2014 TJC H2 Math Promo (Solutions) 2
2014 TJC H2 Math Promo (Questions) 13
DHS_H2Maths_2014_Promo_Qn 18
DHS_H2Maths_2014_Promo_Soln 24
IJC_JC1 Promo_H2 Math_2014_Question Paper 38
IJC_JC1 Promo_H2 Math_2014_Solution 44
RVHS_H2Maths_2014_Promo_Qn 60
RVHS_H2Maths_2014_Promo_Soln 65
VJC_H2Maths_2014_Promo_Qn 76
VJC_H2Maths_2014_Promo_Soln 80

2014 TJC Promotional Examinations H2 Mathematics 9740 (Solutions)

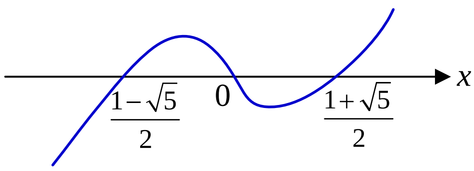
Question 1

$$x-\frac{1}{x}-1\geq 0$$

$$\frac{x^2-1-x}{x}\geq 0$$

$$\frac{\left(x-\frac{1}{2}\right)^2-\frac{5}{4}}{x}\geq 0$$

$$x\left[\left(x-\frac{1}{2}\right)+\frac{\sqrt{5}}{2}\right]\left[\left(x-\frac{1}{2}\right)-\frac{\sqrt{5}}{2}\right]\geq 0$$



$$\therefore \quad \frac{1-\sqrt{5}}{2}\leq x<0 \text{ or } x\geq\frac{1+\sqrt{5}}{2}$$

Question 2

(i) The resulting equation: $y^2 = x\ln\sqrt{x+5}$

Before transformation B: $\left(\frac{y}{2}\right)^2 = x\ln\sqrt{x+5}$ (replace y by $\frac{y}{2}$)

i.e. $y^2 = 4x\ln\sqrt{x+5}$

Before transformation A: $y^2 = 4(x-5)\ln\sqrt{(x-5)+5}$ (replace x by $x-5$)

i.e. $y^2 = 2(x-5)\ln x$

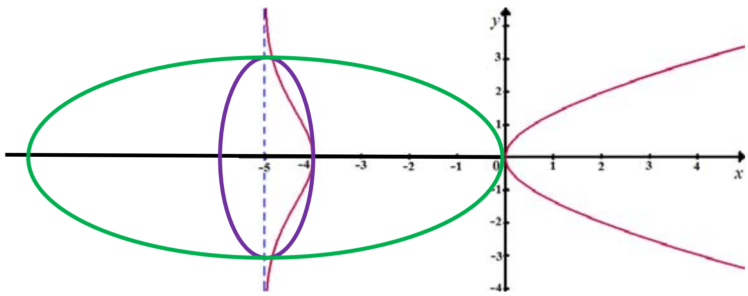
Hence $\mathrm{f}(x) = 2(x-5)\ln x$ or $4(x-5)\ln\sqrt{x}$

(ii) When $-4<x<0$, $\sqrt{x+5}>1 \Rightarrow \ln\sqrt{x+5}>0 \Rightarrow y^2 = x\ln\sqrt{x+5}<0$

Thus y is undefined.

(iii) The graph of $\frac{y^2}{3^2}+\frac{(x+5)^2}{b^2}=1$ is an ellipse with centre at (–5, 0)

To have 3 distinct solutions, $b = 1$ or $b = 5$



Question 3

Let P_n denote the statement $u_n = 2^{\frac{2n+1}{2^n}}$, $n \in \mathbb{Z},\ n \geq 0$

When $n = 0$, LHS = $u_0 = 2$

RHS= $2^{\frac{1}{1}} = 2$

Thus P_0 is true.

Required to prove P_{k+1}: $u_{k+1} = 2^{\frac{2k+3}{2^{k+1}}}$

Assume P_k is true for some $k \in \mathbb{Z},\ k \geq 0$ i.e. $u_k = 2^{\frac{2k+1}{2^k}}$

When $n = k + 1$, $u_{k+1} = u_k^{\frac{2(k+1)+1}{4(k+1)-2}}$ using the recurrence relation

$$= u_k^{\frac{2k+3}{4k+2}}$$

$$= \left(2^{\frac{2k+1}{2^k}}\right)^{\frac{2k+3}{4k+2}} \quad \text{using the inductive hypothesis}$$

$$= \left(2^{\frac{2k+1}{2^k}}\right)^{\frac{2k+3}{2(2k+1)}} \quad \text{(this step or equivalent must be shown)}$$

$$= 2^{\frac{2k+3}{2^{k+1}}} = 2^{\frac{2(k+1)+1}{2^{(k+1)}}}$$

Hence if P_k is true then P_{k+1} is true.

Since P_0 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z},\ n \geq 0$.

Question 4

$$\frac{dP}{dt} = 0.1P - 5000$$

$$\int \frac{1}{0.1P - 5000}\, dP = \int 1\, dt$$

$$\frac{1}{0.1}\ln\left|\, 0.1P - 5000 \,\right| = t + c \quad \text{where } c \text{ is an arbitrary constant}$$

$$\left|\, 0.1P - 5000 \,\right| = e^{\frac{1}{10}(t+c)}$$

$$0.1P - 5000 = \pm e^{\frac{1}{10}c}\, e^{\frac{1}{10}t} = Ae^{\frac{t}{10}} \quad \text{where } A = \pm e^{\frac{1}{10}c}$$

General solution is $P = Be^{\frac{t}{10}} + 50000$

At year 2000, $t = 0$ and $P = 5\times10^6$,

$$5\times10^6 = Be^0 + 50000 \Rightarrow B = 4950000$$

Particular solution is $P = 4950000\, e^{\frac{t}{10}} + 50000$

When $P = 10\times10^6$,

$$10\times10^6 = 4950000\, e^{\frac{t}{10}} + 50000 \Rightarrow t = 6.98$$

The population will be 10 million in the year 2006.

Question 5

et $\dfrac{2r+3}{r(r+1)}=\dfrac{A}{r}+\dfrac{B}{r+1}$

$2r + 3 = A(r + 1) + Br$

Subst $r = 0$, $A=3$

Subst $r = -1$, $B=-1$

$\therefore \quad \dfrac{2r+3}{r(r+1)}=\dfrac{3}{r}-\dfrac{1}{r+1}$

(i) $\quad S_n=\sum_{r=1}^{n}\left[\dfrac{2r+3}{r(r+1)}\left(\dfrac{1}{3^r}\right)\right]=\sum_{r=1}^{n}\left[\left(\dfrac{3}{r}-\dfrac{1}{r+1}\right)\left(\dfrac{1}{3^r}\right)\right]$

$$=\sum_{r=1}^{n}\left[\frac{1}{3^{r-1}r}-\frac{1}{3^r(r+1)}\right]$$

$$=\left[\begin{array}{l} 1-\dfrac{1}{3(2)} \\ +\dfrac{1}{3(2)}-\dfrac{1}{3^2(3)} \\ +\dfrac{1}{3^2(3)}-\dfrac{1}{3^3(4)} \\ +\dfrac{1}{3^3(4)}-\dfrac{1}{3^4(5)} \\ \quad\vdots \\ +\dfrac{1}{3^{n-1}(n)}-\dfrac{1}{3^n(n+1)} \end{array}\right]$$

$$=1-\frac{1}{3^n(n+1)}$$

(ii) $\quad$ As $n\to\infty$, $\dfrac{1}{3^n(n+1)}\to 0$ and so $S_n \to 1$

Thus the series converges to 1.

Question 6

(a) $$\int_1^4 \frac{|x-2|}{x}\,dx = \int_1^2 \frac{-(x-2)}{x}\,dx + \int_2^4 \frac{(x-2)}{x}\,dx$$

If $x \geq 2$, then $|x-2| = (x-2)$
If $x > 2$, then $|x-2| = -(x-2)$

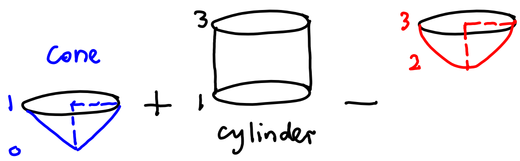
$$= -\int_1^2 \left(1-\frac{2}{x}\right) dx + \int_2^4 \left(1-\frac{2}{x}\right) dx$$

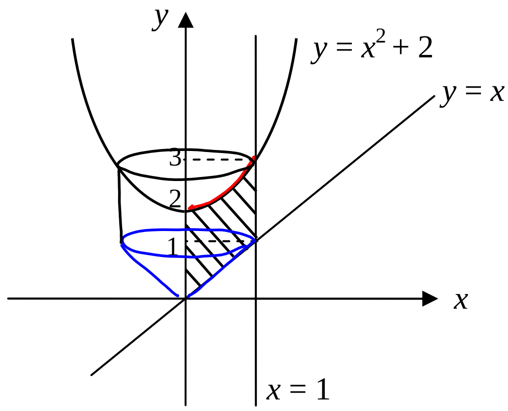
$$= -\left[x-2\ln x\right]_1^2 + \left[x-2\ln x\right]_2^4$$

$$= -\left(2-2\ln 2-1\right)+\left[4-2\ln 4-(2-2\ln 2)\right]$$

$$=1$$

(b) Volume of solid generated





$$= \frac{1}{3}\pi(1)^2(1) + \left[\pi(1)^2(2) - \pi\int_2^3 (y-2)\,dy\right]$$

$$= \frac{7}{3}\pi - \left[\frac{y^2}{2} - 2y\right]_2^3$$

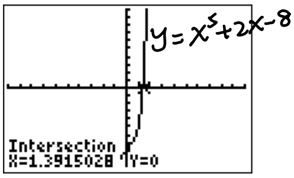
$$= \frac{7}{3}\pi - \frac{1}{2}\pi$$

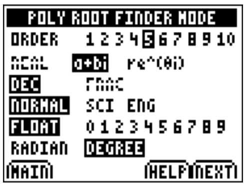
$$= \frac{11}{6}\pi$$

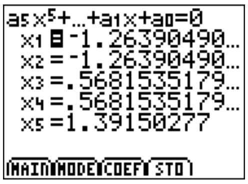
Question 7

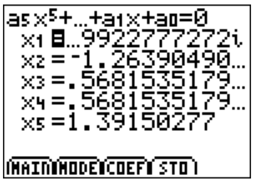
(i) $x^5 + 2x - 8 = 0$

Using GC, $\alpha = 1.392$ (3d.p.)









(ii) If the sequence converges to L (i.e. the limit of the sequence), then as $n \to \infty$, $x_n \to L$ and $x_{n+1} \to L$.

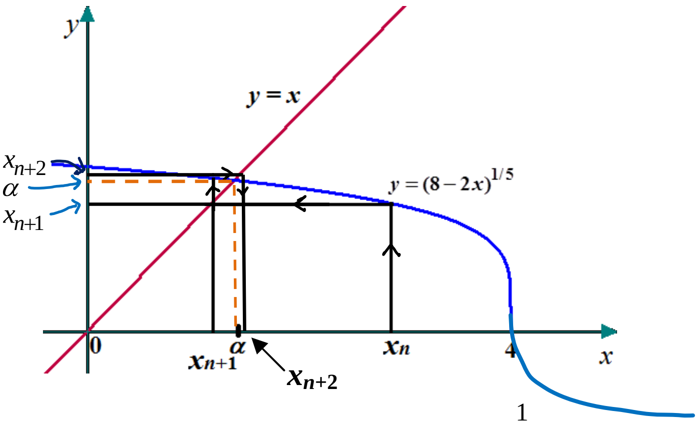
Thus, we have $L = (8-2L)^{\frac{1}{5}}$ [since $x_{n+1} = (8-2x_n)^{\frac{1}{5}}$]

$\Rightarrow \quad L^5 + 2L - 8 = 0$

From part **(i)**, α is the root of the equation $x^5 + 2x - 8 = 0 \Rightarrow L = \alpha$

Hence, if the sequence converges, it will converge to 1.392.

(iii) The graphs of $y = x$ and $y = (8-2x)^{\frac{1}{5}}$ intersect at $x = \alpha$.



Recurrence relation: $x_{n+1} = (8-2x_n)^{\frac{1}{5}}$

From the diagram above, if $\alpha < x_n < 4$, then $0 < x_{n+1} < \alpha$ and $\alpha < x_{n+2} < x_n$

(iv) When $x_1 = 3$, $\alpha < x_1 < 4$

By using results in **(iii)**, we have $0 < x_2 < \alpha$ and $\alpha < x_3 < x_1$.

i.e. $0 < x_2 < \alpha < x_3 < x_1$

Using the GC, the sequence oscillates and converges to 1.392.

(iv) When $x_1 = 3$, $\alpha < x_1 < 4$

By using results in **(a)** and **(b)**, we have $0 < x_2 < \alpha$ and $\alpha < x_3 < x_1$

i.e. $0 < x_2 < \alpha < x_3 < x_1$

Using the GC, the sequence oscillates and converges to 1.392.

Question 8

$$y=(1-\sin x)^{\frac{1}{2}}$$
$$y^2=1-\sin x$$
$$2y\frac{dy}{dx}=-\cos x \qquad \text{---- (1)}$$
$$2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=\sin x=1-y^2$$
$$\therefore 2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2+y^2-1=0 \qquad \text{---- (2)}$$

$$\therefore 2y\frac{d^3y}{dx^3}+2\left(\frac{d^2y}{dx^2}\right)\frac{dy}{dx}+4\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+2y\frac{dy}{dx}=0 \qquad \text{---- (3)}$$

Let $x=0$, $y=1$, $\frac{dy}{dx}=-\frac{1}{2}$, $\frac{d^2y}{dx^2}=-\frac{1}{4}$, $\frac{d^3y}{dx^3}=\frac{1}{8}$

Hence $y=1+\left(-\frac{1}{2}\right)x+\frac{\left(-\frac{1}{4}\right)}{2!}x^2+\frac{\left(\frac{1}{8}\right)}{3!}x^3+\ldots=1-\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{48}x^3+\ldots$

$$(1-\sin x)^{\frac{1}{2}}=1-\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{48}x^3+\ldots$$

Differentiating with respect to x

$$\frac{-\cos x}{2(1-\sin x)^{\frac{1}{2}}}=-\frac{1}{2}-\frac{1}{4}x+\frac{1}{16}x^2+.. \quad \text{or from (1) } \cos x=-2y\frac{dy}{dx}$$

$$\cos x=-2(1-\sin x)^{\frac{1}{2}}\left(-\frac{1}{2}-\frac{1}{4}x+\frac{1}{16}x^2+\ldots\right)$$
$$=-2\left(1-\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{48}x^3+\ldots\right)\left(-\frac{1}{2}-\frac{1}{4}x+\frac{1}{16}x^2+\ldots\right)$$
$$=-2\left(-\frac{1}{2}-\frac{1}{4}x+\frac{1}{16}x^2+\frac{1}{4}x+\frac{1}{8}x^2+\frac{1}{16}x^2+\ldots\right)$$
$$=-2\left(-\frac{1}{2}+\frac{1}{4}x^2+\ldots\right)=1-\frac{x^2}{2}+\ldots$$

[Alternative solution]

$$\cos x=\sqrt{1-\sin^2 x}$$
$$\cos x=\sqrt{(1-\sin x)(1+\sin x)}$$
$$\cos x=(1-\sin x)^{\frac{1}{2}}(1+\sin x)^{-\frac{1}{2}}$$
$$\cos x=\left(1-\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{48}x^3+\ldots\right)\left(1+\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{48}x^3+\ldots\right)$$
$$=1+\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{2}x-\frac{1}{4}x^2-\frac{1}{8}x^2+\ldots \quad =1-\frac{1}{2}x^2+\ldots$$

Question 9

$$y = \frac{px+q}{x^2-5}$$

$$\frac{dy}{dx} = \frac{(x^2-5)p-(px+q)(2x)}{(x^2-5)^2} = \frac{-(px^2+2qx+5p)}{(x^2-5)^2}$$

If C has no turning points and so no stationary points, the equation $\frac{dy}{dx}=0$ has no real roots.

Thus $px^2 + 2qx + 5p = 0$ has no real roots $\Rightarrow$ discriminant < 0

$$(2q)^2 - 4p(5p) < 0$$

$$q^2 < 5p^2$$

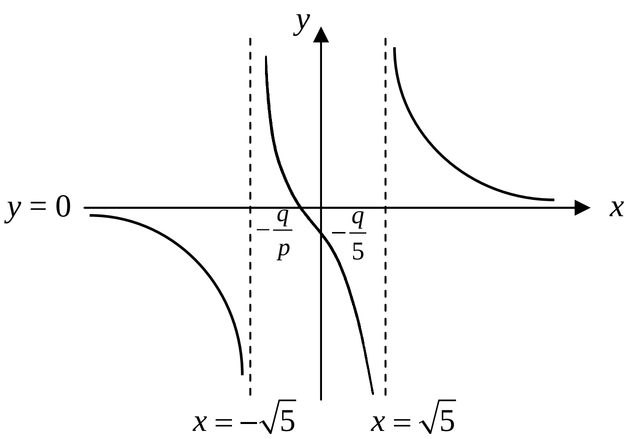
It is given $q^2 < p^2$.

(i) Since $q^2 < 5p^2$ and $p > 0$, using above result, $px^2 + 2qx + 5p > 0$ for all real values of x.

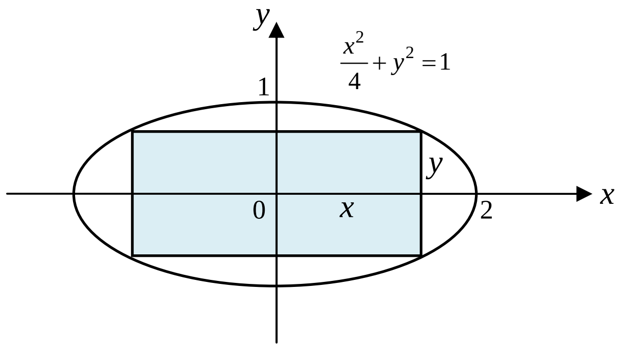
Thus $\frac{dy}{dx} = \frac{-(px^2+2qx+5p)}{(x^2-5)^2} < 0$

(ii) The asymptotes are $x=\sqrt{5},\ x=-\sqrt{5},\ y=0$

(iii)



Question 10



Let the breadth of the rectangle be $2y$.

Area of rectangle, $A = 4xy$

$$= 4x\sqrt{1-\frac{x^2}{4}}$$

$$= 2x\sqrt{4-x^2} \quad \text{(shown)}$$

(a) $\frac{\mathrm{d}A}{\mathrm{d}x} = 2\left(x\times\frac{1}{2}\times\frac{-2x}{\sqrt{4-x^2}}+\sqrt{4-x^2}\right) = \frac{2\left(-x^2+\left(4-x^2\right)\right)}{\sqrt{4-x^2}} = \frac{4\left(2-x^2\right)}{\sqrt{4-x^2}}$

Let $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{4\left(2-x^2\right)}{\sqrt{4-x^2}} = 0$

$\Rightarrow \quad x = \sqrt{2} \quad$ since x is positive

x	$\left(\sqrt{2}\right)^-$	0	$\left(\sqrt{2}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}x}$	+ve	0	–ve

Thus A is maximum when $x = \sqrt{2}$

Maximum value of A = $2\left(\sqrt{2}\right)\sqrt{4-\left(\sqrt{2}\right)^2} = 4$

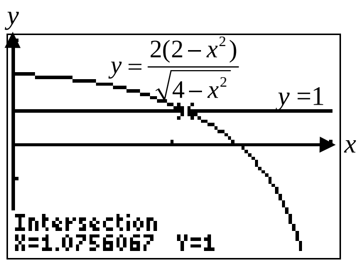
(b) At a particular instant, $\frac{\mathrm{d}x}{\mathrm{d}t} = 2, \quad \frac{\mathrm{d}A}{\mathrm{d}t} = 4$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$4 = \frac{4\left(2-x^2\right)}{\sqrt{4-x^2}}\times 2$$

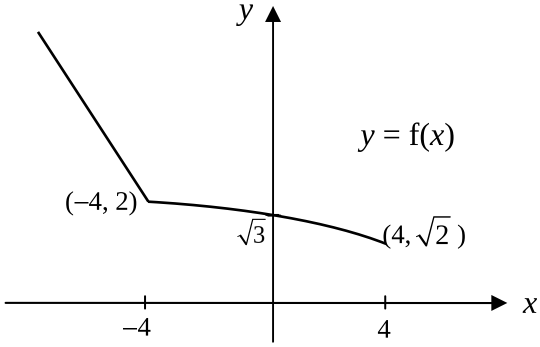
$$\frac{2\left(2-x^2\right)}{\sqrt{4-x^2}} = 1$$

Using GC, $x = 1.08$ as $0 < x < 2$



Question 11

(i)



(ii) For $-4 \le x \le 4$, $y = \sqrt{\left(3 - \frac{x}{4}\right)}$ $\Rightarrow$ $x = 4\left(3 - y^2\right)$

For $x < -4$, $y = -x - 2$ $\Rightarrow$ $x = -y - 2$

Thus $f^{-1}(x) = \begin{cases} 4(3 - x^2) & \text{for } \sqrt{2} \le x \le 2 \\ -x - 2 & \text{for } x > 2 \end{cases}$

(iii) Domain of ff^{-1} = Domain of f^{-1} = Range of f = $[\sqrt{2}, \infty)$

Domain of $f^{-1}f$ = Domain of f = $(-\infty, 4]$

For $ff^{-1}(x) = f^{-1}f(x)$, the set of values of x is $[\sqrt{2}, \infty) \cap (-\infty, 4] = [\sqrt{2}, 4]$

(iv) Since range of f = $[\sqrt{2}, \infty) \not\subseteq (-\infty, 0)$ = domain of g,

gf does not exist as a function.

Question 12

(i) $\dfrac{dx}{dt} = -2a\cos t \sin t$

$\dfrac{dy}{dt} = 3b\sin^2 t\cos t$

$\therefore\ \dfrac{dy}{dx} = \dfrac{3b\sin^2 t\cos t}{-2a\sin t\cos t} = -\dfrac{3b}{2a}\sin t$

(ii) $x = a\cos^2 t = \dfrac{1}{2}a \ \Rightarrow\ \cos t = \pm\dfrac{1}{\sqrt{2}} \ \Rightarrow\ t = \dfrac{\pi}{4}$ since $0 \le t \le \dfrac{\pi}{2}$

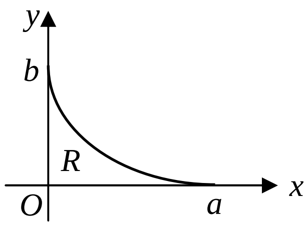
At the point when $t = \dfrac{\pi}{4}$, $y = \dfrac{1}{2\sqrt{2}}b$, $\dfrac{dy}{dx} = -\dfrac{3b}{2a}\sin\dfrac{\pi}{4} = -\dfrac{3b}{2\sqrt{2}a}$

Equation of the tangent is $y - \dfrac{1}{2\sqrt{2}}b = -\dfrac{3b}{2\sqrt{2}a}\left(x - \dfrac{1}{2}a\right)$

At T where $x = 0$, $y = -\dfrac{3b}{2\sqrt{2}a}\left(-\dfrac{1}{2}a\right) + \dfrac{1}{2\sqrt{2}}b = \dfrac{5}{4\sqrt{2}}b$ or $\dfrac{5\sqrt{2}}{8}b$

Coordinates of T are $\left(0, \dfrac{5\sqrt{2}}{8}b\right)$

(iii) Area of R



$= \displaystyle\int_0^a y\ dx$

$= \displaystyle\int_{\frac{\pi}{2}}^{0} \left(b\sin^3 t\right)\left(-2a\sin t\cos t\right)\ dt$

$= -2ab\displaystyle\int_{\frac{\pi}{2}}^{0} \sin^4 t\cos t\ dt$

Let $u = \sin t$, $\dfrac{du}{dt} = \cos t$

When $t = 0$, $u = 0$

When $t = \dfrac{\pi}{2}$, $u = 1$

$= -2ab\displaystyle\int_1^0 u^4\ du = 2ab\int_0^1 u^4\ du$

$= 2ab\left[\dfrac{u^5}{5}\right]_0^1$

$= \dfrac{2}{5}ab$

Question 13

(a) $T_n = S_n - S_{n-1} = \left(5-5^{1-n}\right)-\left(5-5^{2-n}\right)$

$= 5^{2-n} - 5^{1-n} = 5^{1-n}(5-1) = 4\left(5^{1-n}\right)$

$\dfrac{T_n}{T_{n-1}} = \dfrac{4\left(5^{1-n}\right)}{4\left(5^{2-n}\right)} = \dfrac{4\left(5^{1-n}\right)}{20\left(5^{1-n}\right)} = \dfrac{1}{5}$ is a constant independent of n

Thus the series is a geometric series with common ratio = $\dfrac{1}{5}$

$S_n = 5-5^{1-n}$, $T_1 = S_1 = 5-1 = 4$

As $n \to \infty$, $5^{1-n} \to 0$ and $S_n \to 5 = S_\infty$

$S_\infty - S_{k-1} < \dfrac{1}{50}$

$5-\left(5-5^{2-k}\right) < \dfrac{1}{50}$

$\left(\dfrac{1}{5}\right)^k < \dfrac{1}{1250}$

$k > \dfrac{\ln\left(\dfrac{1}{1250}\right)}{\ln\left(\dfrac{1}{5}\right)} = 4.43$

The least value of k is 5

[Alternative solution]

$T_k + T_{k+1} + T_{k+2} + ... < \dfrac{1}{50}$

$\dfrac{4\left(\dfrac{1}{5}\right)^{k-1}}{1-\dfrac{1}{5}} < \dfrac{1}{50}$

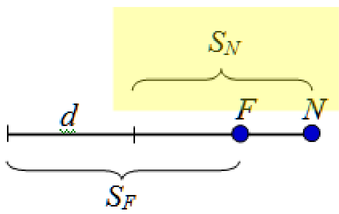
$\left(\dfrac{1}{5}\right)^{k-1} < \dfrac{1}{250}$

$k > 4.43$

(b) After n hops, total distance covered:

$S_N = 0.4n$

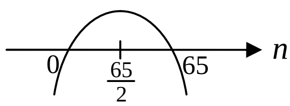
$S_F = \dfrac{n}{2}\left(2(2)+(n-1)(-0.05)\right) = \dfrac{n}{2}\left(4.05-0.05n\right)$



For $S_F < S_N + d$, $\dfrac{n}{2}\left(4.05-0.05n\right) < 0.4n + d$

$0.025n(65-n) < d$

Maximum $y = 0.025n(65-n)$ occurs when $n = \dfrac{65}{2}$



Since $n \in \mathbb{Z}^+$, when $n = 33$, $0.025n(65-n) = 26.4$

Minimum value of $d = 26.5$

[Alternative solution]

For Nicholas's father not to be able to catch up with his son,

$T_n \le 0.4$

$2-\dfrac{1}{20}(n-1) \le 0.4$

$n \ge 33$

$S_{33} - 33\times 0.4 = \dfrac{33}{2}\left(4-\dfrac{1}{20}\times 32\right) - 33\times 0.4 = 26.4$

Hence the minimum value of d is 26.5 m

H

TEMASEK JUNIOR COLLEGE, SINGAPORE
JC One
Promotion Examination 2014
Higher 2

MATHEMATICS 9740

29 September 2014

Additional Materials: Answer paper
List of Formulae (MF15)

3 hours

READ THESE INSTRUCTIONS FIRST

Write your *Civics Group* and *Name* on all the work that you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, fasten all your work securely together.

This document consists of 5 printed pages.

[Turn over

1 Solve the inequality $x \geq \frac{1}{x}+1$, giving your answers in exact form. [4]

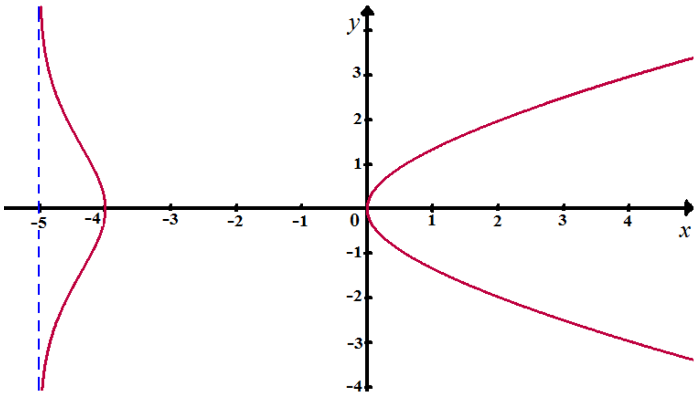
2 **(i)** A graph with equation y^2 = f(x) undergoes transformation A followed by transformation B where A and B are described as follows:

A: a translation of 5 units in the negative direction of the x-axis,

B: a scaling parallel to the y-axis by factor $\frac{1}{2}$.

The equation of the resulting graph is $y^2 = x\ln\sqrt{x+5}$.
Find f(x), showing your workings clearly. [2]

(ii) The diagram below shows a sketch of the graph $y^2 = x\ln\sqrt{x+5}$. Explain clearly why y is undefined when $-4 < x < 0$. [1]



(iii) If there are 3 distinct solutions for the following simultaneous equations

$$y^2 = x\ln\sqrt{x+5},$$

$$\frac{y^2}{3^2}+\frac{(x+5)^2}{b^2}=1, \text{ where } b \in \mathbb{Z}^+,$$

find the possible values of b. [2]

3 A sequence u_0, u_1, u_2,… is such that $u_0 = 2$ and $u_n = \left(u_{n-1}\right)^{\frac{2n+1}{4n-2}}$ for n = 1, 2, 3,….

Prove by induction that $u_n = 2^{\frac{2n+1}{2^n}}$ for all non-negative integers n. [5]

4 The population of a city is P at time t years from a certain date. There is a 10% population growth and five thousand people leave the country every year. Write down a differential equation to relate P and t.
Given that the population was 5 million at the start of year 2000, express P in terms of t. Find the year in which the country will have a population of 10 million. [6]

5 Express $\dfrac{2r+3}{r(r+1)}$ in partial fractions. [2]

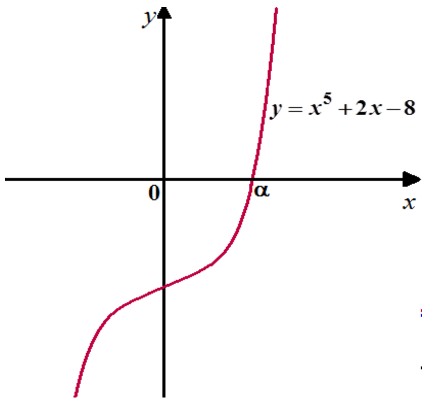
Denoting $S_n = \displaystyle\sum_{r=1}^{n}\left[\frac{2r+3}{r(r+1)}\left(\frac{1}{3^r}\right)\right]$, find S_n in terms of n. [3]

Hence determine whether the series converges. [1]

6 **(a)** Using an algebraic method, find the exact value of $\displaystyle\int_1^4 \frac{|x-2|}{x}\,\mathrm{d}x$. [3]

(b) Sketch and shade the finite region bounded by the curve $y = x^2 + 2$, the lines $y = x$ and $x = 1$, and the y-axis. Find the exact volume of the solid formed when the region is rotated 2π radians about the y-axis. [4]

7 The diagram shows the graph of $y = x^5 + 2x - 8$. The root of the equation $x^5 + 2x - 8 = 0$ is denoted by α.



(i) Find the value of α correct to 3 decimal places. [1]

The real numbers x_n satisfy the recurrence relation $x_{n+1} = (8-2x_n)^{\frac{1}{5}}$ for $n \geq 1$.

(ii) Using the result in **(i)**, show that if the sequence converges, it will converge to α. [2]

(iii) By considering the graphs of $y = x$ and $y = (8-2x)^{\frac{1}{5}}$ on the same diagram, or otherwise, prove that if $\alpha < x_n < 4$, then

(a) $0 < x_{n+1} < \alpha$,

(b) $\alpha < x_{n+2} < x_n$. [3]

(iv) It is given that $x_1 = 3$. Use the results in part **(iii)** to obtain an inequality relating 0, α, x_1, x_2 and x_3. With the help of a graphic calculator, describe the behaviour of the sequence. [2]

8 Given that $y=(1-\sin x)^{\frac{1}{2}}$, show that $2y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2+y^2-1=0$. [2]

By further differentiation, find the Maclaurin's series of y in ascending powers of x up to and including the term in x^3. [4]

Deduce the Maclaurin's series of $\cos x$ up to and including the term in x^2. [3]

9 The curve C has equation $y=\dfrac{px+q}{x^2-5}$, $x\neq\pm\sqrt{5}$ where p and q are positive constants. If C has no turning points, find the condition satisfied by p and q. [4]

It is given that $q^2<p^2$.

(i) Show that C has a negative gradient at all points on the graph. [2]

(ii) Write down the equations of the asymptotes of C. [1]

(iii) Sketch C, giving the coordinates of the points where the graph crosses the axes.[2]

10 A rectangle is inscribed in an ellipse $\dfrac{x^2}{4}+y^2=1$, with its four vertices being in contact with the ellipse. Given that the length of the rectangle is $2x$, show that the area of the rectangle, A, is $2x\sqrt{4-x^2}$. [2]

(a) Using differentiation, find the maximum value of A. [5]

(b) Given that at a particular instant, x is increasing at the rate of 2 units per second and the rate of change of A is 4 units^2 per second, find the value of x at this instant. [3]

11 The functions f and g are defined by

$$\mathrm{f}(x)=\begin{cases}\sqrt{\left(3-\dfrac{x}{4}\right)} & \text{for } -4\leq x\leq 4\\ -x-2 & \text{for } x<-4\end{cases}$$

$$\mathrm{g}(x)=\mathrm{e}^x,\quad x\in\mathbb{R},\ x<0.$$

(i) Sketch the graph of f. [2]

(ii) Define $\mathrm{f}^{-1}(x)$ in a similar form. [4]

(iii) Find the set of values of x for which $\mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)$. [2]

(iv) Explain why the composite function gf does not exist. [2]

12 A curve has parametric equations given by

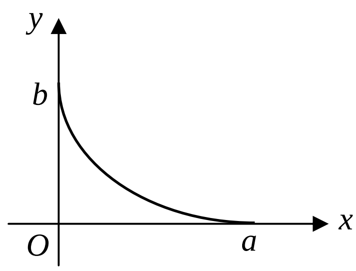
$$x = a\cos^2 t, \quad y = b\sin^3 t,$$

where $0 \le t \le \frac{\pi}{2}$ and a and b are positive constants.

(i) Find $\frac{dy}{dx}$ in terms of a, b and t. [2]

(ii) The tangent to the curve at the point $\left(\frac{1}{2}a, \frac{1}{2\sqrt{2}}b\right)$ cuts the y-axis at T. Find the exact coordinates of T in terms of b. [3]

(iii) A sketch of the curve is shown below and region R is the finite region enclosed between the curve and the axes.



Show that the area of R can be written in the form $\int_\alpha^\beta f(t)\, dt$ where α and β and f are to be determined. By using the substitution $u = \sin t$ or otherwise, find the exact area of R. [5]

13 **(a)** The sum of the first n terms of a series is given by the expression $5 - 5^{1-n}$. Show that the series is a geometric series. [3]

Hence, find the least value of k such that the sum of the series from the k^{th} term onwards is less than $\frac{1}{50}$. [4]

(b) Nicholas and his father start a race at the same time. Nicholas hops at a constant distance of 0.4 m. His father makes a first hop of 2 m and each subsequent hop is 0.05 m less than that of the previous hop. Assume that at the start of the race, Nicholas is d m in front of his father and that they start each hop at the same time. Find the minimum value of d such that Nicholas's father will not be able to catch up with him. Leave your answer correct to 1 decimal place. [4]

Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Promotional Examination
Year 5

MATHEMATICS 9740

(Higher 2)

29 September 2014

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Name, Index Number and Class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer all the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, **attach the question paper to the front of your answer script.**

The total number of marks for this paper is **100**.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	**4**	**6**	**7**	**8**	**8**	**9**	**10**	**11**	**11**	**12**	**14**	**100**

1 A committee consisting of 4 men and 4 women is to be selected from a group of 9 men and 10 women. The youngest of the 9 men is *A* and the youngest of the 10 women is *B*. Find the number of possible ways in which the committee can be formed if

(i) there are no restrictions, [1]

(ii) both *A* and *B* cannot be in the committee together. [2]

The selected committee of eight people sits in a circle for a meeting.

(iii) Find the number of ways that the committee can be seated such that no men are next to each other. [1]

2 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{1}{x} - 1, \qquad x \neq 0,$$

$$\mathrm{g}: x \mapsto (x+2)^2, \qquad x > -2.$$

(i) Show that $\mathrm{fg}(x) = -\dfrac{(x+1)(x+3)}{(x+2)^2}$. State the domain of fg. [3]

(ii) Hence use an algebraic method to solve the inequality $\mathrm{fg}(x) \geq 0$. [3]

3 The function f is defined by

$$\mathrm{f}(x) = \begin{cases} 2x+3 & \text{for } 0 < x \leq 4, \\ -4x+27 & \text{for } 4 < x \leq 6, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+6)$ for all real values of x.

(i) Find the value of $\mathrm{f}(-17) + \mathrm{f}(17)$. [2]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-8 \leq x \leq 13$. [3]

(iii) Hence find the exact value of $\int_{-2}^{6} \mathrm{f}(x)\,\mathrm{d}x$. [2]

4 The equation of a curve C is $y=\dfrac{x}{x+1}$, $x\in\mathbb{R}$, $x\neq -1$.

(i) Use an algebraic method to show that y is strictly increasing for $x>-1$. [1]

The curve C is transformed by a stretch with scale factor $\dfrac{1}{2}$ parallel to the x-axis, followed by a translation of 2 units in the negative y-direction, followed by a reflection in the x-axis.

(ii) Find the equation of the new curve in the form $y=\mathrm{f}(x)$. [3]

(iii) On separate diagrams, sketch the graphs of

(a) $y=\mathrm{f}(x)$, [2]

(b) $y=\mathrm{f}'(x)$. [2]

5 **(a)** Find the general solution of the differential equation

$$2\mathrm{e}^{x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{2\mathrm{e}^{x}}{x^2}+1=0.$$ [3]

(b) Use the substitution $z=2x+y$ to find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x+y+2}{2x+y-1}.$$ [5]

6 A geometric series consisting of positive terms has first term a and common ratio r.

(i) Given that the sum of the first four terms of the series is 16 times that of the sum of the next four terms, find the value of r. [3]

An arithmetic series also has first term a. The ratio of its ninth term to the first term is r.

(ii) Given that the kth term in the arithmetic series is zero, find k. [2]

(iii) Find the least value of n such the sum of the first n terms of the arithmetic series is less than a. [4]

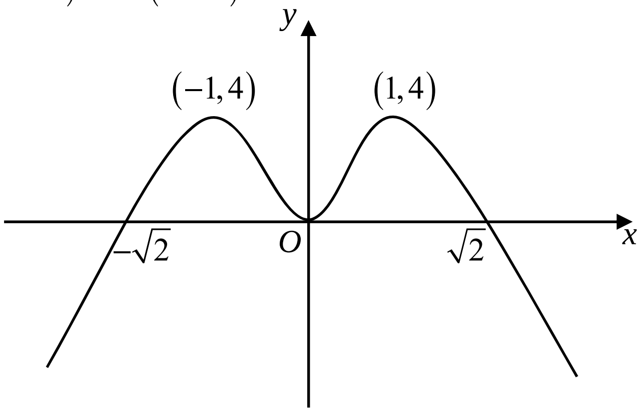
7 **(a)** Find $\displaystyle\int \frac{1}{\sqrt{24x-6x^2}}\,\mathrm{d}x$. [3]

(b) By using integration by parts, find $\displaystyle\int \frac{\ln x}{x^2}\,\mathrm{d}x$. [2]

(c) By using the substitution $u=\cos x$, find

$$\int \frac{\sin x+\sin x\cos x}{1+\cos^2 x}\,\mathrm{d}x.$$ [5]

8 The graph of $y=\mathrm{f}(x)=-4x^2\left(x^2-2\right)$ is given below. The graph is symmetrical about the y-axis, has a minimum point $(0,0)$, maximum points $(-1,4)$ and $(1,4)$, and has roots $(0,0)$, $\left(-\sqrt{2},0\right)$ and $\left(\sqrt{2},0\right)$.



(i) Sketch the graph of $y^2=\mathrm{f}(x)$, showing clearly the stationary points, axial intercepts and behaviour of the graph near the x-intercepts. [3]

(ii) Write down an integral that gives the area of the region enclosed by the curve $y^2=\mathrm{f}(x)$, and evaluate this integral numerically. [2]

(iii) Show that the volume of revolution when the region bounded by the curve $y=\mathrm{f}(x)$ and the x-axis is rotated through π radians about the y-axis is given by $\pi\displaystyle\int_0^4 \sqrt{4-y}\ \mathrm{d}y$. [4]

(iv) Hence find the exact volume in part **(iii)** in terms of π. [2]

9 A sequence $u_1,\ u_2,\ u_3,...$ is such that $u_1 = \dfrac{1}{2}$ and

$$u_{n+1} = u_n - \frac{n+1}{(n+2)!}, \text{ for all } n \geq 1.$$

(i) Prove by mathematical induction that $u_n = \dfrac{1}{(n+1)!}$. [4]

(ii) Hence find $\displaystyle\sum_{r=1}^{n} \frac{r+1}{(r+2)!}$ in terms of n. State the sum to infinity. [4]

(iii) Using your answer for the sum to infinity in part **(ii)** or otherwise, find the exact value of $\displaystyle\sum_{r=1}^{\infty} \frac{r+3}{(r+2)!}$ in terms of e.

[You may use standard results given in the List of Formulae (MF15).] [3]

10 It is given that $y = \ln\left[\sin\left(\dfrac{\pi}{4} + x\right)\right]$.

(i) Show that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1 = 0$. [3]

(ii) By further differentiating the result in part **(i)**, find the Maclaurin series for y up to and including the term in x^3. Give all coefficients in exact form. [3]

(iii) The first three terms in the Maclaurin series for y are equal to the first three terms in the series expansion of $\ln\left[\dfrac{(1+ax)^n}{\sqrt{2}}\right]$, where $|x| < \dfrac{1}{a}$. Using an appropriate expansion from the List of Formulae (MF15), find the constants a and n. [4]

(iv) Use your result in part **(ii)** to deduce the Maclaurin series for $\cot\left(\dfrac{\pi}{4} + x\right)$ up to and including the term in x^2. [2]

11 A curve C has parametric equations

$$x = 4\cos 2\theta, \quad y = 2\sin\theta.$$

(i) Sketch C for $0 \le \theta \le \frac{\pi}{2}$. [2]

(ii) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the form involving a single trigonometric function in terms of θ. [2]

(iii) Find the equation of the normal to C at the point where $\theta = \frac{\pi}{6}$. [3]

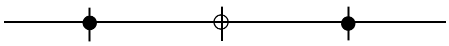
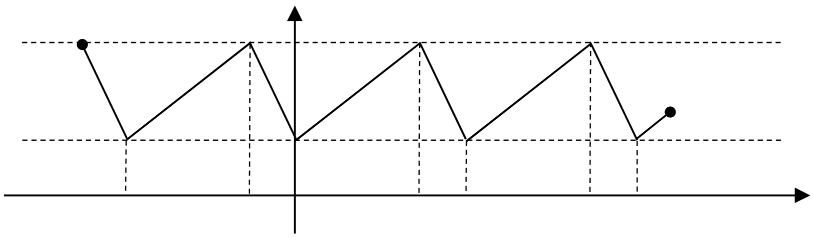
Using a non-calculator method, find the exact coordinates of the point where the normal meets C again. [4]

(iv) Given that θ is increasing at the rate of 4 radians per second when $\theta = \tan^{-1}\left(\frac{3}{4}\right)$, find the exact rate at which $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing at this instant. [3]

END OF PAPER

2014 Year 5 Promotional Examination Solutions

Qn	Suggested Solution
1(i)	Number of ways $= {}^{9}C_4 \times {}^{10}C_4 = 26460$
(ii)	**Method 1** Include both A and B: ${}^{8}C_3 \times {}^{9}C_3 = 4704$ Number of ways required = $26460 - 4704 = 21756$ **Method 2** Include A only: ${}^{1}C_1 \times {}^{8}C_3 \times {}^{9}C_4 = 7056$ Include B only: ${}^{1}C_1 \times {}^{9}C_3 \times {}^{8}C_4 = 5880$ Exclude both A and B: ${}^{8}C_4 \times {}^{9}C_4 = 8820$ Number of ways required = 7056+5880+8820 = 21756 **Method 3** Exclude A: ${}^{8}C_4 \times {}^{10}C_4 = 14700$ Exclude B: ${}^{9}C_4 \times {}^{9}C_4 = 15876$ Exclude both A and B: ${}^{8}C_4 \times {}^{9}C_4 = 8820$ Number of ways required = 14700 + 15876 – 8820 = 21756
(iii)	Number of ways required $= \underbrace{(4-1)!}_{\text{Fix a gender in a circle}} \times \underbrace{4!}_{\text{Arrange the other gender}} = 144$
2(i)	$\text{fg}(x) = \text{f}\left(\left(x+2\right)^2\right)$ $= \dfrac{1}{\left(x+2\right)^2} - 1$ $= \dfrac{1-\left(x+2\right)^2}{\left(x+2\right)^2}$ $= \dfrac{\left(1-x-2\right)\left(1+x+2\right)}{\left(x+2\right)^2}$ $= -\dfrac{\left(x+1\right)\left(x+3\right)}{\left(x+2\right)^2}$ (shown) $\text{D}_{\text{fg}} = \text{D}_{\text{g}} = \left(-2, \infty\right)$

(ii)	$\mathrm{fg}(x) \geq 0$ $-\frac{(x+1)(x+3)}{(x+2)^2} \geq 0$ $\frac{(x+1)(x+3)}{(x+2)^2} \leq 0$  $\therefore -3 \leq x \leq -1, \quad x \neq -2$ Since $D_{\mathrm{fg}} = (-2, \infty)$ Taking intersection, $-2 < x \leq -1$
3(i)	$\mathrm{f}(-17) = \mathrm{f}(-11) = \mathrm{f}(-5) = \mathrm{f}(1) = 5$ $\mathrm{f}(17) = \mathrm{f}(11) = \mathrm{f}(5) = 7$ $\therefore \mathrm{f}(-17) + \mathrm{f}(17) = 5 + 7 = 12$
(ii)	
(iii)	**<u>Method 1</u>** Integral corresponds to the sum of areas of trapeziums. $\int_{-2}^{6} \mathrm{f}(x)\mathrm{d}x = 2\left(\frac{1}{2}(2)(11+3)\right) + \frac{1}{2}(4)(11+3) = 56$ **<u>Method 2</u>** $\int_{0}^{4} \mathrm{f}(x)\mathrm{d}x = \left[x^2 + 3x\right]_0^4 = 28$ $\int_{-2}^{0} \mathrm{f}(x)\mathrm{d}x = \int_{4}^{6} \mathrm{f}(x)\mathrm{d}x = \left[-2x^2 + 27x\right]_4^6 = 14$ Thus $\int_{-2}^{6} \mathrm{f}(x)\mathrm{d}x = 14 + 28 + 14 = 56$
4(i)	$y = \frac{x}{x+1}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$ for $x > -1$ $\therefore y$ is strictly increasing for $x > -1$

(ii)	see below

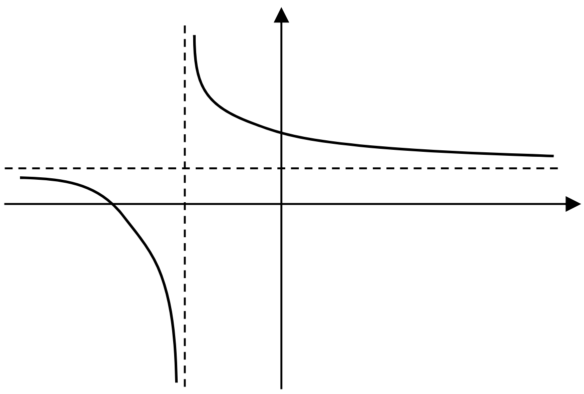
Transformation	Replacement	Eqn of graph
stretch with scale factor $\frac{1}{2}$ parallel to the x-axis	$x \to 2x$	$y = \frac{2x}{2x+1}$
translation of 2 units in the negative y-direction	$y \to y+2$	$y+2 = \frac{2x}{2x+1}$ $y = \frac{2x}{2x+1} - 2$ $= -\frac{2x+2}{2x+1}$
reflection in the x-axis	$y \to -y$	$y = \frac{2x+2}{2x+1}$

$\therefore$ equation of new curve is $y = \mathrm{f}(x) = \frac{2x+2}{2x+1}$

or $2 - \frac{2x}{2x+1}$

or $1 + \frac{1}{2x+1}$

(iii)(a)



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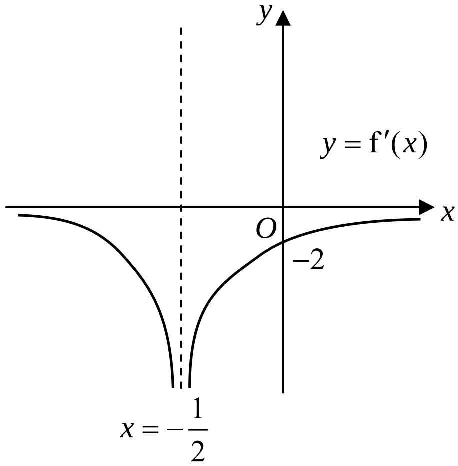
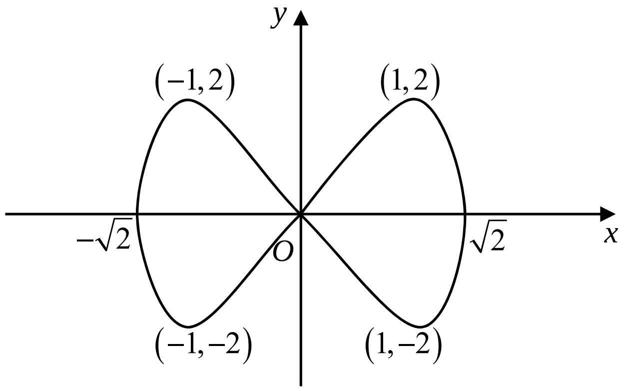
(b)	
5(a)	$2\mathrm{e}^{x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{2\mathrm{e}^{x}}{x^2}+1=0$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{1}{x^2}-\frac{1}{2}\mathrm{e}^{-x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}+\frac{1}{2}\mathrm{e}^{-x}+C,$ where C is an arbitrary constant $y=\ln\lvert x\rvert-\frac{1}{2}\mathrm{e}^{-x}+Cx+D,$ where D is an arbitrary constant
(b)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x+y+2}{2x+y-1}$ -------(1) $z=2x+y$ $\frac{\mathrm{d}z}{\mathrm{d}x}=2+\frac{\mathrm{d}y}{\mathrm{d}x}$ Substituting into (1): $\frac{\mathrm{d}z}{\mathrm{d}x}-2=\frac{z+2}{z-1}$ $\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{z+2}{z-1}+2=\frac{3z}{z-1}$ $\frac{z-1}{z}\frac{\mathrm{d}z}{\mathrm{d}x}=3$ $\left(1-\frac{1}{z}\right)\frac{\mathrm{d}z}{\mathrm{d}x}=3$ $\int\left(1-\frac{1}{z}\right)\mathrm{d}z=\int 3\,\mathrm{d}x$ $z-\ln\lvert z\rvert=3x+C$ $2x+y-\ln\lvert 2x+y\rvert=3x+C$

6(i)	$\dfrac{a\left(1-r^4\right)}{1-r}=16\left[\dfrac{ar^4\left(1-r^4\right)}{1-r}\right]$ $\therefore r^4=\dfrac{1}{16}$ Since the terms are positive, $r=\dfrac{1}{2}$ **<u>Alternative</u>** $S_4=16\left(S_8-S_4\right)$ $17S_4=16S_8$ $17\left(\dfrac{a\left(1-r^4\right)}{1-r}\right)=16\left(\dfrac{a\left(1-r^8\right)}{1-r}\right)$ $17\left(1-r^4\right)=16\left(1-r^4\right)\left(1+r^4\right)$ since $a\neq 0$ & $r\neq 1$ $17=16+16r^4$ $r^4=\dfrac{1}{16}$ Since the terms are positive, $r=\dfrac{1}{2}$
(ii)	$\dfrac{a+8d}{a}=\dfrac{1}{2}$ $2a+16d=a$ $\therefore a+16d=0$ --- (*) $a+16d$ is the 17^{th} term of the AP. Therefore $k=17$ **<u>Alternative</u>** Given $T_k=0$ $a+\left(k-1\right)d=0$ From (*) $a=-16d$ $-16d+\left(k-1\right)d=0$ $\therefore k=17$

(iii)	$\frac{n}{2}\left[2a+(n-1)d\right]<a$ $\frac{n}{2}\left[2a+(n-1)\left(-\frac{1}{16}a\right)\right]<a$ $n\left[2+(n-1)\left(-\frac{1}{16}\right)\right]<2$ since $a>0$ $n(32-n+1)<32$ $n^2-33n+32>0$ $(n-1)(n-32)>0$ $n<1$ or $n>32$ Since $n\in\mathbb{Z}^+$, least value of n is 33.
7(a)	$\int\frac{1}{\sqrt{24x-6x^2}}\,\mathrm{d}x$ $=\int\frac{1}{\sqrt{-6(x^2-4x)}}\,\mathrm{d}x$ $=\int\frac{1}{\sqrt{6\left[4-(x-2)^2\right]}}\,\mathrm{d}x$ $=\frac{1}{\sqrt{6}}\int\frac{1}{\sqrt{2^2-(x-2)^2}}\,\mathrm{d}x$ $=\frac{1}{\sqrt{6}}\sin^{-1}\left(\frac{x-2}{2}\right)+C$
(b)	$\int\frac{1}{x^2}\ln x\,\mathrm{d}x$ $=-\frac{1}{x}\ln x-\int\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)\mathrm{d}x$ $=-\frac{1}{x}\ln x+\int\frac{1}{x^2}\,\mathrm{d}x$ $=-\frac{1}{x}\ln x-\frac{1}{x}+C$

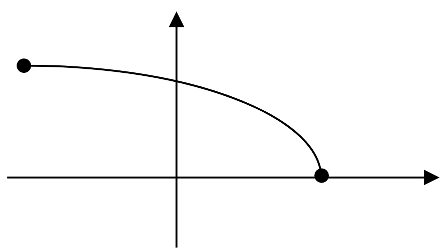
(c)	$\int \frac{\sin x + \sin x \cos x}{1+\cos^2 x}\,\mathrm{d}x$ $= \int \frac{(1+\cos x)}{1+\cos^2 x}\sin x \;\mathrm{d}x$ $= -\int \frac{1+u}{1+u^2}\,\mathrm{d}u$ $= -\int \frac{1}{1+u^2}\,\mathrm{d}u - \frac{1}{2}\int \frac{2u}{1+u^2}\,\mathrm{d}u$ $= -\tan^{-1} u - \frac{1}{2}\ln\left(1+u^2\right) + C$ $= -\tan^{-1}\left(\cos x\right) - \frac{1}{2}\ln\left(1+\cos^2 x\right) + C$ $u = \cos x$ $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$ $\mathrm{d}u = -\sin x \;\mathrm{d}x$
8(i)	
(ii)	Area of $R = 2\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{-4x^2\left(x^2-2\right)}\,\mathrm{d}x$ or $4\int_0^{\sqrt{2}} \sqrt{-4x^2\left(x^2-2\right)}\,\mathrm{d}x$ $= 7.5425 = 7.54$ (3s.f)
(iii)	$y = -4x^4 + 8x^2$ $4x^4 - 8x^2 + y = 0$ Let $h = x^2$ $4h^2 - 8h + y = 0$ $h = \frac{8 \pm \sqrt{8^2 - 4(4)(y)}}{2(4)}$ $\therefore x^2 = 1 \pm \frac{\sqrt{4-y}}{2}$ Volume $= \pi\int_0^4 \left(1+\frac{\sqrt{4-y}}{2}\right) - \left(1-\frac{\sqrt{4-y}}{2}\right)\mathrm{d}y$ $= \pi\int_0^4 \sqrt{4-y}\;\mathrm{d}y$ $= \pi\int_0^4 \sqrt{4-y}\;\mathrm{d}y$ (shown)

(iv)	$\pi\int_0^4 \sqrt{4-y}\ \mathrm{d}y = -\pi\int_0^4 -\left(4-y\right)^{\frac{1}{2}}\ \mathrm{d}y$ $= -\pi\left[\dfrac{(4-y)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$ $= -\dfrac{2}{3}\pi\left[\left(4-4\right)^{\frac{3}{2}} - \left(4-0\right)^{\frac{3}{2}}\right]$ $= \dfrac{16}{3}\pi$
9(i)	Let $\mathrm{P}(n)$ be the proposition $u_n = \dfrac{1}{(n+1)!}$, $n \in \mathbb{Z}^+$. When $n=1$, $\mathrm{LHS} = u_1 = \dfrac{1}{2}$ (given) $\mathrm{RHS} = \dfrac{1}{(1+1)!} = \dfrac{1}{2} = \mathrm{LHS}$ $\therefore$ P(1) is true. Assume $\mathrm{P}(k)$ is true for some $k \in \mathbb{Z}^+$ i.e. $u_k = \dfrac{1}{(k+1)!}$ To prove $\mathrm{P}(k+1)$ is true i.e. $u_{k+1} = \dfrac{1}{(k+1+1)!} = \dfrac{1}{(k+2)!}$ LHS of $\mathrm{P}(k+1) = u_{k+1}$ $= u_k - \dfrac{k+1}{(k+2)!}$ $= \dfrac{1}{(k+1)!} - \dfrac{k+1}{(k+2)!}$ $= \dfrac{k+2-(k+1)}{(k+2)!}$ $= \dfrac{1}{(k+2)!} =$ RHS of $\mathrm{P}(k+1)$ Therefore $\mathrm{P}(k)$ is true $\Rightarrow \mathrm{P}(k+1)$ is true. Since P(1) is true and $\mathrm{P}(k)$ is true $\Rightarrow \mathrm{P}(k+1)$ is true, by mathematical induction, $u_n = \dfrac{1}{(n+1)!}$, $n \in \mathbb{Z}^+$.

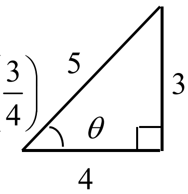
(ii)	$\sum_{r=1}^{n}\frac{r+1}{(r+2)!}=\sum_{r=1}^{n}(u_r-u_{r+1})$ $=[u_1-\cancel{u_2}$ $+\cancel{u_2}-\cancel{u_3}$ $+\ldots$ $+\cancel{u_{n-1}}-\cancel{u_n}$ $+\cancel{u_n}-u_{n+1}]$ $=u_1-u_{n+1}$ $=\frac{1}{2}-\frac{1}{(n+2)!}$ $\sum_{r=1}^{\infty}\frac{r+1}{(r+2)!}=\frac{1}{2}$
(iii)	$\sum_{r=1}^{\infty}\frac{r+3}{(r+2)!}$ $=\sum_{r=1}^{\infty}\frac{r+1}{(r+2)!}+\sum_{r=1}^{\infty}\frac{2}{(r+2)!}=\frac{1}{2}+2\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots\right)$ $=\frac{1}{2}+2\left[\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots\right)-\left(1+\frac{1}{1!}+\frac{1}{2!}\right)\right]$ $=\frac{1}{2}+2(\mathrm{e}-1-1-\frac{1}{2!})$ using $\mathrm{e}^1=1+1+\frac{1^2}{2!}+\frac{1^3}{3!}+\ldots$ $=2\mathrm{e}-\frac{9}{2}$ **<u>(Otherwise)</u>** $\sum_{r=1}^{\infty}\frac{r+3}{(r+2)!}$ $=\sum_{r=1}^{\infty}\frac{r+2}{(r+2)!}+\sum_{r=1}^{\infty}\frac{1}{(r+2)!}=\sum_{r=1}^{\infty}\frac{1}{(r+1)!}+\sum_{r=1}^{\infty}\frac{1}{(r+2)!}$ $=\left(\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\right)+\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots\right)$ $=(\mathrm{e}-1-1)+\left(\mathrm{e}-1-1-\frac{1}{2!}\right)$ using $\mathrm{e}^1=1+1+\frac{1^2}{2!}+\frac{1^3}{3!}+\ldots$ $=2\mathrm{e}-\frac{9}{2}$

Qn	Suggested Solution
10(i)	**Method 1** $y=\ln\left[\sin\left(\frac{\pi}{4}+x\right)\right]$ $e^y=\sin\left(\frac{\pi}{4}+x\right)$ Differentiate wrt x, $e^y\frac{dy}{dx}=\cos\left(\frac{\pi}{4}+x\right)$ Differentiate wrt x, $e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^2=-\sin\left(\frac{\pi}{4}+x\right)=-e^y$ $\Rightarrow e^y\left(\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1\right)=0$ $\Rightarrow \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0 \quad (\because e^y>0)$(shown) **Method 2** $y=\ln\left[\sin\left(\frac{\pi}{4}+x\right)\right]$ $\frac{dy}{dx}=\frac{\cos\left(\frac{\pi}{4}+x\right)}{\sin\left(\frac{\pi}{4}+x\right)}=\cot\left(\frac{\pi}{4}+x\right)$ $\frac{d^2y}{dx^2}=-\operatorname{cosec}^2\left(\frac{\pi}{4}+x\right)$ $=-\left(1+\cot^2\left(\frac{\pi}{4}+x\right)\right)$ $=-\left(\frac{dy}{dx}\right)^2-1$ $\Rightarrow \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0$ (shown)
(ii)	Differentiate wrt x, $\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=0$

	When $x=0$, $y=\ln\left(\sin\left(\frac{\pi}{4}\right)\right)=\ln\left(\frac{1}{\sqrt{2}}\right)=-\frac{1}{2}\ln 2$ $\frac{1}{\sqrt{2}}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{2}}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=1$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1)^2+1=0\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2(1)(-2)=0\Rightarrow\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=4$ $y=-\frac{1}{2}\ln 2+x+(-2)\frac{x^2}{2}+4\frac{x^3}{3!}+\ldots$ $=-\frac{1}{2}\ln 2+x-x^2+\frac{2}{3}x^3+\ldots$ **---(a)**
(iii)	**<u>Method 1</u>** $\ln\left[\frac{(1+ax)^n}{\sqrt{2}}\right]$ $=-\frac{1}{2}\ln 2+n\ln(1+ax)$ $=-\frac{1}{2}\ln 2+n\left(ax-\frac{(ax)^2}{2}+\ldots\right)$ $=-\frac{1}{2}\ln 2+anx-\frac{a^2n}{2}x^2+\ldots$ **---(b)** Compare coefficients of series **(a)** & **(b)**, $an=1$ --- (1) $-\frac{a^2n}{2}=-1$ --- (2) Substitute (1) into (2), $-\frac{a}{2}=-1$ $\Rightarrow a=2$ & $n=\frac{1}{2}$ **<u>Method 2</u>**

	$\ln\left[\frac{(1+ax)^n}{\sqrt{2}}\right]$ $=-\frac{1}{2}\ln 2+\ln(1+ax)^n$ $=-\frac{1}{2}\ln 2+\ln\left(1+nax+\frac{n(n-1)}{2}(ax)^2+...\right)$ $=-\frac{1}{2}\ln 2+nax+\frac{n(n-1)}{2}(ax)^2-\frac{\left(nax+\frac{n(n-1)}{2}(ax)^2\right)^2}{2}+...$ $=-\frac{1}{2}\ln 2+nax+\frac{n(n-1)a^2}{2}x^2-\frac{n^2a^2}{2}x^2+...$ ---**(b)** Compare coefficients of series **(a)** & **(b)**, $an=1$ --- (1) $\frac{n(n-1)a^2}{2}-\frac{n^2a^2}{2}=-1$ $\Rightarrow \frac{a^2n}{2}=1$ --- (2) Substitute (1) into (2), $\frac{a}{2}=1$ $\Rightarrow a=2$ & $n=\frac{1}{2}$
(iv)	$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left[\sin\left(\frac{\pi}{4}+x\right)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{2}\ln 2+x-x^2+\frac{2}{3}x^3+...\right)$ $\Rightarrow \cot\left(\frac{\pi}{4}+x\right)=1-2x+2x^2+...$
11(i)	

(ii)	$x = 4\cos 2\theta, y = 2\sin\theta$ $\frac{dx}{d\theta} = -8\sin 2\theta,\ \frac{dy}{d\theta} = 2\cos\theta$ $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{2\cos\theta}{16\sin\theta\cos\theta} = -\frac{1}{8\sin\theta}$ or $-\frac{1}{8}\operatorname{cosec}\theta.$
(iii)	When $\theta = \frac{\pi}{6},\ x = 4\cos 2\left(\frac{\pi}{6}\right) = 2,\ y = 2\sin\frac{\pi}{6} = 1, \frac{dy}{dx} = -\frac{1}{8}\operatorname{cosec}\frac{\pi}{6} = -\frac{1}{4}$ Equation of normal: $y - 1 = 4(x-2)$ $\therefore y = 4x - 7$ A point on C that lies on the normal is of the form $(4\cos 2\theta, 2\sin\theta)$ and satisfies the equation of the normal. $2\sin\theta = 16\cos 2\theta - 7$ $2\sin\theta = 16(1-2\sin^2\theta) - 7$ $32\sin^2\theta + 2\sin\theta - 9 = 0$ $(2\sin\theta - 1)(16\sin\theta + 9) = 0$ Since $\sin\theta = \frac{1}{2}$ gives the original point, $\sin\theta = -\frac{9}{16}$ $x = 4\cos 2\theta = 4(1-2\sin^2\theta) = \frac{47}{32}$ and $y = 2\left(-\frac{9}{16}\right) = -\frac{9}{8}$ The exact coordinates of the point are $\left(\frac{47}{32}, -\frac{9}{8}\right)$ **<u>Alternative</u>** Cartesian equation of C $x = 4\cos 2\theta = 4(1-2\sin^2\theta) = 4 - 2y^2$ $2y^2 = 4 - x$

	At the intersection, $2y^2 = 4 - x \ \ (-4 \le x \le 4)$ $2(4x-7)^2 = 4 - x$ $32x^2 - 112x + 98 = 4 - x$ $32x^2 - 111x + 94 = 0$ $(x-2)(32x-47) = 0$ $\therefore x = 2$ (gives given point) and $x = \frac{47}{32}$ (between -4 and 4 inclusive) When $x = \frac{47}{32}, y = 4\left(\frac{47}{32}\right) - 7 = -\frac{9}{8}$ Therefore the normal meets C again at $\left(\frac{47}{32}, -\frac{9}{8}\right)$.
(iv)	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{1}{8}\operatorname{cosec}\theta\right).\frac{\mathrm{d}\theta}{\mathrm{d}t}$ $= \frac{4}{8}\operatorname{cosec}\theta\cot\theta \quad \because \frac{\mathrm{d}\theta}{\mathrm{d}t} = 4$ when $\theta = \tan^{-1}\left(\frac{3}{4}\right)$  $= \frac{1}{2}\left(\frac{5}{3}\right)\left(\frac{4}{3}\right) = \frac{10}{9}$ Therefore the rate of increase of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is $\frac{10}{9}$ units per second when $\theta = \tan^{-1}\left(\frac{3}{4}\right)$.

INNOVA JUNIOR COLLEGE
JC 1 MID COURSE EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CLASS | INDEX NUMBER

MATHEMATICS **9740/01**

8 October 2014

3 hours

Additional Materials: Answer Paper
Cover Page

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

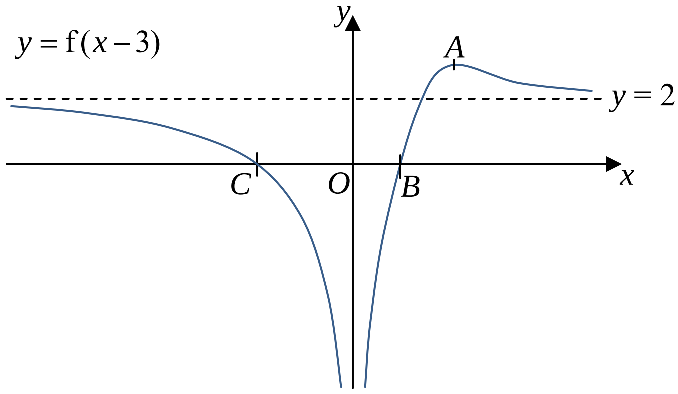
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **1** blank page.

1 Prove by induction that $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}$ for all positive integers n. [4]

2 Find the exact value of $\int_{-2}^{1}\left|e^{2x}-e^{-2x}\right|dx$. [4]

3



The graph of $y=f(x-3)$ is shown above, where the lines $y=2$ and $x=0$ are asymptotes to the curve and $A(2,3)$ is a maximum point. The graph also crosses the x-axis at $B(1,0)$ and $C(-2,0)$. On separate diagrams, sketch the graphs of

(i) $y^2=f(x-3)$, [2]

(ii) $y=f'(x-3)$, [2]

(iii) $y=f(x)$, [2]

showing clearly the equations of any asymptotes, the coordinates of any points of intersection with the axes and the coordinates of the points corresponding to *A*, *B* and *C* (if any).

4 **(i)** Verify that $\frac{1}{r+1}-\frac{1}{r+3}=\frac{2}{(r+1)(r+3)}$. [1]

(ii) Hence find $\sum_{r=1}^{n}\frac{1}{(r+1)(r+3)}$. [3]

(iii) Explain why $\sum_{r=1}^{\infty}\frac{1}{(r+1)(r+3)}$ is a convergent series, and state the value of the sum to infinity. [2]

5 The variables x and y are related by

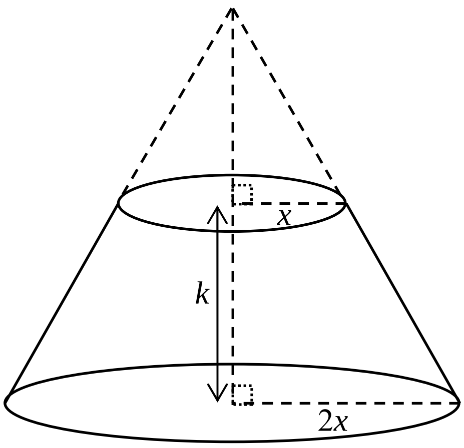
$$\frac{dy}{dx} = \sqrt{xy+1}.$$

(i) Using differentiation, show that $\frac{dy}{dx}\left(2\frac{d^2y}{dx^2} - x\right) - y = 0$. [2]

(ii) Given that the graph of y passes through the point $(0,1)$, obtain the Maclaurin's series for y in ascending powers of x, up to and including the term in x^3. [4]

6 Given that θ is small, show that $\sqrt{2}\sin\left(\frac{\pi}{4} - 2\theta\right) + \tan\theta \approx 1 - \theta - 2\theta^2$. Hence, express $\frac{\sqrt{2}\sin\left(\frac{\pi}{4} - 2\theta\right) + \tan\theta}{3 + \tan\theta}$ as a series in ascending powers of θ up to and including the term in θ^2. [7]

7 The diagram shows a conical frustum, which is created by slicing off the top part of a cone horizontally. The frustum has a horizontal top with radius x cm and a horizontal base with radius $2x$ cm. The volume of the frustum, with height k cm, is 49 cm^3.



(i) Show that the curved surface area of the frustum, S, is given by

$$S^2 = 9\pi^2 x^4 + \frac{3969}{x^2}.$$ [5]

(ii) Hence, use differentiation to find the values of x and k for which S is minimum. [4]

[Volume of cone $= \frac{1}{3}\pi r^2 h$; Curved surface area of cone $= \pi r l$, where l is the length of the slanted edge of a cone]

8 **(a)** Find the exact value of $\int_1^3 2x\ln(2x-1)\,\mathrm{d}x$. [5]

(b) By using the substitution $x = 4\cos\theta$, find $\int \frac{x^2}{\sqrt{16-x^2}}\,\mathrm{d}x$. [6]

SH2 **9** Relative to the origin O, the points A and B have position vectors $\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ and $-6\mathbf{i}+4\mathbf{j}-7\mathbf{k}$ respectively.

(i) Find the size of angle OAB, giving your answer to the nearest 0.1°. [3]

(ii) Find a unit vector perpendicular to $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [3]

(iii) The point N lies on AB such that $AN:NB = \lambda : 1-\lambda$, where $\lambda > 0$. Given that $ON = \frac{1}{3}\sqrt{237}$ units, find the position vector of N. [5]

10 In Gringotts Bank, interest is added to an account at the end of each year at a fixed rate of 2% of the amount in the account at the beginning of that year.

(a) On 1 January 2014, Mr Weasley opened a savings account in the bank. He decided to deposit \$$R$ into his account and he would deposit the same amount on the first day of each subsequent year. In order to support his family financially, he needs to withdraw the interest as soon as it has been added. Find the total amount of interest Mr Weasley would receive on 31 December 2040. [4]

(b) On the same day, Mr Potter also opened a savings account and deposited \$$H$ into it. However for his case, he has decided that he will not withdraw any money out of his account and that he will deposit \$1000 on the first day of each subsequent year. Let S_n denote the total amount in his account at the end of n years.

(i) Write down the expression for S_1 in terms of H and show that $S_n = (1.02)^n H + 51000(1.02^{n-1}-1)$. [5]

(ii) Find the value of n for which the amount in his account exceeds \$25000 for the first time, given that he deposited \$2000 on 1 January 2014. [3]

11 **(a)** **(i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 16x^2 - 25,$$

giving your answer in the form $y = \mathrm{f}(x)$. [3]

(ii) Given that the curve of the general solution of the differential equation in part **(i)** passes through the origin, sketch the family of curves for $x > 0$. [3]

(b) Given that s and t are related by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 16s^2 - 25$$

and that $s = \frac{15}{4}$ when $t = 0$, find s in terms of t, simplifying your answer. [5]

12 A curve C has parametric equations

$$x = \cos 2t, \quad y = \tan t, \quad \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

(i) Sketch the curve C, indicating clearly the equation of any asymptote(s) and intercepts with the axes if any. [2]

(ii) The point P on the curve has parameter $t = \frac{\pi}{4}$. Find the equation of the normal to the curve at P. [3]

(iii) The region R is bounded by the curve C, the normal at P and the x-axis. Find

(a) the exact area of R, [5]

(b) the numerical value of the volume of revolution formed when R is rotated completely about the x-axis, giving your answer correct to 3 decimal places. [3]

END OF PAPER

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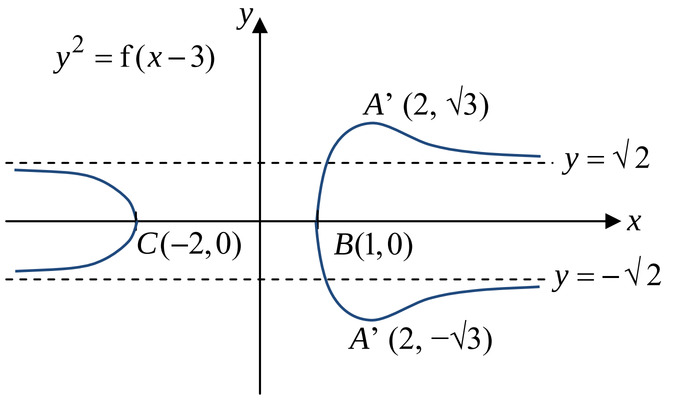
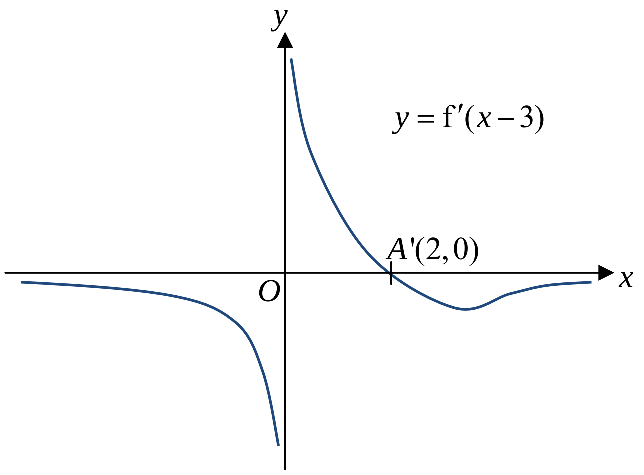
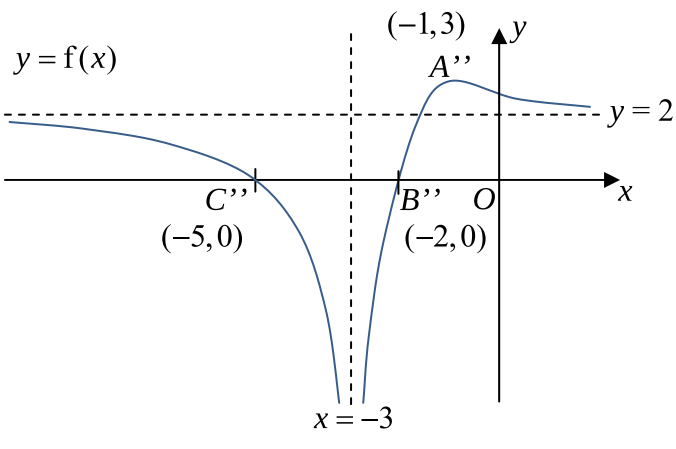
Innova Junior College
H2 Mathematics
JC1 Mid-Course Examination 2014
Suggested Solution

Q1	Solutions
	Step 1: Let P_n denote $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}$ for $n\in\mathbb{Z}^{+}$. Step 2: When n = 1, LHS = $\frac{1}{2!}=\frac{1}{2}$ RHS $=1-\frac{1}{(1+1)!}=\frac{1}{2}$ Therefore, P_1 is true. Step 3: Assume P_k is true for some $k\in\mathbb{Z}^{+}$, i.e. $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{k}{(k+1)!}=1-\frac{1}{(k+1)!}$. Step 4: Want to prove P_{k+1} is true, i.e. $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}=1-\frac{1}{(k+2)!}$. LHS = $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}$ $=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}$ $=1-\frac{1}{(k+2)!}\left[(k+2)-(k+1)\right]$ $=1-\frac{1}{(k+2)!}$ = RHS Thus P_k is true $\Rightarrow$ P_{k+1} is true. Step 5: Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n\in\mathbb{Z}^{+}$.

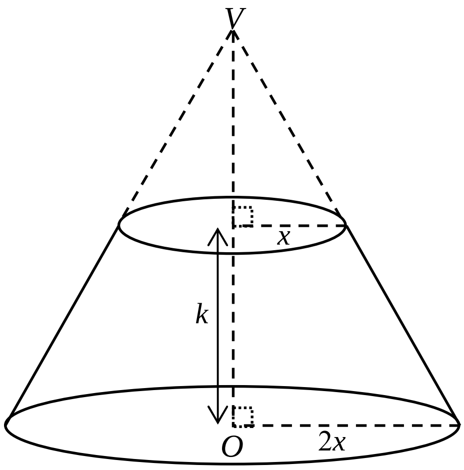
Q2	Solutions
	$\int_{-2}^{1} \left\| e^{2x} - e^{-2x} \right\| \, dx$ $= -\int_{-2}^{0} e^{2x} - e^{-2x} \, dx + \int_{0}^{1} e^{2x} - e^{-2x} \, dx$ $= -\left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x} \right]_{-2}^{0} + \left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x} \right]_{0}^{1}$ $= -\left(\frac{1}{2} + \frac{1}{2} - \frac{1}{2}e^{-4} - \frac{1}{2}e^{4} \right) + \left(\frac{1}{2}e^{2} + \frac{1}{2}e^{-2} - \frac{1}{2} - \frac{1}{2} \right)$ $= \frac{1}{2e^{4}} + \frac{e^{4}}{2} + \frac{1}{2e^{2}} + \frac{e^{2}}{2} - 2$

Q3	Solutions
(i)	
(ii)	
(iii)	

Q4	Solutions
(i)	$\frac{1}{r+1}-\frac{1}{r+3}=\frac{(r+3)-(r+1)}{(r+1)(r+3)}=\frac{2}{(r+1)(r+3)}$
(ii)	$\sum_{r=1}^{n}\frac{1}{(r+1)(r+3)}$ $=\frac{1}{2}\sum_{r=1}^{n}\left[\frac{1}{r+1}-\frac{1}{r+3}\right]$ $=\frac{1}{2}\left[\begin{array}{lcl} & \frac{1}{2} - \cancel{\frac{1}{4}} \\ + & \frac{1}{3} - \cancel{\frac{1}{5}} \\ + & \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} \\ + & \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} \\ + & \ldots\ldots\ldots \\ + & \cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}} \\ + & \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n+1}} \\ + & \cancel{\frac{1}{n}} - \frac{1}{n+2} \\ + & \cancel{\frac{1}{n+1}} - \frac{1}{n+3} \end{array}\right]$ $=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right)$ $=\frac{5}{12}-\frac{1}{2(n+2)}-\frac{1}{2(n+3)}$ or $\frac{5}{12}-\frac{2n+5}{2(n+2)(n+3)}$
(iii)	As $n\to\infty, \frac{1}{2(n+2)}\to 0$ and $\frac{1}{2(n+3)}\to 0.$ So, $\sum_{r=1}^{n}\frac{1}{(r+1)(r+3)}\to\frac{5}{12}$ which is a finite number. Therefore, the series is convergent. $\sum_{r=1}^{\infty}\frac{1}{(r+1)(r+3)}=\frac{5}{12}$

Q5	Solutions
(i)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{xy+1}$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{2}\frac{1}{\sqrt{xy+1}}\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right).$ $2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=y+x\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}\left(2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\right)-y=0$
(ii)	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\left(2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-1\right)-\frac{\mathrm{d}y}{\mathrm{d}x}=0$ At (0,1), $\frac{\mathrm{d}y}{\mathrm{d}x}=1, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{2}, \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{3}{4}$. Maclaurin's series for y $y=1+(1)x+\frac{\left(\frac{1}{2}\right)}{2!}x^2+\frac{\left(\frac{3}{4}\right)}{3!}x^3+\cdots$ $\quad=1+x+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots$

Q6	Solutions
	$\sqrt{2}\sin\left(\frac{\pi}{4}-2\theta\right)+\tan(\theta) \quad =\sqrt{2}\left[\sin\left(\frac{\pi}{4}\right)\cos(2\theta)-\cos\left(\frac{\pi}{4}\right)\sin(2\theta)\right]+\tan(\theta)$ $=\sqrt{2}\left[\frac{\sqrt{2}}{2}\cos(2\theta)-\frac{\sqrt{2}}{2}\sin(2\theta)\right]+\tan(\theta)$ $=\left[\cos(2\theta)-\sin(2\theta)\right]+\tan(\theta)$ $\approx\left(\left(1-\frac{(2\theta)^2}{2}\right)-2\theta\right)+\theta$ $=1-\theta-2\theta^2$
	$\frac{\sqrt{2}\sin\left(\frac{\pi}{4}-2\theta\right)+\tan\theta}{3+\tan\theta}$ $\approx\frac{1-\theta-2\theta^2}{3+\theta}$ $=\left(1-\theta-2\theta^2\right)(3+\theta)^{-1}$ $=\left(1-\theta-2\theta^2\right)(3)^{-1}\left(1+\frac{\theta}{3}\right)^{-1}$ $=\left(1-\theta-2\theta^2\right)\left(\frac{1}{3}\right)\left(1-\frac{\theta}{3}+\frac{\theta^2}{9}+...\right)$ $\approx\frac{1}{3}\left(1-\frac{4}{3}\theta-\frac{14}{9}\theta^2\right)$

Q7	Solutions
(i)	 By similar triangles, $OV = 2k$ Volume of the frustum $= \frac{1}{3}\pi(2x)^2 2k - \frac{1}{3}\pi(x)^2 k$ $= \frac{7}{3}\pi x^2 k$ $49 = \frac{7}{3}\pi x^2 k$ $k = \frac{21}{\pi x^2}$ or $x = \sqrt{\frac{21}{k\pi}}$ Total curved surface area $S = \pi(2x)l - \pi x\left(\frac{l}{2}\right) = \frac{3}{2}\pi x l$ $S^2 = \frac{9}{4}\pi^2 x^2 l^2$ $= \frac{9}{4}\pi^2 x^2\left[(2x)^2 + (2k)^2\right]$ $= \frac{9}{4}\pi^2 x^2\left[(2x)^2 + \left(\frac{42}{\pi x^2}\right)^2\right]$ $= 9\pi^2 x^4 + \frac{3969}{x^2}$ (shown)
(ii)	$2S\frac{\mathrm{d}S}{\mathrm{d}x} = 36\pi^2 x^3 - \frac{7938}{x^3}$ $\frac{\mathrm{d}S}{\mathrm{d}x} = 0$ $36\pi^2 x^3 - \frac{7938}{x^3} = 0$

$36\pi^2 x^6 - 7938 = 0$

$x = 1.67822$ ($x > 0$) and $k = \dfrac{21}{\pi(1.67822)^2} = 2.37340$

$$2\left(\frac{\mathrm{d}S}{\mathrm{d}x}\right)^2 + 2S\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = 108\pi^2x^2 + \frac{23814}{x^4} > 0$$

$\therefore$ $x = 1.68$ and $k = 2.37$ give minimum area.

Alternatively,

$$S = \sqrt{9\pi^2x^4 + \frac{3969}{x^2}}$$

$$\frac{\mathrm{d}S}{\mathrm{d}x} = \frac{36\pi^2x^3 - \dfrac{7938}{x^3}}{2\sqrt{9\pi^2x^4 + \dfrac{3969}{x^2}}} = 0$$

$$36\pi^2x^3 - \frac{7938}{x^3} = 0$$

$x = 1.678$

x	$\left(\sqrt[6]{\dfrac{7938}{36\pi^2}}\right)^-$	$\sqrt[6]{\dfrac{7938}{36\pi^2}}$	$\left(\sqrt[6]{\dfrac{7938}{36\pi^2}}\right)^+$
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	*-ve*	0	*+ve*
Slope			

Therefore x=1.678 and k =2.37 gives minimum area

Q8	**Solutions**

(a)

$$\int_1^3 2x\ln(2x-1)\ \mathrm{d}x$$

$$=\left[x^2\ln(2x-1)\right]_1^3-\int_1^3\frac{2x^2}{2x-1}\ \mathrm{d}x$$

$$=\left[9\ln 5-0\right]-\int_1^3 x+\frac{1}{2}+\frac{1}{2(2x-1)}\ \mathrm{d}x$$

$$=9\ln 5-\left[\frac{1}{2}x^2+\frac{1}{2}x+\frac{1}{4}\ln(2x-1)\right]_1^3$$

$$=9\ln 5-\left[\frac{9}{2}+\frac{3}{2}+\frac{1}{4}\ln 5-\frac{1}{2}-\frac{1}{2}-0\right]$$

$$=\frac{35}{4}\ln 5-5$$

(b)

$$\int\frac{x^2}{\sqrt{16-x^2}}\ \mathrm{d}x$$

Let $x=4\cos\theta$

$\mathrm{d}x=-4\sin\theta\ \ \mathrm{d}\theta$

$$\int\frac{x^2}{\sqrt{16-x^2}}\ \mathrm{d}x=\int\frac{16\cos^2\theta}{\sqrt{16-16\cos^2\theta}}\left(-4\sin\theta\right)\ \mathrm{d}\theta$$

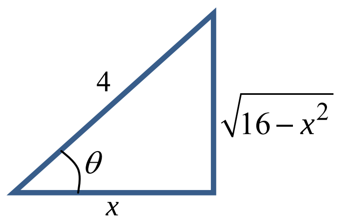
$$=-\int\frac{16\cos^2\theta}{4\sin\theta}\left(4\sin\theta\right)\ \mathrm{d}\theta$$

$$=-8\int 2\cos^2\theta\ \mathrm{d}\theta$$

$$=-8\int\cos 2\theta+1\ \mathrm{d}\theta$$

$$=-8\left(\frac{1}{2}\sin 2\theta+\theta\right)+C$$

$$=-8\left(\theta+\sin\theta\cos\theta\right)+C$$



$$\int\frac{x^2}{\sqrt{16-x^2}}\ \mathrm{d}x=-8\left(\cos^{-1}\left(\frac{x}{4}\right)+\frac{x\sqrt{16-x^2}}{16}\right)+C$$

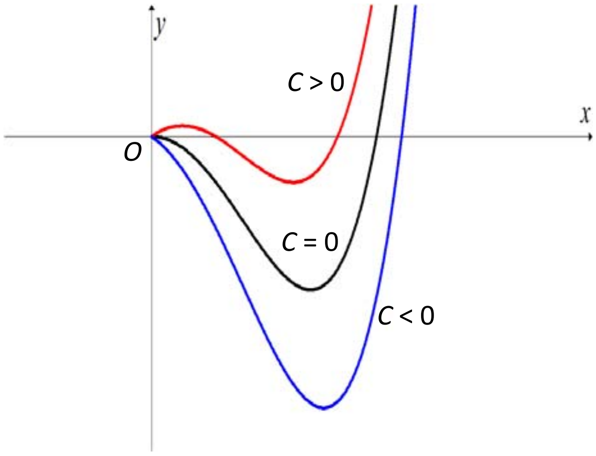
$$=-8\cos^{-1}\left(\frac{x}{4}\right)-\frac{x\sqrt{16-x^2}}{2}+C$$

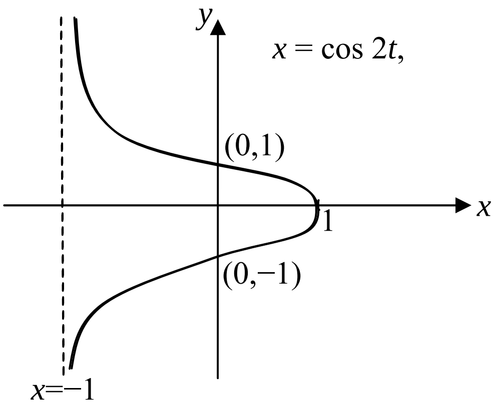
Q9	Solutions
(i)	$\overrightarrow{AO}=\begin{pmatrix}-1\\-3\\2\end{pmatrix}$ and $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-7\\1\\-5\end{pmatrix}$ $\cos\measuredangle OAB=\dfrac{\begin{pmatrix}-1\\-3\\2\end{pmatrix}\bullet\begin{pmatrix}-7\\1\\-5\end{pmatrix}}{\sqrt{14}\sqrt{75}}=\dfrac{7-3-10}{\sqrt{14}\sqrt{75}}=\dfrac{-6}{\sqrt{1050}}$ $\measuredangle OAB=\cos^{-1}\left(\dfrac{-5}{\sqrt{66}}\right)=100.67°=100.7° \quad (1\text{d.p})$
(ii)	Vector perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{AB}$ $=\overrightarrow{OA}\times\overrightarrow{OB}$ $=\begin{pmatrix}1\\3\\-2\end{pmatrix}\times\begin{pmatrix}-6\\4\\-7\end{pmatrix}$ $=\begin{pmatrix}-13\\19\\22\end{pmatrix}$ $\therefore$ Required unit vector $=\dfrac{\begin{pmatrix}-13\\19\\22\end{pmatrix}}{\left\|\begin{pmatrix}-13\\19\\22\end{pmatrix}\right\|}=\dfrac{1}{\sqrt{1014}}\begin{pmatrix}-13\\19\\22\end{pmatrix}$ (alternatively, $=\dfrac{\overrightarrow{OA}\times\overrightarrow{BO}}{\left\|\overrightarrow{OA}\times\overrightarrow{BO}\right\|}=......=\dfrac{1}{\sqrt{1014}}\begin{pmatrix}13\\-19\\-22\end{pmatrix}$)
(iii)	By Ratio Theorem, $\overrightarrow{ON}=\lambda\overrightarrow{OB}+(1-\lambda)\overrightarrow{OA}$ $=\lambda\begin{pmatrix}-6\\4\\-7\end{pmatrix}+(1-\lambda)\begin{pmatrix}1\\3\\-2\end{pmatrix}=\begin{pmatrix}1-7\lambda\\3+\lambda\\-2-5\lambda\end{pmatrix}$

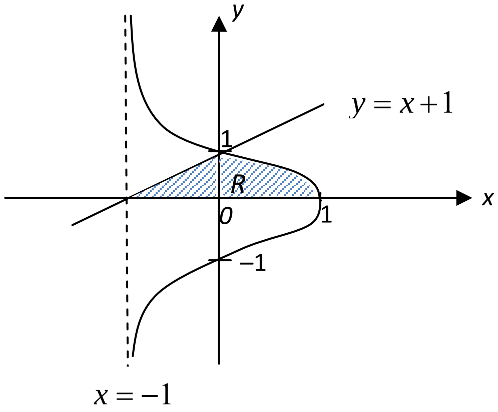
	$\left\|\overrightarrow{ON}\right\| = \dfrac{\sqrt{237}}{3} \Rightarrow \left\|\begin{pmatrix} 1-7\lambda \\ 3+\lambda \\ -2-5\lambda \end{pmatrix}\right\| = \dfrac{\sqrt{237}}{3}$ $(1-14\lambda+49\lambda^2)+(9+6\lambda+\lambda^2)+(4+20\lambda+25\lambda^2) = \dfrac{237}{9}$ $75\lambda^2+12\lambda-\dfrac{37}{3}=0$ $\Rightarrow \lambda = -\dfrac{37}{75}$(reject) or $\lambda = \dfrac{1}{3}$ $\overrightarrow{ON} = \dfrac{1}{3}\begin{pmatrix} -6 \\ 4 \\ -7 \end{pmatrix} + \left(\dfrac{2}{3}\right)\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} -4 \\ 10 \\ -11 \end{pmatrix}$

Q10	Solutions
(a)	Number of years that lapsed = 2040 – 2014 + 1 = 27 Interest from 1st year = $ $0.02R$ Interest from 2nd year = $ $0.02(2R)$ Interest from 3rd year = $ $0.02(3R)$ … Interest from 27th year= $ $0.02(27R)$ Hence, total amount of interest $=\$\ 0.02(R+2R+3R+\cdots+27R)$ $=\$\ 0.02R\left(\frac{27(27+1)}{2}\right)$ $=\$\ 7.56R$
(b) (i)	$S_1=1.02H$ $S_2=(1.02)^2H+1.02(1000)$ $S_3=(1.02)^3H+(1.02+1.02^2)(1000)$ … $S_n=(1.02)^nH+(1.02+1.02^2+\cdots+1.02^{n-1})(1000)$ $(1.02+1.02^2+\cdots+1.02^{n-1})$ is a GP with first term $a=1.02$ and common ratio $r=1.02$. $S_n=(1.02)^nH+1000\left(\frac{1.02(1.02^{n-1}-1)}{1.02-1}\right)$ $=(1.02)^nH+51000(1.02^{n-1}-1)$ (shown)
(b) (ii)	$S_n=2000(1.02)^n+51000(1.02^{n-1}-1)>25000$
	Method 1: From GC, $n=19 \Rightarrow S_n=24754$ $n=20 \Rightarrow S_n=26269$ $n=21 \Rightarrow S_n=27815$ Minimum value of n is 20.
	Method 2: $2000(1.02)^n+51000(1.02^{n-1}-1)>25000$

	$2000(1.02)^n + 51000\left(\dfrac{1.02^n}{1.02}\right) - 51000 > 25000$ $52000(1.02)^n > 76000$ $n > \dfrac{\ln(19/13)}{\ln 1.02}$ $n > 19.2$ Minimum value of n is 20.

Q11	Solutions
(a)(i)	$\frac{dy}{dx}=\int 16x^2-25 \ dx=\frac{16}{3}x^3-25x+C$ $y=\int \frac{16}{3}x^3-25x+C \ dx=\frac{4}{3}x^4-\frac{25}{2}x^2+Cx+D$
(a)(ii)	Since y passes through (0, 0), $D=0$. $\therefore \ y=\frac{4}{3}x^4-\frac{25}{2}x^2+Cx$ 
(b)	$\frac{ds}{dt}=16s^2-25$ $\int \frac{1}{16s^2-25} ds=\int 1 \ dt$ $\int \frac{1}{(4s)^2-5^2} ds=\int 1 \ dt$ $\frac{1}{2(5)(4)}\ln\left\|\frac{4s-5}{4s+5}\right\|=t+C$ $\left\|\frac{4s-5}{4s+5}\right\|=e^{40t+40C}$ $\frac{4s-5}{4s+5}=\pm e^{40t+40C}$ $\frac{4s-5}{4s+5}=Ae^{40t}$, where $A=\pm e^{40C}$ Since $s=\frac{15}{4}$ when $t=0$, $\therefore \ A=\frac{1}{2}$ $\Rightarrow \ \frac{4s-5}{4s+5}=\frac{1}{2}e^{40t}$ $2(4s-5)=e^{40t}(4s+5)$ $s=\frac{5e^{40t}+10}{8-4e^{40t}}$.

Q12	
(i)	
(ii)	$x = \cos 2t$, $y = \tan t$ $\frac{dx}{dt} = -2\sin 2t$, $\frac{dy}{dt} = \sec^2 t$ $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\sec^2 t}{-2\sin 2t}$ When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{\sec^2 \frac{\pi}{4}}{-2\sin\frac{\pi}{2}} = -1$ $\therefore$ Gradient of normal = 1 When $t = \frac{\pi}{4}$, $x = \cos\left(\frac{\pi}{2}\right) = 0$ $y = \tan\left(\frac{\pi}{4}\right) = 1$ P is (0,1) Equation of normal is $y - 1 = (1)(x - 0)$ $y = x + 1$

(iii) (a)	 When $x = 0$, $t = \frac{\pi}{4}$ When $x = 1$, $t = 0$ Required area $= \frac{1}{2}(1)(1) + \int_0^1 y \text{ d}x$ $= \frac{1}{2} + \int_{\frac{\pi}{4}}^{0} (\tan t)(-2\sin 2t) \text{ d}t$ $= \frac{1}{2} + 4\int_0^{\frac{\pi}{4}} (\sin^2 t) \text{ d}t$ $= \frac{1}{2} + 2\int_0^{\frac{\pi}{4}} (1 - \cos 2t) \text{ d}t$ $= \frac{1}{2} + 2\left[t - \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{2} + 2\left[\frac{\pi}{4} - \frac{1}{2} - 0\right]$ $= \frac{\pi}{2} - \frac{1}{2}$
(iii) (b)	Required Volume $= \frac{1}{3}\pi(1)^2(1) + \pi\int_0^1 y^2 \text{ d}x$ $= \frac{\pi}{3} + \pi\int_{\frac{\pi}{4}}^{0} (\tan^2 t)(-2\sin 2t) \text{ d}t$ $= 2.26071$ ≈ 2.261 (3d.p.)

1. Burking would like to purchase cement, iron and sand needed for his construction project. He approached three suppliers *A*, *B* and *C* to enquire about their selling prices for the materials. The total prices quoted are \$9880, \$10090 and \$10260 respectively. The breakdown of unit prices for the materials is shown in the table below.

Supplier	Price per tonne of cement (\$)	Price per tonne of iron (\$)	Price per tonne of sand (\$)
A	29.00	450.00	10.00
B	32.50	460.00	8.00
C	22.50	480.00	7.00

Calculate the amount of each material he needs. [3]

Suppose he triples his order for cement and sand while keeping that for iron the same, determine the supplier he should choose to minimise cost. [2]

2. The sum of the first n terms of a progression is given by $S_n = 1 - e^{2n}$.

(i) Find U_n, the nth term of the progression. [2]

(ii) Prove that $U_1,\ U_2,\ U_3,\ \ldots$ is a geometric progression. [2]

(iii) Determine if the sum S_n converges. [1]

3. A committee consisting of 1 chairperson, 1 vice-chairperson, 1 secretary, 2 treasurers is to be chosen from a group of 5 males and 5 females. Find the number of ways to choose the committee if

(i) there are no restrictions on the gender, [2]

(ii) the chairperson and vice-chairperson must be of different gender. [2]

The committee is to be seated around a round table for one of their discussion. Identical chairs with labels for the different appointments are arranged around the table. Find the number of ways the chairs can be arranged. [2]

4. Prove that $\ln\left(r + \dfrac{r}{r+1}\right) = \ln r - \ln(r+1) + \ln(r+2)$. [2]

Hence, find in terms of n,

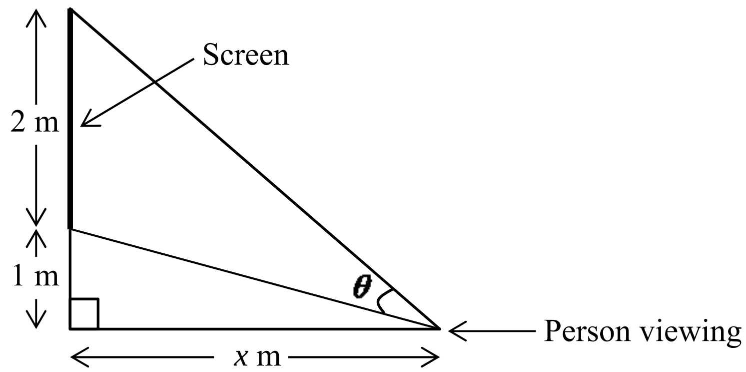
$$\ln\left(2+\frac{2}{3}\right) + \ln\left(3+\frac{3}{4}\right) + \ln\left(4+\frac{4}{5}\right) + \ldots + \ln\left(n-1+\frac{n-1}{n}\right),$$

giving your answer in the form $\ln\left(\dfrac{(n+1)!}{k}\right)$, where k is a constant to be found. [4]

5. Use the method of induction to prove that
$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}.$$ [4]

Hence, find $\sum_{r=0}^{2n} (r+1)^3$. [2]

6. A large meeting room has one of its walls fitted with a projection screen of 2 metres high. The screen is 1 metre above the floor. Let the distance from the screen to someone standing directly in front of it be x metres and the person's viewing angle of the screen be θ (see diagram).



Show that $\theta = \tan^{-1}\frac{3}{x} - \tan^{-1}\frac{1}{x}$. [1]

Show that $\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{a+bx^2}{(x^2+1)(x^2+9)}$ where a and b are constants to be found. Hence find the value of x so that the viewing angle is maximum. [5]

Suggest a reason why the diagram is not realistic. [1]

7. The curve C has equation $y = x + \frac{2}{x-3}$.

(i) State the equations of the asymptotes of C. [2]

(ii) Sketch C, showing its asymptotes and stating the coordinates of the turning points as well as the points of intersection with the axes. [3]

(iii) By drawing a sketch of another suitable curve on the same diagram, show that the equation
$$(x-2)^2 - \left(x - 2 + \frac{2}{x-3}\right)^2 = 9$$
has no real roots. [2]

8. A wet porous substance left in open air loses its moisture m at a rate proportional to the moisture content. If one of such substance has moisture content m_0 initially and loses half of the moisture content in 3 hours, how much longer will it take for the substance to lose 80% of its original moisture content? [7]

9. (i) Given that $y=(1+bx)^n$, find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Use the answer to find the Maclaurin's series for $(1+bx)^n$, up to and including the term in x^2. [4]

(ii) Find the series expansion for $\dfrac{\cos 2x}{1-\sin 2x}$ up to and including the term in x^2, given that x is a sufficiently small angle. [4]

10. (i) Use the substitution $x=2\tan\theta$ to show that

$$\int \frac{1}{(4+x^2)^2}\,\mathrm{d}x=\frac{1}{16}\left(\frac{2x}{x^2+4}+\tan^{-1}\frac{x}{2}\right)+C. \qquad [5]$$

(ii) The region R is bounded by the curve $y=\sqrt{\dfrac{1}{x}-4}$, the line $y=2$, the x-axis and the y-axis. Find the exact volume of the solid formed when R is rotated 2π radians about the y-axis. [4]

11. (a) Find $\int x\mathrm{e}^{-x}\,\mathrm{d}x$. [3]

(b) The curve C has parametric equations

$$x=\frac{(t+2)^2}{2}, \quad y=\mathrm{e}^{-t}, \quad \text{for } t\geq -2.$$

(i) Show that the equation of the normal to the curve at $t=0$ is $y=2x-3$. [4]

(ii) Find the exact value of the area of the region bounded by C, the line $y=2x-3$, the x-axis and the y-axis. [4]

SH2 12. Research is being carried out into how the concentration of a drug in bloodstream varies with time, measured from the time when the drug is given. Observations at successive times give the data shown in the following table.

Time (t minutes)	15	30	60	90	120	150	180	240	300
Concentration (x micrograms per litre)	82	65	43	37	22	19	12	6	2

(i) Sketch a scatter diagram for the data. [2]

(ii) Calculate the product moment correlation coefficient between x and t. Comment on whether a linear model would be appropriate for the relationship between x and t. [2]

It is suggested that the relationship between x and t can be modelled by the formula

$$x = ae^{bt},$$

where a and b are constants.

(iii) For this model, show that the relationship between $\ln x$ and t is linear and calculate its product moment correlation coefficient. [3]

(iv) Using a suitable regression line, estimate, to the nearest minute, the time at which the concentration is at 15μg/l. Give a reason for the choice of the regression line and comment on the reliability of the estimate. [4]

13. Functions f and g are defined by

$$\mathrm{f} : x \mapsto x^2 + 2x - 8, \quad x \geq 2,$$

$$\mathrm{g} : x \mapsto x - \frac{1}{x-1}, \quad x \geq 2.$$

(i) Show that f^{-1} exists. [1]

(ii) Find f^{-1} in similar form. [3]

(iii) Show that the composite function fg does not exist. [2]

Function h is defined as follows

$$\mathrm{h} : x \mapsto k\left(x - \frac{1}{x-1}\right), \quad x \geq 2.$$

(iv) By considering a transformation of the function g, or otherwise, state the range of values of k such that the composite function fh exists. [1]

(v) Find the range of fh when $k = 3$. [2]

(vi) Find the range of values of x for which $\mathrm{f}(x) > \mathrm{g}(x)$. [3]

END OF PAPER

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Markers Comments for 2014 Y5 H2 Maths Promotional Exam

Question 1		
	Let x, y and z be the amount (in tonnes) of cement, iron and sand that Burking needs. $29x+450y+10z=9880-(1)$ $32.5x+460y+8z=10090-(2)$ $22.5x+480y+7z=10260-(3)$ Using GC, $x = 20$, $y = 20$, $z = 30$ Therefore, Burking needs 20 tonnes of cement, 20 tonnes of iron and 30 tonnes of sand. If he triples his order for cement and sand, then the total price quoted by Supplier A = 29(60) + 450(20) + 10(90) = \$11640 Supplier B = 32.5(60) + 460(20) + 8(90) = \$11870 Supplier C = 22.5(60) + 480(20) + 7(90) = \$11580 Thus he should choose Supplier C to minimise cost.	

Question 2		
(i)	$U_n = S_n - S_{n-1}$ $= 1-e^{2n}-\left(1-e^{2(n-1)}\right)$ $= e^{2(n-1)}\left(1-e^{2}\right)$ or $e^{2n}\left(e^{-2}-1\right)$	
(ii)	$\dfrac{U_n}{U_{n-1}}=\dfrac{e^{2(n-1)}\left(1-e^{2}\right)}{e^{2(n-2)}\left(1-e^{2}\right)}=e^{2}$ which is a const (common ratio). Therefore the progression is a geometric progression. or $U_n=\left(1-e^{2}\right)e^{2(n-1)}$ is in the form ar^{n-1}, where $a=\left(1-e^{2}\right)$ and $r=e^{2}$ Therefore the progression is a geometric progression.	
(iii)	Since $\lvert r\rvert=\lvert e^{2}\rvert=7.38906>1$, S_n does not converge.	

Question 3		
(i)	$\binom{10}{5}\times\dfrac{5!}{2!}=15120$ or $\binom{10}{1}\times\binom{9}{1}\times\binom{8}{1}\times\binom{7}{2}=15120$	
(ii)	$\binom{5}{1}\times\binom{5}{1}\times2\times\binom{8}{1}\times\binom{7}{2}=8400$ or $\binom{5}{1}\times\binom{5}{1}\times2\times\binom{8}{3}\times\dfrac{3!}{2!}=8400$	
	Number of ways to arrange the chairs	

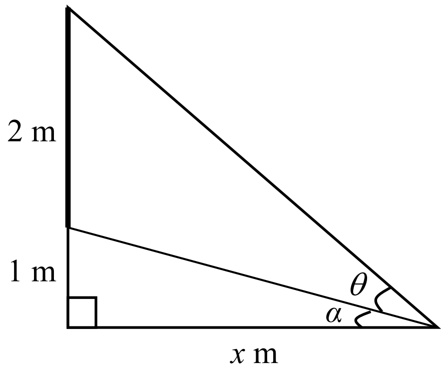
	$=\frac{5!}{5}\div 2=12$	

Question 4		
	$\text{RHS}=\ln r-\ln(r+1)+\ln(r+2)$ $=\ln\left(\frac{r(r+2)}{r+1}\right)$ $=\ln\left(\frac{r^2+2r}{r+1}\right)$ $=\ln\left(\frac{r(r+1)+r}{r+1}\right)$ $=\ln\left(r+\frac{r}{r+1}\right)=\text{LHS}$ (proven)	
	Required series $=\sum_{r=2}^{n-1}\ln\left(r+\frac{r}{r+1}\right)$ $=\sum_{r=2}^{n-1}\left(\ln r-\ln(r+1)+\ln(r+2)\right)$ $=\ln 2-\cancel{\ln 3}+\ln 4$ $+\cancel{\ln 3}-\cancel{\ln 4}+\ln 5$ $+\cancel{\ln 4}-\cancel{\ln 5}+\ln 6$ $\vdots$ $\vdots$ $+\cancel{\ln(n-3)}-\cancel{\ln(n-2)}+\ln(n-1)$ $+\cancel{\ln(n-2)}-\cancel{\ln(n-1)}+\cancel{\ln(n)}$ $+\cancel{\ln(n-1)}-\cancel{\ln(n)}+\ln(n+1)$ $=\ln 2+\ln 4+\ln 5+\ln 6+...+\ln(n-3)+\ln(n-2)+\ln(n-1)+\ln(n+1)$ $=\ln\left(2\cdot 4\cdot 5\cdot 6\cdots(n-3)(n-2)(n-1)(n+1)\right)$ $=\ln\left(\frac{2\cdot 3\cdot 4\cdots(n-1)n(n+1)}{3n}\right)$ $=\ln\left(\frac{(n+1)!}{3n}\right)$, where $k=3n$	

Question 5		
	Let P(n) be the statement "$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$" for $n \in \mathbb{Z}^+$. When $n=1$, LHS = $\sum_{r=1}^{1} r^3 = 1^3 = 1$ RHS = $\frac{1^2(1+1)^2}{4} = 1 = \text{LHS}$ $\therefore$ P(1) is true. Assume P(k) is true for some $k \in \mathbb{Z}^+$, i.e. $\sum_{r=1}^{k} r^3 = \frac{k^2(k+1)^2}{4}$. Want to show P($k$ + 1) is true, i.e. $\sum_{r=1}^{k+1} r^3 = \frac{(k+1)^2(k+2)^2}{4}$. LHS $= \sum_{r=1}^{k+1} r^3$ $= \sum_{r=1}^{k} r^3 + (k+1)^3$ $= \frac{k^2(k+1)^2}{4} + (k+1)^3$ $= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$ $= \frac{(k+1)^2\left(k^2 + 4(k+1)\right)}{4}$ $= \frac{(k+1)^2(k+2)^2}{4} = \text{RHS}$ $\therefore$ P($k+1$) is true. Since P(1) is true, and P(k) is true implies P(k + 1) is true, therefore P(n) is true for all $n \in \mathbb{Z}^+$.	
	$\sum_{r=0}^{2n} (r+1)^3 = \sum_{i-1=0}^{2n} i^3$ Letting $r+1=i$ $= \sum_{i=1}^{2n+1} i^3$ $= \frac{(2n+1)^2(2n+2)^2}{4}$ $= (2n+1)^2(n+1)^2$	

Question 6

Let α be defined as shown.



$$\tan\alpha = \frac{1}{x} \Rightarrow \alpha = \tan^{-1}\frac{1}{x}$$

$$\tan(\theta+\alpha) = \frac{3}{x} \Rightarrow \theta+\alpha = \tan^{-1}\frac{3}{x}$$

$$\therefore \theta = \tan^{-1}\frac{3}{x} - \alpha$$

$$= \tan^{-1}\frac{3}{x} - \tan^{-1}\frac{1}{x} \quad \text{(shown)}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{1+\left(3/x\right)^2}\cdot\frac{-3}{x^2} - \frac{1}{1+\left(1/x\right)^2}\cdot\frac{-1}{x^2}$$

$$= -\frac{x^2}{x^2+3^2}\cdot\frac{3}{x^2} + \frac{x^2}{x^2+1}\cdot\frac{1}{x^2}$$

$$= \frac{1}{x^2+1} - \frac{3}{x^2+9}$$

$$= \frac{x^2+9-3x^2-3}{(x^2+1)(x^2+9)}$$

$$= \frac{6-2x^2}{(x^2+1)(x^2+9)}, \quad a = 6 \text{ and } b = -2 \text{ (shown)}$$

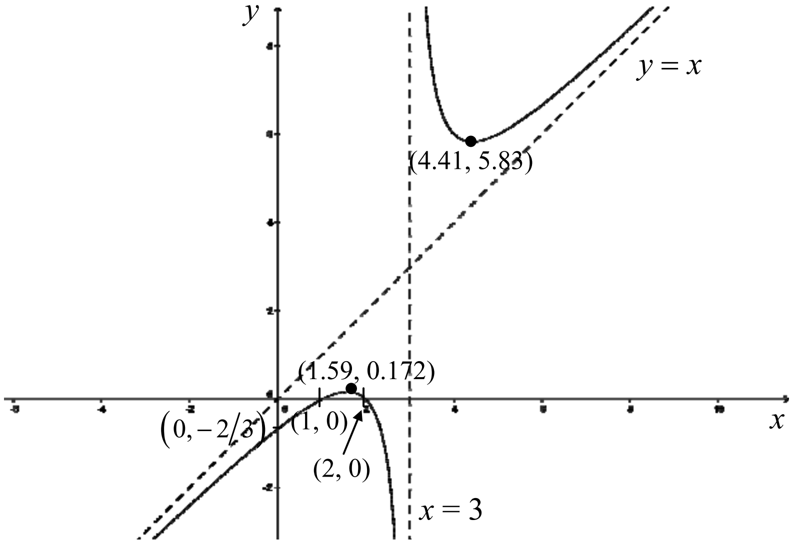
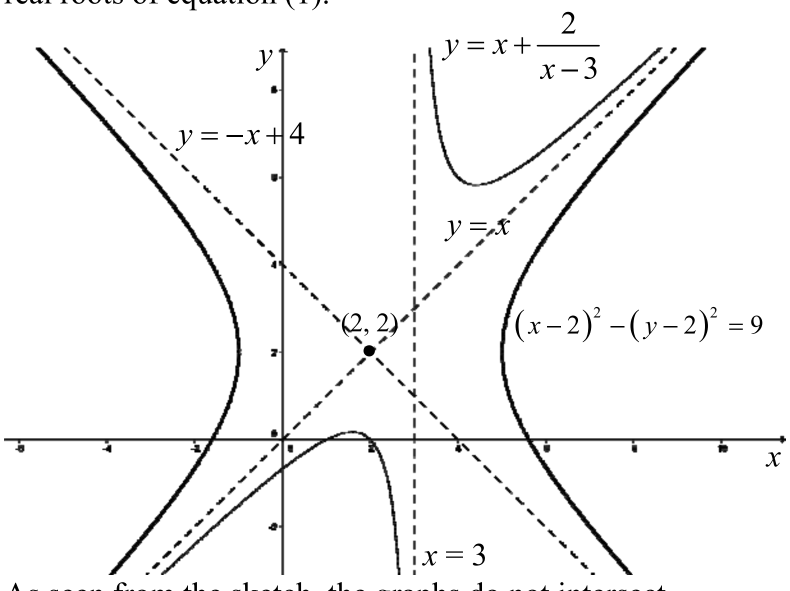
Let $\frac{\mathrm{d}\theta}{\mathrm{d}x} = 0$: $\;6-2x^2 = 0$

$\therefore x = \sqrt{3}$ or $x = -\sqrt{3}$ (rej. as $x > 0$)

x	$(\sqrt{3})^-$	$\sqrt{3}$	$(\sqrt{3})^+$
$\frac{\mathrm{d}\theta}{\mathrm{d}x}$	+ve	0	-ve

$\therefore\ \theta$ is max. at $x = \sqrt{3}$ m.

Unrealistic assumption: The person's eye level is at ground level.
(<u>or</u> The person is viewing from the floor.)

Question 7		
(i)	$y = x$ and $x = 3$.	
(ii)	$y = x + \dfrac{2}{x-3}$ 	
(iii)	$(x-2)^2 - \left(x - 2 + \dfrac{2}{x-3}\right)^2 = 9$ ------ (1) $\Rightarrow (x-2)^2 - \left(\left(x + \dfrac{2}{x-3}\right) - 2\right)^2 = 9$ Hence, sketch the graph of $(x-2)^2 - (y-2)^2 = 9$. The number of intersections between the graphs of $y = x + \dfrac{2}{x-3}$ and $(x-2)^2 - (y-2)^2 = 9$ gives the number of real roots of equation (1).  As seen from the sketch, the graphs do not intersect, implying that equation (1) does not have any real root.	

Question 8		
	Let t denote the time (in hours) taken for the substance to lose its moisture content. $\therefore \frac{\mathrm{d}m}{\mathrm{d}t} = km$ So $\int \frac{1}{m}\ \mathrm{d}m = \int k\ \mathrm{d}t$ $\ln m = kt + C, \quad \text{since } m > 0$ $m = m_0$ when $t = 0$: $\ln m_0 = C$ $\therefore \ln m = kt + \ln m_0$ $m = \frac{1}{2}m_0$ when $t = 3$: $\ln \frac{1}{2}m_0 = 3k + \ln m_0$ $\Rightarrow k = \frac{1}{3}\ln\frac{1}{2}$ or $k = -\frac{\ln 2}{3}$ Thus, $\ln m = -\frac{\ln 2}{3}t + \ln m_0$ or $m = m_0 e^{-\frac{\ln 2}{3}t}$ For $m = 20\% \times m_0 = \frac{m_0}{5}$, $\frac{m_0}{5} = m_0 e^{-\frac{\ln 2}{3}t}$ $\ln\frac{1}{5} = -\frac{\ln 2}{3}t$ $t = 6.96578$ Therefore, it takes 3.97 more hours (from the time when $m = \frac{1}{2}m_0$) for the substance to lose 75% of its original moisture content.	

Question 9		
(a)	Let $y = (1+bx)^n$ $\frac{\mathrm{d}y}{\mathrm{d}x} = n(1+bx)^{n-1}(b)$ $= nb(1+bx)^{n-1}$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = nb(n-1)(1+bx)^{n-2}(b)$ $= n(n-1)b^2(1+bx)^{n-2}$ When $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = nb$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = n(n-1)b^2$ $y = (1+bx)^n = 1 + nbx + \frac{n(n-1)}{2!}(bx)^2 + \ldots$	

(b)

$$\frac{\cos 2x}{1-\sin 2x} \approx \frac{1-\frac{(2x)^2}{2!}}{1-2x}$$

$$=\left(1-2x^2\right)\left(1-2x\right)^{-1}$$

$$=\left(1-2x^2\right)\left(1+(-1)(-2x)+\frac{(-1)(-2)}{2!}(-2x)^2+\ldots\right)$$

$$=\left(1-2x^2\right)\left(1+2x+4x^2+\ldots\right)$$

$$\approx 1+2x+2x^2$$

Question 10

(i)

$$\int \frac{1}{(4+x^2)^2}\,\mathrm{d}x$$

$$=\int \frac{1}{(4+4\tan^2\theta)^2}\;2\sec^2\theta\;\mathrm{d}\theta$$

$$=\int \frac{2\sec^2\theta}{16(1+\tan^2\theta)^2}\;\mathrm{d}\theta$$

$$=\frac{1}{8}\int \frac{1}{\sec^2\theta}\,\mathrm{d}\theta$$

$$=\frac{1}{8}\int \cos^2\theta\;\mathrm{d}\theta$$

$$=\frac{1}{16}\int\left(\cos 2\theta+1\right)\mathrm{d}\theta$$

$$=\frac{1}{16}\left(\frac{\sin 2\theta}{2}+\theta\right)+C$$

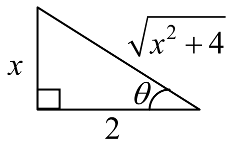
$$=\frac{1}{16}\left(\sin\theta\cos\theta+\theta\right)+C$$

$$=\frac{1}{16}\left(\frac{2x}{x^2+4}+\tan^{-1}\frac{x}{2}\right)+C$$

Let $x=2\tan\theta$

$$\frac{dx}{d\theta}=2\sec^2\theta$$

$$\tan\theta=\frac{x}{2}$$



$$\sin\theta=\frac{x}{\sqrt{x^2+4}}$$

$$\cos\theta=\frac{2}{\sqrt{x^2+4}}$$

(ii)

Required Volume

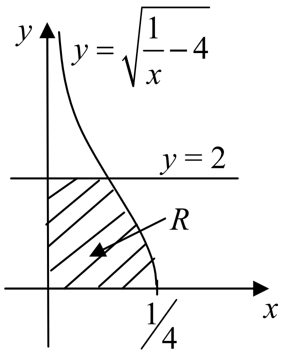
$$=\pi\int_0^2 \frac{1}{(y^2+4)^2}\,\mathrm{d}y$$

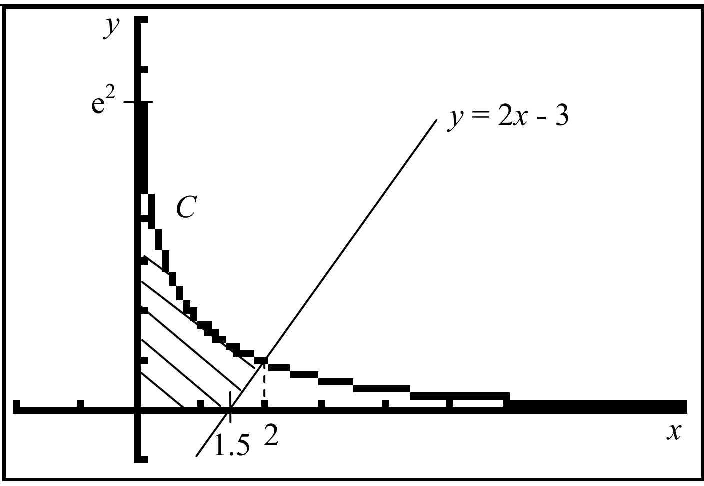
$$=\frac{\pi}{16}\left[\frac{2y}{y^2+4}+\tan^{-1}\left(\frac{y}{2}\right)\right]_0^2$$

$$=\frac{\pi}{16}\left(\frac{1}{2}+\tan^{-1}(1)\right)$$

$$=\frac{\pi}{16}\left(\frac{1}{2}+\frac{\pi}{4}\right)$$

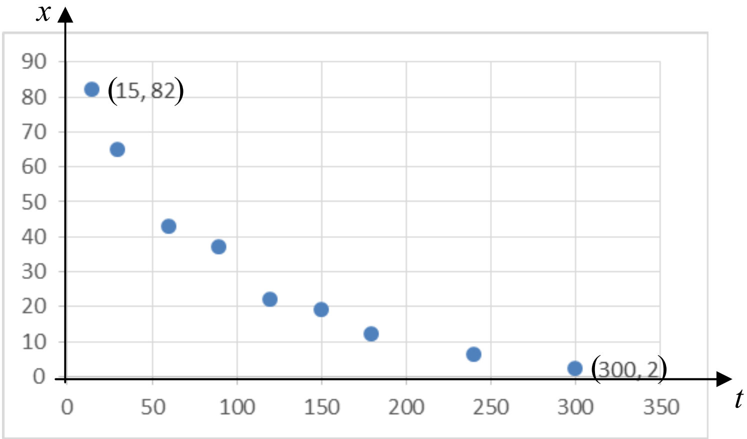
$$=\frac{\pi}{64}\left(2+\pi\right)\;\text{ units}^3$$



Question 11		
(a)	$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx$ $= -xe^{-x} - e^{-x} + C$ $= -e^{-x}(x+1) + C$	
(b) (i)	$x = \frac{(t+2)^2}{2}, \quad y = e^{-t}$ $\frac{dx}{dt} = t+2, \quad \frac{dy}{dt} = -e^{-t}$ $\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ $= \frac{-1}{e^t(t+2)}$ When $t = 0$, $x = 2, y = 1$ and $\frac{dy}{dx} = -\frac{1}{2}$ So, gradient of normal = 2 Hence, equation of normal at $t = 0$: $y - 1 = 2(x-2) \Rightarrow y = 2x - 3$	
(ii)	 Required Area $= \int_0^2 y\, dx - \frac{1}{2}\left(\frac{1}{2}\right)(1)$ $= \int_{-2}^0 y\left(\frac{dx}{dt}\right)dt - \frac{1}{4}$ $= \int_{-2}^0 e^{-t}(t+2)dt - \frac{1}{4}$ $= \int_{-2}^0 te^{-t}dt + 2\int_{-2}^0 e^{-t}dt - \frac{1}{4}$ $= \left[-e^{-t}(t+1)\right]_{-2}^0 - 2\left[e^{-t}\right]_{-2}^0 - \frac{1}{4}$ using result in (a) $= \left[-1 + e^2(-1)\right] - 2\left[1 - e^2\right] - \frac{1}{4}$ $= \left(e^2 - \frac{13}{4}\right)$ units^2	

Question 12

(i)



(ii)

$r = -0.9117118456 = -0.912$ (to 3 s.f.)
Although the value of $|r|$ is close to 1, the scatter diagram shows a curvilinear relationship between x and t. Thus, a linear relationship is not appropriate.

(iii)

$$x = ae^{bt}$$

$$\ln x = \ln\left(ae^{bt}\right)$$

$\ln x = \ln a + bt$, where $\ln a$ and b are constants.
So there is a linear relationship between $\ln x$ and t. (shown)
$r = -0.9938386487 = -0.994$ (to 3 s.f.)

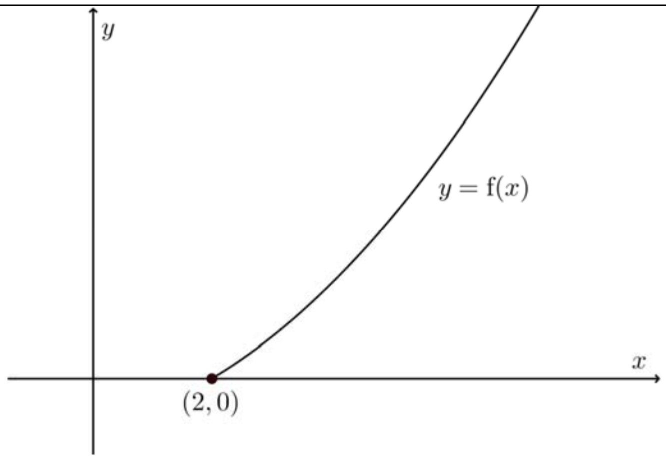
(iv)

From GC, $\ln x = -0.0123434299t + 4.620609296$
When $x = 15$, $t = 154.9455144 \approx 155$ mins (to 3 s.f.)

Since t is the independent variable, the line $\ln x$ on t is used. $r \approx -0.994$ is close to -1 and $x = 15$ is within the data range of x, hence the estimate is reliable.

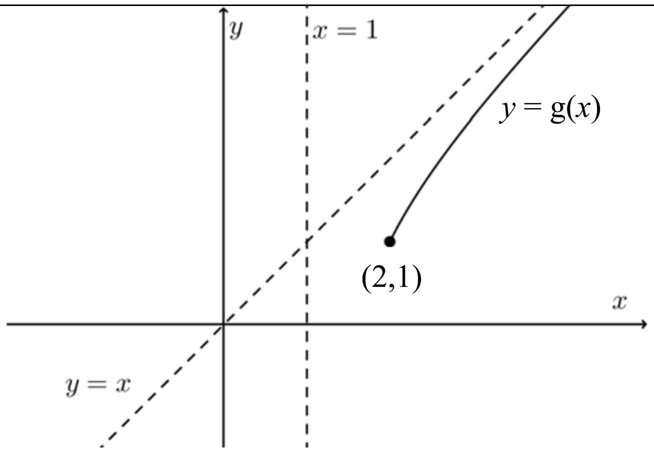
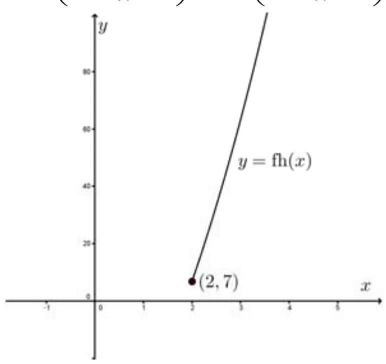
Question 13

(i)



Any horizontal line $y = k$, where $k \in \mathbb{R}$ cuts the graph of $y = f\left(x\right)$ at most once. Therefore **f is one-one function**.

Thus f^{-1} exists.

(ii)	let $y=x^2+2x-8$ $y=(x+1)^2-9$ $y+9=(x+1)^2$ $x=-1+\sqrt{y+9}$ or $-1-\sqrt{y+9}$ (reject since $x\geq 2$) $\therefore \mathrm{f}^{-1}:x\mapsto -1+\sqrt{x+9},\ x\geq 0$	
(iii)	$R_{\mathrm{g}}=[1,\infty)\not\subseteq[2,\infty)=D_{\mathrm{f}}$ $\therefore$ fg does not exist.	
(iv)	 h(x) = k g(x) → varying values of k scales the graph of y = g(x) by factor k parallel to y-axis. For fh to exist, $R_{\mathrm{h}}\subseteq D_{\mathrm{f}}\Rightarrow[k,\infty)\subseteq[2,\infty)$ Graph of y = g(x) should be scaled by a factor of at least 2. Therefore $k\geq 2$	
(v)	When k = 3, $D_{\mathrm{h}}=[2,\infty)\overset{\mathrm{h}}{\mapsto}R_{\mathrm{h}}=[3,\infty)\overset{\mathrm{f}}{\mapsto}R_{\mathrm{fh}}=[7,\infty)$ Or When k = 3, $\mathrm{fh}(x)=\mathrm{f}\left(3\left(x-\frac{1}{x-1}\right)\right)$ $=9\left(x-\frac{1}{x-1}\right)^2+6\left(x-\frac{1}{x-1}\right)-8$  From graph, $R_{\mathrm{fh}}=[7,\infty)$	

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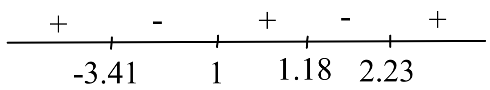
$x^2+2x-8>x-\frac{1}{x-1}$ and $x\geq 2$ (Domain constraint)

$x^2+x-8+\frac{1}{x-1}>0$

$\frac{x^3-9x+9}{x-1}>0$

$\frac{(x-1.184793)(x-2.226682)(x+3.411474)}{x-1}>0$



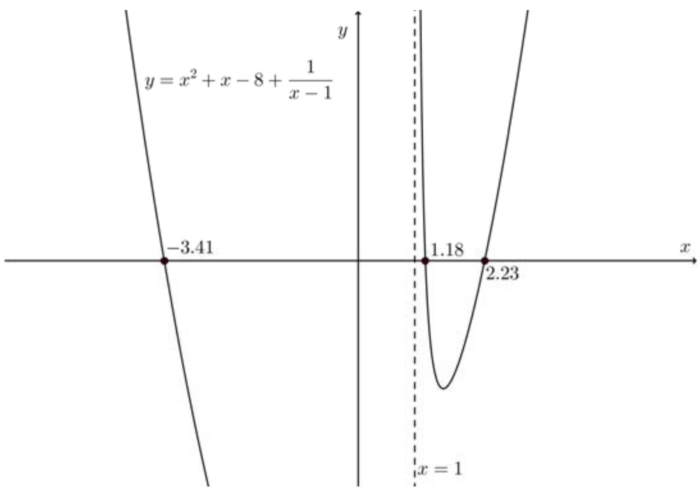
$x<-3.411474$ or $1<x<1.184793$ or $x>2.226682$

but $x\geq 2$

$\therefore x>2.23$ (3sf)

<u>Alternatively</u>:

Consider $\mathrm{f}(x)-\mathrm{g}(x)>0$ where $x\geq 2$.



From graph, $x>2.23$

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VICTORIA JUNIOR COLLEGE
PROMOTIONAL EXAMINATION

MATHEMATICS **9740**

(HIGHER 2)

Friday 8am -11am
26 Sept 2014 3 hours

Additional materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 4 printed pages

VICTORIA JUNIOR COLLEGE **[Turn over**

1 A curve C has equation $y = \dfrac{3x-9}{x^2-x-2}$.

(i) Find the equations of the asymptotes of C. [2]

(ii) Prove, using an algebraic method, that C cannot lie between $\dfrac{1}{t} < y < t$, where t is a constant to be determined. [4]

(iii) Sketch C, indicating its main features. [3]

2 A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{(3\sqrt{3}+1)x_n - 3}{2x_n + 1}$$

for $n \geqslant 1$.

It is given that $x_1 = 2$, and that $x_n \to \lambda$ as $n \to \infty$.

(i) Determine the exact value of λ. [2]

(ii) If $x_n > \sqrt{3}$, show that $x_n > x_{n+1}$. [2]

(iii) Determine the smallest value of n such that $x_n - x_{n+1} < 0.01$. [2]

3 In his later years, the French mathematician Abraham de Moivre noticed that he was getting more lethargic and recorded how many hours he slept daily. He recorded a sleep duration of 6 hours on the first day. On the eighth day, he recorded a sleep duration of 7 hours 45 minutes. He suspected that his daily sleep durations followed an arithmetic progression. On the ninth day, he recorded that he had slept for 8 hours.

(i) Does his latest record support his suspicion? Explain your answer clearly. [3]

(ii) State, with a reason, whether it is possible to conclude that his daily sleep durations followed an arithmetic progression. [1]

Assume, as Abraham de Moivre did, that his daily sleep durations followed an arithmetic progression.

(iii) Find the total number of **waking hours** he had, from the day he started making his records until the 18th day. [3]

4 **(a)** Find $\int \tan^2(3x)\,\mathrm{d}x$. [2]

(b) Find $\int \dfrac{2x+3}{x^2-2x+5}\,\mathrm{d}x$. [4]

(c) Differentiate $\sin^{-1}(x^2)$ with respect to x. [1]

Hence find the exact value of $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\left(\frac{3}{4}\right)^{\frac{1}{4}}} \frac{x}{\sin^{-1}(x^2)\sqrt{1-x^4}}\,\mathrm{d}x$, simplifying your answer. [4]

5 The Maclaurin's series for $\sec x$ is given by

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots .$$

(i) Use the above result to find the series expansion for $\sec^n 2x$, where $n \geq 2$, $n \in \mathbb{Z}^+$, up to and including the term in x^4. Express your answer in the form $a + bnx^2 + \frac{2n}{3}(cn + d)x^4 + \dots$, where a, b, c, d are constants to be determined. [3]

(ii) Hence, obtain an estimate for the value of $\sec^{10} 0.2$ to 3 decimal places. Suggest one possible way to obtain a better estimate. [3]

6 **(i)** Evaluate $\sum_{r=1}^{n} \left(\sin^{r-1} x \cos x \right)$. [2]

(ii) For $0 < x < 2\pi$, $x \neq \pi$, find the values of x such that $\sum_{r=1}^{\infty} \sin^{r-1} x \cos x = 1 + \sin x$. [4]

7 A curve C has parametric equations

$$x = 2t^2, \quad y = e^t .$$

Find $\frac{dy}{dx}$ in terms of t. Sketch C, showing clearly the feature of the curve at the point $t = 0$. [3]

Find the equation of the normal to the curve at the point $\left(8p^2, e^{2p}\right)$, expressing your answer in the form $e^{2p} y = Mx + N$, where M and N are constants in terms of p. [3]

This normal meets the x- and y-axes at points A and B respectively. By using the gradient of the normal or otherwise, find the values of p when triangle OAB is an isosceles triangle, where O is the origin. [3]

8 The function f is defined by

$$f : x \mapsto \frac{-x-2}{5x+1}, \qquad x \in \mathbb{R},\ x \neq -\frac{1}{5} .$$

(i) Explain why both the function f^{-1} and composite function f^2 exist. [4]

(ii) Find $f^2(x)$ and state the range of f^2. [3]

(iii) Determine the solution of the equation $f(x) = f^{-1}(x)$. [2]

The function g is defined by

$$g : x \mapsto \frac{2x-2}{5x+4}, \qquad x \in \mathbb{R},\ x \neq -\frac{4}{5}, x \neq -\frac{1}{5}, x \neq \frac{2}{5} .$$

(iv) Verify that $g^2(x) = f(x)$. [1]

(v) Given that k is an integer, find $g^{49}(k)$, giving your answer in terms of k. [3]

[Turn Over

9 A curve C has equation $x^2-(4y+12)^2=16$. Sketch C, indicating clearly, the coordinates of the axial intercepts, equations of asymptotes and any other relevant features. [4]

Find the range of values of m such that there is no intersection between the line $y=mx-3$ and C. [2]

Another curve C_1 is defined by the parametric equations

$$x=a\sin\theta+4, \qquad y=a\cos\theta-3,$$

where $a>0$.

Find the cartesian equation of C_1. Hence, find the range of values of a such that C_1 intersects C at four distinct points. [3]

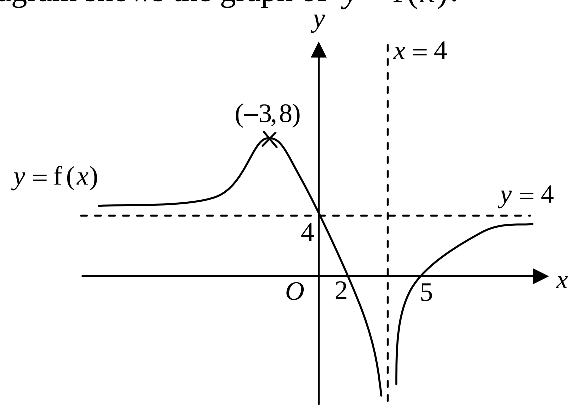
10 **(i)** By expressing $\dfrac{4}{4r^2-1}$ in partial fractions, show that $\displaystyle\sum_{r=1}^{n}\frac{4}{4r^2-1}=2-\frac{2}{2n+1}$. [3]

(ii) Use the method of mathematical induction to prove the result in part **(i)**. [4]

(iii) Explain why $\displaystyle\sum_{r=1}^{\infty}\frac{4}{4r^2-1}$ is a convergent series and state its value. [2]

(iv) Use your answer to part **(iii)** to deduce that $\displaystyle\sum_{r=2}^{\infty}\frac{1}{r^2}<\frac{2}{3}$. [3]

11 **(a)** The diagram shows the graph of $y=\mathrm{f}(x)$.



Sketch, on separate diagrams, the graphs of

(i) $y^2=\mathrm{f}(x)$, [3]

(ii) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

(iii) $y=\mathrm{f}'(x)$. [3]

(b) Describe precisely a sequence of transformations which transforms the graph of $y=x^2+1$ to the graph of $y=-3x^2+1$. [3]

1(i)

$$y=\frac{3x-9}{x^2-x-2}$$
$$=\frac{3x-9}{(x-2)(x+1)}$$

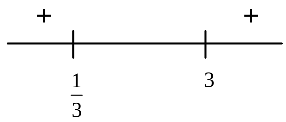
Asymptotes are $x=2$, $x=-1$ and $y=0$

(ii)

$$y=\frac{3x-9}{x^2-x-2}$$
$$yx^2-yx-2y=3x-9$$
$$yx^2+(-y-3)x+9-2y=0$$

Discriminant < 0

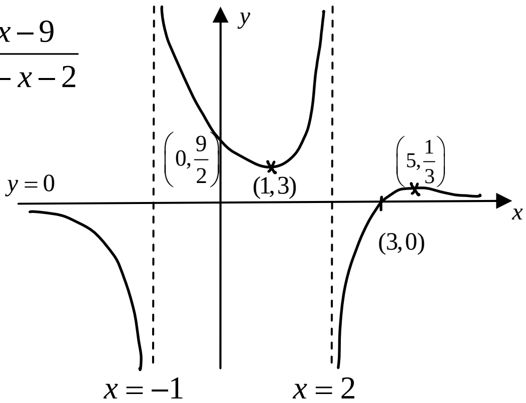
$$(-y-3)^2-4(y)(9-2y)<0$$
$$y^2+6y+9-36y+8y^2<0$$
$$9y^2-30y+9<0$$
$$9(y-3)\left(y-\frac{1}{3}\right)<0$$



$$\frac{1}{3}<y<3 \qquad \therefore t=3$$

(iii)

$$y=\frac{3x-9}{x^2-x-2}$$



2 (i)

As $n\to\infty, x_n\to\lambda$ and $x_{n+1}\to\lambda$

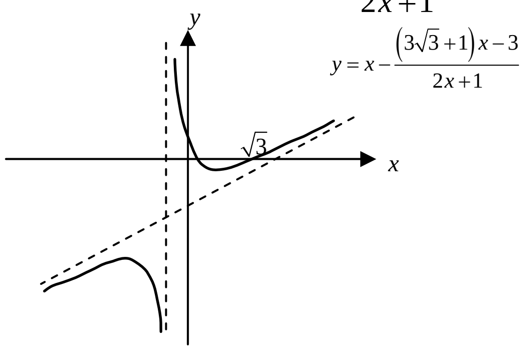
$$\lambda=\frac{\left(3\sqrt{3}+1\right)\lambda-3}{2\lambda+1}$$
$$2\lambda^2+\lambda=3\sqrt{3}\lambda+\lambda-3$$
$$2\lambda^2-3\sqrt{3}\lambda+3=0$$
$$\lambda=\frac{3\sqrt{3}\pm\sqrt{27-4(2)(3)}}{4}$$
$$=\frac{3\sqrt{3}\pm\sqrt{3}}{4}$$
$$=\sqrt{3} \quad \text{or} \quad \frac{1}{2}\sqrt{3}$$

From GC, $\lambda=\sqrt{3}$

(ii)

Consider $\quad x_n - x_{n+1} = x_n - \left[\dfrac{\left(3\sqrt{3}+1\right)x_n - 3}{2x_n+1}\right]$

Sketch the graph of $y = x - \dfrac{\left(3\sqrt{3}+1\right)x - 3}{2x+1}$



From the graph, for $x > \sqrt{3}$,

$x - \dfrac{\left(3\sqrt{3}+1\right)x - 3}{2x+1} > 0$ --(1)

$\therefore$ for $x_n > \sqrt{3}$, $\quad x_n - \dfrac{\left(3\sqrt{3}+1\right)x_n - 3}{2x_n+1} > 0$ --(2)

$x_n - x_{n+1} > 0$

$\therefore x_n > x_{n+1}$

<u>Alternative</u>

Consider $\quad x_n - x_{n+1} = x_n - \left[\dfrac{\left(3\sqrt{3}+1\right)x_n - 3}{2x_n+1}\right]$

$$= \frac{2x_n^{\,2} + x_n - 3\sqrt{3}x_n - x_n + 3}{2x_n+1}$$

$$= \frac{\left(x_n - \sqrt{3}\right)\left(2x_n - \sqrt{3}\right)}{2x_n+1}$$

for $x_n > \sqrt{3}$,

$x_n - \sqrt{3} > 0$, $2x_n - \sqrt{3} > 0$ and $2x_n + 1 > 0$

$\therefore \dfrac{\left(x_n - \sqrt{3}\right)\left(2x_n - \sqrt{3}\right)}{2x_n+1} > 0$

$x_n - x_{n+1} > 0$

$\therefore x_n > x_{n+1}$

<table>
<tr><td>(iii)</td><td>From the GC,

n	x_n
4	1.7816
5	1.7617
6	1.75
7	1.7429

$x_n - x_{n+1} = 0.0199$
$x_n - x_{n+1} = 0.0117$
$x_n - x_{n+1} = 0.0071$

For $x_n - x_{n+1} < 0.01$,
Least $n = 6$</td></tr>
<tr><td>3 (i)

(ii)

(iii)</td><td>$u_1 = 6$ hrs = 360 min
$u_8 = 7$ hrs 45 mins = 465 min
If his sleep duration follows an AP,
Then $u_8 = 360 + 7d = 465$
$\Rightarrow d = 15$
$\therefore u_9 = 360 + 8(15) = 480 = 8$ hrs
Hence his latest record supports his suspicion.

It is not possible to conclude AP as there is insufficient evidence from finite records.

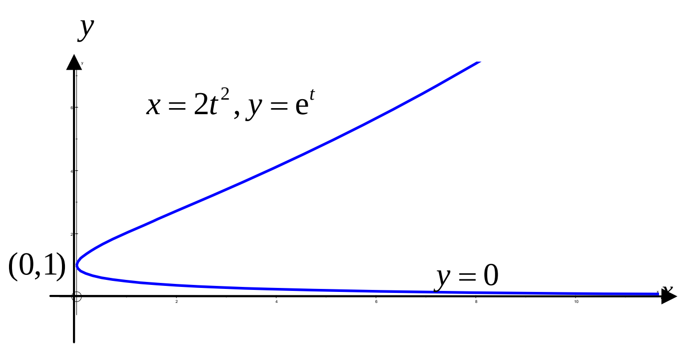
Total number of sleeping hours in the first 18 days
$= S_{18}$
$= \frac{18}{2}\left[2(6) + (18-1)(0.25)\right]$
$= 146.25$

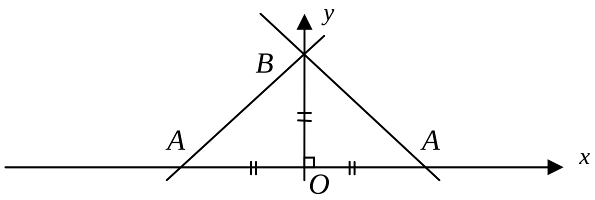
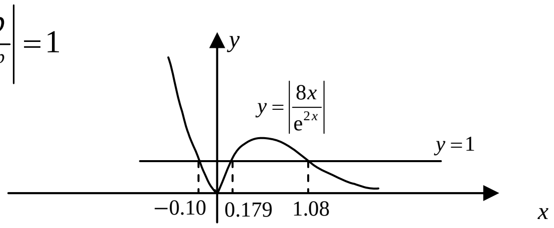
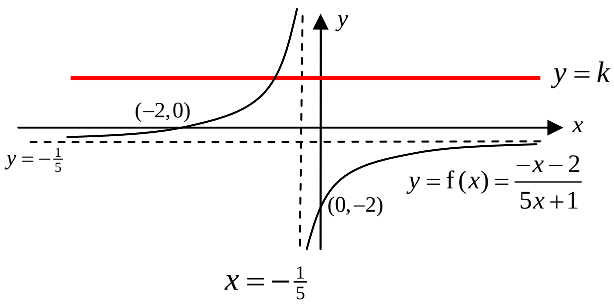
Total number of waking hours in the first 18 days
$= 24(18) - 146.25$
$= 285.75$</td></tr>
<tr><td>4(a)

4(b)</td><td>$\int \tan^2(3x)\,\mathrm{d}x = \int \left[\sec^2(3x) - 1\right] \mathrm{d}x$
$= \frac{1}{3}\tan(3x) - x + C$

$\int \frac{2x+3}{x^2-2x+5}\,\mathrm{d}x$
$= \int \frac{(2x-2)+5}{x^2-2x+5}\,\mathrm{d}x$
$= \int \frac{2x-2}{x^2-2x+5}\,\mathrm{d}x + 5\int \frac{1}{(x-1)^2+4}\,\mathrm{d}x$
$= \ln\left|x^2-2x+5\right| + 5\times\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$</td></tr>
</table>

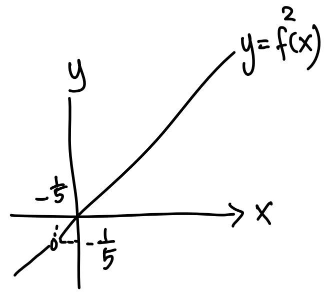
	$=\ln\left(x^2-2x+5\right)+\frac{5}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+C$ $\left(\because (x-1)^2+4>0 \text{ for all real values of } x\right)$
4(c)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}\left(x^2\right)\right)=\frac{2x}{\sqrt{1-x^4}}$ $\int_{\frac{1}{\sqrt{2}}}^{\left(\frac{3}{4}\right)^{\frac{1}{4}}} \frac{x}{\sin^{-1}\left(x^2\right)\sqrt{1-x^4}}\,\mathrm{d}x=\frac{1}{2}\int_{\frac{1}{\sqrt{2}}}^{\left(\frac{3}{4}\right)^{\frac{1}{4}}} \frac{\frac{2x}{\sqrt{1-x^4}}}{\sin^{-1}\left(x^2\right)}\,\mathrm{d}x$ $=\frac{1}{2}\left[\ln\left\|\sin^{-1}\left(x^2\right)\right\|\right]_{\frac{1}{\sqrt{2}}}^{\left(\frac{3}{4}\right)^{\frac{1}{4}}}$ $=\frac{1}{2}\left[\ln\left\|\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\|-\ln\left\|\sin^{-1}\left(\frac{1}{2}\right)\right\|\right]$ $=\frac{1}{2}\left[\ln\frac{\pi}{3}-\ln\frac{\pi}{6}\right]$ $=\frac{1}{2}\ln 2$
5(i) **(ii)**	$\sec^n 2x$ $=\left[1+\frac{(2x)^2}{2}+\frac{5(2x)^4}{24}+\ldots\right]^n$ $=\left[1+\left(2x^2+\frac{10x^4}{3}+\ldots\right)\right]^n$ $=1+\binom{n}{1}\left(2x^2+\frac{10x^4}{3}+\ldots\right)+\binom{n}{2}\left(2x^2+\frac{10x^4}{3}+\ldots\right)^2+\ldots$ $=1+n\left(2x^2+\frac{10x^4}{3}\right)+\frac{n(n-1)}{2}\left(4x^4\right)+\ldots$ $=1+2nx^2+\frac{10nx^4}{3}+2n(n-1)x^4+\ldots$ $=1+2nx^2+\frac{2nx^4}{3}(5+3(n-1))+\ldots$ $=1+2nx^2+\frac{2nx^4}{3}(3n+2)+\ldots$ $\therefore a=1, b=2, c=3, d=2$ Let n = 10, x = 0.1 $\sec^{10}0.2=1+2(10)(0.1)^2+\frac{2(10)(0.1)^4}{3}(30+2)+\ldots$ $=1.221$ (3dp) Use the series expansion for $\sec 2x$ [or $\cos 2x$] up to higher powers of x (e.g. x^6)

6 (i)	$\sum_{r=1}^{n} \sin^{r-1} x \cos x$ $= \cos x + \sin x \cos x + \sin^2 x \cos x + ... + \sin^{n-1} x \cos x$ $= \dfrac{\cos x\left(1-\sin^n x\right)}{1-\sin x}$
(ii)	Sum to infinity, $S = \dfrac{\cos x}{1-\sin x}$ $\dfrac{\cos x}{1-\sin x} = 1+\sin x$ $\cos x = 1-\sin^2 x$ $\cos x = \cos^2 x$ $\cos x(\cos x - 1) = 0$ $\cos x = 0$ or $\cos x = 1$ $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ or $x = 0, 2\pi$ (reject since $0 < x < 2\pi$) Since $\lvert\sin x\rvert < 1$, $x \neq \dfrac{\pi}{2}, \dfrac{3\pi}{2}$. But when $x = \dfrac{3\pi}{2}$, $\sin x = -1$, and $\cos x = 0$ $\therefore \sum_{r=1}^{\infty} \sin^{r-1} x \cos x = 0+0+\ldots = 0$ $1+\sin x = 1-1 = 0$ $\therefore$ LHS $=$ RHS $\therefore x = \dfrac{3\pi}{2}$
7	$\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$ $= \dfrac{e^t}{4t}$ At $t = 0, \dfrac{dy}{dx} \to \infty$ 

	At $t = 2p, \dfrac{dy}{dx} = \dfrac{e^{2p}}{8p}$ $\therefore$Gradient of normal = $-\dfrac{8p}{e^{2p}}$ Equation of normal at $t = 2p$ is $y - e^{2p} = \left(-\dfrac{8p}{e^{2p}}\right)\left(x - 8p^2\right)$ $e^{2p}y = -8px + 64p^3 + e^{4p}$  For ΔOAB to be an isosceles triangle, Gradient of normal = 1 or −1 $\left\|\dfrac{8p}{e^{2p}}\right\| = 1$  $p = -0.102, 0.179, 1.08$ (3s.f.)
8 (i)	 Since every horizontal line $y = k$ ($k \in R_f$) cuts the graph of $y = f(x)$ exactly once, f is 1-1 $\Longrightarrow$ f^{-1} exists. $R_f = (-\infty, -\frac{1}{5}) \cup (-\frac{1}{5}, \infty)$ or $\mathbb{R} \setminus \left\{-\frac{1}{5}\right\}$ or $\left\{y \in \mathbb{R} : y < -\frac{1}{5} \text{ or } y > -\frac{1}{5}\right\}$ or $\left\{y \in \mathbb{R} : y \neq -\frac{1}{5}\right\}$ $D_f = \left(-\infty, -\frac{1}{5}\right) \cup \left(-\frac{1}{5}, \infty\right)$ Since $R_f \subseteq D_f$ $\quad \therefore f^2$ exists.

(ii)

$$\begin{aligned}
f^2(x) &= f\left(\frac{-x-2}{5x+1}\right) \\
&= \frac{-\left(\frac{-x-2}{5x+1}\right)-2}{5\left(\frac{-x-2}{5x+1}\right)+1} \\
&= \frac{x+2-10x-2}{-5x-10+5x+1} \\
&= \frac{-9x}{-9} \\
&= x
\end{aligned}$$



$$R_{f^2} = \left(-\infty, -\frac{1}{5}\right) \cup \left(-\frac{1}{5}, \infty\right)$$

(iii)

$$f(x) = f^{-1}(x) \Rightarrow f^2(x) = x$$

$$\therefore x \in \mathbb{R}, x \neq -\frac{1}{5}$$

(iv)

$$\begin{aligned}
g^2(x) &= g\left(\frac{2x-2}{5x+4}\right) \\
&= \frac{2\left(\frac{2x-2}{5x+4}\right)-2}{5\left(\frac{2x-2}{5x+4}\right)+4} \\
&= \frac{4x-4-10x-8}{10x-10+20x+16} \\
&= \frac{-6x-12}{30x+6} \\
&= \frac{-x-2}{5x+1} = f(x)
\end{aligned}$$

(v)

$$g^2(x) = f(x) \Rightarrow g^4(x) = f^2(x) = x$$

$$\therefore g^{49}(k) = g[\overbrace{g^4 \cdot g^4 \ldots g^4}^{\text{12 times}}(k)] = g(k) = \frac{2k-2}{5k+4}$$

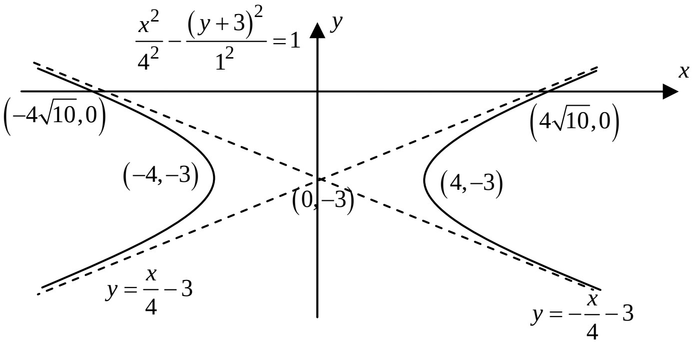
9

$$x^2 - (4y+12)^2 = 4^2$$

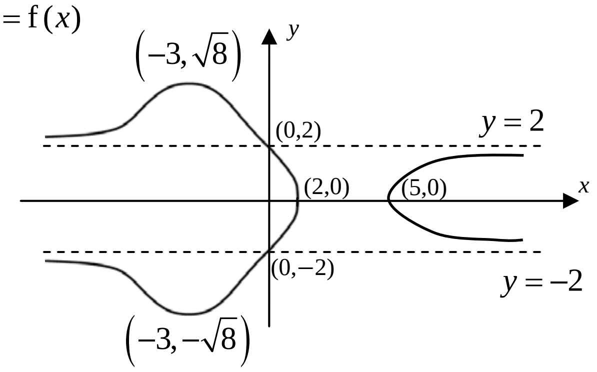
$$x^2 - 4^2(y+3)^2 = 4^2$$

$$\frac{x^2}{4^2} - \frac{(y+3)^2}{1^2} = 1$$

Asymptotes: $(y+3)^2 = \left(\frac{x}{4}\right)^2$

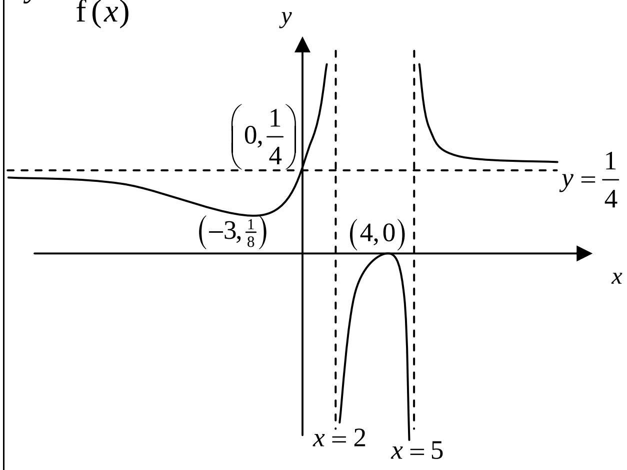
	$y+3=\pm\frac{x}{4}$ $y=\frac{x}{4}-3$ or $y=-\frac{x}{4}-3$ Centre: $(0,-3)$ Vertices: $(4,-3),(-4,-3)$ When $y=0$, $x^2-16(3)^2=4^2$ $x^2=160$ $x=\pm4\sqrt{10}$  Every line $y=mx-3$ passes through (0, -3) and must have a steeper gradient compared to the asymptotes of C. $\therefore m\geq\frac{1}{4}$ or $m\leq-\frac{1}{4}$ $x-4=a\sin\theta$ $-(1)$ $y+3=a\cos\theta$ $-(2)$ $(1)^2+(2)^2$: $a^2\sin^2\theta+a^2\cos^2\theta=(x-4)^2+(y+3)^2$ $\therefore (x-4)^2+(y+3)^2=a^2$ For C_1 to intersect C at four distinct points, $a>8$
10(i)	$\frac{4}{4r^2-1}=\frac{4}{(2r-1)(2r+1)}$ $=\frac{2}{2r-1}-\frac{2}{2r+1}$

	$\sum_{r=1}^{n} \frac{4}{4r^2-1} = \sum_{r=1}^{n} \left(\frac{2}{2r-1} - \frac{2}{2r+1} \right)$ $= \frac{2}{1} - \cancel{\frac{2}{3}}$ $+ \cancel{\frac{2}{3}} - \cancel{\frac{2}{5}}$ $+ \cancel{\cdots}$ $+ \cancel{\frac{2}{2n-3}} - \cancel{\frac{2}{2n-1}}$ $+ \cancel{\frac{2}{2n-1}} - \frac{2}{2n+1}$ $= 2 - \frac{2}{2n+1}$
(ii)	Let P_n be the statement $\sum_{r=1}^{n} \frac{4}{4r^2-1} = 2 - \frac{2}{2n+1}$, $n \in \mathbb{Z}^+$ When $n=1$, LHS of $P_1 = \frac{4}{4(1)^2-1} = \frac{4}{3}$ RHS of $P_1 = 2 - \frac{2}{2(1)+1} = \frac{4}{3}$ $\therefore$ P_1 is true Assume P_k is true for some $k \geq 1$, i.e. $\sum_{r=1}^{k} \frac{4}{4r^2-1} = 2 - \frac{2}{2k+1}$ We want to show P_{k+1} is true i.e. $\sum_{r=1}^{k+1} \frac{4}{4r^2-1} = 2 - \frac{2}{2k+3}$ LHS of $P_{k+1} = \sum_{r=1}^{k+1} \frac{4}{4r^2-1}$ $= 2 - \frac{2}{2k+1} + \frac{4}{4(k+1)^2-1}$ $= 2 + \frac{4}{4k^2+8k+3} - \frac{2}{2k+1}$ $= 2 + \frac{4}{(2k+3)(2k+1)} - \frac{2}{2k+1}$ $= 2 + \frac{4-4k-6}{(2k+3)(2k+1)}$ $= 2 - \frac{4k+2}{(2k+3)(2k+1)}$

	$= 2 - \dfrac{2}{2k+3} =$ RHS of P_{k+1} $\therefore P_k$ true $\Longrightarrow P_{k+1}$ true Since (1) P_1 is true (2) P_k true $\Longrightarrow P_{k+1}$ true By mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$
(iii)	As $n \to \infty$, $\dfrac{2}{2n+1} \to 0$ $\quad\therefore \displaystyle\sum_{r=1}^{n} \frac{4}{4r^2-1} \to 2$ $\therefore$ Series converges. $\displaystyle\sum_{r=1}^{\infty} \frac{4}{4r^2-1} = 2$
(iv)	For all real values of r, $4r^2 > 4r^2 - 1$ $\dfrac{1}{4r^2} < \dfrac{1}{4r^2-1}$ $\dfrac{1}{r^2} < \dfrac{4}{4r^2-1}$ $\displaystyle\sum_{r=2}^{\infty} \frac{1}{r^2} < \sum_{r=2}^{\infty} \frac{4}{4r^2-1}$ $\displaystyle = \sum_{r=1}^{\infty} \frac{4}{4r^2-1} - \frac{4}{4(1)^2-1}$ $= 2 - \dfrac{4}{3}$ $= \dfrac{2}{3}$ (shown)
11(a) **(i)**	$y^2 = \mathrm{f}(x)$ 

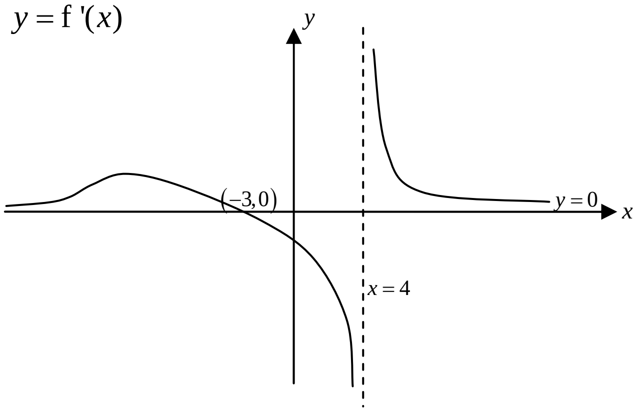
(ii)

$y = \dfrac{1}{\mathrm{f}(x)}$



(iii)

$y = \mathrm{f}'(x)$



(b)

$$y = x^2 + 1 \xrightarrow{(1)} \begin{matrix} -y = x^2 + 1 \\ \text{or} \\ y = -x^2 - 1 \end{matrix} \xrightarrow{(2)} \begin{matrix} \frac{1}{3}y = -x^2 - 1 \\ \text{or} \\ y = -3x^2 - 3 \end{matrix}$$

$$\xrightarrow{(3)} \begin{matrix} y - 4 = -3x^2 - 3 \\ \text{or} \\ y - 4 = -3x^2 + 1 \end{matrix}$$

The series of transformation is

1. Reflection in the *x*-axis
2. Stretch parallel to the *y*-axis, factor 3.
3. Translation 4 units in the positive *y*-direction