r^2, \frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$

Thus $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t} \div \frac{\mathrm{d}A}{\mathrm{d}r} = \frac{kA-4000}{2\pi r} = \frac{k\pi r^2 - 4000}{2\pi r}$

Sub $r = 200$, $\frac{\mathrm{d}r}{\mathrm{d}t} \approx \frac{(0.1111343)\pi(200)^2 - 4000}{2\pi(200)} = 7.93$ (3 s.f.)

Since $A = \pi r^2, \frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$

Thus $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t} \div \frac{\mathrm{d}A}{\mathrm{d}r} = \frac{kA-4000}{2\pi r} = \frac{k\pi r^2 - 4000}{2\pi r}$

Sub $r = 200$, $\frac{\mathrm{d}r}{\mathrm{d}t} \approx \frac{(0.1111343)\pi(200)^2 - 4000}{2\pi(200)} = 7.93$ (3 s.f.)

$\frac{\mathrm{d}A}{\mathrm{d}t} = 0$ means that the rate which Mac needs to cut the weeds is equal to the rate the weeds grow. Thus, the area covered in weeds is unchanged.

6

$$\frac{d^2y}{dx^2}=\frac{2}{x^3}-\frac{1}{x^2}$$

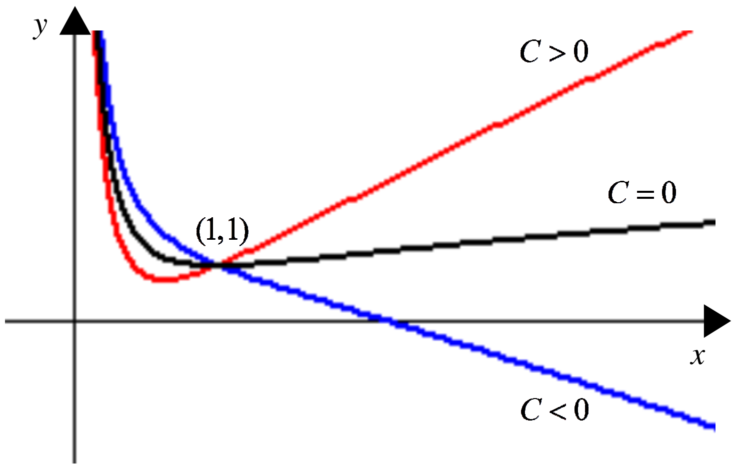
$$\frac{dy}{dx}=-\frac{1}{x^2}+\frac{1}{x}+C$$

$$y=\frac{1}{x}+\ln|x|+Cx+D$$

When $x=1$ and $y=1$,

$1=1+0+C+D \;\Rightarrow\; D=-C$

Hence, $y=\dfrac{1}{x}+\ln|x|+C(x-1)$



7

Given $\dfrac{d\theta}{dt} \propto \theta-25$, $\therefore \dfrac{d\theta}{dt}=-k(\theta-25)$

$$\int \frac{1}{\theta-25}\,d\theta=-\int k\,dt$$

$$\begin{aligned}\ln|\theta-25| &= -kt+c\\ |\theta-25| &= e^{-kt+c}\\ \theta-25 &= \pm e^{c}e^{-kt}\\ \theta &= Ae^{-kt}+25, \text{ where } A=\pm e^{c}\end{aligned}$$

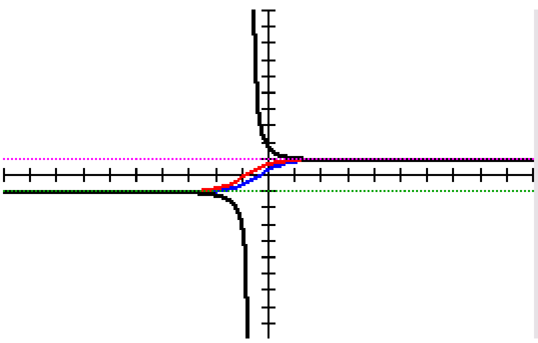
When $t=0$, $\theta=32$

$$32 = Ae^{0}+25 \Rightarrow A=7$$

When $t=1$, $\theta=30$

$$\begin{aligned}30 &= 7e^{-k}+25 \Rightarrow k=-\ln\frac{5}{7}\\ \theta &= 7e^{\left(\ln\frac{5}{7}\right)t}+25\\ \theta &= 7\left(\frac{5}{7}\right)^{t}+25 \qquad \text{(Shown)}\end{aligned}$$

When $\theta = 37$, $t=\dfrac{\ln\left(\frac{12}{7}\right)}{\ln\left(\frac{5}{7}\right)}=-1.60$ (3 sf)

	The time of death is 10.24pm
8	$\frac{\mathrm{d}y}{\mathrm{d}x}=1-y^2$ $\int \frac{1}{1-y^2}\ \frac{\mathrm{d}y}{\mathrm{d}x}\ \mathrm{d}x=\int \mathrm{d}x$ $\frac{1}{2}\ln\left\|\frac{1+y}{1-y}\right\|=x+c$ $\frac{1+y}{1-y}=Ae^{2x},\ A=\pm e^{2c}$ $y=\frac{Ae^{2x}-1}{Ae^{2x}+1}$ When $x=0,\ y=\frac{1}{3}$: $\frac{1}{3}=\frac{A-1}{A+1}$ $A=2$ $y=\frac{2e^{2x}-1}{2e^{2x}+1}$ As $x\to\pm\infty,\ y\to\pm 1,\ \frac{\mathrm{d}y}{\mathrm{d}x}\to 0.$ 
9	**<u>Method 1</u>** $\frac{\mathrm{d}\theta}{\mathrm{d}t}=-k(\theta-25),\qquad k>0$ $\int \frac{1}{\theta-25}\mathrm{d}\theta=\int -k\ \mathrm{d}t$ $\ln\|\theta-25\|=-kt+C$ $\|\theta-25\|=\mathrm{e}^{-kt+C}$ $\theta-25=\pm\mathrm{e}^{-kt}\mathrm{e}^{C}=A\mathrm{e}^{kt}$ where $A=\pm\mathrm{e}^{C}$ $\theta=25+A\mathrm{e}^{-kt}$

Given that when $t=0, \theta=75$

$\Rightarrow 75=25+A \Rightarrow A=50$

$\therefore \theta=25+50\mathrm{e}^{-kt}$

<u>Method 2</u>

$\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=k(\theta-25), \qquad k<0$

$\displaystyle\int \frac{1}{\theta-25}\,\mathrm{d}\theta=\int k\ \mathrm{d}t$

$\ln|\theta-25|=kt+C$

$|\theta-25|=\mathrm{e}^{kt+C}$

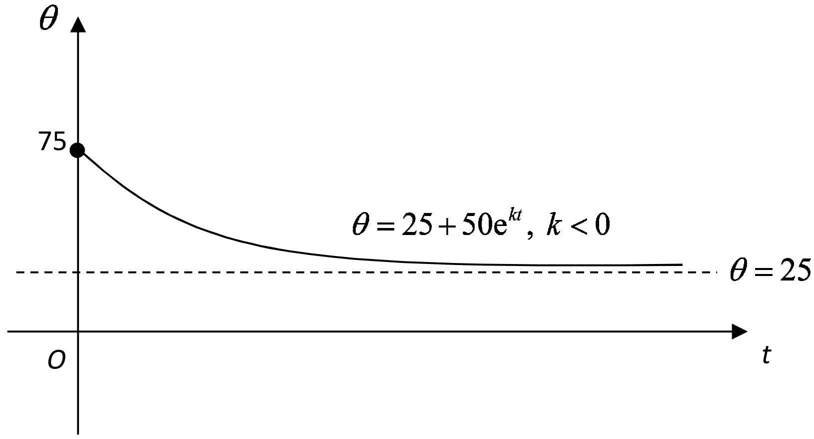
$\theta-25=\pm\mathrm{e}^{kt}\mathrm{e}^{C}=A\mathrm{e}^{kt}$ where $A=\pm\mathrm{e}^{C}$

$\theta=25+A\mathrm{e}^{kt}$

Given that when $t=0, \theta=75$

$\Rightarrow 75=25+A \Rightarrow A=50$

$\therefore \theta=25+50\mathrm{e}^{kt}$



Given that when $t=10, \theta=35$

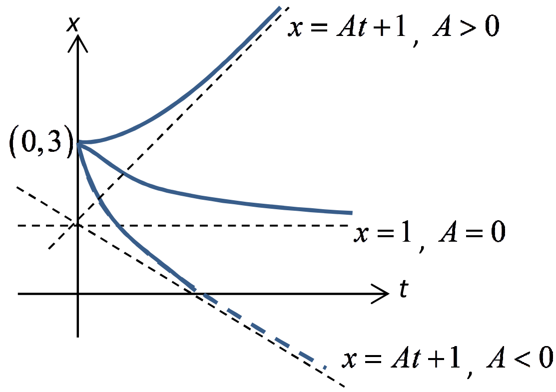
$\Rightarrow 35=25+50\mathrm{e}^{10\mathrm{k}} \Rightarrow k=\dfrac{1}{10}\ln\left(\dfrac{1}{5}\right)$

$\therefore \theta=25+50\mathrm{e}^{\frac{1}{10}\ln\left(\frac{1}{5}\right)t}$

when $\theta=30$,

$30=25+50\mathrm{e}^{\frac{1}{10}\ln\left(\frac{1}{5}\right)t} \Rightarrow t=14.3\approx 14$ (to nearest min)

10 **(i)** $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}=\dfrac{4}{(t+1)^3}$

	$\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{2}{(t+1)^2}+A$ $x=\frac{2}{t+1}+At+B$ When $t=0$, $x=3$ $3=2+B \quad \Rightarrow \quad B=1$ Hence $x=\frac{2}{t+1}+At+1$ **(ii)** 
11	$\frac{\mathrm{d}x}{\mathrm{d}t}=k(200-2t-x)$ When $t=0$, $x=8$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=16$. $16=k(200-8)$ $k=\frac{16}{192}=\frac{1}{12}\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{200-2t-x}{12}$. $u=2t+x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}t}=2+\frac{\mathrm{d}x}{\mathrm{d}t}$ $\frac{\mathrm{d}u}{\mathrm{d}t}-2=\frac{200-u}{12}$ $\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{224-u}{12}$ $\frac{1}{224-u}\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{1}{12}$ Integrating both sides with respect to t, $\int\frac{1}{224-u}\,\mathrm{d}u=\int\frac{1}{12}\,\mathrm{d}t$ $-\ln\lvert 224-u\rvert=\frac{t}{12}+C$ $\ln\lvert 224-u\rvert=-\frac{t}{12}-C$ $\lvert 224-u\rvert=\mathrm{e}^{-\frac{t}{12}-C}$

$224-u=\pm \mathrm{e}^{-\frac{t}{12}-C}=A\mathrm{e}^{-\frac{t}{12}}$ where $A=\pm \mathrm{e}^{-C}$

$u=224-A\mathrm{e}^{-\frac{t}{12}}$

$x=224-2t-A\mathrm{e}^{-\frac{t}{12}}$

When $t=0$, $x=8$

$8=224-A \Rightarrow A=216$

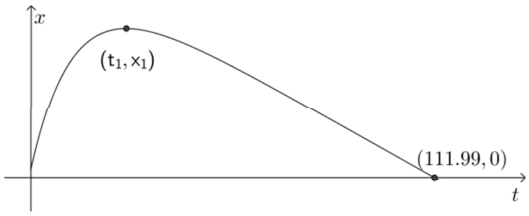
Thus, $x=224-2t-216\mathrm{e}^{-\frac{t}{12}}$

By GC, when $x=0$, $t\approx 112$ (years)

When $\frac{\mathrm{d}x}{\mathrm{d}t}=0$,

$200-2t-x=0$

$x=200-2t$



So, x_1 is the maximum population size of the fish.

t_1 is the number of years for the population to reach its maximum.

12 $\frac{\mathrm{d}V}{\mathrm{d}t}=5-kV$, $k>0$

$\int \frac{1}{5-kV}\,\mathrm{d}v=\int 1\,\mathrm{d}t$

$-\frac{1}{k}\ln|5-kV|=t+c$

$\ln|5-kV|=-kt-kc$

$5-kV=\pm e^{-kc}\mathrm{e}^{-kt}$

$5-kV=A\mathrm{e}^{-kt}$, $\quad A=\pm e^{-kc}$

$V=\frac{1}{k}\left(5-A\mathrm{e}^{-kt}\right)$

$t=0$, $V=0$.

Hence, $0=\frac{1}{k}(5-A)\Rightarrow A=5$.

$V=\frac{1}{k}\left(5-5\mathrm{e}^{-kt}\right)$

$V=\frac{5}{k}\left(1-\mathrm{e}^{-kt}\right)$

As $t\to\infty$, $V\to\frac{5}{k}$

13 When $a=b$, we have

$$\frac{\mathrm{d}u}{\mathrm{d}t}=(a-u)^2$$

$$\int\frac{1}{(a-u)^2}\ \mathrm{d}u=\int 1\mathrm{d}t$$

$$\frac{1}{a-u}=t+C,$$ where C is an arbitrary constant.

$$a-u=\frac{1}{t+C}$$

$$u=a-\frac{1}{t+C}$$

When $t=0$, $u=0$: $C=\frac{1}{a}$

Hence, $u=a-\dfrac{1}{t+\frac{1}{a}}$

$$=a-\frac{a}{at+1}=\frac{a^2t}{at+1}.$$

When $a\neq b$, we have $\dfrac{\mathrm{d}u}{\mathrm{d}t}=(a-u)(b-u)$

$$\int\frac{1}{(a-u)(b-u)}\ \mathrm{d}u=\int 1\mathrm{d}t$$

$$\int\left[\frac{1}{(b-a)(a-u)}+\frac{1}{(a-b)(b-u)}\right]\mathrm{d}u=\int 1\mathrm{d}t$$

Since $u<a$ and $u<b$,

$$-\frac{1}{b-a}\ln(a-u)-\frac{1}{a-b}\ln(b-u)=t+D,$$ where D is an arbitrary constant

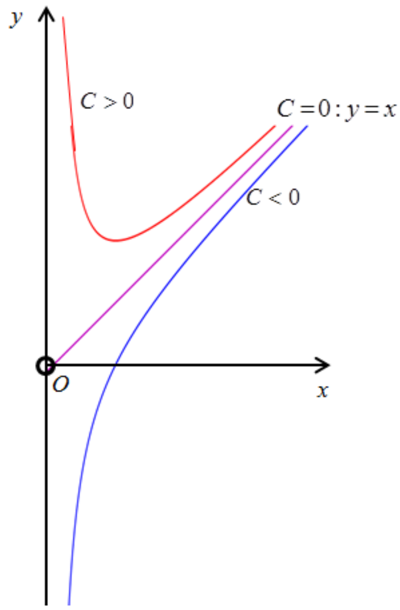
$$\frac{1}{b-a}\ln\left(\frac{b-u}{a-u}\right)=t+D$$

When $t=0$, $u=0$: $D=\frac{1}{b-a}\ln\frac{b}{a}$

Hence, $\dfrac{1}{b-a}\ln\left(\dfrac{b-u}{a-u}\right)=t+\dfrac{1}{b-a}\ln\dfrac{b}{a}$

$$\Rightarrow t=\frac{1}{b-a}\ln\left(\frac{b-u}{a-u}\right)-\frac{1}{b-a}\ln\frac{b}{a}$$

$$=\frac{1}{b-a}\ln\frac{a(b-u)}{b(a-u)}$$

	$y = x + \frac{C}{x}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{C}{x^2} \Rightarrow C = x^2\left(1 - \frac{\mathrm{d}y}{\mathrm{d}x}\right)$ Hence $y = x + \frac{C}{x}$ $= x + \frac{x^2\left(1 - \frac{\mathrm{d}y}{\mathrm{d}x}\right)}{x}$ $= 2x - x\frac{\mathrm{d}y}{\mathrm{d}x}$, where $p = 2$ and $q = -1$. 
14	$y = ux + 2 \quad \Rightarrow \quad \frac{dy}{dx} = x\frac{du}{dx} + u$ Substituting into DE: $\frac{dy}{dx} - \sqrt{4 - (y-2)^2} = 0$ $\frac{dy}{dx} - \sqrt{4 - (y^2 - 4y + 4)} = 0$ $\frac{dy}{dx} = \sqrt{4y - y^2}$ (shown)

$$\therefore \int \frac{1}{\sqrt{4y-y^2}}\,dy = \int 1\,dx$$

$$\int \frac{1}{\sqrt{4-(y-2)^2}}\,dy = \int 1\,dx$$

$$\sin^{-1}\left(\frac{y-2}{2}\right) = x+c$$

$$y-2 = 2\sin(x+c)$$

$$u = \frac{2\sin(x+c)}{x}$$

From part (i), $s = 2\sin(t+c)+2$

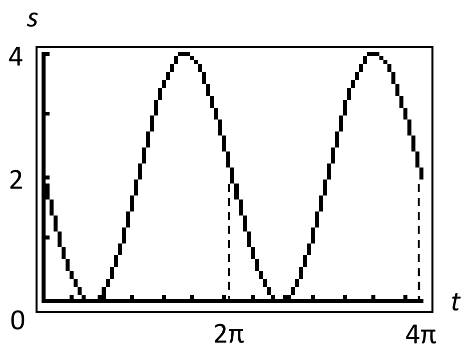
when $t = \dfrac{5\pi}{6}$, $s=1$:

$$\sin\left(\frac{5\pi}{6}+c\right) = -\frac{1}{2}$$

$$\frac{5\pi}{6}+c = -\frac{\pi}{6}$$

$$c = -\pi$$

$$\therefore s = 2\sin(t-\pi)+2$$



The object oscillates about the starting point which is 2m from *O*, with an amplitude of 2m.

The motion assumes the absence of resistance whereby the amplitude remains constant which is unrealistic in real-life.

15 No Qns

16

(i) Time taken $= \dfrac{40}{r}$ mins

(ii) $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta-25)$

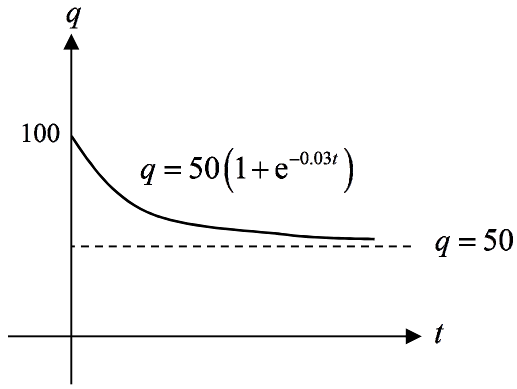
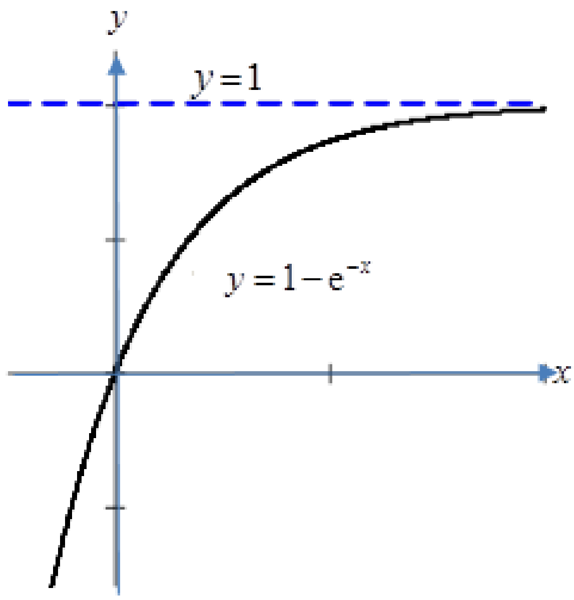
(iii) $\displaystyle\int \frac{1}{\theta-25}\,\mathrm{d}\theta = -\int k\,\mathrm{d}t$

$\ln(\theta-25) = -kt+c$, where c is a constant, since $> 25^{\circ}$C

	$(\theta-25)=\mathrm{e}^{-kt}.\mathrm{e}^{c}$ $\theta=25+A\mathrm{e}^{-kt}$, where A is an arbitrary constant. When $\theta=55$, $\frac{\mathrm{d}\theta}{\mathrm{d}t}=-5$, $k=\frac{1}{6}$ When $t=0$ and $\theta=75$, $A=50$ $\therefore \theta=25+50\mathrm{e}^{-\frac{1}{6}t}$ When $\theta=35$, $35=25+50\mathrm{e}^{-\frac{1}{6}t}$ $\Rightarrow \mathrm{e}^{-\frac{1}{6}t}=\frac{1}{5}$ $\Rightarrow t=-6\ln\left(\frac{1}{5}\right)=6\ln 5$ Total length of time = $\left(\frac{40}{r}+6\ln 5\right)$ mins
17	$\frac{\mathrm{d}v}{\mathrm{d}t}=10-0.02v$ $\int \frac{1}{10-0.02v}\ \mathrm{d}v=\int 1\ \mathrm{d}t$ $\frac{\ln\lvert 10-0.02v\rvert}{-0.02}=t+C$ $\ln\lvert 10-0.02v\rvert=-0.02t+B,\ B=-0.02C$ $\lvert 10-0.02v\rvert=\mathrm{e}^{-0.02t+B}$ $10-0.02v=A\mathrm{e}^{-0.02t},\ \ A=\pm\mathrm{e}^{B}$ $v=\frac{10-A\mathrm{e}^{-0.02t}}{0.02}$ When $t=0$, $v=0$: $A=10$ $v=500(1-\mathrm{e}^{-0.02t})$ (shown) $\frac{\mathrm{d}y}{\mathrm{d}t}=v$ $\frac{\mathrm{d}y}{\mathrm{d}t}=500(1-\mathrm{e}^{-0.02t})$ $y=\int 500(1-\mathrm{e}^{-0.02t})\ \mathrm{d}t$ $=500(t+50\mathrm{e}^{-0.02t})+D$ When $t=0$, $y=0$: $D=-25000$ $y=500(t+50\mathrm{e}^{-0.02t})-25000$ When $v=50$,

	$50 = 500(1-e^{-0.02t})$ $1-e^{-0.02t} = 0.1$ $e^{-0.02t} = 0.9$ $t = -50\ln(0.9)$ (Accept: 5.2680 (5 s.f.)) $y = 500(-50\ln(0.9)+50(0.9)) - 25000$ $= 134$ (to 3 s.f.)
18	**3(a)** $x = 1+\left(\frac{dy}{dx}\right)+\frac{1}{2!}\left(\frac{dy}{dx}\right)^2+\frac{1}{3!}\left(\frac{dy}{dx}\right)^3+\cdots+\frac{1}{r!}\left(\frac{dy}{dx}\right)^r+\cdots.$ $= e^{\frac{dy}{dx}}$ i.e. $x = e^{\frac{dy}{dx}} \Rightarrow \frac{dy}{dx} = \ln x$ Integrate wrt x: $y = \int \ln x \, dx = x\ln x - \int x \cdot \frac{1}{x} \, dx$ $= x\ln x - x + C$ **3(b)** "In" rate: $\frac{H \cdot 1}{T} = \frac{H}{T}$ "Out" rate: px $\frac{dV}{dt} = \frac{dx}{dt} = \frac{H}{T} - px$ (Note $V = 1\times x$ as horizontal base is of unit area) $\therefore \frac{dV}{dt} = \frac{dx}{dt}$ When $x = 1, \frac{dx}{dt} = 0$ $\Rightarrow 0 = \frac{H}{T} - p$ $\therefore p = \frac{H}{T}$ $\frac{dV}{dt} = \frac{dx}{dt} = \frac{H}{T} - \frac{H}{T}x = \frac{H}{T}(1-x)$ i.e. $k = \frac{H}{T}$ $\frac{1}{1-x}\frac{dx}{dt} = \frac{H}{T}$ Integrate wrt t :

	$\int \frac{1}{1-x}\,\mathrm{d}x = \frac{H}{T}\int 1\,\mathrm{d}t$ $\Rightarrow -\ln\lvert 1-x\rvert = \frac{H}{T}t + C$ $1-x = \pm \mathrm{e}^{-C}\mathrm{e}^{-\frac{H}{T}t}$ $\therefore x = 1 - A\mathrm{e}^{-\frac{H}{T}t}$ When $t=0, x=0$ $\Rightarrow 0 = 1-A \quad \therefore A=1$ $\therefore x = 1-\mathrm{e}^{-\frac{H}{T}t}$
19	Rate of salt flowing into tank per minute is $12\times(0.125) = 1.5$ kg Rate of salt flowing out per minute is $\frac{12}{400}\times q = 0.03q$ Therefore, $\frac{\mathrm{d}q}{\mathrm{d}t} = 1.5 - 0.03q$. $\frac{\mathrm{d}q}{\mathrm{d}t} = 1.5 - 0.03q$ $\int \frac{1}{1.5-0.03q}\,\mathrm{d}q = \int 1\,\mathrm{d}t$ $-\frac{1}{0.03}\ln\lvert 1.5-0.03q\rvert = t + C$ $\lvert 1.5-0.03q\rvert = A\mathrm{e}^{-0.03t}$ $1.5-0.03q = B\mathrm{e}^{-0.03t}$ When $t=0$, $q=100$, $1.5-0.03(100) = B$ $B = -1.5$ $1.5-0.03q = -1.5\mathrm{e}^{-0.03t}$ 1.6 kg per litre = 0.16×400 = 64 kg of salt in the tank Thus $1.5-0.03(64) = -1.5\mathrm{e}^{-0.03t}$ $t = 42.4$ min **(3 s.f)** $0.03q = 1.5\left(1+\mathrm{e}^{-0.03t}\right)$ $q = 50\left(1+\mathrm{e}^{-0.03t}\right)$ When t is large, $\mathrm{e}^{-0.03t} \to 0$ Thus, the amount of salt in the tank decreases to 50kg.

	
20	$\frac{d^2y}{dx^2}=-\frac{dy}{dx}$ $\frac{dy}{dx}=-\int\frac{dy}{dx}\,dx$ $\frac{dy}{dx}=-y+C,\ C\in\mathbb{R}$ $\frac{dx}{dy}=\frac{1}{C-y}\Rightarrow x=-\ln\lvert C-y\rvert+D,\ D\in\mathbb{R}$ $\lvert C-y\rvert=e^{-x+D}=Ae^{-x},\ e^{D}=A\in\mathbb{R}^{+}$ $C-y=Be^{-x},\ B\neq 0$ $y=C-Be^{-x}$ When $x=0$, $y=0$ and $\frac{dy}{dx}=1\Rightarrow 0=C-B$ i.e. $C=B$ and $1=0+C\Rightarrow C=1$ Therefore $y=f(x)=1-e^{-x}$ 

Topical Practice: MI solutions

1	Let P_n be the proposition: $\sum_{r=2}^{n}\frac{2}{(r+3)(r+5)}=\frac{11}{30}-\frac{2n+9}{(n+4)(n+5)}$, $n\in\mathbb{Z}^+$, $n\geq 2$. When n = 2, LHS = $\frac{2}{(5)(7)}=\frac{2}{35}$, RHS = $\frac{11}{30}-\frac{2(2)+9}{(6)(7)}=\frac{2}{35}$. Since LHS = RHS, P_2 is true. Assume P_k is true for some $k\in\mathbb{Z}^+$, $k\geq 2$ i.e. $\sum_{r=2}^{k}\frac{2}{(r+3)(r+5)}=\frac{11}{30}-\frac{2k+9}{(k+4)(k+5)}$. Need to show that P_{k+1} is also true. i.e. $\sum_{r=2}^{k+1}\frac{2}{(r+3)(r+5)}=\frac{11}{30}-\frac{2(k+1)+9}{(k+1+4)(k+1+5)}=\frac{11}{30}-\frac{2k+11}{(k+5)(k+6)}$. LHS of $P_{k+1}=\sum_{r=2}^{k+1}\frac{2}{(r+3)(r+5)}$ $=\sum_{r=2}^{k}\frac{2}{(r+3)(r+5)}+\frac{2}{(k+4)(k+6)}$ $=\left[\frac{11}{30}-\frac{2k+9}{(k+4)(k+5)}\right]+\frac{2}{(k+4)(k+6)}$ $=\frac{11}{30}-\frac{(2k+9)(k+6)-2(k+5)}{(k+4)(k+5)(k+6)}$ $=\frac{11}{30}-\frac{2k^2+21k+54-2k-10}{(k+4)(k+5)(k+6)}$ $=\frac{11}{30}-\frac{2k^2+19k+44}{(k+4)(k+5)(k+6)}$ $=\frac{11}{30}-\frac{(k+4)(2k+11)}{(k+4)(k+5)(k+6)}$ $=\frac{11}{30}-\frac{2k+11}{(k+5)(k+6)}$ Since P_2 is true, and P_k is true $\Rightarrow P_{k+1}$ is true , by mathematical induction, P_n is true for all $n\in\mathbb{Z}^+$, $n\geq 2$. $\sum_{r=4}^{n+4}\frac{2}{r(r+2)}=\sum_{r=1}^{n+1}\frac{2}{(r+3)(r+5)}$

	$=\sum_{r=2}^{n+1}\frac{2}{(r+3)(r+5)}+\frac{2}{(4)(6)}$ $=\frac{11}{30}-\frac{2n+11}{(n+5)(n+6)}+\frac{2}{24}$ $=\frac{9}{20}-\frac{2n+11}{(n+5)(n+6)}$ $\sum_{r=4}^{n+4}\frac{1}{(r+1)^2}=\frac{1}{2}\sum_{r=4}^{n+4}\frac{2}{(r+1)^2}$ $<\frac{1}{2}\sum_{r=4}^{n+4}\frac{2}{r(r+2)}$ (Since $(r+1)^2=r^2+2r+1>r^2+2r=r(r+2)$) $=\frac{1}{2}\left[\frac{9}{20}-\frac{2n+11}{(n+5)(n+6)}\right]$ $<\frac{9}{40}$ (since $\frac{2n+11}{(n+5)(n+6)}>0$ for all $n\in Z^+$)
2	Let P_n be the proposition $u_n=2(\ln n-n)$ for $n\in\mathbb{Z}^+$. When n = 1, LHS $=u_1=-2$ (given). RHS $=2(\ln 1-1)=-2$. Since LHS = RHS, $\therefore$ P_1 is true. Assume P_k is true for some $k\in\mathbb{Z}^+$, i.e. $u_k=2(\ln k-k)$. We want to show that P_{k+1} is also true, i.e. $u_{k+1}=2(\ln(k+1)-(k+1))=2\ln(k+1)-2(k+1)$. LHS = $u_{k+1}=u_k+\ln\left(1+\frac{2k+1}{k^2}\right)-2$ $=2(\ln k-k)+\ln\left(1+\frac{2k+1}{k^2}\right)-2$ $=2\ln k-2k+\ln\left(\frac{k^2+2k+1}{k^2}\right)-2$ $=2\ln k-2k+\ln\left((k+1)^2\right)-\ln\left(k^2\right)-2$ $=2\ln(k+1)-2(k+1)$ = RHS. $\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for for all $n\in\mathbb{Z}^+$. $\sum_{r=15}^{n}\frac{e^{u_r}}{r^2}=\sum_{r=15}^{n}\frac{e^{2\ln r-2r}}{r^2}=\sum_{r=15}^{n}\frac{r^2e^{-2r}}{r^2}=\sum_{r=15}^{n}e^{-2r}$.

$$\sum_{r=15}^{n} \mathrm{e}^{-2r} = \sum_{r=15}^{n} \left(\mathrm{e}^{-2}\right)^{r}$$

$$= \frac{\mathrm{e}^{-30}\left(1-\left(\mathrm{e}^{-2}\right)^{n-14}\right)}{1-\mathrm{e}^{-2}}$$

$$= \frac{1-\mathrm{e}^{28-2n}}{\mathrm{e}^{30}-\mathrm{e}^{28}}.$$

$$= \frac{\mathrm{e}^{-28}-\mathrm{e}^{-2n}}{\mathrm{e}^{2}-1}.$$

$\sum_{r=1}^{\infty} \frac{\mathrm{e}^{u_r}}{r^2} = \sum_{r=1}^{\infty} \mathrm{e}^{-2r}$ converges as $\left|\mathrm{e}^{-2}\right| < 1$.

$$\sum_{r=1}^{\infty} \mathrm{e}^{-2r} = \frac{\mathrm{e}^{-2}}{1-\mathrm{e}^{-2}} = \frac{1}{\mathrm{e}^{2}-1}.$$

(Alternatively, students may use part (ii) by taking *n* to infinity, and then adding on the sum of the first 14 terms.)

3

$$\text{R.H.S.} = \frac{1}{3}\left[r(r+1)(r+2)-(r-1)r(r+1)\right]$$

$$= \frac{1}{3}\left[(r+2)-(r-1)\right]r(r+1)$$

$$= \frac{1}{3}(3)r(r+1)$$

$$= r(r+1) = \text{L.H.S.}$$

$$1\times 2 + 2\times 3 + 3\times 4 + \cdots + n(n+1)$$

$$= \sum_{r=1}^{n} r(r+1)$$

$$= \frac{1}{3}\sum_{r=1}^{n}\left[r(r+1)(r+2)-(r-1)r(r+1)\right]$$

$$\begin{aligned} = \frac{1}{3}[&1\cdot 2\cdot 3 - 0\cdot 1\cdot 2 \\ &+2\cdot 3\cdot 4 - 1\cdot 2\cdot 3 \\ &+3\cdot 4\cdot 5 - 2\cdot 3\cdot 4 \\ &\quad \cdots \qquad \cdots \\ &+n(n+1)(n+2)-(n-1)n(n+1)] \end{aligned}$$

$$= \frac{1}{3}\left[n(n+1)(n+2) - 0\cdot 1\cdot 2\right]$$

$$= \frac{1}{3}n(n+1)(n+2)$$

Let P_n be the statement

$\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$, where $n \in \mathbb{Z}^{+}$

When $n=1$, $\text{L.H.S.} = 1\cdot 2\cdot 3 = 6$

	$\text{R.H.S.} = \frac{1}{4} 1 \cdot 2 \cdot 3 \cdot 4 = 6 = \text{L.H.S.}$ $\therefore \mathrm{P}_1$ is true. Suppose P_k is true for some $k \in \mathbb{Z}^+$, i.e. $\sum_{r=1}^{k} r(r+1)(r+2) = \frac{1}{4} k(k+1)(k+2)(k+3)$ R.T.P. P_{k+1} is true, i.e. $\sum_{r=1}^{k+1} r(r+1)(r+2) = \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$ $\text{L.H.S.} = \sum_{r=1}^{k+1} r(r+1)(r+2)$ $= \sum_{r=1}^{k} r(r+1)(r+2) + (k+1)(k+2)(k+3)$ $= \frac{1}{4} k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$ $= \frac{1}{4}(k+1)(k+2)(k+3)(k+4) = \text{R.H.S.}$ $\therefore \mathrm{P}_{k+1}$ is true. Since P_1 is true, P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true. By mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$. $\sum_{r=1}^{n} r(r+1)(r+2)(r+3)$ $= \frac{1}{5} n(n+1)(n+2)(n+3)(n+4)$
4	Given $u_1 = \frac{1}{4}$ and $u_{n+1} = u_n - \frac{4}{n^2(n+1)(n+2)^2}, n \geq 1.$ Let P(n) be the proposition that $u_n = \frac{1}{n^2(n+1)^2}$, $n \geq 1$. Consider P(1): LHS = $u_1 = \frac{1}{4}$ (given) RHS $= \frac{1}{1^2(1+1)^2} = \frac{1}{4} = \text{LHS}$ Hence P(1) is true. Assume P(k) is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \frac{1}{k^2(k+1)^2}$ Consider P(k+1).

	LHS $=u_{k+1}$ $=u_k-\dfrac{4}{k^2(k+1)(k+2)^2}$ $=\dfrac{1}{k^2(k+1)^2}-\dfrac{4}{k^2(k+1)(k+2)^2}$ $=\dfrac{(k+2)^2-4(k+1)}{k^2(k+1)^2(k+2)^2}$ $=\dfrac{k^2}{k^2(k+1)^2(k+2)^2}$ $=\dfrac{1}{(k+1)^2(k+2)^2}$ = RHS Hence P(k) is true $\Rightarrow$ P(k+1) is true. Since P(1) is true & P(k) is true $\Rightarrow$ P(k+1) is true, by mathematical induction, $u_n=\dfrac{1}{n^2(n+1)^2}$ for all $n\geq 1$. $\displaystyle\sum_{n=1}^{N}\frac{1}{n^2(n+1)(n+2)^2}=\sum_{n=1}^{N}\left[\frac{1}{4}(u_n-u_{n+1})\right]$ $=\dfrac{1}{4}\displaystyle\sum_{n=1}^{N}(u_n-u_{n+1})$ $=\begin{cases}\frac{1}{4}[(u_1-\cancel{u_2})\\ +(\cancel{u_2}-\cancel{u_3})\\ \quad\vdots\\ +(\cancel{u_{N-1}}-\cancel{u_N})\\ +(\cancel{u_N}-u_{N+1})]\end{cases}$ $=\dfrac{1}{4}(u_1-u_{N+1})$ $=\dfrac{1}{4}\left(\dfrac{1}{4}-\dfrac{1}{(N+1)^2(N+2)^2}\right)$ $=\dfrac{1}{16}-\dfrac{1}{4(N+1)^2(N+2)^2}$ As $N\to\infty,\dfrac{1}{4(N+1)^2(N+2)^2}\to 0$ So $\displaystyle\sum_{n=1}^{N}\frac{1}{n^2(n+1)(n+2)^2}\to\frac{1}{16}-0=\frac{1}{16}$

	Hence, the series is convergent and $\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)(n+2)^2}=\frac{1}{16}$ $\sum_{n=2}^{N}\frac{1}{n(n^2-1)^2}=\sum_{n=2}^{N}\frac{1}{(n-1)^2n(n+1)^2}$ $=\sum_{k=1}^{N-1}\frac{1}{k^2(k+1)(k+2)^2}$ (Let $k=n-1$) $=\frac{1}{16}-\frac{1}{4(N-1+1)^2(N-1+2)^2}$ $=\frac{1}{16}-\frac{1}{4N^2(N+1)^2}$
5	$\mathrm{f}(r)=(3r-2)(3r+1)=9r^2-3r-2$ $\sum_{r=1}^{n}\mathrm{f}(r)=\sum_{r=1}^{n}\left(9r^2-3r-2\right)$ $=9\sum_{r=1}^{n}r^2-3\sum_{r=1}^{n}r-\sum_{r=1}^{n}2$ $=\frac{3}{2}n(n+1)(2n+1)-\frac{3}{2}n(n+1)-2n$ $=3n^3+3n^2-2n$ $=n\left(3n^2+3n-2\right)$ Hence, $a=3$, $b=3$ and $c=-2$. $S_1=\frac{1}{(3-2)(3+1)}=\frac{1}{4}$ $S_2=\frac{1}{4}+\frac{1}{(6-2)(6+1)}=\frac{2}{7}$ (qed) $S_3=\frac{2}{7}+\frac{1}{(9-2)(9+1)}=\frac{3}{10}$ $S_4=\frac{3}{10}+\frac{1}{(12-2)(12+1)}=\frac{4}{13}$ Conjecture: $S_n=\frac{n}{3n+1}$ Let $\mathrm{P}(n)$ be the proposition that "$S_n=\frac{n}{3n+1}$ for all $n\in\mathbb{Z}^+$". Consider $\mathrm{P}(1)$, $\text{LHS}=S_1=\frac{1}{4}=\text{RHS}$. Therefore $\mathrm{P}(1)$ is true. Assume $\mathrm{P}(k)$ is true for some $k\in\mathbb{Z}^+$, i.e., $S_k=\frac{k}{3k+1}$.

	Want to show $\mathrm{P}(k+1)$ is true, i.e., $S_{k+1}=\dfrac{k+1}{3(k+1)+1}$. Consider $\mathrm{P}(k+1)$, $S_{k+1}=\displaystyle\sum_{r=1}^{k+1}\frac{1}{\mathrm{f}(r)}$ $=\dfrac{k}{3k+1}+\dfrac{1}{(3(k+1)-2)(3(k+1)+1)}$ $=\dfrac{k(3k+4)+1}{(3k+1)(3k+4)}$ $=\dfrac{(3k+1)(k+1)}{(3k+1)(3k+4)}$ $=\dfrac{k+1}{3(k+1)+1}$ $\therefore \mathrm{P}(k+1)$ is true. Since $\mathrm{P}(1)$ is true and $\mathrm{P}(k)$ is true $\Rightarrow \mathrm{P}(k+1)$ is true, by mathematical induction, $\mathrm{P}(n)$ is true for all $n\in\mathbb{Z}^+$.
6	$u_2=\dfrac{4u_1-1}{8}=\dfrac{4(1)-1}{8}=\dfrac{3}{8}$ (or 0.375) $u_3=\dfrac{4u_2-1}{8}=\dfrac{4\left(\frac{3}{8}\right)-1}{8}=\dfrac{1}{16}$ (or 0.0625) $u_n\to l$ as $n\to\infty \Rightarrow u_{n+1}\to l$ as $n\to\infty$ $\because u_{n+1}=\dfrac{4u_n-1}{8}$, as $n\to\infty$, $l=\dfrac{4l-1}{8}$ $8l-4l=-1$ $l=-\dfrac{1}{4}$ Let P_n be the statement $u_n=5\left(\dfrac{1}{2}\right)^{n+1}-\dfrac{1}{4}$ for $n\in\mathbb{Z}^+$. LHS of $P_1=u_1=1$ (given) RHS of $P_1=5\left(\dfrac{1}{2}\right)^2-\dfrac{1}{4}=\dfrac{5}{4}-\dfrac{1}{4}=1$ $\therefore P_1$ is true. Assume that P_k is true for some $k\in\mathbb{Z}^+$, ie. $u_k=5\left(\dfrac{1}{2}\right)^{k+1}-\dfrac{1}{4}$. Need to prove P_{k+1}, ie $u_{k+1}=5\left(\dfrac{1}{2}\right)^{k+2}-\dfrac{1}{4}$

	LHS $= u_{k+1}$ $= \dfrac{4u_k - 1}{8}$ $= \dfrac{1}{2}u_k - \dfrac{1}{8}$ $= \dfrac{1}{2}\left[5\left(\dfrac{1}{2}\right)^{k+1} - \dfrac{1}{4}\right] - \dfrac{1}{8}$ $= 5\left(\dfrac{1}{2}\right)^{k+2} - \dfrac{1}{8} - \dfrac{1}{8}$ $= 5\left(\dfrac{1}{2}\right)^{k+2} - \dfrac{1}{4}$ $=$ RHS $\therefore P_{k+1}$ is true Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, $\therefore$ by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$. $\sum_{n=1}^{N} u_n = \sum_{n=1}^{N}\left(5\left(\dfrac{1}{2}\right)^{n+1} - \dfrac{1}{4}\right)$ $= \dfrac{5\left(\frac{1}{4}\right)\left[1-\left(\frac{1}{2}\right)^{N}\right]}{1-\frac{1}{2}} - \dfrac{1}{4}N$ $= \dfrac{5}{2}\left(1 - \dfrac{1}{2^N}\right) - \dfrac{1}{4}N$ or $\dfrac{5}{2} - \dfrac{5}{2^{N+1}} - \dfrac{1}{4}N$ or equivalent
7	Let P_n be the statement $u_n = \dfrac{1}{3}\left[1 + 8(-2)^n\right]$ for all $n \in \mathbb{Z}_0^+$. LHS of $P_0 = u_0 = 3$ (Given) RHS of $P_0 = \dfrac{1}{3}\left[1 + 8(-2)^0\right] = 3$ $\therefore P_0$ is true. Assume that P_k is true for some $k \in \mathbb{Z}_0^+$, ie $u_k = \dfrac{1}{3}\left[1 + 8(-2)^k\right]$. We want to prove P_{k+1} is true, ie $u_{k+1} = \dfrac{1}{3}\left[1 + 8(-2)^{k+1}\right]$. LHS of $P_{k+1} = u_{k+1} = 1 - 2u_k$ (Given) $= 1 - \dfrac{2}{3}\left[1 + 8(-2)^k\right]$

	$= \frac{1}{3} - \frac{16}{3}(-2)^k$ $= \frac{1}{3} + \frac{8}{3}(-2)(-2)^k$ $= \frac{1}{3}\left[1+8(-2)^{k+1}\right]$ $\therefore P_{k+1}$ is true Since P_0 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}_0^+$. The sequence is **divergent** as $n \to \infty$, $(-2)^n$ does not converge to a finite number
8	$\frac{1}{1+3^{r-1}} - \frac{1}{1+3^r} = \frac{1+3^r-1-3^{r-1}}{(1+3^{r-1})(1+3^r)}$ $= \frac{3^r\left(1-\frac{1}{3}\right)}{(1+3^{r-1})(1+3^r)}$ $= \frac{2(3^r)}{3(1+3^{r-1})(1+3^r)}$ $\sum_{r=1}^{n} \frac{3^r}{(1+3^{r-1})(1+3^r)} = \frac{3}{2}\sum_{r=1}^{n}\left[\frac{1}{1+3^{r-1}} - \frac{1}{1+3^r}\right]$ $= \frac{3}{2}\left[\frac{1}{1+1} - \frac{1}{1+3}\right.$ $+ \frac{1}{1+3} - \frac{1}{1+3^2}$ $+ \frac{1}{1+3^2} - \frac{1}{1+3^3}$ $+ \ldots$ $+ \frac{1}{1+3^{n-3}} - \frac{1}{1+3^{n-2}}$ $+ \frac{1}{1+3^{n-2}} - \frac{1}{1+3^{n-1}}$ $\left. + \frac{1}{1+3^{n-1}} - \frac{1}{1+3^n}\right]$ $= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{1+3^n}\right)$

$$\sum_{r=1}^{\infty}\frac{3^{r+1}}{\left(1+3^{r-1}\right)\left(1+3^{r}\right)}=3\sum_{r=1}^{\infty}\frac{3^{r}}{\left(1+3^{r-1}\right)\left(1+3^{r}\right)}$$

$$=3\left(\frac{3}{2}\left(\frac{1}{2}-0\right)\right)$$

$$=\frac{9}{4}$$

Let P_n be the statement $\sum_{r=1}^{n}\frac{3^{r}}{\left(1+3^{r-1}\right)\left(1+3^{r}\right)}=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{1+3^{n}}\right)\ \forall n\in\mathbb{Z}^{+}$.

When $n=1$, $\text{LHS}=\sum_{r=1}^{1}\frac{3^{r}}{\left(1+3^{r-1}\right)\left(1+3^{r}\right)}=\frac{3^{1}}{\left(1+3^{1-1}\right)\left(1+3^{1}\right)}=\frac{3}{8}$

$\text{RHS}=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{1+3^{1}}\right)=\frac{3}{8}=\text{LHS}$

$\therefore P_1$ is true

Assume P_k true for some $k\in\mathbb{Z}^{+}$ ie $\sum_{r=1}^{k}\frac{3^{r}}{\left(1+3^{r-1}\right)\left(1+3^{r}\right)}=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{1+3^{k}}\right)$

To prove P_{k+1} is true $\sum_{r=1}^{k+1}\frac{3^{r}}{\left(1+3^{r-1}\right)\left(1+3^{r}\right)}=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{1+3^{k+1}}\right)$

When $n=k+1$,

$$\text{LHS}=\sum_{r=1}^{k+1}\frac{3^{r}}{\left(1+3^{r-1}\right)\left(1+3^{r}\right)}$$

$$=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{1+3^{k}}\right)+\frac{3^{k+1}}{\left(1+3^{k}\right)\left(1+3^{k+1}\right)}$$

$$=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{1+3^{k}}+\frac{2\cdot 3^{k}}{\left(1+3^{k}\right)\left(1+3^{k+1}\right)}\right)$$

$$=\frac{3}{2}\left(\frac{1}{2}-\frac{1+3^{k+1}-2\cdot 3^{k}}{\left(1+3^{k}\right)\left(1+3^{k+1}\right)}\right)$$

$$=\frac{3}{2}\left(\frac{1}{2}-\frac{1+3^{k}}{\left(1+3^{k}\right)\left(1+3^{k+1}\right)}\right)$$

$$=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{1+3^{k+1}}\right)=\text{RHS}$$

P_k true $\Rightarrow P_{k+1}$ true.

Since P_1 true and P_k true $\Rightarrow P_{k+1}$ true, P_n is true $\forall n\in\mathbb{Z}^{+}$.

9	$u_1 = 3 = \dfrac{2^2 - 1}{2 - 1}$. $u_2 = 3 - \dfrac{2}{3} = \dfrac{7}{3} = \dfrac{2^3 - 1}{2^2 - 1}$. $u_3 = 3 - \dfrac{2}{\left(\frac{7}{3}\right)} = \dfrac{15}{7} = \dfrac{2^4 - 1}{2^3 - 1}$. Let P_n be the statement that $u_n = \dfrac{2^{n+1} - 1}{2^n - 1}$ for $n \geq 1$. When $n = 1$, LHS = $u_1 = 3$ RHS $= \dfrac{2^2 - 1}{2 - 1} = \dfrac{3}{1} = 3 =$ LHS (shown). $\therefore \mathrm{P}_1$ is true. Assume P_k is true some $k \in \mathbb{Z}^+$, i.e., $u_k = \dfrac{2^{k+1} - 1}{2^k - 1}$. To prove P_{k+1} is also true, i.e. $u_{k+1} = \dfrac{2^{k+2} - 1}{2^{k+1} - 1}$. LHS = u_{k+1} $= 3 - \dfrac{2}{u_k}$ $= 3 - \dfrac{2}{\left(\frac{2^{k+1} - 1}{2^k - 1}\right)}$ $= 3 - \dfrac{2\left(2^k - 1\right)}{2^{k+1} - 1}$ $= \dfrac{3.2^{k+1} - 3 - 2^{k+1} + 2}{2^{k+1} - 1}$ $= \dfrac{2.2^{k+1} - 1}{2^{k+1} - 1}$ $= \dfrac{2^{k+2} - 1}{2^{k+1} - 1}$ $=$ RHS $\therefore \mathrm{P}_k$ is true $\Rightarrow \mathrm{P}_{k+1}$ is true. Since P_1 is true, and P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

	$u_1u_2u_3...u_n=\left(\frac{2^2-1}{2-1}\right)\left(\frac{2^3-1}{2^2-1}\right)\left(\frac{2^4-1}{2^3-1}\right)...\left(\frac{2^{n+1}-1}{2^n-1}\right)$ $=\left(\frac{2^{n+1}-1}{2-1}\right)$ $=2^{n+1}-1\to\infty$ as $n\to\infty$. Hence, the limit does not exist.
10	Let P_n be the statement $\sum_{r=1}^{n}\cos(2r\theta)=\frac{\sin(2n+1)\theta-\sin\theta}{2\sin\theta}$ for $n\in\mathbb{Z}^+$ When n = 1, LHS $=\sum_{r=1}^{1}\cos(2r\theta)=\cos 2\theta$ RHS $=\frac{\sin3\theta-\sin\theta}{2\sin\theta}=\frac{2\cos2\theta\sin\theta}{2\sin\theta}=\cos2\theta$ $\therefore P_1$ is true. Assume that P_k true for some $k\in Z^+$, $k\geq 1$. i.e. $\sum_{r=1}^{k}\cos(2r\theta)=\frac{\sin(2k+1)\theta-\sin\theta}{2\sin\theta}$ For $n=k+1$, we want to prove $\sum_{r=1}^{k+1}\cos(2r\theta)=\frac{\sin(2k+3)\theta-\sin\theta}{2\sin\theta}$ LHS $=\sum_{r=1}^{k+1}\cos(2r\theta)=\sum_{r=1}^{k}\cos(2r\theta)+\cos(2k+2)\theta$ $=\frac{\sin(2k+1)\theta-\sin\theta}{2\sin\theta}+\cos(2k+2)\theta$ $=\frac{\sin(2k+1)\theta-\sin\theta+2\cos(2k+2)\theta\sin\theta}{2\sin\theta}$ $=\frac{\sin(2k+1)\theta-\sin\theta+\sin(2k+3)\theta-\sin(2k+1)\theta}{2\sin\theta}$ $=\frac{\sin(2k+3)\theta-\sin\theta}{2\sin\theta}$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by the principle of Mathematical Induction, P_n is true for all $n\in\boldsymbol{Z}^+$. $\sum_{r=1}^{n}\cos(r\pi)=\sum_{r=1}^{n}\cos\left(2r\cdot\frac{\pi}{2}\right)=\frac{\sin(2n+1)\frac{\pi}{2}-\sin\frac{\pi}{2}}{2\sin\frac{\pi}{2}}=\frac{(-1)^n-1}{2}$

11	Let P_N be the statement $\sum_{n=0}^{N}\frac{3n+2}{(n+1)!3^{n+1}}=1-\frac{1}{(N+1)!3^{N+1}}$ for all $N\geq 0$. When $N=0$, $\text{LHS}=\sum_{n=0}^{0}\frac{3n+2}{(n+1)!3^{n+1}}=\frac{3(0)+2}{(0+1)!3^{0+1}}=\frac{2}{3}$ $\text{RHS}=1-\frac{1}{(0+1)!3^{0+1}}=\frac{2}{3}=\text{LHS}$ Thus, P_0 is true. Assume P_k is true for some $k\geq 0$, i.e. $\sum_{n=0}^{k}\frac{3n+2}{(n+1)!3^{n+1}}=1-\frac{1}{(k+1)!3^{k+1}}$. Consider P_{k+1}: To show $\sum_{n=0}^{k+1}\frac{3n+2}{(n+1)!3^{n+1}}=1-\frac{1}{(k+2)!3^{k+2}}$ $\text{LHS of } \mathrm{P}_{k+1}=\sum_{n=0}^{k+1}\frac{3n+2}{(n+1)!3^{n+1}}$ $=\sum_{n=0}^{k}\frac{3n+2}{(n+1)!3^{n+1}}+\frac{3(k+1)+2}{(k+2)!3^{k+2}}$ $=1-\frac{1}{(k+1)!3^{k+1}}+\frac{3k+5}{(k+2)!3^{k+2}}$ $=1-\frac{(k+2)(3)}{(k+1)!(k+2)3^{k+1}\cdot 3}+\frac{3k+5}{(k+2)!3^{k+2}}$ $=1-\frac{3k+6}{(k+2)!3^{k+2}}+\frac{3k+5}{(k+2)!3^{k+2}}$ $=1+\frac{3k+5-3k-6}{(k+2)!3^{k+2}}$ $=1-\frac{1}{(k+2)!3^{k+2}}=\text{RHS of } \mathrm{P}_{k+1}$ P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true. Since P_0 is true, and P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true, by mathematical induction, P_N is true for all $N\geq 0$.
12	Let P_n be the statement $\sum_{r=1}^{n}\sin(2r\theta)=\frac{\cos\theta-\cos\left[(2n+1)\theta\right]}{2\sin\theta}$, $n=1,2,3,\ldots$ $n=1$:

$$\text{LHS} = \sum_{r=1}^{1} \sin(2r\theta) = \sin(2\theta)$$

$$\text{RHS} = \frac{\cos\theta - \cos(3\theta)}{2\sin\theta}$$

$$= \frac{-2\sin\left[\frac{1}{2}(\theta+3\theta)\right]\sin\left[\frac{1}{2}(\theta-3\theta)\right]}{2\sin\theta}$$

$$= \frac{-2\sin(2\theta)\sin(-\theta)}{2\sin\theta}$$

$$= \sin(2\theta)$$

$\text{LHS} = \text{RHS}$, $\therefore P_1$ is true.

Assume P_k is true for some $k = 1, 2, 3......$

i.e. $\displaystyle\sum_{r=1}^{k} \sin(2r\theta) = \frac{\cos\theta - \cos\left[(2k+1)\theta\right]}{2\sin\theta}$

To show P_{k+1} is true.

i.e. $\displaystyle\sum_{r=1}^{k+1} \sin(2r\theta) = \frac{\cos\theta - \cos\left[(2k+3)\theta\right]}{2\sin\theta}$

$$\text{LHS} = \sum_{r=1}^{k+1} \sin(2r\theta)$$

$$= \sum_{r=1}^{k} \sin(2r\theta) + \sin\left[(2k+2)\theta\right]$$

$$= \frac{\cos\theta - \cos\left[(2k+1)\theta\right]}{2\sin\theta} + \sin\left[(2k+2)\theta\right]$$

$$= \frac{\cos\theta - \cos\left[(2k+1)\theta\right] + 2\sin\left[(2k+2)\theta\right]\sin\theta}{2\sin\theta}$$

$$= \frac{\cos\theta - \cos\left[(2k+1)\theta\right] - \cos\left[(2k+3)\theta\right] + \cos\left[(2k+1)\theta\right]}{2\sin\theta}$$

$$= \frac{\cos\theta - \cos\left[(2k+3)\theta\right]}{2\sin\theta} = \text{RHS}$$

$\therefore P_{k+1}$ is true

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is also true, by the principle of Mathematical Induction, P_n is true for all $n = 1, 2, 3......$

Note:

$$2\sin\theta\sin(2k+2)\theta$$
$$= -\cos\left[(2k+3)\theta\right] + \cos\left[(-2k-1)\theta\right]$$
$$= -\cos\left[(2k+3)\theta\right] + \cos[-(2k+1)\theta]$$
$$= -\cos\left[(2k+3)\theta\right] + \cos[(2k+1)\theta]$$

Factor formula should be applied on $2\sin\theta\sin(2k+2)\theta$ to avoid the negative angle.

	$\sum_{r=1}^{n}\sin(2r\theta)=\frac{\cos\theta-\cos(2n+1)\theta}{2\sin\theta}$ $\sum_{r=1}^{n}2\sin(r\theta)\cos(r\theta)=\frac{\cos\theta-\cos(2n+1)\theta}{2\sin\theta}$ $2(\sin\theta\cos\theta+\sin2\theta\cos2\theta+...+\sin n\theta\cos n\theta)=\frac{\cos\theta-\cos(2n+1)\theta}{2\sin\theta}$ $\sin\theta\cos\theta+\sin2\theta\cos2\theta+...+\sin n\theta\cos n\theta=\frac{\cos\theta-\cos(2n+1)\theta}{4\sin\theta}$
13	Let P_n be the statement $\sum_{r=2}^{n}\frac{2^r(r-1)}{r(r+1)}=\frac{2^{n+1}}{n+1}-2$, $n\in\mathbb{Z}^+$, $n\geq 2$. When $n=2$, LHS $=\frac{2^2(2-1)}{2(2+1)}=\frac{4}{6}=\frac{2}{3}$ RHS $=\frac{2^{2+1}}{2+1}-2=\frac{8}{3}-2=\frac{2}{3}$ Since LHS = RHS, P_2 is true. Assume that P_k is true for some $k\in\mathbb{Z}^+$, $k\geq 2$, i.e. $\sum_{r=2}^{k}\frac{2^r(r-1)}{r(r+1)}=\frac{2^{k+1}}{k+1}-2$. To show P_{k+1} is true, i.e. $\sum_{r=2}^{k+1}\frac{2^r(r-1)}{r(r+1)}=\frac{2^{k+2}}{k+2}-2$. LHS $=\sum_{r=2}^{k+1}\frac{2^r(r-1)}{r(r+1)}=\sum_{r=2}^{k}\frac{2^r(r-1)}{r(r+1)}+\frac{2^{k+1}(k+1-1)}{(k+1)(k+1+1)}$ $=\frac{2^{k+1}}{k+1}-2+\frac{2^{k+1}(k)}{(k+1)(k+2)}$ $=\frac{2^{k+1}(k+2)+2^{k+1}(k)}{(k+1)(k+2)}-2$ $=\frac{2^{k+1}(2k+2)}{(k+1)(k+2)}-2$ $=\frac{2(2^{k+1})(k+1)}{(k+1)(k+2)}-2$ $=\frac{2^{k+2}}{k+2}-2=\text{RHS}$ $\therefore$ P_{k+1} is true whenever P_k is true. Since P_2 is true, by mathematical induction, P_n is true for all $n\in\mathbb{Z}^+$, $n\geq 2$.

	$\sum_{r=3}^{2N} \frac{2^r(r-1)}{r(r+1)} = \sum_{r=2}^{2N} \frac{2^r(r-1)}{r(r+1)} - \frac{2^2(2-1)}{2(2+1)}$ $= \left(\frac{2^{2N+1}}{2N+1} - 2\right) - \frac{2}{3} = \frac{2^{2N+1}}{2N+1} - \frac{8}{3}$
14	Let P_n denote the statement $u_n = \frac{n+1}{n!}$ for $n \geq 0, n \in \mathbb{Z}$. When $n=0$, $\text{LHS} = u_0 = 1$ (given by question) $\text{RHS} = \frac{0+1}{0!} = 1 =$ LHS Therefore, P_0 is true. Assume that P_k is true for some $k \in \mathbb{Z},\ k \geq 0$. i.e. $u_k = \frac{k+1}{k!}$. We need to show that P_{k+1} is also true. i.e. $u_{k+1} = \frac{k+2}{(k+1)!}$. $u_{k+1} = u_k - \frac{k^2+k-1}{(k+1)!}$ $= \frac{k+1}{k!} - \frac{k^2+k-1}{(k+1)!}$ $= \frac{(k+1)^2 - k^2 - k + 1}{(k+1)!}$ $= \frac{k^2+2k+1-k^2-k+1}{(k+1)!}$ $= \frac{k+2}{(k+1)!}$ Therefore P_{k+1} is also true once P_k is true. Since P_0 is true, and P_k is true implies that P_{k+1} is also true, by Mathematical Induction, P_n is true for $n \in \mathbb{Z},\ n \geq 0$.

	$\sum_{r=1}^{N}\frac{(r+1)^2+r}{(r+2)!}=\sum_{j-1=1}^{N}\frac{j^2+j-1}{(j+1)!}$ (replacing r by $j-1$) $=\sum_{j=2}^{N+1}\frac{j^2+j-1}{(j+1)!}$ $=\sum_{j=2}^{N+1}\left(u_j-u_{j+1}\right)$ $=u_2-u_3$ $+u_3-u_4$ $+u_4-u_5$ $+\dots$ $+u_{N-1}-u_N$ $+u_N-u_{N+1}$ $+u_{N+1}-u_{N+2}$ $=u_2-u_{N+2}$ $=\frac{3}{2}-\frac{N+3}{(N+2)!}$
15	Let P_n be the statement $\sin x+\sin 11x+\sin 21x+\ldots+\sin(10n+1)x=\frac{\cos 4x-\cos(10n+6)x}{2\sin 5x}$ for $n=0,1,2,3,\ldots$ When $n=0$, $\text{LHS}=\sin x$ $\text{RHS}=\frac{\cos 4x-\cos 6x}{2\sin 5x}$ $=\frac{-2\sin 5x\sin(-x)}{2\sin 5x}=\frac{2\sin 5x\sin x}{2\sin 5x}$ $=\sin x=\text{LHS}$ Hence P_0 is true. Assume P_k is true for some $k\in\{0,1,2,3,\ldots\}$, i.e. $\sin x+\sin 11x+\sin 21x+\ldots+\sin(10k+1)x=\frac{\cos 4x-\cos(10k+6)x}{2\sin 5x}$. To prove P_{k+1} is true, i.e. $\sin x+\sin 11x+\ldots+\sin(10(k+1)+1)x=\frac{\cos 4x-\cos(10k+16)x}{2\sin 5x}$.

	$\text{LHS} = \sin x + \sin 11x + ... + \sin(10k+1)x + \sin(10k+11)x$ $= \dfrac{\cos 4x - \cos\left(10k+6\right)x}{2\sin 5x} + \sin\left(10k+11\right)x$ $= \dfrac{\cos 4x - \cos\left(10k+6\right)x + 2\sin\left(10k+11\right)x\sin 5x}{2\sin 5x}$ $= \dfrac{\cos 4x - \cos\left(10k+6\right)x + \cos\left(10k+6\right)x - \cos\left(10k+16\right)x}{2\sin 5x}$ $= \dfrac{\cos 4x - \cos\left(10k+16\right)x}{2\sin 5x} = RHS$ Hence P_k is true implies P_{k+1} is true. Since P_0 is true, and P_k is true implies P_{k+1} is true, by Mathematical induction, P_n is true for all $n \in \{0,1,2,3,..\}$
16	Let P_n be the statement that $u_n = 4-(n+2)\left(\dfrac{1}{2}\right)^{n-1}$ for $n \in \mathbb{Z}^+$ For $n = 1$, LHS = $u_1 = 1$ RHS = $4-(1+2)\left(\dfrac{1}{2}\right)^{0}$ $= 4-3$ $= 1$ = LHS Hence, P_1 is true. Assume P_k is true for some $k \in \mathbb{Z}^+$, that is $u_k = 4-(k+2)\left(\dfrac{1}{2}\right)^{k-1}$. To show that P_{k+1} is true, that is $u_{k+1} = 4-(k+3)\left(\dfrac{1}{2}\right)^{k}$ LHS = u_{k+1} $= u_k + (k+1)\left(\dfrac{1}{2}\right)^{k}$ (from recurrence relation) $= 4-(k+2)\left(\dfrac{1}{2}\right)^{k-1} + (k+1)\left(\dfrac{1}{2}\right)^{k}$ (from P_k statement) $= 4-\left[2(k+2)-(k+1)\right]\left(\dfrac{1}{2}\right)^{k}$ $= 4-\left(k+3\right)\left(\dfrac{1}{2}\right)^{k}$ Hence P_{k+1} is true. Since P_1 is true and P_{k+1} is true when P_k is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
17	Let P_n be the statement "$S_n = \dfrac{1}{4} + \dfrac{3^n}{4}\left(2n-1\right)$" for all positive integers n. When $n = 1$, LHS = $S_1 = 1$, (Accept: LHS = $1(3^0) = 1$)

$$\text{RHS} = \frac{1}{4} + \frac{3}{4}(1) = 1$$

LHS = RHS, P_1 is true.

Assume that P_k is true for some positive integer k,

i.e. $S_k = \frac{1}{4} + \frac{3^k}{4}(2k-1)$.

To prove that P_{k+1} is true: i.e. $S_{k+1} = \frac{1}{4} + \frac{3^{k+1}}{4}\left[2(k+1)-1\right]$.

$$\begin{aligned}
S_{k+1} &= \sum_{r=1}^{k+1} r(3)^{r-1} \\
&= \sum_{r=1}^{k} r(3)^{r-1} + (k+1)3^k \\
&= \frac{1}{4} + \frac{3^k}{4}(2k-1) + (k+1)3^k \\
&= \frac{1}{4} + \frac{3^k}{4}(2k-1+4k+4) \\
&= \frac{1}{4} + \frac{3^k}{4}(6k+3) \\
&= \frac{1}{4} + \frac{3^k}{4}(3)(2k+1) \\
&= \frac{1}{4} + \frac{3^{k+1}}{4}[2(k+1)-1)]
\end{aligned}$$

$\therefore$ P_k is true $\Longrightarrow$ P_{k+1} is true.

Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all positive integers n, i.e. $S_n = \frac{1}{4} + \frac{3^n}{4}(2n-1)$

Method 1:

$$\begin{aligned}
u_{n+1} - u_n &= (n+1)3^n - n3^{n-1} \\
&= 3^{n-1}\left[3n+3-n\right] \\
&= 3^{n-1}\left[2n+3\right]
\end{aligned}$$

Hence recurrence relation is:

$$\begin{aligned}
u_{n+1} &= u_n + 3^{n-1}\left[2n+3\right] \\
&= 3u_n + 3^n
\end{aligned}$$

Method 2:

$$\begin{aligned}
u_{n+1} &= (n+1)3^n \\
&= 3(n3^{n-1}) + 3^n \\
&= 3u_n + 3^n
\end{aligned}$$

	Hence recurrence relation is: $u_{n+1} = u_n + 3^{n-1}\left[2n+3\right]$ $= 3u_n + 3^n$ **Method 3:** $\dfrac{u_{n+1}}{u_n} = \dfrac{(n+1)3^n}{n3^{n-1}}$ $= 3\left[1+\dfrac{1}{n}\right]$ Hence recurrence relation is: $u_{n+1} = 3\left[1+\dfrac{1}{n}\right]u_n = 3u_n + 3^n$ The sequence increases and diverges.
18	Let P_n be the statement $u_n = \dfrac{12a}{(n+2)(n+3)}$ for $n \ge 1$. When n = 1, LHS = $u_1 = a$ (given) RHS $= \dfrac{12a}{(1+2)(1+3)} = a$ = LHS $\therefore$ P_1 is true. Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \dfrac{12a}{(k+2)(k+3)}$. When $n = k + 1$, LHS $= u_{k+1} = \dfrac{k+2}{k+4}u_k$ $= \dfrac{k+2}{k+4}\dfrac{12a}{(k+2)(k+3)}$ $= \dfrac{12a}{(k+4)(k+3)}$ $= \dfrac{12a}{((k+1)+2)((k+1)+3)}$ = RHS $\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$. As $n \to \infty$, $n(n+2)\dfrac{u_n}{u_1} = n(n+2)\dfrac{1}{a}\dfrac{12a}{(n+2)(n+3)} = \dfrac{12n}{n+3} = \dfrac{12}{1+\dfrac{3}{n}} \to 12$
19	Let P_n be the statement: $w_n = an + (n-1)$, $n \in \mathbb{Z}^+$.

	LHS of $P_1 = w_1 = a$ (given) RHS of $P_1 = a(1) + (1-1) = a$ $\therefore P_1$ is true. Assume P_k is true for some $k \in \mathbb{Z}^+$ i.e. $w_k = ak + (k-1)$ We want to show P_{k+1} is true i.e. $w_{k+1} = a(k+1) + k$ LHS of $P_{k+1} = w_{k+1}$ $= \frac{1}{k}[(k+1)w_k + 1]$ $= \frac{1}{k}\{(k+1)[ak + (k-1)] + 1\}$ $= a(k+1) + \frac{(k+1)(k-1)+1}{k}$ $= a(k+1) + \frac{k^2}{k}$ $= a(k+1) + k$ $=$ RHS of P_{k+1} $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true Since we have shown that (1) P_1 is true and (2) P_k is true $\Rightarrow P_{k+1}$ is true. $\therefore$ By mathematical induction, P_n is true for all positive integers n.
20	Let P_n be the statement, $\sum_{r=1}^{n} \frac{r(2^r)}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$ for $n \in \mathbb{Z}^+$ When $n = 1$, LHS $= \frac{1(2^1)}{(1+2)!} = \frac{2}{6}$ RHS $= 1 - \frac{2^2}{(1+2)!} = 1 - \frac{4}{6} = \frac{2}{6} =$ LHS i.e. P_1 is true Assume that P_k is true for some $k \in \mathbb{Z}^+$ i.e. $\sum_{r=1}^{k} \frac{r(2^r)}{(r+2)!} = 1 - \frac{2^{k+1}}{(k+2)!}$ Show that P_{k+1} is also true i.e. $\sum_{r=1}^{k+1} \frac{r(2^r)}{(r+2)!} = 1 - \frac{2^{k+2}}{(k+3)!}$ LHS $= \sum_{r=1}^{k} \frac{r(2^r)}{(r+2)!} + \frac{(k+1)2^{k+1}}{(k+3)!}$ $= 1 - \frac{2^{k+1}}{(k+2)!} + \frac{(k+1)2^{k+1}}{(k+3)!}$

	$= 1 - \dfrac{(k+3)2^{k+1} - (k+1)2^{k+1}}{(k+3)!}$ $= 1 - \dfrac{2^{k+1}(k+3-k-1)}{(k+3)!}$ $= 1 - \dfrac{2^{k+1}(2)}{(k+3)!} = 1 - \dfrac{2^{k+2}}{(k+3)!} = \text{RHS}$ i.e. P_{k+1} is true Therefore by mathematical induction, P_n is true for $n \in \mathbb{Z}^+$

Topical Practice Questions: Sequences and Series Solutions (2015 Prelim)

1	To find least n such that $T_n = 15 + (n-1)(0.5) > 39$. $0.5n > 24.5$ $n > 49.$ Therefore, on the 50th day of rental the owner will first have to pay the artist more than \$39 as the daily rental rate. $\mathrm{f}(n) = \dfrac{12000}{4n^2+4n-3}.$ $\mathrm{f}(n) = \dfrac{12000}{(2n-1)(2n+3)}$ $= \dfrac{3000}{2n-1} - \dfrac{3000}{2n+3}.$ $\sum_{r=1}^{r=m} \mathrm{f}(r) = 3000 \sum_{r=1}^{r=m} \dfrac{1}{2r-1} - \dfrac{1}{2r+3}$ $= 3000 \begin{pmatrix} \frac{1}{1} & -\cancel{\frac{1}{5}} \\ +\frac{1}{3} & -\cancel{\frac{1}{7}} \\ +\cancel{\frac{1}{5}} & -\cancel{\frac{1}{9}} \\ \vdots & \\ +\cancel{\frac{1}{2m-5}} & -\cancel{\frac{1}{2m-1}} \\ +\cancel{\frac{1}{2m-3}} & -\frac{1}{2m+1} \\ +\cancel{\frac{1}{2m-1}} & -\frac{1}{2m+3} \end{pmatrix}$ $= 3000\left(1 + \dfrac{1}{3} - \dfrac{1}{2m+1} - \dfrac{1}{2m+3}\right)$ $= 3000\left(\dfrac{4}{3} - \dfrac{4m+4}{(2m+1)(2m+3)}\right)$ $= 4000 - \dfrac{12000(m+1)}{(2m+1)(2m+3)}.$ Given: $\dfrac{k}{2}\left[2(15) + (k-1)(0.5)\right] < 4000 - \dfrac{12000(k+1)}{(2k+1)(2k+3)}$.

Considering

$$Y_1 = \frac{k}{2}\left[2(15)+(k-1)(0.5)\right]-4000+\frac{12000(k+1)}{(2k+1)(2k+3)},$$

k	Y_1
98	-123.2
99	-59.5
100	4.7037

By GC, largest value of k is 99.

2 Let n be the number of intervals between the first hook to the last.

$$\frac{n}{2}\left[2(50)+(n-1)(-2)\right] \le 500$$

$$n^2-51n+500 \ge 0$$

$$n \le 13.2 \text{ or } n \ge 37.8$$

Thus smallest $n=13$

$n \ge 37.8$ is rejected since any additional intervals beyond 13 will give a total length greater than 500.

Since the smallest number of intervals is 13, number of hooks = 13+1 = 14

$$80+80(0.9)+80(0.9)^2+\ldots+80(0.9)^n = \frac{80\left(1-0.9^n\right)}{1-0.9}$$

$$\frac{80\left(1-0.9^n\right)}{1-0.9} > 500$$

$$0.9^n < 0.375$$

$$n\ln(0.9) < \ln(0.375)$$

$$n > 9.31$$

Smallest n = 10

Length of the ribbon between the n^{th} and $(n+1)^{\text{th}}$ hook must satisfy the condition:

$$80(0.9)^{n-1} \ge 50-2(n-1)$$

Length of the ribbon between the n^{th} and $(n+1)^{\text{th}}$ hook must satisfy the condition:

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3 Odd day duration forms AP with $a=20$, $d=9, n=38$

$$U_{38} = 20+(38-1)9$$
$$= 353$$

Odd day duration forms AP with $a=20$, $d=9, n=38$

	$U_{38} = 20 + (38-1)9$ $= 353$ 75^{th} day onwards forms GP with $a = 353$, $r = 0.8$, $n = 26$ $U_{26} = 353(0.8)^{26-1}$ $= 1.33$ Duration of exercise is too low to be effective towards the end portion of the 100 days.
4	$v_{n+1} + 2 = \frac{1}{2}(v_n + 2) + 1$ $v_{n+1} + 2 = \frac{1}{2}v_n + 1 + 1$ $v_{n+1} = \frac{1}{2}v_n$ Since $\frac{v_{n+1}}{v_n} = \frac{1}{2} = \text{constant}$, sequence V is a GP. **<u>Method ①:</u>** $v_1 = u_1 - 2 = -1$ $v_n = (-1)\left(\frac{1}{2}\right)^{n-1}$ when $n \to \infty$, $v_n = (-1)\left(\frac{1}{2}\right)^{n-1} \to 0$. So the limit of sequence V is 0. Since $u_n = v_n + 2$, when $n \to \infty$, $u_n \to 2$. So the limit of sequence U is 2. **<u>Method ②:</u>** Let the limit of U be L. According to the given recurrence relation, when $n \to \infty$, $$L = \frac{1}{2}L + 1$$ So $L = 2$, i.e. the limit of sequence U is 2. Since $v_n = u_n - 2$, when $n \to \infty$, $v_n \to 2 - 2 = 0$. So the limit of sequence V is 0. For V, sum to infinity $= \frac{-1}{1-\frac{1}{2}} = -2$. $\sum_{r=1}^{n} v_r = \frac{(-1)\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = -2\left(1-\left(\frac{1}{2}\right)^n\right) = 2\left(\frac{1}{2}\right)^n - 2$ $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}(v_r + 2) = \sum_{r=1}^{n} v_r + 2n = 2\left(\frac{1}{2}\right)^n - 2 + 2n$ When $n \to \infty$, $2\left(\frac{1}{2}\right)^n \to 0$, $2n \to \infty$,

	thus $\sum_{r=1}^{n} u_r \to \infty$, i.e., sum to infinity doesn't exist.
5	Area covered by weed at the end of the first week $= 0.9(500+80)$ $= 0.9(500)+80(0.9)\ \text{m}^2$ $= 0.9(0.9(500)+80(0.9)+80)$ Area covered by weed at the end of the second week $= 0.9^2(500)+80(0.9+0.9^2)$ $= 541.8\quad \text{m}^2$ Area covered by weed at the end of the nth week $= 0.9^n(500)+80(0.9+0.9^2+...+0.9^n)$ $= 0.9^n(500)+0.9(80)\left(\frac{1-0.9^n}{0.1}\right)$ $= 0.9^n(500)+720\left(1-0.9^n\right)\ \text{m}^2$ Therefore, $k=720$. As $n\to\infty, 0.9^n \to 0$ So $0.9^n(500)+720(1-0.9^n) \to 0(500)+720(1-0)=720$ Hence the area covered with weed at the end of the week in the long run is 720 m^2. Change in area covered with weed in the nth week $= 80-(50+10(n-1))$ $= 40-10n\ \text{m}^2$ Area covered with weed at the end of the nth week $= 500+\sum_{r=1}^{n}(40-10r)$ $= 500+\sum_{r=1}^{n}40-10\sum_{r=1}^{n}r$ $= 500+40n-10\left[\frac{n}{2}(1+n)\right]$ $= 500+35n-5n^2\ \text{m}^2$
6	Amount after 16 days = $1000\times\left(\frac{1}{2}\right)^2 = 250$ mg Amount of I-131 on Day 49 $= \left[\left[1000\times\left(\frac{1}{2}\right)^2+1000\right]\times\left(\frac{1}{2}\right)^2+1000\right]\times\left(\frac{1}{2}\right)^2+1000\cdots\cdots(*)$ $= 1000\left[1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3\right]$

	$=1000\left[\dfrac{1-\left(\frac{1}{4}\right)^4}{1-\frac{1}{4}}\right]=1328.125 \text{ mg}=1328 \text{ mg } (\text{nearest mg})$ $S_\infty=\dfrac{1000}{1-\frac{1}{4}}=1333.33<1334 \text{ mg}$ Amount of I-131 will never exceed 1334 mg. Amount of I-125 on Day 121 $=1000\times\left(\frac{1}{2}\right)^2=250 \text{ mg}$ I-131 is added on Day 17, …, 113, $\Rightarrow$ total 7 times Amount of I-131 on Day 121 $=1000\left[1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\cdots+\left(\frac{1}{4}\right)^6\right]\times\frac{1}{2}=500\left[\dfrac{1-\left(\frac{1}{4}\right)^7}{1-\frac{1}{4}}\right]=666.626 \text{ mg}$ Total amount of radioisotopes $=250+666.626=917 \text{ mg} (\text{nearest mg})$
7	Stage 1: 6 = 6(1) Stage 2: 12 = 6(2) Stage 3: 18 = 6(3) Stage 4: 24 = 6(4) ….. Stage 12: 6(12) A.P. : $6, 12, 18 \ldots\ldots\ldots, 72, \ldots\ldots\ldots\ldots, 6N$ $T_1=a=6 \qquad d=6$ Distance travelled from the 1st stage till the 12th stage $=S_{12}=\frac{6}{2}(12)(1+12)=468$ Stage N: 6n Distance travelled from the 1st stage till the nth stage $=S_n=\frac{6}{2}(n)(1+n)=3n(n+1)$ $S_n\le 10000$ $3n(n+1)\le 10000$ <u>Method 1:</u> $3n^2+3N-10000\le 0$ $\therefore -58.24\le n\le 57.24$

	Greatest $n = 57$. Method 2: (Use table in GC) Stage 1: 3+3 = 6(1) Stage 2: 6+6 = 6(2) Stage 3: 12+12 = 6(4) Stage 4: 24+24 = 6(8) ….. Stage N: $6(2^{N-1})$ G.P. $T_1 = a = 6$ $r = 2$ $S_N = \frac{6\left(2^N - 1\right)}{2-1}$ $S_N = 6\left(2^N - 1\right)$ Suppose $S_N \le 10000$ $6\left(2^N - 1\right) \le 10000$ $\left(2^N - 1\right) \le \frac{10000}{6}$ $N \le \frac{\ln\left(\frac{10006}{6}\right)}{\ln 2}$ or use GC to find $N < 10.704$ Maximum N = 10 for $S_N < 10000$ When n = 10 , $S_{10} = 6138$ Hence remaining distance $= 10000 - 6138$ $= 3862$ cm At stage 11, $U_{11} = 6\left(2^{10}\right) = 6144$ Half of stage 11 is 3072 **Distance from *O*** **= 6144 – 3862 = 2282 cm** The direction of travel is towards O from P_{11} since 3862 > 3072.
8	Let A_n denote the distance ran on the nth training session and S_n denote the total distance

	ran for the n training sessions. $A_n = 7.5 + 0.8(n-1)$ $= 6.7 + 0.8n$ $S_n \geq 475$ $\frac{n}{2}\left[2(7.5)+0.8(n-1)\right] \geq 475$ $7.1n + 0.4n^2 \geq 475$ From GC, $n = 26.7$ Least $n = 27$ For the modified training session, let B_n denote the distance ran on the nth training session and and G_n denote the total distance ran for the n training sessions. $B_6 = 14.93$ $x(1.2)^5 = 14.93$ $x = 6$ (nearest integer) $G_n = \dfrac{6\left[\left(\frac{6}{5}\right)^n - 1\right]}{\frac{6}{5}-1}$ $= 30\left[\left(\frac{6}{5}\right)^n - 1\right]$ $\sum_{n=1}^{N} G_n = \sum_{n=1}^{N} 30\left[\left(\frac{6}{5}\right)^n - 1\right]$ $= 30\sum_{n=1}^{N}\left(\frac{6}{5}\right)^n - \sum_{n=1}^{N} 30$ $= \dfrac{30\cdot\frac{6}{5}\left[\left(\frac{6}{5}\right)^N - 1\right]}{\frac{6}{5}-1} - 30N$ $= 6\left\{30\left[\left(\frac{6}{5}\right)^N - 1\right]\right\} - 30N$ $= 6G_N - 30N$
9	$\mathrm{T}_n = \ln 3x^{n-1}$ $\mathrm{T}_n - \mathrm{T}_{n-1} = \ln 3x^{n-1} - \ln 3x^{n-2}$ $= \ln \dfrac{3x^{n-1}}{3x^{n-2}}$ $= \ln x$, which is a constant Therefore, the sequence is an arithmetic progression. $r = \dfrac{\ln 3x^6 - \ln 3}{\ln 3x^{18} - \ln 3} = \dfrac{1}{3}$ or $r = \dfrac{\ln 3x^2 - \ln 3}{\ln 3x^6 - \ln 3} = \dfrac{1}{3}$

$$\frac{20}{2}\left(2\ln 3+19\ln x\right)>\frac{18\ln x}{1-\frac{1}{3}}$$

$$20\ln 3+190\ln x>27\ln x$$

$$\ln x>-\frac{20}{163}\ln 3$$

$$x>e^{-\frac{20}{163}\ln 3}=0.874$$

10 In week 1, he skips $6\times 50=300$ times and he increases by 60 times for each subsequent week.

Therefore, in week 8, he would have completed $300+(8-1)(60)=720$ (shown)

Total number of skips after n weeks $=\frac{n}{2}[2(300)+(n-1)(60)]$

$$=300n+30n(n-1)$$

$$=30n^2+270n$$

Therefore, given inequality: $30n^2+270n>5000$

Using GC,

When $n=9$, $30n^2+270n=4860<5000$

When $n=10$, $30n^2+270n=5700>5000$

$\therefore$ Least value of n is 10.

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$\therefore$ Least value of n is 10.

11 Amount saved after n months > 20000

$1000+1000(1.05)+\cdots+1000(1.05)^{n-1}>20000$

$$\frac{1000(1.05^n-1)}{1.05-1}>20000$$

$$20000(1.05^n - 1) > 20000$$

$$1.05^n - 1 > 1$$

$$1.05^n > 2$$

$$n \ln 1.05 > \ln 2$$

$$n > \frac{\ln 2}{\ln 1.05} = 14.20669908$$

Using GC, smallest n = 15 i.e. it takes 15 months for Joanne's savings to exceed \$20000.

Joanne's savings at the end of the Nth month

$= 20000(1.05^N - 1)$ (from part **(a)**).

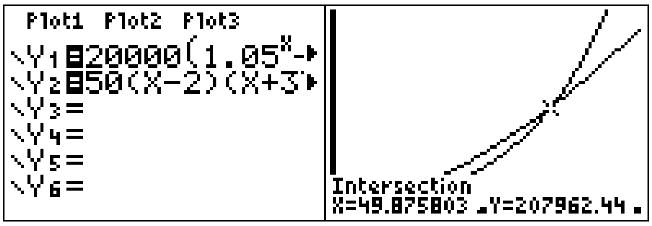
Jim's savings

= 0 + 0 + 2000 + (2000+100) + ….+ [2000 +100(N – 3)]

$$= \frac{N-2}{2}\left[2 \cdot 2000 + (N-3)(100)\right], \text{ for } N \geq 3$$

$$= 50(N-2)(N+37), \text{ for } N \geq 3$$

$$20000(1.05^N - 1) > 50(N-2)(N+37)$$



By GC , we find that n = 50.

Let x = 1 + r/100.

For Joanne's total savings to be at least \$30000

$$1000x + 1000x^2 + ... + 1000x^{24} \geq 30000$$

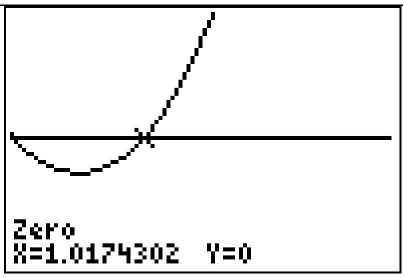
$$\frac{1000x\left(1-x^{24}\right)}{1-x} \geq 30000$$

$$\frac{x\left(1-x^{24}\right)}{1-x} \geq 30$$

$$x\left(1-x^{24}\right) \leq 30\left(1-x\right)$$

$$\left(\because x = 1 + 100r > 1\right)$$

$$x^{25} - 31x + 30 \geq 0$$



Sketching $y = x^{25} - 31x + 30$, we see that $x \geq 1.0174302$,

Hence the interest rate must be at least 1.74%
i.e. least value of r = 1.8 (correct to 1 decimal place)

12 (i) $x_5(x_{21}) = 4096 \Rightarrow ar^4\left(ar^{20}\right) = 4096 \Rightarrow a^2r^{24} = 4096 \cdots (1)$

$$ar^{12} = \sqrt{4096} = 64$$

$$\sum_{k=1}^{25} \log_4 x_k = \log_4 x_1 + \log_4 x_2 + \log_4 x_3 + \ldots + \log_4 x_{25}$$

$$= \log_4\left(x_1 x_2 x_3 \ldots x_{25}\right)$$

$$= \log_4\left(a(ar)\left(ar^2\right)\ldots\left(ar^{24}\right)\right)$$

$$= \log_4\left(a^{25} r^{1+2+\ldots+24}\right)$$

$$= \log_4\left(a^{25} r^{\frac{25(24)}{2}}\right)$$

$$= \log_4\left(a^{25} r^{25\times 12}\right) \cdots (2)$$

$$= \log_4\left(ar^{12}\right)^{25} = 25\log_4\left(ar^{12}\right)$$

$$= 25\log_4(64) = 25\log_4\left(4^3\right)$$

$$= 25\times 3 = 75$$

(ii) $y_n - y_{n-1} = \log_4(x_n) - \log_4(x_{n-1}) = \log_4 \dfrac{x_n}{x_{n-1}}$

$= \log_4 \dfrac{ar^{n-1}}{ar^{n-2}} = \log_4 r$ is a constant free from n.

Hence, $\{ y_n \}$ is an arithmetic sequence.

(b) Let the amount of oil mined in the first year be a.

The maximum total amount of oil mined

$= a + 0.94a + 0.94^2 a + 0.94^3 a + \ldots$

$= \dfrac{a}{1-0.94} \simeq 16.66a < 17a$

Let n be the number of year at which the mine will be in operation.

	$\dfrac{a(1-0.94^n)}{1-0.94} > 16a$ $\Rightarrow \quad 0.94^n < 0.04$ $\Rightarrow \quad n > \dfrac{\lg 0.04}{\lg 0.94} \approx 52.02$ Smallest n is 53.	$\dfrac{(1-0.94^n)}{1-0.94} > 16$ n | $\dfrac{(1-0.94^n)}{1-0.94}$ 52 | 15.999096 53 | 16.03915

n	$\dfrac{(1-0.94^n)}{1-0.94}$
52	15.999096
53	16.03915

The mine will be closed in the 53th year. Therefore, the mine will be closed in 2049.

13 **(i)**

Let the height of the shortest Singa sculpture be x.

Total height of all the Singa sculptures =

$$2\times\frac{14}{2}\left[2x+(14-1)(10)\right]+\left[x+(15-1)(10)\right]=3120$$

$28x+1820+x+140=3120$

$29x=1160$

$x=40$

Height of the shortest Singa sculpture is 40 cm

Height of the tallest Singa sculpture = $40+(15-1)(10)=180$ cm

Alternative

Let y be the height of the tallest Singa sculpture.

Total height of all the Singa sculptures $=2\times\dfrac{14}{2}\left[2(y-10)+(14-1)(-10)\right]+y=3120$

$28y-280-1820+y=3120$

$29y=5220$

$y=180$

The height of the tallest Singa sculpture is 180 cm.

Height of shortest Singa sculpture = $180+(15-1)(-10)=40$ cm

or

Height of shortest Singa sculpture = $170+(14-1)(-10)=40$ cm

Alternative

Let the height of the shortest Singa sculpture be x.

Total height of all the Singa sculptures =

$$\frac{14}{2}\left[2x+(14-1)(10)\right]+\frac{15}{2}\left[2x+(15-1)(10)\right]=3120$$

$x=40$

Height of the shortest Singa sculpture is 40 cm

	Height of the tallest Singa sculpture = $40+(15-1)(10)=180$ cm **(ii)** $T_n < 35$ $210(0.95)^{n-1} < 35$ $(0.95)^{n-1} < \dfrac{1}{6}$ $(n-1)\ln 0.95 < \ln\left(\dfrac{1}{6}\right)$ $n-1 > \dfrac{\ln\left(\frac{1}{6}\right)}{\ln 0.95}$ $n > 35.93$ $n = 36$ The number of sports contested at the 2015 SEA Games is 36. **(iii)** $T_{36} = 210(0.95)^{36-1} = 34.88$ The height of the shortest Nila sculpture is 34.88 cm. $S_{36} = \dfrac{210\left[1-(0.95)^{36}\right]}{1-0.95} = 3537.33$ The total height of all the Nila sculptures is 3537.33 cm.
14	Let the common difference be d. Then $5+24d=\dfrac{31}{5} \Rightarrow d=\dfrac{1}{20}$. To find least value of n such that $\dfrac{n}{2}\left[2(5)+(n-1)\dfrac{1}{20}\right]>1000$ $10n+\dfrac{n(n-1)}{20}>2000$ $200n+n^2-n>40000$ $n^2+199n-40000>0$ From G.C., $n<-322.88$ or $n>123.88$ Since n is positive, least value of n is 124.
15	$\sum_{r=N+1}^{2N}\left(7^{r+1}+3r^2\right)$

	$=\left[7^{N+2}+7^{N+3}+\ldots+7^{2N+1}\right]+3\left[\sum_{r=1}^{2N} r^2-\sum_{r=1}^{N} r^2\right]$ $=\frac{7^{N+2}(7^N-1)}{7-1}+3\left[\frac{2N(4N+1)(2N+1)}{6}-\frac{N(2N+1)(N+1)}{6}\right]$ $=\frac{7^{N+2}(7^N-1)}{6}+3\left[\frac{N(2N+1)[2(4N+1)-(N+1)]}{6}\right]$ $=\frac{7^{N+2}(7^N-1)}{6}+\frac{N(2N+1)(7N+1)}{2}$
16	$S_n=2n^2+3n$ $S_{n-1}=2(n-1)^2+3(n-1)$ $=(2n^2-4n+2)+(3n-3)$ $=2n^2-n-1$ $u_n=S_n-S_{n-1}$ $=(2n^2+3n)-(2n^2-n-1)$ $=4n+1$ $u_{n-1}=4(n-1)+1$ $=4n-3$ $u_n-u_{n-1}=(4n+1)-(4n-3)$ $=4$ Since $u_n-u_{n-1}=$constant, the given sequence is an arithmetic progression. (shown) $S_n>2015$ $2n^2+3n>2015$ $2n^2+3n-2015>0$ Using GC, $n<-32.5$ or $n>31$ Since n is a positive integer, least value of n = 32. Total height of an n-layered cake $=10+10\left(\frac{5k}{100}\right)+10\left(\frac{5k}{100}\right)^2+\ldots+10\left(\frac{5k}{100}\right)^{n-1}$ $=10\left[1+\left(\frac{k}{20}\right)+\left(\frac{k}{20}\right)^2+\ldots+\left(\frac{k}{20}\right)^{n-1}\right]$

$$= 10\left[\frac{1\left(1-\left(\frac{k}{20}\right)^n\right)}{1-\frac{k}{20}}\right]$$

$$= \frac{200}{20-k}\left[1-\left(\frac{k}{20}\right)^n\right] \text{ (shown)}$$

When $k = 19$,

Total height

$$= 200\left[1-\left(\frac{19}{20}\right)^n\right]$$

$$200\left[1-\left(\frac{19}{20}\right)^n\right] \le 120$$

Using GC,

$n \le 17.86375281$

Maximum number of layers is 17.

It is assumed that the thickness of whipped cream used to join the different layers together is negligible.

17 1st Lesson - 40 minutes

2nd Lesson - 45 minutes + 5

3rd Lesson - 50 minutes + 5

$\vdots$ $\vdots$

This is an arithmetic progression:

$u_1 = 40$ and common difference, $d = 5$

60 hours = (60 × 60) minutes = 3600 minutes

Total time (in mins) after n lessons, S_n

$$= \frac{n}{2}[2(40)+(n-1)(5)]$$

$$= \frac{n}{2}[75+5n]$$

$\therefore$ For Adrian to attend at least 60 hours of lessons,

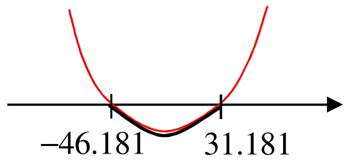
$$\frac{n}{2}[75+5n] \ge 3600$$

Method 1:

$\therefore$ For Adrian to attend ***at least*** 60 hours of lessons,

$$\frac{n}{2}[75+5n] \geq 3600$$

$$\Rightarrow n^2+15n-1440 \geq 0$$



$\Rightarrow n \leq -46.181$ or $n \geq 31.181$

(rejected $\because n>0$)

$\therefore$ Adrian has to attend a minimum of 32 lessons before he is qualified to take the test.

Method 2: Using GC to set up the table of values

Using GC to tabulate a table of values of S_n for various values of n,

n	$S_n = \frac{n}{2}[75+5n]$	Comparison with 3600
31	3565	<3600
32	3760	>3600

$\therefore$ Adrian has to attend a minimum of 32 lessons before he is qualified to take the test.

$u_{n+1} = S_n$, $n \in \mathbb{Z}^+$

For $n \geq 2$,

$$\frac{u_{n+1}}{u_n} = \frac{S_n}{u_n}$$

$$= \frac{u_n + S_{n-1}}{u_n}$$

$$= \frac{u_n + u_n}{u_n} \qquad \because u_n = S_{n-1}$$

$$=2$$

$\because \frac{u_{n+1}}{u_n} =$ constant, $\{u_2, u_3, u_4, \ldots\}$ follows a geometric progression with common ratio 2.

	$\sum_{r=1}^{N+1} u_r = u_1 + \sum_{r=2}^{N+1} u_r$ $= u_1 + \frac{u_2\left(2^N - 1\right)}{2-1}$ $= u_1 + u_1\left(2^N - 1\right) \quad \because u_1 = u_2$ $= u_1 2^N$
18	**(a)(i)** Since they form terms of a GP, the ratio will be a constant. $\frac{a}{a+4d} = \frac{a+4d}{a+13d}$ $a(a+13d) = (a+4d)^2$ $a^2 + 13ad = a^2 + 8ad + 16d^2$ $5ad - 16d^2 = 0$ $d = 0$ (rejected) or $d = \frac{5a}{16}$ Hence, common ratio of G = $\frac{a}{a+4\left(\frac{5a}{16}\right)}$ $= \frac{a}{\frac{9a}{4}} = \frac{4}{9}$ (shown) **(ii)** Let the first term of G be b. Sum to infinity of G = $\frac{b}{1-\frac{4}{9}} = \frac{9}{5}b$ Sum to infinity of odd-numbered terms of G = $\frac{b}{1-\left(\frac{4}{9}\right)^2} = \frac{81}{65}b$ Hence, required ratio = $\frac{9/5}{81/65} = \frac{13}{9}$ **(b)(i)**

	Amount owed at the end of 2004 = $(1.045)^4(29000)$ Since he pays back \$$x$ per month, total paid back before the end of 2005 = \$$12x$ Hence, amount owed on at the end of 2005 = $1.045\left[(1.045)^4(29000)-12x\right]=1.045^5(29000)-1.045(12x)$ <table><tr><td>Year (n)</td><td>Amount owed at the end of the year</td></tr><tr><td>2005 (n = 1)</td><td>$1.045^5(29000)-1.045(12x)$</td></tr><tr><td>2006 (n = 2)</td><td>$1.045\left[1.045^5(29000)-1.045(12x)-12x\right]$ $=1.045^6(29000)-1.045^2(12x)-1.045(12x)$</td></tr><tr><td>2007 (n = 3)</td><td>$1.045\left[1.045^6(29000)-1.045^2(12x)-1.045(12x)-12x\right]$ $=1.045^7(29000)-1.045^3(12x)-1.045^2(12x)-1.045(12x)$</td></tr><tr><td>nth year</td><td>$1.045^{n+4}(29000)-1.045^n(12x)-1.045^{n-1}(12x)-...-1.045^2(12x)-1.045(12x)$</td></tr></table> Amount owed at the end of the nth year $=1.045^{n+4}(29000)-1.045^n(12x)-1.045^{n-1}(12x)-...-1.045^2(12x)-1.045(12x)$ = $1.045^{n+4}(29000)-\dfrac{1.045(12x)(1.045^n-1)}{1.045-1}$ $=1.045^{n+4}(29000)-\dfrac{836x}{3}(1.045^n-1)$ **(ii)** Since the loan must be repaid within 8 years, when n = 8, amount owed must be ≤ 0. $1.045^{12}(29000)-\dfrac{836x}{3}(1.045^8-1)\le 0$ $x\ge 418.11$ Hence, the minimum he must pay each month is \$419 (correct to 3 sf)
19	Let x be the nth term of the given AP. $x=5+6(n-1)$ $=6n-1$ --------- (1) $\dfrac{n}{2}(5+x)=2760$ ---------- (2)

	Sub (1) into (2): $3n^2+2n-2760=0$ $\Rightarrow \quad n=30 \quad$ or $\quad n=\dfrac{-92}{3}$ (reject as $n\in\mathbb{Z}^+$) $\Rightarrow \quad x=179$ **(i)** At the time of withdrawal, the amount in the elder son's account $=1.02^5(80000-x)$ **(ii)** Year — Balance at the end of the year in younger son's account 1(age 7) $\quad 1.02(x)$ 2(age 8) $\quad 1.02(1.02(x)+1000)=1.02^2(x)+1.02(1000)$ 3(age 9) $\quad 1.02(1.02^2(x)+1.02(1000)+1000)$ $=1.02^3(x)+1.02^2(1000)+1.02(1000)$ … 15(age 21) $\quad 1.02^{15}(x)+1.02^{14}(1000)+1.02^{13}(1000)+\ldots+1.02(1000)$ [B1] AEF At the time of withdrawal, the amount in the younger son's account $=1.02^{15}(x)+1.02(1000)\left(\dfrac{1.02^{14}-1}{1.02-1}\right)$ $=1.02^{15}(x)+51000\left(1.02^{14}-1\right)$ If they should receive the same amount of money at the time of withdrawal, $1.02^5(80000-x)=1.02^{15}(x)+51000\left(1.02^{14}-1\right)$ $x=\dfrac{1.02^5(80000)-51000\left(1.02^{14}-1\right)}{1.02^{15}+1.02^5}=29401.85$ Therefore, $\$x=\$29402$ (correct to nearest integer)
20	$100+(15-1)50=800$ m $S_A=\dfrac{n}{2}\left[2(100)+(n-1)50\right]=25n(3+n)$ $25n(3+n)\geq 10000$ Using G.C, $n\leq -21.6$ (rejected $\because n>0$) or $n\geq 18.6$ Least number of stages = 19 $50(1.2)^{14}=642$ m $S_B=\dfrac{50(1.2^n-1)}{1.2-1}=250(1.2^n-1)$

	$S_B > S_A$ Using G.C, $n \geq 22.1$ Least number of stages = 23 When $n = 23$, $S_B - S_A = 1361.8 = 1362$ m (nearest whole no.)
21	(table below)

n	Amount at end of year n
1	1.08(1000)
2	$1.08\left[1000 + 1.08(1000)\right]$ $= 1000\left(1.08 + 1.08^2\right)$
3 :	$1.08\left[1000 + 1000(1.08) + 1000(1.08)^2\right]$ $= 1000\left(1.08 + 1.08^2 + 1.08^3\right)$
n	$1000\left(1.08 + 1.08^2 + \ldots + 1.08^n\right)$

Amount at the end of year 2040

$= 1000\left[(1.08) + (1.08)^2 + \ldots + (1.08)^{26}\right]$

$= 1000\left\{\dfrac{1.08\left[1 - (1.08)^{26}\right]}{1 - 1.08}\right\}$

$= 86351$ (to nearest dollar)

Q6(ii) $S_n = \underbrace{1000 + 1080 + 1160 + \ldots}_{n \text{ terms}} > 86351$

$S_n = \dfrac{n}{2}\left[2(1000) + (n-1)(80)\right] > 86351$

$\Rightarrow 40n^2 + 960n - 86351 > 0$

$\Rightarrow n < -59.987$ (N.A.) or $n > 35.987$

$\therefore$ Least number of years that he still needs to save = 36

The year at which Mr Woo's savings in this savings plan will first exceed \$86351
$= 2015 + 36 - 1 = 2050$

22	Let a_k be the no. of marbles placed in the k^{th} bag and A.P.: $a_1, a_2, ..., a_n$ where $a_1 = 6$ and $d = a_{k+1} - a_k = 6$ Consider $S_n = \dfrac{n}{2}[2(6) + 6(n-1)] \leq 1922$ $0 \leq n \leq 24.816$ When $n = 24$, $S_{24} = 1800$ Using GC,

n	S_n
.	…
24	1800 $\leftarrow$ less than 1922
25	1950
.	…

24 bags contain 1800 marbles
i.e. **122** marbles were left behind

Month	Start ($)	End ($)
1 (Feb)	$A_1 = 10000$	$B_1 = 1.015(10000)$
2	$A_2 = 1.015(10000) - 1200$	$B_2 = 1.015^2(10000) - 1.015(1200)$
3	$A_3 = 1.015^2(10000) - 1.015(1200)$ -1200	$B_3 = 1.015^3(10000) - 1.015^2(1200)$ $-1.015(1200)$
…	…	
k	$A_k = 1.015^{k-1}(10000)$ $-1200(1 + 1.015 + 1.015^2 + ...$ $+1.015^{k-2})$	$B_k = 1.015^k(10000)$ $-1200(1.015)(1 + 1.015 + ...$ $+1.015^{k-2})$

Amount owed at end of month k is $B_k = 1.015^k(10000) - 1200(1.015) \times \frac{1.015^{k-1} - 1}{1.015 - 1}$ Final payment will be at the start of the month after $B_k \leq 1200$
From GC,
$B_8 = 2345.52 > 1200$
$B_9 = 1162.70 \leq 1200$
Amount of final payment $= \$1162.70$, made on start of 10th month
Final payment of $\$1162.70$ (nearest cents) is paid on 1st November 2015.

OR

Amount owed at start of month k is $A_k = 1.015^{k-1}(10000) - 1200 \times \frac{1.015^{k-1} - 1}{1.015 - 1}$

Final payment is made when $A_k \leq 0$
From GC,
$A_9 = 1145.52$

	$A_{10} = -27.30$ Final payment of $-27.30 + 1200 = \$1162.70$ (nearest cents) is paid on 1st November 2015.