ANGLO-CHINESE JUNIOR COLLEGE
MATHEMATICS DEPARTMENT

MATHEMATICS
Higher 2
Paper 1

9740

8 October 2013

JC 1 PROMOTIONAL EXAMINATION

Time allowed**: 3 hours**

Additional Materials: List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Index number, Form Class, graphic and/or scientific calculator model/s on the cover page.
Write your Index number and full name on all the work you hand in.
Write in dark blue or black pen on your answer scripts.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in the question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together

This document consists of **7** printed pages.

THE BEST IS YET TO BE

Anglo-Chinese Junior College

ANGLO-CHINESE JUNIOR COLLEGE
MATHEMATICS DEPARTMENT
JC 1 Promotional Examination 2013

MATHEMATICS 9740
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/ 100

Index No: | | | |

Form Class: ____________

Name: ______________________________

Calculator model: ____________________

Arrange your answers in the same numerical order.

Place this cover sheet on top of them and tie them together with the string provided.

Question no.	Marks
1	/ 2
2	/ 3
3	/ 4
4	/ 4
5	/ 5
6	/ 6
7	/ 6
8	/ 7
9	/ 6
10	/ 9
11	/ 9
12	/ 11
13	/ 13
14	/ 9
15	/ 6

1 The graph of $y = \mathrm{f}(x)$ undergoes, in succession, the following transformations:

Step 1: a translation of 1 unit in the negative *y*-direction; followed by

Step 2: a stretch with scale factor 2 parallel to the *x*-axis.

The equation of the resulting curve is $y = \ln(2x+3)$, $x > -\frac{3}{2}$. Determine the equation of the graph, $y = \mathrm{f}(x)$. **[2]**

2 Given that the curve $y = ax^3 + bx^2 + cx + d$ has turning points at (-4, 258) and (4, 2). Write and solve a system of simultaneous linear equations satisfied by the constants a, b, c and d. **[3]**

3 Differentiate the following with respect to *x*.

(i) $\sqrt{\cos^{-1}\left(\frac{x}{2}\right)}$, **[2]**

(ii) $\ln\sqrt{\frac{(x+1)^3}{x^2-1}}$. **[2]**

4 Find the following integrals:

(i) $\int \frac{1}{x\sqrt{\ln x}}\, dx$; **[2]**

(ii) $\int \frac{e^{-2x}}{\sqrt{4-e^{-4x}}}\, dx$. **[2]**

5 Without the use of a graphing calculator, solve the inequality $\frac{3x^2+6x-10}{x^2+3x-4} \geq 2$. **[3]**

Deduce the range of values of *x* such that $\frac{3x^2+6|x|-10}{x^2+3|x|-4} \geq 2$. **[2]**

[Turn Over

6 A curve C has parametric equations

$$x = 1 - cos\theta\,, \qquad y = \theta + \sin\theta\,,$$

where $0 \le \theta \le 2\pi$,

(i) Show that $\frac{dy}{dx} = \cot\frac{1}{2}\theta$ and find the gradient of C at the point P where $\theta = \pi$. **[3]**

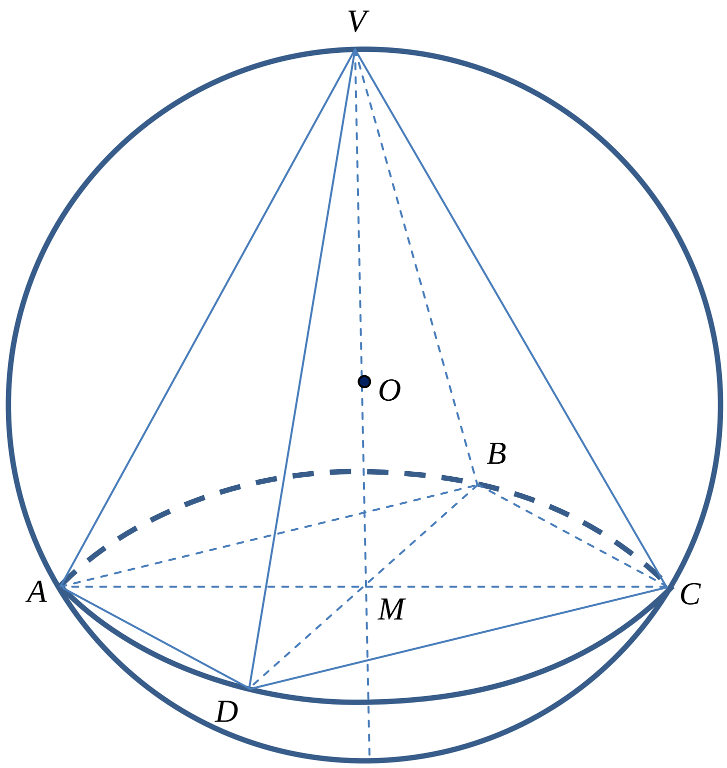
(ii) The tangent at P meets the y-axis at A. The tangent at the point Q, where $\theta = \frac{\pi}{2}$, meets the y-axis at B. Find the area of triangle ABP. **[3]**

7 A right pyramid block has a square base $ABCD$ and its vertical height VM is $(a + x)$ where $0 < x < a$. M is the point where the diagonals AC and BD of the square meet. This right pyramid block is inscribed in a sphere of fixed radius a so that the vertices V, A, B, C and D of the block just touch the interior of the sphere with the vertical height VM passing through the centre O of the sphere.

(i) Show that the length of the side of the square base $ABCD$ is $\sqrt{2(a^2 - x^2)}$. **[2]**

(ii) Hence, find the maximum volume of the block in terms of a. **[4]**

[Volume of a pyramid = $\frac{1}{3} \times$ base area $\times$ height]



8 The function f is defined by $\mathrm{f}: x \mapsto x + \dfrac{1}{x}$ for $x \in \mathbb{R},\ x \geq 1$.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. **[3]**

(ii) Find $\mathrm{fff}^{-1}(x)$ and state its domain and range. **[3]**

(iii) Show that the composite function f^2 exists. **[1]**

9 If $f(k) = \dfrac{1}{k^2}$, show that $f(k) - f(k+2) = \dfrac{4(k+1)}{k^2(k+2)^2}$. **[1]**

Hence, show that the sum to n terms of the series $\dfrac{2}{(1^2)(3^2)} + \dfrac{3}{(2^2)(4^2)} + \dfrac{4}{(3^2)(5^2)} + \ldots$ is

$\dfrac{1}{4}\left(\dfrac{5}{4} - \dfrac{1}{(n+1)^2} - \dfrac{1}{(n+2)^2}\right)$. **[3]**

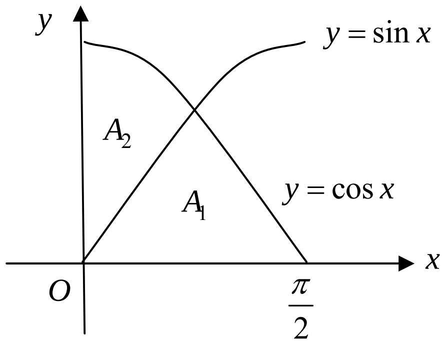
Show that $\displaystyle\sum_{k=2}^{n} \frac{k+1}{k^2(k+2)^2} < \frac{13}{144}$ for all values of $n \geq 2$. **[2]**

10 **(a)** Use integration by parts to find the exact value of $\int_1^e (\ln x)^2\, dx$. **[4]**

(b) By means of the substitution $x = 3\cos^2\theta + 7\sin^2\theta$, where $0 \leq \theta \leq \dfrac{\pi}{2}$, prove that

$\displaystyle\int_3^7 \frac{1}{\sqrt{[(7-x)(x-3)]}}\, dx = \pi$. **[5]**

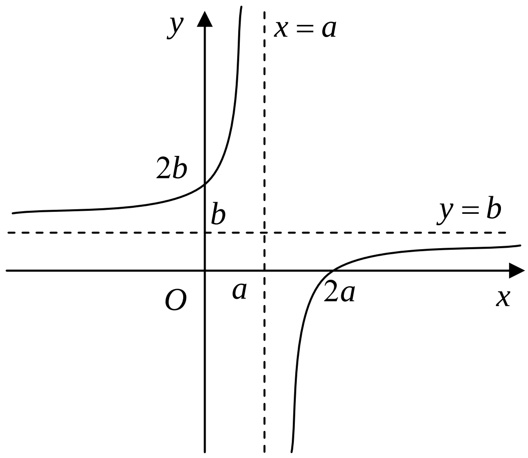
11



The region bounded by the axes and the curve $y = \cos x$ from $x = 0$ to $x = \dfrac{\pi}{2}$ is divided into two parts, of areas A_1 and A_2, by the curve $y = \sin x$ (see diagram). Prove that $A_1 = \left(\sqrt{2}\right) A_2$. **[5]**

The line $y = \dfrac{1}{2}$ meets the curve $y = \sin x$ and the y-axis at P and Q respectively. The region OPQ, bounded by the arc OP and the lines PQ and QO, is rotated through 4 right angles about the x-axis to form a solid of revolution of volume V. Find the exact value of V in terms of π. **[4]**

12 The diagram shows the graph of $y = \mathrm{f}(x)$. The curve crosses the axes at the points $(2a,\ 0)$ and $(0,\ 2b)$. The asymptotes are $x = a$ and $y = b$. The gradient of the curve at the point $(0,\ 2b)$ is 1.



On separate diagrams, sketch the graphs of

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, **[3]**

(ii) $y^2 = \mathrm{f}(x)$, **[3]**

(iii) $y = \mathrm{f}'(x)$, **[2]**

(iv) $y = \mathrm{f}(|x|)$, **[3]**

giving the equations of any asymptotes and the coordinates of any points of intersection with the *x*- and *y*-axes.

13 **(a)** In triangle *ABC*, angle $A = \left(\dfrac{\pi}{2} - \alpha\right)$ radians, $AB = AC = b$ and $BC = a$.

Show that $\dfrac{a}{b} = \dfrac{\cos\alpha}{\sin\left(\dfrac{\pi + 2\alpha}{4}\right)}$. **[1]**

Deduce, for small values of α, $a \approx \sqrt{2}b\left(1 - \dfrac{\alpha}{2} - \dfrac{\alpha^2}{8}\right)$. **[4]**

(b) Given that $y = e^{\sin^{-1} 4x}$, show that

(i) $\sqrt{1-16x^2}\,\dfrac{dy}{dx} = 4y$, **[1]**

(ii) $\left(1-16x^2\right)\dfrac{d^2y}{dx^2} - 16x\dfrac{dy}{dx} = 16y$. **[2]**

By further differentiation of the result, find the Maclaurin series for *y* up to and including the term in x^3. **[3]**

By choosing a suitable value of *x*, show that $e^{-\frac{\pi}{6}} \approx \dfrac{7}{12}$. **[2]**

14 **(a)** Prove by induction that $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$. **[4]**

(b) Use the result in part **(a)** to

(i) evaluate $\sum_{r=1}^{30}(2r+3)^2$, **[2]**

(ii) prove that $\sum_{r=1}^{n}r^2 = \frac{1}{6}n(n+1)(2n+1)$. **[3]**

15 A man met with an accident and went into a coma on 10th January 2013. As a result, he did not pay the bank the outstanding balance of \$$M$ for his credit card bill when it is due for payment on 27th January 2013. On the 27th of each month when the payment for the credit card bill is due, the bank will charge a 2% interest on any outstanding balance that is unpaid. After the 2% interest has been added, the bank will still charge an additional late payment charge of \$$L$ monthly.

(a) Express in terms of L and M, his outstanding balance on his credit card on 1st February 2013. **[1]**

(b) If the man still remains in coma exactly n months later on the day he met with an accident, show that the accumulated outstanding balance on the man's credit card is $1.02^n M + 50L(1.02^n - 1)$. **[3]**

(c) Given that $M = 1000$ and $L = 55$. Find the least value of n when the accumulated outstanding balance on his credit card first exceeds \$2010. **[2]**

~ End of Paper ~

Anglo-Chinese Junior College

H2 Mathematics 9740

2013 JC 1 PROMO Solution

Qn	Solution
1	$y=\ln(2x+3),\ x>-\dfrac{3}{2}$ Before Step 2: $y=\ln[2(2x)+3]=\ln(4x+3)$ Before Step 1: $y=\ln(4x+3)+1$ **OR** Resulting curve: $y=\mathrm{f}\left(\dfrac{1}{2}x\right)-1=\ln(2x+3)$ $\Rightarrow\ \mathrm{f}\left(\dfrac{1}{2}x\right)=\ln\left[4\left(\dfrac{1}{2}x\right)+3\right]+1$ $\therefore\ y=\mathrm{f}(x)=\ln(4x+3)+1$
2	Given $y=ax^3+bx^2+cx+d$ $\dfrac{dy}{dx}=3ax^2+2bx+c$ When $x=-4,\ \dfrac{dy}{dx}=0,\ 3a(-4)^2+2b(-4)+c=0$ $48a-8b+c=0$ (1) When $x=4,\ \dfrac{dy}{dx}=0,\ 3a(4)^2+2b(4)+c=0$ $48a+8b+c=0$ (2) When $x=-4,\ y=258,$ $a(-4)^3+b(-4)^2+c(-4)+d=258$ $-64a+16b-4c+d=258$ (3) When $x=4,\ y=2,$ $a(4)^3+b(4)^2+c(4)+d=2$ $64a+16b+4c+d=2$ (4) Using G.C. $a=1,\ b=0,\ c=-48,\ d=130.$
3i	$\dfrac{d}{dx}\left(\sqrt{\cos^{-1}\left(\dfrac{x}{2}\right)}\right)=\dfrac{1}{2}\left(\cos^{-1}\left(\dfrac{x}{2}\right)\right)^{-\frac{1}{2}}\dfrac{-1}{\sqrt{1-\left(\dfrac{x}{2}\right)^2}}\cdot\dfrac{1}{2}$ $=-\dfrac{1}{2\sqrt{(4-x^2)\cos^{-1}\left(\dfrac{x}{2}\right)}}$

3ii	$$\frac{d}{dx}\left(\ln\sqrt{\frac{(x+1)^3}{x^2-1}}\right)=\frac{d}{dx}\left(\ln\left(\frac{x+1}{\sqrt{x-1}}\right)\right)$$ $$=\frac{d}{dx}\left(\ln(x+1)-\frac{1}{2}\ln(x-1)\right)$$ $$=\frac{1}{x+1}-\frac{1}{2(x-1)}$$ Alternative solution: $$\frac{d}{dx}\left(\ln\sqrt{\frac{(x+1)^3}{x^2-1}}\right)=\frac{1}{\sqrt{\frac{(x+1)^3}{x^2-1}}}\times\frac{1}{2}\frac{1}{\sqrt{\frac{(x+1)^3}{x^2-1}}}\times\frac{3(x+1)^2(x^2-1)-(x+1)^3(2x)}{(x^2-1)^2}$$ $$=\frac{x-3}{2(x^2-1)}$$
4(i)	$$\int\frac{1}{x\sqrt{\ln x}}dx$$ $$=\int\frac{\left(\frac{1}{x}\right)}{\sqrt{\ln x}}dx \quad \text{using } \int[f(x)]^n f'(x)dx=\frac{1}{n+1}[f(x)]^{n+1}+c$$ $$=2\sqrt{\ln x}+c$$
(ii)	$$\int\frac{e^{-2x}}{\sqrt{4-e^{-4x}}}dx$$ $$=-\frac{1}{2}\int\frac{-2e^{-2x}}{\sqrt{2^2-(e^{-2x})^2}}dx \text{ using } \int\frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx=\sin^{-1}\left[\frac{f(x)}{a}\right]+c$$ $$=-\frac{1}{2}\sin^{-1}\left(\frac{e^{-2x}}{2}\right)+c$$

5

$$\frac{3x^2+6x-10}{x^2+3x-4}\geq 2$$

$$\frac{3x^2+6x-10}{x^2+3x-4}-2\geq 0$$

$$\frac{3x^2+6x-10-2x^2-6x+8}{x^2+3x-4}\geq 0$$

$$\frac{x^2-2}{(x+4)(x-1)}\geq 0$$

$$\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)(x+4)(x-1)\geq 0$$

+		-		+		-		+
	-4		$-\sqrt{2}$		1		$\sqrt{2}$	

$x<-4$ or $-\sqrt{2}\leq x<1$ or $x\geq\sqrt{2}$

$$\frac{3x^2+6|x|-10}{x^2+3|x|-4}\geq 2$$

$$\frac{3|x|^2+6|x|-10}{|x|^2+3|x|-4}\geq 2$$

$|x|<-4$ (n.a.) ; $-\sqrt{2}\leq|x|<1$; $|x|\geq\sqrt{2}$

$-1<x<1$ or $x\leq-\sqrt{2}$ or $x\geq\sqrt{2}$

6i

$x=1-cos\theta$ $\qquad$ $y=\theta+\sin\theta$

$\dfrac{dx}{d\theta}=\sin\theta$ $\qquad$ $\dfrac{dy}{d\theta}=1+\cos\theta$

$$\frac{dy}{dx}=\frac{1+\cos\theta}{\sin\theta}$$

$$=\frac{2\cos^2\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}$$

$$=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$=\cot\frac{\theta}{2}$$

When $\theta=\pi$, $\dfrac{dy}{dx}=0$.

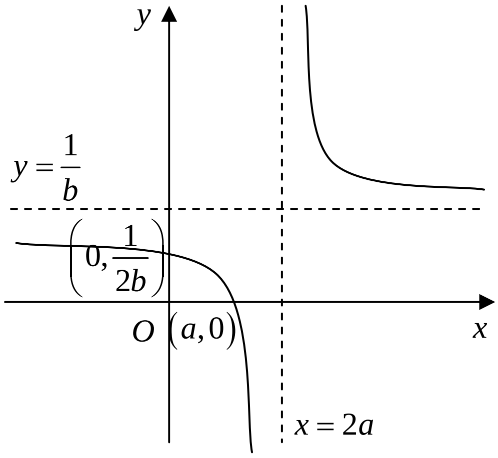
ii)	Coordinate at $A(0,\pi)$ When $\theta=\frac{\pi}{2}$, $\frac{dy}{dx}=1$ Equation of tangent: $\dfrac{y-\frac{\pi}{2}-\sin\frac{\pi}{2}}{x-1+\cos\frac{\pi}{2}}=1$ $y=x+\frac{\pi}{2}$ Coordinate at $B\left(0,\frac{\pi}{2}\right)$ $\therefore$ area of triangle ABP $=\frac{1}{2}\times\left(\pi-\frac{\pi}{2}\right)\times 2$ $=\frac{\pi}{2}$
7i)	Diagonal $DB=2\sqrt{a^2-x^2}$ Length of side of square $=\sin\left(\frac{\pi}{4}\right)2\sqrt{a^2-x^2}$ $=\frac{\sqrt{2}}{2}2\sqrt{a^2-x^2}$ $=\sqrt{2\left(a^2-x^2\right)}$
ii)	Volume of block , $v=\frac{2}{3}\left(a^2-x^2\right)\left(x+a\right)$ Diff. w.r.t.x $\frac{dv}{dx}=\frac{2}{3}\left[\left(a^2-x^2\right)+\left(x+a\right)\left(-2x\right)\right]$ $=\frac{2}{3}\left[\left(a-x\right)\left(a+x\right)+\left(x+a\right)\left(-2x\right)\right]$ $=\frac{2}{3}\left(x+a\right)\left(a-3x\right)$ For stationary point, $\frac{dv}{dx}=0$ $0=\frac{2}{3}\left(x+a\right)\left(a-3x\right)$ $x=-a$ (n.a.) $\quad x=\frac{a}{3}$

	$\frac{d^2v}{dx^2}=\frac{2}{3}\left[(x+a)(-3)+(a-3x)\right]$ $\frac{d^2v}{dx^2}<0$ when $x=\frac{a}{3}$ Max. volume of block , $v=\frac{2}{3}\left(a^2-\left(\frac{a}{3}\right)^2\right)\left(\left(\frac{a}{3}\right)+a\right)$ $=\frac{64a^3}{81}units^3$
8 (i)	$\mathrm{f}:x\mapsto x+\frac{1}{x}$ for $x\in\mathbb{R},\ x\geq 1$ Let $y=x+\frac{1}{x}\ \Rightarrow\ x^2-yx+1=0$ $x=\frac{y+\sqrt{y^2-4}}{2}$ or $x=\frac{y-\sqrt{y^2-4}}{2}$ (rejected since $x\geq 1$ & $y\geq 2$) $\mathrm{f}^{-1}(x)=\frac{x+\sqrt{x^2-4}}{2}$ $D_{\mathrm{f}^{-1}}=[2,\infty)$
8 (ii)	$\mathrm{fff}^{-1}(x)=\mathrm{f}(x)=x+\frac{1}{x}$ Domain of $\mathrm{fff}^{-1}=D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=[2,\infty)$ Range of $\mathrm{fff}^{-1}=\{\mathrm{f}(x):x\in[2,\infty)\}=\left[\frac{5}{2},\infty\right)$
8 (iii)	$\mathrm{f}:x\mapsto x+\frac{1}{x}$ for $x\in\mathbb{R},\ x\geq 1$ $D_f=[1,\infty)$, $R_f=[2,\infty)$ Since $R_f\subseteq D_f$, ff exists.
9	Given $f(k)=\frac{1}{k^2}$ $f(k)-f(k+2)$ $=\frac{1}{k^2}-\frac{1}{(k+2)^2}$ $=\frac{(k+2)^2-k^2}{k^2(k+2)^2}$ $=\frac{(k^2+4k+4)-k^2}{k^2(k+2)^2}$ $=\frac{4(k+1)}{k^2(k+2)^2}$

	$\frac{2}{(1)^2(3)^2}+\frac{3}{(2)^2(4)^2}+\frac{4}{(3)^2(5)^2}+\ldots+\frac{n+1}{(n)^2(n+2)^2}$ $=\sum_{k=1}^{n}\frac{k+1}{k^2(k+2)^2}$ $=\sum_{k=1}^{n}\frac{1}{4}\left(\frac{1}{k^2}-\frac{1}{(k+2)^2}\right)$ $=\frac{1}{4}\left(\sum_{k=1}^{n}\frac{1}{k^2}-\sum_{k=1}^{n}\frac{1}{(k+2)^2}\right)$ $=\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots+\frac{1}{n^2}\right)$ $-\frac{1}{4}\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots+\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}\right)$ $=\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)$ $=\frac{1}{4}\left(\frac{5}{4}-\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)$ $\sum_{k=2}^{n}\frac{k+1}{k^2(k+2)^2}$ $=\sum_{k=1}^{n}\frac{k+1}{k^2(k+2)^2}-\frac{2}{(1^2)(3^2)}$ $=\frac{1}{4}\left(\frac{5}{4}-\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)-\frac{2}{9}$ $=\frac{13}{144}-\frac{1}{4}\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}\right)$ $<\frac{13}{144}$ $(\because \frac{1}{(n+1)^2}>0$ and $\frac{1}{(n+2)^2}>0\ \forall n\in\mathbb{Z}^+)$
10 (a)	$\int_1^e(\ln x)^2\,dx=\left[(\ln x)^2(x)\right]_1^e-\int_1^e x\left(\frac{2\ln x}{x}\right)dx$ $=e-2\int_1^e(\ln x)\,dx$ $=e-2\left\{\left[(\ln x)(x)\right]_1^e-\int_1^e x\left(\frac{1}{x}\right)dx\right\}$ $=e-2e+2(e-1)=e-2$
10 (b)	$x=3\cos^2\theta+7\sin^2\theta$ $\frac{dx}{d\theta}=6\cos\theta(-\sin\theta)+14\sin\theta\cos\theta$ $=8\sin\theta\cos\theta$ when $x=3$, $3\cos^2\theta+7\sin^2\theta=3 \Rightarrow \sin^2\theta=0 \Rightarrow \theta=0$ when $x=7$, $3\cos^2\theta+7\sin^2\theta=7 \Rightarrow \cos^2\theta=0 \Rightarrow \theta=\frac{\pi}{2}$

	$\int_3^7 \frac{1}{\sqrt{[(7-x)(x-3)]}}\,dx$ $= \int_0^{\frac{\pi}{2}} \frac{8\sin\theta\cos\theta}{\sqrt{[(7-3\cos^2\theta-7\sin^2\theta)(3\cos^2\theta+7\sin^2\theta-3)]}}\,d\theta$ $= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{[(4\cos^2\theta)(4\sin^2\theta)]}}(8\sin\theta\cos\theta)\,d\theta$ $= 2\int_0^{\frac{\pi}{2}} 1\,d\theta = 2\left(\frac{\pi}{2}\right) = \pi$ (proved)
11	$A_1 = \int_0^{\frac{\pi}{4}}(\sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\cos x)\,dx = [-\cos x]_0^{\frac{\pi}{4}} + [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $= \left(-\frac{\sqrt{2}}{2}+1\right)+\left(1-\frac{\sqrt{2}}{2}\right)$ $= 2-\sqrt{2}$ $A_2 = \int_0^{\frac{\pi}{4}}(\cos x-\sin x)\,dx = [\sin x+\cos x]_0^{\frac{\pi}{4}}$ $= \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)-(0+1)$ $= \sqrt{2}-1$ **<u>OR</u>** $A_2 = \int_0^{\frac{\pi}{2}}(\cos x)\,dx - A_1 = [\sin x]_0^{\frac{\pi}{2}} - (2-\sqrt{2})$ $= (1+0)-(2-\sqrt{2})$ $= \sqrt{2}-1$ **<u>OR</u>** $A_2 = \int_0^{\frac{\sqrt{2}}{2}}(\sin^{-1} y)\,dy + \int_{\frac{\sqrt{2}}{2}}^{1}(\cos^{-1} y)\,dy$ $\therefore \quad A_1 = 2-\sqrt{2} = \sqrt{2}(\sqrt{2}-1) = \sqrt{2}A_2$ (proved)
	$P = \left(\frac{\pi}{6}, \frac{1}{2}\right)$ $V = \pi\left(\frac{1}{2}\right)^2\frac{\pi}{6} - \pi\int_0^{\frac{\pi}{6}}(\sin x)^2\,dx$ $= \frac{\pi^2}{24} - \frac{\pi}{2}\int_0^{\frac{\pi}{6}}(1-\cos 2x)\,dx$ $= \frac{\pi^2}{24} - \frac{\pi}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{6}}$ $= \frac{\pi^2}{24} - \frac{\pi}{2}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}\pi}{8} - \frac{\pi^2}{24}$

<table>
<tr><td>12
(i)</td><td>$y=\dfrac{1}{\mathrm{f}(x)}$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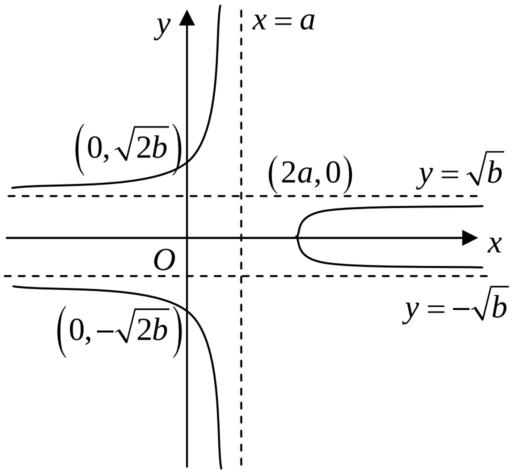
(ii)</td><td>$y^2=\mathrm{f}(x)$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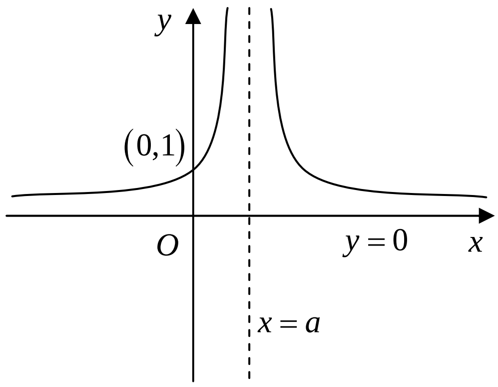
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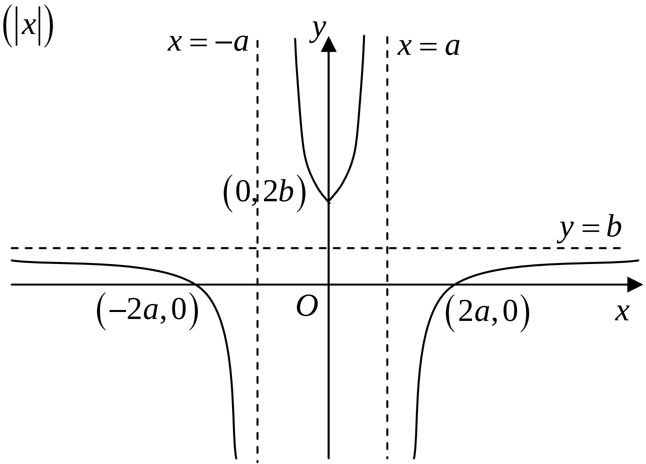
</td></tr>
<tr><td>13a)</td><td>Sine rule</td></tr>
</table>

13b)	$$\frac{\sin A}{a}=\frac{\sin B}{b}$$ $$\frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{a}=\frac{\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}{b}$$ $$\frac{a}{b}=\frac{\cos\alpha}{\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}$$ $$\frac{a}{b}=\frac{\cos\alpha}{\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}$$ $$=\frac{\cos\alpha}{\sin\frac{\pi}{4}\cos\frac{\alpha}{2}+\cos\frac{\pi}{4}\sin\frac{\alpha}{2}}$$ $$=\frac{\cos\alpha}{\frac{\sqrt{2}}{2}\cos\frac{\alpha}{2}+\frac{\sqrt{2}}{2}\sin\frac{\alpha}{2}}$$ For small values of α $$\frac{a}{b}\approx\frac{1-\frac{\alpha^2}{2}}{\frac{\sqrt{2}}{2}\left(1-\frac{\left(\frac{\alpha}{2}\right)^2}{2}\right)+\frac{\sqrt{2}}{2}\left(\frac{\alpha}{2}\right)}$$ $$=\frac{1-\frac{\alpha^2}{2}}{\frac{\sqrt{2}}{2}\left(1+\left(\frac{\alpha}{2}-\frac{\alpha^2}{8}\right)\right)}$$ $$a=\sqrt{2}b\left(1-\frac{\alpha^2}{2}\right)\left(1+\left(\frac{\alpha}{2}-\frac{\alpha^2}{8}\right)\right)^{-1}$$ $$=\sqrt{2}b\left(1-\frac{\alpha^2}{2}\right)\left(1+(-1)\left(\frac{\alpha}{2}-\frac{\alpha^2}{8}\right)+\frac{(-1)(-2)}{2}\left(\frac{\alpha}{2}-\frac{\alpha^2}{8}\right)^2+\ldots\right)$$ $$=\sqrt{2}b\left(1-\frac{\alpha^2}{2}\right)\left(1-\frac{\alpha}{2}+\frac{3\alpha^2}{8}\right)$$ $$=\sqrt{2}b\left(1-\frac{\alpha}{2}-\frac{\alpha^2}{8}\right)$$

i)	$y = e^{\sin^{-1} 4x}$ $\ln y = \sin^{-1} 4x$ diff. w.r.t x $\frac{1}{y}\frac{dy}{dx} = \frac{4}{\sqrt{1-16x^2}}$ $\sqrt{1-16x^2}\frac{dy}{dx} = 4y$
ii)	$\sqrt{1-16x^2}\frac{dy}{dx} = 4y$ diff. w.r.t x $\sqrt{1-16x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{1}{2}\left(1-16x^2\right)^{-\frac{1}{2}}\left(-32x\right)\right) = 4\frac{dy}{dx}$ $\sqrt{1-16x^2}\frac{d^2y}{dx^2} - \frac{16x}{\sqrt{1-16x^2}}\frac{dy}{dx} = 4\frac{dy}{dx}$ $\left(1-16x^2\right)\frac{d^2y}{dx^2} - 16x\frac{dy}{dx} = 4\sqrt{1-16x^2}\frac{dy}{dx}$ $\left(1-16x^2\right)\frac{d^2y}{dx^2} - 16x\frac{dy}{dx} = 16y$ $\left(1-16x^2\right)\frac{d^2y}{dx^2} - 16x\frac{dy}{dx} = 16y$ diff. w.r.t x $\left(1-16x^2\right)\frac{d^3y}{dx^3} + \left(-32x\right)\frac{d^2y}{dx^2} - 16x\frac{d^2y}{dx^2} - 16\frac{dy}{dx} = 16\frac{dy}{dx}$ $\left(1-16x^2\right)\frac{d^3y}{dx^3} - 48x\frac{d^2y}{dx^2} = 32\frac{dy}{dx}$ $f(0) = 1$ $f'(0) = 4$ $f''(0) = 16$ $f'''(0) = 128$ $f(x) = 1 + 4x + 8x^2 + \frac{64}{3}x^3 + \ldots$ $e^{-\frac{\pi}{6}} = e^{\sin^{-1} 4x}$ $-\frac{\pi}{6} = \sin^{-1} 4x$ $\sin\left(-\frac{\pi}{6}\right) = 4x$ $x = -\frac{1}{8}$

	$e^{-\frac{\pi}{6}} \approx 1+4\left(-\frac{1}{8}\right)+8\left(-\frac{1}{8}\right)^2+\frac{64}{3}\left(-\frac{1}{8}\right)^3+\ldots$ $=1-\frac{1}{2}+\frac{1}{8}-\frac{1}{24}$ $=\frac{7}{12}$
14 (a)	Let P(n) denote the statement $\sum_{r=1}^{n}(2r-1)^2=\frac{1}{3}n(2n-1)(2n+1)$. When n = 1, $LHS=\sum_{r=1}^{1}(2r-1)^2=1^2=1$ $RHS=\frac{1}{3}(1)(2-1)(2+1)=\frac{1}{3}(1)(1)(3)=1$ $LHS=RHS$ Hence P(1) is true. Assume P(k) is true for some $k\in\mathbb{Z}^+$. i.e. $\sum_{r=1}^{k}(2r-1)^2=\frac{1}{3}k(2k-1)(2k+1)$. We need to show that P(k+1) is true. i.e. $\sum_{r=1}^{k+1}(2r-1)^2=\frac{1}{3}(k+1)(2k+1)(2k+3)$ $\sum_{r=1}^{k+1}(2r-1)^2$ $=\sum_{r=1}^{k}(2r-1)^2+(2k+1)^2$ $=\frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2$ $=\frac{1}{3}(2k+1)\left[k(2k-1)+3(2k+1)\right]$ $=\frac{1}{3}(2k+1)\left[2k^2+5k+3\right]$ $=\frac{1}{3}(2k+1)\left[(k+1)(2k+3)\right]$ $=\frac{1}{3}(k+1)(2k+1)(2k+3)$ Hence $P(k)\Rightarrow P(k+1)$ is true. Since P(1) is true and $P(k)\Rightarrow P(k+1)$ is true, by the principle of Mathematical induction, P(n) is true $\forall n\in\mathbb{Z}^+$

14 b (i)	<u>Method 1:</u> $\sum_{r=1}^{30}(2r+3)^2$ $=5^2+7^2+9^2+\ldots+63^2$ $=\left(1^2+3^2+5^2+\ldots+63^2\right)-1^2-3^2$ $=\sum_{r=1}^{32}(2r-1)^2-10$ $=\frac{1}{3}(32)(64-1)(64+1)-10$ $=\frac{1}{3}(32)(63)(65)-10$ $=43670$ <u>Method 2:</u> Let $r=k-2$ $2r+3=2(k-2)+3=2k-1$ When $r=1,\ k-2=1\Rightarrow k=3$ When $r=30,\ k-2=30\Rightarrow k=32$ $\sum_{r=1}^{30}(2r+3)$ $=\sum_{k=3}^{32}(2k-1)$ $=\sum_{r=1}^{32}(2r-1)^2-1^2-3^2$ $=\sum_{r=1}^{32}(2r-1)^2-10$ $=\frac{1}{3}(32)(64-1)(64+1)-10$ $=\frac{1}{3}(32)(63)(65)-10$ $=43670$
14b (ii)	To prove: $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$ Proof: $\sum_{r=1}^{n}(2r-1)^2=\frac{1}{3}n(2n-1)(2n+1)$ $\sum_{r=1}^{n}(4r^2-4r+1)=\frac{1}{3}n(2n-1)(2n+1)$

$$4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 = \frac{1}{3}n(2n-1)(2n+1)$$

$$4\sum_{r=1}^{n} r^2 - 4\left[\frac{1}{2}(n)(n+1)\right] + n = \frac{1}{3}n(2n-1)(2n+1)$$

$$4\sum_{r=1}^{n} r^2 = \frac{1}{3}n(2n-1)(2n+1) + 2n(n+1) - n$$

$$4\sum_{r=1}^{n} r^2 = \frac{1}{3}n\left[(2n-1)(2n+1) + 6(n+1) - 3\right]$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{12}n\left[(4n^2-1) + (6n+6) - 3\right]$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{12}n\left[4n^2 + 6n + 2\right]$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{12}n(2n+2)(2n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

15 (a)

Original amount = \$$M$
2% interest charged = \$$0.02M$
Late payment charge = \$$L$
Total outstanding balance = \$$(1.02M + L)$

(b)

No of months later	Outstanding balance left unpaid after 27th of the month
1	$1.02M + L$
2	$1.02^2 M + 1.02L + L$
3	$1.02^3 M + 1.02^2 L + 1.02L + L$
4	$1.02^4 M + 1.02^3 L + 1.02^2 L + 1.02L + L$
. . .	. . .
n	$1.02^n M + 1.02^{n-1} L + \ldots + 1.02^2 L + 1.02L + L$

Outstanding balance left unpaid n months later

$= 1.02^n M + 1.02^{n-1} L + \ldots + 1.02^2 L + 1.02L + L$

$= 1.02^n M + \underbrace{(1.02^{n-1} L + \ldots + 1.02^2 L + 1.02L + L)}$

This is a G.P. with first term $a = L$, common ratio $r = 1.02$ and number of terms is n.

$$= 1.02^n M + \frac{L(1.02^n - 1)}{1.02 - 1}$$

$$= 1.02^n M + \frac{L(1.02^n - 1)}{0.02}$$

	$= 1.02^n M + \frac{100L(1.02^n - 1)}{2}$ $= 1.02^n M + 50L(1.02^n - 1)$
(c)	Putting $1.02^n M + 50L(1.02^n - 1) > 2010$. Given M = 1000 and L = 55. $1.02^n(1000) + 50(55)(1.02^n - 1) > 2010$ $1.02^n(1000) + (2750)(1.02^n) - 2750 > 2010$ $1.02^n(1000) + (2750)(1.02^n) > 4760$ $1.02^n(3750) > 4760$ $1.02^n > \frac{476}{375}$ $\log(1.02^n) > \log(\frac{476}{375})$ $n\log(1.02) > \log(\frac{476}{375})$ $n > \log(\frac{476}{375}) \Big/ \log(1.02)$ $n > 12.04$ Since n is a positive integer, n = 13, 14, 15, … Hence n = 13.

1. **(i)*** Find the expansion of $\dfrac{1-x^2}{\sqrt{4-x}}$ in ascending powers of x, up to and including the term in x^2. [3]

(ii)* State the set of values of x for which this expansion is valid. [1]

(iii)* Hence, by substituting a suitable value of x, find an approximation for $\sqrt{15}$ in the form $\dfrac{a}{b}$, where a and b are integers to be determined. [3]

2. Evaluate $\sum_{r=2}^{n}\left(2^{-r}+2nr+n^2\right)$, giving your answer in terms of n. [4]

3. A curve C is defined by parametric equations

$$x=\mathrm{e}^{\theta}\cos\theta,\quad y=\mathrm{e}^{-\theta}\sin\theta,\qquad \text{for } -\frac{\pi}{2}\le\theta\le 0.$$

(i) Sketch the curve C, indicating the axial intercepts in exact form. [2]

(ii) Show that the area bounded by the curve C and the axes is given by

$$\int_{-\frac{\pi}{2}}^{0}(\sin^2\theta-\sin\theta\cos\theta)\,\mathrm{d}\theta\ .$$

Hence determine its exact value. [5]

4. A sequence u_n, $n=0,1,2,3,\ldots$, is such that $u_0=-\dfrac{1}{2}$ and

$$u_{n+1}=u_n+\ln(n+1)-\frac{1}{4n^2-1}\qquad\text{for all } n\ge 0.$$

(i) Prove by mathematical induction that $u_n=\ln(n!)+\dfrac{1}{2(2n-1)}$. [5]

(ii) Hence find $\sum_{n=0}^{N}\left[\ln(n+1)-\dfrac{1}{4n^2-1}\right]$. [3]

(iii) Does the series found in **(ii)** converge? Give a reason for your answer. [1]

(iv) Using the series found in **(ii)**, evaluate $\sum_{n=2}^{N}\left[\ln(n-1)-\dfrac{1}{4(n-2)^2-1}\right]$. [2]

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5. The curve with equation $y=-\sqrt{-2x}$ is transformed by a translation of 2 units in the positive x-direction, followed by a reflection in the y-axis.

(i) Find the equation of the resultant curve in the form $y=\mathrm{f}(x)$ and the coordinates of the points where this curve crosses the x- and y- axes. On a single diagram, sketch the graph $y=\mathrm{f}(x)$ and its inverse. [5]

(ii) Solve the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, giving your answers in exact form. [3]

(iii) The function g is defined such that $\mathrm{f}^{-1}\mathrm{g}(x)=\dfrac{x^2}{2}-2$. Find $\mathrm{g}(x)$. [2]

6. Without using a calculator, solve $\dfrac{x(4x-1)}{2x-1} < 3x+1$. [3]

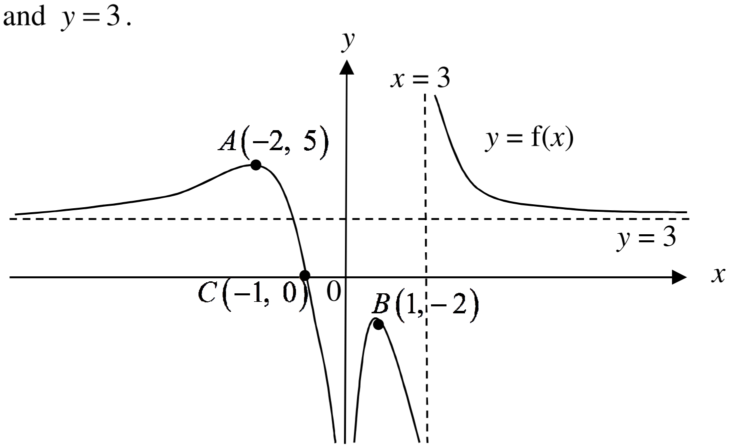
Hence, find the solutions of the inequalities

(a) $x-5 < 3x+1 < \dfrac{x(4x-1)}{2x-1}$,

(b) $\dfrac{\cos x\,(4\cos x+1)}{2\cos x+1} > 3\cos x-1$ for $0 \le x \le \pi$,

leaving your answers in exact form. [6]

7. The diagram shows a sketch of the curve $y=\mathrm{f}(x)$. The curve cuts the x-axis at $C(-1,\ 0)$, has stationary points at $A(-2,\ 5)$ and $B(1,-2)$, and asymptotes $x=0$, $x=3$ and $y=3$.



On separate diagrams, sketch the graphs of

(i) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y=\mathrm{f}'(x)$, [3]

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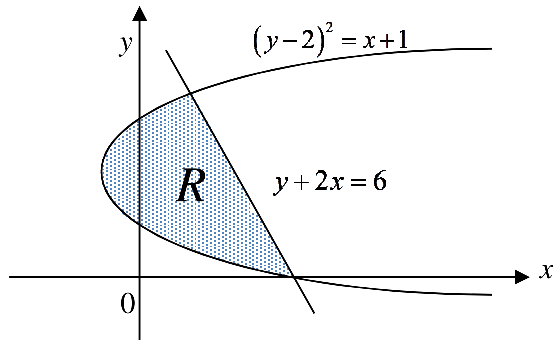
showing, in each case, the asymptotes, the coordinates of the stationary points and the points of intersection with the axes, whenever possible.

8. **(a)*** Find $\int \frac{1}{x^2} \ln(x+1) \, dx$. [3]

(b)*The diagram shows a shaded region R bounded by the curve $(y-2)^2 = x+1$ and the line $y+2x=6$.



Find the volume generated when R is rotated through 2π radians about the x-axis, leaving your answer correct to 3 significant figures. [4]

9. The lines l_1 and l_2 have equations

$$\frac{x-1}{3} = \frac{y-2}{a}, \; z=1 \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}, \; \lambda \in \mathbb{R}$$

respectively, where a is a constant.

(i) Given that l_1 and l_2 intersect at the point N, find N and the value of a. [3]

(ii) Show that the position vector of F, the foot of the perpendicular from the point $P(2,1,1)$ to the line l_2 is $\frac{4}{3}\mathbf{i}+\frac{5}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$. [3]

(iii) Find the position vector of the point P', the reflection of P in the line l_2. [2]

(iv) The point Q has coordinates (1, 2, 0). Find the ratio of the area of triangle NQP to the area of triangle FQP'. [3]

10. A curve C has equation $y = \frac{x^2}{x+3\lambda}$, $x \neq -3\lambda$ and λ is a positive constant.

(i) Find the coordinates of the stationary points of C. [3]

(ii) Draw a sketch of C, labeling clearly, in terms of λ, the asymptotes and the stationary points. [2]

(iii) Use the graph in **(ii)**, find the number of roots of the equation $x^4 - 2\lambda x - 6\lambda^2 = 0$. [3]

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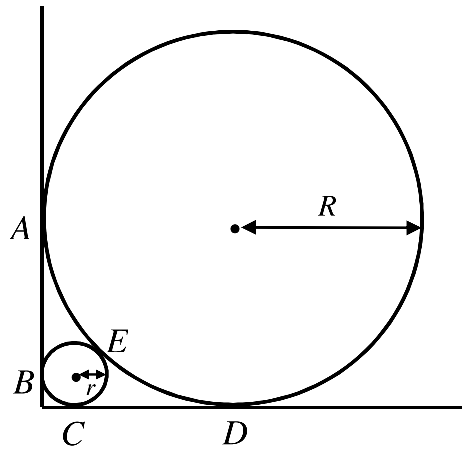
The function f is defined by $\mathrm{f}: x \mapsto \dfrac{x^2}{x+3\lambda}$, $x \le -6\lambda$.

(iv) Show that f^2 exists and find the value of $\mathrm{f}^2(-6\lambda)$. [4]

11. Two solid cylinders of the same height are placed at a corner of the wall such that the vertices *A*, *B*, *C* and *D* touch the wall. At point *E*, the two cylinders touch each other. The diagram below shows a cross section of the cylinders.



Let *r* be the radius of the small cylinder and *R* be the radius of the big cylinder.

(i) Show that $R = \left(\sqrt{2}+1\right)^2 r$ [2]

(ii) Given that the volume of the small cylinder is $\dfrac{16\pi}{\sqrt{2}+1}\text{cm}^3$, find the **exact value** of the radius *r* such that the surface area of the big cylinder is a minimum. [5]

12. Mary has a monthly income of \$4000. She is considering applying for a car loan of \$40,000 for 6 years which charges an interest rate of 3.00% per annum, compounded monthly. Interest is chargeable immediately when the loan sum is drawn out. The monthly repayment, $\$m$, is fixed throughout the loan tenure.

(i) Show that the calculated loan balance at the end of the n^{th} loan month, after the monthly repayment is made, is given by

$$40000\left(\frac{401}{400}\right)^n - 400m\left[\left(\frac{401}{400}\right)^n - 1\right].$$ [3]

(ii) By legislation, banks can approve a car loan only if the monthly repayment does not exceed 15% of an applicant's monthly income. Prove that Mary will not be able to apply for the car loan. [3]

(iii) If the interest rate for all car loans by the banks is compounded monthly, find the range of interest rates chargeable which will enable Mary to apply for the

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car loan successfully. Give your answer in the form $r\%$ per annum, correct to 1 decimal place. [3]

END OF PAPER

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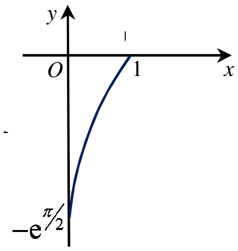
Qn	Solutions
1(i)	$\dfrac{1-x^2}{\sqrt{4-x}}=\left(1-x^2\right)\left(4-x\right)^{-1/2}$ $=\dfrac{1}{2}\left(1-x^2\right)\left(1-\dfrac{x}{4}\right)^{-1/2}$ $=\dfrac{1}{2}\left(1-x^2\right)\left(1+\dfrac{x}{8}+\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\dfrac{x}{4}\right)^2+\ldots\right)$ $=\dfrac{1}{2}\left(1-x^2\right)\left(1+\dfrac{x}{8}+\dfrac{3x^2}{128}+\ldots\right)$ $=\dfrac{1}{2}+\dfrac{1}{16}x-\dfrac{125}{256}x^2+\ldots$
(ii)	Expansion is valid for $\left\{x:-4<x<4,\ x\in\mathbb{R}\right\}$.
(iii)	By letting $x=\dfrac{1}{4}$, $\dfrac{1-\left(\frac{1}{4}\right)^2}{\sqrt{4-\frac{1}{4}}}\approx\dfrac{1}{2}+\dfrac{1}{16}\left(\dfrac{1}{4}\right)-\dfrac{125}{256}\left(\dfrac{1}{16}\right)$ $\dfrac{\frac{15}{16}}{\sqrt{\frac{15}{4}}}\approx\dfrac{1987}{4096}$ $\sqrt{15}\approx\dfrac{1987}{512}$ where $a=1987$ and $b=512$ or $\sqrt{15}\approx\dfrac{7680}{1987}$ where $a=7680$ and $b=1987$
2	$\sum_{r=2}^{n}\left(2^{-r}+2nr+n^2\right)$ $=\sum_{r=2}^{n}\left(2^{-r}\right)+\sum_{r=2}^{n}2nr+\sum_{r=2}^{n}n^2$ $=\dfrac{\left(\frac{1}{2}\right)^2\left(1-\left(\frac{1}{2}\right)^{n-1}\right)}{1-\frac{1}{2}}+2n\cdot\dfrac{n-1}{2}\left(2+n\right)+n^2\left(n-1\right)$ $=\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{n-1}\right)+n(n-1)\left[(2+n)+n\right]$ $=\frac{1}{2}-\left(\frac{1}{2}\right)^{n}+2n(n^2-1)$

3(i)*	$x = e^{\theta}\cos\theta, \quad y = e^{-\theta}\sin\theta, \text{ for } -\frac{\pi}{2} \le \theta \le 0$ When $\theta = 0$, x-intercept: $(1, 0)$ When $\theta = -\frac{\pi}{2}$, y-intercept: $(0, -e^{\pi/2})$ 
(ii)*	Area $= -\int_0^1 y \, dx$ $= -\int_{-\frac{\pi}{2}}^{0} (e^{-\theta}\sin\theta)\left\{e^{\theta}[\cos\theta - \sin\theta]\right\} d\theta$ $= \int_{-\frac{\pi}{2}}^{0} (\sin^2\theta - \sin\theta\cos\theta)\, d\theta$ $= \int_{-\frac{\pi}{2}}^{0} \left\{\left(\frac{1-\cos 2\theta}{2}\right) - \frac{\sin 2\theta}{2}\right\} d\theta$ $= \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_{-\pi/2}^{0}$ $= \frac{\pi + 2}{4}$
4(i)	Let P_n be the statement $u_n = \ln(n!) + \frac{1}{2(2n-1)}$ for $n \ge 0, n \in \mathbb{Z}$. When $n = 0$, LHS $= u_0 = -\frac{1}{2}$ RHS $= \ln(0!) + \frac{1}{2(-1)} = -\frac{1}{2}$ Since LHS = RHS, $\therefore P_0$ is true. Assume that P_k is true for some $k \ge 0$, $k \in \mathbb{Z}$, i.e. $u_k = \ln(k!) + \frac{1}{2(2k-1)}$, need to prove that P_{k+1} is true, i.e., to show that $u_{k+1} = \ln(k+1)! + \frac{1}{2(2(k+1)-1)} = \ln(k+1)! + \frac{1}{2(2k+1)}$. LHS of P_{k+1} $= u_{k+1}$ $= u_k + \ln(k+1) - \frac{1}{4k^2-1}$ $= \ln(k!) + \frac{1}{2(2k-1)} + \ln(k+1) - \frac{1}{4k^2-1}$ $= \ln[(k+1)k!] + \frac{1}{2(2k-1)} - \frac{1}{(2k-1)(2k+1)}$ $= \ln(k+1)! + \frac{2k-1}{2(2k-1)(2k+1)}$ $= \ln(k+1)! + \frac{1}{2(2k+1)} =$ RHS of P_{k+1}

Since P_0 is true and P_k is true $\Rightarrow P_{k+1}$ is true,

$\therefore$ by the principle of mathematical induction, P_n is true for all non-negative integers *n*.

(ii)

$$u_{n+1} = u_n + \ln(n+1) - \frac{1}{4n^2-1}$$

$$\Rightarrow u_{n+1} - u_n = \ln(n+1) - \frac{1}{4n^2-1}.$$

$$\begin{aligned}
\therefore \sum_{n=0}^{N}\left[\ln(n+1) - \frac{1}{4n^2-1}\right] &= \sum_{n=0}^{N}(u_{n+1} - u_n) \\
&= \quad (u_1 - u_0 \\
&\quad + u_2 - u_1 \\
&\quad + u_3 - u_2 \\
&\quad + u_4 - u_3 \\
&\quad + \quad \vdots \\
&\quad + u_{N+1} - u_N \;) \\
&= u_{N+1} - u_0 \\
&= \ln(N+1)! + \frac{1}{2(2(N+1)-1)} - \left(-\frac{1}{2}\right) \\
&= \ln(N+1)! + \frac{1}{2(2N+1)} + \frac{1}{2}
\end{aligned}$$

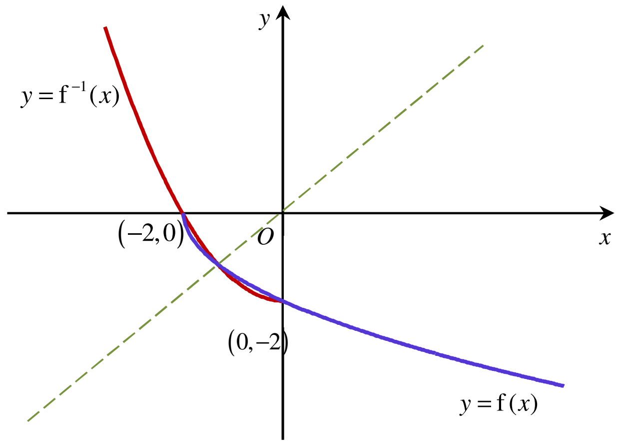
(iii)

$$\therefore \sum_{n=0}^{N}\left[\ln(n+1) - \frac{1}{4n^2-1}\right] = \ln(N+1)! + \frac{1}{2(2N+1)} + \frac{1}{2}.$$

The series is divergent since $\ln(N+1)! \to \infty$ when $N \to \infty$.

(iv) Replace *n* with *n* + 2,

$$\begin{aligned}
&\sum_{n=2}^{N}\left[\ln(n-1) - \frac{1}{4(n-2)^2-1}\right] \\
&= \sum_{n+2=2}^{N}\left[\ln(n+2-1) - \frac{1}{4(n+2-2)^2-1}\right] \\
&= \sum_{n=0}^{N-2}\left[\ln(n+1) - \frac{1}{4n^2-1}\right] \\
&= \ln(N-2+1)! + \frac{1}{2(2(N-2)+1)} + \frac{1}{2} \\
&= \ln(N-1)! + \frac{1}{2(2N-3)} + \frac{1}{2}
\end{aligned}$$

5(i)	$y=-\sqrt{-2x} \xrightarrow{x\to x-2} y=-\sqrt{4-2x} \xrightarrow{x\to -x} y=-\sqrt{4+2x}$ Coordinates of points: $(-2,0)$, $(0,-2)$. 
(ii)	From the diagram, the graphs intersect at $x=-2,0$, and where $\mathrm{f}(x)=x \Rightarrow -\sqrt{4+2x}=x \Rightarrow x^2-2x-4=0$ $x=\dfrac{2\pm\sqrt{4+16}}{2}=1\pm\sqrt{5}$ Since the graphs intersect where $x\le 0$, solutions for $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ are $x=-2,\ 1-\sqrt{5},\ 0$.
(iii)	$\mathrm{f}^{-1}\mathrm{g}(x)=\dfrac{x^2}{2}-2$ $\Rightarrow \mathrm{f}\left(\mathrm{f}^{-1}\mathrm{g}(x)\right)=\mathrm{f}\left(\dfrac{x^2}{2}-2\right)$ $\Rightarrow \quad \mathrm{g}(x)=-\sqrt{4+x^2-4}=-\sqrt{x^2}=-\lvert x\rvert$
6	$\dfrac{x(4x-1)}{2x-1}<3x+1$ $\dfrac{4x^2-x-(2x-1)(3x+1)}{2x-1}<0$ $\dfrac{-2x^2+1}{2x-1}<0$ $\dfrac{x^2-\frac{1}{2}}{2x-1}>0$ $\dfrac{(x+\frac{1}{\sqrt{2}})(x-\frac{1}{\sqrt{2}})}{2x-1}>0$ $\therefore\ -\dfrac{1}{\sqrt{2}}<x<\dfrac{1}{2}$ or $x>\dfrac{1}{\sqrt{2}}$.
(a)	Solution of $3x+1<\dfrac{x(4x-1)}{2x-1}$ is $x<-\dfrac{1}{\sqrt{2}}$ or $\dfrac{1}{2}<x<\dfrac{1}{\sqrt{2}}$. Also, $x-5<3x+1 \Rightarrow x>-3$. Taking the intersection of the solutions, $-3<x<-\dfrac{1}{\sqrt{2}}$ or $\dfrac{1}{2}<x<\dfrac{1}{\sqrt{2}}$.

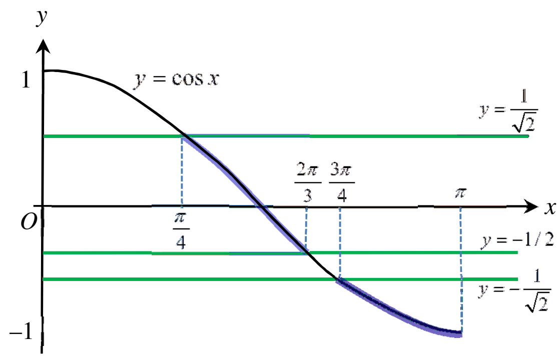
(b) Replace x with $-\cos x$,

$$\frac{-\cos x(-4\cos x-1)}{-2\cos x-1}<-3\cos x+1$$

$$\Rightarrow \frac{\cos x(4\cos x+1)}{2\cos x+1}>3\cos x-1.$$

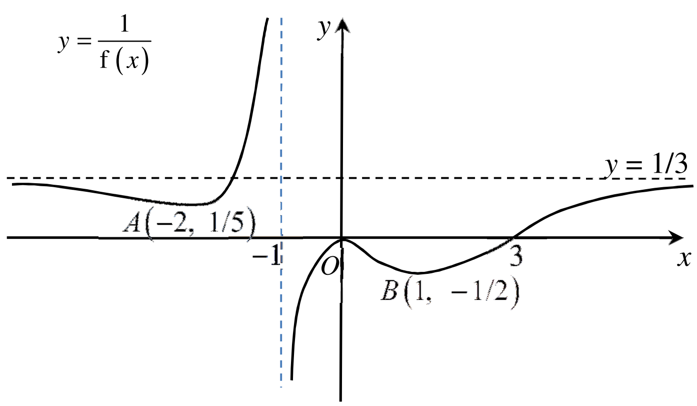
$$\therefore \quad -\frac{1}{\sqrt{2}}<-\cos x<\frac{1}{2} \quad \text{or} \quad -\cos x>\frac{1}{\sqrt{2}}$$

$$-\frac{1}{2}<\cos x<\frac{1}{\sqrt{2}} \quad \text{or} \quad \cos x<-\frac{1}{\sqrt{2}}.$$

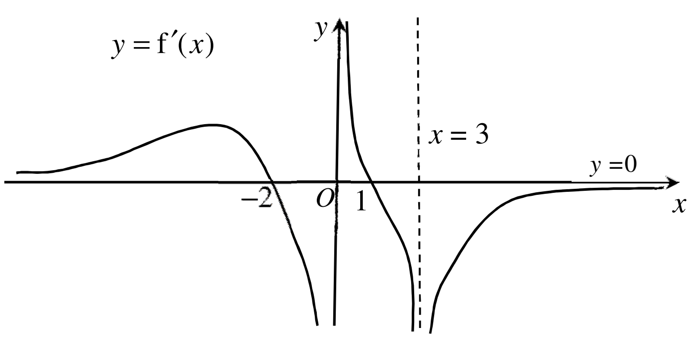


$$\therefore \quad \frac{\pi}{4}<x<\frac{2\pi}{3} \quad \text{or} \quad \frac{3\pi}{4}<x\le\pi.$$

7(i)



7(ii)



8(a)*	$\int \frac{1}{x^2}\ \ln(x+1)\,\mathrm{d}x = -\frac{1}{x}\ln(x+1)+\int \frac{1}{x}\cdot\frac{1}{x+1}\,\mathrm{d}x$ $= -\frac{1}{x}\ln(x+1)+\int \left(\frac{1}{x}-\frac{1}{x+1}\right)\mathrm{d}x$ $= -\frac{1}{x}\ln(x+1)+\ln\lvert x\rvert-\ln(x+1)+c$ <u>Alternative</u> <u>method for</u> $\int \frac{1}{x}\cdot\frac{1}{x+1}\,\mathrm{d}x$ $=\int \frac{1}{(x+\frac{1}{2})^2-(\frac{1}{2})^2}\,\mathrm{d}x = \ln\left\lvert\frac{x}{x+1}\right\rvert+c$
8(b)*	Points of intersection of $(y-2)^2 = x+1$ and $y+2x=6$ $(4-2x)^2 = x+1 \Rightarrow 4x^2-17x+15=0 \Rightarrow x=3$ or $x=\frac{5}{4}$ or GC. Also, $(y-2)^2 = x+1 \Rightarrow y = 2\pm\sqrt{x+1}$ Volume generated $= \int_{-1}^{\frac{5}{4}}\pi\left(2+\sqrt{x+1}\right)^2\,\mathrm{d}x+\int_{\frac{5}{4}}^{3}\pi(6-2x)^2\,\mathrm{d}x-\int_{-1}^{3}\pi\left(2-\sqrt{x+1}\right)^2\,\mathrm{d}x$ $\approx 78.57254 = 78.6$ (3 s.f.)
9(i)	$l_1 : \frac{x-1}{3}=\frac{y-2}{a}, z=1 \quad\Rightarrow\quad l_1 : \mathbf{r}=\begin{pmatrix}1\\2\\1\end{pmatrix}+\mu\begin{pmatrix}3\\a\\0\end{pmatrix}, \mu\in\mathbb{R}$ $l_2 : \mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}, \lambda\in\mathbb{R}$ If l_1 intersects with l_2, $\begin{pmatrix}1+3\mu\\2+a\mu\\1\end{pmatrix}=\begin{pmatrix}1-\lambda\\2+\lambda\\\lambda\end{pmatrix}$ $1+3\mu = 1-\lambda$ ---------- (1) $2+a\mu = 2+\lambda$ ---------- (2) $1 = \lambda$ ---------- (3) Solving for (1) and (3): $\lambda=1$ and $\mu=-\frac{1}{3}$ Therefore, point N is (0, 3, 1). Substitute the values of λ and μ into (2): $2+a\left(-\frac{1}{3}\right)=2+1$ $a=-3$.
(ii)	Let F be the foot of the perpendicular from point $P(2,1,1)$ to the line l_2. Since F lies on l_2, $\overrightarrow{OF}=\begin{pmatrix}1-\lambda\\2+\lambda\\\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$

$$\overrightarrow{PF}=\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}-1-\lambda\\1+\lambda\\-1+\lambda\end{pmatrix}$$

$$PF\perp l_2\Rightarrow\begin{pmatrix}-1-\lambda\\1+\lambda\\-1+\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=0$$

$$\Rightarrow 1+\lambda+1+\lambda-1+\lambda=0$$

$$\Rightarrow \lambda=-\frac{1}{3}$$

Thus $\overrightarrow{OF}=\begin{pmatrix}1-\left(-\frac{1}{3}\right)\\2-\frac{1}{3}\\-\frac{1}{3}\end{pmatrix}=\begin{pmatrix}4/3\\5/3\\-1/3\end{pmatrix}=\frac{4}{3}\mathbf{i}+\frac{5}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$. (shown)

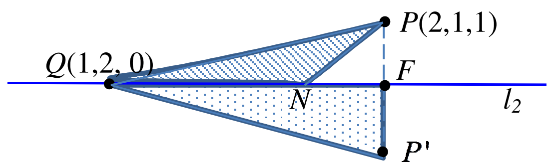
(iii) Let P' be the point of reflection of P about the line l_2.

$\overrightarrow{PF}=\overrightarrow{FP'}\Rightarrow$ By the mid-point theorem, $\overrightarrow{OF}=\dfrac{\overrightarrow{OP'}+\overrightarrow{OP}}{2}$.

$$\Rightarrow \overrightarrow{OP'}=2\overrightarrow{OF}-\overrightarrow{OP}$$

$$=2\begin{pmatrix}4/3\\5/3\\-1/3\end{pmatrix}-\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}2/3\\7/3\\-5/3\end{pmatrix}$$

(iv)



Note that Q lies on l_2.

$$\frac{\text{Area of }\Delta NQP}{\text{Area of }\Delta FQP'}=\left(\frac{\frac{1}{2}PF\times NQ}{\frac{1}{2}FP'\times QF}\right)=\left(\frac{NQ}{FQ}\right)\text{ since } PF=FP'.$$

$$\overrightarrow{NQ}=\begin{pmatrix}1\\2\\0\end{pmatrix}-\begin{pmatrix}0\\3\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}\quad\Rightarrow NQ=\sqrt{3}$$

$$\overrightarrow{FQ}=\begin{pmatrix}1\\2\\0\end{pmatrix}-\begin{pmatrix}4/3\\5/3\\-1/3\end{pmatrix}=\begin{pmatrix}-1/3\\1/3\\1/3\end{pmatrix}=1/3\begin{pmatrix}-1\\1\\1\end{pmatrix}\quad\Rightarrow FQ=\tfrac{1}{3}\sqrt{3}$$

$$\frac{\text{Area of }\Delta NQP}{\text{Area of }\Delta FQP'}=\left(\frac{\sqrt{3}}{\frac{1}{3}\sqrt{3}}\right)=3.$$

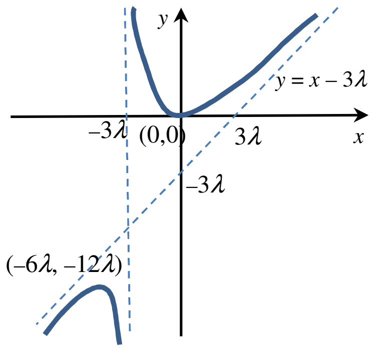
Therefore, the ratio is 3:1.

10(i) $y=\dfrac{x^2}{x+3\lambda} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x(x+3\lambda)-x^2}{(x+3\lambda)^2}=\dfrac{x^2+6\lambda x}{(x+3\lambda)^2}$

At stationary point, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow x=0$ or $x=-6\lambda$. Stationary points: $(0, 0)$, $(-6\lambda, -12\lambda)$.

(ii) $y=\dfrac{x^2}{x+3\lambda} \Rightarrow y=x-3\lambda+\dfrac{9\lambda^2}{x+3\lambda}$

Asymptotes : $x=-3\lambda$, $y=x-3\lambda$



(ii) $x^4-2\lambda x-6\lambda^2=0 \Rightarrow \dfrac{x^2}{x+3\lambda}=\dfrac{2\lambda}{x^2}$

By sketching the graph of $y=\dfrac{2\lambda}{x^2}$ on the diagram, there are 2 points of intersections, hence there are 2 roots to the equation.

(iii) $R_f=(-\infty,-12\lambda]$, and $D_f=(-\infty,-6\lambda]$.

Since $R_f \subseteq D_f$, hence f^2 exists.

$$f^2(-6\lambda)=f\left(\frac{36\lambda^2}{-6\lambda+3\lambda}\right)=f(-12\lambda)=\frac{144\lambda^2}{-12\lambda+3\lambda}=-16\lambda.$$

11(i)

$$(R-r)^2+(R-r)^2=(R+r)^2$$

$$\frac{(R-r)^2}{(R+r)^2}=\frac{1}{2}$$

$$\Rightarrow \frac{R-r}{R+r}=\frac{1}{\sqrt{2}}$$

$$R\left(\sqrt{2}-1\right)=\left(\sqrt{2}+1\right)r$$

$$R=\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}r$$

$$=\frac{\left(\sqrt{2}+1\right)^2 r}{2-1} \Rightarrow R=\left(\sqrt{2}+1\right)^2 r$$

(ii) Volume of small cylinder $=V=\pi r^2 h=\dfrac{16\pi}{\sqrt{2}+1}$.

	$h=\dfrac{16}{r^2\left(\sqrt{2}+1\right)}$ Surface area of big cylinder $=A=2\pi Rh+2\pi R^2$. $A=2\pi\left(\sqrt{2}+1\right)^2 rh+2\pi\left(\sqrt{2}+1\right)^4 r^2$ $=2\pi\left(\sqrt{2}+1\right)^2 r\left(\dfrac{16}{r^2\left(\sqrt{2}+1\right)}\right)+2\pi\left(\sqrt{2}+1\right)^4 r^2$ $=\dfrac{32\pi\left(\sqrt{2}+1\right)}{r}+2\pi\left(\sqrt{2}+1\right)^4 r^2$ $\dfrac{\mathrm{d}A}{\mathrm{d}r}=4\pi\left(\sqrt{2}+1\right)^4 r-\dfrac{32\pi\left(\sqrt{2}+1\right)}{r^2}$ Let $\dfrac{\mathrm{d}A}{\mathrm{d}r}=0$, then $4\pi\left(\sqrt{2}+1\right)^4 r=\dfrac{32\pi\left(\sqrt{2}+1\right)}{r^2}$ $\Rightarrow\quad r^3=\dfrac{32\pi\left(\sqrt{2}+1\right)}{4\pi\left(\sqrt{2}+1\right)^4}$ $=\dfrac{8}{\left(\sqrt{2}+1\right)^3}$ $\Rightarrow\quad r=\dfrac{2}{\sqrt{2}+1}$ or $2\left(\sqrt{2}-1\right)$ $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2}=4\pi\left(\sqrt{2}+1\right)^4+\dfrac{64\pi\left(\sqrt{2}+1\right)}{r^3}$ When $r=2\left(\sqrt{2}-1\right)$, $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2}>0$. Hence, $r=2\left(\sqrt{2}-1\right)$ gives the minimum surface area of the big cylinder.
12(i)	Monthly interest chargeable = $\dfrac{3}{12}\%=\dfrac{1}{4}\%$. Let monthly repayment amount = $\$m$.

Loan Mth	Loan balance at beginning of loan month	Loan Balance at end of loan month (after monthly repayment)
1	$40000\left(\frac{401}{400}\right)$	$40000\left(\frac{401}{400}\right)-m$
2	$40000\left(\frac{401}{400}\right)^2-\left(\frac{401}{400}\right)m$	$40000\left(\frac{401}{400}\right)^2-\left(\frac{401}{400}\right)m-m$
⋮	⋮	⋮
n	$40000\left(\frac{401}{400}\right)^n-\left(\frac{401}{400}\right)^{n-1}m-\left(\frac{401}{400}\right)^{n-2}m-\ldots-\left(\frac{401}{400}\right)m$	$40000\left(\frac{401}{400}\right)^n-\left(\frac{401}{400}\right)^{n-1}m-\left(\frac{401}{400}\right)^{n-2}m-\ldots-\left(\frac{401}{400}\right)m-m$

	Loan balance at the end of n^{th} loan month after monthly repayment $= 40000\left(\frac{401}{400}\right)^n - \left(\frac{401}{400}\right)^{n-1} m - \left(\frac{401}{400}\right)^{n-2} m - \ldots - \left(\frac{401}{400}\right) m - m$ $= 40000\left(\frac{401}{400}\right)^n - m\left[\frac{\left(\frac{401}{400}\right)^n - 1}{\frac{401}{400} - 1}\right]$ $= 40000\left(\frac{401}{400}\right)^n - 400m\left[\left(\frac{401}{400}\right)^n - 1\right]$
(ii)	Let $40000\left(\frac{401}{400}\right)^{72} - 400m\left[\left(\frac{401}{400}\right)^{72} - 1\right] = 0$ $\Rightarrow \quad m = 607.75$ 15% of \$4000 = \$600. Since $m = 607.75 > 600$, Mary is not able to take up the car loan.
(iii)	Let $40000a^{72} - 600\left[\frac{a^{72} - 1}{a - 1}\right] \le 0.$ From the GC, using the graph of $y = 40000x^{72} - 600\left[\frac{x^{72} - 1}{x - 1}\right]$, $1 < a \le 1.0021378.$ $12 \times (1.0021378 - 1) \times 100\% = 2.56536\%$ $\therefore \; 0\% < r\% \le 2.5\%$ (to 1 decimal place)

CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC1 Promotional Examination

MATHEMATICS 9740/01

Paper 1

04 October 2013

3 hours

Additional Materials: List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER.
Place this cover sheet in front and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

Name: ____________________________ **Class:** ________________

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**Total**
Marks														
Total	3	4	5	6	6	6	6	8	8	9	11	13	15	**100**

This document consists of 5 printed pages.

Catholic Junior College

1. Sketch the curve $(y-5)^2-(x+3)^2=4$, indicating clearly the coordinates of the turning point(s) and equations of the asymptotes. **[3]**

2. Expand $\dfrac{1}{\sqrt[3]{2x-1}}$ in ascending powers of x, up to and including the term in x^2.
State the range of values of x for which this expansion is valid. **[4]**

3. The graph of $y=\mathrm{f}(x)$, where $\mathrm{f}(x)$ is a cubic polynomial, passes through the points $(1, 6)$, $(-2, 15)$ and has two stationary points at $x=\dfrac{1}{3}$ and $x=-2$. Find the equation of the curve and hence, find its x-intercept. **[5]**

4. **(a)** Given that $y=\tan^{-1}\sqrt{x}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. **[2]**

(b) Given that $\sqrt[x]{y}=\sqrt[y]{x}$, where $x>0$, $y>0$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. **[4]**

5. The parametric equations of a curve are

$$x=t^3, \quad y=\frac{7}{t}, \quad t\neq 0.$$

(i) Find the equation of the tangent to the curve at the point where $t=k$, simplifying your answer. **[3]**

(ii) Hence find the coordinates of the points X and Y where this tangent meets the x- and y-axes respectively. **[2]**

(iii) Hence or otherwise, find the area of the triangle OXY, where O is the origin. **[1]**

6. Prove by the method of differences that $\displaystyle\sum_{r=2}^{n}\frac{1}{r^2-1}=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}$. **[4]**

Hence, or otherwise, give a reason why the series $\displaystyle\sum_{r=2}^{n}\frac{1}{r^2-1}$ is convergent and state the sum to infinity. **[2]**

7. Prove by the method of mathematical induction that

$$\sum_{r=1}^{n}\cos[(2r-1)\theta]=\frac{\sin 2n\theta}{2\sin\theta}$$ for all positive integers n. **[6]**

8. **(a)** **(i)** Without using a calculator, solve the inequality $\dfrac{x+6}{x^2-3x-4} \le \dfrac{1}{4-x}$. **[3]**

(ii) Hence, deduce the range of values of x that satisfies

$$\frac{|x|+6}{x^2-3|x|-4} \le \frac{1}{4-|x|}.$$ **[2]**

(b) Solve the inequality $\ln(x+6) \le -\dfrac{x}{3}$. **[3]**

9. Charis Insurance provides an investment linked savings insurance plan with two options of premium payment, monthly and yearly.
For the monthly premium plan, premiums of \$500 are collected on the first day of each month and an interest of 0.5% per month is earned on the last day of each month, such that there is \$502.50 in the account at the end of the first month and \$1007.51 in the account at the end of the second month.

(i) Show that the total amount in the monthly premium account at the end of n complete months can be expressed as $M(1.005^n - 1)$, where M is an integer to be found. **[4]**

For the yearly premium plan, premiums of \$6000 are made on the first day of each year and an interest of 6% per year is earned on the last day of each year.

(ii) Given that the total amount in the yearly premium account at the end of k complete years is $\$\left[106000\left(1.06^k - 1\right)\right]$, find the number of complete years it will take for the total amount to first exceed \$120 000. **[2]**

A young couple who just had their first child would like to take up a savings plan for a period of 20 years to prepare for their child's university education. A friend of the couple stated that "0.5% a month is the same as 6% a year since 12 × 0.5 = 6". With reference to evidence obtained from the expressions from (i) and (ii), comment on the validity of the statement. **[2]**

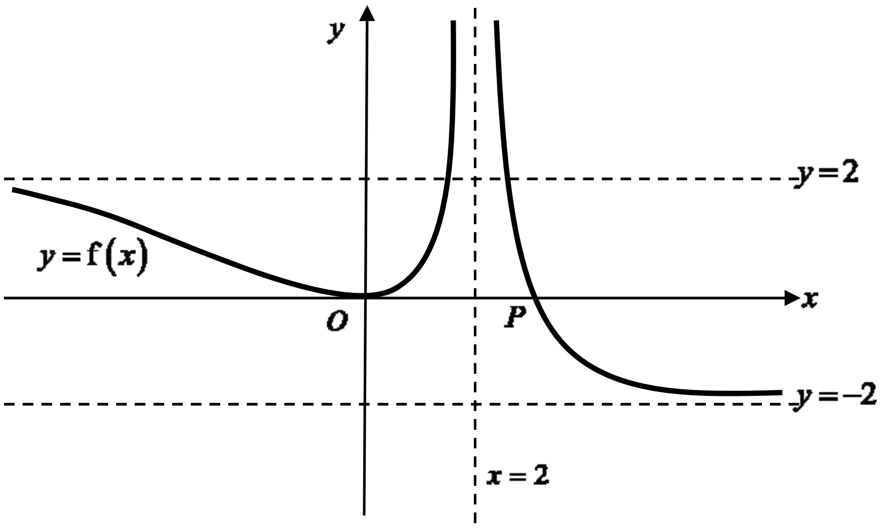
10. **(i)** Given that $\mathrm{f}(x) = \mathrm{e}^{\cos x + k\sin x}$, where k is a constant, find $\mathrm{f}(0)$, $\mathrm{f}'(0)$, $\mathrm{f}''(0)$. Hence write down the first three terms in the Maclaurin series for $\mathrm{f}(x)$. Give the coefficients in terms of e and k. **[5]**

(ii) Find the value of k such that $\sqrt{2}\sin(x+\frac{\pi}{4}) = \cos x + k\sin x$ for all x. **[2]**

(iii) By considering the series in part **(i)**, show that

$\mathrm{e}^{\sqrt{2}\,\sin(x+\frac{\pi}{4})}\sin x \approx \mathrm{e}\left(x^2+x\right)$, where x is a small angle. **[2]**

11. **(a)** The diagram below shows the graph of $y = \mathrm{f}(x)$. It passes through the origin O and P (3, 0), and has asymptotes $x = 2$, $y = 2$ .and $y = -2$.



On separate diagrams, sketch the graph of

(i) $y = \mathrm{f}'(x)$, **[3]**

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, **[3]**

indicating clearly any asymptote(s) and axial intercept(s).

(b) The graph of $y = \dfrac{1}{2x+3}$ is transformed by a reflection in the y-axis, followed by a translation of 1 unit in the negative x-direction, followed by a stretch with scale factor 2 parallel to the x-axis.

(i) Find the equation of the new graph in the form $y = \mathrm{f}(x)$. **[3]**

(ii) Hence, or otherwise, sketch the new graph with any axial intercept(s) and asymptote(s) indicated clearly. **[2]**

12. Functions f and g are defined by

$$\mathrm{f} : x \mapsto (4+2x)^{\frac{1}{2}}, \quad x \in \mathbb{R},\ 0 \le x \le 16$$
$$\mathrm{g} : x \mapsto 3x+1, \qquad x \in \mathbb{R}$$

(i) State the range of f. **[1]**

(ii) With the aid of a diagram, show that f^{-1} exists and define f^{-1} in a similar form. **[4]**

(iii) On the same diagram as in part **(ii)**, sketch the graphs of f^{-1} and $\mathrm{f}^{-1}\mathrm{f}$, indicating their endpoints. **[3]**

(iv) Explain why the *x*-coordinates of the point(s) of intersection between the graphs in part **(iii)** satisfies the equation $x^2 - 2x - 4 = 0$. **[1]**

(v) State whether the composite function fg exists, justifying your answer. **[2]**

(vi) Find the largest possible domain of g in the form [*m*, *n*], *m*, *n* $\in \mathbb{R}$, for which the composite function fg exists. **[2]**

13. **(a)** Relative to the origin *O*, two points *A* and *B* have position vectors given by **a =** 4**i** + 2**j** + 3**k** and **b** = 4**i** – 2**j** + 7**k** respectively. The point *P* divides *AB* in the ratio 3 : 1.

(i) Find the coordinates of *P*. **[2]**

(ii) The vector **c** is a unit vector in the direction of $\overrightarrow{OP}$.
Write **c** as a column vector, and give the geometrical meaning of $|\mathbf{a} \bullet \mathbf{c}|$. **[2]**

(iii) By using vector cross product, find the exact area of triangle *OAP*. **[3]**

(b) The line *l* has equation $\dfrac{x-3}{-3} = y+3 = \dfrac{z-1}{-2}$ and the plane *p* has equation $3x - y + 2z = 0$.

(i) Show that *l* is perpendicular to *p*. **[2]**

(ii) Find the coordinates of the point of intersection of *l* and *p*. **[3]**

(iii) Show that the point *C* with coordinates $(-9,1,-7)$ lies on *l*.

Find the coordinates of the point *C'* which is the mirror image of *C* in *p*. **[3]**

— End of Paper —

CATHOLIC JUNIOR COLLEGE
H2 MATHEMATICS
JC1 PROMOTIONAL EXAMINATION 2013

Solutions

1

$$(y-5)^2-(x+3)^2=4$$

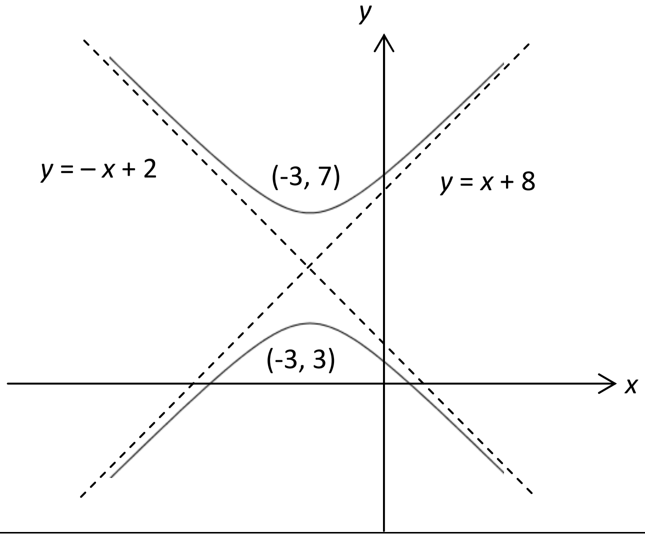
$$-\frac{(x+3)^2}{2^2}+\frac{(y-5)^2}{2^2}=1$$

Asymptotes:

$$(y-5)^2=(x+3)^2$$

$$y-5=\pm(x+3)$$

$$y=x+8 \quad \text{or} \quad y=-x+2$$



2

$$\frac{1}{\sqrt[3]{2x-1}}=(2x-1)^{-\frac{1}{3}}=-1(1-2x)^{-\frac{1}{3}}$$

$$=-\left(1+\left(-\frac{1}{3}\right)(-2x)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2}(-2x)^2+\ldots\right)$$

$$\approx-(1+\frac{2}{3}x+\frac{8}{9}x^2)$$

Validity: $-\frac{1}{2}<x<\frac{1}{2}$

3

Let $y=Ax^3+Bx^2+Cx+D$

$$\therefore\frac{dy}{dx}=3Ax^2+2Bx+C$$

$$A+B+C+D=6$$

$$-8A+4B-2C+D=15$$

$$A+2B+3C=0$$

$$12A-4B+C=0$$

Solving,

$$A=2,B=5,C=-4,D=3$$

$$y=2x^3+5x^2-4x+3$$

	When $y = 0$, $x = -3.26$ (3sf) x-intercept = $(-3.26, 0)$
4	(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\sqrt{x}\right) = \frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2}x^{-\frac{1}{2}}$ $= \frac{1}{2\sqrt{x}(1+x)}$ (b) $\sqrt[x]{y} = \sqrt[y]{x}$ Taking logarithm on both sides, $\frac{1}{x}\ln y = \frac{1}{y}\ln x$ $y\ln y = x\ln x$ Differentiating both sides, $y\cdot\frac{1}{y}\cdot\frac{dy}{dx} + \frac{dy}{dx}\cdot\ln y = x\cdot\frac{1}{x} + 1\cdot\ln x$ $(1+\ln y)\frac{dy}{dx} = 1 + \ln x$ $\frac{dy}{dx} = \frac{1+\ln x}{1+\ln y}$
5	(i) $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{dt}} \div \frac{\mathrm{dx}}{\mathrm{dt}} = \left(-\frac{7}{t^2}\right) \div \left(3t^2\right) = -\frac{7}{3t^4}$ $y - \frac{7}{k} = -\frac{7}{3k^4}\left(x - k^3\right)$ $y = -\frac{7}{3k^4}x + \frac{28}{3k}$ (ii) $y - \frac{7}{k} = -\frac{7}{3k^4}\left(x - k^3\right)$ $y = -\frac{7}{3k^4}x + \frac{28}{3k}$ $y = 0, \quad x = 4k^3 \quad \Rightarrow \quad X \text{ is } \left(4k^3, 0\right)$ $x = 0, \quad y = \frac{28}{3k} \quad \Rightarrow \quad Y \text{ is } \left(0, \frac{28}{3k}\right)$ (iii) Area of OXY $= \frac{1}{2}\left(OX\right)\left(OY\right)$ $= \frac{1}{2}\left(4k^3\right)\left(\frac{28}{3k}\right)$ $= \frac{56}{3}k^2 \text{ units}^2$
6	$\sum_{r=2}^{n}\frac{1}{r^2-1} = \sum_{r=2}^{n}\frac{1}{(r-1)(r+1)}$

	$=\frac{1}{2}\sum_{r=2}^{n}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$ $=\frac{1}{2}[\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+$ $\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+$ $\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+$ $\vdots$ $\left(\frac{1}{n-3}-\cancel{\frac{1}{n-1}}\right)+$ $\left(\cancel{\frac{1}{n-2}}-\frac{1}{n}\right)+$ $\left(\cancel{\frac{1}{n-1}}-\frac{1}{n+1}\right)$ $=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{(n+1)}\right)$ $=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}$ $\sum_{r=2}^{n}\frac{1}{r^2-1}=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}$. $n\to\infty$,. $\frac{1}{2n}\to 0$, $\frac{1}{2(n+1)}\to 0$, so $\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}\to\frac{3}{4}$ so $\sum_{r=2}^{\infty}\frac{1}{r^2-1}=\frac{3}{4}$
7	Let P_n be the statement $\sum_{r=1}^{n}\cos[(2r-1)\theta]=\frac{\sin 2n\theta}{2\sin\theta}$ for $n\in Z^+, n\ge 1$ When $n=1$, L.H.S. = $\cos\theta$ R.H.S. = $\frac{\sin 2\theta}{2\sin\theta}=\frac{2\sin\theta\cos\theta}{2\sin\theta}=\cos\theta$ =L.H.S. Assume P_k is true, i.e. $\sum_{r=1}^{k}\cos[(2r-1)\theta]=\frac{\sin 2k\theta}{2\sin\theta}$ for some $k\in Z^+$, $k\ge 1$. Required to prove P_{k+1} is true, i.e. $\sum_{r=1}^{k+1}\cos[(2r-1)\theta]=\frac{\sin[2(k+1)\theta]}{2\sin\theta}$ L.H.S. = $\sum_{r=1}^{k}\cos[(2r-1)\theta]+u_{k+1}$ $=\frac{\sin 2k\theta}{2\sin\theta}+\cos[(2k+1)\theta]$ $=\frac{\sin 2k\theta+2\cos[(2k+1)\theta]\sin\theta}{2\sin\theta}$

$$=\frac{\sin 2k\theta+\sin[2(k+1)\theta]-\sin 2k\theta}{2\sin\theta}$$

$$=\frac{\sin[2(k+1)\theta]}{2\sin\theta}=\text{R.H.S.}$$

P_k is true $\Rightarrow P_{k+1}$ is true.

Hence, by Mathematical Induction, P_n is true for all $n\in Z^+,\ n\geq 1$.

8 (a)(i)

$$\frac{x+6}{x^2-3x-4}\leq\frac{1}{4-x}$$

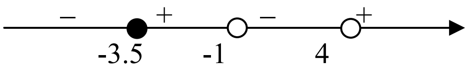
$$\frac{x+6}{(x+1)(x-4)}-\frac{1}{4-x}\leq 0$$

$$\frac{x+6}{(x+1)(x-4)}+\frac{1}{x-4}\leq 0$$

$$\frac{x+6+(x+1)}{(x+1)(x-4)}\leq 0$$

$$\frac{2x+7}{(x+1)(x-4)}\leq 0$$

Using test-point method,



$\therefore x\leq -3.5$ or $-1<x<4$

(ii)

$$\frac{|x|+6}{x^2-3|x|-4}\leq\frac{1}{4-|x|}$$

Replace x by $|x|$

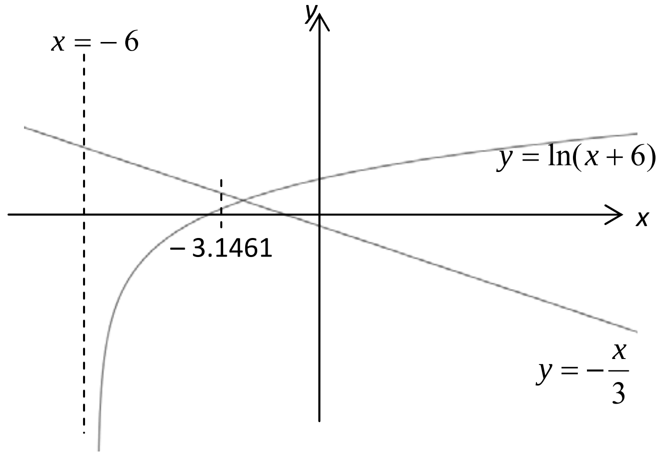
$\therefore |x|\leq -3.5$ or $-1<|x|<4$

(no real solution) $|x|<4$

$-4<x<4$

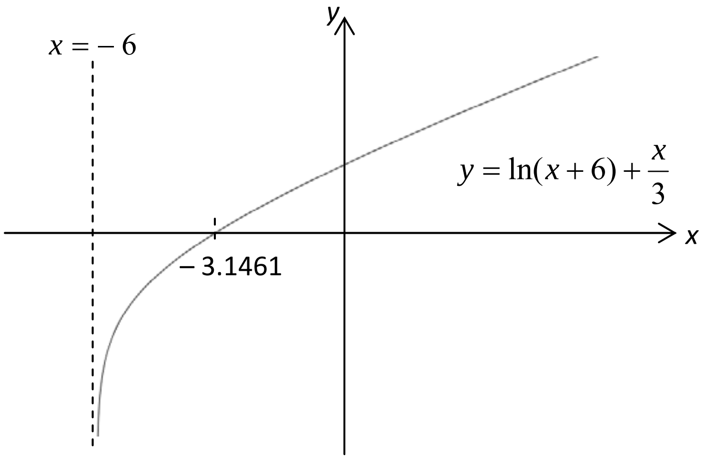
(b)

Draw the graphs of $y=\ln(x+6)$ and $y=-\dfrac{x}{3}$.



Ans: $-6 < x \le -3.15$

Alternative solution: Draw the graph of $y = \ln(x+6) + \dfrac{x}{3}$.



Ans: $-6 < x \le -3.15$

9 (i)

Total amount after 1 month = $1.005(500)$

Total amount after 2 month = $1.005^2(500) + 1.005(500)$

Total amount after 3 month

$= 1.005^3(500) + 1.005^2(500) + 1.005(500)$

Total amount after n months $= 1.005^n(500) + 1.005^{n-1}(500) + \cdots + 1.005(500)$

$$= \frac{1.005(500)\left(1.005^n - 1\right)}{1.005 - 1}$$

$$= 100500\left(1.005^n - 1\right)$$

M = 100500

(ii)

$$106000\left(1.06^k - 1\right) > 120000$$

Solving, $k > 12.99$

$\therefore k = 13$ complete years.

From (i) and (ii), the final amount after 20 years is

$100500\left(1.005^{240}-1\right)=\$232175.55$ for monthly account

$106000\left(1.06^{20}-1\right)=\$233956.36$ for yearly account

Hence the statement is invalid as the final total amount differs quite significantly

10

(i)

We are given that $f(x)=e^{\cos x+k\sin x}$.

Differentiating,

$f'(x)=e^{\cos x+k\sin x}(-\sin x+k\cos x)$.

Differentiating,

$f''(x)=e^{\cos x+k\sin x}(-\cos x-k\sin x)+e^{\cos x+k\sin x}(-\sin x+k\cos x)^2$.

So we have

$f'(0)=e^{\cos 0+k\sin 0}(-\sin 0+k\cos 0)$

$=ke$,

$f''(0)=e^{\cos 0+k\sin 0}(-\cos 0-k\sin 0)+e^{\cos 0+k\sin 0}(-\sin 0+k\cos 0)^2$

$=(k^2-1)e$,

$f(0)=e^{\cos 0+k\sin 0}=e$.

Hence,

$$f(x)=e+kex+\frac{1}{2}(k^2-1)ex^2+\cdots.$$

(ii)

Since

$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$

$=\sqrt{2}\left(\sin x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x\right)$

$=\sqrt{2}\cdot\frac{\sqrt{2}}{2}(\sin x+\cos x)$

$=\cos x+\sin x$,

we have $k=1$.

(iii)

Since x is a small angle,

$\sin x\approx x$,

then

$\sin x\, e^{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}$

$=\sin x\, e^{\cos x+\sin x}$

$$\approx x\left[e+1\cdot ex+\frac{1}{2}(1^2-1)ex^2\right]$$

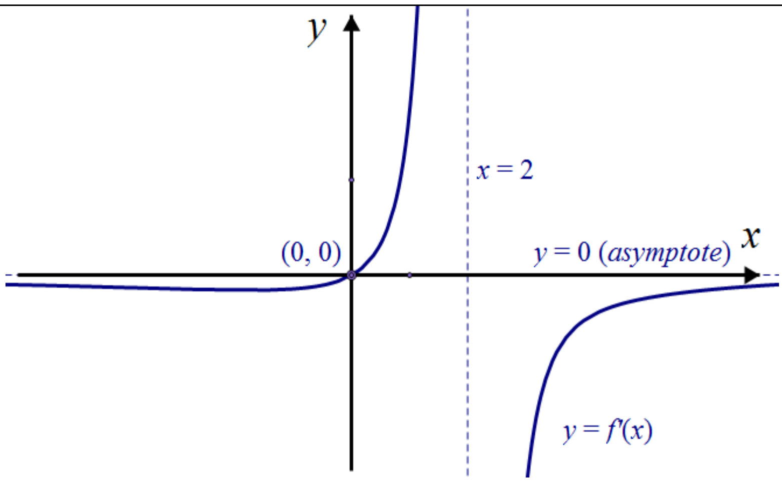
$=x(e+ex)$

$=(x^2+x)e$.

11

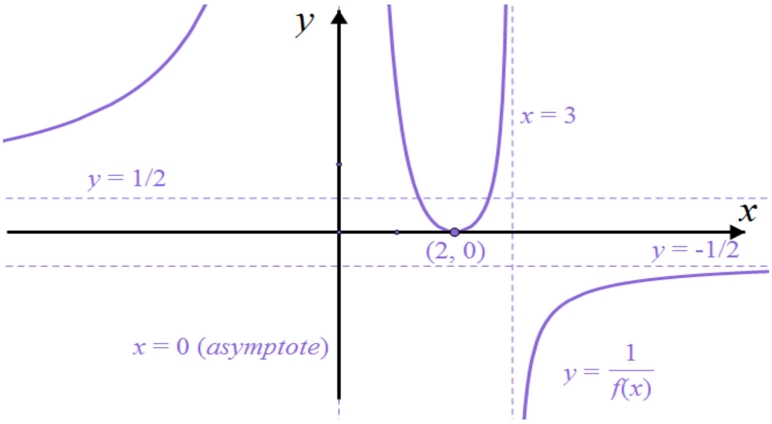
(a)(i)

$y=f'(x)$



(a)(ii)

$$y=\frac{1}{\mathrm{f}(x)}$$



(b)(i)

$$y=\frac{1}{2x+3}$$

$\downarrow$

$$y=\frac{1}{2(-x)+3}=\frac{1}{-2x+3}$$

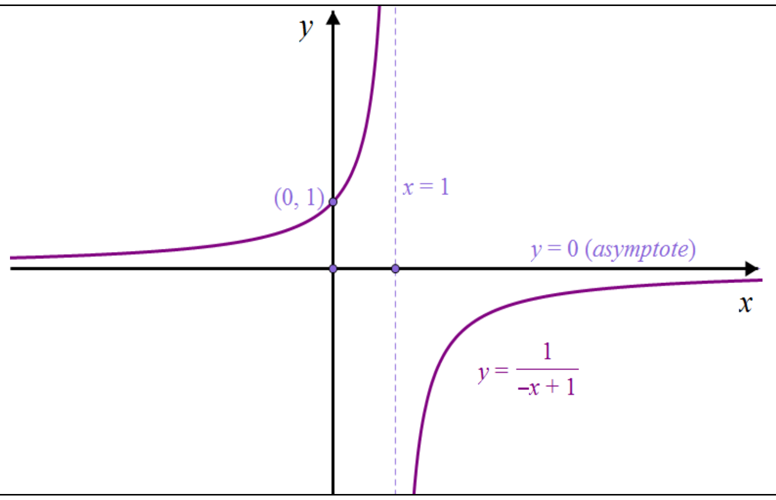
$\downarrow$

$$y=\frac{1}{-2(x+1)+3}=\frac{1}{-2x+1}$$

$\downarrow$

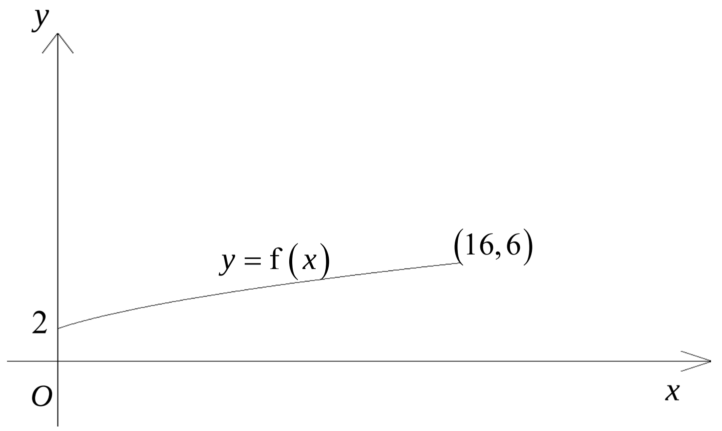
$$y=\frac{1}{-2\left(\frac{1}{2}x\right)+1}=\frac{1}{-x+1}$$

(b)(ii)



12

(i)
As f is an increasing function,
$f(0) = (4)^{½} = 2$
$f(16) = (36)^{½} = 6$
Range of f, $R_f = [2,6]$
(ii)

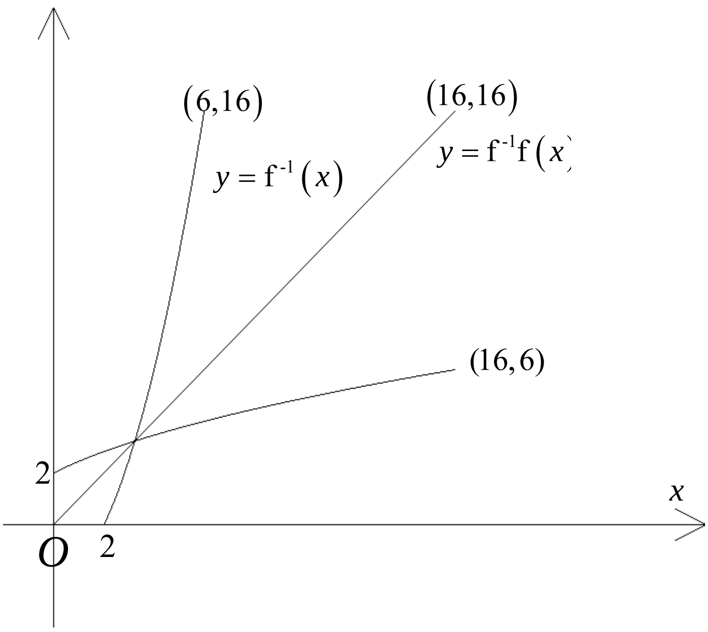


f is a 1-1 function as the line $y = k$, $2 \leq k \leq 6$ intersects the graph of f exactly once.
(OR: f is a 1-1 function as any line $y = k$ intersects the graph of f at most once.)
Hence f^{-1} exists.

Let $y=f(x) = (4+2x)^{1/2}$
$y^2 = 4+2x$
$x = \frac{1}{2}(y^2 - 4)$
$f^{-1}(x) = \frac{1}{2}(x^2 - 4)]$
$D_{f^{-1}} = R_f = [2,6]$

Hence $f^{-1} : x \to \frac{1}{2}(x^2 - 4),\ 2 \leq x \leq 6$

(iii)



(iv)
By considering f(x) = x, $(4 + 2x)^{1/2} = x$

$$x^2 - 2x - 4 = 0$$

The x-coordinates of the points of intersection satisfy the equation $x^2 - 2x - 4 = 0$.

(v)
$R_g = \mathbb{R}$
$D_f = [0, 16]$
$R_g \nsubseteq D_f$
=> fg does not exist.

(vi)
Consider $R_g = D_f$
$3x+1 = 0 \Rightarrow x = -1/3$
$3x+1 = 16 \Rightarrow x = 5$
Hence $[-\frac{1}{3}, 5]$ is the largest possible domain of g for fg to exist.

13 (a)(i)

$$\overrightarrow{OP} = \frac{\overrightarrow{OA} + 3\overrightarrow{OB}}{4}$$

$$= \frac{1}{4}\left[\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix}$$

(a)(ii)

$$\mathbf{c} = \frac{1}{\sqrt{4^2+(-1)^2+6^2}}\begin{pmatrix}4\\-1\\6\end{pmatrix} = \frac{1}{\sqrt{53}}\begin{pmatrix}4\\-1\\6\end{pmatrix}$$

Geometrically, $|\mathbf{a}\bullet\mathbf{c}|$ is the length of projection of the vector **a** on $\overrightarrow{OP}$ or **c**.

(a)(iii)

$$\mathbf{a}\times\mathbf{p} = \begin{pmatrix}4\\2\\3\end{pmatrix}\times\begin{pmatrix}4\\-1\\6\end{pmatrix} = \begin{pmatrix}15\\-12\\-12\end{pmatrix}$$

Area of triangle *OAP*

$$= \frac{1}{2}|\mathbf{a}\times\mathbf{p}|$$

$$= \frac{1}{2}\sqrt{15^2+(-12)^2+(-12)^2}$$

$$= \frac{1}{2}\sqrt{513}$$

(b)(i)

Line l: $r = \begin{pmatrix}3\\-3\\1\end{pmatrix} + \mu\begin{pmatrix}-3\\1\\-2\end{pmatrix}, \quad \mu\in\mathbb{R}$

Plane p: $\mathbf{r}\bullet\begin{pmatrix}3\\-1\\2\end{pmatrix} = 0$

Since $\begin{pmatrix}-3\\1\\-2\end{pmatrix} = -\begin{pmatrix}3\\-1\\2\end{pmatrix}$, the normal of the plane p is parallel to the line l, the line l is perpendicular to p.

(b)(ii)

When l intersects p, $\begin{pmatrix}3-3\mu\\-3+\mu\\1-2\mu\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\2\end{pmatrix} = 0$

$9-9\mu+3-\mu+2-4\mu = 0$

$\mu = 1$

Coordinates of point of intersection = (0, -2, -1)

(b)(iii)

Suppose C with coordinates $(-9,1,-7)$ lies on l, $\begin{pmatrix}-9\\1\\-7\end{pmatrix} = \begin{pmatrix}3-3\mu\\-3+\mu\\1-2\mu\end{pmatrix}$

	$-9 = 3 - 3\mu$ $\mu = 4$ Since C satisfies the parametric equations of l with $\mu = 4$, therefore C lies on l. We note that C lies on l, l is perpendicular to p and l meets p at (0, -2, -1), By Ratio Theorem, $\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = \dfrac{\begin{pmatrix} -9 \\ 1 \\ -7 \end{pmatrix} + \overrightarrow{OC'}}{2}$ $\overrightarrow{OC'} = 2\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} -9 \\ 1 \\ -7 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \\ 5 \end{pmatrix}$

HWA CHONG INSTITUTION
JC1 Promotional Examinations
Higher 2

MATHEMATICS 9740

7 October 2013

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

1 Sophia has a total saving of $90 million in three accounts A, B and C with $\$x$ million, $\$y$ million and $\$z$ million respectively. She transfers funds among the accounts based on the table below.

Percentage of Fund transferred from initial amount in	To Account A	To Account B	To Account C
Account A	–	37.5%	12.5%
Account B	5%	–	5%
Account C	10%	20%	–

For instance, 37.5% and 12.5% of the initial amount in Account A are transferred to Account B and Account C respectively.
As a result of the funds transfer, the amount in Account A decreases by $16 million and the amount in Account B increases by $19 million.

(i) By considering the amount in Account A, show that

$$0.5x - 0.05y - 0.1z = 16.$$ [1]

(ii) By forming a system of linear equations, find the values of x, y and z. [3]

2 It is given that the expansion of $(2+px)^{-q}$ in ascending powers of x, up to and including the term in x, is $\frac{1}{4} - x$. Find the values of p and q.

Find, in terms of n, the coefficient of x^n in the above expansion. [4]

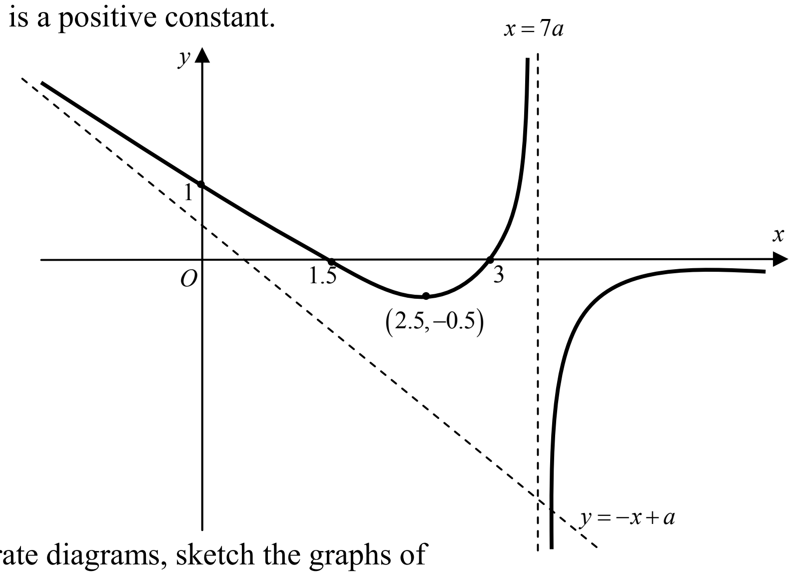
3 A water tank contains 8000 litres of water initially. At the beginning of each day, 500 litres of water is added to the tank. At the end of each day, 10% of the amount of water in the tank will be used.

(i) Show that the amount of water in the tank after 3 days is 7051.5 litres. [1]

(ii) Find the least number of days it will take for the water in the tank to be less than 5000 litres. [3]

(iii) Will the tank ever dry up? Justify your answer. [1]

4 The diagram below shows the graph of $y = \mathrm{f}(x)$. It cuts the axes at the points $(0,\ 1)$, $(1.5,\ 0)$ and $(3,\ 0)$. It has a minimum point at $(2.5, -0.5)$. The horizontal, vertical and oblique asymptotes are $y = 0$, $x = 7a$ and $y = -x + a$ respectively, where a is a positive constant.



On separate diagrams, sketch the graphs of

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \mathrm{f}'(x)$, [3]

showing clearly the axial intercepts, the stationary points and the equations of the asymptotes where applicable.

5 A sequence of real numbers $\{u_n\}$, for $n \in \mathbb{Z}^+$, satisfies the recurrence relation $\dfrac{u_{n+1} + a}{u_n + b} = \dfrac{a}{b}$, with $u_1 = a$, where a and b are fixed non-zero real constants and $a \neq b$.

(i) Given that the limit l of the sequence $\{u_n\}$ exists, find the value of l. [2]

(ii) By expressing u_{n+1} in terms of u_n, find an expression for u_n, leaving your answer in terms of a, b and n. [2]

(iii) Given that the sum to infinity S for the sequence $\{u_n\}$ exists, state an inequality satisfied by a and b. Find S in terms of a and b. [2]

6 **(a)** By using the substitution $u = 9 + 4x^2$, find $\int x^3 \sqrt{9 + 4x^2}\ \mathrm{d}x$. [4]

(b) Evaluate $\int_0^1 x^2 \tan^{-1} x\ \mathrm{d}x$, giving your answer in exact form. [4]

7 The coordinates of 3 points A, B and C are $(2,\ 0, -1)$, $(-3,\ 1,\ 2)$ and $(1, -2, -4)$ respectively.

(a) Find the point D on the x-axis such that there exists a point P on line AB where C, D and P are collinear. [4]

(b) Find two possible points E on the x-y plane, such that $\overrightarrow{OE}$ is a unit vector and $\measuredangle AOE = 150^\circ$. [4]

8 **(i)** Express $\dfrac{2}{r(r+1)(r+3)}$ in partial fractions. [2]

(ii) Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{2r(r+1)(r+3)}$. [3]

(iii) Using the result in part **(ii)**, determine the value of $\displaystyle\sum_{r=5}^{\infty} \frac{1}{2r(r-2)(r-3)}$. [3]

9 Prove by mathematical induction that for all $n \in \mathbb{Z}^+$,

$$1 + (1+2) + (1+2+3) + (1+2+3+4) + \ldots + (1+2+3+\ldots+n) = \frac{1}{6}n(n+1)(n+2).$$

[5]

Hence find, in terms of n,

(i) $3 + (3+6) + (3+6+9) + (3+6+9+12) + \ldots + (3+6+9+\ldots+(6n-3))$, [2]

(ii) $3 \times (3 \times 9) \times (3 \times 9 \times 27) \times \ldots \times (3 \times 9 \times 27 \times 81 \times \ldots \times 3^n)$. [2]

10 The functions f and g are defined as follows.

$$\mathrm{f}(x) = \sqrt{|2-x|+1}\ , \qquad x \in \mathbb{R},$$

$$\mathrm{g}(x) = \begin{cases} -\dfrac{1}{3}x + \dfrac{2}{3}\ , & 0 \le x < 2, \\ 1-(x-3)^2, & x \ge 2. \end{cases}$$

(i) Show that f^{-1} does not exist. [1]

(ii) If the domain of f is restricted to $[k, \infty)$ such that f^{-1} exists, state the least value of k and define f^{-1} in a similar form. [3]

Use the new domain of f found in part **(ii)** for the following parts.

(iii) Show algebraically that there is no value of x for which $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. [2]

(iv) Find the range of the composite function gf. [2]

(v) Find the value of x such that $\mathrm{gf}(x) = 1$. [1]

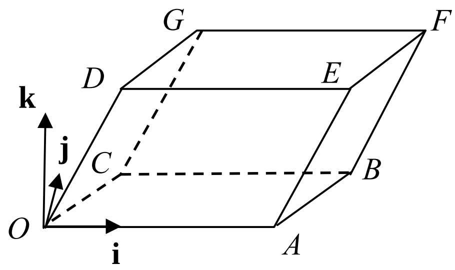
11 Sketch the graph of $y = \dfrac{2x^2-3}{x-2}$, showing clearly the axial intercepts, the stationary points and the equations of the asymptotes where applicable. [3]

(a) Solve the inequality $\dfrac{2x^2-3}{x-2} \ge 1$. [2]

Deduce the solution of the inequality $\dfrac{2\sin^2 x-3}{\sin x-2} \ge 1$, where $0 \le x \le 2\pi$. [2]

(b) Describe fully a sequence of transformations which would transform the graph of $y = 2x + \dfrac{5}{x}$ to the graph of $y = \dfrac{2x^2-3}{x-2}$. [3]

12 An art structure, which is a parallelpiped (made of 6 faces of parallelograms) has a horizontal base *OABC*, with *OA*, *OC* and *OD* as its three sides and remaining vertices are *B*, *E*, *F*, and *G* as shown in the diagram below.



It is given that $\overrightarrow{OA} = 5\mathbf{i}$ and $\overrightarrow{OC} = \mathbf{i} + 7\mathbf{j}$. The lines l_1 and l_2 have equations given by $l_1 :\ \underset{\sim}{r} = (5+\lambda)\mathbf{i} + (7\lambda - 14)\mathbf{j} + 6\mathbf{k}$, where λ is a real parameter and $l_2 :\ 3x = z + 15$, $y = 0$. *E* and *F* are on line l_1, and *A* and *E* are on line l_2.

(i) Find the position vector of *E*. [2]

(ii) Find the equation, in scalar product form, of the plane *ABFE*. [3]

(iii) Find the projection vector of $\overrightarrow{AE}$ onto the base *OABC*. Hence, or otherwise, find the area of the projection of the plane *ABFE* onto the base. [2]

(iv) Find the equation of the line l_3, which is the reflection of line *AE* about the base *OABC*. [2]

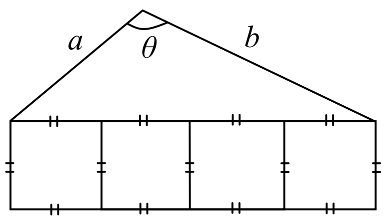
(v) An architect wants to add a shelter which has the plane equation $x + ay + bz = c$, where *a*, *b* and *c* are unknown constants. He wants the shelter to meet the plane *ABFE* at *EF*. What can be said about the values of *a*, *b* and *c* ? [2]

13 **(a)** Using differentiation, find the equation of the tangent at the point $(-2,\ 1)$ on the curve $x^3 - y^3 = 3(x-y)$. [3]

(b) A spherical balloon is inflated such that $0.1\ \mathrm{m}^3$ of air is pumped into the balloon every second. Find the rate of change of its surface area when the diameter is $1\ \mathrm{m}$. [4]

[Volume of sphere = $\dfrac{4}{3}\pi r^3$ and surface area of sphere = $4\pi r^2$.]

(c) When designing the floor plan of his new house, Mr Lim wants to build a triangular garage with 2 adjacent walls of fixed lengths a and b meters and making an angle of θ radians. On the third side of his triangular garage, he intends to build 4 square-shaped rooms of equal size (see diagram). Find the value of θ when the total area covered by the garage and the 4 rooms is a maximum. [5]



Suggested Solutions 2013 C1 H2 Math Promotional Examination

Qtn	Solutions
1(i)	Funds transferred into Account *A*: $0.05y+0.1z$ Funds transferred from Account *A*: $0.375x+0.125x=0.5x$ So we have $0.5x-\left(0.05y+0.1z\right)=16$ i.e. $0.5x-0.05y-0.1z=16$ ----(1)
(ii)	Similarly, for Account *B*, we have $-0.375x+0.1y-0.2z=-19$ ----(2) We also know $x+y+z=90$ ----(3) Solving (1), (2), (3) using GC, we have $x=40, y=20, z=30$
2	$\left(2+px\right)^{-q}$ $=2^{-q}\left(1+\dfrac{px}{2}\right)^{-q}$ $=2^{-q}\left(1+\left(-q\right)\left(\dfrac{px}{2}\right)+...\right)$ $=2^{-q}\left(1-\dfrac{pqx}{2}+...\right)$ $\approx\dfrac{1}{4}-x$ $\Rightarrow 2^{-q}=\dfrac{1}{4}$ ---(1) & $\dfrac{1}{4}\left(\dfrac{-2p}{2}\right)=-1$ ---(2) $q=2, p=4$
	$\left(2+4x\right)^{-2}$ $=\dfrac{1}{4}\left(1+2x\right)^{-2}$ $=\dfrac{1}{4}\left(1+\left(-2\right)\left(2x\right)+\dfrac{\left(-2\right)\left(-3\right)}{2!}\left(2x\right)^{2}+\dfrac{\left(-2\right)\left(-3\right)\left(-4\right)}{3!}\left(2x\right)^{3}+...\right)$ x^n coefficient $=\dfrac{1}{4}\left(\dfrac{\left(-2\right)\left(-3\right)\left(-4\right)...\left(-\left(n+1\right)\right)}{n!}\right)\left(2\right)^{n}$ $=\dfrac{1}{4}\left(-1\right)^{n}\left(n+1\right)2^{n}=\left(-1\right)^{n}\left(n+1\right)2^{n-2}$
3(i)	Vol of water at end of Day 1 $=0.9(8500)$ Vol of water at end of Day 2

$= 0.9\left(500 + 0.9(8500)\right) = 0.9\left(500\right) + 0.9^2(8500)$

Vol of water at end of Day 3

$= 0.9(500) + 0.9^2(500) + 0.9^3(8500)$

= 7051.5 litres

(ii) Vol of water at end of Day *n*, *V*

$= 0.9(500) + 0.9^2(500) + ... + 0.9^{n-1}(500) + 0.9^n(8500)$

$= 500\left(0.9 + 0.9^2 + ... + 0.9^{n-1}\right) + 0.9^n(8500)$

$$= 500\left[\frac{0.9\left(1 - 0.9^{n-1}\right)}{1 - 0.9}\right] + 0.9^n(8500)$$

$= 4500\left[1 - 0.9^{n-1}\right] + 0.9^n(8500)$

For $V < 5000$,

$4500\left[1 - 0.9^{n-1}\right] + 0.9^n(8500) < 5000$

From G.C,

n	V
18	5025.3
19	4972.8
20	4925.5

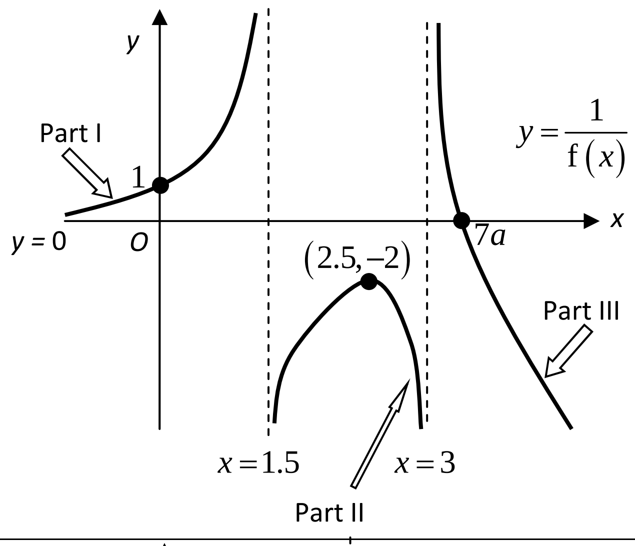
Least $n = 19$

Least number of days = 19.

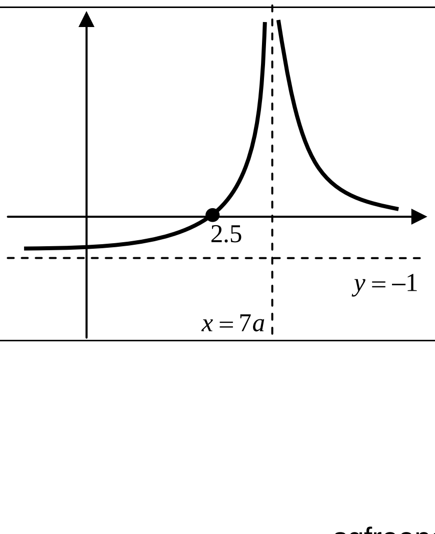
(iii) As $n \to \infty$, $\quad V \to 4500$

Therefore, water tank will never dry up.

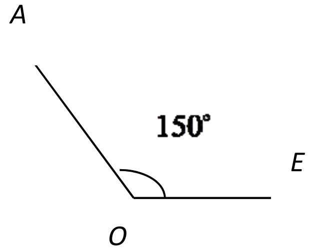
4i



ii



5	(i) Since l is the limit, As $n \to \infty$, $u_n \to l$, $u_{n+1} \to l$ $\therefore \frac{l+a}{l+b} = \frac{a}{b}$ $\Rightarrow \mathrm{b}(l+a) = a(l+b)$ $\Rightarrow bl = al$ $\Rightarrow l(b-a) = 0 \quad (\because a \neq b)$ $\Rightarrow l = 0$ (ii) $\frac{u_{n+1}+a}{u_n+b} = \frac{a}{b}$ $\Rightarrow b(u_{n+1}+a) = a(u_n+b)$ $\Rightarrow bu_{n+1} = au_n$ $\Rightarrow u_{n+1} = \frac{a}{b}u_n$ Hence $\{u_n\}$ is a GP with ratio $\frac{a}{b}$ and since $u_1 = a$, $u_n = a\left(\frac{a}{b}\right)^{n-1}$ **(ii)** Since S exists, $\lvert r \rvert < 1 \Rightarrow \left\lvert \frac{a}{b} \right\rvert < 1$ $S = \frac{a}{1-\frac{a}{b}}$ $= \frac{ab}{b-a}$
6(i)	$\frac{\mathrm{d}u}{\mathrm{d}x} = 8x$ $\int x^3\sqrt{9+4x^2}\ \mathrm{d}x = \int \frac{1}{8}x^2(8x)(9+4x^2)^{1/2}\ \mathrm{d}x$ $= \frac{1}{8}\int \left(\frac{u-9}{4}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)(u)^{1/2}\ \mathrm{d}x$ $= \int \frac{1}{32}u^{3/2} - \frac{9}{32}u^{1/2}\ \mathrm{d}u$ $= \frac{1}{80}u^{5/2} - \frac{3}{16}u^{3/2} + C$ $= \frac{1}{80}(9+4x^2)^{5/2} - \frac{3}{16}(9+4x^2)^{3/2} + C$

(ii)	$\int_0^1 x^2 \tan^{-1} x \, dx = \left[\left(\frac{1}{3}x^3\right)\tan^{-1} x\right]_0^1 - \int_0^1 \left(\frac{1}{3}x^3\right)\left(\frac{1}{1+x^2}\right) dx$ $= \left[\left(\frac{1}{3}x^3\right)\tan^{-1} x\right]_0^1 - \frac{1}{3}\int_0^1 \left(x - \frac{x}{1+x^2}\right) dx$ $= \left[\left(\frac{1}{3}x^3\right)\tan^{-1} x - \frac{1}{3}\left(\frac{1}{2}x^2 - \frac{1}{2}\ln\left(1+x^2\right)\right)\right]_0^1$ $= \left(\frac{1}{3}\right)\left(\frac{\pi}{4}\right) - \frac{1}{3}\left(\frac{1}{2} - \frac{1}{2}\ln(2)\right)$ $= \frac{\pi}{12} - \frac{1}{6}(1 - \ln 2)$
7(a)	AB line $\Rightarrow \underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$ $\overrightarrow{OP} = \begin{pmatrix} 2-5\lambda \\ \lambda \\ -1+3\lambda \end{pmatrix}$ $\overrightarrow{OD} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ C, P, D are collinear. $\overrightarrow{CP} = k\overrightarrow{CD}$ $\begin{pmatrix} 2-5\lambda-a \\ \lambda \\ -1+3\lambda \end{pmatrix} = k\begin{pmatrix} a-1 \\ 2 \\ 4 \end{pmatrix}$ $\Rightarrow \lambda = 1, k = \frac{1}{2}, a = -\frac{5}{3}$ $\overrightarrow{OD} = \begin{pmatrix} -\frac{5}{3} \\ 0 \\ 0 \end{pmatrix}$
(b)	$E(a, b, 0)$ $\overrightarrow{OE} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$  $a^2 + b^2 = 1$ $\cos 150^\circ = \frac{\overrightarrow{OA} \bullet \overrightarrow{OE}}{\left\|\overrightarrow{OA}\right\|(1)}$

$$-\frac{\sqrt{3}}{2}=\frac{\begin{pmatrix}2\\0\\-1\end{pmatrix}\bullet\begin{pmatrix}a\\b\\0\end{pmatrix}}{\sqrt{5}}$$

$$-\frac{\sqrt{3}}{2}=\frac{2a}{\sqrt{5}}\Rightarrow a=-\frac{\sqrt{15}}{4}\text{ or }0.968\text{ (3 s.f.)}$$

$$\frac{15}{16}+b^2=1\Rightarrow b=\pm\frac{1}{4}$$

$$E\left(-\frac{\sqrt{15}}{4},\frac{1}{4},0\right)\text{ or }E\left(-\frac{\sqrt{15}}{4},-\frac{1}{4},0\right)$$

8

(i)

$$\frac{2}{r(r+1)\left(r+3\right)}\equiv\frac{A}{r}+\frac{B}{r+1}+\frac{C}{r+3}$$

$$2\equiv A\left(r+1\right)\left(r+3\right)+B+r\left(r+3\right)+Cr\left(r+1\right)$$

$$r=0,\quad A=\frac{2}{3}\qquad r=-1,\quad B=-1\qquad r=0,\quad C=\frac{1}{3}$$

$$\therefore\frac{2}{r(r+1)\left(r+3\right)}\equiv\frac{2}{3r}-\frac{1}{r+1}+\frac{1}{3(r+3)}$$

$$\frac{1}{4}\sum_{r=1}^{n}\frac{2}{r(r+1)\left(r+3\right)}=\frac{1}{4}\sum_{r=1}^{n}\left(\frac{2}{3r}-\frac{1}{r+1}+\frac{1}{3(r+3)}\right)$$

$$\begin{aligned}
=\frac{1}{4}[\frac{2}{3}\quad &-\frac{1}{2}\quad &&+\cancel{\frac{1}{12}}\\
+\frac{2}{6}\quad &-\frac{1}{3}\quad &&+\cancel{\frac{1}{15}}\\
+\frac{2}{9}\quad &-\cancel{\frac{1}{4}}\quad &&+\cancel{\frac{1}{18}}\\
+\cancel{\frac{2}{12}}\quad &-\cancel{\frac{1}{5}}\quad &&+\cancel{\frac{1}{21}}\\
+\cancel{\frac{2}{15}}\quad &-\cancel{\frac{1}{6}}\quad &&+\cancel{\frac{1}{24}}\\
&\vdots\\
+\cancel{\frac{2}{3(n-3)}}\quad &-\cancel{\frac{1}{n-2}}\quad &&+\cancel{\frac{1}{3n}}\\
+\cancel{\frac{2}{3(n-2)}}\quad &-\cancel{\frac{1}{n-1}}\quad &&+\frac{1}{3\left(n+1\right)}\\
+\cancel{\frac{2}{3(n-1)}}\quad &-\cancel{\frac{1}{n}}\quad &&+\frac{1}{3\left(n+2\right)}\\
+\cancel{\frac{2}{3n}}\quad &-\frac{1}{n+1}\quad &&+\frac{1}{3\left(n+3\right)}]
\end{aligned}$$

	$=\frac{1}{4}\left[\frac{7}{18}-\frac{1}{n+1}+\frac{1}{3(n+1)}+\frac{1}{3(n+2)}+\frac{1}{3(n+3)}\right]$ $=\frac{1}{12}\left[\frac{7}{6}-\frac{2}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right]$ (iii) $\sum_{r=5}^{\infty}\frac{1}{2}\frac{1}{r(r-2)(r-3)}$ Replace r by $r+3$, $=\sum_{r=2}^{\infty}\frac{1}{2r(r+1)(r+3)}$ $=\sum_{r=1}^{\infty}\frac{1}{2r(r+1)(r+3)}-\frac{1}{2(1)(2)(4)}$ $=\lim_{n\to\infty}\left(\frac{1}{12}\left[\frac{7}{6}-\frac{2}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right]\right)-\frac{1}{16}$ $=\lim_{n\to\infty}\left(\frac{7}{72}-\frac{1}{6(n+1)}+\frac{1}{12(n+2)}+\frac{1}{12(n+3)}\right)-\frac{1}{16}$ $=\frac{7}{72}-\frac{1}{16}=\frac{5}{144}$
9	*(See alternative solution below)* Let P(n) be the statement " 1+ (1+2) + (1+2+3) + (1+2+3+…+n) $=\frac{1}{6}n(n+1)(n+2)$, $n\in\mathbb{Z}^+$ " When $n=1$, LHS of P(1) = 1, RHS of P(1) = $\frac{(1)(2)(3)}{6}=1$ Since LHS = RHS, P(1) is true. Assume P(k) is true for some $k\in\mathbb{Z}^+$, i.e. 1+ (1+2) + (1+2+3) + (1+2+3+…+k) $=\frac{1}{6}k(k+1)(k+2)$ To show P(k+1) is true, i.e. 1+ (1+2) + (1+2+3) + (1+2+3+…+k+k+1) $=\frac{1}{6}(k+1)(k+2)(k+3)$ LHS of P(k+1) =1+(1+2)+(1+2+3)+(1+2+3+…+k)+(1+2+3+…+k+k+1) $=\frac{1}{6}k(k+1)(k+2)+(1+2+3+\ldots+k+k+1)$ $=\frac{1}{6}k(k+1)(k+2)+\frac{1}{2}(k+1)(k+2)$

$= \frac{1}{6}(k+1)(k+2)(k+3)$

= RHS of P(k+1)

Since P(1) is true, and P(k) is true => P(k+1) is true, by mathematical induction, P(n) is true for $n \in \mathbb{Z}^+$.

Alternative Solution:

Let $P(n)$ be the statement "$\sum_{r=1}^{n} U_r = \frac{1}{6}n(n+1)(n+2)$, where $U_r = 1+2+3+...+r$, $n \in Z^+$"

When $n=1$, LHS of P(1) = $\sum_{r=1}^{1} U_r = U_1 = 1$,

RHS of P(1) = $\frac{6}{6} = 1$

Since LHS = RHS, P(1) is true.

Assume P(k) is true for some $k \in \mathbb{Z}^+$,

i.e. $\sum_{r=1}^{k} U_r = \frac{1}{6}k(k+1)(k+2)$

To show P(k+1) is true,

i.e. $\sum_{r=1}^{k+1} U_r = \frac{1}{6}(k+1)(k+2)(k+3)$

LHS of P(k+1)

$= \sum_{r=1}^{k+1} U_r$

$= \sum_{r=1}^{k} U_r + U_{k+1}$

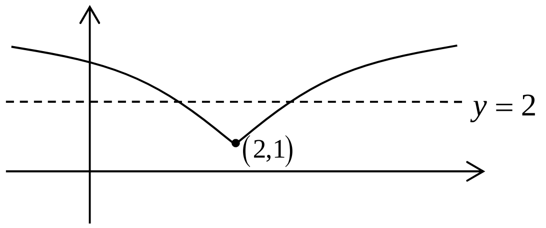
$= \frac{1}{6}k(k+1)(k+2) + (1+2+3+\ldots+k+k+1)$

$= \frac{1}{6}k(k+1)(k+2) + \frac{1}{2}(k+1)(k+2)$

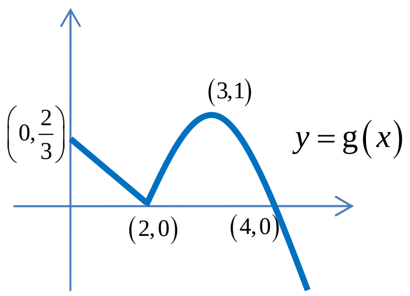
$= \frac{1}{6}k(k+1)(k+2)(k+3)$

= RHS of P(k+1)

Since P(1) is true, and P(k) is true => P(k+1) is true, by mathematical induction, P(n) is true for $n \in \mathbb{Z}^+$.

	(i) $3+(3+6)+(3+6+9)+...+(3+6+9+...+(6n-3))$ $=3[1+(1+2)+(1+2+3)+...+(1+2+3+...+(2n-1))]=3\left[\frac{1}{6}(2n-1)(2n)(2n+1)\right]$ $=n(2n-1)(2n+1)$ (ii) $3\times(3\times9)\times(3\times9\times27)\times...\times(3\times9\times27\times81\times...\times3^n)$ $=3\times(3^{1+2})\times(3^{1+2+3})\times...\times(3^{1+2+3+...+n})$ $=3^{1+(1+2)+(1+2+3)+...+(1+2+3+...+n)}$ $=3^{\frac{n(n+1)(n+2)}{6}}$
10 (i)	$\mathrm{f}(x)=\sqrt{\lvert 2-x\rvert}+1,\ x\in\mathbb{R}$  The horizontal line $y=2$ cuts the curve at more than one point, hence f is not one-to-one and f^{-1} does not exist. <u>OR</u> $\mathrm{f}(1)=\mathrm{f}(3)=2$, hence f is not one-to-one and f^{-1} does not exist.
(ii)	The minimum value is $k=2$. Let $y=\mathrm{f}(x)=\sqrt{\lvert 2-x\rvert}+1=\sqrt{x-2}+1(\because x\geq 2)$ $\Rightarrow x=2+(y-1)^2$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=[1,\infty)$ $\quad\therefore \mathrm{f}^{-1}(x)=2+(x-1)^2, x\geq 1$
(iii)	If there exists a solution for $\mathrm{f}^{-1}(x)=\mathrm{f}(x)$ $\Rightarrow$ there exists a solution for $\mathrm{f}^{-1}(x)=x$ $\Rightarrow 2+(x-1)^2=x$ $\Rightarrow x^2-3x+3=0$ $\Rightarrow\left(x-\frac{3}{2}\right)^2+\frac{3}{4}=0$ $\Rightarrow$ no solution for x $\Rightarrow \mathrm{f}^{-1}(x)=\mathrm{f}(x)$ has no solution.

<table>
<tr><td>(iv)</td><td>



$[2,\infty)\xrightarrow{\mathrm{f}}[1,\infty)\xrightarrow{\mathrm{g}}(-\infty,1]$ $\therefore R_{\mathrm{gf}}=(-\infty,1]$

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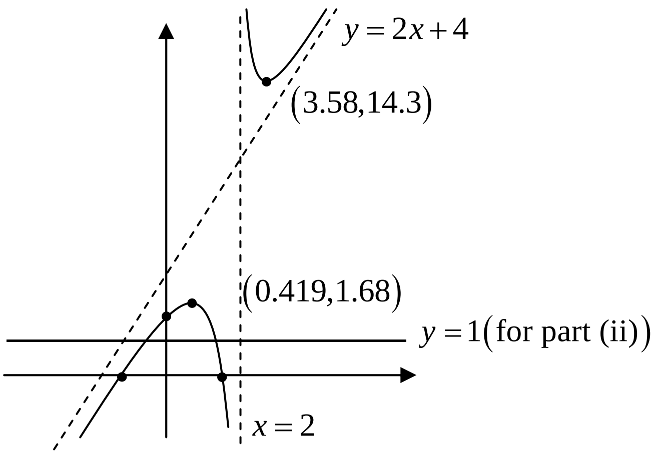
<tr><td>(v)</td><td>

$\mathrm{g}\,\mathrm{f}(x)=1$
$\mathrm{f}(x)=3$
$\sqrt{x-2}+1=3$
$\sqrt{x-2}=2$
$x-2=4$
$x=6$

</td></tr>
<tr><td>11</td><td>



<u>3 axial intercepts</u>
$\left(0,\frac{3}{2}\right),\left(\pm\sqrt{\frac{3}{2}},0\right)$ OR $(0,1.5),(\pm1.22,0)$
<u>2 turning points</u>
$(0.419,1.68),(3.58,14.3)$
<u>2 asymptotes</u>
$x=2, y=2x+4$

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<tr><td>(a)</td><td>

Using the graph, the intersections of the curve with the line y = 1 are $(-0.5,1),(1,1)$, so the solution is

$-\frac{1}{2}\le x\le 1$ or $x>2$

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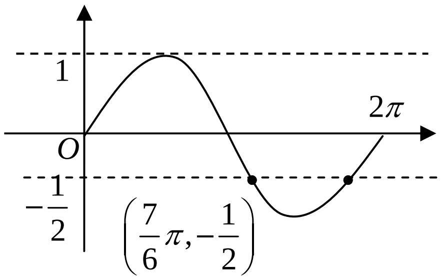
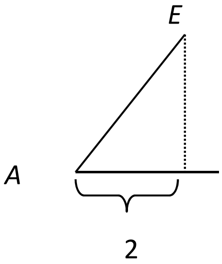
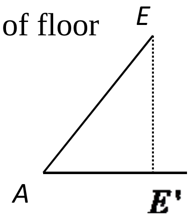
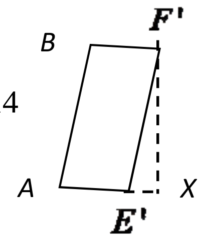
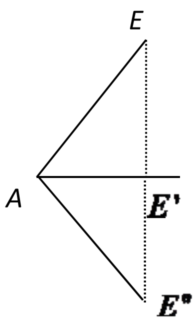
	$\dfrac{2\sin^2 x-3}{\sin x-2}\geq 1$ So the solution is $-\dfrac{1}{2}\leq \sin x\leq 1$ or $\sin x>2\left(\text{rej}\right)$  $\therefore 0\leq x\leq \dfrac{7}{6}\pi$ or $\dfrac{11}{6}\pi\leq x\leq 2\pi$
(b)	$y=\dfrac{2x^2-3}{x-2}=2x+4+\dfrac{5}{x-2}$ Translation of 2 units in the positive *x*-direction, followed by translation of 8 units in the positive *y*-direction.
12 (i)	$l_{EF}:\underset{\sim}{r}=\begin{pmatrix}5\\-14\\6\end{pmatrix}+\lambda\begin{pmatrix}1\\7\\0\end{pmatrix},\lambda\in\mathbb{R}$ $l_{AE}:3x=z+15$ $\dfrac{x-0}{1}=\dfrac{z-(-15)}{3},y=0$ $l_{AE}:\underset{\sim}{r}=\begin{pmatrix}0\\0\\-15\end{pmatrix}+\mu\begin{pmatrix}1\\0\\3\end{pmatrix},\mu\in\mathbb{R}$ $\lambda=2$ $\overrightarrow{OE}=\begin{pmatrix}5\\-14\\6\end{pmatrix}+2\begin{pmatrix}1\\7\\0\end{pmatrix}=\begin{pmatrix}7\\0\\6\end{pmatrix}$
(ii)	$\underset{\sim}{n}=\begin{pmatrix}1\\7\\0\end{pmatrix}\times\begin{pmatrix}1\\0\\3\end{pmatrix}=\begin{pmatrix}21\\-3\\-7\end{pmatrix}$

	$\begin{pmatrix}5\\0\\0\end{pmatrix}.\begin{pmatrix}21\\-3\\-7\end{pmatrix}=105$ $\underset{\sim}{r}.\begin{pmatrix}21\\-3\\-7\end{pmatrix}=105$
(iii)	Method 1: By Observation, Projection vector of $\overrightarrow{AE}$ onto $\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}$  Method 2: Projection of of $\overrightarrow{AE}$ onto normal of floor $\overrightarrow{AE'}=\left(\begin{pmatrix}2\\0\\6\end{pmatrix}.\widehat{\begin{pmatrix}1\\0\\0\end{pmatrix}}\right)\widehat{\begin{pmatrix}1\\0\\0\end{pmatrix}}=\begin{pmatrix}2\\0\\0\end{pmatrix}$  Method 1: $F'X=7$ (Deduce from $\overrightarrow{OC}$) $Area=(AE')(F'X)=2\times 7=14$  Method 2: $Area=\left\|\overrightarrow{AB}\times\overrightarrow{AE'}\right\|=\left\|\begin{pmatrix}1\\7\\0\end{pmatrix}\times\begin{pmatrix}2\\0\\0\end{pmatrix}\right\|=\left\|\begin{pmatrix}0\\0\\14\end{pmatrix}\right\|=14$
(iv)	Let E'' be the reflection of E about and plane $OABC$. $\overrightarrow{OE}=\begin{pmatrix}7\\0\\6\end{pmatrix},\overrightarrow{OE''}=\begin{pmatrix}7\\0\\-6\end{pmatrix}$, $\overrightarrow{AE''}=\overrightarrow{OE''}-\overrightarrow{OA}=\begin{pmatrix}2\\0\\-6\end{pmatrix}$ 

	$l_3 : \underset{\sim}{r} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \beta \in \mathbb{R}$
(v)	Let $\prod$ be plane $x + ay + bz = c$. EF is // $\prod$. $\begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix}$ is $\perp$ to $\underset{\sim}{n}_{\prod}$. $\begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} . \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 0 \Rightarrow 1 + 7a = 0 \Rightarrow a = -\frac{1}{7}$ E is on plane $\prod$. $\begin{pmatrix} 7 \\ 0 \\ 6 \end{pmatrix} . \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = c \Rightarrow 7 + 6b = c.$
13 (a)	$x^3 - y^3 = 3x - 3y$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 - y^3\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(3x - 3y\right)$ $3x^2 - 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 3 - 3\frac{\mathrm{d}y}{\mathrm{d}x}$ $3x^2 - 3 = 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} - 3\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{x^2 - 1}{y^2 - 1} = \frac{\mathrm{d}y}{\mathrm{d}x}$ Substitute $x = -2$ and $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{0}$ (undefined) Therefore, the tangent is a vertical line. Thus, the tangent is $x = -2$.
(b)	Let the radius be r . We want to find $\frac{\mathrm{d}S}{\mathrm{d}t}$, $\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}r} \times \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}r}$ $= \left(8\pi r\right) \times \left(0.1\right) \div \left(4\pi r^2\right)$ $= \frac{1}{5r}$ Sub $r = \frac{1}{2}$ into $\frac{\mathrm{d}S}{\mathrm{d}t}$, we get $\frac{2}{5}\mathrm{m}^2 / \mathrm{s}$.

(c)	Let the side of each room be x. By cosine rule, $(4x)^2 = a^2 + b^2 - 2ab\cos\theta$ Total area, $A = \frac{1}{2}ab\sin\theta + 4x^2$ $A = \frac{1}{2}ab\sin\theta + \frac{1}{4}\left(a^2 + b^2 - 2ab\cos\theta\right)$ $\quad = \frac{1}{2}ab\sin\theta + \frac{1}{4}a^2 + \frac{1}{4}b^2 - \frac{1}{2}ab\cos\theta$ To find max area, we let $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 0$. $\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{2}ab\sin\theta + \frac{1}{4}a^2 + \frac{1}{4}b^2 - \frac{1}{2}ab\cos\theta\right)$ $\quad = \frac{1}{2}ab\cos\theta + \frac{1}{2}ab\sin\theta$ $\frac{1}{2}ab\cos\theta + \frac{1}{2}ab\sin\theta = 0$ $\tan\theta = -1$ $\theta = \frac{3\pi}{4}$ $\left(\text{since } 0 < \theta < \pi\right)$ Therefore, stationary point at $\theta = \frac{3\pi}{4}$. $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = \frac{1}{2}ab\cos\theta - \frac{1}{2}ab\sin\theta$ $\left.\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}\right\|_{\theta=\frac{3\pi}{4}} < 0$ Thus, the stationary point is maximum.

INNOVA JUNIOR COLLEGE
JC 1 MID COURSE EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CLASS INDEX NUMBER

MATHEMATICS 9740/01

8 October 2013

3 hours

Additional Materials: Answer Paper
Cover Page
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1* **(i)** Find the expansion of $\dfrac{1+x^2}{\sqrt{(4+2x)}}$ in ascending powers of x, up to and including the term in x^2. [3]

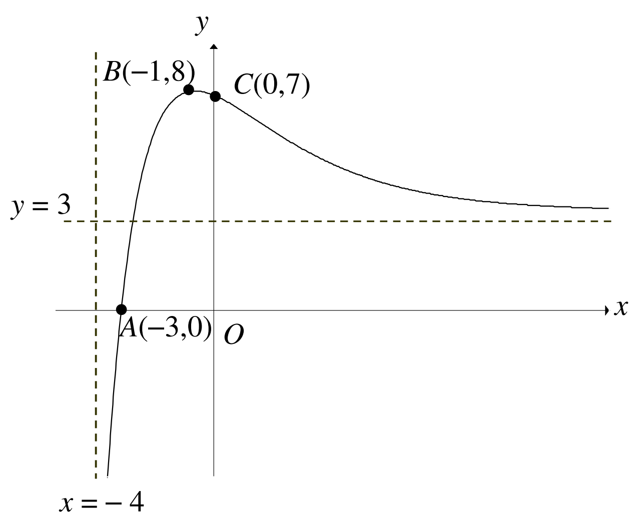
(ii) State the range of values of x for which this expansion is valid. [1]

(iii) Write down the equation of the tangent to the curve

$$y=\frac{1+x^2}{\sqrt{(4+2x)}}$$

at the point where $x = 0$. [1]

2



The diagram shows the graph of $y=\mathrm{f}\left(x\right)$. There is a maximum point $B\left(-1,8\right)$ and the curve cuts the axes at the points $A\left(-3,0\right)$ and $C\left(0,7\right)$. The lines $x=-4$ and $y=3$ are asymptotes of the curve.

Sketch, on separate diagrams, the graphs of

(i) $y=\mathrm{f}'\left(x\right)$, [2]

(ii) $y=-\sqrt{\left\{\mathrm{f}\left(\frac{1}{2}x\right)\right\}}$, [3]

stating the equations of the asymptotes and the coordinates of the points corresponding to A, B and C where possible.

3 **(i)** Using the method of difference, show that

$$\sum_{r=1}^{n} \frac{k}{(r+1)(r+3)} = \frac{k}{2}\left(a - \frac{1}{n+2} - \frac{1}{n+3}\right),$$

where a is a constant to be determined. [4]

(ii) Hence find the range of values of k such that $\sum_{r=1}^{\infty} \frac{k}{(r+1)(r+3)}$ is at most 1. [2]

4 **(i)** Prove by induction that $\sum_{r=1}^{n} \frac{r(2^r)}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$ for all positive integers n. [5]

(ii) Hence find an expression in terms of n for $\sum_{r=n}^{2n} \frac{r(2^r)}{(r+2)!}$. [2]

5* Find

(i) $\int \frac{4}{\sqrt{(5+4x-4x^2)}}\, dx$, [3]

(ii) $\int (3\sin 2\theta - \sec\theta)^2 \, d\theta$. [4]

6 Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P on AB is such that $AP : PB = 2 : 3$. It is given that $|\mathbf{a}| = \sqrt{5}$, $|\mathbf{b}| = 3$ and OP is perpendicular to AB.

(i) Show that $\mathbf{a} \cdot \mathbf{b} = -3$. [3]

(ii) Find the size of angle AOB. [2]

(iii) Find the length of projection of $\overrightarrow{OB}$ onto OA. [1]

7 A water tank in the shape of an inverted cone has a height twice that of its radius. Water is poured into the cone. Given that, when the depth of the water is 10 cm, the volume of water is increasing at a rate of 10π cm^{3}s^{-1}, find the rate of increase at this instant of

(i) the slant height of the cone in contact with the water, [5]

(ii) the curved surface area of the cone in contact with the water. [2]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$ and the curved surface area is πrl .]

8 The equation of a curve is $x^2 - 2xy + 2y^2 = -12$.

(i) Find the equations of the tangent and normal to the curve at the point $P(2,4)$. [5]

(ii) The tangent at *P* meets the *y*-axis at *A* and the normal at *P* meets the *x*-axis at *B*. Find the area of triangle *APB*. [3]

9 **(a)** An arithmetic progression *A* has first term 3 and the sum of the terms from the 16th term to the 30th term inclusive is 2025. Show that the common difference is 6. [3]

If S_n is the sum of the first *n* terms of *A*, show that the sum of the first *n* even-numbered terms of *A*, that is, the second, fourth, sixth, … terms, is given by $\left(2+\frac{1}{n}\right)S_n$. [2]

(b) A geometric series *G* has first term 30 and common ratio $-\frac{4}{5}$. Write down the sum, S_n , of the first *n* terms of the series. [1]

Find the least value of *n* for which the magnitude of the difference between S_n and the sum to infinity of the series is less than 0.004. [3]

A new series is formed by taking the reciprocal of the corresponding terms of *G*. Determine if the new series is convergent. [1]

10* **(i)** By successively differentiating $\ln(3+x)$, find the Maclaurin's series for $\ln(3+x)$, up to and including the term in x^3. [3]

(ii) Given that θ is small, find the expansion of $\left(2-\cos 5\theta^2\right)^{\frac{1}{2}}$ in ascending powers of θ, up to and including the term in θ^4. [2]

Two particles A and B produce y units of energy when they are x units away from their original position at $x=0$. The energy produced by particles A and B can be found by the equations

$$y=\ln(3+x) \text{ and}$$
$$y=\left(2-\cos 5x^2\right)^{\frac{1}{2}}$$

respectively, where $x \geq 0$.

(iii) Explain in the context of the question, what is meant by the solution to the equation

$$\ln(3+x)=\left(2-\cos 5x^2\right)^{\frac{1}{2}}.$$ [1]

(iv) Using your answers from parts **(i)** and **(ii)**, find an estimate for the maximum distance from the original position such that the difference in energy produced by both particles is at most 0.4 units. [You may assume that both particles are at the same distance from the original position.] [2]

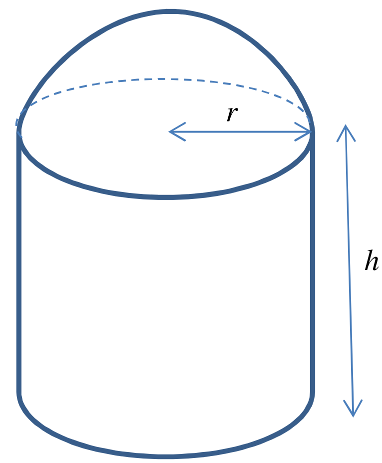
11 **(i)** Find a vector equation of the line through the points A and B with position vectors $3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ and $-\mathbf{i}+12\mathbf{j}+9\mathbf{k}$ respectively. [2]

(ii) The perpendicular to this line from the point C with position vector $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ meets the line at the point N. Find the position vector of N. [3]

(iii) Find a Cartesian equation of the line AC. [2]

(iv) Use a vector product to find the exact area of triangle OAB. [3]

12 A container is made up of an open cylinder of varying height h cm and varying radius r cm, and a hollow hemispherical lid of varying radius r cm. It costs 5 cents per square centimetre to manufacture the base, 3 cents per square centimetre to manufacture the curved surface of the cylinder and 4 cents per square centimetre to manufacture the curved surface of the hemisphere.



(i) Given that the cylinder is of fixed volume V cm^3, show that the manufacturing cost of the container is minimum when r is $\left(\frac{3V}{13\pi}\right)^{\frac{1}{3}}$. [7]

(ii) Using the value of r in part **(i)** and taking V to be 30, find the maximum number of containers that a person can buy if he has \$22. [2]

[The surface area of a sphere is $4\pi r^2$.]

13 The function f is defined as follows:

$$\mathrm{f}: x \mapsto \frac{1}{x^2-4} \quad \text{for } x \in \mathbb{R},\ x \neq -2,\ x \neq 2.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$. [2]

The function g is defined as follows:

$$\mathrm{g}: x \mapsto \frac{1}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq a,\ x \neq 3,\ x \neq b.$$

It is given that the function fg exists.

(ii) Find the values of a and b. [2]

(iii) Show that $\mathrm{fg}(x) = \frac{(x-3)^2}{(2x-5)(7-2x)}$. [2]

(iv) Solve the inequality $\mathrm{fg}(x) > 0$. [3]

(v) Find the range of fg. [3]

2013 H2 Maths MCE_Marking Scheme

1* **(i)** Find the expansion of $\dfrac{1+x^2}{\sqrt{(4+2x)}}$ in ascending powers of x, up to and including the term in x^2. [3]

(ii) State the range of values of x for which this expansion is valid. [1]

(iii) Write down the equation of the tangent to the curve

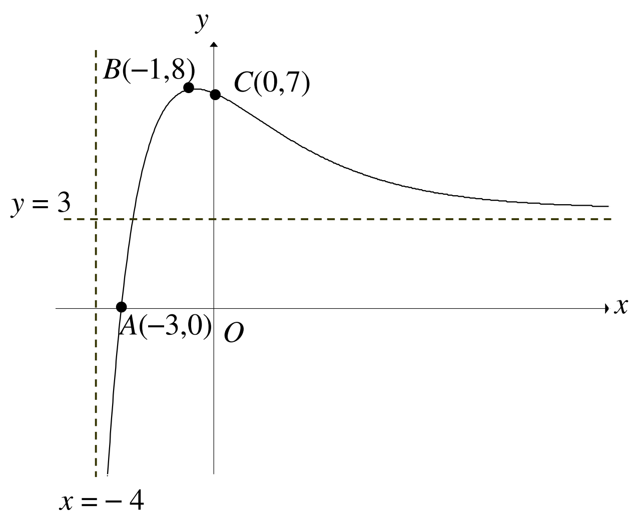
$$y=\frac{1+x^2}{\sqrt{(4+2x)}}$$

at the point where $x = 0$. [1]

1(i)	$\dfrac{1+x^2}{\sqrt{(4+2x)}}$ $=\left(1+x^2\right)(4+2x)^{-\frac{1}{2}}$ $=\dfrac{1}{2}\left(1+x^2\right)\left(1+\dfrac{x}{2}\right)^{-\frac{1}{2}}$ $=\dfrac{1}{2}\left(1+x^2\right)\left(1+\left(-\dfrac{1}{2}\right)\left(\dfrac{x}{2}\right)+\left(\dfrac{-\frac{1}{2}\times-\frac{3}{2}}{2!}\right)\left(\dfrac{x}{2}\right)^2+\ldots\right)$ $=\dfrac{1}{2}\left(1+x^2\right)\left(1-\dfrac{x}{4}+\dfrac{3}{32}x^2+\ldots\right)$ $=\dfrac{1}{2}-\dfrac{1}{8}x+\dfrac{3}{64}x^2+\dfrac{1}{2}x^2+\ldots$ $=\dfrac{1}{2}-\dfrac{1}{8}x+\dfrac{35}{64}x^2+\ldots$
(ii)	$\left\lvert\dfrac{x}{2}\right\rvert<1$ $-1<\dfrac{x}{2}<1$ $-2<x<2$
(iii)	$y=\dfrac{1}{2}-\dfrac{1}{8}x$

*: Not in topics tested for 2014 SRJC Promo

2



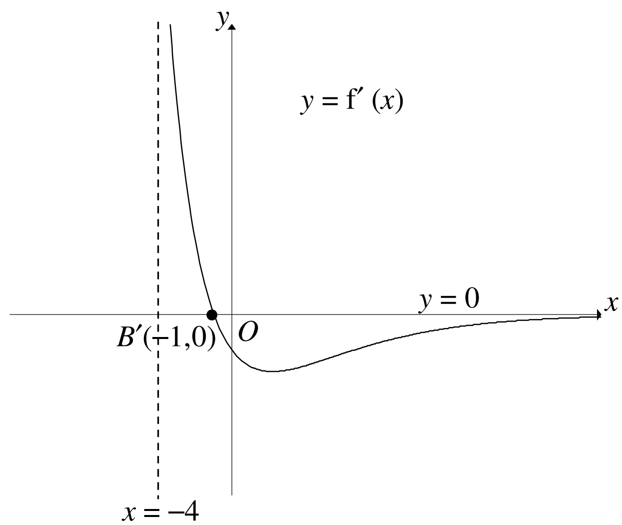
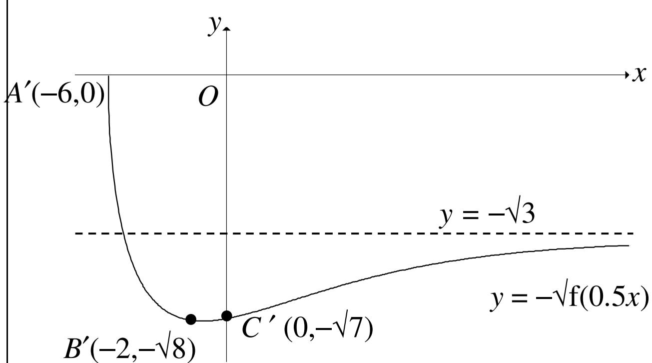
The diagram shows the graph of $y = \mathrm{f}(x)$. There is a maximum point $B(-1,8)$ and the curve cuts the axes at the points $A(-3,0)$ and $C(0,7)$. The lines $x = -4$ and $y = 3$ are asymptotes of the curve.

Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}'(x)$, [2]

(ii) $y = -\sqrt{\left\{\mathrm{f}\left(\frac{1}{2}x\right)\right\}}$, [3]

stating the equations of the asymptotes and the coordinates of the points corresponding to *A*, *B* and *C* where possible.

2(i)	y y = f′ (x) y = 0 x B′(−1,0) O x = −4
(ii)	y x A′(−6,0) O y = −√3 y = −√f(0.5x) C ′ (0,−√7) B′(−2,−√8)

3 **(i)** Using the method of difference, show that

$$\sum_{r=1}^{n}\frac{k}{(r+1)(r+3)}=\frac{k}{2}\left(a-\frac{1}{n+2}-\frac{1}{n+3}\right),$$

where a is a constant to be determined. [4]

(ii) Hence find the range of values of k such that $\displaystyle\sum_{r=1}^{\infty}\frac{k}{(r+1)(r+3)}$ is at most 1. [2]

3(i)	$\dfrac{k}{(r+1)(r+3)}=\dfrac{k}{2}\left(\dfrac{1}{r+1}-\dfrac{1}{r+3}\right)$ $\displaystyle\sum_{r=1}^{n}\frac{k}{(r+1)(r+3)}=\frac{k}{2}\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+3}\right)$ $=\dfrac{k}{2}\Big(\dfrac{1}{2}-\cancel{\dfrac{1}{4}}$ $+\dfrac{1}{3}-\cancel{\dfrac{1}{5}}$ $+\cancel{\dfrac{1}{4}}-\cancel{\dfrac{1}{6}}$ $+\cancel{\dfrac{1}{5}}-\cancel{\dfrac{1}{7}}$ $+\ \ldots$ $+\ \ldots$ $+\cancel{\dfrac{1}{n-1}}-\cancel{\dfrac{1}{n+1}}$ $+\cancel{\dfrac{1}{n}}-\dfrac{1}{n+2}$ $+\cancel{\dfrac{1}{n+1}}-\dfrac{1}{n+3}\Big)$ $=\dfrac{k}{2}\left(\dfrac{5}{6}-\dfrac{1}{n+2}-\dfrac{1}{n+3}\right)$ $a=\dfrac{5}{6}$

(ii)	$\sum_{r=1}^{\infty}\frac{k}{(r+1)(r+3)}=\frac{k}{2}\left(\frac{5}{6}\right)=\frac{5k}{12}$ $\frac{5k}{12}\le 1$ $\Rightarrow k\le\frac{12}{5}$

4 **(i)** Prove by induction that $\sum_{r=1}^{n}\frac{r\left(2^r\right)}{(r+2)!}=1-\frac{2^{n+1}}{(n+2)!}$ for all positive integers n. [5]

(ii) Hence find an expression in terms of n for $\sum_{r=n}^{2n}\frac{r\left(2^r\right)}{(r+2)!}$. [2]

4(i)	Let P_n denote $\sum_{r=1}^{n}\frac{r\left(2^r\right)}{(r+2)!}=1-\frac{2^{n+1}}{(n+2)!}$ for $n\in\mathbb{Z}^+$. When n = 1, LHS = $\sum_{r=1}^{1}\frac{r\left(2^r\right)}{(r+2)!}$ $=\frac{(1)\left(2^1\right)}{(1+2)!}$ $=\frac{2}{3!}$ $=\frac{1}{3}$ RHS = $1-\frac{2^{1+1}}{(1+2)!}$ $=1-\frac{4}{3!}$ $=1-\frac{2}{3}$ $=\frac{1}{3}$ Therefore, P_1 is true. Assume P_k is true for some $k\in\mathbb{Z}^+$,

	i.e. $\sum_{r=1}^{k} \frac{r\left(2^r\right)}{(r+2)!}=1-\frac{2^{k+1}}{(k+2)!}$. Want to prove P_{k+1} is true, i.e. $\sum_{r=1}^{k+1} \frac{r\left(2^r\right)}{(r+2)!}=1-\frac{2^{k+2}}{(k+3)!}$. LHS = $\sum_{r=1}^{k+1} \frac{r\left(2^r\right)}{(r+2)!}$ $=\sum_{r=1}^{k} \frac{r\left(2^r\right)}{(r+2)!}+\frac{(k+1)\left(2^{k+1}\right)}{(k+3)!}$ $=\left[1-\frac{2^{k+1}}{(k+2)!}\right]+\frac{(k+1)\left(2^{k+1}\right)}{(k+3)!}$ $=1-\left[\frac{\left(2^{k+1}\right)(k+3)}{(k+3)!}-\frac{(k+1)\left(2^{k+1}\right)}{(k+3)!}\right]$ $=1-\left[\frac{\left(2^{k+1}\right)\left[(k+3)-(k+1)\right]}{(k+3)!}\right]$ $=1-\left[\frac{\left(2^{k+1}\right)(2)}{(k+3)!}\right]$ $=1-\frac{2^{k+2}}{(k+3)!}$ = RHS Thus P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
(ii)	$\sum_{r=n}^{2n} \frac{r\left(2^r\right)}{(r+2)!}$ $=\sum_{r=1}^{2n} \frac{r\left(2^r\right)}{(r+2)!}-\sum_{r=1}^{n-1} \frac{r\left(2^r\right)}{(r+2)!}$ $=\left[1-\frac{2^{2n+1}}{(2n+2)!}\right]-\left[1-\frac{2^n}{(n+1)!}\right]$ $=\frac{2^n}{(n+1)!}-\frac{2^{2n+1}}{(2n+2)!}$

5* Find

(i) $\displaystyle\int \frac{4}{\sqrt{(5+4x-4x^2)}}\,\mathrm{d}x$, [3]

(ii) $\displaystyle\int (3\sin 2\theta - \sec\theta)^2\,\mathrm{d}\theta$. [4]

5(i)	$5+4x-4x^2$ $=-4\left(x^2-x-\frac{5}{4}\right)$ $=-4\left(\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-\frac{5}{4}\right)$ $=-4\left(\left(x-\frac{1}{2}\right)^2-\frac{6}{4}\right)=4\left[\frac{3}{2}-\left(x-\frac{1}{2}\right)^2\right]$ $\displaystyle\int \frac{4}{\sqrt{5+4x-4x^2}}\,\mathrm{d}x$ $\displaystyle=\int \frac{4}{\sqrt{4\left[\frac{3}{2}-\left(x-\frac{1}{2}\right)^2\right]}}\,\mathrm{d}x$ or $\displaystyle\int \frac{4}{\sqrt{6-(2x-1)^2}}\,\mathrm{d}x$ $\displaystyle=\int \frac{4}{2\sqrt{\frac{3}{2}-\left(x-\frac{1}{2}\right)^2}}\,\mathrm{d}x$ $=2\sin^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{3}{2}}}\right)+C$ or $2\sin^{-1}\left(\frac{2x-1}{\sqrt{6}}\right)+C$
(ii)	$\int (3\sin 2\theta-\sec\theta)^2\,\mathrm{d}\theta$ $=\int 9\sin^2 2\theta - 6\sin 2\theta\sec\theta+\sec^2\theta\,\mathrm{d}\theta$ $=\frac{9}{2}\int(1-\cos 4\theta)d\theta-6\int 2\sin\theta\cos\theta\sec\theta\,\mathrm{d}\theta+\int\sec^2\theta\,\mathrm{d}\theta$ $=\frac{9}{2}\int(1-\cos 4\theta)d\theta-12\int\sin\theta\,\mathrm{d}\theta+\int\sec^2\theta\,\mathrm{d}\theta$ $=\frac{9}{2}\left(\theta-\frac{1}{4}\sin 4\theta\right)-12(-\cos\theta)+\tan\theta+c$ $=\frac{9}{2}\theta-\frac{9}{8}\sin 4\theta+12\cos\theta+\tan\theta+c$

*: Not in topics tested for 2014 SRJC Promo

6 Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P on AB is such that $AP : PB = 2 : 3$. It is given that $|\mathbf{a}| = \sqrt{5}$, $|\mathbf{b}| = 3$ and OP is perpendicular to AB.

(i) Show that $\mathbf{a} \cdot \mathbf{b} = -3$. [3]

(ii) Find the size of angle AOB. [2]

(iii) Find the exact length of projection of $\overrightarrow{OB}$ onto OA. [1]

6(i)	By Ratio Theorem, $\overrightarrow{OP} = \frac{1}{5}(3\mathbf{a} + 2\mathbf{b})$. Since $OP \perp AB$, $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$. $\frac{1}{5}(3\mathbf{a} + 2\mathbf{b}) \cdot (\mathrm{b} - \mathrm{a}) = 0$ $3\mathbf{a} \cdot \mathbf{b} - 3\mathbf{a} \cdot \mathbf{a} + 2\mathbf{b} \cdot \mathbf{b} - 2\mathbf{b} \cdot \mathbf{a} = 0$ $\mathbf{a} \cdot \mathbf{b} - 3\lvert\mathbf{a}\rvert^2 + 2\lvert\mathbf{b}\rvert^2 = 0$ $\mathbf{a} \cdot \mathbf{b} - 15 + 18 = 0$ $\mathbf{a} \cdot \mathbf{b} = -3$
(ii)	$\cos \angle AOB = \frac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert \lvert\mathbf{b}\rvert}$ $= \frac{-3}{3\sqrt{5}}$ $= -\frac{1}{\sqrt{5}}$ $\angle AOB = 116.6^\circ$ (or 2.03 rad)
(iii)	Length of projection of $\overrightarrow{OB}$ onto OA $= \left\lvert \frac{\mathbf{b} \cdot \mathbf{a}}{\lvert\mathbf{a}\rvert} \right\rvert$ $= \frac{3}{\sqrt{5}}$

7 A water tank in the shape of an inverted cone has a height twice that of its radius. Water is poured into the cone. Given that, when the depth of the water is 10 cm, the volume of water is increasing at a rate of 10π cm^{3}s^{-1}, find the rate of increase at this instant of

(i) the slant height of the cone in contact with the water, [5]

(ii) the curved surface area of the cone in contact with the water. [2]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$ and the curved surface area is πrl .]

7(i)	Let the radius of the water surface, the depth of the water, the slant height of the water and the volume of the water at time t seconds be r cm, h cm, l cm and V cm^3 respectively. $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi r^2(2r)=\frac{2}{3}\pi r^3$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}r}\cdot\frac{\mathrm{d}r}{\mathrm{d}t}=2\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}$ $h=2r=10\Rightarrow r=5$ When $\frac{\mathrm{d}V}{\mathrm{d}t}=10\pi$ and $r=5$, $10\pi=2\pi(5)^2\frac{\mathrm{d}r}{\mathrm{d}t}$ $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{1}{5}$ Using Pythagoras' theorem, $l^2=(2r)^2+r^2$ $l=\sqrt{5}r$ $\frac{\mathrm{d}l}{\mathrm{d}t}=\frac{\mathrm{d}l}{\mathrm{d}r}\cdot\frac{\mathrm{d}r}{\mathrm{d}t}=\sqrt{5}\frac{\mathrm{d}r}{\mathrm{d}t}=\sqrt{5}\left(\frac{1}{5}\right)=\frac{\sqrt{5}}{5}$ or 0.44721 The rate of increase of the slant height of the cone in contact with the water is $\frac{\sqrt{5}}{5}$cms^{-1} (or 0.447 cms^{-1}) .

7(ii)	Let the curved surface area of the water at time t seconds be A cm^2. $A=\pi rl=\pi r\left(\sqrt{5}r\right)=\sqrt{5}\pi r^2$ $\dfrac{\mathrm{d}A}{\mathrm{d}t}=\dfrac{\mathrm{d}A}{\mathrm{d}r}\cdot\dfrac{\mathrm{d}r}{\mathrm{d}t}=2\sqrt{5}\pi r\dfrac{\mathrm{d}r}{\mathrm{d}t}$ When $r=5$, $\dfrac{\mathrm{d}r}{\mathrm{d}t}=\dfrac{1}{5}$. $\dfrac{\mathrm{d}A}{\mathrm{d}t}=2\sqrt{5}\pi(5)\left(\dfrac{1}{5}\right)=2\sqrt{5}\pi=14.0496$ The rate of increase of the curved surface area of the cone in contact with the water is $2\sqrt{5}\pi$ cm^{2}s^{-1} (or 14.0 cm^{2}s^{-1}).

8 The equation of a curve is $x^2-2xy+2y^2=-12$.

(i) Find the equations of the tangent and normal to the curve at the point $P(2,4)$. [5]

(ii) The tangent at P meets the y-axis at A and the normal at P meets the x-axis at B. Find the area of triangle APB. [3]

8(i)	$x^2-2xy+2y^2=-12$ $2x-\left(2x\frac{\mathrm{d}y}{\mathrm{d}x}+2y\right)+4y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $2x-2y=2x\frac{\mathrm{d}y}{\mathrm{d}x}-4y\frac{\mathrm{d}y}{\mathrm{d}x}$ $2x-2y=\frac{\mathrm{d}y}{\mathrm{d}x}(2x-4y)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x-2y}{2x-4y}$ $=\frac{x-y}{x-2y}$ At $P(2,4)$: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2-4}{2-8}$ $=\frac{1}{3}$ Equation of tangent: $y-4=\frac{1}{3}(x-2)$ $y=\frac{1}{3}x+\frac{10}{3}$ Gradient of normal $=-3$ Equation of normal: $y-4=-3(x-2)$ $y=-3x+10$

8(ii)	When tangent meets y-axis at A, $x=0$ $y=\dfrac{10}{3}$ $\therefore A\left(0,\dfrac{10}{3}\right)$ When normal meets x-axis at B, $y=0$ $3x=10$ $x=\dfrac{10}{3}$ $\therefore B\left(\dfrac{10}{3},0\right)$ Area of triangle APB $=\dfrac{1}{2}\times AP\times BP$ $=\dfrac{1}{2}\times\sqrt{\dfrac{40}{9}}\times\sqrt{\dfrac{160}{9}}$ $=\dfrac{40}{9}\text{units}^2$ (or 4.44 units^2)

9 **(a)** An arithmetic progression A has first term 3 and the sum of the terms from the 16^{th} term to the 30^{th} term inclusive is 2025. Show that the common difference is 6. [3]

If S_n is the sum of the first n terms of A, show that the sum of the first n even-numbered terms of A, that is, the second, fourth, sixth, … terms, is given by $\left(2+\frac{1}{n}\right)S_n$. [2]

9(a)	$S_{30}-S_{15}=2025$ $\frac{30}{2}\left[2(3)+29d\right]-\frac{15}{2}\left[2(3)+14d\right]=2025$ $330d=1980$ $d=6$ $S_n=\frac{n}{2}\left[6+(n-1)6\right]=3n^2$ Sum of 1^{st} n even-numbered terms $=\frac{n}{2}\left[2(3+6)+(n-1)12\right]$ $=\frac{n}{2}\left[6+12n\right]$ $=3n^2\left(\frac{1}{n}+2\right)$ $=\left(2+\frac{1}{n}\right)S_n$

9(b) A geometric series G has first term 30 and common ratio $-\frac{4}{5}$. Write down the sum, S_n, of the first n terms of the series. [1]

Find the least value of n for which the magnitude of the difference between S_n and the sum to infinity of the series is less than 0.004. [3]

A new series is formed by taking the reciprocal of the corresponding terms of G. Determine if the new series is convergent. [1]

9(b)	$S_n = \dfrac{30\left[1-\left(-\frac{4}{5}\right)^n\right]}{1-\left(-\frac{4}{5}\right)} = \dfrac{50}{3}\left[1-\left(-\frac{4}{5}\right)^n\right]$ $\left\|S_n - S_\infty\right\| < 0.004$ $\left\|\dfrac{50}{3}\left[1-\left(-\frac{4}{5}\right)^n\right]-\dfrac{50}{3}\right\| < 0.004$ $\dfrac{50}{3}\left(\dfrac{4}{5}\right)^n < 0.004$ $\left(\dfrac{4}{5}\right)^n < 0.004\times\dfrac{3}{50}$ $n > \dfrac{\ln\left(0.004\times\frac{3}{50}\right)}{\ln\left(\frac{4}{5}\right)}$ $n > 37.352$ Least value of n is 38. New series $\dfrac{1}{a}+\dfrac{1}{ar}+\dfrac{1}{ar^2}+\dfrac{1}{ar^3}+\dfrac{1}{ar^4}+\ldots$ is a geometric series with common ratio $\dfrac{1}{r} = -\dfrac{5}{4}$. Since $\left\|\dfrac{1}{r}\right\| = \dfrac{5}{4} > 1$, the new series is not convergent.

10* **(i)** By successively differentiating $\ln(3+x)$, find the Maclaurin's series for $\ln(3+x)$, up to and including the term in x^3. [3]

(ii) Given that θ is small, find the expansion of $\left(2-\cos 5\theta^2\right)^{\frac{1}{2}}$ in ascending powers of θ, up to and including the term in θ^4. [2]

Two particles A and B produce y units of energy when they are x units away from their original position at $x=0$. The energy produced by particles A and B can be found by the equations

$$y=\ln(3+x) \text{ and}$$

$$y=\left(2-\cos 5x^2\right)^{\frac{1}{2}}$$

respectively, where $x \geq 0$.

(iii) Explain in the context of the question, what is meant by the solution to the equation

$$\ln(3+x)=\left(2-\cos 5x^2\right)^{\frac{1}{2}}.$$ [1]

(iv) Using your answers from parts **(i)** and **(ii)**, find an estimate for the maximum distance from the original position such that the difference in energy produced by both particles is at most 0.4 units. [2]

[You may assume that both particles are at the same distance from the original position.]

10(i)	Let $y=\ln(3+x)$ $\frac{dy}{dx}=(3+x)^{-1}$ $\frac{d^2y}{dx^2}=-(3+x)^{-2}$ $\frac{d^3y}{dx^3}=2(3+x)^{-3}$ When $x=0$, $y=\ln 3,\quad \frac{dy}{dx}=\frac{1}{3},\quad \frac{d^2y}{dx^2}=-\frac{1}{9},\quad \frac{d^3y}{dx^3}=\frac{2}{27}$ $\therefore y=\ln 3+\frac{x}{3}-\frac{x^2}{18}+\frac{x^3}{81}+\ldots$
(ii)	Given that θ is small, $\left(2-\cos 5\theta^2\right)^{\frac{1}{2}}=\left[2-\left(1-\frac{\left(5\theta^2\right)^2}{2}\right)+\ldots\right]^{\frac{1}{2}}$ $=\left(1+\frac{25}{2}\theta^4+\ldots\right)^{\frac{1}{2}}$ $=1+\left(\frac{1}{2}\right)\frac{25}{2}\theta^4+\ldots$ $=1+\frac{25}{4}\theta^4+\ldots$
(iii)	The solution (x value) denotes the distance in units where both particles produce the same number of units of energy.
(iv)	$\left\|\ln 3+\frac{x}{3}-\frac{x^2}{18}+\frac{x^3}{81}-\left(1+\frac{25}{4}x^4\right)\right\|\le 0.4$ Or $-0.4\le \ln 3+\frac{x}{3}-\frac{x^2}{18}+\frac{x^3}{81}-\left(1+\frac{25}{4}x^4\right)\le 0.4$ From GC, $x\le 0.57298752$ (given $x\ge 0$) An estimate for the maximum distance is 0.572 units. (3 s.f.)

11 **(i)** Find a vector equation of the line through the points A and B with position vectors $3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ and $-\mathbf{i}+12\mathbf{j}+9\mathbf{k}$ respectively. [2]

(ii) The perpendicular to this line from the point C with position vector $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ meets the line at the point N. Find the position vector of N. [3]

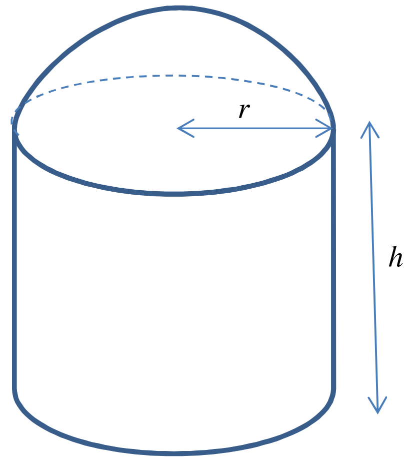
(iii) Find a Cartesian equation of the line AC. [2]

(iv) Use a vector product to find the exact area of triangle OAB. [3]

11(i)	$\overrightarrow{AB}=\begin{pmatrix}-1\\12\\9\end{pmatrix}-\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}-4\\8\\4\end{pmatrix}=4\begin{pmatrix}-1\\2\\1\end{pmatrix}$ $l_{AB}:\mathbf{r}=\begin{pmatrix}3\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix},\lambda\in\mathbb{R}$ or $l_{AB}:\mathbf{r}=\begin{pmatrix}-1\\12\\9\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix},\lambda\in\mathbb{R}$
(ii)	Since N lies on line AB, $\overrightarrow{ON}=\begin{pmatrix}3\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{CN}=\begin{pmatrix}3\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}-\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}1\\3\\7\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}$ Since $CN\perp AB$, $\overrightarrow{CN}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=0$ $\left[\begin{pmatrix}1\\3\\7\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}\right]\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=0$ $\begin{pmatrix}1\\3\\7\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=0$ $12+6\lambda=0$ $\lambda=-2$ $\overrightarrow{ON}=\begin{pmatrix}3\\4\\5\end{pmatrix}-2\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}5\\0\\3\end{pmatrix}$

11(iii)	$\overrightarrow{AC}=\begin{pmatrix}2\\1\\-2\end{pmatrix}-\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}-1\\-3\\-7\end{pmatrix}=-\begin{pmatrix}1\\3\\7\end{pmatrix}$ Cartesian eqn of line AC: $x-3=\dfrac{y-4}{3}=\dfrac{z-5}{7}$ or $x-2=\dfrac{y-1}{3}=\dfrac{z+2}{7}$
(iv)	Area of triangle OAB $=\dfrac{1}{2}\left\lvert\overrightarrow{OA}\times\overrightarrow{OB}\right\rvert$ $=\dfrac{1}{2}\left\lvert\begin{pmatrix}3\\4\\5\end{pmatrix}\times\begin{pmatrix}-1\\12\\9\end{pmatrix}\right\rvert$ $=\dfrac{1}{2}\left\lvert\begin{pmatrix}-24\\-32\\40\end{pmatrix}\right\rvert$ $=\dfrac{1}{2}\times 8\left\lvert\begin{pmatrix}-3\\-4\\5\end{pmatrix}\right\rvert$ $=4\sqrt{9+16+25}$ $=4\sqrt{50}$ $=20\sqrt{2}$

12 A container is made up of an open cylinder of varying height h cm and varying radius r cm, and a hollow hemispherical lid of varying radius r cm. It costs 5 cents per square centimetre to manufacture the base, 3 cents per square centimetre to manufacture the curved surface of the cylinder and 4 cents per square centimetre to manufacture the curved surface of the hemisphere.



(i) Given that the cylinder is of fixed volume V cm^3, show that the manufacturing cost of the container is minimum when r is $\left(\frac{3V}{13\pi}\right)^{\frac{1}{3}}$. [7]

(ii) Using the value of r in part **(i)** and taking V to be 30, find the maximum number of containers that a person can buy if he has \$22. [2]

[The surface area of a sphere is $4\pi r^2$.]

12(i)	$V = \pi r^2 h$ $\therefore h = \frac{V}{\pi r^2}$

	Let C cents be the manufacturing cost of the container. $C=4\left(2\pi r^2\right)+3\left(2\pi rh\right)+5\left(\pi r^2\right)$ $=13\pi r^2+6\pi r\left(\dfrac{V}{\pi r^2}\right)$ $=13\pi r^2+\dfrac{6V}{r}$ $\dfrac{\mathrm{d}C}{\mathrm{d}r}=13\pi\left(2r\right)+6V\left(-r^{-2}\right)$ $=26\pi r-\dfrac{6V}{r^2}$ Let $\dfrac{\mathrm{d}C}{\mathrm{d}r}=0$ $26\pi r-\dfrac{6V}{r^2}=0$ $26\pi r^3=6V$ $r^3=\dfrac{6V}{26\pi}$ $=\dfrac{3V}{13\pi}$ $r=\sqrt[3]{\dfrac{3V}{13\pi}}$ $\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2}=26\pi-6V\left(-2r^{-3}\right)$ $=26\pi+\dfrac{12V}{r^3}$ $=26\pi+\dfrac{12V}{\left(\dfrac{3V}{13\pi}\right)}$ $=26\pi+52\pi$ $=78\pi>0$ Hence, the manufacturing cost is minimum when $r=\sqrt[3]{\dfrac{3V}{13\pi}}$. [Shown]

(ii)	$C = 13\pi r^2 + \dfrac{6V}{r}$ $= 13\pi\left(\sqrt[3]{\dfrac{3V}{13\pi}}\right)^2 + \dfrac{6V}{\sqrt[3]{\dfrac{3V}{13\pi}}}$ $= 13\pi\left(\dfrac{90}{13\pi}\right)^{\frac{2}{3}} + \dfrac{180}{\left(\dfrac{90}{13\pi}\right)^{\frac{1}{3}}}$ $= 207.48$ cents $= \$2.0748$ $\dfrac{22}{2.0748}$ $= 10.603$ $\therefore$ Maximum number of containers he can buy is 10.

13 The function f is defined as follows:

$$\mathrm{f} : x \mapsto \frac{1}{x^2 - 4} \quad \text{for } x \in \mathbb{R},\ x \neq -2,\ x \neq 2.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$. [2]

The function g is defined as follows:

$$\mathrm{g} : x \mapsto \frac{1}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq a,\ x \neq 3,\ x \neq b.$$

It is given that the function fg exists.

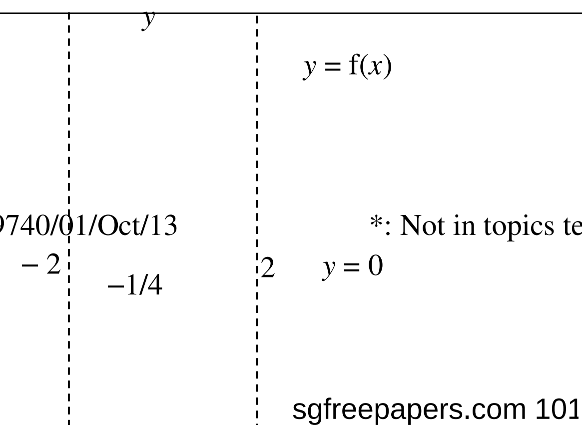
(ii) Find the values of a and b. [2]

(iii) Show that $\mathrm{fg}(x) = \dfrac{(x-3)^2}{(2x-5)(7-2x)}$. [2]

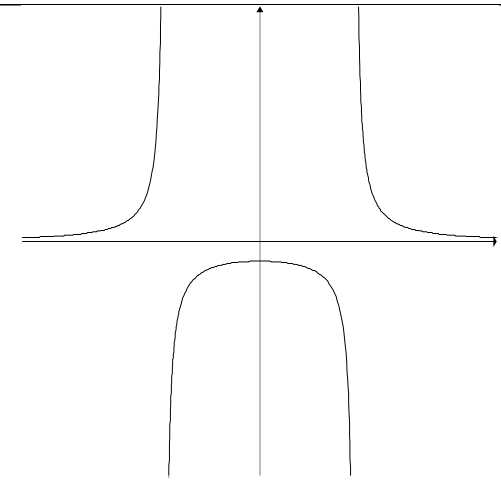
(iv) Solve the inequality $\mathrm{fg}(x) > 0$. [3]

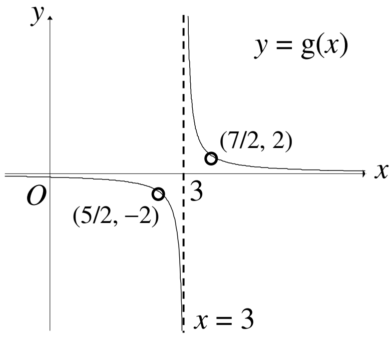
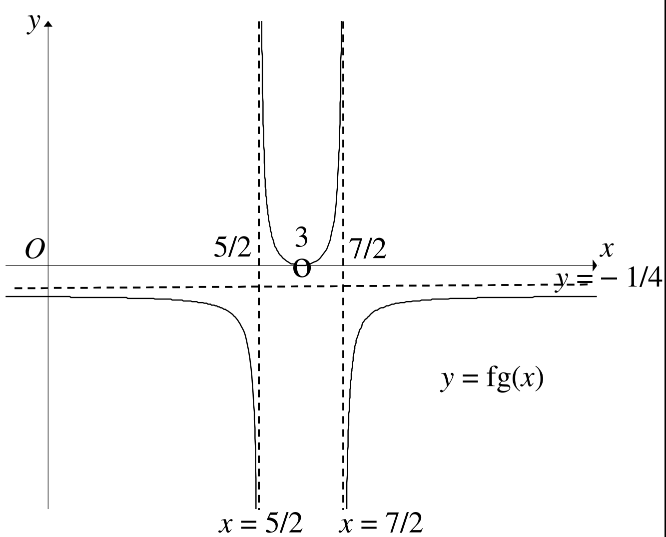
(v) Find the range of fg. [3]

13(i)	



*: Not in topics tested for 2014 SRJC Promo

<table>
<tr><td></td><td>
</td></tr>
<tr><td>(ii)</td><td>For fg to exist, $R_{\text{g}} \subseteq D_{\text{f}}$.
Hence, $\text{g}(x)$ cannot take the values – 2 and 2.
$\frac{1}{x-3} = -2 \Rightarrow x = \frac{5}{2}$
$\frac{1}{x-3} = 2 \Rightarrow x = \frac{7}{2}$
The values of a and b are $\frac{5}{2}$ and $\frac{7}{2}$.</td></tr>
<tr><td>(iii)</td><td>$\text{f g}(x) = \frac{1}{\left(\frac{1}{x-3}\right)^2 - 4}$
$= \frac{1}{\frac{1-4(x-3)^2}{(x-3)^2}}$
$= \frac{(x-3)^2}{1^2 - \left[2(x-3)\right]^2}$
$= \frac{(x-3)^2}{\left[1+2(x-3)\right]\left[1-2(x-3)\right]}$
$= \frac{(x-3)^2}{(2x-5)(7-2x)}$ (shown)</td></tr>
<tr><td>(iv)</td><td>$\frac{(x-3)^2}{(7-2x)(2x-5)} > 0$
−   +   +   −</td></tr>
</table>

	2.5 3 3.5 Solving, $\frac{5}{2}<x<3$ or $3<x<\frac{7}{2}$ or $\frac{5}{2}<x<\frac{7}{2},\ x\neq 3$
(v)	Sketching the graph of $y = \mathrm{g}(x)$,  $R_{\mathrm{g}}=\{y\in\mathbb{R}: y\neq -2,0,2\}$ Referring to the graph of $y = \mathrm{f}(x)$ in part (i), $R_{\mathrm{fg}}=\left\{y\in\mathbb{R}: y<-\frac{1}{4}\ \text{ or }\ y>0\right\}$ OR Sketch the graph of $y = \mathrm{fg}(x)$. 

	From the graph of $y = \mathrm{fg}(x)$, $R_{\mathrm{fg}} = \left\{ y \in \mathbb{R} : y < -\frac{1}{4} \text{ or } y > 0 \right\}$.

1* Expand

$$(1+2x)\sqrt{4+3x}$$

in ascending powers of x, up to and including the term in x^2. [3]

Determine the range of values of x for which the expansion is valid. [1]

2 **(i)** Given that $\dfrac{2n-1}{(n-1)^2 n^2}$ can be written in the form $\dfrac{A}{(n-1)^2}+\dfrac{B}{n^2}$, find the values of the constants A and B. [2]

(ii) Hence find $\displaystyle\sum_{r=2}^{N}\frac{2r-1}{(r-1)^2 r^2}$. [3]

(iii) Using your answer in (ii), find $\displaystyle\sum_{r=1}^{N}\frac{2r+1}{r^2(r+1)^2}$. [2]

3 Machines *A* and *B* are used to cut metal bars of length 30m into pieces of decreasing lengths.

(i) The lengths of all the pieces cut by machine *A* form an arithmetic progression with common difference *d* m. If the total length of the first 25 pieces cut is 25m and the length of the 25th piece is 0.5m, find the value of *d*. [3]

(ii) The length of the first piece cut by machine *B* is 2m and the lengths of all the pieces cut form a geometric progression. The 25th piece cut by machine *B* has length 0.5m. Find the maximum number of pieces of metal bars cut. [4]

4 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = 1 \text{ and } u_{n+1} = \frac{4+2u_n}{5} \quad \text{for } n \geq 1.$$

(i) Find the values of u_2 and u_3. [2]

(ii) It is given that $u_n \to l$ as $n \to \infty$. Showing your working, find the exact value of l. [2]

(iii) For this value of l, use the method of mathematical induction to prove that

$$u_n = l - \frac{1}{3}\left(\frac{2}{5}\right)^{n-1} \quad \text{for } n \geq 1.$$ [4]

*: Not in the topics tested in 2014 SRJC Promo

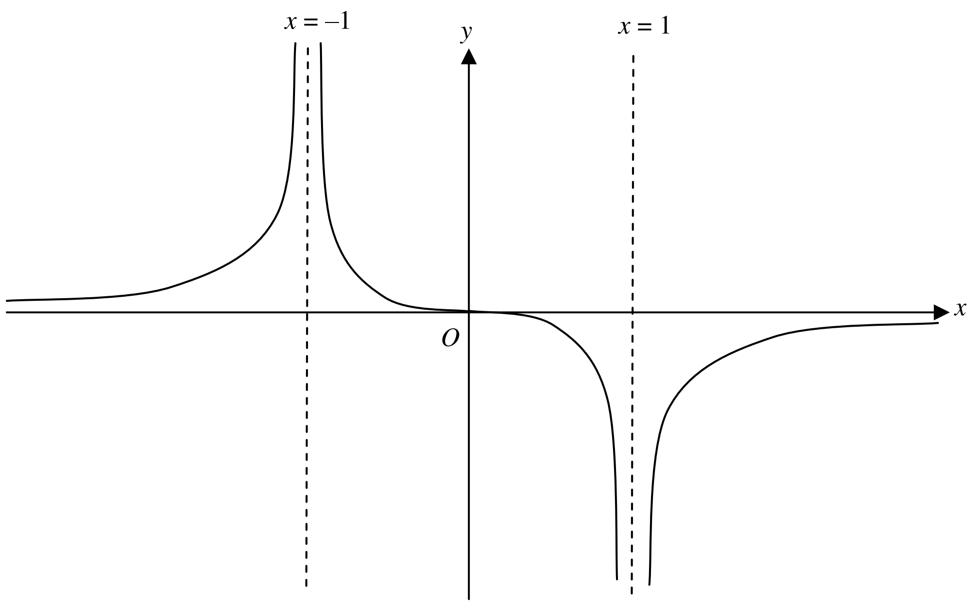
5 The curve C has equation $y = \dfrac{x^2 - 3x + 3}{1 - x}$.

(i) Find the equations of the asymptotes of C. [2]

(ii) Prove using an algebraic method, that y cannot lie between two certain values (to be determined). [3]

(iii) Sketch the curve C clearly labeling all asymptotes, turning points and axial intercepts. [3]

6 The diagram shows the graph of $y = \mathrm{f}(x)$. It has a vertical asymptotes at $x = 1$ and $x = -1$. It has a stationary point of inflexion at the origin.



Sketch on separate diagrams, the graphs of

(i) $y = \mathrm{f}(2 - x)$, [3]

(ii) $y = -\left|\mathrm{f}(x)\right|$, [2]

(iii) $y = \mathrm{f}'(x)$. [2]

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[Turn over

7 **(a)** Show that $x^2 - 3x + 5$ is always positive and solve the inequality

$$\frac{x^2 - 3x + 5}{(4-x)(x-2)} < 0.$$ [4]

Hence find the solution for the inequality $\frac{(x+2)^2 - 3x - 1}{x(2-x)} < 0$. [2]

(b) A factory produces 3 brands of drinks, *A*, *B* and *C*. The total price of 1 litre of *A*, 1 litre of *B* and 2 litres of *C* is $9. The total price of 1 litre of *B* and 1 litre of *C* is $3.50. The total price of 2.5 litres of *B* and 2 litres of *C* is twice the price of 1 litre of *A*.
Write down and solve the equations to find the price of each litre of *A*, *B* and *C*. [4]

8 The functions f and g are defined by

$$\mathrm{f}: x \mapsto 3\ln\left(x^2+1\right), \quad 0 \le x \le 2,$$
$$\mathrm{g}: x \mapsto \mathrm{e}^x + 1, \quad x \ge 0.$$

(i) Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. [3]

(ii) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on a single diagram. State the geometrical relationship between the graphs and hence state the number of solutions to $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [4]

(iii) Show that gf exists, define it in a similar form and find its range. [4]

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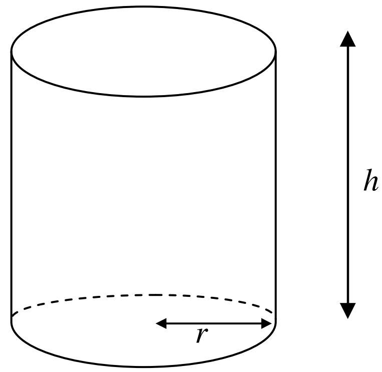
9 **(a)**



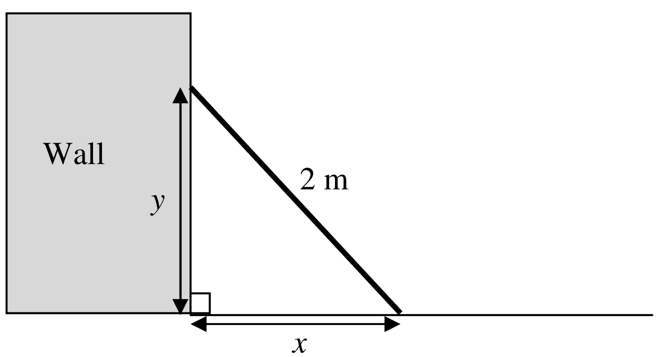
A closed cylindrical can with base radius r and height h has a fixed volume V.

(i) Show that the total surface area of the can, A, is given by

$$A = 2\pi r^2 + \frac{2V}{r}.$$ [1]

(ii) Find h in terms of r when the minimum surface area is achieved. [4]

(b)



A ladder of length 2 m, leaning against the wall, slips in such a way that x increases at a rate of 0.02 ms^{-1}. Find the rate of decrease of y at the instant when x is 1 m. [4]

*: Not in the topics tested in 2014 SRJC Promo

[Turn over

10 **(a)** The curve C is defined by

$$x = e^{3t}, \qquad y = t^2, \qquad \text{where } t \geq 0.$$

(i) Find $\frac{dy}{dx}$ in terms of t and determine the value of t for which $\frac{dy}{dx}$ is zero. [3]

(ii) Sketch the graph of C. [2]

(b) The equation of a curve C is $x^2 - 2xy + 2y^2 = k$, where k is a constant.

Find $\frac{dy}{dx}$ in terms of x and y. [3]

Given that C has two points for which the tangents are parallel to the line $y = x$, find the range of values of k. [3]

Given that $k = 4$, find the exact coordinates of each point on the curve C at which the tangent is parallel to the y-axis. [4]

11* **(a)** Find

(i) $\int x^2 e^x \, dx$, [3]

(ii) $\int_0^{\frac{\pi}{3}} \sin^2 2x \, dx$, leaving your answer in exact form. [3]

(b) Using the substitution $u = 3x - 1$, find

$$\int \frac{9x}{(3x-1)^2} \, dx.$$ [3]

(c) Given that $x + 1 = A(2x - 4) + B$ for all values of x, find the constants A and B.

Hence, find

$$\int \frac{x+1}{x^2 - 4x + 13} \, dx.$$ [5]

[End of Paper]

*: Not in the topics tested in 2014 SRJC Promo

Jurong Junior College
2013 JC1 H2 Mathematics Promo Solutions
(Qn 1 and 11 are not in topics tested for 2014 SRJC Promo)

Qn	Solution
1	$(1+2x)\sqrt{4+3x}$ $=(1+2x)2\left(1+\frac{3x}{4}\right)^{\frac{1}{2}}$ $=2(1+2x)\left(1+\frac{3x}{8}-\frac{9x^2}{128}+...\right)$ $=2+\frac{19x}{4}+\frac{87x^2}{64}...$
	Validity: $\left\lvert\frac{3x}{4}\right\rvert<1$ $-\frac{4}{3}<x<\frac{4}{3}$

Qn	Solution
2	(i) $\frac{2n-1}{(n-1)^2n^2}=\frac{A}{(n-1)^2}+\frac{B}{n^2}$ $=\frac{An^2+B(n-1)^2}{(n-1)^2n^2}$ $2n-1=An^2+B(n-1)^2$ When $n=0,\ B=-1$. When $n=1,\ A=1$. $\therefore \frac{2n-1}{(n-1)^2n^2}=\frac{1}{(n-1)^2}-\frac{1}{n^2}$
	(ii) $\sum_{r=2}^{N}\frac{2r-1}{(r-1)^2r^2}=\sum_{r=2}^{N}\left[\frac{1}{(r-1)^2}-\frac{1}{r^2}\right]$ $=\Big[\frac{1}{1^2}-\frac{1}{2^2}$ $+\frac{1}{2^2}-\frac{1}{3^2}$ $+\frac{1}{3^2}-\frac{1}{4^2}$ $+...$ $+\frac{1}{(N-1)^2}-\frac{1}{N^2}\Big]=1-\frac{1}{N^2}$

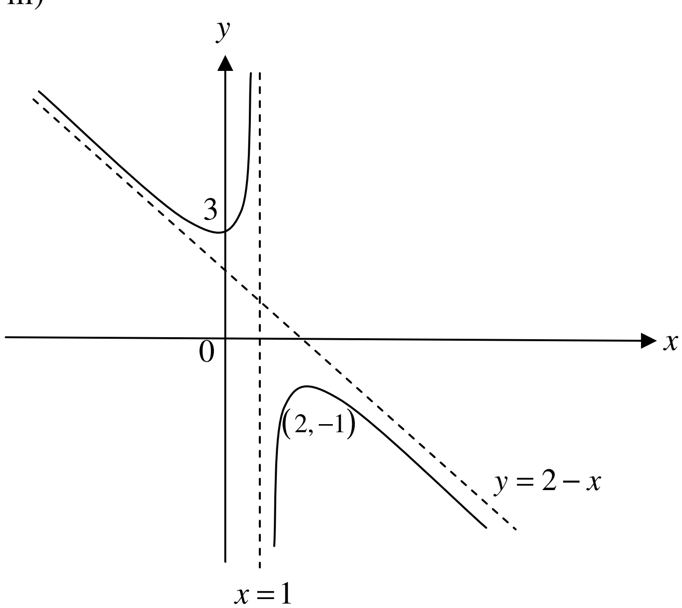
	$\sum_{r=1}^{N} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=2}^{N+1} \frac{2r-1}{(r-1)^2 r^2}$ $= 1 - \frac{1}{(N+1)^2}$

Qn	Solution
3	(i) $S_{25} = 25$ $\frac{25}{2}[a+0.5] = 25$ $\Rightarrow \quad a = 1.5$ $a + 24d = 0.5$ Subst $a = 1.5$, $d = -\frac{1}{24}$ = 0.0417 (to 3 s.f)
	(ii) GP $a = 2$ $ar^{24} = 0.5$ $2r^{24} = 0.5$ $r^{24} = \frac{1}{4}$ $r = \sqrt[24]{\frac{1}{4}}$ =0.94387 (to 5 s.f) $S_n \leq 30$ $\frac{2\left[1-\left(\sqrt[24]{\frac{1}{4}}\right)^n\right]}{1-\left(\sqrt[24]{\frac{1}{4}}\right)} \leq 30$ $1-\left(\sqrt[24]{\frac{1}{4}}\right)^n \leq 0.84195$ $\left(\sqrt[24]{\frac{1}{4}}\right)^n \geq 0.15805$ $n \leq \frac{\ln 0.15805}{\ln \sqrt[24]{\frac{1}{4}}}$ $n \leq 31.931$ Therefore maximum number of pieces cut = 31.

Alternative Solution	
	$S_n \le 30$ $\dfrac{2\left[1-(0.94387)^n\right]}{1-(0.94387)} \le 30$ $1-(0.94387)^n \le 0.84195$ $(0.94387)^n \ge 0.15805$ $n \le \dfrac{\ln 0.15805}{\ln 0.94387}$ $n \le 31.9$
	Therefore maximum number of pieces cut = 31.

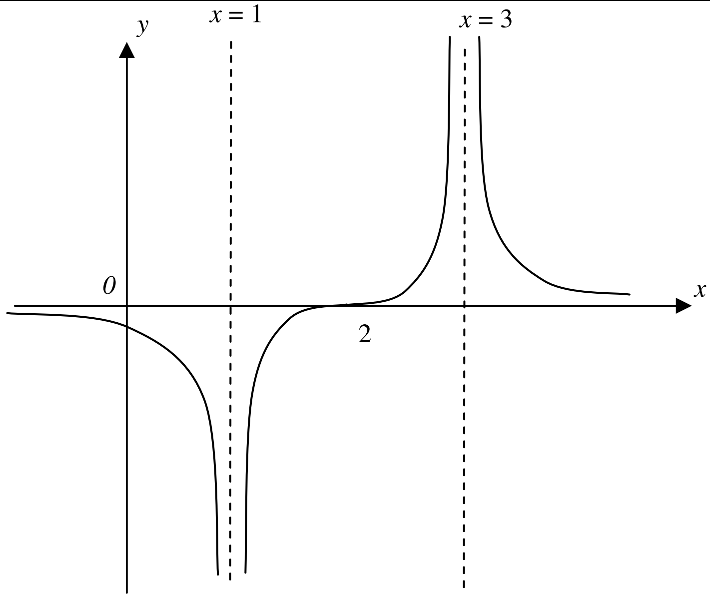
Qn	Solution
4	(i) $u_2 = \dfrac{4+2(1)}{5} = \dfrac{6}{5} = 1.2$ $u_3 = \dfrac{4+2(\frac{6}{5})}{5} = \dfrac{32}{25} = 1.28$
	(ii) As $n \to \infty,\ u_n \to l\ ,\ u_{n+1} \to l$. $l = \dfrac{4+2l}{5}$ $l = \dfrac{4}{3}$
	(iii) Let P_n be the statement $u_n = \dfrac{4}{3} - \dfrac{1}{3}\left(\dfrac{2}{5}\right)^{n-1}$ for all $n \ge 1$. . LHS of $P_1 = u_1 = 1$ (by defn) RHS of $P_1 = \dfrac{4}{3} - \dfrac{1}{3}\left(\dfrac{2}{5}\right)^{1-1} = \dfrac{3}{3} = 1$ $\therefore P_1$ is true. Assume that P_k is true for some $k \ge 1$, ie $u_k = \dfrac{4}{3} - \dfrac{1}{3}\left(\dfrac{2}{5}\right)^{k-1}$ We want to prove P_{k+1}, ie $u_{k+1} = \dfrac{4}{3} - \dfrac{1}{3}\left(\dfrac{2}{5}\right)^{k}$ LHS of $P_{k+1} = u_{k+1}$ $= \dfrac{4+2u_k}{5}$ $= \dfrac{4}{5} + \dfrac{2}{5}\left[\dfrac{4}{3} - \dfrac{1}{3}\left(\dfrac{2}{5}\right)^{k-1}\right]$

	$=\frac{12}{15}+\frac{8}{15}-\frac{1}{3}\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^{k-1}$ $=\frac{4}{3}-\frac{1}{3}\left(\frac{2}{5}\right)^{k}$ $=\text{RHS of } P_{k+1}$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true. $\therefore$ By Mathematical Induction, P_n is true for all $n \geq 1$.

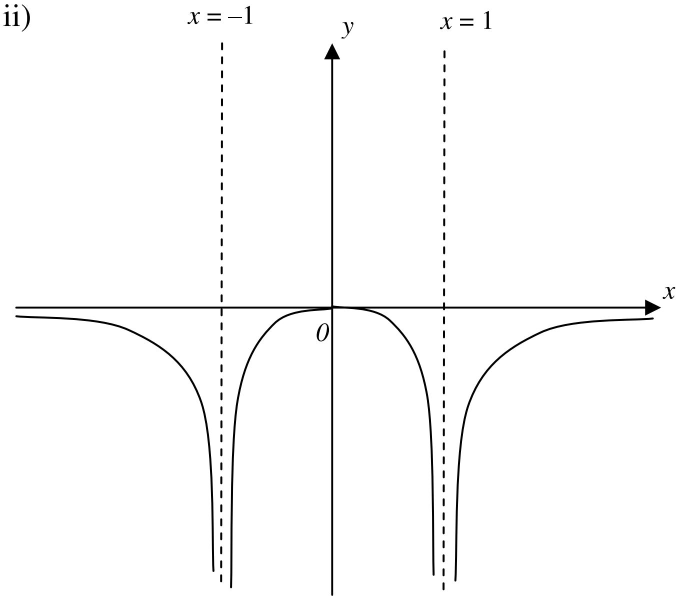
5	i) Asymptotes: By Long Division, $y=\frac{x^2-3x+3}{1-x}=2-x+\frac{1}{1-x}$ Asymptotes: $x=1$, $y=2-x$
	ii) $y=\frac{x^2-3x+3}{1-x}$ $y(1-x)=x^2-3x+3$ $x^2+(y-3)x+3-y=0$ For no solutions, Discriminant <0 $(y-3)^2-4(3-y)<0$ $(y^2-6y+9)-(12-4y)<0$ $y^2-2y-3<0$ $(y-3)(y+1)<0$ $\therefore -1<y<3$
	iii) 

6

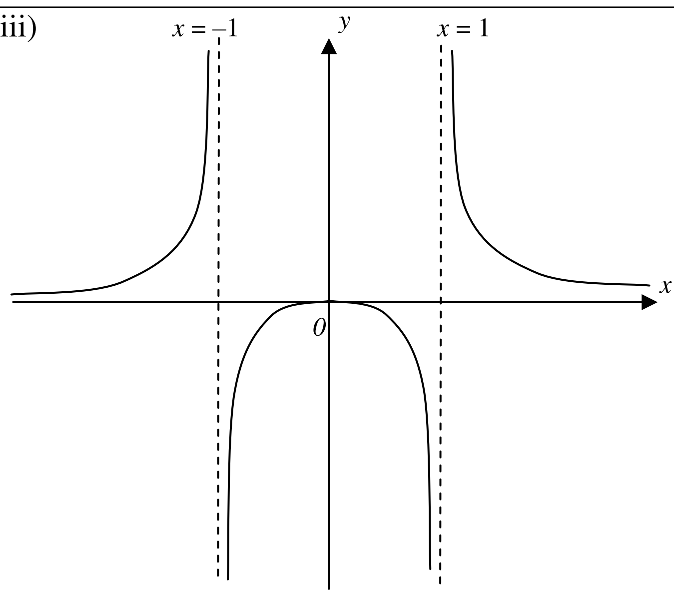
i)

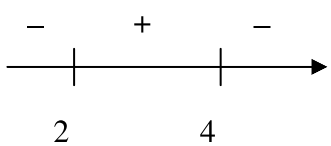


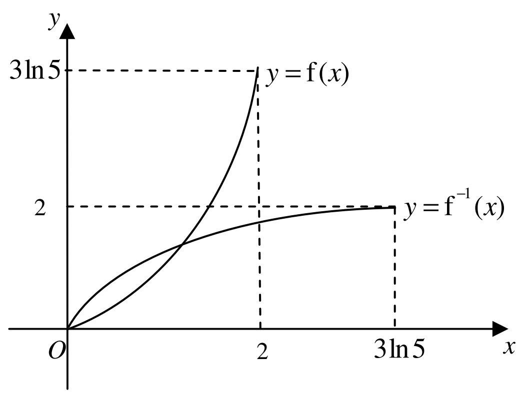
ii)



iii)

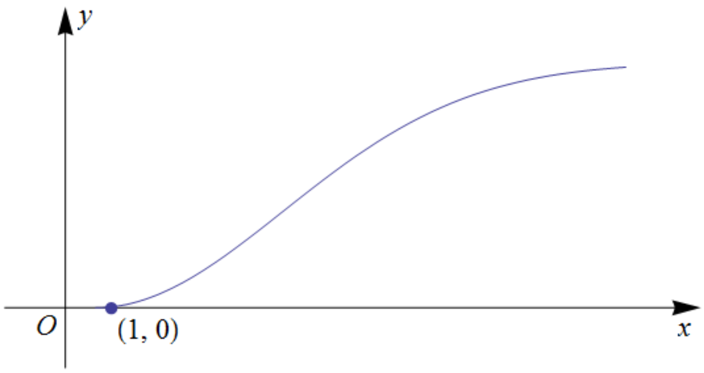


Qn	Solution
7	(a) $x^2-3x+5=\left(x-\frac{3}{2}\right)^2-\left(-\frac{3}{2}\right)^2+5$ $=\left(x-\frac{3}{2}\right)^2+\frac{11}{4}$ Since $\left(x-\frac{3}{2}\right)^2\geq 0$ for all real values of x, $\therefore$ x^2-3x+5 is always positive. $\frac{x^2-3x+5}{(4-x)(x-2)}<0$ Since x^2-3x+5 is always positive, $(4-x)(x-2)<0$  $\therefore$ $x<2$ or $x>4$ -------(1) $\frac{(x+2)^2-3x-1}{x(2-x)}<0$ Replace x in eqn (1) with $(x+2)$, $\therefore$ $x+2<2$ or $x+2>4$ $\Rightarrow$ $x<0$ or $x>2$
	(b) Let the price of 1 litre of A, B and C be a, b and c respectively. Given that $a+b+2c=9$ $b+c=3.50$ $2.5b+2c=2a$ $\Rightarrow 2a-2.5b-2c=0$ Using GC, $a=\$4$, $b=\$2$, $c=\$1.50$.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>8</td><td>i)
$y=3\ln\left(x^2+1\right)$
$x=\pm\sqrt{e^{\frac{y}{3}}-1}$
$x=\sqrt{e^{\frac{y}{3}}-1}$ since $0\le x\le 2$
$\therefore f^{-1}\left(x\right)=\sqrt{e^{\frac{x}{3}}-1},\quad 0\le x\le 3\ln 5$</td></tr>
<tr><td></td><td>ii)


They are reflections about $y=x$ and
there are 2 solutions.</td></tr>
<tr><td></td><td>iii)
$R_f=\left[0,3\ln 5\right]$
$D_g=\left[0,\infty\right)$
$R_f\subseteq D_g$
$\therefore$ gf exists
$gf\left(x\right)=\left(x^2+1\right)^3+1,\qquad 0\le x\le 2$
$R_{gf}=\left[2,126\right]$</td></tr>
</table>

Qn	Solution
9 (a)	(i) $V=\pi r^2 h$ $h=\dfrac{V}{\pi r^2}$ $A=2\pi r^2+2\pi rh$ $=2\pi r^2+2\pi r\left(\dfrac{V}{\pi r^2}\right)$ $=2\pi r^2+\dfrac{2V}{r}$ (shown)
	(ii) For min A, $\dfrac{\mathrm{d}A}{\mathrm{d}r}=4\pi r-\dfrac{2V}{r^2}=0$ $4\pi r^3=2V$ $r=\left(\dfrac{V}{2\pi}\right)^{\frac{1}{3}}$ $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2}=4\pi+\dfrac{4V}{r^3}>0$ Thus, A is minimum. Substitute $V=\pi r^2 h$, $r=\left(\dfrac{\pi r^2 h}{2\pi}\right)^{\frac{1}{3}}$ $r^3=\dfrac{r^2 h}{2}$ $h=2r$
(b)	$y=\sqrt{2^2-x^2}$ $=\sqrt{4-x^2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2\sqrt{4-x^2}}(-2x)$ $=-\dfrac{x}{\sqrt{4-x^2}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\mathrm{d}y}{\mathrm{d}x}\times\dfrac{\mathrm{d}x}{\mathrm{d}t}$ $=-\dfrac{x}{\sqrt{4-x^2}}\times(0.02)$ $=-\dfrac{1}{\sqrt{4-1^2}}\times(0.02)$ $=-0.011547$ $=-0.0115$ $\therefore$ y decreases at a rate of 0.0115 ms^{-1}.

Qn	Solution
10(a) (i)	$x = e^{3t} \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t} = 3e^{3t}$ $y = t^2 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}t} = 2t$ $\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2t}{3e^{3t}}$ When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\dfrac{2t}{3e^{3t}} = 0$ $t = 0$
(ii)	
(b)	$x^2 - 2xy + 2y^2 = k \qquad \ldots (1)$ Differentiate throughout w.r.t. x. $2x - 2\left(x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y\right) + 4y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y-x}{2y-x}$
	For tangents which are parallel to the line $y = x$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$. $\dfrac{y-x}{2y-x} = 1$ $y - x = 2y - x$ $y = 0$ Subst. $y = 0$ into (1): $x^2 - 2x(0) + 2(0)^2 = k$ $x^2 = k$ Given that there are 2 tangents parallel to the line $y = x$, $k > 0$

	For tangents which are parallel to the y-axis, $\frac{dy}{dx}$ is undefined. $$2y-x=0$$ $$x=2y$$ Subst. $x=2y$ and $k=4$ into (1): $$(2y)^2-2(2y)y+2y^2=4$$ $$y=\pm\sqrt{2}$$ $$x=\pm2\sqrt{2}$$ The coordinates are $\left(-2\sqrt{2},-\sqrt{2}\right)$ and $\left(2\sqrt{2},\sqrt{2}\right)$.

Qn	Solution
11(a) (i)	$\int x^2e^x\ dx = x^2e^x-2\int xe^x\ dx$ $= x^2e^x-2\left[xe^x-\int e^x dx\right]$ $= x^2e^x-2\left[xe^x-e^x\right]+c$ $= e^x\left(x^2-2x+2\right)+c$
(ii)	$\int_0^{\frac{\pi}{3}}\sin^2 2x\ dx=\frac{1}{2}\int_0^{\frac{\pi}{3}}1-\cos 4x\ dx$ $=\frac{1}{2}\left[x-\frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{3}}$ $=\frac{1}{2}\left[\frac{\pi}{3}-\frac{1}{4}\sin\frac{4\pi}{3}\right]$ $=\frac{1}{2}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right]$
(b)	$\int\frac{9x}{\left(3x-1\right)^2}\ dx=\int\frac{u+1}{u^2}\ du$ $=\int\frac{1}{u}+u^{-2}du$ $=\ln\lvert u\rvert-\frac{1}{u}+c$ $=\ln\lvert 3x-1\rvert-\frac{1}{3x-1}+c$

(c)	$x+1=A(2x-4)+B$ $\quad=2Ax-4A+B$ By comparing coefficients, $2A=1 \Rightarrow A=\frac{1}{2}$ $-4A+B=1 \Rightarrow B=3$
	$\int \frac{x+1}{x^2-4x+13}\,\mathrm{d}x$ $=\int \frac{\frac{1}{2}(2x-4)+3}{x^2-4x+13}\,\mathrm{d}x$ $=\frac{1}{2}\int \frac{2x-4}{x^2-4x+13}\,\mathrm{d}x+3\int \frac{1}{(x-2)^2+3^2}\,\mathrm{d}x$ $=\frac{1}{2}\ln\left\|x^2-4x+13\right\|+3\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{x-2}{3}\right)+c$ $=\frac{1}{2}\ln\left(x^2-4x+13\right)+\tan^{-1}\left(\frac{x-2}{3}\right)+c$

2013 MJC H2 MATH (9740) JC 1 PROMOTIONAL EXAM – MARKING SCHEME

Qn	Solution
1	**Inequalities**

$$x^2 - x + 7 = \left(x - \frac{1}{2}\right)^2 + 7 - \left(\frac{1}{2}\right)^2$$

$$= \left(x - \frac{1}{2}\right)^2 + \frac{27}{4}$$

Since $\left(x - \frac{1}{2}\right)^2 \geq 0$ for all real values of x, $\left(x - \frac{1}{2}\right)^2 + \frac{27}{4} > 0$ (shown).

$$\frac{3}{(x-2)^2} > \frac{-1}{x+1}, \ x \neq -1, \ x \neq 2$$

$$\frac{3}{(x-2)^2} + \frac{1}{x+1} > 0$$

$$\frac{3(x+1) + (x^2 - 4x + 4)}{(x+1)(x-2)^2} > 0$$

$$\frac{x^2 - x + 7}{(x+1)(x-2)^2} > 0$$

Since $x^2 - x + 7 = \left(x - \frac{1}{2}\right)^2 + \frac{27}{4} > 0$ and $(x-2)^2 > 0$ for all $x \in \mathbb{R} \setminus \{2\}$

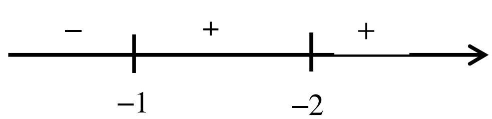
$\Rightarrow (x+1) > 0$

$\therefore x > -1$, $x \neq 2$

<u>Alternatively</u>

Since $x^2 - x + 7 = \left(x - \frac{1}{2}\right)^2 + \frac{27}{4} > 0$ for all real values of x,

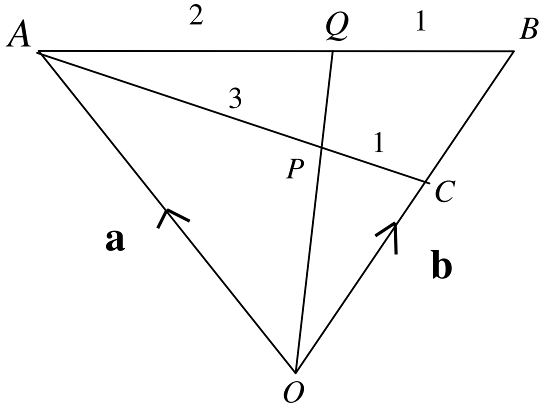
$$\frac{1}{(x+1)(x-2)^2} > 0$$



$\therefore x > -1$, $x \neq 2$

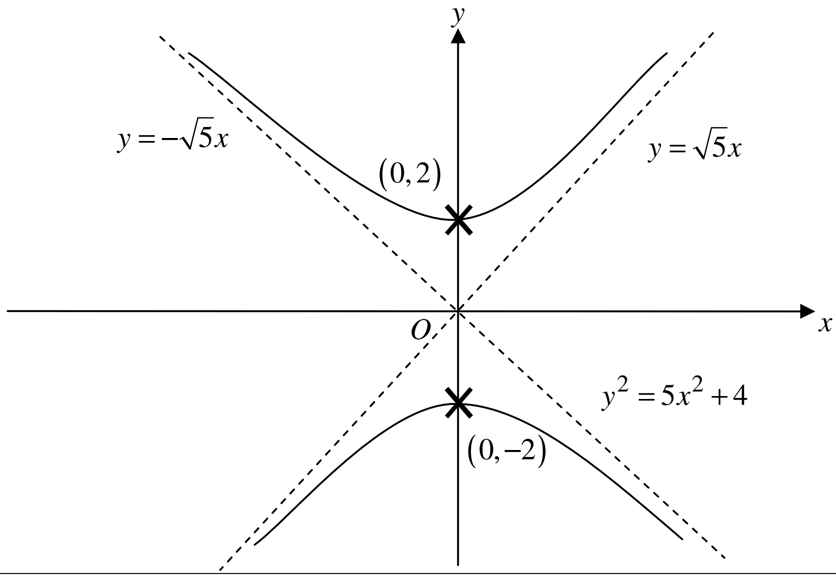
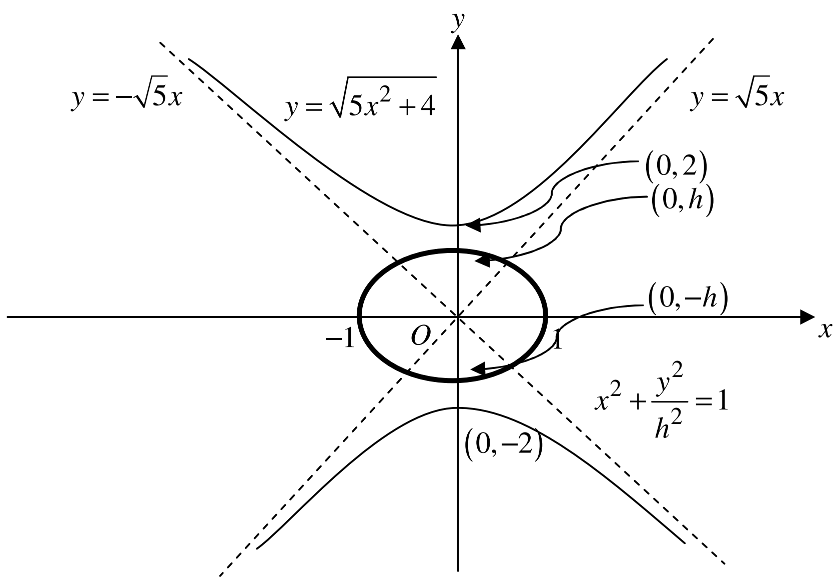
Qn	Solution
2	**Techniques of Differentiation**
	$x=\sin^{-1}(1-t)$ $\qquad$ $y=\mathrm{e}^{\sqrt{2t-t^2}}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{\sqrt{1-(1-t)^2}}(-1)$ $\qquad$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^{\sqrt{2t-t^2}}\frac{1}{2}(2t-t^2)^{-\frac{1}{2}}(2-2t)$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{\sqrt{2t-t^2}}$ $\qquad$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{e}^{\sqrt{2t-t^2}}(1-t)}{\sqrt{2t-t^2}}$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\mathrm{e}^{\sqrt{2t-t^2}}(t-1)$

Qn	Solution
3	**SLE**
(i)	At A, $b+c=a+d$. At B, $a+b+c=48$. At C, $a+c=2b$. At D, $d=b+2a$. After simplifying, $-a+b+c-d=0$. $a+b+c=48$. $a-2b+c=0$. $2a+b-d=0$. Using GC, $a=8, b=16, c=24$ and $d=32$.
(ii)	Total amount collected $=\$0.50(2c+b)$ $=\$0.50(48+16)$ $=\$32$

Qn	Solution
4	**Vectors I**
(i)	 $\overrightarrow{OC} = k\mathbf{b}$ Using Ratio Theorem, $\overrightarrow{OP} = \dfrac{\mathbf{a}+3k\mathbf{b}}{4}$ $\overrightarrow{OQ} = \dfrac{\mathbf{a}+2\mathbf{b}}{3}$
(ii)	Given that O, P and Q are collinear, $\overrightarrow{OP} = \lambda\overrightarrow{OQ}$ for some $\lambda \in \mathbb{R}$ $\dfrac{1}{4}\mathbf{a}+\dfrac{3k}{4}\mathbf{b} = \lambda\left(\dfrac{1}{3}\mathbf{a}+\dfrac{2}{3}\mathbf{b}\right)$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors, $\dfrac{1}{4} = \dfrac{\lambda}{3}$ ------ (1) and $\dfrac{3}{4}k = \dfrac{2}{3}\lambda$ ------ (2) From (1): $\lambda = \dfrac{3}{4}$ ------ (3) Substitute (3) into (2) $k = \dfrac{2}{3}\left(\dfrac{3}{4}\right)\left(\dfrac{4}{3}\right)$ $= \dfrac{2}{3}$ $\therefore k = \dfrac{2}{3}$ **<u>Alternatively,</u>** Given that O, P and Q are collinear, $\overrightarrow{OQ} = \lambda\overrightarrow{OP}$ for some $\lambda \in \mathbb{R}$

	$\frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{b}=\lambda\left(\frac{1}{4}\mathbf{a}+\frac{3k}{4}\mathbf{b}\right)$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors, $\frac{1}{3}=\lambda\left(\frac{1}{4}\right)$ ------ (1) and $\frac{2}{3}=\lambda\left(\frac{3k}{4}\right)$ ------ (2) From (1): $\lambda=\frac{4}{3}$ ------ (3) Substitute (3) into (2) $k=\frac{2}{3}\left(\frac{4}{3\lambda}\right)$ $=\frac{2}{3}\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)=\frac{2}{3}$ $\therefore k=\frac{2}{3}$

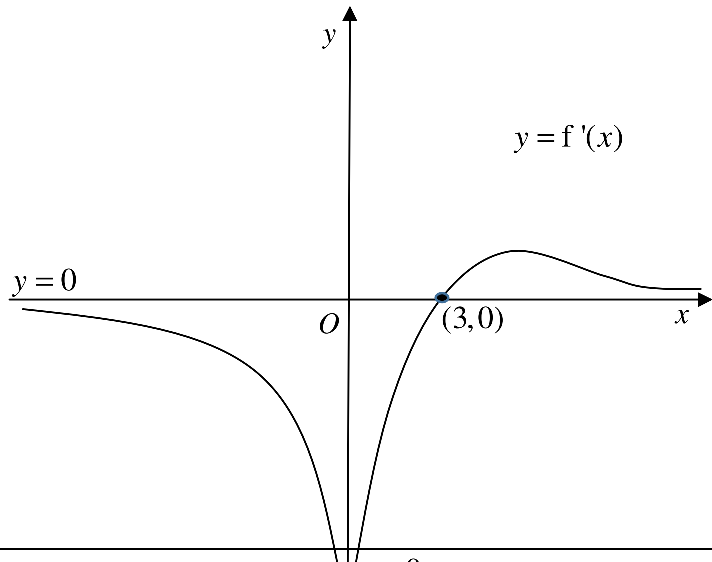
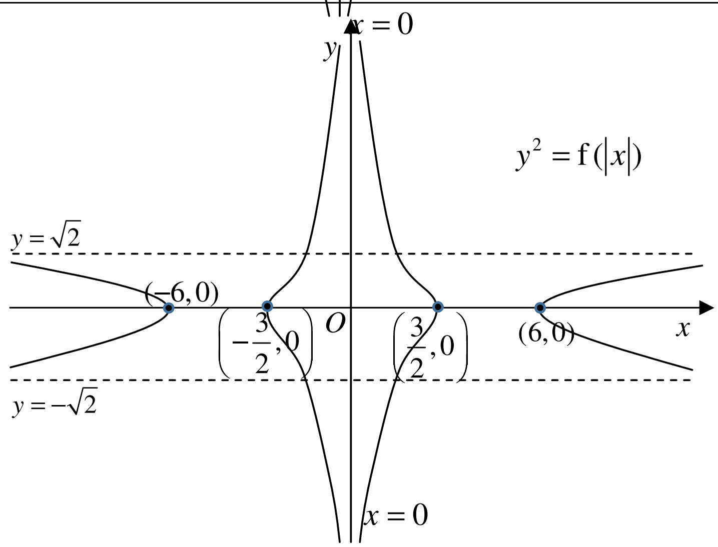
Qn	Solution
5*	**Maclaurin's Series and Binomial Theorem [Not in topics tested for SRJC 2014 Promo]**
(i)	$\mathrm{e}^x\sin 2x$ $=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\left(2x-\frac{(2x)^3}{3!}+\ldots\right)$ $=\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\right)\left(2x-\frac{8x^3}{6}+\ldots\right)$ $=2x-\frac{8x^3}{6}+2x^2+x^3+\ldots$ $=2x+2x^2-\frac{1}{3}x^3+\ldots$
(ii)	$\frac{\mathrm{e}^x\sin 2x}{\sqrt{4-x}}=\left[\mathrm{e}^x\sin 2x\right](4-x)^{-\frac{1}{2}}$ $=\left(2x+2x^2-\frac{1}{3}x^3+\ldots\right)(4)^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$ $=\frac{1}{2}\left(2x+2x^2-\frac{1}{3}x^3+\ldots\right)\left(1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\ldots\right)$ $=\frac{1}{2}\left(2x+2x^2-\frac{1}{3}x^3+\ldots\right)\left(1+\frac{x}{8}+\frac{3}{8}\left(\frac{x^2}{16}\right)+\ldots\right)$ $=\frac{1}{2}\left(2x+\frac{2x^2}{8}+2x^2+\frac{3x^3}{64}-\frac{x^3}{3}+\frac{2x^3}{8}\ldots\right)$ $=x+\frac{9x^2}{8}-\frac{7x^3}{384}+..$

Qn	Solution
6	**Graphing Techniques 1**
(i)	
(ii)	$5x^2+4=h^2\left(1-x^2\right)$ $y^2=h^2\left(1-x^2\right)$ $y^2+x^2h^2=h^2$ $\frac{y^2}{h^2}+x^2=1$  Graph to be inserted is $x^2+\frac{y^2}{h^2}=1$. From the graphs, $0<h<2$.

Qn	Solution
7	**Application of Differentiation (Tangent/ Normal)**

	$y=\dfrac{x^2}{x-1}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x(x-1)-x^2}{(x-1)^2}$ $=\dfrac{x^2-2x}{(x-1)^2}$ Since gradient of tangent at A is $\dfrac{8}{9}$ $\dfrac{x^2-2x}{(x-1)^2}=\dfrac{8}{9}$ Using GC, $x=4$ or $x=-2$ Since $x_2<x_1$, x coordinate at point B is $x_2=-2$ Sub $x_2=-2$ into C we have $y_2=-\dfrac{4}{3}$ $\therefore$ coordinates of B is $\left(-2,-\dfrac{4}{3}\right)$ Since gradient of normal at B is $-\dfrac{9}{8}$ $y-\left(-\dfrac{4}{3}\right)=-\dfrac{9}{8}\big(x-(-2)\big)----(*)$ $y=-\dfrac{9}{8}x-\dfrac{43}{12}$

Qn	Solution
8	**Transformation of graphs**
(a)	$y=\dfrac{x-1}{3x^2-5}$ $\downarrow$ 1. Replace y by $-y$ $-y=\dfrac{x-1}{3x^2-5}$ $\downarrow$ 2. Replace y by $y-1$ $1-y=\dfrac{x-1}{3x^2-5}$ $\downarrow$ 3. Replace x by $2x$ $1-y=\dfrac{2x-1}{12x^2-5}$ The transformations are in the following order: 1. Reflection in the x-axis. 2. Translation of 1 unit in the positive y-direction. 3. Scaling parallel to the x-axis by factor ½. (or 3-1-2, 1-3-2, 1-3-2) **Alternatively,** The transformations are in the following order:

	1. Translation of 1 unit in the negative y-direction. 2. Reflection in the x-axis. 3. Scaling parallel to the x-axis by factor ½. (or 1-3-2)
(b)(i)	
(b)(ii)	

Qn	Solution
9	**Mathematical Induction (RR) and MOD**
(i)	Let P_n be the statement $u_n = \dfrac{1}{2n^2}$ for $n \in \mathbb{Z}^+$. When $n = 1$, LHS = $u_1 = \dfrac{1}{2}$ RHS $= \dfrac{1}{2(1)^2} = \dfrac{1}{2} = \text{LHS}$ $\therefore \mathrm{P}_1$ is true. Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \dfrac{1}{2k^2}$ -------------------- (*)

	To prove P_{k+1} is also true, i.e. $u_{k+1}=\dfrac{1}{2(k+1)^2}$. LHS = $u_{k+1}=u_k-\dfrac{2(k+1)-1}{2k^2(k+1)^2}$ (from the recurrence relation) $=u_k-\dfrac{2k+1}{2k^2(k+1)^2}$ $=\dfrac{1}{2k^2}-\dfrac{2k+1}{2k^2(k+1)^2}$ from (*) $=\dfrac{(k+1)^2-2k-1}{2k^2(k+1)^2}$ $=\dfrac{k^2}{2k^2(k+1)^2}$ $=\dfrac{1}{2(k+1)^2}=\text{RHS}$ Thus P_k is true $\Rightarrow P_{k+1}$ is true. Since P_1 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n\in\mathbb{Z}^+$.
(ii)	$\displaystyle\sum_{n=1}^{N}\frac{2n+1}{2n^2(n+1)^2}=\sum_{n=1}^{N}(u_n-u_{n+1})$ $= u_1-u_2$ $+u_2-u_3$ $\vdots$ $+u_N-u_{N+1}$ $= u_1-u_{N+1}$ $=\dfrac{1}{2}-\dfrac{1}{2(N+1)^2}=\dfrac{1}{2}\left(1-\dfrac{1}{(N+1)^2}\right)$
(iii)	$\displaystyle\sum_{n=0}^{N}\frac{2n+3}{2(n+1)^2(n+2)^2}=\sum_{n=1}^{N+1}\frac{2n+1}{2n^2(n+1)^2}$ $=\dfrac{1}{2}-\dfrac{1}{2(N+2)^2}$

Qn	Solution
10	**Vectors**
(i)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$ $\Rightarrow$ a direction vector for the line is $\begin{pmatrix}2\\-1\\1\end{pmatrix}$ vector equation of the line AB : $r=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$ To determine whether point C lies on the line: Let $\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$. Then $\begin{cases}2=1+2\lambda\Rightarrow\lambda=\frac{1}{2}\\1=1-\lambda\Rightarrow\lambda=0\\5=1+\lambda\Rightarrow\lambda=4\end{cases}$ Since the values of λ are inconsistent, i.e. no value of λ satisfies all the equations, hence shown that point C does not lie on the line AB .
(ii)	Let N be the foot of the perpendicular from C to line AB line AB : $r=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$ Since N lies on line AB then $\overrightarrow{ON}=\begin{pmatrix}1+2\lambda\\1-\lambda\\1+\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. $\overrightarrow{CN}=\overrightarrow{ON}-\overrightarrow{OC}=\begin{pmatrix}1+2\lambda\\1-\lambda\\1+\lambda\end{pmatrix}-\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}-1+2\lambda\\-\lambda\\-4+\lambda\end{pmatrix}$ $\overrightarrow{CN}\perp$ line AB, $\overrightarrow{CN}\cdot\mathbf{d}=0\Rightarrow\begin{pmatrix}-1+2\lambda\\-\lambda\\-4+\lambda\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix}=0$ $\Rightarrow-2+4\lambda+\lambda-4+\lambda=0\Rightarrow\quad\lambda=1$ Therefore, the position vector of the foot of the perpendicular from point C to line AB. $\overrightarrow{ON}=\begin{pmatrix}1+2(1)\\1-(1)\\1+(1)\end{pmatrix}=\begin{pmatrix}3\\0\\2\end{pmatrix}$ Since $\overrightarrow{ON}=\overrightarrow{OB}$, the angle ABC is 90 degrees.
(iii)	The position vector of C', the reflection of point C in the line AB

	$\overrightarrow{ON}=\dfrac{\overrightarrow{OC}+\overrightarrow{OC'}}{2}$ $\overrightarrow{OC'}=2\overrightarrow{ON}-\overrightarrow{OC}$ $=2\begin{pmatrix}3\\0\\2\end{pmatrix}-\begin{pmatrix}2\\1\\5\end{pmatrix}$ $=\begin{pmatrix}4\\-1\\-1\end{pmatrix}$

Qn	Solution
11	**AP GP**
(a)	Let T_1, T_3, T_6, be the first, third and sixth term of an arithmetic series with first term a and common difference d. $T_1=a,\quad T_3=a+2d\ ,\ T_6=a+5d$ $\dfrac{a+5d}{a+2d}=\dfrac{a+2d}{a}$ $a\left(a+5d\right)=\left(a+2d\right)^2$ $a^2+5ad=a^2+4ad+4d^2$ $ad=4d^2$ Since $d\neq 0 \Rightarrow a=4d$ Common ratio $r=\dfrac{T_3}{T_1}=\dfrac{a+2d}{a}=\dfrac{6d}{4d}=\dfrac{3}{2}$ Since $\lvert r\rvert>1$, the geometric progression is not convergent. $S_{15}=\dfrac{15}{2}\left[2a+14d\right]$ $=\dfrac{15}{2}\left[2(4d)+14d\right]$ $=165d$ $=\dfrac{165}{4}a$
(b)	$a=2, r=\dfrac{9}{10}$ $S_\infty=\dfrac{a}{1-r}$ $=\dfrac{2}{1-\dfrac{9}{10}}$ $=20$

	$S_n \geq 15$ $\dfrac{2}{1-\frac{9}{10}}\left(1-\left(\dfrac{9}{10}\right)^n\right) \geq 15$ $\left(1-\left(\dfrac{9}{10}\right)^n\right) \geq 0.75$ $\left(\dfrac{9}{10}\right)^n \leq 0.25$ $n \geq 13.158$ The minimum number of days required is 14 days.

Qn	Solution
12	**Applications of Differentiation**
(i)	$\sin\alpha = \dfrac{h}{PQ} \quad \therefore PQ = h \text{ cosec } \alpha$ $QR = k - PQ - RS$ $= k - 2PQ$ $= k - 2h \text{ cosec } \alpha$ (shown) $A = \dfrac{h}{2}(QR + PS)$ $= \dfrac{h}{2}\left(2QR + 2\dfrac{h}{\tan\alpha}\right)$ $= h(k - 2h \text{ cosec } \alpha + h\cot\alpha)$ $= hk + h^2(\cot\alpha - 2 \text{ cosec } \alpha)$ (shown)
(ii)	$A = hk + h^2(\cot\alpha - 2 \text{ cosec } \alpha)$ $\dfrac{\mathrm{d}A}{\mathrm{d}\alpha} = h^2(-\text{cosec}^2\ \alpha + 2\text{cosec } \alpha \cot\alpha)$ $= h^2\text{cosec } \alpha(-\text{cosec } \alpha + 2\cot\alpha)$ When $\dfrac{\mathrm{d}A}{\mathrm{d}\alpha} = 0, \quad h^2\text{cosec } \alpha(-\text{cosec } \alpha + 2\cot\alpha) = 0$ Since $h^2\text{cosec } \alpha \neq 0$, $-\text{cosec } \alpha + 2\cot\alpha = 0$ $\dfrac{-1+2\cos\alpha}{\sin\alpha} = 0$ $-1 + 2\cos\alpha = 0$ $\cos\alpha = \dfrac{1}{2}$ $\alpha = \dfrac{\pi}{3}$

α	$\left(\frac{\pi}{3}\right)^-$	$\frac{\pi}{3}$	$\left(\frac{\pi}{3}\right)^+$
$\frac{dA}{d\alpha}$	/	—	\

Alternatively

$$\frac{dA}{d\alpha} = h^2(-\text{cosec}^2\,\alpha + 2\text{cosec}\,\alpha\cot\alpha)$$

$$\frac{d^2A}{d\alpha^2} = h^2(2\text{cosec}^2\,\alpha\cot\alpha - 2\text{cosec}^3\,\alpha - 2\text{cosec}\,\alpha\cot^2\alpha)$$

$$= 2h^2\text{cosec}\,\alpha(\text{cosec}\,\alpha\cot\alpha - \text{cosec}^2\,\alpha - \cot^2\alpha)$$

When $\alpha = \frac{\pi}{3}$,

$$\frac{d^2A}{d\alpha^2} = 2h^2\text{cosec}\,\frac{\pi}{3}(\text{cosec}\,\frac{\pi}{3}\cot\frac{\pi}{3} - \text{cosec}^2\,\frac{\pi}{3} - \cot^2\frac{\pi}{3})$$

$$= 2h^2\,\frac{2}{\sqrt{3}}\left(\frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) - \left(\frac{2}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right)$$

$$= \frac{4}{\sqrt{3}}h^2\left(\frac{2}{3} - \frac{4}{3} - \frac{1}{3}\right) < 0$$

$$= -\frac{4}{\sqrt{3}}h^2 < 0$$

$\alpha = \frac{\pi}{3}$ gives max A

When $\alpha = \frac{\pi}{3}$

Max $A = hk + h^2(\cot\alpha - 2\ \text{cosec}\,\alpha)$

$$= hk + h^2(\cot\frac{\pi}{3} - 2\ \text{cosec}\,\frac{\pi}{3})$$

$$= hk + h^2\left(\frac{1}{\sqrt{3}} - 2\left(\frac{2}{\sqrt{3}}\right)\right)$$

$$= hk - \sqrt{3}h^2$$

MERIDIAN JUNIOR COLLEGE
JC1 Promotional Examination
Higher 2

H2 Mathematics 9740/01

Paper 1 **08 October 2013**

2 Hours 30 Minutes

Additional Materials: Writing paper
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages and **2** blank page

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***: Not in topics tested for SRJC 2014 Promotional Exam**

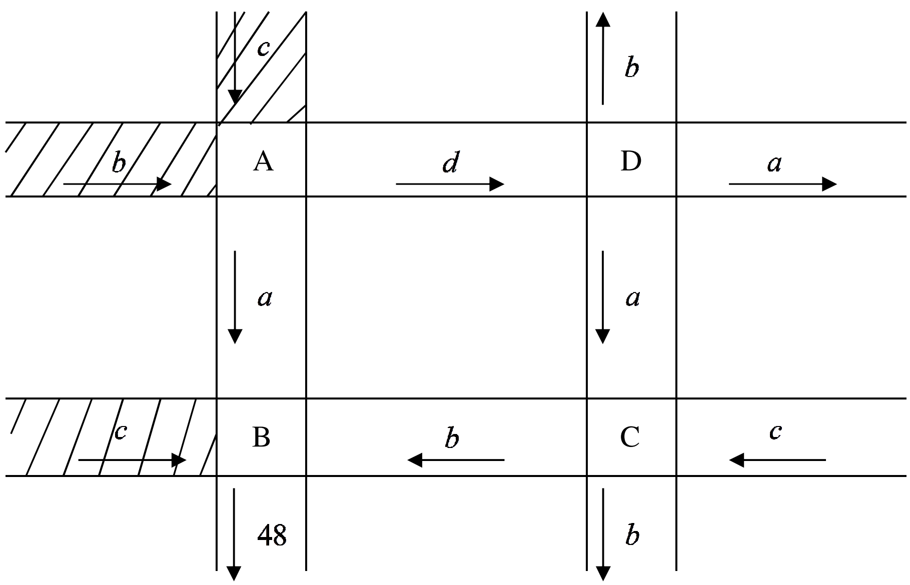
1 Show that $x^2 - x + 7$ is always positive for all real values of x. [1]

Hence, using an algebraic method, solve the inequality

$$\frac{3}{(x-2)^2} > -\frac{1}{x+1}.$$ [3]

2 The parametric equations of a curve C are $x = \sin^{-1}(1-t)$, $y = \mathrm{e}^{\sqrt{2t-t^2}}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [4]

3 The diagram below shows the traffic flow of vehicles in four traffic junctions A, B, C and D. Each arrow indicates the direction of the vehicles entering or leaving the junction. The unknown constants *a, b, c* and *d* indicate the number of vehicles entering or leaving a particular junction. It is given that the total number of vehicles entering a traffic junction must be equal to the total number of vehicles leaving that same junction. There are 48 vehicles leaving junction B.



(i) Determine the values of a, b, c and d. [3]

(ii) The shaded region indicates the presence of an Electronic Road Pricing (ERP) gantry located at that road. It is known that each gantry charges a fixed price of \$0.50 per vehicle. How much revenue will be collected in total by the gantries in these regions? [1]

[Turn Over

4 Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where **a** and **b** are non-zero and non-parallel vectors. The point C lies on OB such that $\overrightarrow{OC} = k\overrightarrow{OB}$, where k is a constant. P is on AC such that $AP : PC = 3 : 1$, and Q is on AB such that $AQ : AB = 2 : 3$.

(i) Find $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of **a**, **b** and k. [2]

(ii) Given that O, P and Q are collinear, find the value of k. [3]

5 **(i)*** Obtain the series expansion for $e^x \sin 2x$, up to and including the term in x^3. [3]

(ii)* Hence deduce the first three non-zero terms in the series expansion of $\dfrac{e^x \sin 2x}{\sqrt{4-x}}$. [3]

6 The curve C has equation $y^2 = 5x^2 + 4$.

(i) Sketch C, indicating clearly the axial intercepts, the equations of the asymptotes and the coordinates of the stationary points. [3]

(ii) Hence by inserting a suitable graph, determine the range of values of h, where h is a positive constant, such that the equation $5x^2 + 4 = h^2\left(1 - x^2\right)$ has no real roots. [3]

7 The curve C has equation

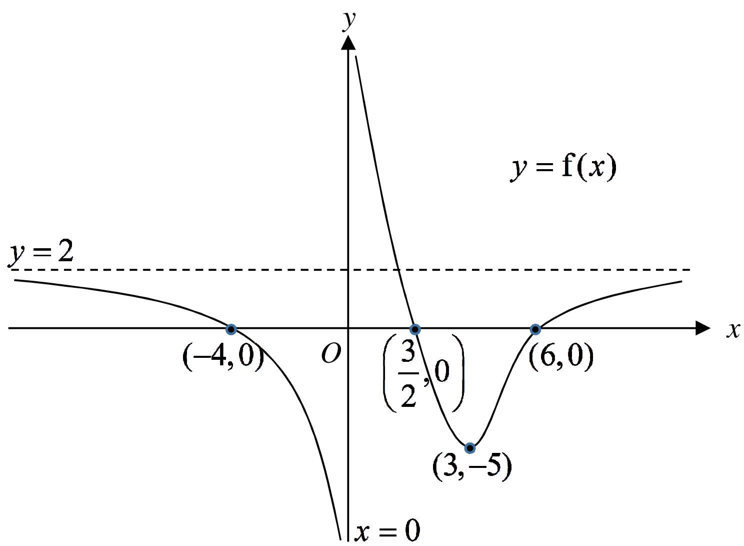
$$y = \frac{x^2}{x-1}.$$

Points $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ lie on curve C such that the tangent at A is parallel to tangent at B where $x_2 < x_1$. Given further that the equation of tangent at A is $y = \frac{8}{9}x + \frac{16}{9}$, find the coordinates of B, and hence find the equation of normal at point B. [6]

***: Not in topics tested for SRJC 2014 Promotional Exam**

8 **(a)** State a sequence of transformations which transform the graph of $y=\dfrac{x-1}{3x^2-5}$ to the graph of $1-y=\dfrac{2x-1}{12x^2-5}$. [3]

(b) The diagram below shows the graph of $y=\mathrm{f}(x)$.



Sketch, on separate clearly labeled diagrams, the graphs of

(i) $y=\mathrm{f}\,'(x)$, [2]

(ii) $y^2=\mathrm{f}(|x|)$. [3]

9 A sequence $u_1, u_2, u_3, \cdots$ is such that $u_1 = \dfrac{1}{2}$ and

$$u_{n+1} = u_n - \frac{2n+1}{2n^2(n+1)^2}, \text{ for all } n \geq 1.$$

(i) Use the method of mathematical induction to prove that $u_n = \dfrac{1}{2n^2}$ for $n \in \mathbb{Z}^+$. [4]

(ii) Hence find $\displaystyle\sum_{n=1}^{N} \frac{2n+1}{2n^2(n+1)^2}$. [3]

(iii) Use your answer to part **(ii)** to find $\displaystyle\sum_{n=0}^{N} \frac{2n+3}{2(n+1)^2(n+2)^2}$. [2]

10 Referred to the origin *O*, the position vectors of two points *A* and *B* are given by $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}+2\mathbf{k}$ respectively. Also, the position vector of *C* is given by $2\mathbf{i}+\mathbf{j}+5\mathbf{k}$.

(i) Find a vector equation of the line *AB* and show that point *C* does not lie on the line. [3]

(ii) Find the position vector of the foot of the perpendicular from point *C* to line *AB*.
Hence write down the size of angle *ABC*. [5]

(iii) Find the position vector of C', the reflection of point *C* in the line *AB*. [2]

***: Not in topics tested for SRJC 2014 Promotional Exam**

11 **(a)** The first, third and sixth terms of an arithmetic progression with non-zero common difference d and first term a, are three consecutive terms of a geometric progression. Determine if the geometric series is convergent, justifying your answer. Find also the sum of the first 15 terms of the arithmetic progression in terms of a. [5]

(b) A pile driver is used to drive piles into the soil at a new condominium site. On the first day, the depth piled into the soil is 2 m. On each subsequent day, the depth piled into the soil is $\frac{9}{10}$ of the depth piled into the soil on the previous day. Find the maximum theoretical depth that can possibly be piled into the soil. Find the minimum number of days required to drive the piles to a depth of at least 15m into the soil. [5]

12 A student wants to construct a model of a roof structure of fixed height h cm from a rectangular piece of cardboard of width k cm. The cardboard is to be bent in such a way that the cross-section $PQRS$ is as shown in the diagram, with $PQ+QR+RS=k$ and with PQ and RS each inclined to the horizontal at an angle α.

(i) Show that $QR = k - 2h \operatorname{cosec} \alpha$ and that the area $A\,\text{cm}^2$ of the cross-section $PQRS$ is given by $A = hk + h^2(\cot\alpha - 2\operatorname{cosec}\alpha)$. [3]

(ii) Use differentiation to find, in terms of k and h, the maximum value of A as α varies. [5]

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*: Not in topics tested for SRJC 2014 Promotional Exam

NANYANG JUNIOR COLLEGE
JC1 PROMOTIONAL EXAMINATION

Higher 2

MATHEMATICS **9740/01**

Paper 1 **1st October 2013**

3 Hours

Additional Materials: Cover Sheet
Answer Papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

1 Solve the inequality $\left|x^2-2x-3\right|>x+1$. **[4]**

2 Differentiate the following expressions with respect to x, simplifying your answers as far as possible:

(a) $\tan^{-1}\left(\frac{2}{x}\right)$, **[3]**

(b) $\ln\sqrt{\frac{1+x}{1-x}}$. **[3]**

3 A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1=\frac{1}{4}$ and $u_{n+1}=u_n+\frac{1}{n(n+1)}+2^{-n}$, for $n\in\mathbb{Z}^+$.

(i) Prove by mathematical induction that $u_n=\frac{9}{4}-\frac{1}{n}-2^{-n+1}$ for $n\in\mathbb{Z}^+$. **[5]**

(ii) Explain why $\{u_n\}$ is convergent. **[1]**

(iii) Show that u_n is less than $\frac{9}{4}$ for $n\in\mathbb{Z}^+$. **[1]**

4 Show that $r!(r^2+1)=(r+2)!-3(r+1)!+2r!$ where $r\in\mathbb{Z}^+$. **[1]**

Hence, using method of difference, show that the sum of the first n terms of the series

$(5)(2!)+(10)(3!)+(17)(4!)+\cdots$ is $(n+2)!(n+1)-2$. **[4]**

Using the above result, explain why $\sum_{r=1}^{n} r!(r^2)$ is less than $(n+1)!n$. **[2]**

5 **(a)** The points A and B relative to the origin O have position vectors $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $-4\mathbf{i}+5\mathbf{j}+2\mathbf{k}$ respectively. The point P lies on line AB such that $\dfrac{AP}{PB}=\dfrac{\lambda}{1-\lambda}$.

(i) Show that $\overrightarrow{OP}=(1-5\lambda)\mathbf{i}+(2+3\lambda)\mathbf{j}+(4\lambda-2)\mathbf{k}$. **[1]**

(ii) Given further that C is a point with position vector $-5\mathbf{i}+\alpha\mathbf{j}-2\mathbf{k}$ and that O, P and C are collinear, find the values of λ and α. **[3]**

(b) The equations of three planes π_1, π_2, π_3 are

$$\pi_1 : 2x-2y+z=-4\ ,$$
$$\pi_2 : 2x+3y-4z=1\ ,$$
$$\pi_3 : \beta x-3y+z=\gamma\ ,$$

respectively.

(i) The planes π_1 and π_2 intersect in a line l. Find a vector equation of l. **[1]**

(ii) Hence, find the values of β and γ such that there are infinitely many points of intersection between π_1, π_2 and π_3. **[2]**

6 The curve C_1 has equation $x^2-\dfrac{y^2}{4}=1$. The curve C_2 has parametric equations

$$x=a\sin t,\quad y=a\cos t\ \text{, where } 0\le t\le 2\pi \text{ and } a>0.$$

(i) Write down the Cartesian equation of C_2. Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes. **[5]**

(ii) State the range of values of a such that there are 4 points of intersection between C_1 and C_2. Show algebraically, that the x-coordinates of the points of intersection satisfy the equation $5x^2=4+a^2$. **[2]**

(iii) Explain geometrically why there are only 2 values for the x-coordinates when there are 4 points of intersection between C_1 and C_2. Find the exact values of x if $a=3$. **[2]**

7 The function f is defined by

$$\mathrm{f}: x \rightarrow x^2 - \frac{1}{x}, \quad x \in \mathbb{R},\ 1 \leq x < 2.$$

(i) Show, by differentiation, that f is strictly increasing. **[2]**

(ii) State the range of f. **[1]**

(iii) Solve the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, giving your answer to two decimal places. **[2]**

The function g is defined by

$$\mathrm{g}: x \rightarrow 1 + \sin x, \quad x \in \mathbb{R},\ 0 \leq x < \frac{\pi}{2}.$$

(iv) Only one of the composite functions fg and gf exists. Give a definition (including the domain) of the composite that exists, and explain why the other composite does not exist. **[3]**

(v) For the composite function which exists, state its range. **[1]**

8 The equation of a curve is

$$y(x+2)^2 + 2y^2(x+2) - 12x = 0, \text{ where } x \text{ and } y \text{ are positive variables.}$$

(i) Show that the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is $\frac{1}{16}$ when $x = 2$. **[5]**

(ii) Find the equation of the normal to the curve at the point where $x = 2$. **[2]**

(iii) Given that the normal in **(ii)** meets the line $x = 2$ at the point P and the line $x = 0$ at the point S. Find the exact area of triangle OSP, where O is the origin. **[2]**

9 There are 16 boys and 10 girls in a JC1 class. It so happens that within the class, the heights of all the girls form a geometric progression, while the heights of all the boys form an arithmetic progression. The two shortest students in the class, a boy and a girl, both have a height of 150.0 cm, while the tallest boy in the class has a height of 180.0 cm. The fourth shortest girl in the class has a height of 157.5 cm.

(i) Show that the common ratio r between the heights of the girls is $1.05^{\frac{1}{3}}$ and find the height of the tallest girl in the class, giving your answer in cm correct to one decimal place. **[2]**

(ii) Find the number of girls in the class taller than 164.0 cm. **[3]**

(iii) Find the average height of the girls in the class, giving your answer in cm correct to one decimal place. **[3]**

(iv) Find the average height of the entire class, giving your answer in cm correct to one decimal place. **[2]**

10 The position vectors of the points A, B and C with respect to the origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}-2\mathbf{b}$ respectively. Plane π contains the point A and has $\mathbf{b}$ as its normal vector. If the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ is 60° and $|\mathbf{a}| = 2|\mathbf{b}|$, find in terms of $\mathbf{b}$,

(i) the length of projection of $\mathbf{a}$ onto $\mathbf{b}$, **[2]**

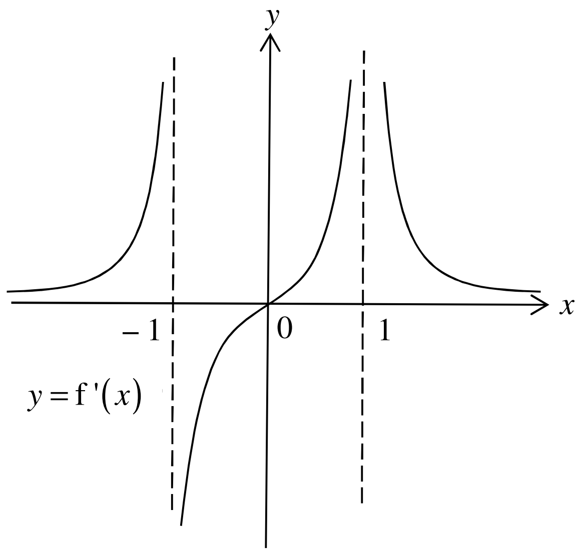
(ii) the distance between point C and the plane π. **[3]**

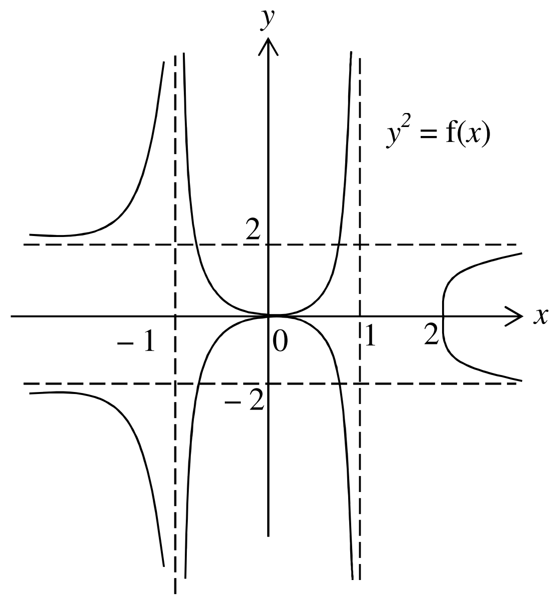
Given that $\mathbf{a} = \mathbf{i}+5\mathbf{j}+2\mathbf{k}$ and $\mathbf{b} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$,

(iii) find the position vector of the foot of perpendicular from point C to the plane π, **[5]**

(iv) show that the position vector of the point of the reflection of point C in the plane π is $3\mathbf{i}+9\mathbf{j}$. **[2]**

11 The graphs of $y = \mathrm{f}'(x)$ and $y^2 = \mathrm{f}(x)$ are shown in the diagrams below.





(a) On separate diagrams, sketch the graphs of

(i) $y = \mathrm{f}'(1-x)$, **[3]**

(ii) $y = \mathrm{f}(x)$, **[4]**

showing clearly the x-intercepts and asymptotes (if any).

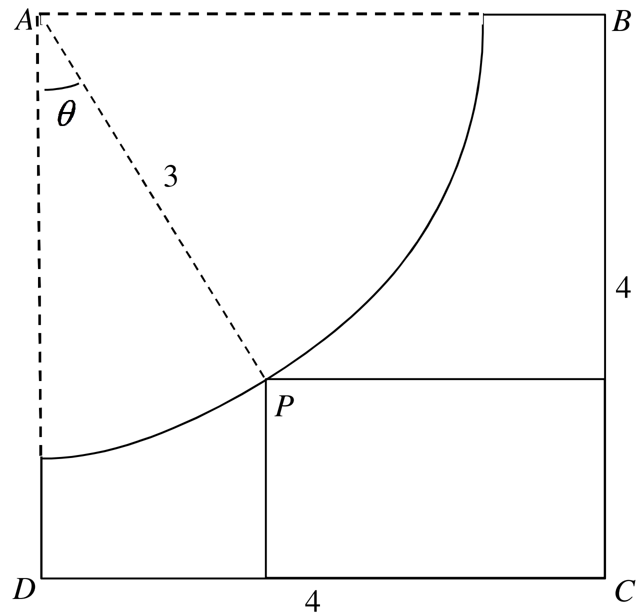
(b) State the set of values of x for which the graph of $y = \mathrm{f}(x)$ is concave upwards. **[2]**

12 **(a)** The curve C has parametric equations

$$x = \theta^2 + 4\theta, \qquad y = \frac{2}{\theta}, \qquad \text{for } \theta > 0.$$

A point $P(x, y)$ moves on the curve C in such a way that the x-coordinate of P decreases at a constant rate of 4 units per second. Find the rate at which the y-coordinate of P is changing when $x = 4$. **[4]**

(b)



The diagram above shows the floor plan of a storeroom. The floor plan consists of a square $ABCD$ of side 4 units from which a quadrant of a circle with centre A and radius 3 units has been removed. The owner intends to store a rectangular crate with one corner of the base at C, and the opposite corner of the base at P against the curved wall. The base of the crate has area y unit^2 and angle DAP is θ radians, where $0 \le \theta \le \frac{\pi}{4}$.

Show that $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 3(\sin\theta - \cos\theta)(4 - 3\sin\theta - 3\cos\theta)$. **[2]**

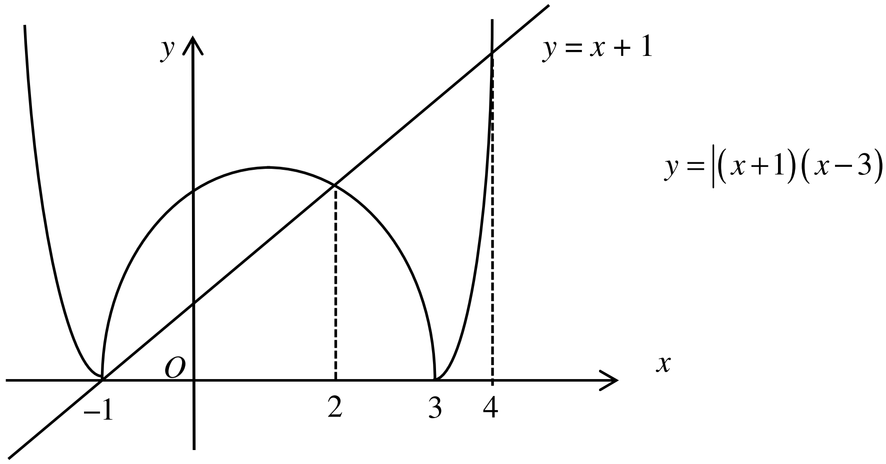
Hence, find the least possible value of y. **[5]**

-----END OF PAPER-----

2013 NYJC JC1 Promo 9740/1 Solutions

Qn

1



$x < -1$ or $-1 < x < 2$ or $x > 4$.

2

(a) $\dfrac{d}{dx}[\tan^{-1}\left(\dfrac{2}{x}\right)] = \dfrac{2(-x^{-2})}{1+(\frac{2}{x})^2} = \dfrac{-2}{x^2+4}$

(b) $\dfrac{d}{dx}(\ln\sqrt{\dfrac{1+x}{1-x}}) = \dfrac{d}{dx}[\dfrac{1}{2}\left(\ln(1+x)-\ln(1-x)\right)]$

$= \dfrac{1}{2}(\dfrac{1}{1+x} - \dfrac{-1}{1-x}) = \dfrac{1}{1-x^2}$ or $\dfrac{1}{(1+x)(1-x)}$

Alternative Solution

$$\frac{d}{dx}\left(\text{In}\sqrt{\frac{1+x}{1-x}}\right) = \left(\frac{1}{\sqrt{\frac{1+x}{1-x}}}\right)\left(\frac{1}{2}.\sqrt{\frac{1-x}{1+x}}\left(\frac{1-x+1+x}{(1-x)^2}\right)\right)$$

$$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)\left(\frac{2}{(1-x)^2}\right)$$

$$= \frac{1}{(1-x)(1+x)}$$

3

(i) Let P_n denote the proposition $u_n = \dfrac{9}{4} - \dfrac{1}{n} - 2^{-n+1}$ for all $n \in \mathbb{Z}^+$.

For $n = 1$, LHS $= u_1 = \dfrac{1}{4}$

RHS $= \dfrac{9}{4} - \dfrac{1}{1} - 2^{-1+1} = \dfrac{9}{4} - 1 - 1 = \dfrac{1}{4} =$ LHS.

$\therefore P_1$ is true.

Qn

Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e., $u_k = \frac{9}{4} - \frac{1}{k} - 2^{-k+1}$.

To prove that that P_{k+1} is true, i.e., $u_{k+1} = \frac{9}{4} - \frac{1}{k+1} - 2^{-(k+1)+1}$

For $n = k + 1$,

$$\begin{aligned}
\text{LHS} = u_{k+1} &= u_k + \frac{1}{k(k+1)} + 2^{-k} \\
&= \frac{9}{4} - \frac{1}{k} - 2^{-k+1} + \frac{1}{k(k+1)} + 2^{-k} \\
&= \frac{9}{4} - \left(\frac{1}{k} - \frac{1}{k(k+1)}\right) - \left(2^{-k}\right)(2-1) \\
&= \frac{9}{4} - \frac{k+1-1}{k(k+1)} - 2^{-k-1+1} \\
&= \frac{9}{4} - \frac{1}{k+1} - 2^{-(k+1)+1}
\end{aligned}$$

Hence P_{k+1} is true

Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, hence by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

(ii) As $n \to \infty$, $\frac{1}{n} \to 0$, $2^{-n} \to 0$, hence $u_n \to \frac{9}{4}$, i.e. $\{u_n\}$ is convergent

(iii) Since $\frac{1}{n} > 0$, $2^{-n} > 0$ for $n \geq 1$, $u_n = \frac{9}{4} - \frac{1}{n} - 2^{-n+1} < \frac{9}{4}$

4

$$\begin{aligned}
(r+2)! - 3(r+1)! + 2r! &= r!\big((r+2)(r+1) - 3(r+1) + 2\big) \\
&= r!(r^2 + 3r + 2 - 3r - 3 + 2) \\
&= r!\left(r^2 + 1\right) \quad \text{(Shown)}
\end{aligned}$$

$$\begin{aligned}
\sum_{r=2}^{n+1} r!(r^2+1) = \sum_{r=2}^{n+1} &\left[(r+2)! - 3(r+1)! + 2r!\right] \\
= \; & 4! - 3(3!) + 2(2!) \\
& + 5! - 3(4!) + 2(3!) \\
& + 6! - 3(5!) + 2(4!) \\
& \quad \vdots \\
& + (n+1)! - 3(n)! + 2(n-1)! \\
& + (n+2)! - 3(n+1)! + 2(n)!
\end{aligned}$$

Qn	
	$+(n+3)!-3(n+2)!+2(n+1)!$ $=(n+3)!-2(n+2)!-3!+2(2!)$ $=(n+2)!(n+3-2)-2$ $=(n+2)!(n+1)-2$ (Shown) $\sum_{r=1}^{n} r!(r^2+1)=(n+1)!(n)-2+(1!)(1^1+1)=(n+1)!n$ Since $r!(r^2)<r!(r^2+1)$ for $r\in\mathbb{Z}^+$ Therefore $\sum_{r=1}^{n} r!(r^2)<\sum_{r=1}^{n} r!(r^2+1)=(n+1)!n$
5a **b**	$\overrightarrow{OP}=\frac{(1-\lambda)\overrightarrow{OA}+\lambda\overrightarrow{OB}}{1-\lambda+\lambda}$ $=(1-\lambda)(\mathbf{i}+2\mathbf{j}-2\mathbf{k})+\lambda(-4\mathbf{i}+5\mathbf{j}+2\mathbf{k})$ $(1-5\lambda)\mathbf{i}+(2+3\lambda)\mathbf{j}+(4\lambda-2)\mathbf{k}$ $\overrightarrow{OP}=\mu\overrightarrow{OC}$ $\begin{pmatrix}1-5\lambda\\2+3\lambda\\4\lambda-2\end{pmatrix}=\mu\begin{pmatrix}-5\\\alpha\\-2\end{pmatrix}$ Solving, $\lambda=\frac{2}{5}$, $\mu=\frac{1}{5}$, $\alpha=16$ $\pi_1: 2x-2y+z=-4$, $\pi_2: 2x+3y-4z=1$, $\pi_3: \beta x-3y+z=\gamma$. Line of intersection of π_1 and π_2, l: $\mathbf{r}=\begin{pmatrix}-1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$, $\lambda\in\mathbb{R}$. For infinite points of intersection between 3 planes, l is on π_3.

Qn

$$\begin{pmatrix}1\\2\\2\end{pmatrix}\bullet\begin{pmatrix}\beta\\-3\\1\end{pmatrix}=0 \quad \Rightarrow \beta = 4$$

$$\begin{pmatrix}-1\\1\\0\end{pmatrix}\bullet\begin{pmatrix}\beta\\-3\\1\end{pmatrix}=\gamma \quad \Rightarrow \gamma = -7$$

6(i)

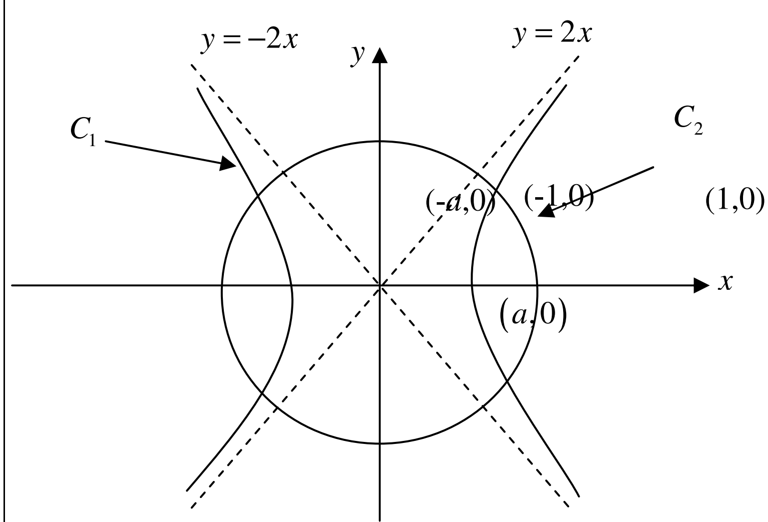
$$x = a\sin t, \quad y = a\cos t$$

$$\sin t = \frac{x}{a}, \quad \cos t = \frac{y}{a}$$

$$\sin^2 t + \cos^2 t = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

$$x^2 + y^2 = a^2$$



(ii)

$a > 1$

$$x^2 - \frac{y^2}{4} = 1 \quad ....(1)$$

$$x^2 + y^2 = a^2$$

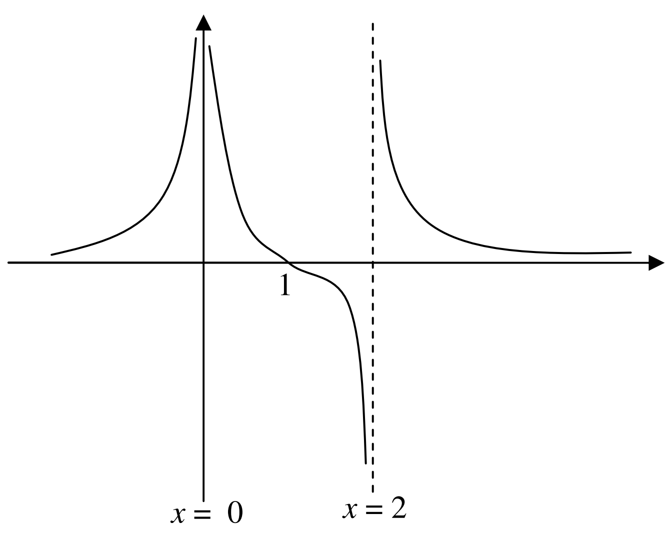
$$y^2 = a^2 - x^2 \quad ....(2)$$

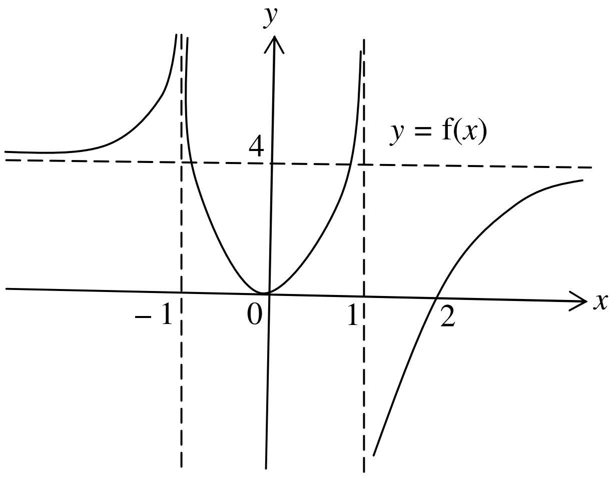
Qn	
(iii)	$x^2-\left[\dfrac{a^2-x^2}{4}\right]=1$ $4x^2-a^2+x^2=4$ $5x^2=4+a^2$ (shown) The points of intersection between the 2 curves are **symmetrical about the *x*-axis**, thus there are only 2 values for the x-coordinates. $5x^2=13$ $x=\pm\sqrt{\dfrac{13}{5}}$
7(i)	$f'(x)=2x+\dfrac{1}{x^2}>0$ for $1\le x<2$ $\Rightarrow$ f is strictly increasing.
(ii)	Since f is strictly increasing, its minimum and maximum values correspond to the minimum and maximum x values. Thus $R_f=\left[1-1,4-\dfrac{1}{2}\right)=\left[0,\dfrac{7}{2}\right)$.
(iii)	$f(x)=f^{-1}(x)\Rightarrow f(x)=x$ $\Rightarrow x^2-\dfrac{1}{x}=x$ $\Rightarrow x^3-x^2-1=0$ $\Rightarrow x=1.47$.
(iv)	Since $R_g=[1,2)=D_f$, fg exists. Since $R_f=\left[0,\dfrac{7}{2}\right)\not\subset\left[0,\dfrac{\pi}{2}\right)=D_g$, gf does not exist. $fg(x)=f(\sin x+1)=(\sin x+1)^2-\dfrac{1}{\sin x+1}$. $D_{fg}=D_g=\left[0,\dfrac{\pi}{2}\right)$.
(v)	$fg: x\to(\sin x+1)^2-\dfrac{1}{\sin x+1}$, $x\in\mathbb{R}$, $0\le x<\dfrac{\pi}{2}$. $R_{fg}=\left[0,\dfrac{7}{2}\right)$.

Qn	
8(i)	$(x+2)^2 y+2(x+2)y^2-12x=0$ Differentiating wrt x, $\frac{\mathrm{d}y}{\mathrm{d}x}(x+2)^2+2y(x+2)+4y\frac{\mathrm{d}y}{\mathrm{d}x}(x+2)+2y^2-12=0$ -------(1) When $x=2$, $16y+8y^2-24=0$ $y^2+2y-3=0$ $(y+3)(y-1)=0$ $y=-3(\text{rejected} \because y>0)$ or $y=1$ Subst (2 , 1) into equation (1), $16\frac{\mathrm{d}y}{\mathrm{d}x}+8+16\frac{\mathrm{d}y}{\mathrm{d}x}+2-12=0$ $32\frac{\mathrm{d}y}{\mathrm{d}x}=2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{16}$
(ii)	Equation of normal: $y-1=-16(x-2)$ $y=-16x+33$
(iii)	Points P and S has coordinates (2 , 1) and (0 , 33) respectively Area of triangle OSP $=\frac{1}{2}\times 33\times 2=33$
9(i)	Let u_n denote the height of the nth shortest girl in the class in cm, and r denote the common ratio between the heights of the girls. Then $u_n=ar^{n-1}$ where $u_1=a=150.0$ and $u_4=ar^3=157.5$ $\Rightarrow r^3=\frac{157.5}{150.0}=1.05 \quad \Rightarrow \quad r=1.05^{\frac{1}{3}}$ Also, $u_{10}=ar^9=a(r^3)^3=(150.0)(1.05)^3=173.6$ (to 1 d.p.) $\therefore$ The height of the tallest girl is 173.6 cm.
ii	$u_n>164.0$ $\Rightarrow (150.0)(1.05)^{\frac{n-1}{3}}-164.0>0$ Using GC,

n	$(150.0)(1.05)^{\frac{n-1}{3}}-164.0$
6	-1.29
7	1.38
8	4.09

Qn	
	Hence $n \geq 7$. Since there are 10 girls in the class, the number of girls who are taller than 164.0 cm is 10-7+1=4. Thus there are 4 girls in the class taller than 164.0 cm.
iii	Average height of girls $=\frac{1}{10}S_{10}=\frac{1}{10}\frac{a(1-r^{10})}{1-r}$ $=\frac{(150.0)(1-1.05^{\frac{10}{3}})}{10(1-1.05^{\frac{1}{3}})}$ $=161.57$ $=161.6$ cm (to 1 d.p.)
iv	Average height of boys $=\frac{1}{16}S_{16}$ $=\frac{1}{16}\times\frac{16}{2}(150.0+180.0)$ $=165.0$ cm Average height of class $=\frac{16(165.0)+10(161.57)}{16+10}$ $=163.7$ cm (to 1 d.p.)
10(i)	length of projection $=\lvert \mathbf{a}\cdot\hat{\mathbf{b}}\rvert = \lvert\mathbf{a}\rvert\lvert\hat{\mathbf{b}}\rvert\cos 60°$ $=2\lvert\mathbf{b}\rvert\left(\frac{1}{2}\right)=\lvert\mathbf{b}\rvert$
(ii)	distance between C and the plane = $\left\lvert\frac{(\mathbf{a}-2\mathbf{b}-\mathbf{a})\cdot(\mathbf{b})}{\lvert\mathbf{b}\rvert}\right\rvert = \left\lvert\frac{-2\mathbf{b}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert}\right\rvert$ $=\left\lvert\frac{-2\lvert\mathbf{b}\rvert^2}{\lvert\mathbf{b}\rvert}\right\rvert$ $=2\lvert\mathbf{b}\rvert$
(iii)	$\mathbf{c}=\begin{pmatrix}1\\5\\2\end{pmatrix}-2\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\\4\end{pmatrix}$

Qn	
	$\pi : \mathbf{r}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}1\\5\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}=9,$ $l : \mathbf{r}=\begin{pmatrix}-1\\1\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$ $\begin{pmatrix}-1+\lambda\\1+2\lambda\\4-\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}=9$ $-1+\lambda+2+4\lambda-4+\lambda=9\Rightarrow\lambda=2$
(iv)	position vector of the foot of perpendicular from **c** to plane = $\begin{pmatrix}-1\\1\\4\end{pmatrix}+2\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}1\\5\\2\end{pmatrix}$ position vector of point of reflection of C in plane $=2\begin{pmatrix}1\\5\\2\end{pmatrix}-\begin{pmatrix}-1\\1\\4\end{pmatrix}=\begin{pmatrix}3\\9\\0\end{pmatrix}$
11a	

Qn	
	
b	$(-\infty,-1)\cup(-1,1)$
12a	$x=\theta^2+4\theta\,,\qquad y=\dfrac{2}{\theta}$ $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\mathrm{d}y}{\mathrm{d}x}\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t}$ $=\left(\dfrac{\mathrm{d}y}{\mathrm{d}\theta}\cdot\dfrac{\mathrm{d}\theta}{\mathrm{d}x}\right)\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t}$ $=\dfrac{-2}{\theta^2}\cdot\dfrac{1}{2\theta+4}\cdot(-4)$ $=\dfrac{4}{\theta^2(\theta+2)}$ When $x=4$, $\theta^2+4\theta=4\Rightarrow\theta=0.82843$ since $\theta>0$ $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{4}{\theta^2(\theta+2)}=2.06$ units/sec Rate of change of *y*-coordinate is 2.06 units/sec.

Qn	
12b	$y=(4-3\cos\theta)(4-3\sin\theta)$ $\frac{dy}{d\theta}=(4-3\sin\theta)(3\sin\theta)+(4-3\cos\theta)(-3\cos\theta)$ $=3\left[4\sin\theta-3\sin^2\theta-4\cos\theta+3\cos^2\theta\right]$ $=3\left[3(\cos^2\theta-\sin^2\theta)+4\sin\theta-4\cos\theta\right]$ $=3\left[3(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)+4(\sin\theta-\cos\theta)\right]$ $=3(\sin\theta-\cos\theta)(4-3\sin\theta-3\cos\theta)$ $\frac{dy}{d\theta}=0$ $3(\sin\theta-\cos\theta)(4-3\sin\theta-3\cos\theta)=0$ $\sin\theta-\cos\theta=0$ or $4-3\sin\theta-3\cos\theta=0$ $\sin\theta+\cos\theta=\frac{4}{3}$ $\theta=\frac{\pi}{4}$ or $\theta=0.44556$ θ=1.1252 (rej $0\le\theta\le\frac{\pi}{4}$) $\frac{d^2y}{d\theta^2}=3(\sin\theta-\cos\theta)(3\sin\theta-3\cos\theta)+3(\sin\theta+\cos\theta)(4-3\sin\theta-3\cos\theta)$ When $\theta=\frac{\pi}{4}$, $\frac{d^2y}{d\theta^2}<0\Rightarrow y$ is max When $\theta=0.44556$, $\frac{d^2y}{d\theta^2}>0\Rightarrow y$ is min Min $y=(4-3\cos 0.44556)(4-3\sin 0.44556)$=3.50

Name	( )	Class	

RIVER VALLEY HIGH SCHOOL
2013 Year 5 Promotional Examination
Higher 2

MATHEMATICS 9740/01

Paper 1

19 September 2013

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)
Cover Page

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, class and index number in the space at the top of this page.

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
Up to **2 marks may be deducted** for poor presentation in your answers.

At the end of the examination, place the cover page on top of your answer paper and fasten all your work securely together.

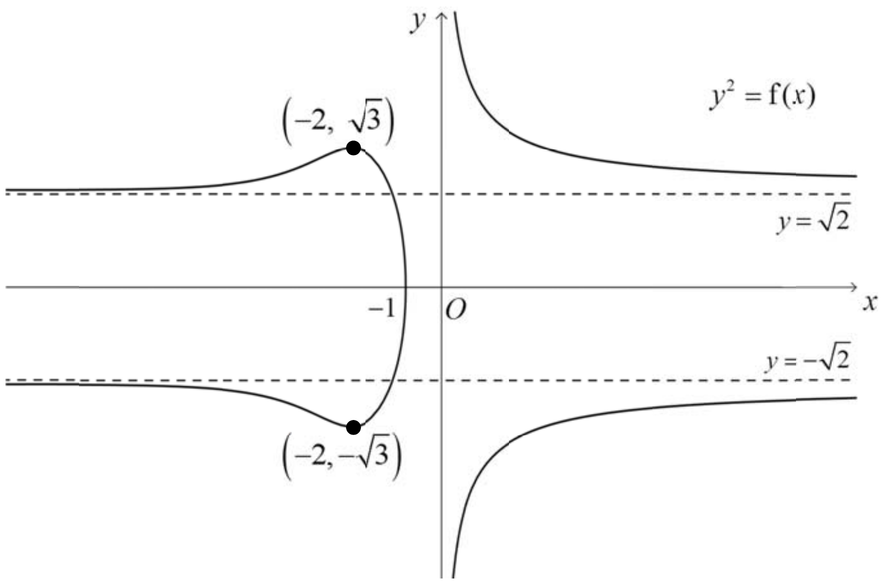
The number of marks is given in brackets [ ] at the end of each question or part question.

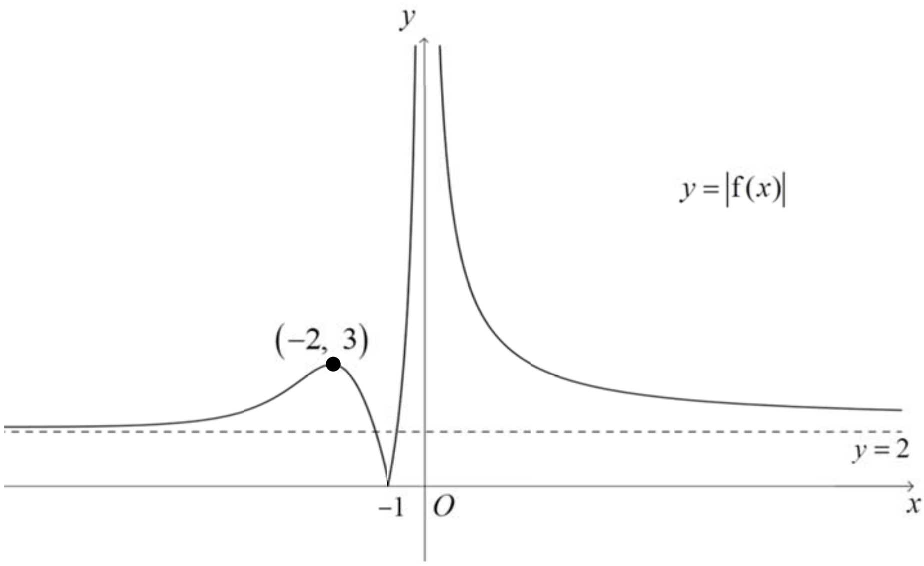
This document consists of **7** printed pages and **1** blank page.

1. (i) Let $f(x) = (x+3)(9-4x)^{-\frac{1}{2}}$. Find the series expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [3]

(ii) Denote the answer to part **(i)** by $g(x)$. Find, for $-\frac{9}{4} \le x \le \frac{9}{4}$, the set of values of x for which the value of $g(x)$ is within ± 0.2 of $f(x)$. [2]

2. The graphs of $y^2 = f(x)$ and $y = |f(x)|$ are given below.





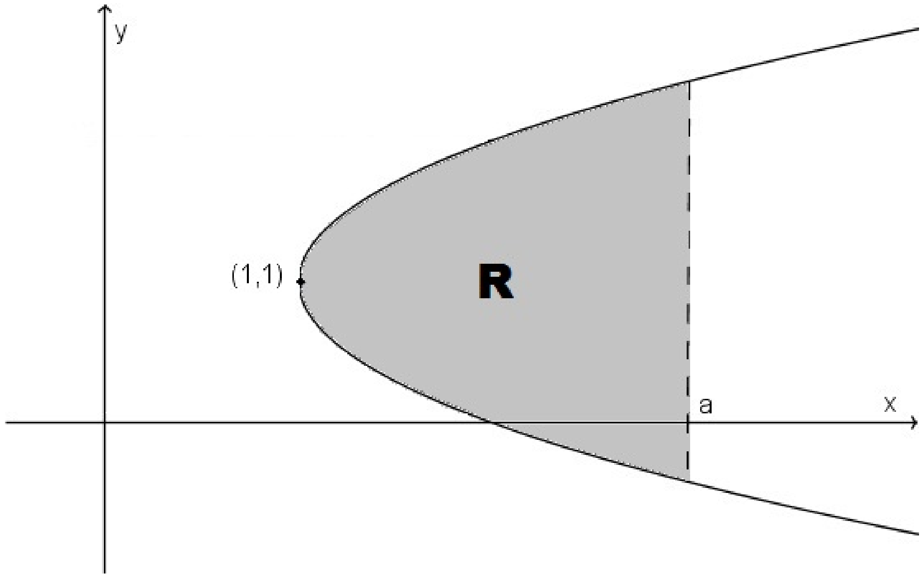
Deduce the graphs of

(i) $y = f(x)$, [3]

(ii) $y = f'(x)$, [2]

clearly indicating any asymptotes, intersections with the axes and stationary points.

3. The diagram shows the sketch of the curve $C, (y-1)^2 = x\sqrt{x^2-1}$, with the vertex at $(1,1)$.



(i) Write down the equation of the graph when C is translated 1 unit in the negative y-direction. [1]

(ii) The shaded region R, bounded by C and the vertical line, $x = a$, is rotated through π radians about the line $y = 1$. By using the substitution $u = \sqrt{x^2-1}$, or otherwise, find the exact volume obtained in terms of a. [5]

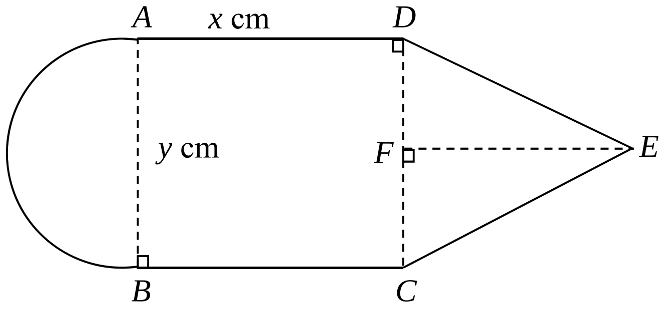
4. (a) A theme park sells day passes at different prices depending on the age of the customer. The age categories are senior citizens (ages 60 and above), adult (ages 13 to 59) and child (ages 4 to 12). Three tour groups visited the theme park on the same day. The numbers in each category for each group together with the total cost of the day passes for each group are given as follows.

Group	Senior Citizens	Adult	Child	Total Cost
1	2	19	9	$196.40
2	0	10	3	$90.20
3	1	7	4	$77.00

Write down and solve equations to find the cost of a day pass for each of the age category. [3]

(b) Without using a GC, solve $\dfrac{4x^2-4|x|+1}{x^2-2|x|-8} \geq 0$. [4]

5. The cross section of an open container consists of a semicircle, a rectangle *ABCD* and an isosceles triangle *CED* as shown in the diagram below. Given that $AD = BC = x$ cm, $AB = DC = FE = y$ cm, $DE = CE$ and the height of the container is $\frac{5}{3}$ cm.



The interior vertical walls of this container, *ADECB*, need to be painted. The time needed to paint the walls will be 1 minute per 10 cm^2 for the straight parts and 1 minute per 8 cm^2 for the semicircular part. Given that a total time of 200 minutes is required to paint all the walls, find, by differentiation, the values of x and y which gives a maximum cross-sectional area, giving your answers correct to the nearest integers. [7]

6. It is given that the curve $y^3 + \tan^{-1} y = \ln(\cos x)$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$, passes through the origin.

(i) Show that $\left(3y^4 + 3y^2 + 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(1 + y^2\right)\tan x$. [2]

(ii) Find the Maclaurin series for y, up to and including the term x^2. [3]

(iii) Hence, find an approximation to the value of $\int_0^{\pi/4} \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x$, in terms of π. [2]

7. In a particular river in Brazil, a sudden surge in the number of piranhas (a type of fish known for their sharp teeth and a voracious appetite for meat) is observed and has affected the livelihood of the villagers living along the river. A group of fishermen is engaged to catch these piranhas and the piranhas are caught at a rate inversely proportional to the number of piranhas left. Furthermore, due to aggressive nature, the number of piranhas is reduced at a rate of one-tenth of the piranhas remaining.

(i) If x (in thousands) is the number of piranhas remaining at time t (in days) after the group of fishermen is deployed to catch the piranhas, show that $x^2 + 10k = Ae^{-0.2t}$ where k is a positive constant. [4]

(ii) If there are 5000 piranhas at the start of the deployment of the fishermen and after 5 days, the number of piranhas remaining is 3000. Calculate the number of days required to remove all the piranhas. [3]

8. (a) Five out of the six digits, 0, 1, 2, 3, 4 and 5 are chosen and arranged randomly to form a five-digit number. No digit is repeated.

Find the number of five-digit numbers that are
(i) greater than 10000, [2]
(ii) greater than 10000 and even. [3]

(b) An ice-cream shop has 4 different flavours of ice-cream, vanilla, chocolate, strawberry and durian and 3 different toppings containing peanuts, raisins and berries. Assuming Peter decides to visit the ice-cream shop and make a selection of at least 1 flavour and at least 1 topping, find how many different selections can he make? [3]

9. (a) The function f and g are defined by

$$\mathrm{f}: x \mapsto x^2 - 6x + 11, \quad x > 3$$
$$\mathrm{g}: x \mapsto \frac{1}{x^2}, \quad x \geq k, \text{ where } k \text{ is a positive constant.}$$

(i) Show that the inverse function of f exists. [1]
(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]
(iii) State the greatest value of k for which the composite function gf exists and find the range of gf for this value of k. [3]

(b) Given that h is a one-one function, determine, with reasons, if hh^{-1} exists. [2]

10. (a) The sum, S_n of the first n terms of a sequence $u_1,\ u_2,\ u_3,\ \ldots$ is given by

$$S_n = \ln a^n b^{\frac{1}{2}\left(n^2 - n\right)}, \text{ where } 0 < a < 1,\ b > 1.$$

(i) Find u_n in terms of a and b. [2]
(ii) Prove that the sequence is an arithmetic progression. [2]
(iii) Given that $0 < ab^{n-1} < 1$ when $n < 7$, find the sum of the negative terms of the sequence. [1]

(b) By considering $\sin(n\theta)\sin\left(\frac{1}{2}\theta\right)$, show, using the method of differences,

$$\sum_{n=1}^{N} \sin(n\theta) = \frac{1}{2}\cot\left(\frac{1}{2}\theta\right) - \frac{\cos\left[\left(N + \frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{1}{2}\theta\right)}.$$ [4]

11. (a) A and B are events such that $\mathrm{P}(B)=0.3$, $\mathrm{P}(A' \cup B')=0.9$ and $\mathrm{P}(A \cap B')=0.45$.

(i) $\mathrm{P}(A)$, [2]

(ii) $\mathrm{P}(A' \cap B)$. [2]

(b) In a cooking school, all students must take a theory and practical test. It is reported that 95% of the students pass the theory test. Of those who pass, 85% also pass the practical test. Of those who fail the theory test, 60% pass the practical test.

Draw a tree diagram to show the above information. [2]

Find the probability that a student, randomly chosen from the cooking school,

(i) passes the practical test, [1]

(ii) passes the theory test, given that he fails the practical test. [2]

12. A curve C has parametric equations

$$x=\mathrm{e}^{t},\ y=t^{2}.$$

(i) Sketch the curve C. [2]

The normal to C at point A with coordinates $\left(\mathrm{e}^{2}, 4\right)$ is denoted by l.

(ii) Find the Cartesian equation of l, expressing y in terms of x. [3]

(iii) Find the exact area of the region bounded by l, C and the x-axis. Express your answer in the form $\frac{a}{\mathrm{e}^{2}}+b\mathrm{e}^{2}+c$ where a, b and c are constants to be determined.

[5]

13. It is thought that the pH value of water may affect the size of pearl in pearl oyster farming. A pearl farmer wished to investigate whether there was any correlation between the pH value of the water and the size of the pearl cultivated. The size of the pearls and the pH value of the water where the oysters are cultivated are shown in the table below.

pH value of water, x	7.7	7.8	7.9	8.0	8.1	8.2	8.3
Size of pearl, y (in cm)	6.82	7.28	7.61	7.79	7.91	8.02	8.05

(i) Draw a scatter diagram to illustrate the data, labeling the axes clearly. [2]

(ii) Comment on whether a linear model would be appropriate. [1]

It is thought that the size of pearl can be modeled by one of the formulae

$$y = a + bx^2 \quad \text{or} \quad y^2 = c + dx$$

where a, b, c and d are constants.

(iii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) x^2 and y,

(b) x and y^2. [2]

(iv) Use your answer to parts **(i)** and **(iii)** to explain which of $y = a + bx^2$ or $y^2 = c + dx$ is the better model. [2]

(v) The pearl farmer will like to have pearls which are exactly 8.00 cm. Find the equation of a suitable regression line, and use it to find the required pH value of the water, correct to 1 decimal place. Comment on the reliability of your answer. [4]

END OF PAPER

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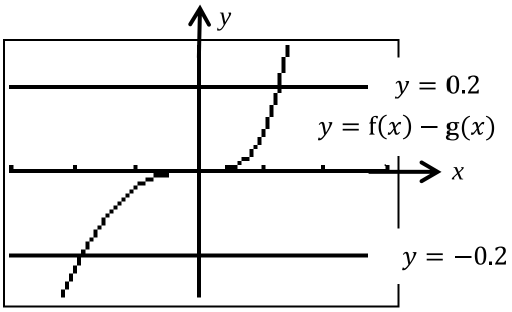
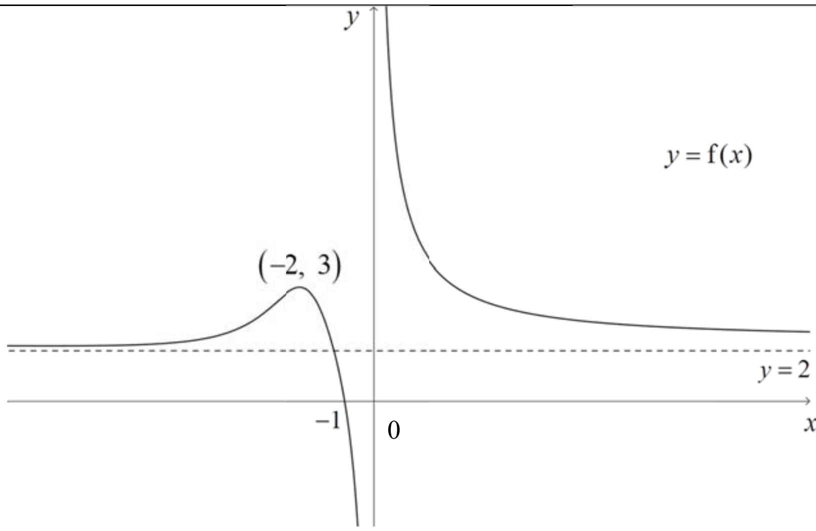
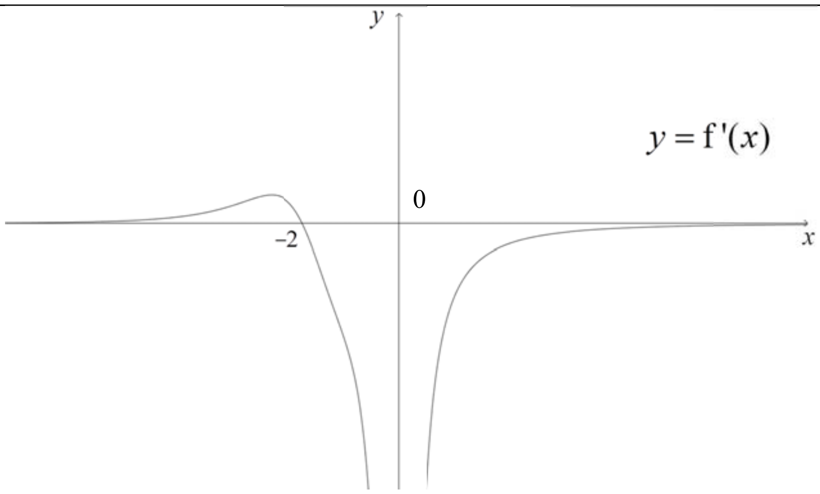
2013 Year 5 H2 Maths Promotional Examination Marking Scheme

1(i)	$\mathrm{f}(x)=(x+3)(9-4x)^{-\frac{1}{2}}$ $=(x+3)9^{-\frac{1}{2}}\left(1-\frac{4}{9}x\right)^{-\frac{1}{2}}$ $=\frac{1}{3}(x+3)\left[1+\frac{\left(-\frac{1}{2}\right)}{1}\left(-\frac{4}{9}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{4}{9}x\right)^2+\dots\right]$ $=\frac{1}{3}(x+3)\left(1+\frac{2}{9}x+\frac{2}{27}x^2+\dots\right)$ $\approx 1+\frac{5}{9}x+\frac{4}{27}x^2$	
(ii)	$-0.2<\mathrm{f}(x)-\mathrm{g}(x)<0.2$ or $\left\|\mathrm{f}(x)-\mathrm{g}(x)\right\|<0.2$  Using GC, $\{x\in R,-1.87<x<1.25\}$	

2(i)	 	
(ii)		

<table>
<tr><td></td><td>
</td><td></td></tr>
</table>

<table>
<tr><td>3 (i)</td><td>Graph to be translated 1 unit in negative y- direction
$\Rightarrow y=\mathrm{f}\left(x\right)-1 \Rightarrow y+1=\mathrm{f}\left(x\right)$
Replace y with $y+1$,
$(y+1-1)^2 = x\sqrt{x^2-1}$
$y^2 = x\sqrt{x^2-1}$</td><td></td></tr>
<tr><td>(ii)</td><td>Volume obtained
$=\pi\int_1^a x\sqrt{x^2-1}\ \mathrm{d}x$
$=\pi\int_0^{\sqrt{a^2-1}} xu\left(\frac{u}{x}\right)\ \mathrm{d}u$ $u=\sqrt{x^2-1}$
$=\pi\int_0^{\sqrt{a^2-1}} u^2\ \mathrm{d}u$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{x}{u}$
$=\pi\left[\frac{u^3}{3}\right]_0^{\sqrt{a^2-1}}$ $\frac{\mathrm{d}x}{\mathrm{d}u}=\frac{u}{x}$
$=\frac{\pi}{3}\left(a^2-1\right)^{\frac{3}{2}}$</td><td></td></tr>
</table>

<table>
<tr><td>4(a)</td><td>Let x, y and z be the cost of a day pass for a senior, adult and child respectively.

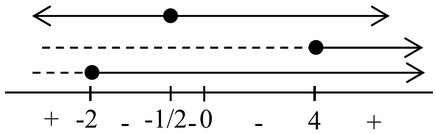
$2x+19y+9z=196.4$
$10y+3z=90.2$
$x+7y+4z=77$

Using GC,</td><td></td></tr>
</table>

	$x = 3.60$ $y = 7.40$ $z = 5.40$ Thus, the cost of a day pass for a senior is \$3.60, for an adult is \$7.40 and for a child is \$5.40.	
(b)	$\dfrac{4x^2 - 4\lvert x\rvert + 1}{x^2 - 2\lvert x\rvert - 8} \geq 0$ Let $y = \lvert x\rvert$ $\dfrac{4y^2 - 4y + 1}{y^2 - 2y - 8} \geq 0$ $\dfrac{(2y-1)^2}{(y+2)(y-4)} \geq 0$ Since $(2y-1)^2 \geq 0$, $(2y-1)^2 = 0$ satisfy the inequality $y = \dfrac{1}{2}$ $\lvert x\rvert = \dfrac{1}{2}$ $x = \dfrac{1}{2}$ or $x = -\dfrac{1}{2}$ $(y+2)(y-4) > 0$ $y < -2$ or $y > 4$ $\lvert x\rvert < -2$ or $\lvert x\rvert > 4$ (no solution) $x > 4$ or $x < -4$ Answer: $x < -4$ or $x > 4$ <u>Alternatively(Method 2),</u> $\dfrac{4\lvert x\rvert^2 - 4\lvert x\rvert + 1}{\lvert x\rvert^2 - 2\lvert x\rvert - 8} \geq 0$	

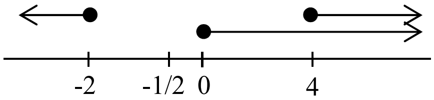
When $x \geq 0$,

$$\frac{4x^2-4x+1}{x^2-2x-8} \geq 0 \text{ and } x \geq 0$$



$$\frac{(2x-1)^2}{(x+2)(x-4)} \geq 0$$

$x < -2$ or $x > 4$ and $x \geq 0$



$x = \frac{1}{2}$ or $x > 4$

Or when $x < 0$,

$$\frac{4x^2+4x+1}{x^2+2x-8} \geq 0 \text{ and } x < 0$$

$$\frac{(2x+1)^2}{(x-2)(x+4)} \geq 0$$

$x > 2$ or $x < -4$ and $x < 0$

$x = -\frac{1}{2}$ or $x < -4$

Answer: $x = \frac{1}{2}$ or $x = -\frac{1}{2}$ or $x < -4$ or $x > 4$

<u>Alternatively(Method 3),</u>

$$\frac{4|x|^2-4|x|+1}{|x|^2-2|x|-8} \geq 0$$

$$\frac{(2|x|-1)^2}{(|x|+2)(|x|-4)} \geq 0$$

$(2|x|-1)^2 = 0$ satisfy the inequality $\Rightarrow |x| = \frac{1}{2} \Rightarrow x = \pm\frac{1}{2}$

$(|x|+2) > 0$ for all values of x,

$(|x|-4) > 0 \Rightarrow |x| > 4 \Rightarrow x < -4$ or $x > 4$

Answer: $x = \frac{1}{2}$ or $x = -\frac{1}{2}$ or $x < -4$ or $x > 4$

5	side of triangle $= \sqrt{\left(\frac{y}{2}\right)^2 + y^2} = \frac{\sqrt{5}y}{2}$	

	$\left[\dfrac{x+x+2\left(\dfrac{y\sqrt{5}}{2}\right)}{10}+\dfrac{\frac{\pi y}{2}}{8}\right]\left(\dfrac{5}{3}\right)=200$ $\dfrac{x}{5}+\dfrac{\sqrt{5}y}{10}+\dfrac{\pi y}{16}=120$ $\dfrac{x}{5}=120-\dfrac{\sqrt{5}y}{10}-\dfrac{\pi y}{16}$ $x=600-\dfrac{\sqrt{5}y}{2}-\dfrac{5\pi y}{16}$ Cross sectional area, W $=\pi\left(\dfrac{y}{2}\right)^2\left(\dfrac{1}{2}\right)+xy+\dfrac{1}{2}y^2$ $=\dfrac{\pi y^2}{8}+\dfrac{y^2}{2}+y\left(600-\dfrac{\sqrt{5}y}{2}-\dfrac{5\pi y}{16}\right)$ $=\dfrac{\pi y^2}{8}+\dfrac{y^2}{2}+600y-\dfrac{\sqrt{5}y^2}{2}-\dfrac{5\pi y^2}{16}$ $=600y-\dfrac{3\pi y^2}{16}+\dfrac{y^2}{2}-\dfrac{\sqrt{5}y^2}{2}$ For maximum W, $\dfrac{\mathrm{d}W}{\mathrm{d}y}=0$ $600-\dfrac{3\pi y}{8}+y-\sqrt{5}y=0$ $y\left(-\dfrac{3\pi}{8}+1-\sqrt{5}\right)=-60$ $y=248.533\approx 249$ $x=78.1347\approx 78$ $\dfrac{\mathrm{d}^2W}{\mathrm{d}y^2}=-\dfrac{3\pi}{8}+1-\sqrt{5}=-2.414$ $y=249$ and $x=78$ will result in a maximum cross sectional area.	

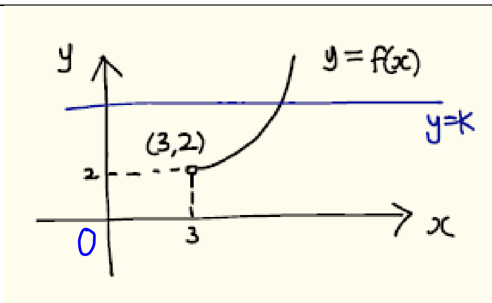
6(i)	$y^3 + \tan^{-1} y = \ln(\cos x)$ Differentiating both sides w.r.t x, $3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sin x}{\cos x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}\left(3y^2\left(1+y^2\right)+1\right) = -\left(1+y^2\right)\tan x$ $\left(3y^4+3y^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(1+y^2\right)\tan x$ (shown)	
(ii)	Differentiating both sides w.r.t x, $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\left(12y^3+6y\right) + \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(3y^4+3y^2+1\right)$ $= -\left(1+y^2\right)\sec^2 x - \left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)\tan x$ When $x = 0$, $y = 0,\ \frac{\mathrm{d}y}{\mathrm{d}x} = 0,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1$ $\therefore y = 0 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \ldots = -\frac{1}{2}x^2 + \ldots$	
(iii)	$\int_0^{\pi/4} \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x \approx \left[-\frac{x^2}{2}\right]_0^{\pi/4}$ $= -\frac{\left(\pi/4\right)^2}{2} - 0$ $= -\frac{\pi^2}{32}$ or $-0.03125\pi^2$	

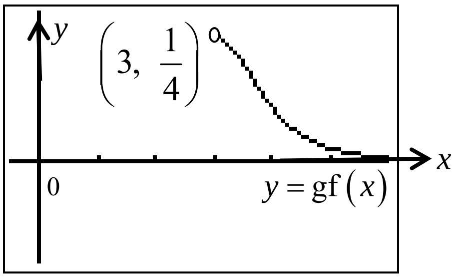
7 (i)	Due to the fishermen catching the fishes, $\frac{\mathrm{d}x}{\mathrm{d}t} \propto \frac{1}{x}$ $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{k}{x}$,where k is a positive constant Due to the aggressive nature of the fishes, $\frac{\mathrm{d}x}{\mathrm{d}t} = -0.1x$ Rate of change of fishes,	

	$\frac{dx}{dt} = -\frac{k}{x} - 0.1x$ $= -\frac{k+0.1x^2}{x}$ $\int \frac{x}{k+0.1x^2}\ dx = \int -1\ dt$ $\frac{1}{0.2}\int \frac{0.2x}{k+0.1x^2}\ dx = \int -1\ dt$ $\frac{1}{0.2}\ln\left\|k+0.1x^2\right\| = -t + c$ $\ln\left\|k+0.1x^2\right\| = -0.2t + c_1$ $\left\|k+0.1x^2\right\| = e^{-0.2t+c_1}$ $k+0.1x^2 = \pm e^{c_1} e^{-0.2t}$ $x^2 + 10k = \pm 10 e^{c_1} e^{-0.2t}$ $x^2 + 10k = Ae^{-0.2t}$ Alternatively, $\int \frac{x}{k+0.1x^2}\ dx = \int -1\ dt$ $\frac{1}{0.2}\int \frac{0.2x}{k+0.1x^2}\ dx = \int -1\ dt$ $\frac{1}{0.2}\ln\left(k+0.1x^2\right) = -t + c$ since $k+0.1x^2 > 0$ $\ln\left(k+0.1x^2\right) = -0.2t + c_1$ $k+0.1x^2 = e^{-0.2t+c_1}$ $x^2 + 10k = Ae^{-0.2t}$	
(ii)	When $t = 0$, $x = 5$ $25 + 10k = A$ When $t = 5$, $x = 3$ $9 + 10k = Ae^{-1}$ Solving, $A = 25.3116$ and $k = 0.0311627$ When $x = 0$, $t = 21.986$ Number of days required = 22	

8(a) (i)	No. of five-digit numbers greater than 10000 $=5\times5\times4\times3\times2$ $=600$ Alternatively, No restrictions – case where 0 is the first digit ${}^6P_5-{}^5P_4=600$	
(ii)	<u>Method 1</u> Case 1: First digit is 1 or 3 or 5 (odd) $3\times4\times3\times2\times3=216$ Case 2: First digit is 2 or 4 (even) $2\times4\times3\times2\times2=96$ No. of five-digit numbers greater than 10000 and even $=216+96$ $=312$ <u>Method 2</u> Case 1: Last digit is 2 or 4 $4\times4\times3\times2\times2=192$ Case 2: Last digit is 0 $5\times4\times3\times2\times1=120$ No. of five-digit numbers greater than 10000 and even $=192+120$ $=312$ <u>Method 3</u> No. of five digit numbers greater than 10000 – No. of five digit numbers greater than 10000 that are odd $=600-4\times4\times3\times2\times3$ $=312$	
(b)	Total number of selections $=\left(2^4-1\right)\times\left(2^3-1\right)=105$ Alternatively, <u>Method 2:</u> Listing 12 Cases $\left({}^4C_1+{}^4C_2+{}^4C_3+{}^4C_4\right)\times\left({}^3C_1+{}^3C_2+{}^3C_3\right)=105$ <u>Method 3:</u> Complement	

	No restrictions – 0 flavors or 0 toppings $=2^4\times2^3-\left(2^3+2^4-1\right)$ $=128-23$ $=105$	

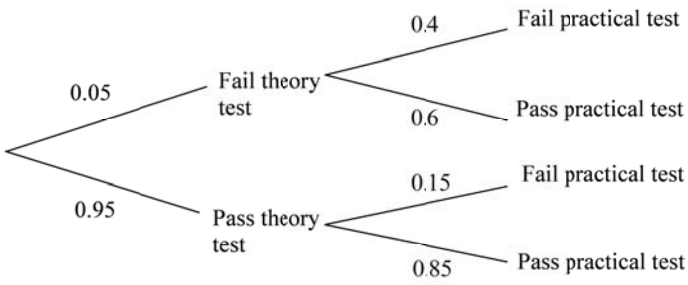
9(i)	 Any horizontal line, $y=k$, $k>2$ cuts the graph of $y=\mathrm{f}(x)$ at most once.	
(ii)	Let $y=\mathrm{f}(x)=x^2-6x+11$ $y=(x-3)^2+2$ $(x-3)^2=y-2$ $x=3\pm\sqrt{y-2}$ Since $x>3$, $x=3+\sqrt{y-2}$ $\therefore \mathrm{f}^{-1}(x)=3+\sqrt{x-2},\ x>2$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=\left(2,\ \infty\right)$	
(iii)	For gf to exist, $R_{\mathrm{f}}\subseteq D_{\mathrm{g}}$ i.e. $\left(2,\ \infty\right)\subseteq\left[k,\ \infty\right)$ $\therefore$ greatest value of $k=2$ $\underset{(3,\infty)}{D_{\mathrm{f}}}\to\underset{(2,\infty)}{R_{\mathrm{f}}}\to\underset{\left(0,\frac{1}{4}\right)}{R_{\mathrm{gf}}}$ $R_{\mathrm{gf}}=\left(0,\ \dfrac{1}{4}\right)$ Alternative method, $\mathrm{gf}\left(x\right)=g\left(x^2-6x+11\right)$ $\quad=\dfrac{1}{\left(x^2-6x+11\right)^2},x>3$	

	 $R_{\mathrm{gf}} = \left(0,\ \frac{1}{4}\right)$	
(b)	Since h is a one-one function, h^{-1} exists. Since $R_{\mathrm{h}^{-1}} = D_{\mathrm{h}}$, the rule for composite function, $R_{\mathrm{h}^{-1}} \subseteq D_{\mathrm{h}}$ is fulfilled. Therefore hh^{-1} exists.	

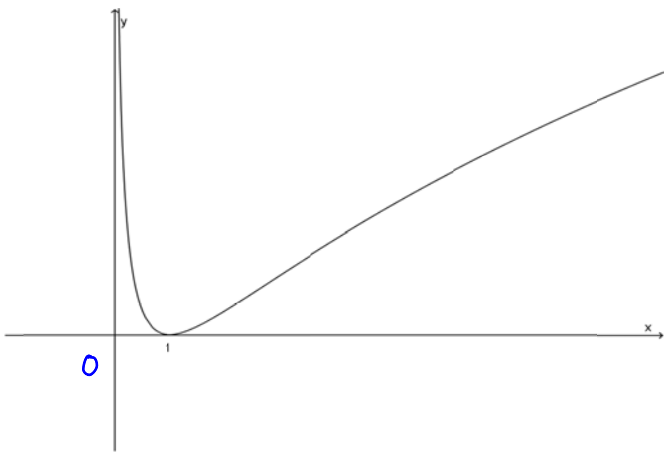
10 (a)(i)	$u_n = S_n - S_{n-1}$ $= \ln a^n b^{\frac{1}{2}(n^2-n)} - \ln a^{n-1} b^{\frac{1}{2}\left((n-1)^2-(n-1)\right)}$ $= \ln ab^{\frac{1}{2}\left(n^2-n-(n^2-3n+2)\right)}$ $= \ln ab^{n-1}$	
(ii)	$u_n - u_{n-1} = \ln ab^{n-1} - \ln ab^{n-1-1}$ $= \ln b$ Since $\ln b$ is a constant, the sequence is an AP.	
(iii)	For $n < 7$, $0 < ab^{n-1} < 1 \Rightarrow \ln ab^{n-1} < 0$ Therefore, sum of negative terms is $S_6 = \ln a^6 b^{\frac{1}{2}(6^2-6)} = \ln a^6 b^{15}$	
(b)	Using factor formula, $\sin(n\theta)\sin\left(\frac{1}{2}\theta\right) = \frac{-1}{2}\left(\cos\left[\left(n+\frac{1}{2}\right)\theta\right] - \cos\left[\left(n-\frac{1}{2}\right)\theta\right]\right)$ $\sin(n\theta) = \frac{-1}{2\sin\left(\frac{1}{2}\theta\right)}\left(\cos\left[\left(n+\frac{1}{2}\right)\theta\right] - \cos\left[\left(n-\frac{1}{2}\right)\theta\right]\right)$	

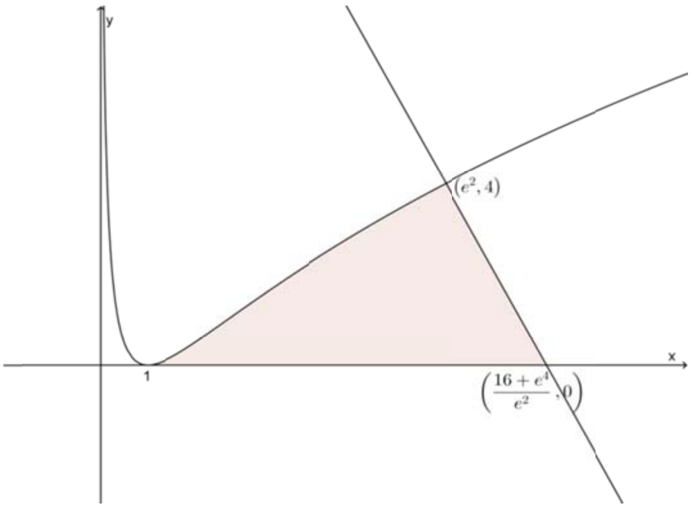
	$\sum_{n=1}^{N}\sin(n\theta)=\frac{-1}{2\sin\left(\frac{1}{2}\theta\right)}\sum_{n=1}^{N}\left(\cos\left[\left(n+\frac{1}{2}\right)\theta\right]-\cos\left[\left(n-\frac{1}{2}\right)\theta\right]\right)$ $=\frac{-1}{2\sin\left(\frac{1}{2}\theta\right)}\left(\begin{array}{l}\cancel{\cos\frac{3\theta}{2}}-\cos\frac{\theta}{2}\\ \cancel{+\cos\frac{5\theta}{2}}\cancel{-\cos\frac{3\theta}{2}}\\ \cancel{+\cos\frac{7\theta}{2}}\cancel{-\cos\frac{5\theta}{2}}\\ +\cdots\\ \cancel{+\cos\left[\left(N-\frac{1}{2}\right)\theta\right]}\cancel{-\cos\left[\left(N-\frac{3}{2}\right)\theta\right]}\\ +\cos\left[\left(N+\frac{1}{2}\right)\theta\right]\cancel{-\cos\left[\left(N-\frac{1}{2}\right)\theta\right]}\end{array}\right)$ $=\frac{-1}{2\sin\left(\frac{1}{2}\theta\right)}\left(\cos\left[\left(N+\frac{1}{2}\right)\theta\right]-\cos\frac{\theta}{2}\right)$ $=\frac{\cos\frac{\theta}{2}}{2\sin\left(\frac{1}{2}\theta\right)}-\frac{\cos\left[\left(N+\frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{1}{2}\theta\right)}$ $=\frac{1}{2}\cot\left(\frac{1}{2}\theta\right)-\frac{\cos\left[\left(N+\frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{1}{2}\theta\right)}$ (shown)	

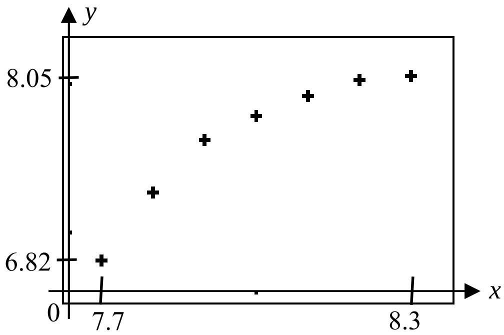
<table>
<tr><td>11
(a)(i)</td><td>$\mathrm{P}(A\cap B)=1-\mathrm{P}(A'\cup B')=1-0.9=0.1$

$\mathrm{P}(A)=\mathrm{P}(A\cap B')+\mathrm{P}(A\cap B)=0.45+0.1=0.55$</td><td></td></tr>
<tr><td>(ii)</td><td>$\mathrm{P}(A'\cap B)=\mathrm{P}(B)-\mathrm{P}(A\cap B)$
$=0.3-0.1$
$=0.2$</td><td></td></tr>
<tr><td>(b)</td><td>
</td><td></td></tr>
<tr><td>(i)</td><td>$\mathrm{P}(\text{passes the practical test})$

$=0.05\times0.6+0.95\times0.85=0.8375$</td><td></td></tr>
<tr><td>(ii)</td><td>$\mathrm{P}(\text{passes the theory test} \mid \text{he fails the practical test})$
$=\dfrac{0.95\times0.15}{1-0.8375}$
$=0.877$</td><td></td></tr>
</table>

<table>
<tr><td>12
(i)</td><td>
</td><td></td></tr>
</table>

(ii)	$\frac{dy}{dt}=2t,\ \frac{dx}{dt}=e^{t}$ $\frac{dy}{dx}=2te^{-t}$ At point A, $t=2$, gradient of normal $=\frac{-1}{2(2)e^{-2}}=\frac{-e^{2}}{4}$ Equation of line l, $y-4=-\frac{e^{2}}{4}\left(x-e^{2}\right)$ $y=-\frac{e^{2}}{4}x+\frac{16+e^{4}}{4}$	
(iii)	 Required area = area of triangle + area under curve C $=\frac{1}{2}(4)\left(\frac{16+e^{4}}{e^{2}}-e^{2}\right)+\int_{t=0}^{t=2} y\frac{dx}{dt}dt$ $=2\left(\frac{16}{e^{2}}\right)+\int_{0}^{2} t^{2}e^{t}dt$ $=\frac{32}{e^{2}}+\left\{\left[t^{2}e^{t}\right]_{0}^{2}-2\int_{0}^{2} te^{t}dt\right\}$ $=\frac{32}{e^{2}}+4e^{2}-2\left\{\left[te^{t}\right]_{0}^{2}-\int_{0}^{2} e^{t}dt\right\}$ $=\frac{32}{e^{2}}+4e^{2}-2\left\{2e^{2}-\left[e^{t}\right]_{0}^{2}\right\}$ $=\frac{32}{e^{2}}+4e^{2}-4e^{2}+2e^{2}-2$ $=\frac{32}{e^{2}}+2e^{2}-2$	

13(i)		
(ii)	The scatter diagram shows y is increasing at a decreasing rate and hence a linear model is not appropriate.	
(iii)	(a) $r \approx 0.9358$ (b) $r \approx 0.9464$	
(iv)	Since the product moment correlation coefficient between x and y^2 is closer to 1 compared to that between x^2 and y and y increases as x increases but at a decreasing rate, hence $y^2 = c + dx$ is the better model.	
(v)	Using the GC, $y^2 = -176.23 + 29.347x$ When $y = 8.00$, $x \approx 8.2$ From (iii), $r \approx 0.9464$ is close to 1. Since $y = 8.00$ is within the data range of y and x is the independent variable, hence the answer is reliable.	

TEMASEK JUNIOR COLLEGE, SINGAPORE
JC One
Promotion Examination 2013
Higher 2

MATHEMATICS **9740**

Solutions

1 Find the general solution of the following differential equation

$$\frac{1}{1+x}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{1+x^2}=0, \quad \text{where } x \neq -1.$$ **[4]**

Solution:

$$\left(\frac{1}{1+x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{1+x^2}=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1+x}{1+x^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{1+x^2}-\frac{x}{1+x^2}$$

$$y=-\int\frac{1}{1+x^2}\,\mathrm{d}x-\frac{1}{2}\int\frac{2x}{1+x^2}\,\mathrm{d}x$$

$$=-\tan^{-1}x-\frac{1}{2}\ln\left(1+x^2\right)+c$$

(or $-\tan^{-1}x-\frac{1}{2}\ln\left|1+x^2\right|+c$)

2 **(i)** The first three terms of a sequence are given by $u_1 = 19$, $u_2 = 34$, $u_3 = 52$. Given that u_n is a quadratic polynomial in n, find u_n in terms of n. **[4]**

(ii) Find the smallest value of n for which u_n is greater than 200. **[2]**

Solution:

(i) Let $u_n = an^2 + bn + c$ where a, b, c are constants.

When $n=1$, $a+b+c=19$ ----- (1)
When $n=2$, $4a+2b+c=34$ ----- (2)
When $n=3$, $9a+3b+c=52$ ----- (3)

Using GC to solve the system of equations, we get

$$a=\frac{3}{2},\quad b=\frac{21}{2},\quad c=7$$

$$\therefore\quad u_n=\frac{3}{2}n^2+\frac{21}{2}n+7$$

(ii)

Method I: For $u_n > 200$,

$$\frac{3}{2}n^2+\frac{21}{2}n+7>200$$

$\Rightarrow \quad n<-15.4 \quad$ or $\quad n>8.37 \quad$ (3sf)

$\therefore$ the smallest value of n is 9.

Method II:

For $u_n > 200$,
By GC
$U_8 = 187 < 200$
$U_9 = 223 > 200$

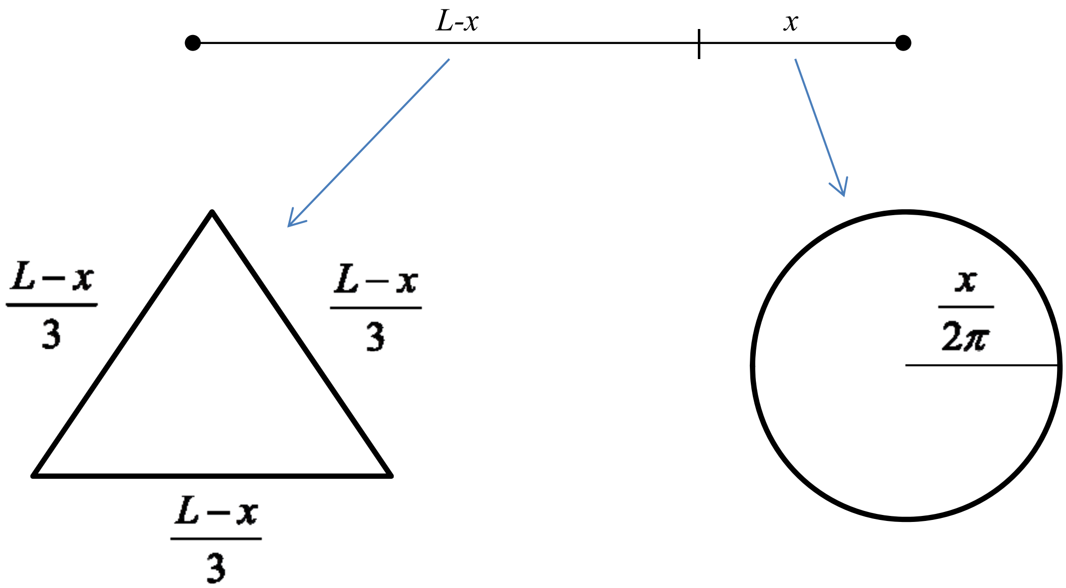
$\therefore$ The smallest value of n is 9.

3 A wire of length L cm is cut into two pieces. One piece is used to form a circle while the other piece is used to form an equilateral triangle.
Show that, with the total area of the circle and triangle being the smallest, the proportion of the length of the smaller piece to the length of the bigger piece is $\dfrac{\sqrt{3}\pi}{9}$.

[6]

Solution:

Let one of the pieces be x cm and use it for form the circle.
So the other piece is L-x and it's used to for the equilateral triangle.



For area of circle (radius r): $2\pi r = x \Rightarrow r = \dfrac{x}{2\pi}$

Therefore area is $\pi\left(\dfrac{x}{2\pi}\right)^2 = \dfrac{x^2}{4\pi}$

For area of equilateral triangle:

Area = $\dfrac{1}{2}\left(\dfrac{L-x}{3}\right)^2 \sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{36}(L-x)^2$

Hence total area, $A = \dfrac{x^2}{4\pi} + \dfrac{\sqrt{3}}{36}(L-x)^2$ [the other form $\dfrac{(L-x)^2}{4\pi} + \dfrac{\sqrt{3}}{36}x^2$ also accepted]

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{x}{2\pi} - \frac{\sqrt{3}}{18}(L-x)$$

Method I:

For max/min, $\frac{\mathrm{d}A}{\mathrm{d}x}=0$,

$$\Rightarrow \frac{1x}{2\pi}-\frac{2\sqrt{3}}{36}\left(L-x\right)=0 \Rightarrow \frac{x}{2\pi}=\frac{\sqrt{3}}{18}\left(L-x\right) \Rightarrow \frac{x}{\left(L-x\right)}=\frac{\sqrt{3}}{9}\pi<1$$

Hence the ratio of the length of the smaller piece to the length of the bigger piece is $\frac{\sqrt{3}\pi}{9}$

(shown)

And $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=\frac{1}{2\pi}+\frac{\sqrt{3}}{18}>0 \Rightarrow A$ is minimum.

Method II:

For max/min, $\frac{\mathrm{d}A}{\mathrm{d}x}=0$,

$$\Rightarrow \frac{2x}{4\pi}-\frac{2\sqrt{3}}{36}\left(L-x\right)=0 \Rightarrow \frac{x}{2\pi}-\frac{\sqrt{3}}{18}\left(L-x\right)=0 \text{ --- } (*)$$

$$\Rightarrow x\left(\frac{1}{2\pi}+\frac{\sqrt{3}}{18}\right)=\frac{\sqrt{3}}{18}L \Rightarrow x=\frac{\sqrt{3}\pi L}{9+\sqrt{3}\pi}$$

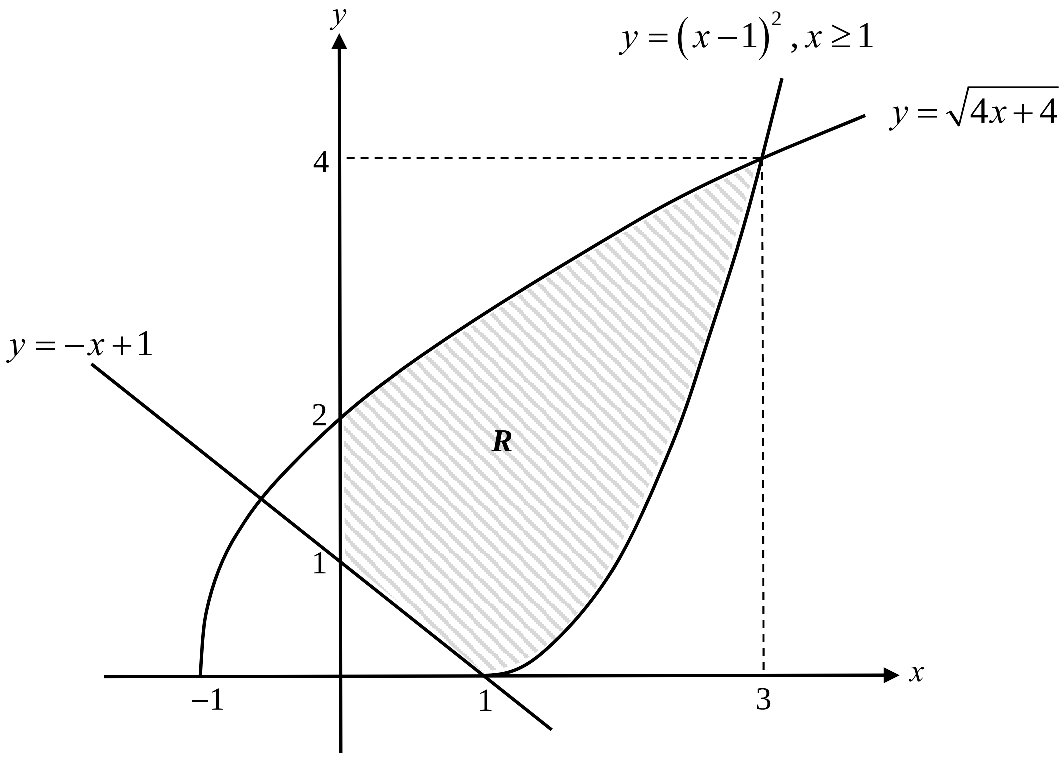
$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=\frac{1}{2\pi}+\frac{\sqrt{3}}{18}>0 \Rightarrow A$ is minimum at $x=\frac{\sqrt{3}\pi L}{9+\sqrt{3}\pi}$

From (*) $\frac{x}{2\pi}-\frac{\sqrt{3}}{18}(L-x)=0 \Rightarrow \frac{x}{2\pi}=\frac{\sqrt{3}}{18}(L-x) \Rightarrow \frac{x}{L-x}=\frac{\sqrt{3}\pi}{9}(<1)$

Hence the ratio of the length of the smaller piece to the length of the bigger piece is $\frac{\sqrt{3}\pi}{9}$

(shown)

4



The shaded region R in the diagram above is bounded by the y-axis, the line $y=-x+1$ and the curves $y=(x-1)^2$ for $x \geq 1$ and $y=\sqrt{4x+4}$.

Find the volume of the solid of revolution formed when R is rotated completely about the y-axis. **[6]**

Solution:

Required volume = $\pi\int_0^4 \left(1+\sqrt{y}\right)^2 \text{ dy} - \pi\int_2^4 \left(\frac{y^2-4}{4}\right)^2 \text{ dy} - \frac{\pi}{3}(1)^2(1)$

$\approx 17.26666709\pi \approx 54.24483447 \approx 54.2 \text{ unit}^2$

5 Given that $y=\ln\left(2+\tan^{-1}x\right)$, show that

$$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1+x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0\,.$$ **[3]**

Hence find the Maclaurin's expansion for y, up to and including the term in x^2. **[3]**

Solution:

$y=\ln\left(2+\tan^{-1}x\right)\Rightarrow e^y=2+\tan^{-1}x$

Differentiate wrt x

$\Rightarrow e^y\frac{dy}{dx}=\frac{1}{1+x^2}\Rightarrow\left(1+x^2\right)\frac{dy}{dx}=e^{-y}\;---(1)$

Differentiate (1) wrt x

$\Rightarrow\left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=-e^{-y}\frac{dy}{dx}=-\left(1+x^2\right)\left(\frac{dy}{dx}\right)^2$ [From (1)]

$\Rightarrow\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1+x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$

When $x=0,y=\ln 2,\frac{dy}{dx}=\frac{1}{2},\frac{d^2y}{dx^2}=-\frac{1}{4}$

$\Rightarrow y=\ln 2+\frac{\left(\frac{1}{2}\right)}{1!}x+\frac{\left(-\frac{1}{4}\right)}{2!}x^2+\cdots\approx\ln 2+\frac{1}{2}x-\frac{1}{8}x^2$

6 Prove by mathematical induction $\displaystyle\sum_{r=1}^{n}\frac{r}{3^{r-1}}=\frac{9}{4}-\frac{1}{3^{n-1}}\left(\frac{3}{4}+\frac{n}{2}\right)$ for all positive integers of n. **[5]**

Hence show that

$$\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\frac{4}{4^4}\cdots<\frac{9}{16}.$$ **[2]**

Solution:

Let $P(n)$ be the statement $\displaystyle\sum_{r=1}^{n}\frac{r}{3^{r-1}}=\frac{9}{4}-\frac{1}{3^{n-1}}\left(\frac{3}{4}+\frac{n}{2}\right)$ for $n=1,2,3,4,\ldots$

When n = 1, LHS = $\displaystyle\sum_{r=1}^{1}\frac{r}{3^{r-1}}$ = 1 ; RHS = $\displaystyle\frac{9}{4}-\left(\frac{3}{4}+\frac{1}{2}\right)=1$

So $P(1)$ is true.

Assume $P(k)$ is true for some $k\in\mathbb{Z}^+$, i.e. $\displaystyle\sum_{r=1}^{k}\frac{r}{3^{r-1}}=\frac{9}{4}-\frac{1}{3^{k-1}}\left(\frac{3}{4}+\frac{k}{2}\right)$

To show $P(k+1)$ is true i.e. $\displaystyle\sum_{r=1}^{k+1}\frac{r}{3^{r-1}}=\frac{9}{4}-\frac{1}{3^{k}}\left(\frac{3}{4}+\frac{k+1}{2}\right)$

$$\text{LHS} = \sum_{r=1}^{k+1}\frac{r}{3^{r-1}} = \sum_{r=1}^{k}\frac{r}{3^{r-1}} + \frac{k+1}{3^k}$$

$$= \left[\frac{9}{4}-\frac{1}{3^{k-1}}\left(\frac{3}{4}+\frac{k}{2}\right)\right]+\frac{k+1}{3^k}$$

$$= \frac{9}{4}-\frac{1}{3^{k}}\left(\frac{9}{4}+\frac{3k}{2}-k-1\right)$$

$$=\frac{9}{4}-\frac{1}{3^{k}}\left(\frac{9}{4}+\frac{3k-2k-2}{2}\right)=\frac{9}{4}-\frac{1}{3^{k}}\left(\frac{9}{4}+\frac{k-2}{2}\right)=\frac{9}{4}-\frac{1}{3^{k}}\left(\frac{5}{4}+\frac{k}{2}\right)$$

$$=\frac{9}{4}-\frac{1}{3^{k}}\left(\frac{3}{4}+\frac{k+1}{2}\right) = \text{RHS}$$

So $P(k+1)$ is true.

Since $P(1)$ is true, and $P(k)$ is true $\Rightarrow$ $P(k+1)$ is true.

$\therefore$ By mathematical induction, $P(n)$ is true for all $n\in\mathbb{Z}^+$, ie. $\displaystyle\sum_{r=1}^{n}\frac{r}{3^{r-1}}=\frac{9}{4}-\frac{1}{3^{n-1}}\left(\frac{3}{4}+\frac{n}{2}\right)$ $n\in\mathbb{Z}^+$

Since $\displaystyle\sum_{r=1}^{\infty}\frac{r}{3^{r-1}}=\frac{9}{4}$

Hence $\displaystyle\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\cdots=\sum_{r=1}^{\infty}\frac{r}{4^r}=\frac{1}{4}\sum_{r=1}^{\infty}\frac{r}{4^{r-1}}<\frac{1}{4}\sum_{r=1}^{\infty}\frac{r}{3^{r-1}}=\frac{9}{16}$ (deduced)

7 Functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{2x-2}{x-2}, \quad \text{for } x \in \mathbb{R}, x < 1,$$

$$\mathrm{g}: x \mapsto \sqrt{2-x}, \quad \text{for } x \in \mathbb{R}, x \le 2.$$

(i) Given that f has an inverse, show that the composite function gf^{-1} exists. Find gf^{-1} and state its range. **[5]**

(ii) Find the value(s) of x such that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. **[2]**

Solution:

(i) $\mathrm{R}_{\mathrm{f}^{-1}} = \mathrm{D}_{\mathrm{f}} = (-\infty, 1)$

$\mathrm{D}_g = (-\infty, 2]$

Since $\mathrm{R}_{\mathrm{f}^{-1}} \subset \mathrm{D}_g$, the composite function gf^{-1} exists. (Shown)

Let $y = \dfrac{2x-2}{x-2}$.

$\Rightarrow \quad xy - 2y = 2x - 2$

$\Rightarrow \quad xy - 2x = 2y - 2$

$\Rightarrow \quad x(y-2) = 2y - 2$

$\Rightarrow \quad x = \dfrac{2y-2}{y-2}$

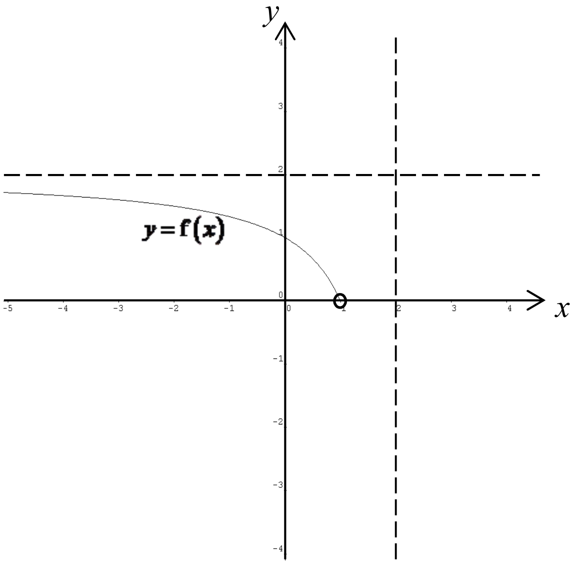
$\Rightarrow \quad \mathrm{f}^{-1}(x) = \dfrac{2x-2}{x-2}$.

$$\mathrm{gf}^{-1}(x) = \mathrm{g}\left(\frac{2x-2}{x-2}\right)$$

$$= \sqrt{2-\left(\frac{2x-2}{x-2}\right)} = \sqrt{2-\left(2+\frac{2}{x-2}\right)} = \sqrt{-\frac{2}{x-2}}$$

$\mathrm{D}_{\mathrm{gf}^{-1}} = \mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = (0, 2)$

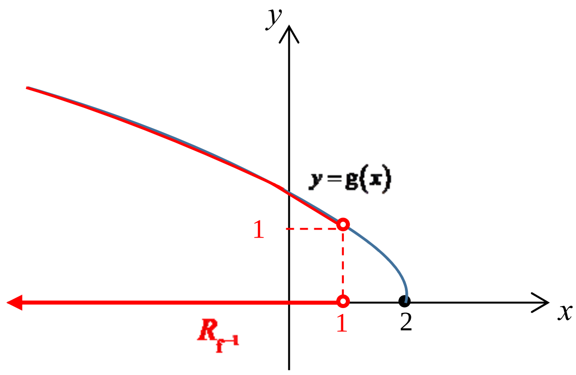
So, $\mathrm{gf}^{-1}: x \mapsto \sqrt{-\dfrac{2}{x-2}}$, $x \in \mathbb{R}, 0 < x < 2$



For range of gf^{-1}:

<u>M1 - By mapping method</u>

$$\underset{D_{gf^{-1}}=D_{f^{-1}}}{(0,2)} \xrightarrow{f^{-1}} \underset{R_{f^{-1}}}{(-\infty,1)} \xrightarrow{g} \underset{R_{gf^{-1}}}{(1,\infty)}$$

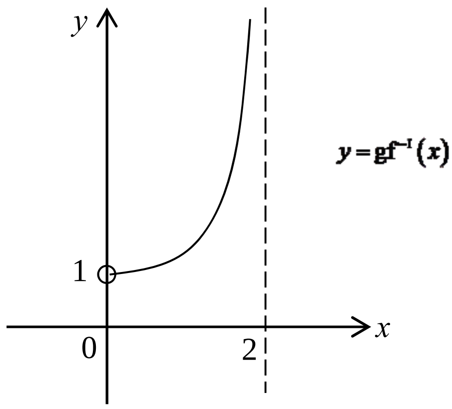


Thus, $R_{gf^{-1}}=(1,\infty)$.

<u>M2 - By direct sketching method</u>

$D_{gf^{-1}}=D_{f^{-1}}=R_f=(0,2)$

Therefore $R_{gf^{-1}}=(1,\infty)$

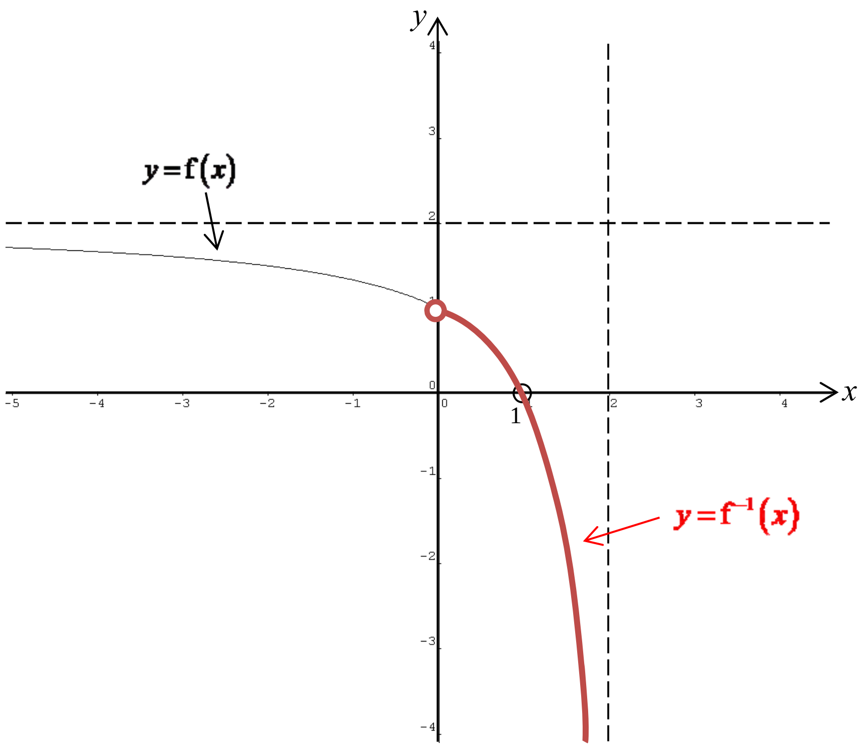


(ii)

From the graph,

$f(x)=f^{-1}(x)$

$\Rightarrow$ $0<x<1$



8 Prove that

$$\ln\left(\frac{(r-1)(r+2)}{r(r+1)}\right) \equiv \ln\left((r-1)(r)\right) - 2\ln\left((r)(r+1)\right) + \ln\left((r+1)(r+2)\right). \qquad \textbf{[2]}$$

Hence, find in terms of n,

$$\ln\left(\frac{1\times 4}{2\times 3}\right) + \ln\left(\frac{2\times 5}{3\times 4}\right) + \ln\left(\frac{3\times 6}{4\times 5}\right) + \cdots + \ln\left(\frac{(n-1)(n+2)}{(n)(n+1)}\right) + \ln\left(\frac{(n)(n+3)}{(n+1)(n+2)}\right),$$

leaving your answer as a single logarithmic function. **[5]**

Solution:

(i)

$$\begin{aligned} RHS &\equiv \ln\left((r-1)(r)\right) - 2\ln\left((r)(r+1)\right) + \ln\left((r+1)(r+2)\right) \\ &\equiv \ln\left(\frac{(r-1)(r)(r+1)(r+2)}{(r)^2(r+1)^2}\right) \\ &\equiv \ln\left(\frac{(r-1)(r+2)}{(r)(r+1)}\right) \equiv LHS \end{aligned}$$

(ii)

$$\ln\left(\frac{1\times 4}{2\times 3}\right) + \ln\left(\frac{2\times 5}{3\times 4}\right) + \ln\left(\frac{3\times 6}{4\times 5}\right) + \cdots + \ln\left(\frac{(n-1)(n+2)}{(n)(n+1)}\right) + \ln\left(\frac{(n)(n+3)}{(n+1)(n+2)}\right)$$

$$= \sum_{r=2}^{n+1}\left[\ln\frac{(r-1)(r+2)}{r(r+1)}\right] = \sum_{r=2}^{n+1}\left[\ln(r-1)r - 2\ln\left[r(r+1)\right] + \ln(r+1)(r+2)\right]$$

$$\begin{aligned} &= \ln(1)(2) - 2\ln(2)(3) + \ln(3)(4) \\ &+ \ln(2)(3) - 2\ln(3)(4) + \ln(4)(5) \\ &+ \ln(3)(4) - 2\ln(4)(5) + \ln(5)(6) \\ &+ \ln(4)(5) - 2\ln(5)(6) + \ln(6)(7) \\ &+ \cdots \\ &+ \ln(n-2)(n-1) - 2\ln(n-1)(n) + \ln(n)(n+1) \\ &+ \ln(n-1)(n) - 2\ln(n)(n+1) + \ln(n+1)(n+2) \\ &+ \ln(n)(n+1) - 2\ln(n+1)(n+2) + \ln(n+2)(n+3) \\ &= \ln(1)(2) - \ln(2)(3) - \ln(n+1)(n+2) + \ln(n+2)(n+3) \quad \textbf{[} \end{aligned}$$

$$= \ln\left[\frac{2(n+2)(n+3)}{6(n+1)(n+2)}\right] = \ln\left[\frac{n+3}{3(n+1)}\right]$$

9 Jessie wishes to take up a loan of \$20,000 on the 1st day of the Year 2014. She intends to pay an instalment of \$300 on the 1st day of each month, beginning from February 2014. She sources out two banks, *XYZ* Bank and *ABC* Bank, which offer such loans. The two banks have different ways of charging interest. *XYZ* Bank charges a monthly interest of 0.5% on the outstanding amount owed at the end of each month, while *ABC* Bank charges a fixed interest of \$60 at the end of each month until the loan is repaid.

(a) If Jessie takes up the loan from *XYZ* Bank, show that the outstanding loan at the end of February 2014 after the interest has been added will be \$19899. **[2]**

Hence, find the number of months Jessie will take to repay her loan. **[4]**

(b) Which bank should Jessie take a loan from if she wishes to clear her loan as soon as possible? Justify your answers. **[3]**

Solution:

kth month	**Outstanding loan at the beginning of kth month from 2014**	**Outstanding loan at the end of kth month from 2014**
1	20000	$1.005(20000)$
2	$1.005(20000)-300$	$1.005^2(20000)-300(1.005)$
3	$1.005^2(20000)-300(1.005)-300$	
$\vdots$	$\vdots$	
n	$1.005^{n-1}(20000)-300(1.005)^{n-2}-300(1.005)^{n-3}$ $-\cdots-300(1.005)^2-300(1.005)-300$	

(a) Outstanding loan at the end of February 2014 $=1.005^2(20000)-300(1.005)=\$19899$

[Shown]

Hence

Let $1.005^{n-1}(20000)-300(1.005)^{n-2}-\cdots-300(1.005)-300\le 0$

$$\Rightarrow 1.005^{n-1}(20000)-300\left[1+(1.005)+(1.005)^2+\cdots+(1.005)^{n-2}\right]\le 0$$

$$\Rightarrow 1.005^{n-1}(20000)-300\left[\frac{1\bullet(1.005)^{n-1}-1}{1.005-1}\right]\le 0$$

$$\Rightarrow 1.005^{n-1}(20000)-60000\left[(1.005)^{n-1}-1\right]\le 0$$

$$\Rightarrow 40000(1.005)^{n-1}\ge 60000$$

$$\Rightarrow (n-1)\ge\frac{\ln\frac{60000}{40000}}{\ln(1.005)}\Rightarrow n\ge 82.29558565$$

$\Rightarrow$ Jessie will repay her loan on the 1st day of 83rd month. Therefore, she will take 82 months to repay her loan.

(b)

Method I:

For Bank *ABC*,

k^{th} month	**Outstanding loan at the beginning of k^{th} month from 2014**	**Outstanding loan at the end of k^{th} month from 2014**
1	20000	$20000+60$
2	$20000+60-300$	$20000+60-300+60$
3	$20000+60-300+60-300=20000+60(2)-300(2)$ $=20000-240(2)$	
$\vdots$	$\vdots$	
n	$20000-240(n-1)$	

For $20000-240(n-1)\le 0 \Rightarrow n \ge 84.33333$

$\Rightarrow$ Jessie will repay her loan on the 1st day of 85th month if she takes up bank *ABC*.
Hence, she should take the loan from bank *XYZ*.

Method II:
When n = 83, $20000-240(83-1)=320>0$
$\Rightarrow$ Jessie will not be able to clear her loan by the 83rd month if she takes up bank *ABC*.
Hence, she should take the loan from bank *XYZ*.

10 A curve C is given parametrically by the equations

$$x = 2\cos^3\theta, \quad y = 2\sin^3\theta$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Show that the normal at the point with parameter θ has equation

$$y\sin\theta = x\cos\theta + 2\left(\sin^4\theta - \cos^4\theta\right).$$ **[4]**

The normal at the point Q where $\theta = \frac{\pi}{6}$, cuts C again at the point P, where $\theta = p$. Show that $\sin^3 p - \sqrt{3}\cos^3 p + 1 = 0$ and hence find the coordinates of P. **[5]**

Solution:

$x = 2\cos^3\theta,$ $\qquad y = 2\sin^3\theta$

$\frac{dx}{dt} = 3(2)\cos^2\theta(-\sin\theta)$ $\qquad \frac{dy}{dt} = 3(2)\sin^2\theta\cos\theta$

$= -6\sin\theta\cos^2\theta$ $\qquad = 6\sin^2\theta\cos\theta$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\tan\theta$

$\Rightarrow$ Gradient of normal to the curve $= \cot\theta$

Eqn. of normal to the curve at $\left(2\cos^3\theta, 2\sin^3\theta\right)$:

$$\frac{y - 2\sin^3\theta}{x - 2\cos^3\theta} = \frac{\cos\theta}{\sin\theta}$$

$\Rightarrow \quad y\sin\theta - 2\sin^4\theta = x\cos\theta - 2\cos^4\theta$

$\Rightarrow \quad y\sin\theta = x\cos\theta + 2\left(\sin^4\theta - \cos^4\theta\right)$ (shown)

Eqn. of normal to the curve at Q, i.e. $\theta = \frac{\pi}{6}$:

$$y\left(\frac{1}{2}\right) = x\left(\frac{\sqrt{3}}{2}\right) + 2\left(\left(\frac{1}{2}\right)^4 - \left(\frac{\sqrt{3}}{2}\right)^4\right)$$

$\Rightarrow \quad y = \sqrt{3}x - 2$

When the normal to the curve at Q cuts C again at P, i.e. $\theta = p$,

$$2\sin^3 p = \sqrt{3}\left(2\cos^3 p\right) - 2$$

$\Rightarrow \quad \sin^3 p - \sqrt{3}\cos^3 p + 1 = 0$ (shown)

$\Rightarrow \quad p = -0.7445633$ or 0.52359878 (rejected, $\because$ point Q)

$\therefore$ The coordinates of P is $(0.795, -0.622)$. (3sf)

11 A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \sqrt{\frac{2(x_n^2 - x_n)}{3}} + 1, \quad x_1 = k, \text{ where } k \ge 1.$$

(a) When $k = 5$, state the value of x_9 and describe the behavior of the sequence. **[2]**

(b) Prove algebraically that, if the sequence converges, then it converges to either 1 or 3. **[3]**

(c) State a value of k such that the sequence converges to 1. **[1]**

(d) When $k = 2$, state the integer m such that $m \le x_n < m+1$ for all integers $n \ge 1$. **[1]**

Hence, by considering $\frac{x_{n+1}-1}{x_n - 1}$, show that $x_{n+1} > x_n$ for all integers $n \ge 1$. **[3]**

Solution:

(a) $x_9 = 3.44$

The sequence converges to 3 decreasingly.

(b) If the sequence converges to l. So when $n \to \infty$, $x_{n+1} \to l$ and $x_n \to l$.

Solving, we have

$$l = \sqrt{\frac{2(l^2 - l)}{3}} + 1 \Rightarrow 3(l-1)^2 = 2l^2 - 2l \Rightarrow l^2 - 4l + 3 = 0 \Rightarrow l = 1 \text{ or } l = 3.$$

Hence, if the sequence converges, then it converges to either 1 or 3. [Proven]

(c) The sequence converges to 1 when k = 1

(d) From GC, $m = 2$.

Method I:

$$\frac{x_{n+1}-1}{x_n-1} = \frac{\sqrt{\frac{2}{3}\left(x_n{}^2 - x_n\right)}}{x_n - 1} = \sqrt{\frac{2x_n}{3\left(x_n - 1\right)}} = \sqrt{\frac{2}{3}}\sqrt{1 + \frac{1}{x_n - 1}}$$

$$2 \le x_n < 3 \Rightarrow \frac{1}{x_n - 1} > \frac{1}{2} \quad .$$

$$\Rightarrow \sqrt{\frac{2}{3}}\sqrt{1 + \frac{1}{x_n - 1}} > \sqrt{\frac{2}{3}}\sqrt{\frac{3}{2}} = 1$$

$$\Rightarrow \frac{x_{n+1}-1}{x_n - 1} > 1 \Rightarrow x_{n+1} > x_n$$

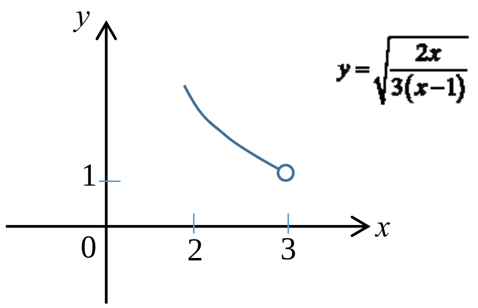
Or $$\frac{x_{n+1}-1}{x_n-1} = \frac{\sqrt{\frac{2}{3}\left(x_n{}^2 - x_n\right)}}{x_n - 1} = \sqrt{\frac{2x_n}{3\left(x_n - 1\right)}}$$

Now $x_n < 3 \Rightarrow -2x_n < 3 - 3x_n \Rightarrow \frac{-2x_n}{3 - 3x_n} > 1 \left(\because x_n > 1\right) \Rightarrow \sqrt{\frac{2x_n}{3\left(x_n - 1\right)}} > 1$

Method II:

$$\frac{x_{n+1}-1}{x_n-1}=\frac{\sqrt{\frac{2}{3}\left(x_n{}^2-x_n\right)}}{x_n-1}=\sqrt{\frac{2x_n}{3\left(x_n-1\right)}}$$

From the graph of $y=\sqrt{\frac{2x}{3\left(x-1\right)}}$, when $2\le x<3, y>1$



Since $2\le x_n<3$

$$\frac{x_{n+1}-1}{x_n-1}=\sqrt{\frac{2x_n}{3\left(x_n-1\right)}}>1\Rightarrow x_{n+1}>x_n$$

12 **(a)** Find $\int_1^e \frac{1}{x^2}\ln\left(\frac{1}{x^2}\right)\,\mathrm{d}x$, leaving your answer in exact form. **[4]**

(b) Using the substitution $u=\sqrt{t}$, find $\int \frac{\sqrt{t}}{t-1}\,\mathrm{d}t$. **[6]**

Solution:

(a) Method I (simplify using Laws of Log before integration):

$$\int_1^{\mathrm{e}} \frac{1}{x^2}\ln\left(\frac{1}{x^2}\right)\,\mathrm{d}x$$

$$=-2\int_1^{\mathrm{e}} x^{-2}\ln x\,\mathrm{d}x$$

$$=-2\left\{\left[-x^{-1}\ln x\right]_1^{\mathrm{e}}-\int_1^{\mathrm{e}} -x^{-1}\frac{1}{x}\,\mathrm{d}x\right\}$$

$$=-2\left\{\left[-\mathrm{e}^{-1}-0\right]-\int_1^{\mathrm{e}} x^{-2}\,\mathrm{d}x\right\}$$

$$=-2\left\{\mathrm{e}^{-1}-\left[-x^{-1}\right]_1^{\mathrm{e}}\right\}$$

$$=-2\left\{\mathrm{e}^{-1}-\left[-\mathrm{e}^{-1}+1\right]\right\}=-2\left\{2\mathrm{e}^{-1}-1\right\}$$

$$=4\mathrm{e}^{-1}-2$$

Method II (apply By Parts formula without simplification):

$$\int_1^{\mathrm{e}} \frac{1}{x^2}\ln\left(\frac{1}{x^2}\right)\,\mathrm{d}x$$

$$=\left[-\frac{1}{x}\ln\left(\frac{1}{x^2}\right)\right]_1^{\mathrm{e}}-\int_1^{\mathrm{e}} -\frac{1}{x}\left(\frac{1}{\frac{1}{x^2}}\right)\left(-\frac{2}{x^3}\right)\,\mathrm{d}x$$

$$=\left[-\frac{1}{\mathrm{e}}\ln\left(\frac{1}{\mathrm{e}^2}\right)+\ln 1\right]-\int_1^{\mathrm{e}} \frac{2}{x^2}\,\mathrm{d}x$$

$$=\left[-\frac{1}{\mathrm{e}}(-2)+0\right]-\int_1^{\mathrm{e}} \frac{2}{x^2}\,\mathrm{d}x$$

$$=\frac{2}{\mathrm{e}}-\left[-\frac{2}{x}\right]_1^{\mathrm{e}}$$

$$=\frac{2}{\mathrm{e}}-\left[-\frac{2}{\mathrm{e}}+2\right]$$

$$=\frac{4}{\mathrm{e}}-2$$

(b) $u = \sqrt{t} \quad \Rightarrow \quad t = u^2$

Diff. wrt u, $\dfrac{\mathrm{d}t}{\mathrm{d}u} = 2u$

$$\int \frac{\sqrt{t}}{t-1}\,\mathrm{d}t$$

$$= \int \frac{u}{u^2-1}(2u)\,\mathrm{d}u = 2\int \frac{u^2}{u^2-1}\,\mathrm{d}u$$

$$= 2\int \left(1 + \frac{1}{u^2-1}\right)\mathrm{d}u$$

$$= 2\left[u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right] + C$$

$$= 2\sqrt{t} + \ln\left|\frac{\sqrt{t}-1}{\sqrt{t}+1}\right| + C$$

13 It is given that $\mathrm{f}(x) = -x - 1 + \dfrac{k^2 - 1}{x - 1}$ where $k > 1$.

(i) Show by differentiation that the graph of $y = \mathrm{f}(x)$ has no turning points. **[3]**

(ii) On separate diagrams, draw sketches of the graphs of

(a) $y = \mathrm{f}(x)$, **[4]**

(b) $y = \mathrm{f}'(x)$. **[2]**

You should indicate where possible, numerically or in terms of *k*, any asymptotes and axial intercepts for each of the curves.

(iii) Find in terms of *k*, the range of *x* that satisfies the inequality

$$k\,\mathrm{f}(x) \le (x-k)^2 (x+k)$$ **[4]**

Solution:

(i) $\mathrm{f}(x) = -x - 1 + \dfrac{k^2 - 1}{x - 1} \Rightarrow \mathrm{f}'(x) = -1 - \dfrac{k^2 - 1}{(x-1)^2}$

Since $k > 1$, $\therefore k^2 - 1 < 0$

Since $(x-1)^2$ is also always > 0, $-1 - \dfrac{k^2 - 1}{(x-1)^2} < 0$

$\therefore \mathrm{f}'(x) \ne 0$ for all $x \in \mathbb{R}$

$\therefore$ $y = \mathrm{f}(x)$ has no turning points.

Hence $y = \mathrm{f}(x)$ has no turning point.

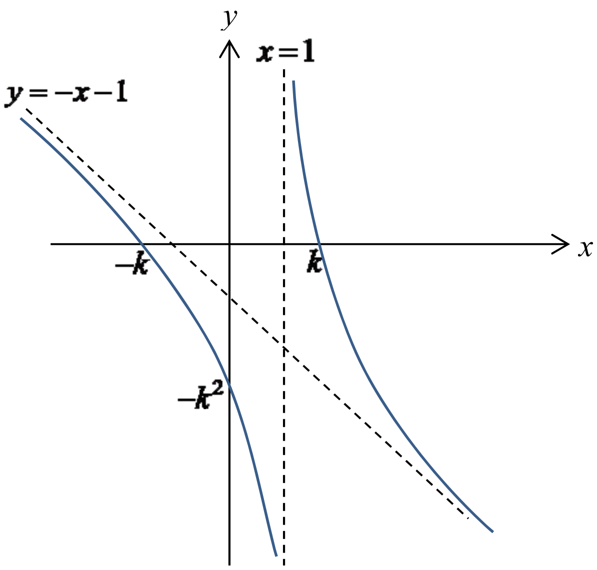
(ii)(a) When $x = 0$, $y=-1+\dfrac{k^2-1}{-1}=-k^2$

When $y = 0$, $-x-1+\dfrac{k^2-1}{x-1}=0$

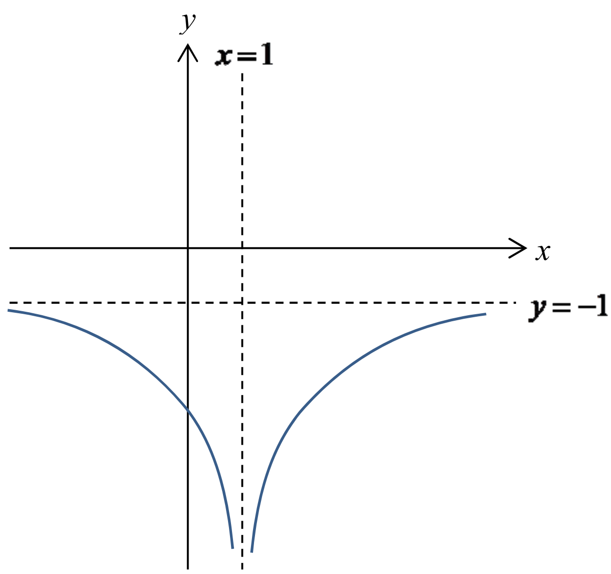
$k^2-1=(x+1)(x-1)$

$k^2-1=x^2-1$

$x=\pm k$



(ii)(b)



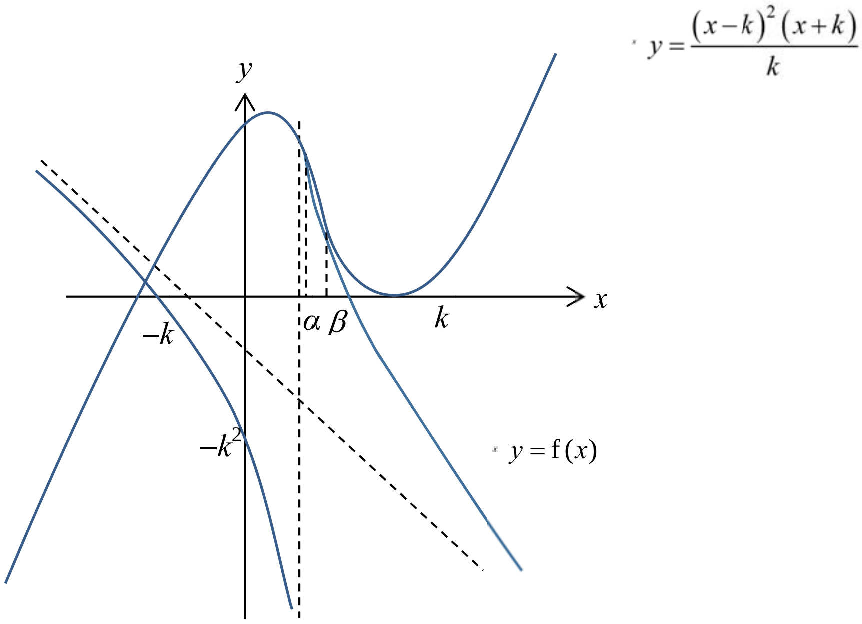
(iii)
Method 1:

$$k\,\mathrm{f}(x) \le (x-k)^2(x+k)$$

$$\Rightarrow \quad \mathrm{f}(x) \le \frac{(x-k)^2(x+k)}{k}$$

$\therefore$ Sketch the curves $y=\mathrm{f}(x)$ and $y=\dfrac{(x-k)^2(x+k)}{k}$

Case 1:



To find α and β, set

$$-x-1+\frac{k^2-1}{x-1}=\frac{(x-k)^2(x+k)}{k}$$

$$\Rightarrow \quad \frac{-x^2+k^2}{x-1}=\frac{(x-k)^2(x+k)}{k}$$

$$\Rightarrow \quad (x-k)(x+k)\left[\frac{(x-k)}{k}+\frac{1}{x-1}\right]=0$$

$$\Rightarrow \quad (x-k)(x+k)\left[\frac{x^2-(k+1)x+2k}{k(x-1)}\right]=0$$

$$\Rightarrow \quad x=\pm k \quad \text{or} \quad x^2-(k+1)x+2k=0$$

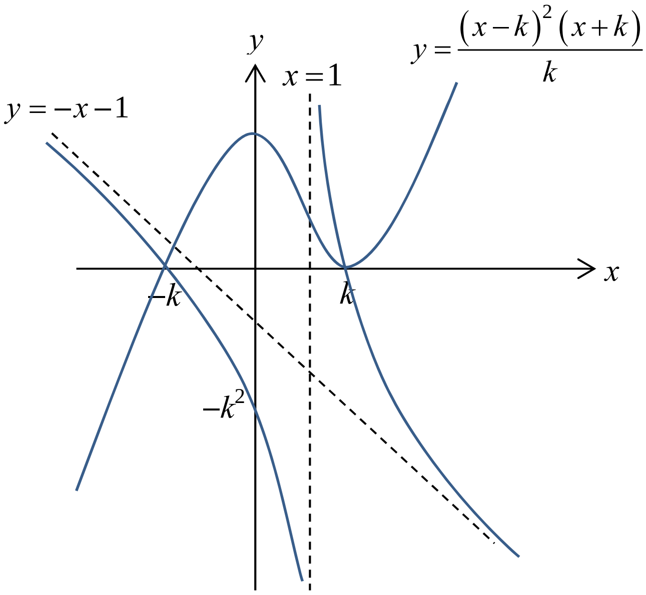
$$\Rightarrow \quad x=\pm k \quad \text{or} \quad x=\frac{(k+1)\pm\sqrt{(k+1)^2-8k}}{2}$$

$$\therefore \alpha=\frac{(k+1)-\sqrt{k^2-6k+1}}{2} \text{ and } \beta=\frac{(k+1)+\sqrt{k^2-6k+1}}{2}$$

$$\therefore -k \le x < 1 \text{ or } \frac{(k+1)-\sqrt{k^2-6k+1}}{2} \le x \le \frac{(k+1)+\sqrt{k^2-6k+1}}{2} \text{ or } x \ge k$$

This case is valid if $k^2-6k+1 \ge 0$, i.e. $(k-3)^2-8 \ge 0$, i.e. $k \ge 3+2\sqrt{2}$ (since $k>1$)

Case 2 ($1 < k < 3+2\sqrt{2}$):



From the diagram, we have

$$-k \le x < 1 \quad \text{or} \quad x \ge k\ .$$

Method 2:

$$k\left(-x-1+\frac{k^2-1}{x-1}\right)\le(x-k)^2(x+k)$$

$$k\left(\frac{-x^2+k^2}{x-1}\right)\le(x-k)^2(x+k)$$

$$(x-k)(x+k)\left(\frac{-k}{x-1}-(x-k)\right)\le 0$$

$$(x-k)(x+k)\left(\frac{x^2-(k+1)+2k}{x-1}\right)\ge 0$$

$$(x-k)(x+k)(x-1)\left(x^2-(k+1)+2k\right)\ge 0 \quad , \qquad x\ne 1$$

Case 1($\left(x^2-(k+1)+2k\right)$ can be factorized, i.e. when $(k+1)^2-4(1)(2k)\ge 0$,

i.e. $k^2-6k+1\ge 0$,

i.e. $k\ge\dfrac{6+\sqrt{36-4}}{2}$,

i.e. $k\ge 3+2\sqrt{2}$)

We have

$$(x-k)(x+k)(x-1)\left(x-\left(\frac{-(k+1)-\sqrt{(k+1)^2-8k}}{2}\right)\right)\left(x-\left(\frac{-(k+1)+\sqrt{(k+1)^2-8k}}{2}\right)\right)\ge 0$$

$$\therefore -k\le x<1 \text{ or } \frac{(k+1)-\sqrt{k^2-6k+1}}{2}\le x\le\frac{(k+1)+\sqrt{k^2-6k+1}}{2} \text{ or } x\ge k$$

Case 2 ($1<k<3+2\sqrt{2}$)

Since $\left(x^2-(k+1)+2k\right)>0$,

$\therefore (x-k)(x+k)(x-1)\ge 0$

$\therefore -k\le x<1$ or $x\ge k$

H

TEMASEK JUNIOR COLLEGE, SINGAPORE
JC One
Promotion Examination 2013
Higher 2

MATHEMATICS **9740**

4 October 2013

Additional Materials: Answer paper
List of Formulae (MF15)

3 hours

READ THESE INSTRUCTIONS FIRST

Write your *Civics Group* and *Name* on all the work that you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, fasten all your work securely together.

This document consists of 5 printed pages.

TEMASEK JUNIOR COLLEGE, SINGAPORE
PASSION PURPOSE DRIVE

[Turn over

1 Find the general solution of the following differential equation

$$\frac{1}{1+x}\frac{dy}{dx}+\frac{1}{1+x^2}=0, \quad \text{where } x\neq -1. \qquad \textbf{[4]}$$

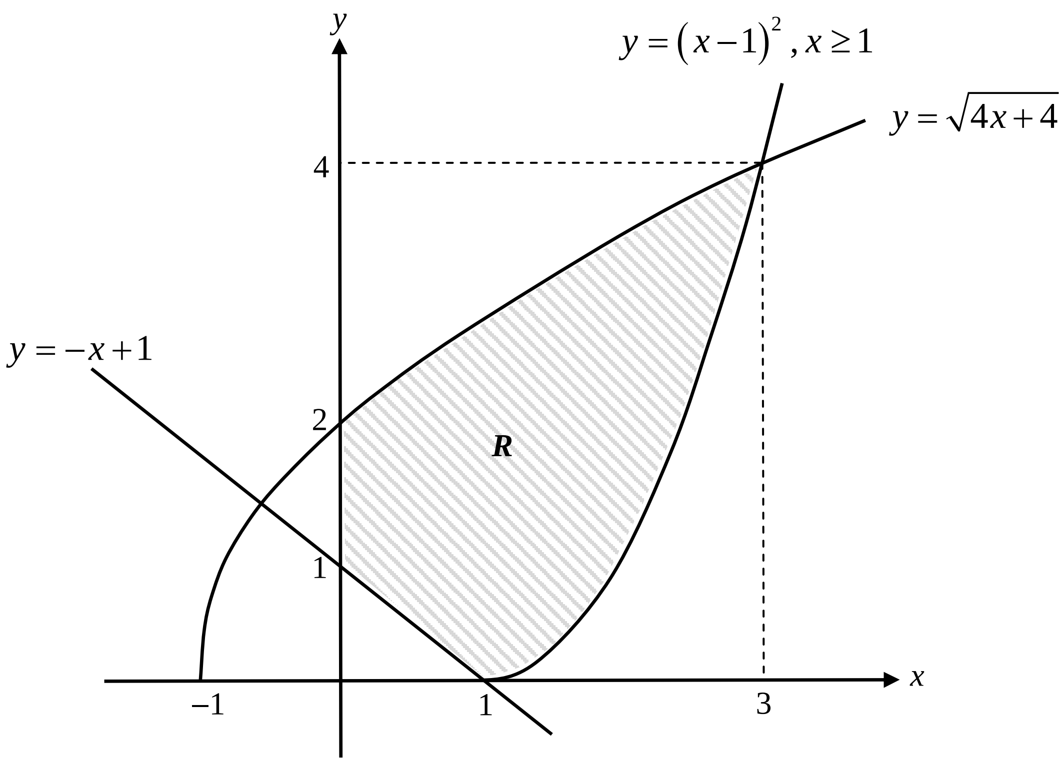
2 **(i)** The first three terms of a sequence are given by $u_1 = 19$, $u_2 = 34$, $u_3 = 52$. Given that u_n is a quadratic polynomial in *n*, find u_n in terms of *n*. **[4]**

(ii) Find the smallest value of *n* for which u_n is greater than 200. **[2]**

3 A wire of length *L* cm is cut into two pieces. One piece is used to form a circle while the other piece is used to form an equilateral triangle.
Show that, with the total area of the circle and triangle being the smallest, the ratio of the length of the smaller piece to the length of the bigger piece is $\frac{\sqrt{3}\pi}{9}$.

[6]

4



The shaded region *R* in the diagram above is bounded by the *y*-axis, the line $y=-x+1$ and the curves $y=(x-1)^2$ for $x\geq 1$ and $y=\sqrt{4x+4}$.

Find the volume of the solid of revolution formed when *R* is rotated completely about the *y*-axis. **[6]**

5 Given that $y = \ln\left(2 + \tan^{-1} x\right)$, show that

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \left(1+x^2\right)\left(\frac{dy}{dx}\right)^2 = 0 .$$ **[3]**

Hence find the Maclaurin's expansion for y, up to and including the term in x^2. **[3]**

6 Prove by mathematical induction $\sum_{r=1}^{n} \frac{r}{3^{r-1}} = \frac{9}{4} - \frac{1}{3^{n-1}}\left(\frac{3}{4} + \frac{n}{2}\right)$ for all positive integers of n. **[5]**

Hence show that

$$\frac{1}{4} + \frac{2}{4^2} + \frac{3}{4^3} + \frac{4}{4^4} \cdots < \frac{9}{16} .$$ **[2]**

7 Functions f and g are defined by

$$\text{f} : x \mapsto \frac{2x-2}{x-2}, \quad \text{for } x \in \mathbb{R}, x < 1,$$

$$\text{g} : x \mapsto \sqrt{2-x}, \quad \text{for } x \in \mathbb{R}, x \leq 2 .$$

(i) Given that f has an inverse, show that the composite function gf^{-1} exists. Find gf^{-1} and state its range. **[5]**

(ii) Find the value(s) of x such that $\text{f}(x) = \text{f}^{-1}(x)$. **[2]**

8 Prove that

$$\ln\left(\frac{(r-1)(r+2)}{r(r+1)}\right) \equiv \ln\left((r-1)(r)\right) - 2\ln\left((r)(r+1)\right) + \ln\left((r+1)(r+2)\right).$$ **[2]**

Hence, find in terms of n,

$$\ln\left(\frac{1\times4}{2\times3}\right) + \ln\left(\frac{2\times5}{3\times4}\right) + \ln\left(\frac{3\times6}{4\times5}\right) + \cdots + \ln\left(\frac{(n-1)(n+2)}{(n)(n+1)}\right) + \ln\left(\frac{(n)(n+3)}{(n+1)(n+2)}\right),$$

leaving your answer as a single logarithmic function. **[5]**

9 Jessie wishes to take up a loan of \$20,000 on the 1st day of the Year 2014. She intends to pay an instalment of \$300 on the 1st day of each month, beginning from February 2014. She sources out two banks, *XYZ* Bank and *ABC* Bank, which offer such loans. The two banks have different ways of charging interest. *XYZ* Bank charges a monthly interest of 0.5% on the outstanding amount owed at the end of each month, while *ABC* Bank charges a fixed interest of \$60 at the end of each month until the loan is repaid.

(a) If Jessie takes up the loan from *XYZ* Bank, show that the outstanding loan at the end of February 2014 after the interest has been added will be \$19899. **[2]**

Hence, find the number of months Jessie will take to repay her loan. **[4]**

(b) Which bank should Jessie take a loan from if she wishes to clear her loan as soon as possible? Justify your answers. **[3]**

10 A curve C is given parametrically by the equations

$$x = 2\cos^3\theta, \quad y = 2\sin^3\theta$$

where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

Show that the normal at the point with parameter θ has equation

$$y\sin\theta = x\cos\theta + 2\left(\sin^4\theta - \cos^4\theta\right).$$ **[4]**

The normal at the point Q where $\theta = \dfrac{\pi}{6}$, cuts C again at the point P, where $\theta = p$. Show that $\sin^3 p - \sqrt{3}\cos^3 p + 1 = 0$ and hence find the coordinates of P. **[5]**

11 A sequence of real numbers $x_1, x_2, x_3, ...$ satisfies the recurrence relation

$$x_{n+1} = \sqrt{\frac{2(x_n^2 - x_n)}{3} + 1}\,, \quad x_1 = k, \text{ where } k \geq 1.$$

(a) When $k = 5$, state the value of x_9 and describe the behavior of the sequence. **[2]**

(b) Prove algebraically that, if the sequence converges, then it converges to either 1 or 3. **[3]**

(c) State a value of k such that the sequence converges to 1. **[1]**

(d) When $k = 2$, state the integer m such that $m \leq x_n < m + 1$ for all integers $n \geq 1$. **[1]**

Hence, by considering $\dfrac{x_{n+1} - 1}{x_n - 1}$, show that $x_{n+1} > x_n$ for all integers $n \geq 1$. **[3]**

12 **(a)** Find $\int_1^e \frac{1}{x^2}\ln\left(\frac{1}{x^2}\right)\,\mathrm{d}x$, leaving your answer in exact form. **[4]**

(b) Using the substitution $u=\sqrt{t}$, find $\int \frac{\sqrt{t}}{t-1}\,\mathrm{d}t$. **[6]**

13 It is given that $\mathrm{f}(x)=-x-1+\frac{k^2-1}{x-1}$ where $k>1$.

(i) Show by differentiation that the graph of $y=\mathrm{f}(x)$ has no turning points. **[3]**

(ii) On separate diagrams, draw sketches of the graphs of

(a) $y=\mathrm{f}(x)$, **[4]**

(b) $y=\mathrm{f}'(x)$. **[2]**

You should indicate where possible, numerically or in terms of *k*, any asymptotes and axial intercepts for each of the curves.

(iii) Find in terms of *k*, the range of *x* that satisfies the inequality

$$k\,\mathrm{f}(x)\le(x-k)^2(x+k)$$ **[4]**

End of Paper

VICTORIA JUNIOR COLLEGE
PROMOTIONAL EXAMINATION

MATHEMATICS **9740**
(HIGHER 2)

Monday 8 am -11 am
23 September 2013 3 hours

Additional materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
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Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
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You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 6 printed pages

 VICTORIA JUNIOR COLLEGE **[Turn over**

1 A sequence with its first four terms given is shown below.

$$1,\ (1+2),\ \ (1+2+2^2),\ \ (1+2+2^2+2^3),\ \ \cdots$$

Show that the n^{th} term of this sequence is 2^n-1. **[2]**

Find the sum of the first n terms of the sequence. **[3]**

2 A sequence of positive real numbers $x_1, x_2, x_3, \ldots$ satisfies the relation

$$x_{n+1}=\frac{3-x_n}{2x_n+3} \quad \text{for } n \geq 1.$$

(i) Given that the sequence converges to α, find the exact value of α. **[3]**

(ii) By using a graphical approach, prove that

$$x_{n+1}>x_n \text{ if } 0<x_n<\alpha.$$

[2]

3 A curve is defined by the parametric equations

$$x=2at^2, \qquad y=3at,$$

where a is a non–zero constant.

Given that B is the point $\left(\frac{17a}{4},0\right)$, find the coordinates of the points on the curve which are nearest to B. **[5]**

4 **(i)** Given that $\mathrm{f}(r)=(r-1)r^2$, show that $\mathrm{f}(r+1)-\mathrm{f}(r)=r(3r+1)$. **[1]**

(ii) Use the method of differences to find $\sum_{r=1}^{N} r(3r+1)$ in terms of N. Hence find the limit of $\sum_{r=1}^{N}\frac{r(3r+1)}{N^3}$ as N approaches infinity. **[3]**

(iii) Use your first answer in **part (ii)** to find $\sum_{r=3}^{N}(r-1)(3r-2)$ in the form aN^3+bN^2+cN+d, where a, b, c and d are constants to be found. **[2]**

5 **(a)** **(i)** Prove that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x}{x^2+1}\right)=\dfrac{2}{\left(x^2+1\right)^2}-\dfrac{1}{x^2+1}$. **[2]**

(ii) Find the exact value of $\displaystyle\int_0^1 \frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x$. **[3]**

(b) Find the constant A such that $\dfrac{1}{1-\mathrm{e}^{2x}}=A+\dfrac{\mathrm{e}^{2x}}{1-\mathrm{e}^{2x}}$. Hence find $\displaystyle\int \frac{1}{1-\mathrm{e}^{2x}}\,\mathrm{d}x$. **[3]**

6 **(i)** Find the expansion of $\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{(1+x)^2}$ in ascending powers of x, up to and including the term in x^2. **[3]**

Let $y=\sin^{-1}(x)+\dfrac{1}{(1+x)}$.

(ii) By successively differentiating y, find the Maclaurin's series for y, up to and including the term in x^3. **[4]**

(iii) Show that the same result in **part (i)** can be obtained by using your answer in **part (ii)**. **[2]**

7 A sequence $u_0, u_1, u_2, \ldots$ is such that $u_0=b$ and $u_{n+1}=ru_n+a$, for all $n\geq 0$, where a, b and r are constants.

(a) For the case where $r\neq 1$,

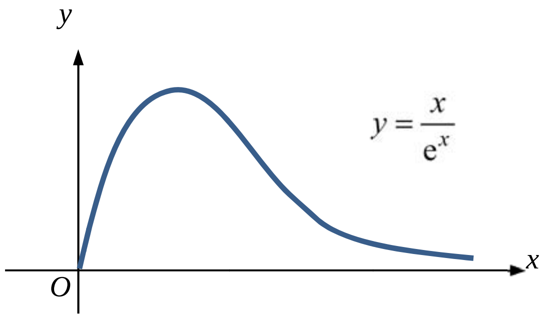
(i) prove by induction that $u_n=r^n b+a\dfrac{1-r^n}{1-r}$ for $n\geq 0$, **[4]**

(ii) write down the set of values of r for which the sequence $u_0, u_1, u_2, \ldots$ converges, and state the limit of this sequence. **[2]**

(b) For the case where $r=1$, find u_1, u_2, u_3, and hence find $\displaystyle\sum_{n=0}^{N} u_n$ in terms of a, b, N.
Give your answer in the form $\dfrac{N+1}{k_1}\left(k_2 b+Na\right)$, where k_1 and k_2 are integers to be determined. **[3]**

[Turn over

8



The above diagram shows a sketch of the curve C with equation $y = \dfrac{x}{e^x}$, $x \geq 0$.

(a) **(i)** Find the exact coordinates of the maximum point on C. **[3]**

(ii) Hence show that $\ln x \leq x - 1$ for all $x > 0$. **[2]**

(b) A particle is constrained to move along C, starting from the origin O, such that its x-coordinate increases at a constant rate. The particle took 2 seconds to reach the point $\left(4, \dfrac{4}{e^4}\right)$. When it is at the point $\left(a, \dfrac{a}{e^a}\right)$, the y-coordinate of the particle is decreasing at a rate of 0.25 unit per second. Find a given that $a < 2$. **[4]**

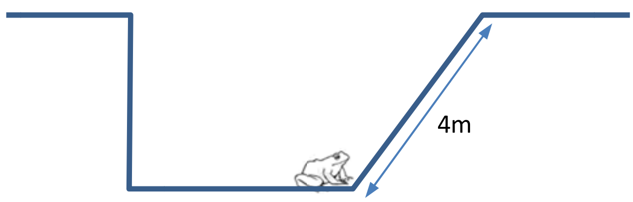
9 **(a)** The sum, S_{n-1}, of the first $n - 1$ terms of a sequence $u_1, u_2, u_3, \ldots$ is given by

$$S_{n-1} = 8n^2 - 19n + 11 .$$

(i) Find u_n and show that the sequence is an arithmetic progression. **[4]**

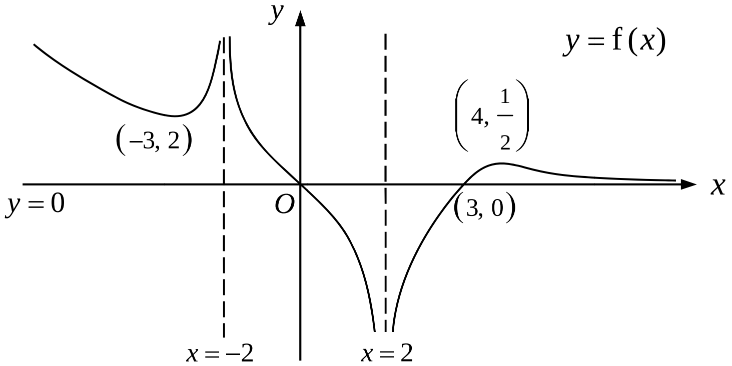
(ii) Find the least value of n, such that sum of the first n terms is at least 4000 less than the sum of the next n terms. **[3]**

(b)



A frog falls into a muddy drain with a slant wall measuring 4m in length. It tries to escape from the drain by leaping successively on the slant wall. Though it can cover 0.7 m in its first leap, the wall is so slippery that for subsequent attempts it can only cover 4/5 the distance of its previous leap. Determine if the frog will be able to escape form the drain, justifying your answer. **[3]**

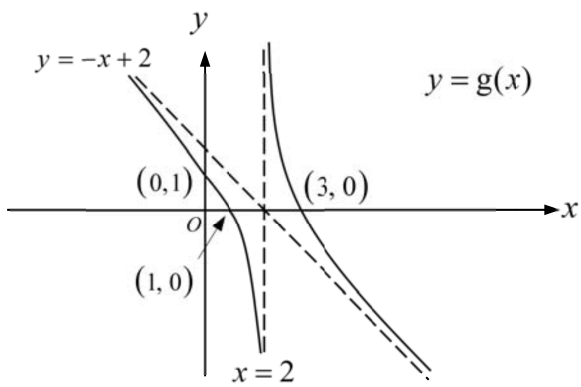
10 **(i)**



The diagram above shows the graph of $y = f(x)$. It has a non-stationary point of inflexion $(0, 0)$, an intersection with the x-axis at $(3, 0)$, a minimum point $(-3, 2)$ and a maximum point $\left(4, \frac{1}{2}\right)$. The vertical asymptotes of the graph are $x = -2$ and $x = 2$. The horizontal asymptote is $y = 0$.

Sketch the graph of $y = \sqrt{f(2x)}$, making clear the main relevant features and the shape of the graph near the points where $y = 0$. **[3]**

(ii)



The diagram above shows the graph of $y = g(x)$. The intersections of the graph with the axes have coordinates $(0,1)$, $(1,0)$ and $(3,0)$. The asymptotes of the graph are the lines $x = 2$ and $y = -x + 2$.

Sketch the graph of $y = g'(x)$, making clear the main relevant features. **[3]**

(iii) The function h is defined as

$$h(x) = \begin{cases} g(x) & \text{for} \quad x \le 2, \\ f(x) & \text{for} \quad x > 2. \end{cases}$$

Sketch the graphs of

(a) $y = h(x)$, **[1]**

(b) $y = \dfrac{1}{h(x)}$, making clear the main relevant features. **[4]**

[Turn over

11 The function f is defined as follows.

$$\mathrm{f} : x \mapsto x - \frac{4}{x} \text{ for } x \in \mathbb{R},\ x < 0.$$

(i) Find $\mathrm{f}^{-1}(x)$. **[3]**

(ii) Show that $\mathrm{f}'(x) > 0$. **[1]**

(iii) Solve the inequality $\mathrm{f}^{-1}(x) < -6$, giving your answer in exact form. **[2]**

(iv) Sketch the graph of $y = \mathrm{f}^{-1}\mathrm{f}(x)$. **[1]**

Functions h and g are defined by

$$\mathrm{h} : x \mapsto x - \frac{4}{x} \qquad \text{for } x \in \mathbb{R},\ x \neq -2,\ x \neq 0,\ x \neq 2,$$

$$\mathrm{g} : x \mapsto \frac{1}{x} - 1 \qquad \text{for } x \in \mathbb{R},\ x \neq 0.$$

(v) Show that $\mathrm{gh}(x) = -\frac{(x^2 - x - 4)}{(x^2 - 4)}$. **[1]**

(vi) Solve the inequality $\mathrm{gh}(x) \geq 0$, giving your answer in an exact form. **[3]**

12 The curve C_1 has equation $\frac{(x-1)^2}{4} = \frac{y^2}{9} + 4$.

Sketch C_1, making clear the main relevant features, and state the set of values that x can take. **[4]**

Another curve C_2 is defined by the parametric equations

$$x = \frac{2}{t^2 + 1}, \quad y = 3\sqrt{t}\ln t, \quad \text{where } t > 1.$$

Use a non-graphical method to determine the set of possible values of x. **[2]**

Sketch the curve C_2, labelling all axial intercepts and asymptotes (if any) clearly. **[2]**

Hence, **without solving the equation**, state the number of real roots to the equation

$$9\left(\frac{2}{t^2+1} - 1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144,$$

explaining your reason(s) clearly. **[2]**

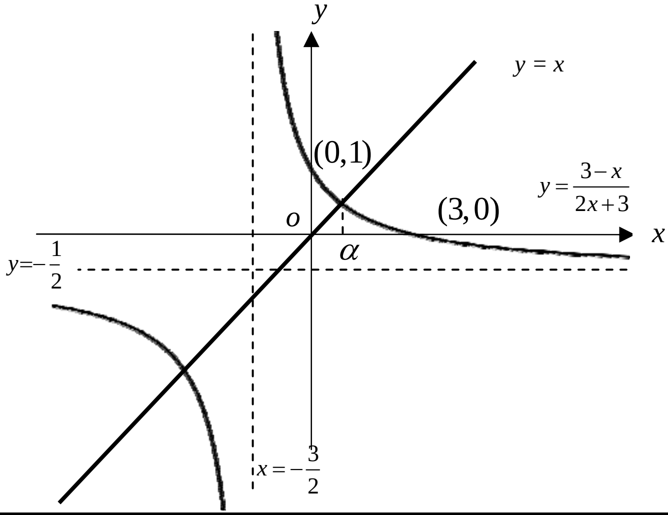
Given that $k > 0$, state the smallest integer value of k such that the equation

$$9\left(\frac{2}{t^2+1} + k - 1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144$$

has exactly one real root which is positive. **[2]**

Victoria Junior College

Mathematics H2 (9740) – JC 1 Promotional Examination 2013

Solutions	
1. (i) The nth term $=1+2+2^2+...+2^{n-1}$ $=\dfrac{1-2^n}{1-2}$ $=2^n-1$ (ii) $S_n=\sum_{r=1}^{n}\left(2^r-1\right)=\sum_{r=1}^{n}2^r-\sum_{r=1}^{n}1$ $=\dfrac{2(1-2^n)}{1-2}-n$ $=2^{n+1}-n-2$	
2. (i) As $n\to\infty$, $x_n\to\alpha$ and $x_{n+1}\to\alpha$. $\alpha=\dfrac{3-\alpha}{2\alpha+3}$ $2\alpha^2+4\alpha-3=0$ $\alpha=\dfrac{-4\pm\sqrt{16+24}}{4}$ $\alpha=-1\pm\frac{1}{2}\sqrt{10}$ Since $x_n>0$ for all n, $\alpha=-1+\frac{1}{2}\sqrt{10}$. (ii) Sketch $y=\dfrac{3-x}{2x+3}=-\dfrac{1}{2}+\dfrac{9}{2(2x+3)}$ and $y=x$. 	

2. When $0 < x < \alpha$, the graph of $y = \dfrac{3-x}{2x+3}$ is above the graph of $y = x$. $\quad \therefore \quad \dfrac{3-x}{2x+3} > x$.

Hence for $0 < x_n < \alpha$, $\quad \dfrac{3-x_n}{2x_n+3} > x_n$

$$\Rightarrow \quad x_{n+1} > x_n.$$

3 (i) Let A be a point on the curve.

$$AB^2 = \left(\frac{17a}{4} - 2at^2\right)^2 + \left(0 - 3at\right)^2$$

$$= \frac{289a^2}{16} + 4a^2t^4 - 17a^2t^2 + 9a^2t^2$$

$$= 4a^2t^4 - 8a^2t^2 + \frac{289a^2}{16}$$

$$AB = \sqrt{4a^2t^4 - 8a^2t^2 + \frac{289a^2}{16}}$$

Let $S = AB$.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{16a^2t^3 - 16a^2t}{2\sqrt{4a^2t^4 - 8a^2t^2 + \frac{289a^2}{16}}}$$

Let $\dfrac{\mathrm{d}S}{\mathrm{d}t} = 0$, then

$$\frac{16a^2t^3 - 16a^2t}{2\sqrt{4a^2t^4 - 8a^2t^2 + \frac{289a^2}{16}}} = 0$$

$$16a^2t^3 - 16a^2t = 0 \Rightarrow t(t^2 - 1) = 0$$

$\Rightarrow t = 0$ or $t = 1$ or $t = -1$

At $t = 0$, $S = AB = \dfrac{17a}{4}$.

At $t = \pm 1$, $S = AB = \dfrac{15a}{4}$ (nearer)

Hence, substitute $t = \pm 1$ (which correspond to points nearest to B) into x and y.

The coordinates are: ($2a$, $3a$) and ($2a$, -$3a$).

4. (i) $$\begin{aligned}&\mathrm{f}(r+1)-\mathrm{f}(r)\\&=r(r+1)^2-(r-1)r^2\\&=r\left[(r+1)^2-(r-1)r\right]\\&=r\left(r^2+2r+1-r^2+r\right)\\&=r(3r+1)\end{aligned}$$ (ii) $\sum_{r=1}^{N} r(3r+1)$ $$\begin{aligned}&=\sum_{r=1}^{N}\left(\mathrm{f}(r+1)-\mathrm{f}(\mathrm{r})\right)\\&=\ \mathrm{f}(2)-\mathrm{f}(1)+\\&\quad\ \mathrm{f}(3)-\mathrm{f}(2)+\\&\qquad\vdots\\&\quad\ \mathrm{f}(N)-\mathrm{f}(N-1)+\\&\quad\ \mathrm{f}(N+1)-\mathrm{f}(N)+\\&=\ \mathrm{f}(N+1)-\mathrm{f}(1)\\&=\ N(N+1)^2-0\\&=\ N(N+1)^2\end{aligned}$$ $$\sum_{r=1}^{N}\frac{r(3r+1)}{N^3}=\frac{N(N+1)^2}{N^3}=\left(\frac{N+1}{N}\right)^2=\left(1+\frac{1}{N}\right)^2.$$ As $N\to\infty$, $\frac{1}{N}\to 0$. $\therefore$ the limit of $\sum_{r=1}^{N}\frac{r(3r+1)}{N^3}$ is 1. (iii) $\sum_{r=3}^{N}(r-1)(3r-2)$ $$=2\times 7+3\times 10+...+(N-1)(3N-2)$$ $$\sum_{r=1}^{N} r(3r+1)$$ $$=1\times 4+\left[2\times 7+...+(N-1)(3N-2)\right]+N(3N+1)$$ $$\begin{aligned}\therefore\sum_{r=3}^{N}(r-1)(3r-2)&=\sum_{r=1}^{N} r(3r+1)-4-N(3N+1)\\&=N(N+1)^2-4-N(3N+1)\\&=N^3+2N^2+N-4-3N^2-N\\&=N^3-N^2-4\end{aligned}$$	

5. (a)(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x}{x^2+1}\right)=\dfrac{x^2+1-x(2x)}{\left(x^2+1\right)^2}$

$$=\frac{1-x^2}{\left(x^2+1\right)^2}$$

$$=\frac{2-1-x^2}{\left(x^2+1\right)^2}$$

$$=\frac{2}{\left(x^2+1\right)^2}-\frac{1+x^2}{\left(x^2+1\right)^2}$$

$$=\frac{2}{\left(x^2+1\right)^2}-\frac{1}{x^2+1}$$

(ii) $\displaystyle\int_0^1\left[\frac{2}{\left(x^2+1\right)^2}-\frac{1}{x^2+1}\right]\mathrm{d}x=\left[\frac{x}{x^2+1}\right]_0^1$

$$2\int_0^1\frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x-\left[\tan^{-1}x\right]_0^1=\frac{1}{2}$$

$$2\int_0^1\frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x=\frac{1}{2}+\frac{\pi}{4}$$

$$\int_0^1\frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x=\frac{1}{4}+\frac{\pi}{8}$$

(b) RHS $= A+\dfrac{e^{2x}}{1-e^{2x}}$

$$=\frac{A-Ae^{2x}+e^{2x}}{1-e^{2x}}$$

Comparing the numerator to that of the LHS,

$A-Ae^{2x}+e^{2x}=1$

$\Rightarrow A=1$

$$\int\frac{1}{1-e^{2x}}\mathrm{d}x=\int\left(1+\frac{e^{2x}}{1-e^{2x}}\right)\mathrm{d}x$$

$$=x-\frac{1}{2}\ln\left|1-e^{2x}\right|+C$$

6 (i)

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{(1+x)^2}=(1-x^2)^{-1/2}-(1+x)^{-2}$$

$$=\left(1+\frac{1}{2}x^2+\cdots\right)-\left(1-2x+\frac{(-2)(-3)}{2!}x^2+\cdots\right)$$

$$=2x-\frac{5}{2}x^2+\cdots$$

(ii) $$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-(1+x)^{-2}$$

$$\frac{d^2y}{dx^2}=\left(-\frac{1}{2}\right)(1-x^2)^{-3/2}(-2x)+2(1+x)^{-3}$$

$$=x(1-x^2)^{-3/2}+2(1+x)^{-3}$$

$$\frac{d^3y}{dx^3}=(1-x^2)^{-3/2}+x\left(-\frac{3}{2}\right)(1-x^2)^{-5/2}(-2x)-6(1+x)^{-4}$$

When $x = 0$,

$$y=1$$

$$\frac{dy}{dx}=1-1=0$$

$$\frac{d^2y}{dx^2}=0+2=2$$

$$\frac{d^3y}{dx^3}=1+0-6=-5$$

Hence, $y=1+x^2-\frac{5}{6}x^3+\cdots$

(iii) $y=\sin^{-1}(x)+\frac{1}{(1+x)}=1+x^2-\frac{5}{6}x^3+\cdots$

Differentiating both sides w.r.t x,

$\frac{1}{\sqrt{1-x^2}}-\frac{1}{(1+x)^2}=2x-\frac{5}{2}x^2+\cdots$ (verified).

7. (a)(i)	
Let P_n be the statement: $u_n = r^n b + a\dfrac{1-r^n}{1-r}$ for $n \geq 0$. Consider P_0 : L.H.S. of $P_0 = u_0 = b$ R.H.S. of $P_0 = r^0 b + a\dfrac{1-r^0}{1-r} = b$ $\therefore$ P_0 is true. Assume P_k is true for some $k \geq 0$. i.e. $u_k = r^k b + a\dfrac{1-r^k}{1-r}$. Consider P_{k+1} : R.H.S. of P_{k+1} = $r^{k+1} b + a\dfrac{1-r^{k+1}}{1-r}$ L.H.S. of P_{k+1} = u_{k+1} $= ru_k + a$ $= r\left(r^k b + a\dfrac{1-r^k}{1-r}\right) + a$ $= r^{k+1} b + \dfrac{ar\left(1-r^k\right)}{1-r} + \dfrac{a(1-r)}{1-r}$ $= r^{k+1} b + \dfrac{ar - ar^{k+1} + a - ar}{1-r}$ $= r^{k+1} b + \dfrac{a\left(1-r^{k+1}\right)}{1-r}$ $\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true. Hence, $\begin{cases} P_0 \text{ is true} \\ P_k \text{ is true} \Rightarrow P_{k+1} \text{ is true.} \end{cases}$ By induction, $u_n = r^n b + a\dfrac{1-r^n}{1-r}$ for $n \geq 0$.	

7(ii) The sequence converges for $\{r \in \mathbb{R} : -1 < r < 1\}$. The limit of the sequence is $\dfrac{a}{1-r}$. (b) $u_0 = b$ $u_1 = b + a$ $u_2 = b + 2a$ $u_3 = b + 3a$ $\vdots \quad \vdots$ $u_N = b + Na$ $\therefore \sum_{n=0}^{N} u_n = (N+1)b + \dfrac{N}{2}(a + Na)$ $= (N+1)b + \dfrac{N}{2}(1+N)a$ $= \dfrac{N+1}{2}(2b + Na)$	
8(a)(i) $y = \dfrac{x}{e^x}$ $\dfrac{dy}{dx} = \dfrac{e^x - xe^x}{e^{2x}}$ $= \dfrac{1-x}{e^x}$ $\dfrac{dy}{dx} = 0 \Rightarrow x = 1$ Substitute $x = 1$ into y. Maximum point is $\left(1, \dfrac{1}{e}\right)$. (ii) For $x > 0$, $y \le \dfrac{1}{e}$ i.e. $\dfrac{x}{e^x} \le \dfrac{1}{e}$ Since ln is an increasing function, $\ln\left(\dfrac{x}{e^x}\right) \le \ln\left(e^{-1}\right)$ $\Rightarrow \ln x - \ln e^x \le -1$ $\Rightarrow \ln x - x \le -1$ $\Rightarrow \ln x \le x - 1$	

8(*b*) The particle took 2 seconds to move from $x=0$ to $x=4$,

so $\dfrac{dx}{dt}=2$.

At $x=a$,

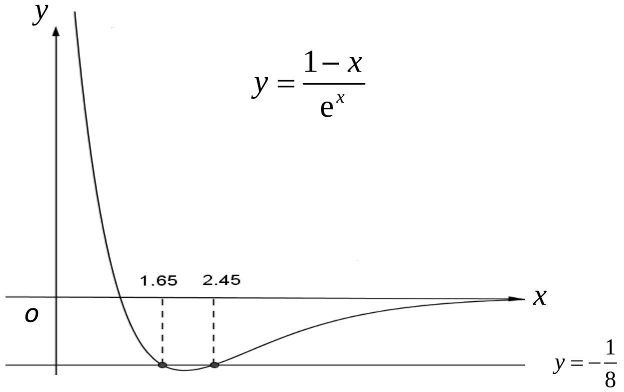
$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$
$$=-0.25\times\frac{1}{2}=-\frac{1}{8}$$

At $\left(a,\dfrac{a}{e^a}\right), \dfrac{dy}{dx}=\dfrac{1-a}{e^a}$

$$\therefore \frac{1-a}{e^a}=-\frac{1}{8}$$

$y=\dfrac{1-x}{e^x}$

From GC, $a=1.65$ (reject 2.45 as $a<2$).



$\mathbf{9}(\mathrm{a})(\mathrm{i})$ Replacing n with $n+1$,

$$S_n=8(n+1)^2-19(n+1)+11$$
$$=8n^2+16n+8-19n-19+11$$
$$=8n^2-3n$$

$$u_n=S_n-S_{n-1}$$
$$=\left(8n^2-3n\right)-\left(8n^2-19n+11\right)$$
$$=16n-11$$
$$u_n-u_{n-1}=\left(16n-11\right)-\left(16(n-1)-11\right)$$
$$=16$$

Since the difference between 2 consecutive terms is a constant, the sequence is an AP.

(ii) $\left(S_{2n} - S_n\right) - S_n \geq 4000$

$$\left(8(2n)^2 - 3(2n)\right) - 2\left(8n^2 - 3n\right) \geq 4000$$

$$32n^2 - 6n - 16n^2 + 6n \geq 4000$$

$$n^2 \geq 250$$

$\Rightarrow n \leq -15.8$ (reject as $n \in \mathbb{Z}^+$) *or* $n \geq 15.8$

Thus, least n is 16.

(b) The distance covered by frog is a GP with a =0.7 and r = 0.8

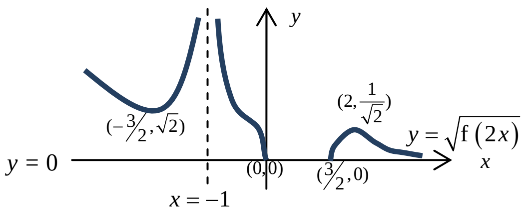
Total distance covered after n leaps is given by

$$S_n = \frac{0.7\left(1-0.8^n\right)}{1-0.8}$$
$$= 3.5\left(1-0.8^n\right)$$

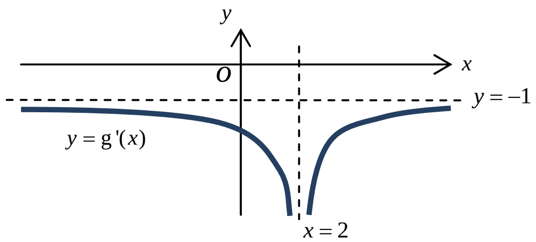
As $n \to \infty$, $(0.8)^n \to 0 \quad \Rightarrow S_n \to 3.5$, that is, $S_\infty = 3.5$

Since $S_\infty < 4$, the frog will never be able to escape from the drain.

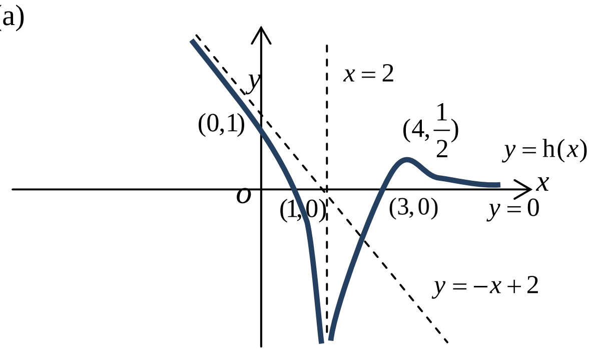
10 (i)



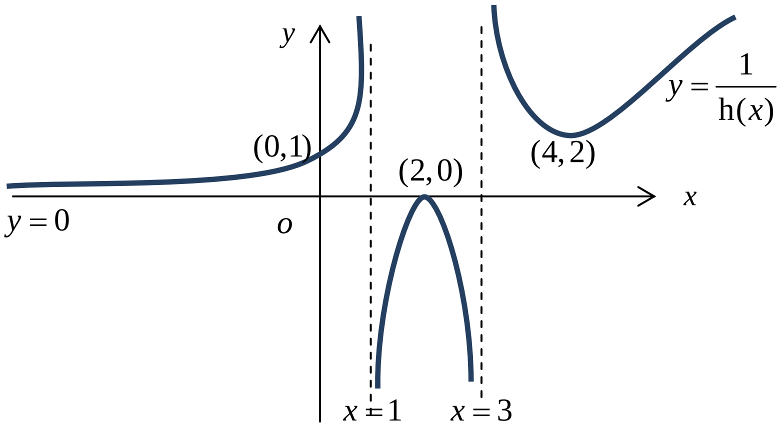
(ii)



(iii) (a)



10(b)



11 (i) $y = x - \frac{4}{x} \Rightarrow y = \frac{x^2-4}{x}$

$x^2 - xy - 4 = 0$

$x = \frac{y \pm \sqrt{y^2+16}}{2}$

Since $x < 0$, $x = \frac{y - \sqrt{y^2+16}}{2}$

$\Rightarrow \mathrm{f}^{-1}(y) = \frac{1}{2}y - \frac{1}{2}\sqrt{y^2+16} \Rightarrow \mathrm{f}^{-1}(x) = \frac{1}{2}x - \frac{1}{2}\sqrt{x^2+16}$.

(ii) $\mathrm{f}'(x) = 1 + \frac{4}{x^2}$. Since $\frac{4}{x^2} > 0$ for all real $x<0$, $\mathrm{f}'(x) > 1$

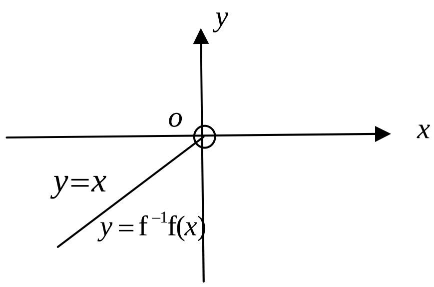
Hence $\mathrm{f}'(x) > 0$.

(iii) Since f is an increasing function,

$\mathrm{f}^{-1}(x) < -6 \Rightarrow \mathrm{f}\left(\mathrm{f}^{-1}(x)\right) < \mathrm{f}(-6)$

$x < -6 - \frac{4}{-6} \Rightarrow x < -\frac{16}{3}$

(iv)

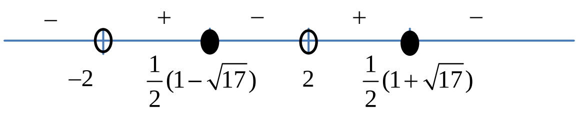


(v) $\mathrm{gh}(x) = \mathrm{g}\left[\mathrm{h}(x)\right] = \frac{1}{\frac{x^2-4}{x}} - 1$

$= \frac{x}{x^2-4} - 1 = \frac{x-(x^2-4)}{x^2-4} \quad = -\frac{(x^2-x-4)}{x^2-4}$

11(vi) **Test Point method:**

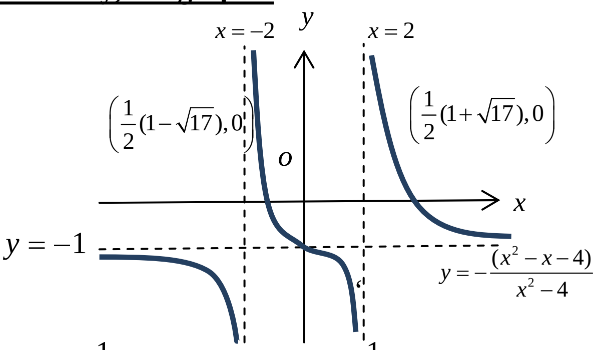
$$x^2 - x - 4 = 0 \Rightarrow x = \frac{1}{2}\left(1 \pm \sqrt{17}\right)$$



Sign of $-\dfrac{(x^2 - x - 4)}{x^2 - 4}$

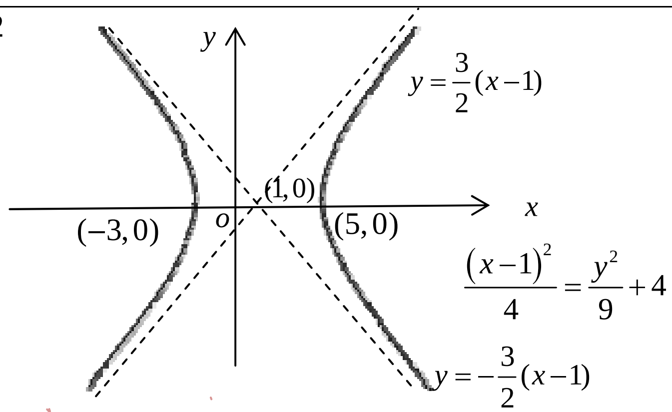
$$\therefore -2 < x \le \frac{1}{2}(1-\sqrt{17}) \text{ or } 2 < x \le \frac{1}{2}(1+\sqrt{17})$$

Alternatively, use graphs:



$$\therefore -2 < x \le \frac{1}{2}(1-\sqrt{17}) \text{ or } 2 < x \le \frac{1}{2}(1+\sqrt{17})$$

12



$$\frac{(x-1)^2}{4} = \frac{y^2}{9} + 4 \Rightarrow \frac{(x-1)^2}{4^2} - \frac{y^2}{6^2} = 1$$

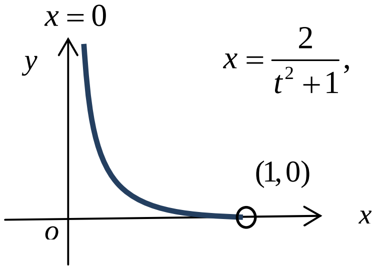
$\therefore$ the set of values of x = $\{x \in \mathbb{R} : x \le -3 \text{ or } x \ge 5\}$

$$t^2 > 1 \Rightarrow t^2 + 1 > 2 \Rightarrow 0 < \frac{1}{t^2+1} < \frac{1}{2}$$

$0 < \dfrac{2}{t^2+1} < 1$, that is, $0 < x < 1$

$\therefore$ the set of values of x = $\{x \in \mathbb{R} : 0 < x < 1\}$

12



$$x = \frac{2}{t^2+1}, \quad y = 3\sqrt{t}\ln t, \quad t > 1.$$

$$9\left(\frac{2}{t^2+1}-1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144 \quad - \quad (1)$$

$$\frac{\left(\frac{2}{t^2+1}-1\right)^2}{4} = \frac{\left(3\sqrt{t}\ln t\right)^2}{9} + 4$$

Since $C_1 : \frac{(x-1)^2}{4} = \frac{y^2}{9} + 4$ and $C_2 : x = \frac{2}{t^2+1}, \quad y = 3\sqrt{t}\ln t,$

the number of roots of the above equation can then be found by the number of intersections between C_1 and C_2. However, since C_1 is only defined for $x \le -3$ or $x \ge 5$ and C_2 is defined for $0 < x < 1$, there is no point of intersection.

Hence $9\left(\frac{2}{t^2+1}-1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144$ has no real root.

$$9\left(\frac{2}{t^2+1}+k-1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144$$

Since x is replaced with $x + k$ in the equation of C_1, C_1 is translated k units in the negative x-direction. Hence smallest integer value of k is 5.

OR

Since x is replaced with $x - k$ in the equation of C_2, C_2 is translated k units in the positive x-direction. Hence smallest integer value of k is 5.

ANGLO-CHINESE JUNIOR COLLEGE
MATHEMATICS DEPARTMENT

MATHEMATICS
Higher 2
Paper 1

9740

8 October 2013

JC 1 PROMOTIONAL EXAMINATION

Time allowed**: 3 hours**

Additional Materials: List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Index number, Form Class, graphic and/or scientific calculator model/s on the cover page.
Write your Index number and full name on all the work you hand in.
Write in dark blue or black pen on your answer scripts.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in the question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together

This document consists of **7** printed pages.

Anglo-Chinese Junior College

ANGLO-CHINESE JUNIOR COLLEGE
MATHEMATICS DEPARTMENT
JC 1 Promotional Examination 2013

MATHEMATICS 9740
Higher 2
Paper 1

/ 100

Index No: | | | | |

Form Class: ____________

Name: ____________________________

Calculator model: ____________________

Arrange your answers in the same numerical order.

Place this cover sheet on top of them and tie them together with the string provided.

Question no.	Marks
1	/ 2
2	/ 3
3	/ 4
4	/ 4
5	/ 5
6	/ 6
7	/ 6
8	/ 7
9	/ 6
10	/ 9
11	/ 9
12	/ 11
13	/ 13
14	/ 9
15	/ 6

1 The graph of $y=\mathrm{f}(x)$ undergoes, in succession, the following transformations:

Step 1: a translation of 1 unit in the negative y-direction; followed by

Step 2: a stretch with scale factor 2 parallel to the x-axis.

The equation of the resulting curve is $y=\ln(2x+3)$, $x>-\dfrac{3}{2}$. Determine the equation of the graph, $y=\mathrm{f}(x)$. **[2]**

2 Given that the curve $y=ax^3+bx^2+cx+d$ has turning points at (-4, 258) and (4, 2). Write and solve a system of simultaneous linear equations satisfied by the constants a, b, c and d. **[3]**

3 Differentiate the following with respect to x.

(i) $\sqrt{\cos^{-1}\left(\dfrac{x}{2}\right)}$, **[2]**

(ii) $\ln\sqrt{\dfrac{(x+1)^3}{x^2-1}}$. **[2]**

4 Find the following integrals:

(i) $\displaystyle\int \frac{1}{x\sqrt{\ln x}}\,dx$; **[2]**

(ii) $\displaystyle\int \frac{e^{-2x}}{\sqrt{4-e^{-4x}}}\,dx$. **[2]**

5 Without the use of a graphing calculator, solve the inequality $\dfrac{3x^2+6x-10}{x^2+3x-4}\geq 2$. **[3]**

Deduce the range of values of x such that $\dfrac{3x^2+6|x|-10}{x^2+3|x|-4}\geq 2$. **[2]**

[Turn Over

6 A curve C has parametric equations

$$x = 1 - cos\theta, \qquad y = \theta + \sin\theta,$$

where $0 \le \theta \le 2\pi$,

(i) Show that $\frac{dy}{dx} = \cot\frac{1}{2}\theta$ and find the gradient of C at the point P where $\theta = \pi$. **[3]**

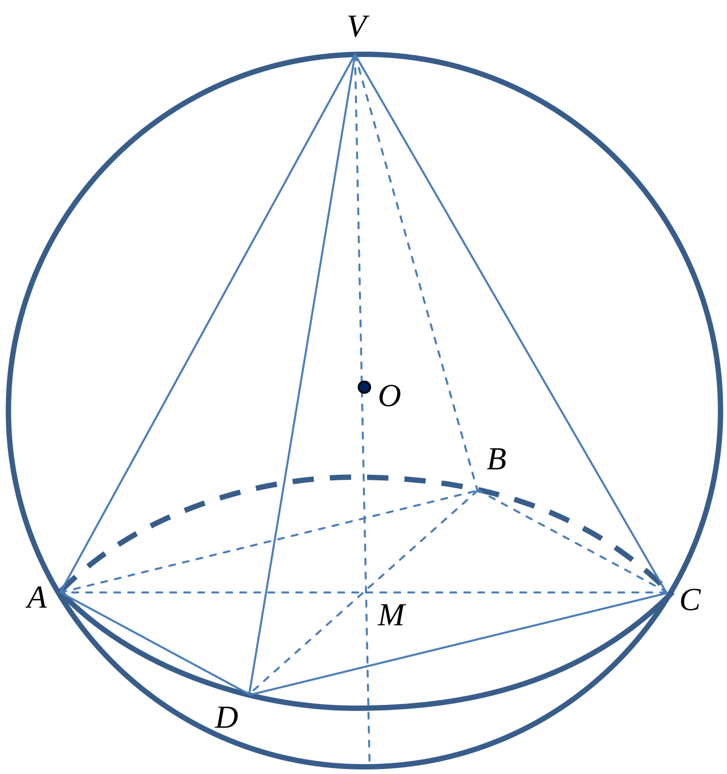
(ii) The tangent at P meets the y-axis at A. The tangent at the point Q, where $\theta = \frac{\pi}{2}$, meets the y-axis at B. Find the area of triangle ABP. **[3]**

7 A right pyramid block has a square base $ABCD$ and its vertical height VM is $(a+x)$ where $0 < x < a$. M is the point where the diagonals AC and BD of the square meet. This right pyramid block is inscribed in a sphere of fixed radius a so that the vertices V, A, B, C and D of the block just touch the interior of the sphere with the vertical height VM passing through the centre O of the sphere.

(i) Show that the length of the side of the square base $ABCD$ is $\sqrt{2(a^2 - x^2)}$. **[2]**

(ii) Hence, find the maximum volume of the block in terms of a. **[4]**

[Volume of a pyramid = $\frac{1}{3} \times$ base area $\times$ height]



8 The function f is defined by $\mathrm{f} : x \mapsto x + \dfrac{1}{x}$ for $x \in \mathbb{R},\ x \ge 1$.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. **[3]**

(ii) Find $\mathrm{fff}^{-1}(x)$ and state its domain and range. **[3]**

(iii) Show that the composite function f^2 exists. **[1]**

9 If $f(k) = \dfrac{1}{k^2}$, show that $f(k) - f(k+2) = \dfrac{4(k+1)}{k^2(k+2)^2}$. **[1]**

Hence, show that the sum to n terms of the series $\dfrac{2}{(1^2)(3^2)} + \dfrac{3}{(2^2)(4^2)} + \dfrac{4}{(3^2)(5^2)} + \dots$ is

$\dfrac{1}{4}\left(\dfrac{5}{4} - \dfrac{1}{(n+1)^2} - \dfrac{1}{(n+2)^2}\right)$. **[3]**

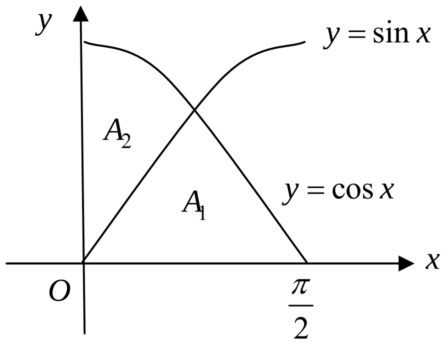
Show that $\displaystyle\sum_{k=2}^{n} \frac{k+1}{k^2(k+2)^2} < \frac{13}{144}$ for all values of $n \ge 2$. **[2]**

10 **(a)** Use integration by parts to find the exact value of $\int_1^e (\ln x)^2\, dx$. **[4]**

(b) By means of the substitution $x = 3\cos^2\theta + 7\sin^2\theta$, where $0 \le \theta \le \dfrac{\pi}{2}$, prove that

$\displaystyle\int_3^7 \frac{1}{\sqrt{[(7-x)(x-3)]}}\, dx = \pi$. **[5]**

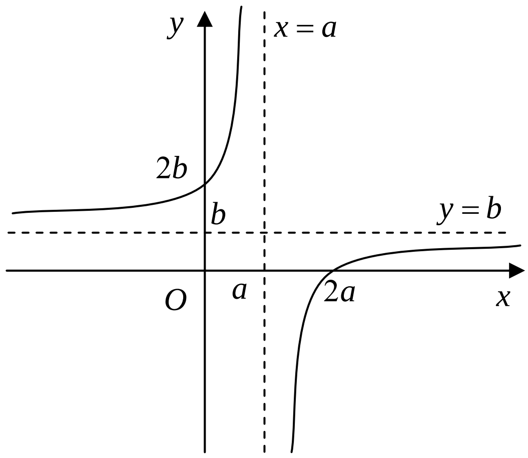
11



The region bounded by the axes and the curve $y = \cos x$ from $x = 0$ to $x = \dfrac{\pi}{2}$ is divided into two parts, of areas A_1 and A_2, by the curve $y = \sin x$ (see diagram). Prove that $A_1 = \left(\sqrt{2}\right)A_2$. **[5]**

The line $y = \dfrac{1}{2}$ meets the curve $y = \sin x$ and the *y*-axis at *P* and *Q* respectively. The region *OPQ*, bounded by the arc *OP* and the lines *PQ* and *QO*, is rotated through 4 right angles about the *x*-axis to form a solid of revolution of volume *V*. Find the exact value of *V* in terms of π. **[4]**

12 The diagram shows the graph of $y = \mathrm{f}(x)$. The curve crosses the axes at the points $(2a, 0)$ and $(0, 2b)$. The asymptotes are $x = a$ and $y = b$. The gradient of the curve at the point $(0, 2b)$ is 1.



On separate diagrams, sketch the graphs of

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, **[3]**

(ii) $y^2 = \mathrm{f}(x)$, **[3]**

(iii) $y = \mathrm{f}'(x)$, **[2]**

(iv) $y = \mathrm{f}(|x|)$, **[3]**

giving the equations of any asymptotes and the coordinates of any points of intersection with the x- and y-axes.

13 **(a)** In triangle ABC, angle $A = \left(\dfrac{\pi}{2} - \alpha\right)$ radians, $AB = AC = b$ and $BC = a$.

Show that $\dfrac{a}{b} = \dfrac{\cos\alpha}{\sin\left(\dfrac{\pi + 2\alpha}{4}\right)}$. **[1]**

Deduce, for small values of α, $a \approx \sqrt{2}b\left(1 - \dfrac{\alpha}{2} - \dfrac{\alpha^2}{8}\right)$. **[4]**

(b) Given that $y = e^{\sin^{-1}4x}$, show that

(i) $\sqrt{1-16x^2}\,\dfrac{dy}{dx} = 4y$, **[1]**

(ii) $\left(1-16x^2\right)\dfrac{d^2y}{dx^2} - 16x\dfrac{dy}{dx} = 16y$. **[2]**

By further differentiation of the result, find the Maclaurin series for y up to and including the term in x^3. **[3]**

By choosing a suitable value of x, show that $e^{-\frac{\pi}{6}} \approx \dfrac{7}{12}$. **[2]**

14 **(a)** Prove by induction that $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$. **[4]**

(b) Use the result in part **(a)** to

(i) evaluate $\sum_{r=1}^{30}(2r+3)^2$, **[2]**

(ii) prove that $\sum_{r=1}^{n}r^2 = \frac{1}{6}n(n+1)(2n+1)$. **[3]**

15 A man met with an accident and went into a coma on 10th January 2013. As a result, he did not pay the bank the outstanding balance of $\$M$ for his credit card bill when it is due for payment on 27th January 2013. On the 27th of each month when the payment for the credit card bill is due, the bank will charge a 2% interest on any outstanding balance that is unpaid. After the 2% interest has been added, the bank will still charge an additional late payment charge of $\$L$ monthly.

(a) Express in terms of L and M, his outstanding balance on his credit card on 1st February 2013. **[1]**

(b) If the man still remains in coma exactly n months later on the day he met with an accident, show that the accumulated outstanding balance on the man's credit card is $1.02^n M + 50L(1.02^n - 1)$. **[3]**

(c) Given that $M = 1000$ and $L = 55$. Find the least value of n when the accumulated outstanding balance on his credit card first exceeds \$2010. **[2]**

~ End of Paper ~

Anglo-Chinese Junior College

H2 Mathematics 9740

2013 JC 1 PROMO Solution

Qn	Solution
1	$y=\ln(2x+3),\ x>-\frac{3}{2}$ Before Step 2: $y=\ln\left[2(2x)+3\right]=\ln(4x+3)$ Before Step 1: $y=\ln(4x+3)+1$ **<u>OR</u>** Resulting curve: $y=\mathrm{f}\left(\frac{1}{2}x\right)-1=\ln(2x+3)$ $\Rightarrow\ \mathrm{f}\left(\frac{1}{2}x\right)=\ln\left[4\left(\frac{1}{2}x\right)+3\right]+1$ $\therefore\ y=\mathrm{f}(x)=\ln(4x+3)+1$
2	Given $y=ax^3+bx^2+cx+d$ $\frac{dy}{dx}=3ax^2+2bx+c$ When $x=-4,\ \frac{dy}{dx}=0,\ 3a(-4)^2+2b(-4)+c=0$ $48a-8b+c=0$ (1) When $x=4,\ \frac{dy}{dx}=0,\ 3a(4)^2+2b(4)+c=0$ $48a+8b+c=0$ (2) When $x=-4,\ y=258,$ $a(-4)^3+b(-4)^2+c(-4)+d=258$ $-64a+16b-4c+d=258$ (3) When $x=4,\ y=2,$ $a(4)^3+b(4)^2+c(4)+d=2$ $64a+16b+4c+d=2$ (4) Using G.C. $a=1,\ b=0,\ c=-48,\ d=130.$
3i	$\frac{d}{dx}\left(\sqrt{\cos^{-1}\left(\frac{x}{2}\right)}\right)=\frac{1}{2}\left(\cos^{-1}\left(\frac{x}{2}\right)\right)^{-\frac{1}{2}}\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{1}{2}$ $=-\frac{1}{2\sqrt{\left(4-x^2\right)\cos^{-1}\left(\frac{x}{2}\right)}}$

3ii	$$\frac{d}{dx}\left(\ln\sqrt{\frac{(x+1)^3}{x^2-1}}\right)=\frac{d}{dx}\left(\ln\left(\frac{x+1}{\sqrt{x-1}}\right)\right)$$ $$=\frac{d}{dx}\left(\ln(x+1)-\frac{1}{2}\ln(x-1)\right)$$ $$=\frac{1}{x+1}-\frac{1}{2(x-1)}$$ Alternative solution: $$\frac{d}{dx}\left(\ln\sqrt{\frac{(x+1)^3}{x^2-1}}\right)=\frac{1}{\sqrt{\frac{(x+1)^3}{x^2-1}}}\times\frac{1}{2}\frac{1}{\sqrt{\frac{(x+1)^3}{x^2-1}}}\times\frac{3(x+1)^2(x^2-1)-(x+1)^3(2x)}{(x^2-1)^2}$$ $$=\frac{x-3}{2(x^2-1)}$$
4(i)	$$\int\frac{1}{x\sqrt{\ln x}}dx$$ $$=\int\frac{\left(\frac{1}{x}\right)}{\sqrt{\ln x}}dx \quad \text{using } \int[f(x)]^n f'(x)dx=\frac{1}{n+1}[f(x)]^{n+1}+c$$ $$=2\sqrt{\ln x}+c$$
(ii)	$$\int\frac{e^{-2x}}{\sqrt{4-e^{-4x}}}dx$$ $$=-\frac{1}{2}\int\frac{-2e^{-2x}}{\sqrt{2^2-\left(e^{-2x}\right)^2}}dx \text{ using } \int\frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx=\sin^{-1}\left[\frac{f(x)}{a}\right]+c$$ $$=-\frac{1}{2}\sin^{-1}\left(\frac{e^{-2x}}{2}\right)+c$$

5

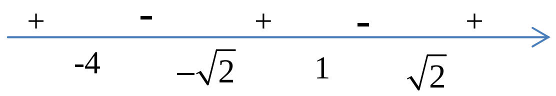
$$\frac{3x^2+6x-10}{x^2+3x-4}\geq 2$$

$$\frac{3x^2+6x-10}{x^2+3x-4}-2\geq 0$$

$$\frac{3x^2+6x-10-2x^2-6x+8}{x^2+3x-4}\geq 0$$

$$\frac{x^2-2}{(x+4)(x-1)}\geq 0$$

$$\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)(x+4)(x-1)\geq 0$$



$x<-4$ or $-\sqrt{2}\leq x<1$ or $x\geq\sqrt{2}$

$$\frac{3x^2+6|x|-10}{x^2+3|x|-4}\geq 2$$

$$\frac{3|x|^2+6|x|-10}{|x|^2+3|x|-4}\geq 2$$

$|x|<-4$ (n.a.) ; $-\sqrt{2}\leq|x|<1$; $|x|\geq\sqrt{2}$

$-1<x<1$ or $x\leq-\sqrt{2}$ or $x\geq\sqrt{2}$

6i

$x=1-cos\theta \qquad y=\theta+\sin\theta$

$$\frac{dx}{d\theta}=\sin\theta \qquad \frac{dy}{d\theta}=1+\cos\theta$$

$$\frac{dy}{dx}=\frac{1+\cos\theta}{\sin\theta}$$

$$=\frac{2\cos^2\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}$$

$$=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$=\cot\frac{\theta}{2}$$

When $\theta=\pi$, $\dfrac{dy}{dx}=0$.

ii)	Coordinate at $A(0,\pi)$ When $\theta=\frac{\pi}{2}$, $\frac{dy}{dx}=1$ Equation of tangent: $\frac{y-\frac{\pi}{2}-\sin\frac{\pi}{2}}{x-1+\cos\frac{\pi}{2}}=1$ $y=x+\frac{\pi}{2}$ Coordinate at $B\left(0,\frac{\pi}{2}\right)$ $\therefore$ area of triangle ABP $=\frac{1}{2}\times\left(\pi-\frac{\pi}{2}\right)\times 2$ $=\frac{\pi}{2}$
7i)	Diagonal $DB=2\sqrt{a^2-x^2}$ Length of side of square $=\sin\left(\frac{\pi}{4}\right)2\sqrt{a^2-x^2}$ $=\frac{\sqrt{2}}{2}2\sqrt{a^2-x^2}$ $=\sqrt{2\left(a^2-x^2\right)}$
ii)	Volume of block , $v=\frac{2}{3}\left(a^2-x^2\right)\left(x+a\right)$ Diff. w.r.t.x $\frac{dv}{dx}=\frac{2}{3}\left[\left(a^2-x^2\right)+\left(x+a\right)\left(-2x\right)\right]$ $=\frac{2}{3}\left[\left(a-x\right)\left(a+x\right)+\left(x+a\right)\left(-2x\right)\right]$ $=\frac{2}{3}\left(x+a\right)\left(a-3x\right)$ For stationary point, $\frac{dv}{dx}=0$ $0=\frac{2}{3}\left(x+a\right)\left(a-3x\right)$ $x=-a$ (n.a.) $\quad x=\frac{a}{3}$

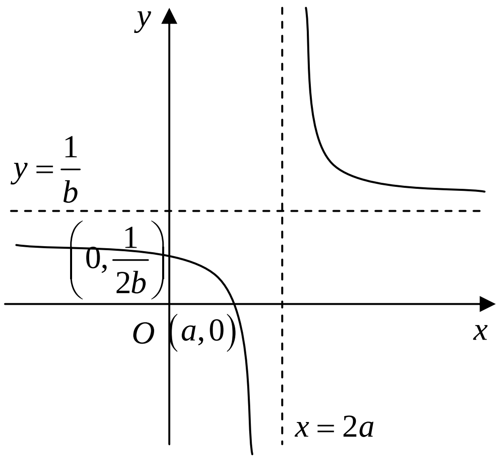
	$\frac{d^2v}{dx^2}=\frac{2}{3}\left[(x+a)(-3)+(a-3x)\right]$ $\frac{d^2v}{dx^2}<0$ when $x=\frac{a}{3}$ Max. volume of block , $v=\frac{2}{3}\left(a^2-\left(\frac{a}{3}\right)^2\right)\left(\left(\frac{a}{3}\right)+a\right)$ $=\frac{64a^3}{81}units^3$
8 (i)	$\text{f}:x\mapsto x+\frac{1}{x}$ for $x\in\mathbb{R},\ x\geq 1$ Let $y=x+\frac{1}{x}\ \Rightarrow\ x^2-yx+1=0$ $x=\frac{y+\sqrt{y^2-4}}{2}$ or $x=\frac{y-\sqrt{y^2-4}}{2}$ (rejected since $x\geq 1$ & $y\geq 2$) $\text{f}^{-1}(x)=\frac{x+\sqrt{x^2-4}}{2}$ $D_{\text{f}^{-1}}=[2,\infty)$
8 (ii)	$\text{fff}^{-1}(x)=\text{f}(x)=x+\frac{1}{x}$ Domain of $\text{fff}^{-1}=D_{\text{f}^{-1}}=R_{\text{f}}=[2,\infty)$ Range of $\text{fff}^{-1}=\{\text{f}(x):x\in[2,\infty)\}=\left[\frac{5}{2},\infty\right)$
8 (iii)	$\text{f}:x\mapsto x+\frac{1}{x}$ for $x\in\mathbb{R},\ x\geq 1$ $D_f=[1,\infty)$, $R_f=[2,\infty)$ Since $R_f\subseteq D_f$, ff exists.
9	Given $f(k)=\frac{1}{k^2}$ $f(k)-f(k+2)$ $=\frac{1}{k^2}-\frac{1}{(k+2)^2}$ $=\frac{(k+2)^2-k^2}{k^2(k+2)^2}$ $=\frac{(k^2+4k+4)-k^2}{k^2(k+2)^2}$ $=\frac{4(k+1)}{k^2(k+2)^2}$

	$\frac{2}{(1)^2(3)^2}+\frac{3}{(2)^2(4)^2}+\frac{4}{(3)^2(5)^2}+\ldots+\frac{n+1}{(n)^2(n+2)^2}$ $=\sum_{k=1}^{n}\frac{k+1}{k^2(k+2)^2}$ $=\sum_{k=1}^{n}\frac{1}{4}\left(\frac{1}{k^2}-\frac{1}{(k+2)^2}\right)$ $=\frac{1}{4}\left(\sum_{k=1}^{n}\frac{1}{k^2}-\sum_{k=1}^{n}\frac{1}{(k+2)^2}\right)$ $=\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots+\frac{1}{n^2}\right)$ $-\frac{1}{4}\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots+\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}\right)$ $=\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)$ $=\frac{1}{4}\left(\frac{5}{4}-\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)$ $\sum_{k=2}^{n}\frac{k+1}{k^2(k+2)^2}$ $=\sum_{k=1}^{n}\frac{k+1}{k^2(k+2)^2}-\frac{2}{(1^2)(3^2)}$ $=\frac{1}{4}\left(\frac{5}{4}-\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)-\frac{2}{9}$ $=\frac{13}{144}-\frac{1}{4}\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}\right)$ $<\frac{13}{144}$ $\left(\because \frac{1}{(n+1)^2}>0 \text{ and } \frac{1}{(n+2)^2}>0\ \forall n\in\mathbb{Z}^+\right)$
10 (a)	$\int_1^e(\ln x)^2\,dx=\left[(\ln x)^2(x)\right]_1^e-\int_1^e x\left(\frac{2\ln x}{x}\right)dx$ $=e-2\int_1^e(\ln x)\,dx$ $=e-2\left\{\left[(\ln x)(x)\right]_1^e-\int_1^e x\left(\frac{1}{x}\right)dx\right\}$ $=e-2e+2(e-1)=e-2$
10 (b)	$x=3\cos^2\theta+7\sin^2\theta$ $\frac{dx}{d\theta}=6\cos\theta(-\sin\theta)+14\sin\theta\cos\theta$ $=8\sin\theta\cos\theta$ when $x=3$, $3\cos^2\theta+7\sin^2\theta=3 \Rightarrow \sin^2\theta=0 \Rightarrow \theta=0$ when $x=7$, $3\cos^2\theta+7\sin^2\theta=7 \Rightarrow \cos^2\theta=0 \Rightarrow \theta=\frac{\pi}{2}$

	$\int_3^7 \frac{1}{\sqrt{[(7-x)(x-3)]}}dx$ $= \int_0^{\frac{\pi}{2}} \frac{8\sin\theta\cos\theta}{\sqrt{[(7-3\cos^2\theta-7\sin^2\theta)(3\cos^2\theta+7\sin^2\theta-3)]}}d\theta$ $= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{[(4\cos^2\theta)(4\sin^2\theta)]}}(8\sin\theta\cos\theta)d\theta$ $= 2\int_0^{\frac{\pi}{2}} 1d\theta = 2\left(\frac{\pi}{2}\right) = \pi$ (proved)
11	$A_1 = \int_0^{\frac{\pi}{4}}(\sin x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\cos x)dx = [-\cos x]_0^{\frac{\pi}{4}} + [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $= \left(-\frac{\sqrt{2}}{2}+1\right)+\left(1-\frac{\sqrt{2}}{2}\right)$ $= 2-\sqrt{2}$ $A_2 = \int_0^{\frac{\pi}{4}}(\cos x - \sin x)dx = [\sin x + \cos x]_0^{\frac{\pi}{4}}$ $= \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)-(0+1)$ $= \sqrt{2}-1$ **<u>OR</u>** $A_2 = \int_0^{\frac{\pi}{2}}(\cos x)dx - A_1 = [\sin x]_0^{\frac{\pi}{2}} - (2-\sqrt{2})$ $= (1+0)-(2-\sqrt{2})$ $= \sqrt{2}-1$ **<u>OR</u>** $A_2 = \int_0^{\frac{\sqrt{2}}{2}}(\sin^{-1}y)dy + \int_{\frac{\sqrt{2}}{2}}^1(\cos^{-1}y)dy$ $\therefore\quad A_1 = 2-\sqrt{2} = \sqrt{2}(\sqrt{2}-1) = \sqrt{2}A_2$ (proved)
	$P = \left(\frac{\pi}{6},\frac{1}{2}\right)$ $V = \pi\left(\frac{1}{2}\right)^2\frac{\pi}{6} - \pi\int_0^{\frac{\pi}{6}}(\sin x)^2dx$ $= \frac{\pi^2}{24} - \frac{\pi}{2}\int_0^{\frac{\pi}{6}}(1-\cos 2x)dx$ $= \frac{\pi^2}{24} - \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{6}}$ $= \frac{\pi^2}{24} - \frac{\pi}{2}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}\pi}{8} - \frac{\pi^2}{24}$

<table>
<tr><td>12
(i)</td><td>

$y=\dfrac{1}{\mathrm{f}(x)}$



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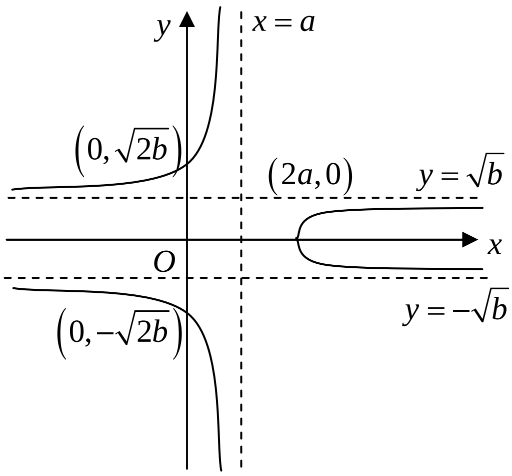
<tr><td>12
(ii)</td><td>

$y^2=\mathrm{f}(x)$



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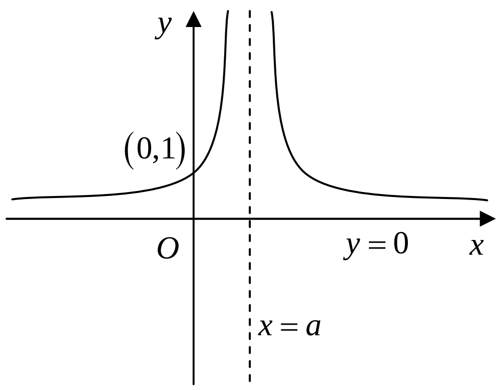
<tr><td>12
(iii)</td><td>

$y=\mathrm{f}'(x)$



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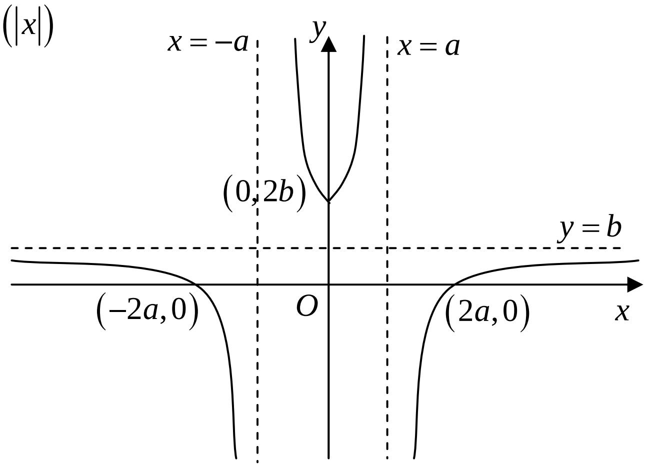
<tr><td>12
(iv)</td><td>

$y=\mathrm{f}(|x|)$



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<tr><td>13a)</td><td>Sine rule</td></tr>
</table>

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$

$$\frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{a}=\frac{\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}{b}$$

$$\frac{a}{b}=\frac{\cos\alpha}{\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}$$

$$\frac{a}{b}=\frac{\cos\alpha}{\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}$$

$$=\frac{\cos\alpha}{\sin\frac{\pi}{4}\cos\frac{\alpha}{2}+\cos\frac{\pi}{4}\sin\frac{\alpha}{2}}$$

$$=\frac{\cos\alpha}{\frac{\sqrt{2}}{2}\cos\frac{\alpha}{2}+\frac{\sqrt{2}}{2}\sin\frac{\alpha}{2}}$$

For small values of α

$$\frac{a}{b}\approx\frac{1-\frac{\alpha^2}{2}}{\frac{\sqrt{2}}{2}\left(1-\frac{\left(\frac{\alpha}{2}\right)^2}{2}\right)+\frac{\sqrt{2}}{2}\left(\frac{\alpha}{2}\right)}$$

$$=\frac{1-\frac{\alpha^2}{2}}{\frac{\sqrt{2}}{2}\left(1+\left(\frac{\alpha}{2}-\frac{\alpha^2}{8}\right)\right)}$$

$$a=\sqrt{2}b\left(1-\frac{\alpha^2}{2}\right)\left(1+\left(\frac{\alpha}{2}-\frac{\alpha^2}{8}\right)\right)^{-1}$$

$$=\sqrt{2}b\left(1-\frac{\alpha^2}{2}\right)\left(1+(-1)\left(\frac{\alpha}{2}-\frac{\alpha^2}{8}\right)+\frac{(-1)(-2)}{2}\left(\frac{\alpha}{2}-\frac{\alpha^2}{8}\right)^2+\ldots\right)$$

$$=\sqrt{2}b\left(1-\frac{\alpha^2}{2}\right)\left(1-\frac{\alpha}{2}+\frac{3\alpha^2}{8}\right)$$

$$=\sqrt{2}b\left(1-\frac{\alpha}{2}-\frac{\alpha^2}{8}\right)$$

13b)

i)	$y = e^{\sin^{-1} 4x}$ $\ln y = \sin^{-1} 4x$ diff. w.r.t x $\frac{1}{y}\frac{dy}{dx} = \frac{4}{\sqrt{1-16x^2}}$ $\sqrt{1-16x^2}\frac{dy}{dx} = 4y$
ii)	$\sqrt{1-16x^2}\frac{dy}{dx} = 4y$ diff. w.r.t x $\sqrt{1-16x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{1}{2}\left(1-16x^2\right)^{-\frac{1}{2}}\left(-32x\right)\right) = 4\frac{dy}{dx}$ $\sqrt{1-16x^2}\frac{d^2y}{dx^2} - \frac{16x}{\sqrt{1-16x^2}}\frac{dy}{dx} = 4\frac{dy}{dx}$ $\left(1-16x^2\right)\frac{d^2y}{dx^2} - 16x\frac{dy}{dx} = 4\sqrt{1-16x^2}\frac{dy}{dx}$ $\left(1-16x^2\right)\frac{d^2y}{dx^2} - 16x\frac{dy}{dx} = 16y$ $\left(1-16x^2\right)\frac{d^2y}{dx^2} - 16x\frac{dy}{dx} = 16y$ diff. w.r.t x $\left(1-16x^2\right)\frac{d^3y}{dx^3} + \left(-32x\right)\frac{d^2y}{dx^2} - 16x\frac{d^2y}{dx^2} - 16\frac{dy}{dx} = 16\frac{dy}{dx}$ $\left(1-16x^2\right)\frac{d^3y}{dx^3} - 48x\frac{d^2y}{dx^2} = 32\frac{dy}{dx}$ $f(0) = 1$ $f'(0) = 4$ $f''(0) = 16$ $f'''(0) = 128$ $f(x) = 1 + 4x + 8x^2 + \frac{64}{3}x^3 + \ldots$ $e^{-\frac{\pi}{6}} = e^{\sin^{-1} 4x}$ $-\frac{\pi}{6} = \sin^{-1} 4x$ $\sin\left(-\frac{\pi}{6}\right) = 4x$ $x = -\frac{1}{8}$

	$e^{-\frac{\pi}{6}} \approx 1+4\left(-\frac{1}{8}\right)+8\left(-\frac{1}{8}\right)^2+\frac{64}{3}\left(-\frac{1}{8}\right)^3+\ldots$ $=1-\frac{1}{2}+\frac{1}{8}-\frac{1}{24}$ $=\frac{7}{12}$
14 (a)	Let P(n) denote the statement $\sum_{r=1}^{n}(2r-1)^2=\frac{1}{3}n(2n-1)(2n+1)$. When n = 1, $LHS=\sum_{r=1}^{1}(2r-1)^2=1^2=1$ $RHS=\frac{1}{3}(1)(2-1)(2+1)=\frac{1}{3}(1)(1)(3)=1$ $LHS=RHS$ Hence P(1) is true. Assume P(k) is true for some $k\in\mathbb{Z}^+$. i.e. $\sum_{r=1}^{k}(2r-1)^2=\frac{1}{3}k(2k-1)(2k+1)$. We need to show that P(k+1) is true. i.e. $\sum_{r=1}^{k+1}(2r-1)^2=\frac{1}{3}(k+1)(2k+1)(2k+3)$ $\sum_{r=1}^{k+1}(2r-1)^2$ $=\sum_{r=1}^{k}(2r-1)^2+(2k+1)^2$ $=\frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2$ $=\frac{1}{3}(2k+1)\left[k(2k-1)+3(2k+1)\right]$ $=\frac{1}{3}(2k+1)\left[2k^2+5k+3\right]$ $=\frac{1}{3}(2k+1)\left[(k+1)(2k+3)\right]$ $=\frac{1}{3}(k+1)(2k+1)(2k+3)$ Hence $P(k)\Rightarrow P(k+1)$ is true. Since P(1) is true and $P(k)\Rightarrow P(k+1)$ is true, by the principle of Mathematical induction, P(n) is true $\forall n\in\mathbb{Z}^+$

14 b (i)	<u>Method 1:</u> $\sum_{r=1}^{30}(2r+3)^2$ $=5^2+7^2+9^2+\ldots+63^2$ $=\left(1^2+3^2+5^2+\ldots+63^2\right)-1^2-3^2$ $=\sum_{r=1}^{32}(2r-1)^2-10$ $=\frac{1}{3}(32)(64-1)(64+1)-10$ $=\frac{1}{3}(32)(63)(65)-10$ $=43670$ <u>Method 2:</u> Let $r=k-2$ $2r+3=2(k-2)+3=2k-1$ When $r=1,\ k-2=1\Rightarrow k=3$ When $r=30,\ k-2=30\Rightarrow k=32$ $\sum_{r=1}^{30}(2r+3)$ $=\sum_{k=3}^{32}(2k-1)$ $=\sum_{r=1}^{32}(2r-1)^2-1^2-3^2$ $=\sum_{r=1}^{32}(2r-1)^2-10$ $=\frac{1}{3}(32)(64-1)(64+1)-10$ $=\frac{1}{3}(32)(63)(65)-10$ $=43670$
14b (ii)	To prove: $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$ Proof: $\sum_{r=1}^{n}(2r-1)^2=\frac{1}{3}n(2n-1)(2n+1)$ $\sum_{r=1}^{n}(4r^2-4r+1)=\frac{1}{3}n(2n-1)(2n+1)$

$$4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 = \frac{1}{3}n(2n-1)(2n+1)$$

$$4\sum_{r=1}^{n} r^2 - 4\left[\frac{1}{2}(n)(n+1)\right] + n = \frac{1}{3}n(2n-1)(2n+1)$$

$$4\sum_{r=1}^{n} r^2 = \frac{1}{3}n(2n-1)(2n+1) + 2n(n+1) - n$$

$$4\sum_{r=1}^{n} r^2 = \frac{1}{3}n\left[(2n-1)(2n+1) + 6(n+1) - 3\right]$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{12}n\left[(4n^2-1) + (6n+6) - 3\right]$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{12}n\left[4n^2 + 6n + 2\right]$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{12}n(2n+2)(2n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

15 (a)

Original amount = \$$M$
2% interest charged = \$$0.02M$
Late payment charge = \$$L$
Total outstanding balance = \$$(1.02M + L)$

(b)

No of months later	Outstanding balance left unpaid after 27th of the month
1	$1.02M + L$
2	$1.02^2 M + 1.02L + L$
3	$1.02^3 M + 1.02^2 L + 1.02L + L$
4	$1.02^4 M + 1.02^3 L + 1.02^2 L + 1.02L + L$
. . .	. . .
n	$1.02^n M + 1.02^{n-1} L + \ldots + 1.02^2 L + 1.02L + L$

Outstanding balance left unpaid n months later

$= 1.02^n M + 1.02^{n-1} L + \ldots + 1.02^2 L + 1.02L + L$

$= 1.02^n M + \underbrace{(1.02^{n-1} L + \ldots + 1.02^2 L + 1.02L + L)}$

This is a G.P. with first term $a = L$, common ratio $r = 1.02$ and number of terms is n.

$$= 1.02^n M + \frac{L(1.02^n - 1)}{1.02 - 1}$$

$$= 1.02^n M + \frac{L(1.02^n - 1)}{0.02}$$

	$= 1.02^n M + \dfrac{100L(1.02^n - 1)}{2}$ $= 1.02^n M + 50L(1.02^n - 1)$
(c)	Putting $1.02^n M + 50L(1.02^n - 1) > 2010$. Given M = 1000 and L = 55. $1.02^n(1000) + 50(55)(1.02^n - 1) > 2010$ $1.02^n(1000) + (2750)(1.02^n) - 2750 > 2010$ $1.02^n(1000) + (2750)(1.02^n) > 4760$ $1.02^n(3750) > 4760$ $1.02^n > \dfrac{476}{375}$ $\log(1.02^n) > \log(\dfrac{476}{375})$ $n\log(1.02) > \log(\dfrac{476}{375})$ $n > \log(\dfrac{476}{375}) \Big/ \log(1.02)$ $n > 12.04$ Since n is a positive integer, n = 13, 14, 15, … Hence n = 13.

Anderson Junior College
JC1 Promotional Examination 2013
H2 Mathematics (9740)

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1. **(i)*** Find the expansion of $\dfrac{1-x^2}{\sqrt{4-x}}$ in ascending powers of x, up to and including the term in x^2. [3]

(ii)* State the set of values of x for which this expansion is valid. [1]

(iii)* Hence, by substituting a suitable value of x, find an approximation for $\sqrt{15}$ in the form $\dfrac{a}{b}$, where a and b are integers to be determined. [3]

2. Evaluate $\displaystyle\sum_{r=2}^{n}\left(2^{-r}+2nr+n^2\right)$, giving your answer in terms of n. [4]

3. A curve C is defined by parametric equations

$$x=\mathrm{e}^{\theta}\cos\theta,\quad y=\mathrm{e}^{-\theta}\sin\theta,\qquad \text{for } -\frac{\pi}{2}\le\theta\le 0.$$

(i) Sketch the curve C, indicating the axial intercepts in exact form. [2]

(ii) Show that the area bounded by the curve C and the axes is given by

$$\int_{-\frac{\pi}{2}}^{0}(\sin^2\theta-\sin\theta\cos\theta)\,\mathrm{d}\theta\,.$$

Hence determine its exact value. [5]

4. A sequence u_n, $n=0,1,2,3,\ldots$, is such that $u_0=-\dfrac{1}{2}$ and

$$u_{n+1}=u_n+\ln(n+1)-\frac{1}{4n^2-1}\qquad \text{for all } n\ge 0.$$

(i) Prove by mathematical induction that $u_n=\ln(n!)+\dfrac{1}{2(2n-1)}$. [5]

(ii) Hence find $\displaystyle\sum_{n=0}^{N}\left[\ln(n+1)-\frac{1}{4n^2-1}\right]$. [3]

(iii) Does the series found in **(ii)** converge? Give a reason for your answer. [1]

(iv) Using the series found in **(ii)**, evaluate $\displaystyle\sum_{n=2}^{N}\left[\ln(n-1)-\frac{1}{4(n-2)^2-1}\right]$. [2]

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5. The curve with equation $y=-\sqrt{-2x}$ is transformed by a translation of 2 units in the positive x-direction, followed by a reflection in the y-axis.

(i) Find the equation of the resultant curve in the form $y=\mathrm{f}(x)$ and the coordinates of the points where this curve crosses the x- and y- axes. On a single diagram, sketch the graph $y=\mathrm{f}(x)$ and its inverse. [5]

(ii) Solve the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, giving your answers in exact form. [3]

(iii) The function g is defined such that $\mathrm{f}^{-1}\mathrm{g}(x)=\dfrac{x^2}{2}-2$. Find $\mathrm{g}(x)$. [2]

6. Without using a calculator, solve $\dfrac{x(4x-1)}{2x-1} < 3x+1$. [3]

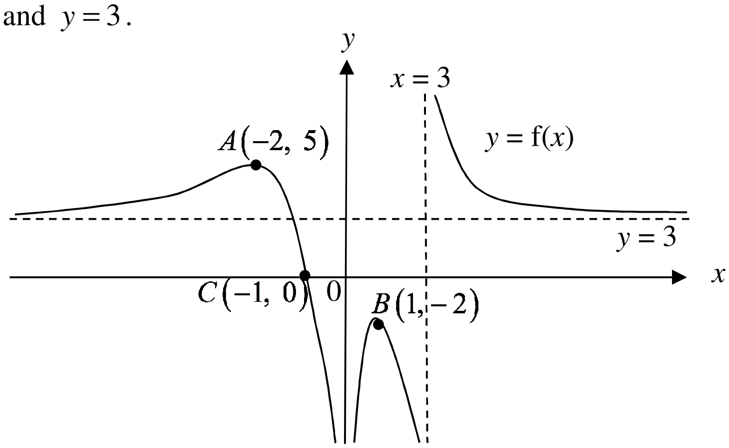
Hence, find the solutions of the inequalities

(a) $x-5 < 3x+1 < \dfrac{x(4x-1)}{2x-1}$,

(b) $\dfrac{\cos x\,(4\cos x+1)}{2\cos x+1} > 3\cos x-1$ for $0 \le x \le \pi$,

leaving your answers in exact form. [6]

7. The diagram shows a sketch of the curve $y=\mathrm{f}(x)$. The curve cuts the x-axis at $C(-1,\ 0)$, has stationary points at $A(-2,\ 5)$ and $B(1,-2)$, and asymptotes $x=0$, $x=3$ and $y=3$.



On separate diagrams, sketch the graphs of

(i) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y=\mathrm{f}\,'(x)$, [3]

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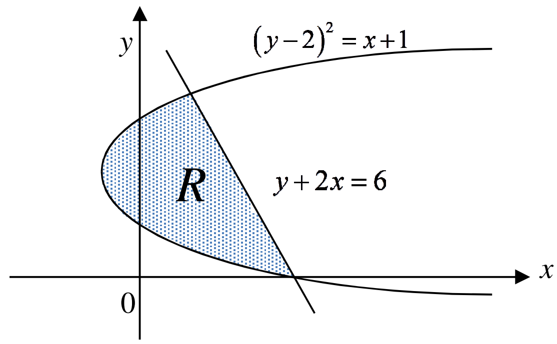
showing, in each case, the asymptotes, the coordinates of the stationary points and the points of intersection with the axes, whenever possible.

8. **(a)*** Find $\int \frac{1}{x^2} \ln(x+1)\ \mathrm{d}x$. [3]

(b)*The diagram shows a shaded region R bounded by the curve $(y-2)^2 = x+1$ and the line $y+2x=6$.



Find the volume generated when R is rotated through 2π radians about the x-axis, leaving your answer correct to 3 significant figures. [4]

9. The lines l_1 and l_2 have equations

$$\frac{x-1}{3} = \frac{y-2}{a},\ z=1 \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}$$

respectively, where a is a constant.

(i) Given that l_1 and l_2 intersect at the point N, find N and the value of a. [3]

(ii) Show that the position vector of F, the foot of the perpendicular from the point $P(2, 1, 1)$ to the line l_2 is $\frac{4}{3}\mathbf{i} + \frac{5}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$. [3]

(iii) Find the position vector of the point P', the reflection of P in the line l_2. [2]

(iv) The point Q has coordinates (1, 2, 0). Find the ratio of the area of triangle NQP to the area of triangle FQP'. [3]

10. A curve C has equation $y = \frac{x^2}{x+3\lambda}$, $x \neq -3\lambda$ and λ is a positive constant.

(i) Find the coordinates of the stationary points of C. [3]

(ii) Draw a sketch of C, labeling clearly, in terms of λ, the asymptotes and the stationary points. [2]

(iii) Use the graph in **(ii)**, find the number of roots of the equation $x^4 - 2\lambda x - 6\lambda^2 = 0$. [3]

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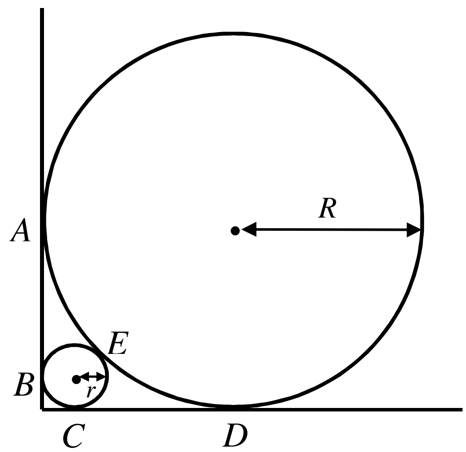
The function f is defined by $\mathrm{f}: x \mapsto \dfrac{x^2}{x+3\lambda}$, $x \le -6\lambda$.

(iv) Show that f^2 exists and find the value of $\mathrm{f}^2(-6\lambda)$. [4]

11. Two solid cylinders of the same height are placed at a corner of the wall such that the vertices *A*, *B*, *C* and *D* touch the wall. At point *E*, the two cylinders touch each other. The diagram below shows a cross section of the cylinders.



Let *r* be the radius of the small cylinder and *R* be the radius of the big cylinder.

(i) Show that $R = \left(\sqrt{2}+1\right)^2 r$ [2]

(ii) Given that the volume of the small cylinder is $\dfrac{16\pi}{\sqrt{2}+1}\text{cm}^3$, find the **exact value** of the radius *r* such that the surface area of the big cylinder is a minimum. [5]

12. Mary has a monthly income of \$4000. She is considering applying for a car loan of \$40,000 for 6 years which charges an interest rate of 3.00% per annum, compounded monthly. Interest is chargeable immediately when the loan sum is drawn out. The monthly repayment, \$$m$, is fixed throughout the loan tenure.

(i) Show that the calculated loan balance at the end of the n$^{\text{th}}$ loan month, after the monthly repayment is made, is given by

$$40000\left(\frac{401}{400}\right)^n - 400m\left[\left(\frac{401}{400}\right)^n - 1\right].$$ [3]

(ii) By legislation, banks can approve a car loan only if the monthly repayment does not exceed 15% of an applicant's monthly income. Prove that Mary will not be able to apply for the car loan. [3]

(iii) If the interest rate for all car loans by the banks is compounded monthly, find the range of interest rates chargeable which will enable Mary to apply for the

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car loan successfully. Give your answer in the form $r\%$ per annum, correct to 1 decimal place. [3]

END OF PAPER

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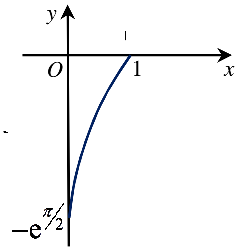
Qn	Solutions
1(i)	$\frac{1-x^2}{\sqrt{4-x}}=(1-x^2)(4-x)^{-1/2}$ $=\frac{1}{2}(1-x^2)\left(1-\frac{x}{4}\right)^{-1/2}$ $=\frac{1}{2}(1-x^2)\left(1+\frac{x}{8}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\ldots\right)$ $=\frac{1}{2}(1-x^2)\left(1+\frac{x}{8}+\frac{3x^2}{128}+\ldots\right)$ $=\frac{1}{2}+\frac{1}{16}x-\frac{125}{256}x^2+\ldots$
(ii)	Expansion is valid for $\{x:-4<x<4,\ x\in\mathbb{R}\}$.
(iii)	By letting $x=\frac{1}{4}$, $\frac{1-\left(\frac{1}{4}\right)^2}{\sqrt{4-\frac{1}{4}}}\approx\frac{1}{2}+\frac{1}{16}\left(\frac{1}{4}\right)-\frac{125}{256}\left(\frac{1}{16}\right)$ $\frac{\frac{15}{16}}{\sqrt{\frac{15}{4}}}\approx\frac{1987}{4096}$ $\sqrt{15}\approx\frac{1987}{512}$ where $a=1987$ and $b=512$ or $\sqrt{15}\approx\frac{7680}{1987}$ where $a=7680$ and $b=1987$
2	$\sum_{r=2}^{n}\left(2^{-r}+2nr+n^2\right)$ $=\sum_{r=2}^{n}\left(2^{-r}\right)+\sum_{r=2}^{n}2nr+\sum_{r=2}^{n}n^2$ $=\frac{\left(\frac{1}{2}\right)^2\left(1-\left(\frac{1}{2}\right)^{n-1}\right)}{1-\frac{1}{2}}+2n\cdot\frac{n-1}{2}(2+n)+n^2(n-1)$ $=\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{n-1}\right)+n(n-1)\left[(2+n)+n\right]$ $=\frac{1}{2}-\left(\frac{1}{2}\right)^n+2n(n^2-1)$

3(i)*

$x = e^{\theta}\cos\theta, \quad y = e^{-\theta}\sin\theta, \text{ for } -\frac{\pi}{2} \le \theta \le 0$

When $\theta = 0$, x-intercept: $(1, 0)$

When $\theta = -\frac{\pi}{2}$, y-intercept: $(0, -e^{\pi/2})$



(ii)*

$$\text{Area} = -\int_0^1 y\,dx$$

$$= -\int_{-\frac{\pi}{2}}^{0} (e^{-\theta}\sin\theta)\left\{e^{\theta}[\cos\theta - \sin\theta]\right\}d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0} (\sin^2\theta - \sin\theta\cos\theta)\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0} \left\{\left(\frac{1-\cos 2\theta}{2}\right) - \frac{\sin 2\theta}{2}\right\} d\theta$$

$$= \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_{-\pi/2}^{0}$$

$$= \frac{\pi + 2}{4}$$

4(i)

Let P_n be the statement $u_n = \ln(n!) + \frac{1}{2(2n-1)}$ for $n \ge 0, n \in \mathbb{Z}$.

When $n = 0$, LHS = $u_0 = -\frac{1}{2}$

RHS = $\ln(0!) + \frac{1}{2(-1)} = -\frac{1}{2}$ Since LHS = RHS, $\therefore P_0$ is true.

Assume that P_k is true for some $k \ge 0,\ k \in \mathbb{Z}$, i.e. $u_k = \ln(k!) + \frac{1}{2(2k-1)}$,

need to prove that P_{k+1} is true, i.e., to show that

$$u_{k+1} = \ln(k+1)! + \frac{1}{2(2(k+1)-1)} = \ln(k+1)! + \frac{1}{2(2k+1)}.$$

LHS of P_{k+1}

$= u_{k+1}$

$= u_k + \ln(k+1) - \frac{1}{4k^2 - 1}$

$= \ln(k!) + \frac{1}{2(2k-1)} + \ln(k+1) - \frac{1}{4k^2-1}$

$= \ln[(k+1)k!] + \frac{1}{2(2k-1)} - \frac{1}{(2k-1)(2k+1)}$

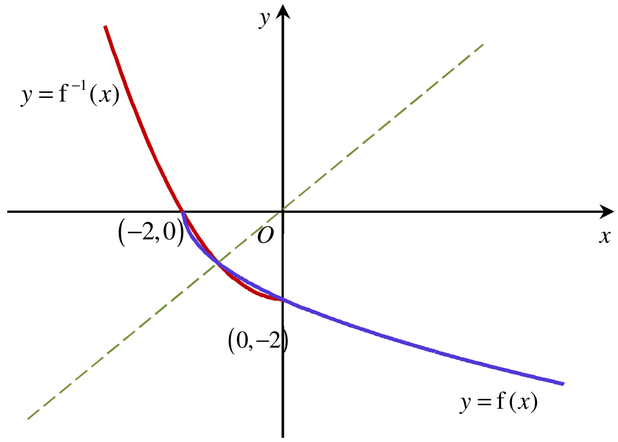
$= \ln(k+1)! + \frac{2k-1}{2(2k-1)(2k+1)}$

$= \ln(k+1)! + \frac{1}{2(2k+1)} = \text{RHS of } P_{k+1}$

	Since P_0 is true and P_k is true $\Rightarrow P_{k+1}$ is true, $\therefore$ by the principle of mathematical induction, P_n is true for all non-negative integers *n*.
(ii)	$u_{n+1} = u_n + \ln(n+1) - \frac{1}{4n^2-1}$ $\Rightarrow u_{n+1} - u_n = \ln(n+1) - \frac{1}{4n^2-1}$. $\therefore \sum_{n=0}^{N}\left[\ln(n+1) - \frac{1}{4n^2-1}\right] = \sum_{n=0}^{N}(u_{n+1} - u_n)$ $= (u_1 - u_0$ $+ u_2 - u_1$ $+ u_3 - u_2$ $+ u_4 - u_3$ $+ \vdots$ $+ u_{N+1} - u_N)$ $= u_{N+1} - u_0$ $= \ln(N+1)! + \frac{1}{2(2(N+1)-1)} - \left(-\frac{1}{2}\right)$ $= \ln(N+1)! + \frac{1}{2(2N+1)} + \frac{1}{2}$
(iii)	$\therefore \sum_{n=0}^{N}\left[\ln(n+1) - \frac{1}{4n^2-1}\right] = \ln(N+1)! + \frac{1}{2(2N+1)} + \frac{1}{2}$. The series is divergent since $\ln(N+1)! \to \infty$ when $N \to \infty$.
(iv)	Replace *n* with *n* + 2, $\sum_{n=2}^{N}\left[\ln(n-1) - \frac{1}{4(n-2)^2-1}\right]$ $= \sum_{n+2=2}^{N}\left[\ln(n+2-1) - \frac{1}{4(n+2-2)^2-1}\right]$ $= \sum_{n=0}^{N-2}\left[\ln(n+1) - \frac{1}{4n^2-1}\right]$ $= \ln(N-2+1)! + \frac{1}{2(2(N-2)+1)} + \frac{1}{2}$ $= \ln(N-1)! + \frac{1}{2(2N-3)} + \frac{1}{2}$

5(i) $y=-\sqrt{-2x} \xrightarrow{x\to x-2} y=-\sqrt{4-2x} \xrightarrow{x\to -x} y=-\sqrt{4+2x}$

Coordinates of points: $(-2,0)$, $(0,-2)$.



(ii) From the diagram, the graphs intersect at $x=-2, 0$, and where

$\mathrm{f}(x)=x \Rightarrow -\sqrt{4+2x}=x \Rightarrow x^2-2x-4=0$

$$x=\frac{2\pm\sqrt{4+16}}{2}=1\pm\sqrt{5}$$

Since the graphs intersect where $x\le 0$, solutions for $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ are $x=-2,\ 1-\sqrt{5},\ 0$.

(iii)

$$\mathrm{f}^{-1}\mathrm{g}(x)=\frac{x^2}{2}-2$$

$$\Rightarrow \mathrm{f}\left(\mathrm{f}^{-1}\mathrm{g}(x)\right)=\mathrm{f}\left(\frac{x^2}{2}-2\right)$$

$$\Rightarrow \quad \mathrm{g}(x)=-\sqrt{4+x^2-4}=-\sqrt{x^2}=-|x|$$

6

$$\frac{x(4x-1)}{2x-1}<3x+1$$

$$\frac{4x^2-x-(2x-1)(3x+1)}{2x-1}<0$$

$$\frac{-2x^2+1}{2x-1}<0$$

$$\frac{x^2-\frac{1}{2}}{2x-1}>0$$

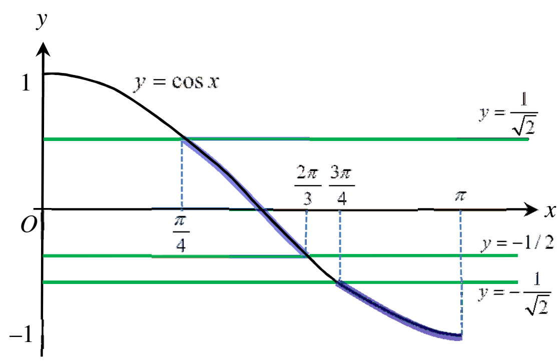
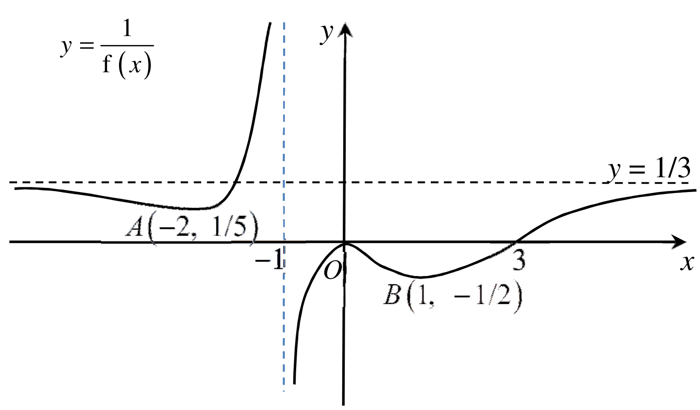
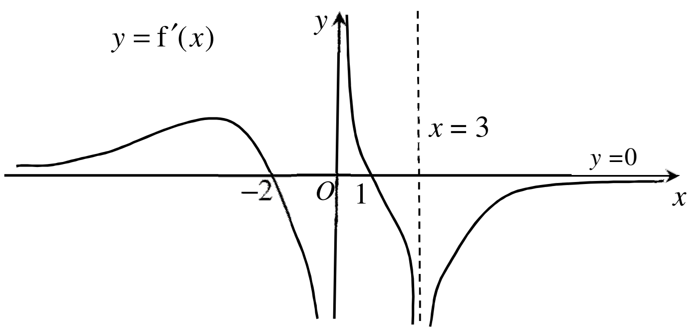
$$\frac{(x+\frac{1}{\sqrt{2}})(x-\frac{1}{\sqrt{2}})}{2x-1}>0$$

$$\therefore\ -\frac{1}{\sqrt{2}}<x<\frac{1}{2} \quad \text{or} \quad x>\frac{1}{\sqrt{2}}.$$

(a) Solution of $3x+1<\dfrac{x(4x-1)}{2x-1}$ is $x<-\dfrac{1}{\sqrt{2}}$ or $\dfrac{1}{2}<x<\dfrac{1}{\sqrt{2}}$.

Also, $x-5<3x+1 \Rightarrow x>-3.$

Taking the intersection of the solutions, $-3<x<-\dfrac{1}{\sqrt{2}}$ or $\dfrac{1}{2}<x<\dfrac{1}{\sqrt{2}}$.

(b)	Replace x with $-\cos x$, $\frac{-\cos x(-4\cos x-1)}{-2\cos x-1} < -3\cos x+1$ $\Rightarrow \frac{\cos x(4\cos x+1)}{2\cos x+1} > 3\cos x-1.$ $\therefore \quad -\frac{1}{\sqrt{2}} < -\cos x < \frac{1}{2} \quad \text{or} \quad -\cos x > \frac{1}{\sqrt{2}}$ $-\frac{1}{2} < \cos x < \frac{1}{\sqrt{2}} \quad \text{or} \quad \cos x < -\frac{1}{\sqrt{2}}.$  $\therefore \quad \frac{\pi}{4} < x < \frac{2\pi}{3} \quad \text{or} \quad \frac{3\pi}{4} < x \le \pi.$
7(i)	
7(ii)	

8(a)*	$\int \frac{1}{x^2}\ \ln(x+1)\,\mathrm{d}x = -\frac{1}{x}\ln(x+1)+\int \frac{1}{x}\cdot\frac{1}{x+1}\,\mathrm{d}x$ $= -\frac{1}{x}\ln(x+1)+\int \left(\frac{1}{x}-\frac{1}{x+1}\right)\mathrm{d}x$ $= -\frac{1}{x}\ln(x+1)+\ln\lvert x\rvert-\ln(x+1)+c$ Alternative method for $\int \frac{1}{x}\cdot\frac{1}{x+1}\,\mathrm{d}x$ $=\int \frac{1}{(x+\frac{1}{2})^2-(\frac{1}{2})^2}\,\mathrm{d}x = \ln\left\lvert\frac{x}{x+1}\right\rvert+c$
8(b)*	Points of intersection of $(y-2)^2 = x+1$ and $y+2x=6$ $(4-2x)^2 = x+1 \Rightarrow 4x^2-17x+15=0 \Rightarrow x=3$ or $x=\frac{5}{4}$ or GC. Also, $(y-2)^2 = x+1 \Rightarrow y = 2\pm\sqrt{x+1}$ Volume generated $=\int_{-1}^{\frac{5}{4}}\pi\left(2+\sqrt{x+1}\right)^2\,\mathrm{d}x+\int_{\frac{5}{4}}^{3}\pi(6-2x)^2\,\mathrm{d}x-\int_{-1}^{3}\pi\left(2-\sqrt{x+1}\right)^2\,\mathrm{d}x$ $\approx 78.57254 = 78.6$ (3 s.f.)
9(i)	$l_1: \frac{x-1}{3}=\frac{y-2}{a}, z=1 \quad\Rightarrow\quad l_1: \mathbf{r}=\begin{pmatrix}1\\2\\1\end{pmatrix}+\mu\begin{pmatrix}3\\a\\0\end{pmatrix}, \mu\in\mathbb{R}$ $l_2: \mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}, \lambda\in\mathbb{R}$ If l_1 intersects with l_2, $\begin{pmatrix}1+3\mu\\2+a\mu\\1\end{pmatrix}=\begin{pmatrix}1-\lambda\\2+\lambda\\\lambda\end{pmatrix}$ $1+3\mu = 1-\lambda$ ---------- (1) $2+a\mu = 2+\lambda$ ---------- (2) $1 = \lambda$ ---------- (3) Solving for (1) and (3): $\lambda=1$ and $\mu=-\frac{1}{3}$ Therefore, point N is (0, 3, 1). Substitute the values of λ and μ into (2): $2+a\left(-\frac{1}{3}\right)=2+1$ $a=-3.$
(ii)	Let F be the foot of the perpendicular from point $P(2,1,1)$ to the line l_2. Since F lies on l_2, $\overrightarrow{OF}=\begin{pmatrix}1-\lambda\\2+\lambda\\\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$

$$\overrightarrow{PF}=\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}-1-\lambda\\1+\lambda\\-1+\lambda\end{pmatrix}$$

$$PF\perp l_2\Rightarrow\begin{pmatrix}-1-\lambda\\1+\lambda\\-1+\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=0$$

$$\Rightarrow 1+\lambda+1+\lambda-1+\lambda=0$$

$$\Rightarrow \lambda=-\frac{1}{3}$$

Thus $\overrightarrow{OF}=\begin{pmatrix}1-\left(-\frac{1}{3}\right)\\2-\frac{1}{3}\\-\frac{1}{3}\end{pmatrix}=\begin{pmatrix}4/3\\5/3\\-1/3\end{pmatrix}=\frac{4}{3}\mathbf{i}+\frac{5}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$. (shown)

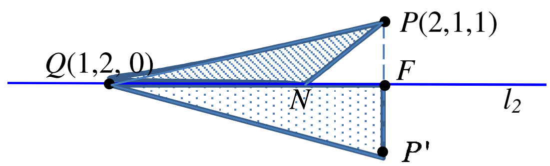
(iii) Let P' be the point of reflection of P about the line l_2.

$\overrightarrow{PF}=\overrightarrow{FP'}\Rightarrow$ By the mid-point theorem, $\overrightarrow{OF}=\dfrac{\overrightarrow{OP'}+\overrightarrow{OP}}{2}$.

$$\Rightarrow \overrightarrow{OP'}=2\overrightarrow{OF}-\overrightarrow{OP}$$

$$=2\begin{pmatrix}4/3\\5/3\\-1/3\end{pmatrix}-\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}2/3\\7/3\\-5/3\end{pmatrix}$$

(iv)



Note that Q lies on l_2.

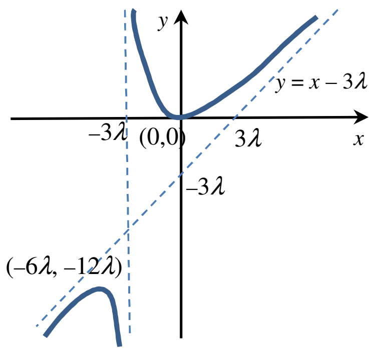
$$\frac{\text{Area of }\Delta NQP}{\text{Area of }\Delta FQP'}=\left(\frac{\frac{1}{2}PF\times NQ}{\frac{1}{2}FP'\times QF}\right)=\left(\frac{NQ}{FQ}\right)\text{ since } PF=FP'.$$

$$\overrightarrow{NQ}=\begin{pmatrix}1\\2\\0\end{pmatrix}-\begin{pmatrix}0\\3\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}\quad\Rightarrow NQ=\sqrt{3}$$

$$\overrightarrow{FQ}=\begin{pmatrix}1\\2\\0\end{pmatrix}-\begin{pmatrix}4/3\\5/3\\-1/3\end{pmatrix}=\begin{pmatrix}-1/3\\1/3\\1/3\end{pmatrix}=1/3\begin{pmatrix}-1\\1\\1\end{pmatrix}\quad\Rightarrow FQ=\tfrac{1}{3}\sqrt{3}$$

$$\frac{\text{Area of }\Delta NQP}{\text{Area of }\Delta FQP'}=\left(\frac{\sqrt{3}}{\frac{1}{3}\sqrt{3}}\right)=3.$$

Therefore, the ratio is 3:1.

10(i)	$y=\dfrac{x^2}{x+3\lambda} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x(x+3\lambda)-x^2}{(x+3\lambda)^2}=\dfrac{x^2+6\lambda x}{(x+3\lambda)^2}$ At stationary point, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow x=0$ or $x=-6\lambda$. Stationary points: $(0, 0)$, $(-6\lambda, -12\lambda)$.
(ii)	$y=\dfrac{x^2}{x+3\lambda} \Rightarrow y=x-3\lambda+\dfrac{9\lambda^2}{x+3\lambda}$ Asymptotes : $x=-3\lambda$, $y=x-3\lambda$ 
(ii)	$x^4-2\lambda x-6\lambda^2=0 \Rightarrow \dfrac{x^2}{x+3\lambda}=\dfrac{2\lambda}{x^2}$ By sketching the graph of $y=\dfrac{2\lambda}{x^2}$ on the diagram, there are 2 points of intersections, hence there are 2 roots to the equation.
(iii)	$R_f=(-\infty,-12\lambda]$, and $D_f=(-\infty,-6\lambda]$. Since $R_f \subseteq D_f$, hence f^2 exists. $f^2(-6\lambda)=f\left(\dfrac{36\lambda^2}{-6\lambda+3\lambda}\right)=f(-12\lambda)=\dfrac{144\lambda^2}{-12\lambda+3\lambda}=-16\lambda$.
11(i)	$(R-r)^2+(R-r)^2=(R+r)^2$ $\dfrac{(R-r)^2}{(R+r)^2}=\dfrac{1}{2}$ $\Rightarrow \dfrac{R-r}{R+r}=\dfrac{1}{\sqrt{2}}$ $R(\sqrt{2}-1)=(\sqrt{2}+1)r$ $R=\dfrac{\sqrt{2}+1}{\sqrt{2}-1}\times\dfrac{\sqrt{2}+1}{\sqrt{2}+1}r$ $=\dfrac{(\sqrt{2}+1)^2 r}{2-1} \Rightarrow R=(\sqrt{2}+1)^2 r$
(ii)	Volume of small cylinder $=V=\pi r^2 h=\dfrac{16\pi}{\sqrt{2}+1}$.

	$h=\dfrac{16}{r^2\left(\sqrt{2}+1\right)}$ Surface area of big cylinder $=A=2\pi Rh+2\pi R^2$. $A=2\pi\left(\sqrt{2}+1\right)^2 rh+2\pi\left(\sqrt{2}+1\right)^4 r^2$ $=2\pi\left(\sqrt{2}+1\right)^2 r\left(\dfrac{16}{r^2\left(\sqrt{2}+1\right)}\right)+2\pi\left(\sqrt{2}+1\right)^4 r^2$ $=\dfrac{32\pi\left(\sqrt{2}+1\right)}{r}+2\pi\left(\sqrt{2}+1\right)^4 r^2$ $\dfrac{\mathrm{d}A}{\mathrm{d}r}=4\pi\left(\sqrt{2}+1\right)^4 r-\dfrac{32\pi\left(\sqrt{2}+1\right)}{r^2}$ Let $\dfrac{\mathrm{d}A}{\mathrm{d}r}=0$, then $4\pi\left(\sqrt{2}+1\right)^4 r=\dfrac{32\pi\left(\sqrt{2}+1\right)}{r^2}$ $\Rightarrow \quad r^3=\dfrac{32\pi\left(\sqrt{2}+1\right)}{4\pi\left(\sqrt{2}+1\right)^4}$ $=\dfrac{8}{\left(\sqrt{2}+1\right)^3}$ $\Rightarrow \quad r=\dfrac{2}{\sqrt{2}+1}$ or $2\left(\sqrt{2}-1\right)$ $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2}=4\pi\left(\sqrt{2}+1\right)^4+\dfrac{64\pi\left(\sqrt{2}+1\right)}{r^3}$ When $r=2\left(\sqrt{2}-1\right)$, $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2}>0$. Hence, $r=2\left(\sqrt{2}-1\right)$ gives the minimum surface area of the big cylinder.
12(i)	Monthly interest chargeable $=\dfrac{3}{12}\%=\dfrac{1}{4}\%$. Let monthly repayment amount $=\$m$.

Loan Mth	Loan balance at beginning of loan month	Loan Balance at end of loan month (after monthly repayment)
1	$40000\left(\dfrac{401}{400}\right)$	$40000\left(\dfrac{401}{400}\right)-m$
2	$40000\left(\dfrac{401}{400}\right)^2-\left(\dfrac{401}{400}\right)m$	$40000\left(\dfrac{401}{400}\right)^2-\left(\dfrac{401}{400}\right)m-m$
⋮	⋮	⋮
n	$40000\left(\dfrac{401}{400}\right)^n-\left(\dfrac{401}{400}\right)^{n-1}m-\left(\dfrac{401}{400}\right)^{n-2}m-\ \dots\ -\left(\dfrac{401}{400}\right)m$	$40000\left(\dfrac{401}{400}\right)^n-\left(\dfrac{401}{400}\right)^{n-1}m-\left(\dfrac{401}{400}\right)^{n-2}m-\ \dots\ -\left(\dfrac{401}{400}\right)m-m$

	Loan balance at the end of n^{th} loan month after monthly repayment $= 40000\left(\frac{401}{400}\right)^n - \left(\frac{401}{400}\right)^{n-1} m - \left(\frac{401}{400}\right)^{n-2} m - \ldots - \left(\frac{401}{400}\right) m - m$ $= 40000\left(\frac{401}{400}\right)^n - m\left[\frac{\left(\frac{401}{400}\right)^n - 1}{\frac{401}{400} - 1}\right]$ $= 40000\left(\frac{401}{400}\right)^n - 400m\left[\left(\frac{401}{400}\right)^n - 1\right]$
(ii)	Let $40000\left(\frac{401}{400}\right)^{72} - 400m\left[\left(\frac{401}{400}\right)^{72} - 1\right] = 0$ $\Rightarrow \quad m = 607.75$ 15% of \$4000 = \$600. Since $m = 607.75 > 600$, Mary is not able to take up the car loan.
(iii)	Let $40000a^{72} - 600\left[\frac{a^{72} - 1}{a - 1}\right] \leq 0.$ From the GC, using the graph of $y = 40000x^{72} - 600\left[\frac{x^{72} - 1}{x - 1}\right]$, $1 < a \leq 1.0021378.$ $12 \times (1.0021378 - 1) \times 100\% = 2.56536\%$ $\therefore \quad 0\% < r\% \leq 2.5\%$ (to 1 decimal place)

CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC1 Promotional Examination

MATHEMATICS 9740/01

Paper 1

04 October 2013

3 hours

Additional Materials: List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER.
Place this cover sheet in front and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

Name: ____________________________ **Class:** ________________

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**Total**
Marks														
Total	3	4	5	6	6	6	6	8	8	9	11	13	15	**100**

This document consists of 5 printed pages.

Catholic Junior College

1. Sketch the curve $(y-5)^2-(x+3)^2=4$, indicating clearly the coordinates of the turning point(s) and equations of the asymptotes. **[3]**

2. Expand $\dfrac{1}{\sqrt[3]{2x-1}}$ in ascending powers of x, up to and including the term in x^2.
State the range of values of x for which this expansion is valid. **[4]**

3. The graph of $y=\mathrm{f}(x)$, where $\mathrm{f}(x)$ is a cubic polynomial, passes through the points $(1, 6)$, $(-2, 15)$ and has two stationary points at $x=\dfrac{1}{3}$ and $x=-2$. Find the equation of the curve and hence, find its x-intercept. **[5]**

4. **(a)** Given that $y=\tan^{-1}\sqrt{x}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. **[2]**

(b) Given that $\sqrt[x]{y}=\sqrt[y]{x}$, where $x>0$, $y>0$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. **[4]**

5. The parametric equations of a curve are

$$x=t^3, \quad y=\frac{7}{t}, \quad t\neq 0.$$

(i) Find the equation of the tangent to the curve at the point where $t=k$, simplifying your answer. **[3]**

(ii) Hence find the coordinates of the points X and Y where this tangent meets the x- and y-axes respectively. **[2]**

(iii) Hence or otherwise, find the area of the triangle OXY, where O is the origin. **[1]**

6. Prove by the method of differences that $\displaystyle\sum_{r=2}^{n}\frac{1}{r^2-1}=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}$. **[4]**

Hence, or otherwise, give a reason why the series $\displaystyle\sum_{r=2}^{n}\frac{1}{r^2-1}$ is convergent and state the sum to infinity. **[2]**

7. Prove by the method of mathematical induction that

$$\sum_{r=1}^{n}\cos[(2r-1)\theta]=\frac{\sin 2n\theta}{2\sin\theta}$$ for all positive integers n. **[6]**

8. **(a)** **(i)** Without using a calculator, solve the inequality $\frac{x+6}{x^2-3x-4} \le \frac{1}{4-x}$. **[3]**

(ii) Hence, deduce the range of values of x that satisfies

$$\frac{|x|+6}{x^2-3|x|-4} \le \frac{1}{4-|x|}.$$

[2]

(b) Solve the inequality $\ln(x+6) \le -\frac{x}{3}$. **[3]**

9. Charis Insurance provides an investment linked savings insurance plan with two options of premium payment, monthly and yearly.
For the monthly premium plan, premiums of \$500 are collected on the first day of each month and an interest of 0.5% per month is earned on the last day of each month, such that there is \$502.50 in the account at the end of the first month and \$1007.51 in the account at the end of the second month.

(i) Show that the total amount in the monthly premium account at the end of n complete months can be expressed as $M(1.005^n - 1)$, where M is an integer to be found. **[4]**

For the yearly premium plan, premiums of \$6000 are made on the first day of each year and an interest of 6% per year is earned on the last day of each year.

(ii) Given that the total amount in the yearly premium account at the end of k complete years is $\$\left[106000\left(1.06^k - 1\right)\right]$, find the number of complete years it will take for the total amount to first exceed \$120 000. **[2]**

A young couple who just had their first child would like to take up a savings plan for a period of 20 years to prepare for their child's university education. A friend of the couple stated that "0.5% a month is the same as 6% a year since 12 × 0.5 = 6". With reference to evidence obtained from the expressions from (i) and (ii), comment on the validity of the statement. **[2]**

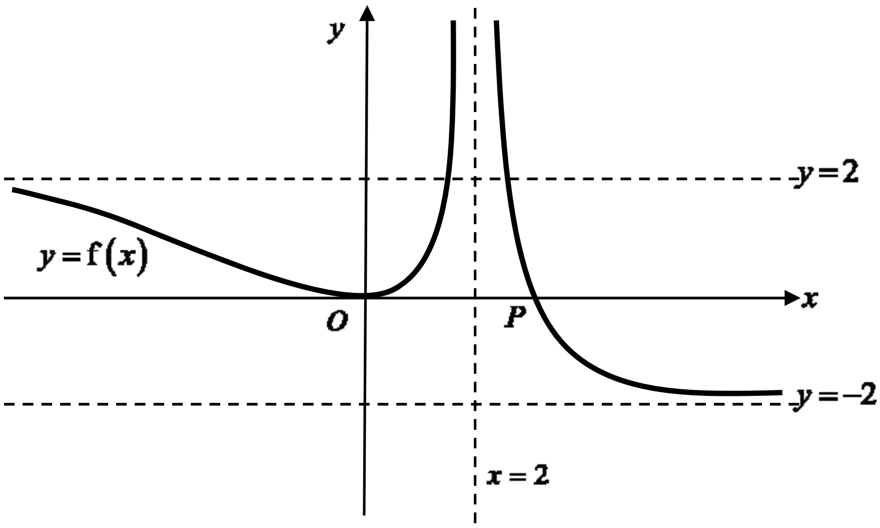
10. **(i)** Given that $f(x) = e^{\cos x + k\sin x}$, where k is a constant, find $f(0)$, $f'(0)$, $f''(0)$. Hence write down the first three terms in the Maclaurin series for $f(x)$. Give the coefficients in terms of e and k. **[5]**

(ii) Find the value of k such that $\sqrt{2}\sin(x + \frac{\pi}{4}) = \cos x + k\sin x$ for all x. **[2]**

(iii) By considering the series in part **(i)**, show that

$e^{\sqrt{2}\sin(x+\frac{\pi}{4})}\sin x \approx e\left(x^2 + x\right)$, where x is a small angle. **[2]**

11. **(a)** The diagram below shows the graph of $y = \mathrm{f}(x)$. It passes through the origin O and P (3, 0), and has asymptotes $x = 2$, $y = 2$ .and $y = -2$.



On separate diagrams, sketch the graph of

(i) $y = \mathrm{f}'(x)$, **[3]**

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, **[3]**

indicating clearly any asymptote(s) and axial intercept(s).

(b) The graph of $y = \dfrac{1}{2x+3}$ is transformed by a reflection in the y-axis, followed by a translation of 1 unit in the negative x-direction, followed by a stretch with scale factor 2 parallel to the x-axis.

(i) Find the equation of the new graph in the form $y = \mathrm{f}(x)$. **[3]**

(ii) Hence, or otherwise, sketch the new graph with any axial intercept(s) and asymptote(s) indicated clearly. **[2]**

12. Functions f and g are defined by

$$\mathrm{f}: x \mapsto (4+2x)^{\frac{1}{2}}, \quad x \in \mathbb{R},\ 0 \le x \le 16$$
$$\mathrm{g}: x \mapsto 3x+1, \quad x \in \mathbb{R}$$

(i) State the range of f. **[1]**

(ii) With the aid of a diagram, show that f^{-1} exists and define f^{-1} in a similar form. **[4]**

(iii) On the same diagram as in part **(ii)**, sketch the graphs of f^{-1} and $\mathrm{f}^{-1}\mathrm{f}$, indicating their endpoints. **[3]**

(iv) Explain why the *x*-coordinates of the point(s) of intersection between the graphs in part **(iii)** satisfies the equation $x^2 - 2x - 4 = 0$. **[1]**

(v) State whether the composite function fg exists, justifying your answer. **[2]**

(vi) Find the largest possible domain of g in the form [*m*, *n*], *m*, *n* $\in \mathbb{R}$, for which the composite function fg exists. **[2]**

13. **(a)** Relative to the origin *O*, two points *A* and *B* have position vectors given by **a** = 4**i** + 2**j** + 3**k** and **b** = 4**i** – 2**j** + 7**k** respectively. The point *P* divides *AB* in the ratio 3 : 1.

(i) Find the coordinates of *P*. **[2]**

(ii) The vector **c** is a unit vector in the direction of $\overrightarrow{OP}$.
Write **c** as a column vector, and give the geometrical meaning of $|\mathbf{a} \bullet \mathbf{c}|$. **[2]**

(iii) By using vector cross product, find the exact area of triangle *OAP*. **[3]**

(b) The line *l* has equation $\dfrac{x-3}{-3} = y+3 = \dfrac{z-1}{-2}$ and the plane *p* has equation $3x - y + 2z = 0$.

(i) Show that *l* is perpendicular to *p*. **[2]**

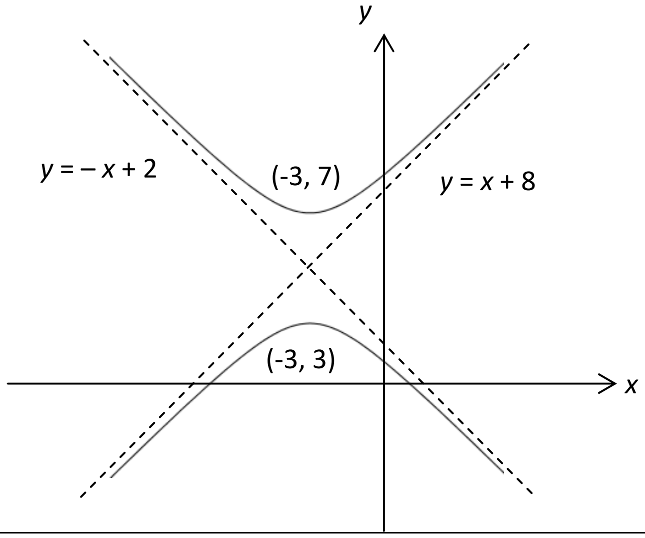
(ii) Find the coordinates of the point of intersection of *l* and *p*. **[3]**

(iii) Show that the point *C* with coordinates $(-9, 1, -7)$ lies on *l*.

Find the coordinates of the point *C’* which is the mirror image of *C* in *p*. **[3]**

— End of Paper —

Solutions

1	$(y-5)^2-(x+3)^2=4$ $-\dfrac{(x+3)^2}{2^2}+\dfrac{(y-5)^2}{2^2}=1$ Asymptotes: $(y-5)^2=(x+3)^2$ $y-5=\pm(x+3)$ $y=x+8$ or $y=-x+2$ 
2	$\dfrac{1}{\sqrt[3]{2x-1}}=(2x-1)^{-\frac{1}{3}}=-1(1-2x)^{-\frac{1}{3}}$ $=-\left[1+\left(-\frac{1}{3}\right)(-2x)+\dfrac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2}(-2x)^2+....\right]$ $\approx-(1+\frac{2}{3}x+\frac{8}{9}x^2)$ Validity: $-\frac{1}{2}<x<\frac{1}{2}$
3	Let $y=Ax^3+Bx^2+Cx+D$ $\therefore \dfrac{dy}{dx}=3Ax^2+2Bx+C$ $A+B+C+D=6$ $-8A+4B-2C+D=15$ $A+2B+3C=0$ $12A-4B+C=0$ Solving, $A=2, B=5, C=-4, D=3$ $y=2x^3+5x^2-4x+3$

	When $y = 0$, $x = -3.26$ (3sf) x-intercept = $(-3.26, 0)$
4	(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\sqrt{x}\right) = \frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2}x^{-\frac{1}{2}}$ $= \frac{1}{2\sqrt{x}(1+x)}$ (b) $\sqrt[x]{y} = \sqrt[y]{x}$ Taking logarithm on both sides, $\frac{1}{x}\ln y = \frac{1}{y}\ln x$ $y\ln y = x\ln x$ Differentiating both sides, $y\cdot\frac{1}{y}\cdot\frac{dy}{dx} + \frac{dy}{dx}\cdot\ln y = x\cdot\frac{1}{x} + 1\cdot\ln x$ $(1+\ln y)\frac{dy}{dx} = 1+\ln x$ $\frac{dy}{dx} = \frac{1+\ln x}{1+\ln y}$
5	(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t} = \left(-\frac{7}{t^2}\right)\div\left(3t^2\right) = -\frac{7}{3t^4}$ $y - \frac{7}{k} = -\frac{7}{3k^4}\left(x-k^3\right)$ $y = -\frac{7}{3k^4}x + \frac{28}{3k}$ (ii) $y - \frac{7}{k} = -\frac{7}{3k^4}\left(x-k^3\right)$ $y = -\frac{7}{3k^4}x + \frac{28}{3k}$ $y = 0,\quad x = 4k^3 \quad\Rightarrow\quad X \text{ is } \left(4k^3, 0\right)$ $x = 0,\quad y = \frac{28}{3k} \quad\Rightarrow\quad Y \text{ is } \left(0, \frac{28}{3k}\right)$ (iii) Area of OXY $= \frac{1}{2}(OX)(OY)$ $= \frac{1}{2}\left(4k^3\right)\left(\frac{28}{3k}\right)$ $= \frac{56}{3}k^2 \text{ units}^2$
6	$\sum_{r=2}^{n}\frac{1}{r^2-1} = \sum_{r=2}^{n}\frac{1}{(r-1)(r+1)}$

	$=\frac{1}{2}\sum_{r=2}^{n}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$ $=\frac{1}{2}[\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+$ $\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+$ $\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+$ $\vdots$ $\left(\frac{1}{n-3}-\cancel{\frac{1}{n-1}}\right)+$ $\left(\cancel{\frac{1}{n-2}}-\frac{1}{n}\right)+$ $\left(\cancel{\frac{1}{n-1}}-\frac{1}{n+1}\right)$ $=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{(n+1)}\right)$ $=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}$ $\sum_{r=2}^{n}\frac{1}{r^2-1}=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}$. $n\to\infty$,. $\frac{1}{2n}\to 0$, $\frac{1}{2(n+1)}\to 0$, so $\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}\to\frac{3}{4}$ so $\sum_{r=2}^{\infty}\frac{1}{r^2-1}=\frac{3}{4}$
7	Let P_n be the statement $\sum_{r=1}^{n}\cos[(2r-1)\theta]=\frac{\sin 2n\theta}{2\sin\theta}$ for $n\in Z^+, n\geq 1$ When $n=1$, L.H.S. = $\cos\theta$ R.H.S. = $\frac{\sin 2\theta}{2\sin\theta}=\frac{2\sin\theta\cos\theta}{2\sin\theta}=\cos\theta$ =L.H.S. Assume P_k is true, i.e. $\sum_{r=1}^{k}\cos[(2r-1)\theta]=\frac{\sin 2k\theta}{2\sin\theta}$ for some $k\in Z^+$, $k\geq 1$. Required to prove P_{k+1} is true, i.e. $\sum_{r=1}^{k+1}\cos[(2r-1)\theta]=\frac{\sin[2(k+1)\theta]}{2\sin\theta}$ L.H.S. = $\sum_{r=1}^{k}\cos[(2r-1)\theta]+u_{k+1}$ $=\frac{\sin 2k\theta}{2\sin\theta}+\cos[(2k+1)\theta]$ $=\frac{\sin 2k\theta+2\cos[(2k+1)\theta]\sin\theta}{2\sin\theta}$

$$= \frac{\sin 2k\theta + \sin[2(k+1)\theta] - \sin 2k\theta}{2\sin\theta}$$

$$= \frac{\sin[2(k+1)\theta]}{2\sin\theta} = \text{R.H.S.}$$

P_k is true $\Rightarrow P_{k+1}$ is true.

Hence, by Mathematical Induction, P_n is true for all $n \in Z^+$, $n \geq 1$.

8

(a)(i)

$$\frac{x+6}{x^2-3x-4} \leq \frac{1}{4-x}$$

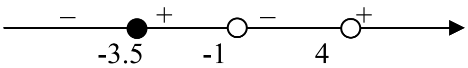
$$\frac{x+6}{(x+1)(x-4)} - \frac{1}{4-x} \leq 0$$

$$\frac{x+6}{(x+1)(x-4)} + \frac{1}{x-4} \leq 0$$

$$\frac{x+6+(x+1)}{(x+1)(x-4)} \leq 0$$

$$\frac{2x+7}{(x+1)(x-4)} \leq 0$$

Using test-point method,



$\therefore x \leq -3.5$ or $-1 < x < 4$

(ii)

$$\frac{|x|+6}{x^2-3|x|-4} \leq \frac{1}{4-|x|}$$

Replace x by $|x|$

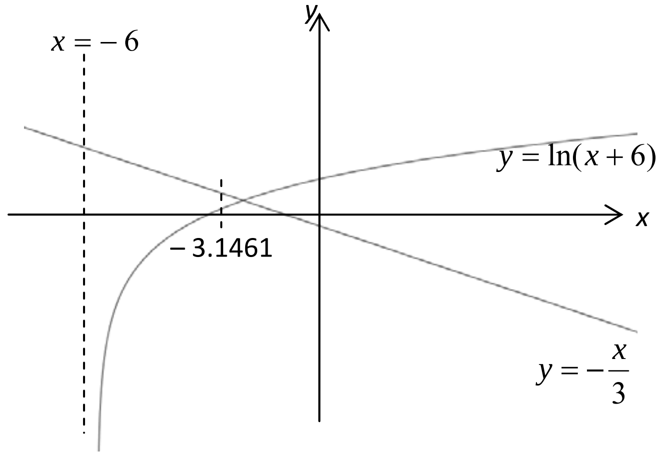
$\therefore |x| \leq -3.5$ or $-1 < |x| < 4$

(no real solution) $|x| < 4$

$-4 < x < 4$

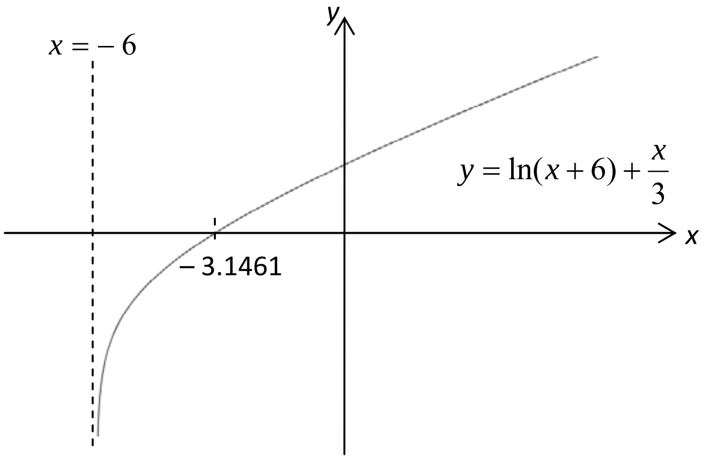
(b)

Draw the graphs of $y = \ln(x+6)$ and $y = -\frac{x}{3}$.



Ans: $-6 < x \leq -3.15$

Alternative solution: Draw the graph of $y = \ln(x+6) + \dfrac{x}{3}$.



Ans: $-6 < x \leq -3.15$

9 (i)

Total amount after 1 month = 1.005(500)

Total amount after 2 month = $1.005^2(500) + 1.005(500)$

Total amount after 3 month

$= 1.005^3(500) + 1.005^2(500) + 1.005(500)$

Total amount after n months $= 1.005^n(500) + 1.005^{n-1}(500) + \cdots + 1.005(500)$

$$= \frac{1.005(500)\left(1.005^n - 1\right)}{1.005 - 1}$$

$$= 100500\left(1.005^n - 1\right)$$

M = 100500

(ii)

$$106000\left(1.06^k - 1\right) > 120000$$

Solving, $k > 12.99$

$\therefore k = 13$ complete years.

From (i) and (ii), the final amount after 20 years is

$100500\left(1.005^{240}-1\right)=\$232175.55$ for monthly account

$106000\left(1.06^{20}-1\right)=\$233956.36$ for yearly account

Hence the statement is invalid as the final total amount differs quite significantly

10 (i)

We are given that $f(x)=e^{\cos x+k\sin x}$.

Differentiating,

$f'(x)=e^{\cos x+k\sin x}(-\sin x+k\cos x)$.

Differentiating,

$f''(x)=e^{\cos x+k\sin x}(-\cos x-k\sin x)+e^{\cos x+k\sin x}(-\sin x+k\cos x)^2$.

So we have

$f'(0)=e^{\cos 0+k\sin 0}(-\sin 0+k\cos 0)$

$=ke$,

$f''(0)=e^{\cos 0+k\sin 0}(-\cos 0-k\sin 0)+e^{\cos 0+k\sin 0}(-\sin 0+k\cos 0)^2$

$=(k^2-1)e$,

$f(0)=e^{\cos 0+k\sin 0}=e$.

Hence,

$f(x)=e+kex+\frac{1}{2}(k^2-1)ex^2+\cdots$.

(ii)

Since

$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$

$=\sqrt{2}\left(\sin x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x\right)$

$=\sqrt{2}\cdot\frac{\sqrt{2}}{2}(\sin x+\cos x)$

$=\cos x+\sin x$,

we have $k=1$.

(iii)

Since x is a small angle,

$\sin x\approx x$,

then

$\sin x e^{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}$

$=\sin x e^{\cos x+\sin x}$

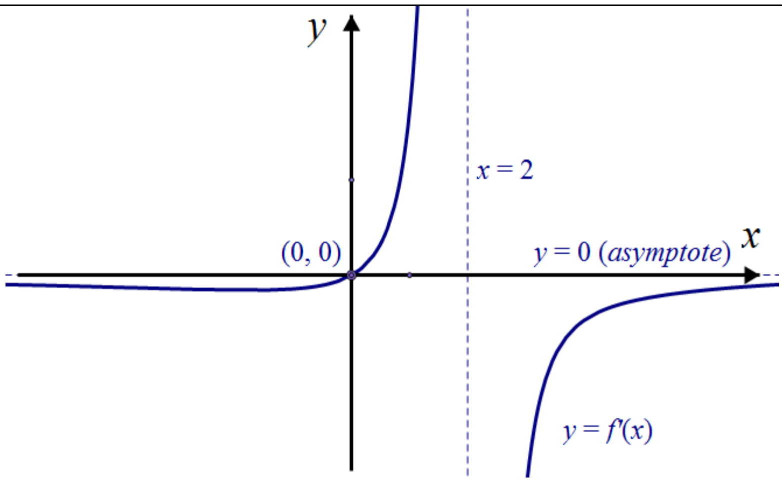
$\approx x\left[e+1\cdot ex+\frac{1}{2}(1^2-1)ex^2\right]$

$=x(e+ex)$

$=(x^2+x)e$.

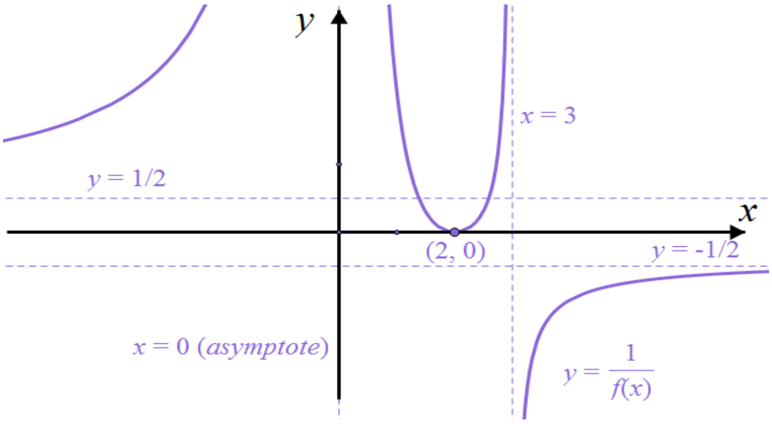
11 (a)(i)

$y=f'(x)$



(a)(ii)

$$y = \frac{1}{f(x)}$$



(b)(i)

$$y = \frac{1}{2x+3}$$

$\downarrow$

$$y = \frac{1}{2(-x)+3} = \frac{1}{-2x+3}$$

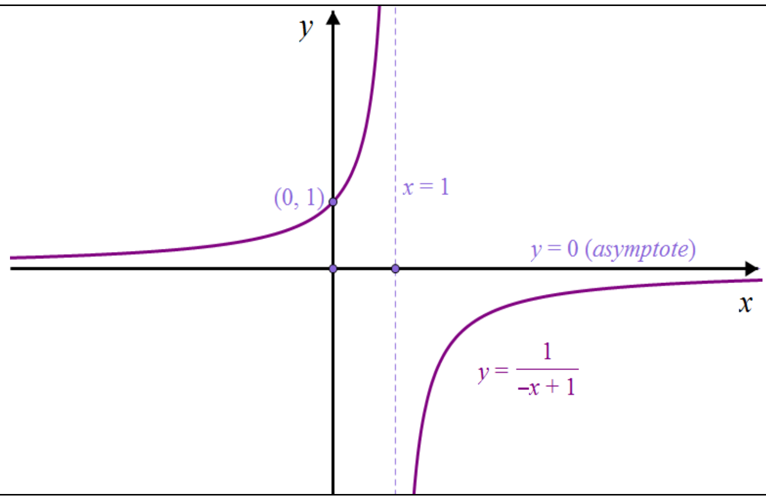
$\downarrow$

$$y = \frac{1}{-2(x+1)+3} = \frac{1}{-2x+1}$$

$\downarrow$

$$y = \frac{1}{-2\left(\frac{1}{2}x\right)+1} = \frac{1}{-x+1}$$

(b)(ii)



12 (i)

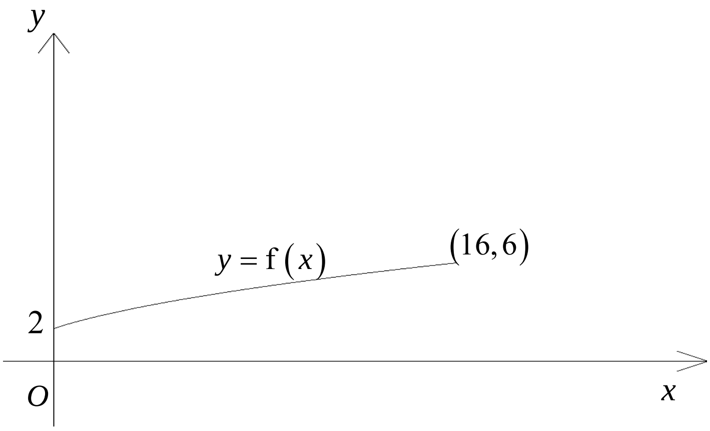
As f is an increasing function,

$f(0) = (4)^{½} = 2$

$f(16) = (36)^{½} = 6$

Range of f, $R_f = [2,6]$

(ii)



f is a 1-1 function as the line $y = k$, $2 \le k \le 6$ intersects the graph of f exactly once.

(OR: f is a 1-1 function as any line $y = k$ intersects the graph of f at most once.)

Hence f^{-1} exists.

Let $y = f(x) = (4+2x)^{1/2}$

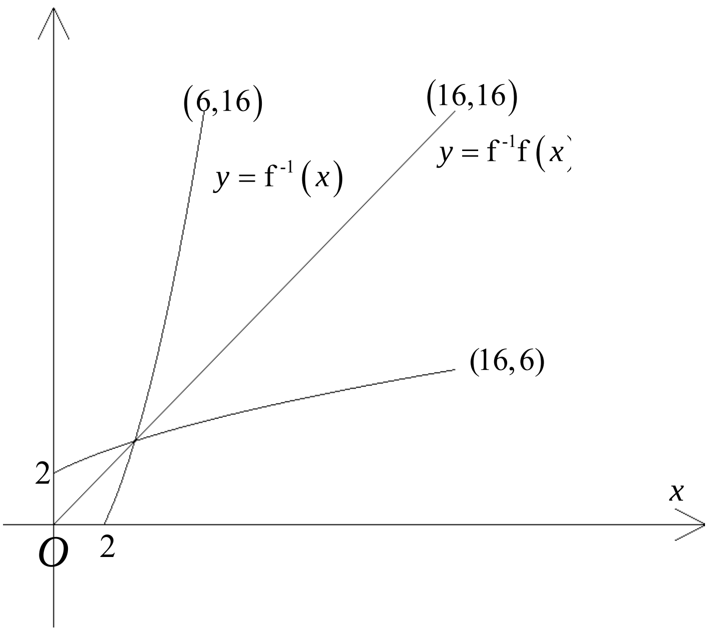
$y^2 = 4+2x$

$x = \frac{1}{2}(y^2 - 4)$

$f^{-1}(x) = \frac{1}{2}(x^2 - 4)]$

$D_{f^{-1}} = R_f = [2,6]$

Hence $f^{-1} : x \to \frac{1}{2}(x^2 - 4),\ 2 \le x \le 6$

	(iii) 
	(iv) By considering f(x) = x, $(4 + 2x)^{1/2} = x$ $x^2 - 2x - 4 = 0$ The x-coordinates of the points of intersection satisfy the equation $x^2 - 2x - 4 = 0$.
	(v) $R_g = \mathbb{R}$ $D_f = [0, 16]$ $R_g \nsubseteq D_f$ => fg does not exist.
	(vi) Consider $R_g = D_f$ $3x+1 = 0 \Rightarrow x = -1/3$ $3x+1 = 16 \Rightarrow x = 5$ Hence $[-\frac{1}{3}, 5]$ is the largest possible domain of g for fg to exist.
13	(a)(i) $\overrightarrow{OP} = \dfrac{\overrightarrow{OA} + 3\overrightarrow{OB}}{4}$ $= \dfrac{1}{4}\left[\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}\right]$ $= \begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix}$ (a)(ii)

$$\mathbf{c} = \frac{1}{\sqrt{4^2+(-1)^2+6^2}}\begin{pmatrix}4\\-1\\6\end{pmatrix} = \frac{1}{\sqrt{53}}\begin{pmatrix}4\\-1\\6\end{pmatrix}$$

Geometrically, $|\mathbf{a}\bullet\mathbf{c}|$ is the length of projection of the vector **a** on $\overrightarrow{OP}$ or **c**.

(a)(iii)

$$\mathbf{a}\times\mathbf{p} = \begin{pmatrix}4\\2\\3\end{pmatrix}\times\begin{pmatrix}4\\-1\\6\end{pmatrix} = \begin{pmatrix}15\\-12\\-12\end{pmatrix}$$

Area of triangle *OAP*

$$= \frac{1}{2}|\mathbf{a}\times\mathbf{p}|$$
$$= \frac{1}{2}\sqrt{15^2+(-12)^2+(-12)^2}$$
$$= \frac{1}{2}\sqrt{513}$$

(b)(i)

Line l: $r = \begin{pmatrix}3\\-3\\1\end{pmatrix} + \mu\begin{pmatrix}-3\\1\\-2\end{pmatrix}, \quad \mu\in\mathbb{R}$

Plane p: $\mathbf{r}\bullet\begin{pmatrix}3\\-1\\2\end{pmatrix} = 0$

Since $\begin{pmatrix}-3\\1\\-2\end{pmatrix} = -\begin{pmatrix}3\\-1\\2\end{pmatrix}$, the normal of the plane p is parallel to the line l, the line l is perpendicular to p.

(b)(ii)

When l intersects p, $\begin{pmatrix}3-3\mu\\-3+\mu\\1-2\mu\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\2\end{pmatrix} = 0$

$$9-9\mu+3-\mu+2-4\mu = 0$$
$$\mu = 1$$

Coordinates of point of intersection = (0, -2, -1)

(b)(iii)

Suppose C with coordinates $(-9,1,-7)$ lies on l, $\begin{pmatrix}-9\\1\\-7\end{pmatrix} = \begin{pmatrix}3-3\mu\\-3+\mu\\1-2\mu\end{pmatrix}$

	$-9 = 3 - 3\mu$ $\mu = 4$ Since C satisfies the parametric equations of l with $\mu = 4$, therefore C lies on l. We note that C lies on l, l is perpendicular to p and l meets p at (0, -2, -1), By Ratio Theorem, $\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = \dfrac{\begin{pmatrix} -9 \\ 1 \\ -7 \end{pmatrix} + \overrightarrow{OC'}}{2}$ $\overrightarrow{OC'} = 2\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} -9 \\ 1 \\ -7 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \\ 5 \end{pmatrix}$

HWA CHONG INSTITUTION
JC1 Promotional Examinations
Higher 2

MATHEMATICS 9740

7 October 2013

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

1 Sophia has a total saving of \$90 million in three accounts A, B and C with \$$x$ million, \$$y$ million and \$$z$ million respectively. She transfers funds among the accounts based on the table below.

Percentage of Fund transferred from initial amount in	To Account A	To Account B	To Account C
Account A	–	37.5%	12.5%
Account B	5%	–	5%
Account C	10%	20%	–

For instance, 37.5% and 12.5% of the initial amount in Account A are transferred to Account B and Account C respectively.

As a result of the funds transfer, the amount in Account A decreases by \$16 million and the amount in Account B increases by \$19 million.

(i) By considering the amount in Account A, show that

$$0.5x - 0.05y - 0.1z = 16.$$ [1]

(ii) By forming a system of linear equations, find the values of x, y and z. [3]

2 It is given that the expansion of $(2+px)^{-q}$ in ascending powers of x, up to and including the term in x, is $\frac{1}{4} - x$. Find the values of p and q.

Find, in terms of n, the coefficient of x^n in the above expansion. [4]

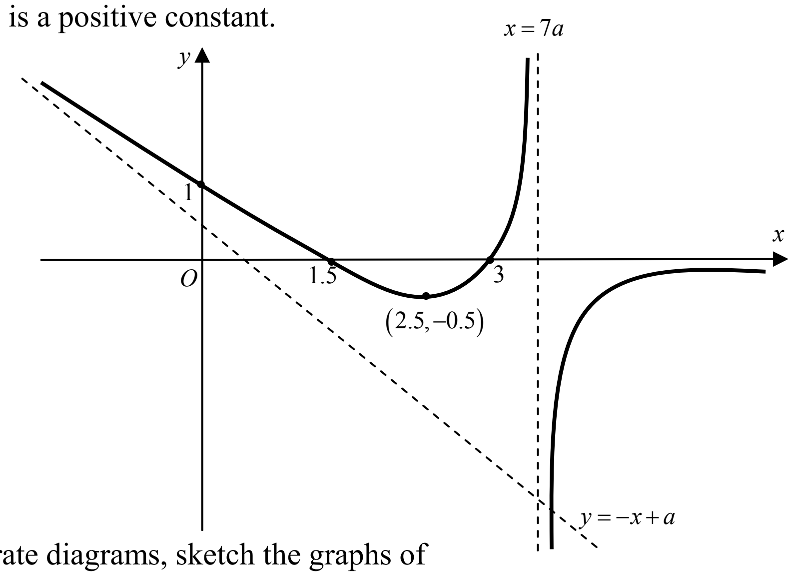
3 A water tank contains 8000 litres of water initially. At the beginning of each day, 500 litres of water is added to the tank. At the end of each day, 10% of the amount of water in the tank will be used.

(i) Show that the amount of water in the tank after 3 days is 7051.5 litres. [1]

(ii) Find the least number of days it will take for the water in the tank to be less than 5000 litres. [3]

(iii) Will the tank ever dry up? Justify your answer. [1]

4 The diagram below shows the graph of $y = \mathrm{f}(x)$. It cuts the axes at the points $(0,\ 1)$, $(1.5,\ 0)$ and $(3,\ 0)$. It has a minimum point at $(2.5, -0.5)$. The horizontal, vertical and oblique asymptotes are $y = 0$, $x = 7a$ and $y = -x + a$ respectively, where a is a positive constant.



On separate diagrams, sketch the graphs of

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \mathrm{f}'(x)$, [3]

showing clearly the axial intercepts, the stationary points and the equations of the asymptotes where applicable.

5 A sequence of real numbers $\{u_n\}$, for $n \in \mathbb{Z}^+$, satisfies the recurrence relation $\dfrac{u_{n+1} + a}{u_n + b} = \dfrac{a}{b}$, with $u_1 = a$, where a and b are fixed non-zero real constants and $a \neq b$.

(i) Given that the limit l of the sequence $\{u_n\}$ exists, find the value of l. [2]

(ii) By expressing u_{n+1} in terms of u_n, find an expression for u_n, leaving your answer in terms of a, b and n. [2]

(iii) Given that the sum to infinity S for the sequence $\{u_n\}$ exists, state an inequality satisfied by a and b. Find S in terms of a and b. [2]

6 **(a)** By using the substitution $u=9+4x^2$, find $\int x^3\sqrt{9+4x^2}\ \mathrm{d}x$. [4]

(b) Evaluate $\int_0^1 x^2 \tan^{-1} x\ \mathrm{d}x$, giving your answer in exact form. [4]

7 The coordinates of 3 points A, B and C are $(2,\ 0,-1)$, $(-3,\ 1,\ 2)$ and $(1,-2,-4)$ respectively.

(a) Find the point D on the x-axis such that there exists a point P on line AB where C, D and P are collinear. [4]

(b) Find two possible points E on the x-y plane, such that $\overrightarrow{OE}$ is a unit vector and $\measuredangle AOE = 150^\circ$. [4]

8 **(i)** Express $\dfrac{2}{r(r+1)(r+3)}$ in partial fractions. [2]

(ii) Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{2r(r+1)(r+3)}$. [3]

(iii) Using the result in part **(ii)**, determine the value of $\displaystyle\sum_{r=5}^{\infty} \frac{1}{2r(r-2)(r-3)}$. [3]

9 Prove by mathematical induction that for all $n \in \mathbb{Z}^+$,

$$1+(1+2)+(1+2+3)+(1+2+3+4)+\ldots+(1+2+3+\ldots+n)=\frac{1}{6}n(n+1)(n+2).$$ [5]

Hence find, in terms of n,

(i) $3+(3+6)+(3+6+9)+(3+6+9+12)+\ldots+(3+6+9+\ldots+(6n-3))$, [2]

(ii) $3\times(3\times9)\times(3\times9\times27)\times\ldots\times(3\times9\times27\times81\times\ldots\times3^n)$. [2]

10 The functions f and g are defined as follows.

$$\mathrm{f}(x) = \sqrt{|2-x|+1}\ , \qquad x \in \mathbb{R},$$

$$\mathrm{g}(x) = \begin{cases} -\dfrac{1}{3}x + \dfrac{2}{3}\ , & 0 \le x < 2, \\ 1-(x-3)^2, & x \ge 2. \end{cases}$$

(i) Show that f^{-1} does not exist. [1]

(ii) If the domain of f is restricted to $[k, \infty)$ such that f^{-1} exists, state the least value of k and define f^{-1} in a similar form. [3]

Use the new domain of f found in part **(ii)** for the following parts.

(iii) Show algebraically that there is no value of x for which $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. [2]

(iv) Find the range of the composite function gf. [2]

(v) Find the value of x such that $\mathrm{gf}(x) = 1$. [1]

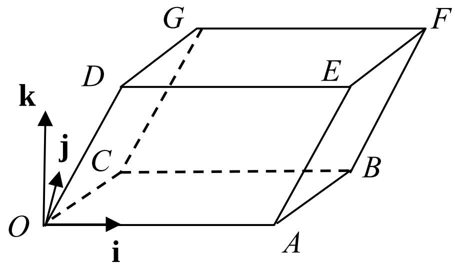
11 Sketch the graph of $y = \dfrac{2x^2-3}{x-2}$, showing clearly the axial intercepts, the stationary points and the equations of the asymptotes where applicable. [3]

(a) Solve the inequality $\dfrac{2x^2-3}{x-2} \ge 1$. [2]

Deduce the solution of the inequality $\dfrac{2\sin^2 x-3}{\sin x-2} \ge 1$, where $0 \le x \le 2\pi$. [2]

(b) Describe fully a sequence of transformations which would transform the graph of $y = 2x + \dfrac{5}{x}$ to the graph of $y = \dfrac{2x^2-3}{x-2}$. [3]

12 An art structure, which is a parallelpiped (made of 6 faces of parallelograms) has a horizontal base *OABC*, with *OA*, *OC* and *OD* as its three sides and remaining vertices are *B*, *E*, *F*, and *G* as shown in the diagram below.



It is given that $\overrightarrow{OA} = 5\mathbf{i}$ and $\overrightarrow{OC} = \mathbf{i} + 7\mathbf{j}$. The lines l_1 and l_2 have equations given by $l_1 : \underset{\sim}{r} = (5+\lambda)\mathbf{i} + (7\lambda - 14)\mathbf{j} + 6\mathbf{k}$, where λ is a real parameter and $l_2 :\ 3x = z + 15$, $y = 0$. *E* and *F* are on line l_1, and *A* and *E* are on line l_2.

(i) Find the position vector of *E*. [2]

(ii) Find the equation, in scalar product form, of the plane *ABFE*. [3]

(iii) Find the projection vector of $\overrightarrow{AE}$ onto the base *OABC*. Hence, or otherwise, find the area of the projection of the plane *ABFE* onto the base. [2]

(iv) Find the equation of the line l_3, which is the reflection of line *AE* about the base *OABC*. [2]

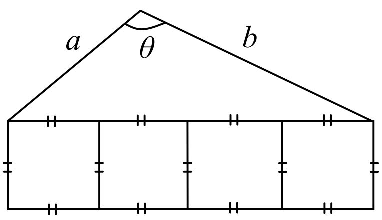
(v) An architect wants to add a shelter which has the plane equation $x + ay + bz = c$, where *a*, *b* and *c* are unknown constants. He wants the shelter to meet the plane *ABFE* at *EF*. What can be said about the values of *a*, *b* and *c* ? [2]

13 **(a)** Using differentiation, find the equation of the tangent at the point $(-2,\ 1)$ on the curve $x^3 - y^3 = 3(x - y)$. [3]

(b) A spherical balloon is inflated such that $0.1\ \text{m}^3$ of air is pumped into the balloon every second. Find the rate of change of its surface area when the diameter is $1\ \text{m}$. [4]

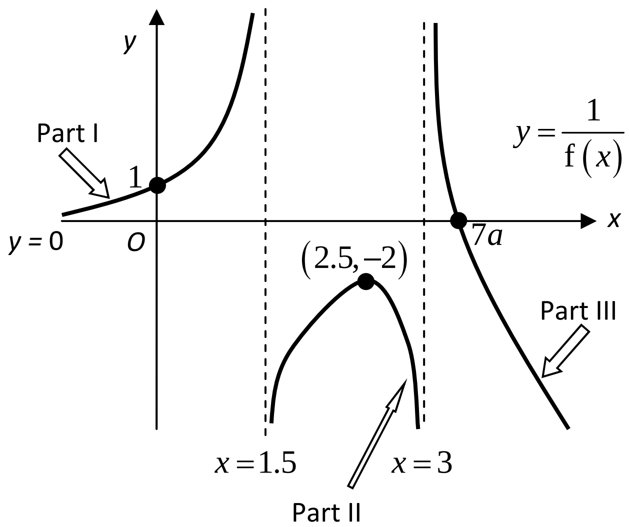
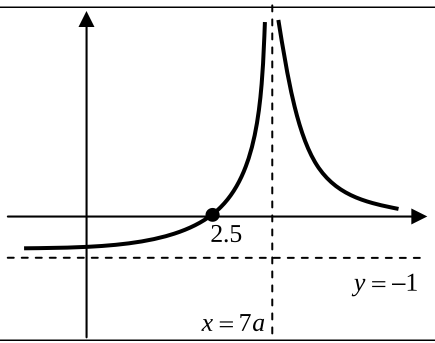
[Volume of sphere = $\frac{4}{3}\pi r^3$ and surface area of sphere = $4\pi r^2$.]

(c) When designing the floor plan of his new house, Mr Lim wants to build a triangular garage with 2 adjacent walls of fixed lengths a and b meters and making an angle of θ radians. On the third side of his triangular garage, he intends to build 4 square-shaped rooms of equal size (see diagram). Find the value of θ when the total area covered by the garage and the 4 rooms is a maximum. [5]



Suggested Solutions 2013 C1 H2 Math Promotional Examination

Qtn	Solutions
1(i)	Funds transferred into Account *A*: $0.05y+0.1z$ Funds transferred from Account *A*: $0.375x+0.125x=0.5x$ So we have $0.5x-\left(0.05y+0.1z\right)=16$ i.e. $0.5x-0.05y-0.1z=16$ ----(1)
(ii)	Similarly, for Account *B*, we have $-0.375x+0.1y-0.2z=-19$ ----(2) We also know $x+y+z=90$ ----(3) Solving (1), (2), (3) using GC, we have $x=40, y=20, z=30$
2	$\left(2+px\right)^{-q}$ $=2^{-q}\left(1+\dfrac{px}{2}\right)^{-q}$ $=2^{-q}\left(1+\left(-q\right)\left(\dfrac{px}{2}\right)+...\right)$ $=2^{-q}\left(1-\dfrac{pqx}{2}+...\right)$ $\approx\dfrac{1}{4}-x$ $\Rightarrow 2^{-q}=\dfrac{1}{4}$ ---(1) & $\dfrac{1}{4}\left(\dfrac{-2p}{2}\right)=-1$ ---(2) $q=2, p=4$
	$\left(2+4x\right)^{-2}$ $=\dfrac{1}{4}\left(1+2x\right)^{-2}$ $=\dfrac{1}{4}\left(1+\left(-2\right)\left(2x\right)+\dfrac{\left(-2\right)\left(-3\right)}{2!}\left(2x\right)^2+\dfrac{\left(-2\right)\left(-3\right)\left(-4\right)}{3!}\left(2x\right)^3+...\right)$ x^n coefficient $=\dfrac{1}{4}\left(\dfrac{\left(-2\right)\left(-3\right)\left(-4\right)...\left(-\left(n+1\right)\right)}{n!}\right)\left(2\right)^n$ $=\dfrac{1}{4}\left(-1\right)^n\left(n+1\right)2^n=\left(-1\right)^n\left(n+1\right)2^{n-2}$
3(i)	Vol of water at end of Day 1 $=0.9(8500)$ Vol of water at end of Day 2

	$= 0.9\left(500 + 0.9(8500)\right) = 0.9\left(500\right) + 0.9^2(8500)$ Vol of water at end of Day 3 $= 0.9(500) + 0.9^2(500) + 0.9^3(8500)$ = 7051.5 litres
(ii)	Vol of water at end of Day *n*, *V* $= 0.9(500) + 0.9^2(500) + ... + 0.9^{n-1}(500) + 0.9^n(8500)$ $= 500\left(0.9 + 0.9^2 + ... + 0.9^{n-1}\right) + 0.9^n(8500)$ $= 500\left[\frac{0.9\left(1-0.9^{n-1}\right)}{1-0.9}\right] + 0.9^n(8500)$ $= 4500\left[1-0.9^{n-1}\right] + 0.9^n(8500)$ For $V < 5000$, $4500\left[1-0.9^{n-1}\right] + 0.9^n(8500) < 5000$ From G.C, n = 18: V = 5025.3; n = 19: V = 4972.8; n = 20: V = 4925.5 Least $n = 19$ Least number of days = 19.
(iii)	As $n \to \infty$, $V \to 4500$ Therefore, water tank will never dry up.
4i	
ii	

n	V
18	5025.3
19	4972.8
20	4925.5

<table>
<tr><td>5</td><td>

(i) Since l is the limit,

As $n \to \infty, \quad u_n \to l, \ \ u_{n+1} \to l$

$$\therefore \frac{l+a}{l+b} = \frac{a}{b}$$

$$\Rightarrow \mathrm{b}(l+a) = a(l+b)$$

$$\Rightarrow bl = al$$

$$\Rightarrow l(b-a) = 0 \quad (\because a \neq b)$$

$$\Rightarrow l = 0$$

(ii)

$$\frac{u_{n+1}+a}{u_n+b} = \frac{a}{b}$$

$$\Rightarrow b(u_{n+1}+a) = a(u_n+b)$$

$$\Rightarrow bu_{n+1} = au_n$$

$$\Rightarrow u_{n+1} = \frac{a}{b}u_n$$

Hence $\{u_n\}$ is a GP with ratio $\frac{a}{b}$ and since $u_1 = a$,

$$u_n = a\left(\frac{a}{b}\right)^{n-1}$$

(ii) Since S exists, $|r| < 1 \Rightarrow \left|\frac{a}{b}\right| < 1$

$$S = \frac{a}{1-\frac{a}{b}}$$

$$= \frac{ab}{b-a}$$

</td></tr>
<tr><td>6(i)</td><td>

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 8x$$

$$\int x^3\sqrt{9+4x^2}\ \mathrm{d}x = \int \frac{1}{8}x^2(8x)(9+4x^2)^{1/2}\ \mathrm{d}x$$

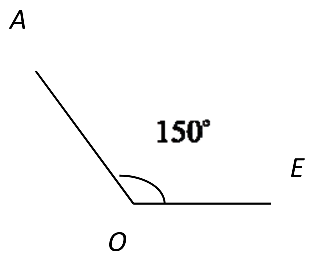
$$= \frac{1}{8}\int\left(\frac{u-9}{4}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)(u)^{1/2}\ \mathrm{d}x$$

$$= \int \frac{1}{32}u^{3/2} - \frac{9}{32}u^{1/2}\ \mathrm{d}u$$

$$= \frac{1}{80}u^{5/2} - \frac{3}{16}u^{3/2} + C$$

$$= \frac{1}{80}(9+4x^2)^{5/2} - \frac{3}{16}(9+4x^2)^{3/2} + C$$

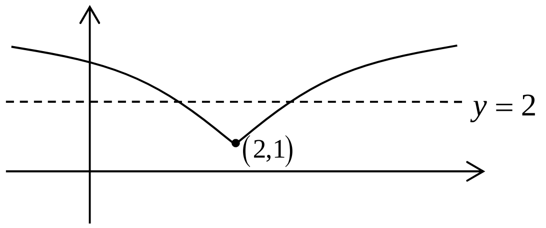
</td></tr>
</table>

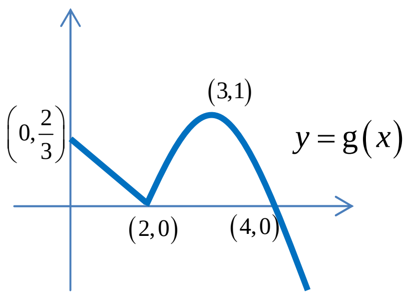
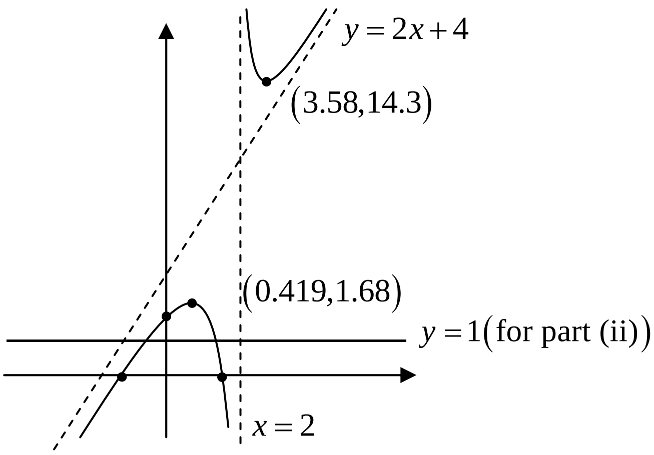
(ii)	$\int_0^1 x^2 \tan^{-1} x \, dx = \left[\left(\frac{1}{3}x^3\right)\tan^{-1} x\right]_0^1 - \int_0^1 \left(\frac{1}{3}x^3\right)\left(\frac{1}{1+x^2}\right) dx$ $= \left[\left(\frac{1}{3}x^3\right)\tan^{-1} x\right]_0^1 - \frac{1}{3}\int_0^1 \left(x - \frac{x}{1+x^2}\right) dx$ $= \left[\left(\frac{1}{3}x^3\right)\tan^{-1} x - \frac{1}{3}\left(\frac{1}{2}x^2 - \frac{1}{2}\ln\left(1+x^2\right)\right)\right]_0^1$ $= \left(\frac{1}{3}\right)\left(\frac{\pi}{4}\right) - \frac{1}{3}\left(\frac{1}{2} - \frac{1}{2}\ln(2)\right)$ $= \frac{\pi}{12} - \frac{1}{6}(1 - \ln 2)$
7(a)	$AB \text{ line} \Rightarrow \underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$ $\overrightarrow{OP} = \begin{pmatrix} 2-5\lambda \\ \lambda \\ -1+3\lambda \end{pmatrix}$ $\overrightarrow{OD} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ C, P, D are collinear. $\overrightarrow{CP} = k\overrightarrow{CD}$ $\begin{pmatrix} 2-5\lambda-a \\ \lambda \\ -1+3\lambda \end{pmatrix} = k\begin{pmatrix} a-1 \\ 2 \\ 4 \end{pmatrix}$ $\Rightarrow \lambda = 1, k = \frac{1}{2}, a = -\frac{5}{3}$ $\overrightarrow{OD} = \begin{pmatrix} -\frac{5}{3} \\ 0 \\ 0 \end{pmatrix}$
(b)	$E(a,b,0)$ $\overrightarrow{OE} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ $a^2 + b^2 = 1$ $\cos 150^\circ = \frac{\overrightarrow{OA} \bullet \overrightarrow{OE}}{\left\lvert\overrightarrow{OA}\right\rvert(1)}$ 

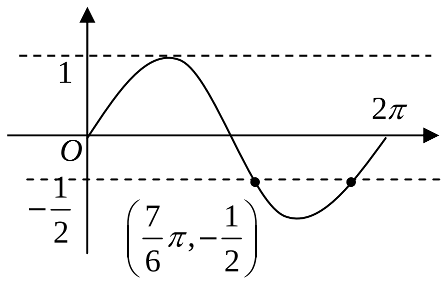
	$-\dfrac{\sqrt{3}}{2}=\dfrac{\begin{pmatrix}2\\0\\-1\end{pmatrix}\bullet\begin{pmatrix}a\\b\\0\end{pmatrix}}{\sqrt{5}}$ $-\dfrac{\sqrt{3}}{2}=\dfrac{2a}{\sqrt{5}}\Rightarrow a=-\dfrac{\sqrt{15}}{4}$ or 0.968 (3 s.f.) $\dfrac{15}{16}+b^2=1\Rightarrow b=\pm\dfrac{1}{4}$ $E\left(-\dfrac{\sqrt{15}}{4},\dfrac{1}{4},0\right)$ or $E\left(-\dfrac{\sqrt{15}}{4},-\dfrac{1}{4},0\right)$
8	(i) $\dfrac{2}{r(r+1)(r+3)}\equiv\dfrac{A}{r}+\dfrac{B}{r+1}+\dfrac{C}{r+3}$ $2\equiv A(r+1)(r+3)+B+r(r+3)+Cr(r+1)$ $r=0,\quad A=\dfrac{2}{3}\qquad r=-1,\ \ B=-1\qquad r=0,\quad C=\dfrac{1}{3}$ $\therefore\dfrac{2}{r(r+1)(r+3)}\equiv\dfrac{2}{3r}-\dfrac{1}{r+1}+\dfrac{1}{3(r+3)}$ $\dfrac{1}{4}\sum_{r=1}^{n}\dfrac{2}{r(r+1)(r+3)}=\dfrac{1}{4}\sum_{r=1}^{n}\left(\dfrac{2}{3r}-\dfrac{1}{r+1}+\dfrac{1}{3(r+3)}\right)$ $=\dfrac{1}{4}[\dfrac{2}{3}\quad-\dfrac{1}{2}\quad+\cancel{\dfrac{1}{12}}$ $+\dfrac{2}{6}\quad-\dfrac{1}{3}\quad+\cancel{\dfrac{1}{15}}$ $+\dfrac{2}{9}\quad-\cancel{\dfrac{1}{4}}\quad+\cancel{\dfrac{1}{18}}$ $+\cancel{\dfrac{2}{12}}\quad-\cancel{\dfrac{1}{5}}\quad+\cancel{\dfrac{1}{21}}$ $+\cancel{\dfrac{2}{15}}\quad-\cancel{\dfrac{1}{6}}\quad+\cancel{\dfrac{1}{24}}$ $\vdots$ $+\cancel{\dfrac{2}{3(n-3)}}\quad-\cancel{\dfrac{1}{n-2}}\quad+\cancel{\dfrac{1}{3n}}$ $+\cancel{\dfrac{2}{3(n-2)}}\quad-\cancel{\dfrac{1}{n-1}}\quad+\dfrac{1}{3(n+1)}$ $+\cancel{\dfrac{2}{3(n-1)}}\quad-\cancel{\dfrac{1}{n}}\quad+\dfrac{1}{3(n+2)}$ $+\quad\cancel{\dfrac{2}{3n}}\quad-\dfrac{1}{n+1}\quad+\dfrac{1}{3(n+3)}]$

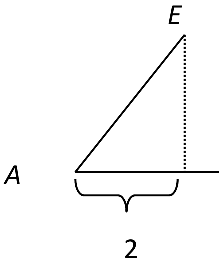
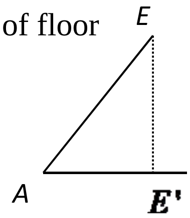
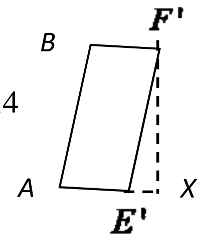
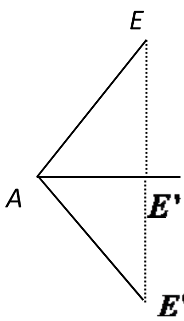
	$=\frac{1}{4}\left[\frac{7}{18}-\frac{1}{n+1}+\frac{1}{3(n+1)}+\frac{1}{3(n+2)}+\frac{1}{3(n+3)}\right]$ $=\frac{1}{12}\left[\frac{7}{6}-\frac{2}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right]$ (iii) $\sum_{r=5}^{\infty}\frac{1}{2}\frac{1}{r(r-2)(r-3)}$ Replace r by $r+3$, $=\sum_{r=2}^{\infty}\frac{1}{2r(r+1)(r+3)}$ $=\sum_{r=1}^{\infty}\frac{1}{2r(r+1)(r+3)}-\frac{1}{2(1)(2)(4)}$ $=\lim_{n\to\infty}\left(\frac{1}{12}\left[\frac{7}{6}-\frac{2}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right]\right)-\frac{1}{16}$ $=\lim_{n\to\infty}\left(\frac{7}{72}-\frac{1}{6(n+1)}+\frac{1}{12(n+2)}+\frac{1}{12(n+3)}\right)-\frac{1}{16}$ $=\frac{7}{72}-\frac{1}{16}=\frac{5}{144}$
9	*(See alternative solution below)* Let $P(n)$ be the statement " 1+ (1+2) + (1+2+3) + (1+2+3+…+n) $=\frac{1}{6}n(n+1)(n+2)$, $n\in Z^{+}$ " When $n=1$, LHS of P(1) = 1, RHS of P(1) = $\frac{(1)(2)(3)}{6}=1$ Since LHS = RHS, P(1) is true. Assume P(k) is true for some $k \in Z^{+}$, i.e. 1+ (1+2) + (1+2+3) + (1+2+3+…+k) $=\frac{1}{6}k(k+1)(k+2)$ To show P(k+1) is true, i.e. 1+ (1+2) + (1+2+3) + (1+2+3+…+k+k+1) $=\frac{1}{6}(k+1)(k+2)(k+3)$ LHS of P(k+1) =1+(1+2)+(1+2+3)+(1+2+3+…+k)+(1+2+3+…+k+k+1) $=\frac{1}{6}k(k+1)(k+2)+(1+2+3+\ldots+k+k+1)$ $=\frac{1}{6}k(k+1)(k+2)+\frac{1}{2}(k+1)(k+2)$

	$=\frac{1}{6}(k+1)(k+2)(k+3)$ = RHS of P(k+1) Since P(1) is true, and P(k) is true => P(k+1) is true, by mathematical induction, P(n) is true for $n \in \mathbb{Z}^+$. **Alternative Solution:** Let P(n) be the statement "$\sum_{r=1}^{n} U_r = \frac{1}{6}n(n+1)(n+2)$, where $U_r = 1+2+3+...+r$, $n \in Z^+$" When $n=1$, LHS of P(1) = $\sum_{r=1}^{1} U_r = U_1 = 1$, RHS of P(1) = $\frac{6}{6} = 1$ Since LHS = RHS, P(1) is true. Assume P(k) is true for some $k \in \mathbb{Z}^+$, i.e. $\sum_{r=1}^{k} U_r = \frac{1}{6}k(k+1)(k+2)$ To show P(k+1) is true, i.e. $\sum_{r=1}^{k+1} U_r = \frac{1}{6}(k+1)(k+2)(k+3)$ LHS of P(k+1) $=\sum_{r=1}^{k+1} U_r$ $=\sum_{r=1}^{k} U_r + U_{k+1}$ $=\frac{1}{6}k(k+1)(k+2) + (1+2+3+\ldots+k+k+1)$ $=\frac{1}{6}k(k+1)(k+2) + \frac{1}{2}(k+1)(k+2)$ $=\frac{1}{6}k(k+1)(k+2)(k+3)$ = RHS of P(k+1) Since P(1) is true, and P(k) is true => P(k+1) is true, by mathematical induction, P(n) is true for $n \in \mathbb{Z}^+$.

	(i) $3+(3+6)+(3+6+9)+...+(3+6+9+...+(6n-3))$ $=3[1+(1+2)+(1+2+3)+...+(1+2+3+...+(2n-1))]=3\left[\frac{1}{6}(2n-1)(2n)(2n+1)\right]$ $=n(2n-1)(2n+1)$ (ii) $3\times(3\times9)\times(3\times9\times27)\times...\times(3\times9\times27\times81\times...\times3^n)$ $=3\times(3^{1+2})\times(3^{1+2+3})\times...\times(3^{1+2+3+...+n})$ $=3^{1+(1+2)+(1+2+3)+...+(1+2+3+...+n)}$ $=3^{\frac{n(n+1)(n+2)}{6}}$
10 (i)	$\mathrm{f}(x)=\sqrt{\lvert 2-x\rvert}+1,\ x\in\mathbb{R}$  The horizontal line $y=2$ cuts the curve at more than one point, hence f is not one-to-one and f^{-1} does not exist. <u>OR</u> $\mathrm{f}(1)=\mathrm{f}(3)=2$, hence f is not one-to-one and f^{-1} does not exist.
(ii)	The minimum value is $k=2$. Let $y=\mathrm{f}(x)=\sqrt{\lvert 2-x\rvert}+1=\sqrt{x-2}+1(\because x\ge 2)$ $\Rightarrow x=2+(y-1)^2$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=[1,\infty)\qquad \therefore \mathrm{f}^{-1}(x)=2+(x-1)^2,\ x\ge1$
(iii)	If there exists a solution for $\mathrm{f}^{-1}(x)=\mathrm{f}(x)$ $\Rightarrow$ there exists a solution for $\mathrm{f}^{-1}(x)=x$ $\Rightarrow 2+(x-1)^2=x$ $\Rightarrow x^2-3x+3=0$ $\Rightarrow \left(x-\frac{3}{2}\right)^2+\frac{3}{4}=0$ $\Rightarrow$ no solution for x $\Rightarrow \mathrm{f}^{-1}(x)=\mathrm{f}(x)$ has no solution.

(iv)	 $[2,\infty)\xrightarrow{\mathrm{f}}[1,\infty)\xrightarrow{\mathrm{g}}(-\infty,1]\ \therefore R_{\mathrm{gf}}=(-\infty,1]$
(v)	$\mathrm{g}\,\mathrm{f}(x)=1$ $\mathrm{f}(x)=3$ $\sqrt{x-2}+1=3$ $\sqrt{x-2}=2$ $x-2=4$ $x=6$
11	 <u>3 axial intercepts</u> $\left(0,\frac{3}{2}\right),\left(\pm\sqrt{\frac{3}{2}},0\right)$ OR $(0,1.5),(\pm1.22,0)$ <u>2 turning points</u> $(0.419,1.68),(3.58,14.3)$ <u>2 asymptotes</u> $x=2, y=2x+4$
(a)	Using the graph, the intersections of the curve with the line y = 1 are $(-0.5,1),(1,1)$, so the solution is $-\frac{1}{2}\le x\le 1$ or $x>2$

	$\dfrac{2\sin^2 x-3}{\sin x-2}\geq 1$ So the solution is $-\dfrac{1}{2}\leq \sin x\leq 1$ or $\sin x>2(\text{rej})$  $\therefore 0\leq x\leq \dfrac{7}{6}\pi$ or $\dfrac{11}{6}\pi\leq x\leq 2\pi$
(b)	$y=\dfrac{2x^2-3}{x-2}=2x+4+\dfrac{5}{x-2}$ Translation of 2 units in the positive x-direction, followed by translation of 8 units in the positive y-direction.
12 (i)	$l_{EF}:\underset{\sim}{r}=\begin{pmatrix}5\\-14\\6\end{pmatrix}+\lambda\begin{pmatrix}1\\7\\0\end{pmatrix},\lambda\in\mathbb{R}$ $l_{AE}:3x=z+15$ $\dfrac{x-0}{1}=\dfrac{z-(-15)}{3},y=0$ $l_{AE}:\underset{\sim}{r}=\begin{pmatrix}0\\0\\-15\end{pmatrix}+\mu\begin{pmatrix}1\\0\\3\end{pmatrix},\mu\in\mathbb{R}$ $\lambda=2$ $\overrightarrow{OE}=\begin{pmatrix}5\\-14\\6\end{pmatrix}+2\begin{pmatrix}1\\7\\0\end{pmatrix}=\begin{pmatrix}7\\0\\6\end{pmatrix}$
(ii)	$\underset{\sim}{n}=\begin{pmatrix}1\\7\\0\end{pmatrix}\times\begin{pmatrix}1\\0\\3\end{pmatrix}=\begin{pmatrix}21\\-3\\-7\end{pmatrix}$

	$\begin{pmatrix}5\\0\\0\end{pmatrix}.\begin{pmatrix}21\\-3\\-7\end{pmatrix}=105$ $\underset{\sim}{r}.\begin{pmatrix}21\\-3\\-7\end{pmatrix}=105$
(iii)	Method 1: By Observation, Projection vector of $\overrightarrow{AE}$ onto $\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}$  Method 2: Projection of of $\overrightarrow{AE}$ onto normal of floor $\overrightarrow{AE'}=\left(\begin{pmatrix}2\\0\\6\end{pmatrix}.\widehat{\begin{pmatrix}1\\0\\0\end{pmatrix}}\right)\widehat{\begin{pmatrix}1\\0\\0\end{pmatrix}}=\begin{pmatrix}2\\0\\0\end{pmatrix}$  Method 1: $F'X=7$ (Deduce from $\overrightarrow{OC}$) $Area=(AE')(F'X)=2\times7=14$  Method 2: $Area=\lvert\overrightarrow{AB}\times\overrightarrow{AE'}\rvert=\left\lvert\begin{pmatrix}1\\7\\0\end{pmatrix}\times\begin{pmatrix}2\\0\\0\end{pmatrix}\right\rvert=\left\lvert\begin{pmatrix}0\\0\\14\end{pmatrix}\right\rvert=14$
(iv)	Let E'' be the reflection of *E* about and plane *OABC*. $\overrightarrow{OE}=\begin{pmatrix}7\\0\\6\end{pmatrix},\overrightarrow{OE''}=\begin{pmatrix}7\\0\\-6\end{pmatrix}$, $\overrightarrow{AE''}=\overrightarrow{OE''}-\overrightarrow{OA}=\begin{pmatrix}2\\0\\-6\end{pmatrix}$ 

	$l_3 : \underset{\sim}{r} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \beta \in \mathbb{R}$
(v)	Let $\prod$ be plane $x + ay + bz = c$. EF is // $\prod$. $\begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix}$ is $\perp$ to $\underset{\sim}{n}_{\prod}$. $\begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} . \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 0 \Rightarrow 1 + 7a = 0 \Rightarrow a = -\frac{1}{7}$ E is on plane $\prod$. $\begin{pmatrix} 7 \\ 0 \\ 6 \end{pmatrix} . \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = c \Rightarrow 7 + 6b = c.$
13 (a)	$x^3 - y^3 = 3x - 3y$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 - y^3\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(3x - 3y\right)$ $3x^2 - 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 3 - 3\frac{\mathrm{d}y}{\mathrm{d}x}$ $3x^2 - 3 = 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} - 3\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{x^2 - 1}{y^2 - 1} = \frac{\mathrm{d}y}{\mathrm{d}x}$ Substitute $x = -2$ and $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{0}$ (undefined) Therefore, the tangent is a vertical line. Thus, the tangent is $x = -2$.
(b)	Let the radius be r . We want to find $\frac{\mathrm{d}S}{\mathrm{d}t}$, $\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}r} \times \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}r}$ $= \left(8\pi r\right) \times \left(0.1\right) \div \left(4\pi r^2\right)$ $= \frac{1}{5r}$ Sub $r = \frac{1}{2}$ into $\frac{\mathrm{d}S}{\mathrm{d}t}$, we get $\frac{2}{5} \mathrm{m}^2 / \mathrm{s}$.

(c)	Let the side of each room be x. By cosine rule, $(4x)^2 = a^2 + b^2 - 2ab\cos\theta$ Total area, $A = \frac{1}{2}ab\sin\theta + 4x^2$ $A = \frac{1}{2}ab\sin\theta + \frac{1}{4}\left(a^2 + b^2 - 2ab\cos\theta\right)$ $= \frac{1}{2}ab\sin\theta + \frac{1}{4}a^2 + \frac{1}{4}b^2 - \frac{1}{2}ab\cos\theta$ To find max area, we let $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 0$. $\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{2}ab\sin\theta + \frac{1}{4}a^2 + \frac{1}{4}b^2 - \frac{1}{2}ab\cos\theta\right)$ $= \frac{1}{2}ab\cos\theta + \frac{1}{2}ab\sin\theta$ $\frac{1}{2}ab\cos\theta + \frac{1}{2}ab\sin\theta = 0$ $\tan\theta = -1$ $\theta = \frac{3\pi}{4}$ $\left(\text{since } 0 < \theta < \pi\right)$ Therefore, stationary point at $\theta = \frac{3\pi}{4}$. $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = \frac{1}{2}ab\cos\theta - \frac{1}{2}ab\sin\theta$ $\left.\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}\right\|_{\theta=\frac{3\pi}{4}} < 0$ Thus, the stationary point is maximum.

INNOVA JUNIOR COLLEGE
JC 1 MID COURSE EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CLASS

INDEX NUMBER

MATHEMATICS

9740/01

8 October 2013

3 hours

Additional Materials: Answer Paper
Cover Page
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1* **(i)** Find the expansion of $\dfrac{1+x^2}{\sqrt{(4+2x)}}$ in ascending powers of x, up to and including the term in x^2. [3]

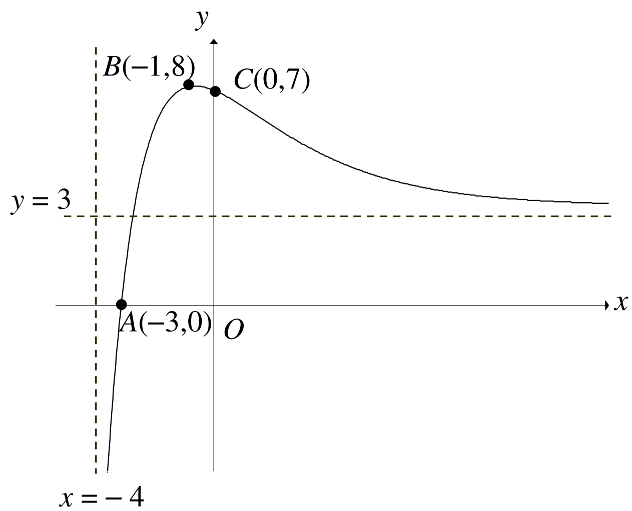
(ii) State the range of values of x for which this expansion is valid. [1]

(iii) Write down the equation of the tangent to the curve

$$y=\frac{1+x^2}{\sqrt{(4+2x)}}$$

at the point where $x = 0$. [1]

2



The diagram shows the graph of $y=\mathrm{f}(x)$. There is a maximum point $B(-1,8)$ and the curve cuts the axes at the points $A(-3,0)$ and $C(0,7)$. The lines $x=-4$ and $y=3$ are asymptotes of the curve.

Sketch, on separate diagrams, the graphs of

(i) $y=\mathrm{f}'(x)$, [2]

(ii) $y=-\sqrt{\left\{\mathrm{f}\left(\frac{1}{2}x\right)\right\}}$, [3]

stating the equations of the asymptotes and the coordinates of the points corresponding to A, B and C where possible.

3 **(i)** Using the method of difference, show that

$$\sum_{r=1}^{n} \frac{k}{(r+1)(r+3)} = \frac{k}{2}\left(a - \frac{1}{n+2} - \frac{1}{n+3}\right),$$

where a is a constant to be determined. [4]

(ii) Hence find the range of values of k such that $\sum_{r=1}^{\infty} \frac{k}{(r+1)(r+3)}$ is at most 1. [2]

4 **(i)** Prove by induction that $\sum_{r=1}^{n} \frac{r\left(2^r\right)}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$ for all positive integers n. [5]

(ii) Hence find an expression in terms of n for $\sum_{r=n}^{2n} \frac{r\left(2^r\right)}{(r+2)!}$. [2]

5* Find

(i) $\int \frac{4}{\sqrt{(5+4x-4x^2)}}\,dx$, [3]

(ii) $\int (3\sin 2\theta - \sec\theta)^2\,d\theta$. [4]

6 Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P on AB is such that $AP : PB = 2 : 3$. It is given that $|\mathbf{a}| = \sqrt{5}$, $|\mathbf{b}| = 3$ and OP is perpendicular to AB.

(i) Show that $\mathbf{a} \cdot \mathbf{b} = -3$. [3]

(ii) Find the size of angle AOB. [2]

(iii) Find the length of projection of $\overrightarrow{OB}$ onto OA. [1]

7 A water tank in the shape of an inverted cone has a height twice that of its radius. Water is poured into the cone. Given that, when the depth of the water is 10 cm, the volume of water is increasing at a rate of 10π cm^{3}s^{-1}, find the rate of increase at this instant of

(i) the slant height of the cone in contact with the water, [5]

(ii) the curved surface area of the cone in contact with the water. [2]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$ and the curved surface area is πrl.]

8 The equation of a curve is $x^2 - 2xy + 2y^2 = -12$.

(i) Find the equations of the tangent and normal to the curve at the point $P(2,4)$. [5]

(ii) The tangent at P meets the y-axis at A and the normal at P meets the x-axis at B. Find the area of triangle APB. [3]

9 **(a)** An arithmetic progression A has first term 3 and the sum of the terms from the 16$^{\text{th}}$ term to the 30$^{\text{th}}$ term inclusive is 2025. Show that the common difference is 6. [3]

If S_n is the sum of the first n terms of A, show that the sum of the first n even-numbered terms of A, that is, the second, fourth, sixth, … terms, is given by $\left(2+\frac{1}{n}\right)S_n$. [2]

(b) A geometric series G has first term 30 and common ratio $-\frac{4}{5}$. Write down the sum, S_n, of the first n terms of the series. [1]

Find the least value of n for which the magnitude of the difference between S_n and the sum to infinity of the series is less than 0.004. [3]

A new series is formed by taking the reciprocal of the corresponding terms of G. Determine if the new series is convergent. [1]

10* **(i)** By successively differentiating $\ln(3+x)$, find the Maclaurin's series for $\ln(3+x)$, up to and including the term in x^3. [3]

(ii) Given that θ is small, find the expansion of $\left(2-\cos 5\theta^2\right)^{\frac{1}{2}}$ in ascending powers of θ, up to and including the term in θ^4. [2]

Two particles A and B produce y units of energy when they are x units away from their original position at $x=0$. The energy produced by particles A and B can be found by the equations

$$y=\ln(3+x) \text{ and}$$
$$y=\left(2-\cos 5x^2\right)^{\frac{1}{2}}$$

respectively, where $x \geq 0$.

(iii) Explain in the context of the question, what is meant by the solution to the equation

$$\ln(3+x)=\left(2-\cos 5x^2\right)^{\frac{1}{2}}.$$ [1]

(iv) Using your answers from parts **(i)** and **(ii)**, find an estimate for the maximum distance from the original position such that the difference in energy produced by both particles is at most 0.4 units. [You may assume that both particles are at the same distance from the original position.] [2]

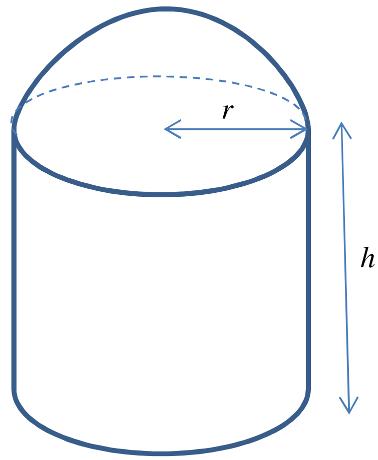
11 **(i)** Find a vector equation of the line through the points A and B with position vectors $3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ and $-\mathbf{i}+12\mathbf{j}+9\mathbf{k}$ respectively. [2]

(ii) The perpendicular to this line from the point C with position vector $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ meets the line at the point N. Find the position vector of N. [3]

(iii) Find a Cartesian equation of the line AC. [2]

(iv) Use a vector product to find the exact area of triangle OAB. [3]

12 A container is made up of an open cylinder of varying height h cm and varying radius r cm, and a hollow hemispherical lid of varying radius r cm. It costs 5 cents per square centimetre to manufacture the base, 3 cents per square centimetre to manufacture the curved surface of the cylinder and 4 cents per square centimetre to manufacture the curved surface of the hemisphere.



(i) Given that the cylinder is of fixed volume V cm^3, show that the manufacturing cost of the container is minimum when r is $\left(\frac{3V}{13\pi}\right)^{\frac{1}{3}}$. [7]

(ii) Using the value of r in part **(i)** and taking V to be 30, find the maximum number of containers that a person can buy if he has \$22. [2]

[The surface area of a sphere is $4\pi r^2$.]

13 The function f is defined as follows:

$$\mathrm{f}: x \mapsto \frac{1}{x^2-4} \quad \text{for } x \in \mathbb{R},\ x \neq -2,\ x \neq 2.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$. [2]

The function g is defined as follows:

$$\mathrm{g}: x \mapsto \frac{1}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq a,\ x \neq 3,\ x \neq b.$$

It is given that the function fg exists.

(ii) Find the values of a and b. [2]

(iii) Show that $\mathrm{fg}(x) = \frac{(x-3)^2}{(2x-5)(7-2x)}$. [2]

(iv) Solve the inequality $\mathrm{fg}(x) > 0$. [3]

(v) Find the range of fg. [3]

2013 H2 Maths MCE_Marking Scheme

1* **(i)** Find the expansion of $\dfrac{1+x^2}{\sqrt{(4+2x)}}$ in ascending powers of x, up to and including the term in x^2. [3]

(ii) State the range of values of x for which this expansion is valid. [1]

(iii) Write down the equation of the tangent to the curve

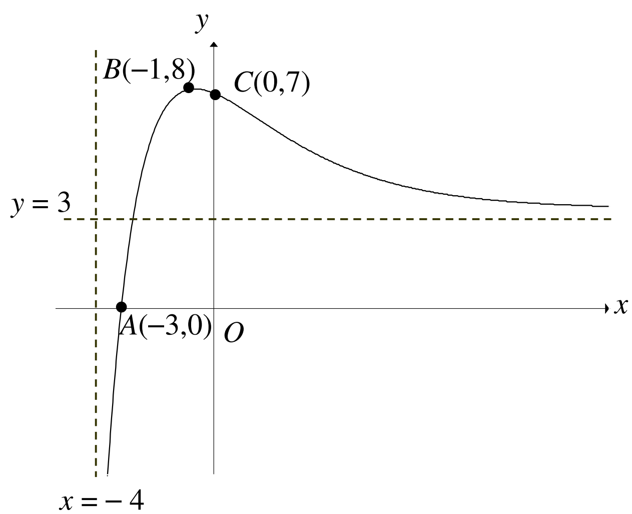
$$y=\frac{1+x^2}{\sqrt{(4+2x)}}$$

at the point where $x = 0$. [1]

1(i)	$\dfrac{1+x^2}{\sqrt{(4+2x)}}$ $=\left(1+x^2\right)(4+2x)^{-\frac{1}{2}}$ $=\dfrac{1}{2}\left(1+x^2\right)\left(1+\dfrac{x}{2}\right)^{-\frac{1}{2}}$ $=\dfrac{1}{2}\left(1+x^2\right)\left(1+\left(-\dfrac{1}{2}\right)\left(\dfrac{x}{2}\right)+\left(\dfrac{-\frac{1}{2}\times-\frac{3}{2}}{2!}\right)\left(\dfrac{x}{2}\right)^2+\ldots\right)$ $=\dfrac{1}{2}\left(1+x^2\right)\left(1-\dfrac{x}{4}+\dfrac{3}{32}x^2+\ldots\right)$ $=\dfrac{1}{2}-\dfrac{1}{8}x+\dfrac{3}{64}x^2+\dfrac{1}{2}x^2+\ldots$ $=\dfrac{1}{2}-\dfrac{1}{8}x+\dfrac{35}{64}x^2+\ldots$
(ii)	$\left\lvert\dfrac{x}{2}\right\rvert<1$ $-1<\dfrac{x}{2}<1$ $-2<x<2$
(iii)	$y=\dfrac{1}{2}-\dfrac{1}{8}x$

*: Not in topics tested for 2014 SRJC Promo

2



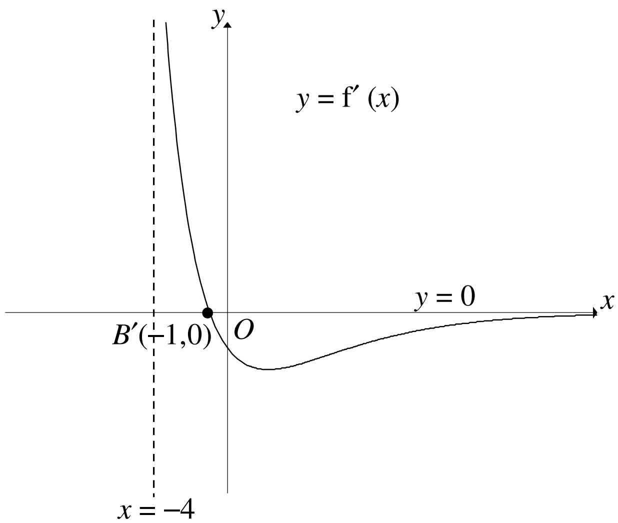
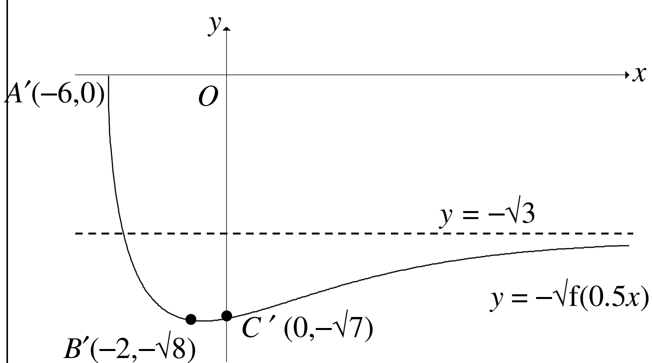
The diagram shows the graph of $y = \mathrm{f}(x)$. There is a maximum point $B(-1,8)$ and the curve cuts the axes at the points $A(-3,0)$ and $C(0,7)$. The lines $x = -4$ and $y = 3$ are asymptotes of the curve.

Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}'(x)$, [2]

(ii) $y = -\sqrt{\left\{\mathrm{f}\left(\frac{1}{2}x\right)\right\}}$, [3]

stating the equations of the asymptotes and the coordinates of the points corresponding to A, B and C where possible.

2(i)	 y y = f′ (x) y = 0 x B′(−1,0) O x = −4
(ii)	 y x A′(−6,0) O y = −√3 y = −√f(0.5x) C′ (0,−√7) B′(−2,−√8)

3 **(i)** Using the method of difference, show that

$$\sum_{r=1}^{n} \frac{k}{(r+1)(r+3)} = \frac{k}{2}\left(a - \frac{1}{n+2} - \frac{1}{n+3}\right),$$

where a is a constant to be determined. [4]

(ii) Hence find the range of values of k such that $\sum_{r=1}^{\infty} \frac{k}{(r+1)(r+3)}$ is at most 1. [2]

3(i)	$\frac{k}{(r+1)(r+3)} = \frac{k}{2}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$ $\sum_{r=1}^{n} \frac{k}{(r+1)(r+3)} = \frac{k}{2}\sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$ $\begin{aligned} &= \frac{k}{2}\left(\frac{1}{2} - \cancel{\frac{1}{4}} \right. \\ &+ \frac{1}{3} - \cancel{\frac{1}{5}} \\ &+ \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} \\ &+ \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} \\ &+ \ldots \\ &+ \ldots \\ &+ \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n+1}} \\ &+ \cancel{\frac{1}{n}} - \frac{1}{n+2} \\ &+ \left. \cancel{\frac{1}{n+1}} - \frac{1}{n+3} \right) \\ &= \frac{k}{2}\left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right) \end{aligned}$ $a = \frac{5}{6}$

(ii)	$\displaystyle\sum_{r=1}^{\infty}\frac{k}{(r+1)(r+3)}=\frac{k}{2}\left(\frac{5}{6}\right)=\frac{5k}{12}$ $\dfrac{5k}{12}\le 1$ $\Rightarrow k\le\dfrac{12}{5}$

4 **(i)** Prove by induction that $\displaystyle\sum_{r=1}^{n}\frac{r\left(2^r\right)}{(r+2)!}=1-\frac{2^{n+1}}{(n+2)!}$ for all positive integers n. [5]

(ii) Hence find an expression in terms of n for $\displaystyle\sum_{r=n}^{2n}\frac{r\left(2^r\right)}{(r+2)!}$. [2]

4(i)	Let P_n denote $\displaystyle\sum_{r=1}^{n}\frac{r\left(2^r\right)}{(r+2)!}=1-\frac{2^{n+1}}{(n+2)!}$ for $n\in\mathbb{Z}^+$. When n = 1, LHS = $\displaystyle\sum_{r=1}^{1}\frac{r\left(2^r\right)}{(r+2)!}$ $=\dfrac{(1)\left(2^1\right)}{(1+2)!}$ $=\dfrac{2}{3!}$ $=\dfrac{1}{3}$ RHS = $1-\dfrac{2^{1+1}}{(1+2)!}$ $=1-\dfrac{4}{3!}$ $=1-\dfrac{2}{3}$ $=\dfrac{1}{3}$ Therefore, P_1 is true. Assume P_k is true for some $k\in\mathbb{Z}^+$,

	i.e. $\sum_{r=1}^{k} \frac{r(2^r)}{(r+2)!} = 1 - \frac{2^{k+1}}{(k+2)!}$. Want to prove P_{k+1} is true, i.e. $\sum_{r=1}^{k+1} \frac{r(2^r)}{(r+2)!} = 1 - \frac{2^{k+2}}{(k+3)!}$. LHS = $\sum_{r=1}^{k+1} \frac{r(2^r)}{(r+2)!}$ $= \sum_{r=1}^{k} \frac{r(2^r)}{(r+2)!} + \frac{(k+1)(2^{k+1})}{(k+3)!}$ $= \left[1 - \frac{2^{k+1}}{(k+2)!}\right] + \frac{(k+1)(2^{k+1})}{(k+3)!}$ $=1 - \left[\frac{(2^{k+1})(k+3)}{(k+3)!} - \frac{(k+1)(2^{k+1})}{(k+3)!}\right]$ $=1 - \left[\frac{(2^{k+1})\left[(k+3)-(k+1)\right]}{(k+3)!}\right]$ $=1 - \left[\frac{(2^{k+1})(2)}{(k+3)!}\right]$ $=1 - \frac{2^{k+2}}{(k+3)!}$ = RHS Thus P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
(ii)	$\sum_{r=n}^{2n} \frac{r(2^r)}{(r+2)!}$ $= \sum_{r=1}^{2n} \frac{r(2^r)}{(r+2)!} - \sum_{r=1}^{n-1} \frac{r(2^r)}{(r+2)!}$ $= \left[1 - \frac{2^{2n+1}}{(2n+2)!}\right] - \left[1 - \frac{2^n}{(n+1)!}\right]$ $= \frac{2^n}{(n+1)!} - \frac{2^{2n+1}}{(2n+2)!}$

5* Find

(i) $\displaystyle\int \frac{4}{\sqrt{(5+4x-4x^2)}}\,\mathrm{d}x$, [3]

(ii) $\displaystyle\int (3\sin 2\theta - \sec\theta)^2\,\mathrm{d}\theta$. [4]

5(i)	$5+4x-4x^2$ $=-4\left(x^2-x-\frac{5}{4}\right)$ $=-4\left(\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-\frac{5}{4}\right)$ $=-4\left(\left(x-\frac{1}{2}\right)^2-\frac{6}{4}\right)=4\left[\frac{3}{2}-\left(x-\frac{1}{2}\right)^2\right]$ $\displaystyle\int \frac{4}{\sqrt{5+4x-4x^2}}\,\mathrm{d}x$ $\displaystyle=\int \frac{4}{\sqrt{4\left[\frac{3}{2}-\left(x-\frac{1}{2}\right)^2\right]}}\,\mathrm{d}x$ or $\displaystyle\int \frac{4}{\sqrt{6-(2x-1)^2}}\,\mathrm{d}x$ $\displaystyle=\int \frac{4}{2\sqrt{\frac{3}{2}-\left(x-\frac{1}{2}\right)^2}}\,\mathrm{d}x$ $=2\sin^{-1}\left(\dfrac{x-\frac{1}{2}}{\sqrt{\frac{3}{2}}}\right)+C$ or $2\sin^{-1}\left(\dfrac{2x-1}{\sqrt{6}}\right)+C$
(ii)	$\int(3\sin 2\theta-\sec\theta)^2\,\mathrm{d}\theta$ $=\int 9\sin^2 2\theta-6\sin 2\theta\sec\theta+\sec^2\theta\,\mathrm{d}\theta$ $=\frac{9}{2}\int(1-\cos 4\theta)d\theta-6\int 2\sin\theta\cos\theta\sec\theta\,\mathrm{d}\theta+\int\sec^2\theta\,\mathrm{d}\theta$ $=\frac{9}{2}\int(1-\cos 4\theta)d\theta-12\int\sin\theta\,\mathrm{d}\theta+\int\sec^2\theta\,\mathrm{d}\theta$ $=\frac{9}{2}\left(\theta-\frac{1}{4}\sin 4\theta\right)-12(-\cos\theta)+\tan\theta+c$ $=\frac{9}{2}\theta-\frac{9}{8}\sin 4\theta+12\cos\theta+\tan\theta+c$

 *: Not in topics tested for 2014 SRJC Promo

6 Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P on AB is such that $AP : PB = 2:3$. It is given that $|\mathbf{a}| = \sqrt{5}$, $|\mathbf{b}| = 3$ and OP is perpendicular to AB.

(i) Show that $\mathbf{a} \cdot \mathbf{b} = -3$. [3]

(ii) Find the size of angle AOB. [2]

(iii) Find the exact length of projection of $\overrightarrow{OB}$ onto OA. [1]

6(i)	By Ratio Theorem, $\overrightarrow{OP} = \frac{1}{5}(3\mathbf{a} + 2\mathbf{b})$. Since $OP \perp AB$, $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$. $\frac{1}{5}(3\mathbf{a} + 2\mathbf{b}) \cdot (\mathrm{b} - \mathrm{a}) = 0$ $3\mathbf{a} \cdot \mathbf{b} - 3\mathbf{a} \cdot \mathbf{a} + 2\mathbf{b} \cdot \mathbf{b} - 2\mathbf{b} \cdot \mathbf{a} = 0$ $\mathbf{a} \cdot \mathbf{b} - 3\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2 = 0$ $\mathbf{a} \cdot \mathbf{b} - 15 + 18 = 0$ $\mathbf{a} \cdot \mathbf{b} = -3$
(ii)	$\cos \angle AOB = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$ $= \frac{-3}{3\sqrt{5}}$ $= -\frac{1}{\sqrt{5}}$ $\angle AOB = 116.6^\circ$ (or 2.03 rad)
(iii)	Length of projection of $\overrightarrow{OB}$ onto OA $= \left\| \frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|} \right\|$ $= \frac{3}{\sqrt{5}}$

7 A water tank in the shape of an inverted cone has a height twice that of its radius. Water is poured into the cone. Given that, when the depth of the water is 10 cm, the volume of water is increasing at a rate of 10π cm^{3}s^{-1}, find the rate of increase at this instant of

(i) the slant height of the cone in contact with the water, [5]

(ii) the curved surface area of the cone in contact with the water. [2]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$ and the curved surface area is πrl .]

7(i)	Let the radius of the water surface, the depth of the water, the slant height of the water and the volume of the water at time t seconds be r cm, h cm, l cm and V cm^3 respectively. $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi r^2(2r)=\frac{2}{3}\pi r^3$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}r}\cdot\frac{\mathrm{d}r}{\mathrm{d}t}=2\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}$ $h=2r=10\Rightarrow r=5$ When $\frac{\mathrm{d}V}{\mathrm{d}t}=10\pi$ and $r=5$, $10\pi=2\pi(5)^2\frac{\mathrm{d}r}{\mathrm{d}t}$ $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{1}{5}$ Using Pythagoras' theorem, $l^2=(2r)^2+r^2$ $l=\sqrt{5}r$ $\frac{\mathrm{d}l}{\mathrm{d}t}=\frac{\mathrm{d}l}{\mathrm{d}r}\cdot\frac{\mathrm{d}r}{\mathrm{d}t}=\sqrt{5}\frac{\mathrm{d}r}{\mathrm{d}t}=\sqrt{5}\left(\frac{1}{5}\right)=\frac{\sqrt{5}}{5}$ or 0.44721 The rate of increase of the slant height of the cone in contact with the water is $\frac{\sqrt{5}}{5}$ cms^{-1} (or 0.447 cms^{-1}) .

7(ii)	Let the curved surface area of the water at time t seconds be A cm^2. $A=\pi rl=\pi r\left(\sqrt{5}r\right)=\sqrt{5}\pi r^2$ $\frac{dA}{dt}=\frac{dA}{dr}\cdot\frac{dr}{dt}=2\sqrt{5}\pi r\frac{dr}{dt}$ When $r=5$, $\frac{dr}{dt}=\frac{1}{5}$. $\frac{dA}{dt}=2\sqrt{5}\pi(5)\left(\frac{1}{5}\right)=2\sqrt{5}\pi=14.0496$ The rate of increase of the curved surface area of the cone in contact with the water is $2\sqrt{5}\pi$ cm^{2}s^{-1} (or 14.0 cm^{2}s^{-1}).

8 The equation of a curve is $x^2 - 2xy + 2y^2 = -12$.

(i) Find the equations of the tangent and normal to the curve at the point $P(2,4)$. [5]

(ii) The tangent at P meets the y-axis at A and the normal at P meets the x-axis at B. Find the area of triangle APB. [3]

8(i)	$x^2 - 2xy + 2y^2 = -12$ $2x - \left(2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\right) + 4y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $2x - 2y = 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y\frac{\mathrm{d}y}{\mathrm{d}x}$ $2x - 2y = \frac{\mathrm{d}y}{\mathrm{d}x}(2x - 4y)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - 2y}{2x - 4y}$ $= \frac{x - y}{x - 2y}$ At $P(2,4)$: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2-4}{2-8}$ $= \frac{1}{3}$ Equation of tangent: $y - 4 = \frac{1}{3}(x - 2)$ $y = \frac{1}{3}x + \frac{10}{3}$ Gradient of normal $= -3$ Equation of normal: $y - 4 = -3(x - 2)$ $y = -3x + 10$

8(ii)	When tangent meets y-axis at A, $x=0$ $y=\frac{10}{3}$ $\therefore A\left(0,\frac{10}{3}\right)$ When normal meets x-axis at B, $y=0$ $3x=10$ $x=\frac{10}{3}$ $\therefore B\left(\frac{10}{3},0\right)$ Area of triangle APB $=\frac{1}{2}\times AP\times BP$ $=\frac{1}{2}\times\sqrt{\frac{40}{9}}\times\sqrt{\frac{160}{9}}$ $=\frac{40}{9}\text{ units}^2$ (or 4.44 units^2)

9 **(a)** An arithmetic progression A has first term 3 and the sum of the terms from the 16^{th} term to the 30^{th} term inclusive is 2025. Show that the common difference is 6. [3]

If S_n is the sum of the first n terms of A, show that the sum of the first n even-numbered terms of A, that is, the second, fourth, sixth, … terms, is given by $\left(2+\frac{1}{n}\right)S_n$. [2]

9(a)	$S_{30}-S_{15}=2025$ $\frac{30}{2}\left[2(3)+29d\right]-\frac{15}{2}\left[2(3)+14d\right]=2025$ $330d=1980$ $d=6$ $S_n=\frac{n}{2}\left[6+(n-1)6\right]=3n^2$ Sum of 1^{st} n even-numbered terms $=\frac{n}{2}\left[2(3+6)+(n-1)12\right]$ $=\frac{n}{2}\left[6+12n\right]$ $=3n^2\left(\frac{1}{n}+2\right)$ $=\left(2+\frac{1}{n}\right)S_n$

9(b) A geometric series G has first term 30 and common ratio $-\frac{4}{5}$. Write down the sum, S_n, of the first n terms of the series. [1]

Find the least value of n for which the magnitude of the difference between S_n and the sum to infinity of the series is less than 0.004. [3]

A new series is formed by taking the reciprocal of the corresponding terms of G. Determine if the new series is convergent. [1]

9(b)	$S_n = \dfrac{30\left[1-\left(-\frac{4}{5}\right)^n\right]}{1-\left(-\frac{4}{5}\right)} = \dfrac{50}{3}\left[1-\left(-\frac{4}{5}\right)^n\right]$ $\left\|S_n - S_\infty\right\| < 0.004$ $\left\|\frac{50}{3}\left[1-\left(-\frac{4}{5}\right)^n\right]-\frac{50}{3}\right\| < 0.004$ $\frac{50}{3}\left(\frac{4}{5}\right)^n < 0.004$ $\left(\frac{4}{5}\right)^n < 0.004\times\frac{3}{50}$ $n > \dfrac{\ln\left(0.004\times\frac{3}{50}\right)}{\ln\left(\frac{4}{5}\right)}$ $n > 37.352$ Least value of n is 38. New series $\frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}+\frac{1}{ar^3}+\frac{1}{ar^4}+\ldots$ is a geometric series with common ratio $\frac{1}{r}=-\frac{5}{4}$. Since $\left\|\frac{1}{r}\right\|=\frac{5}{4}>1$, the new series is not convergent.

10* **(i)** By successively differentiating $\ln(3+x)$, find the Maclaurin's series for $\ln(3+x)$, up to and including the term in x^3. [3]

(ii) Given that θ is small, find the expansion of $\left(2-\cos 5\theta^2\right)^{\frac{1}{2}}$ in ascending powers of θ, up to and including the term in θ^4. [2]

Two particles A and B produce y units of energy when they are x units away from their original position at $x=0$. The energy produced by particles A and B can be found by the equations

$$y=\ln(3+x) \text{ and}$$

$$y=\left(2-\cos 5x^2\right)^{\frac{1}{2}}$$

respectively, where $x \geq 0$.

(iii) Explain in the context of the question, what is meant by the solution to the equation

$$\ln(3+x)=\left(2-\cos 5x^2\right)^{\frac{1}{2}}.$$ [1]

(iv) Using your answers from parts **(i)** and **(ii)**, find an estimate for the maximum distance from the original position such that the difference in energy produced by both particles is at most 0.4 units. [2]

[You may assume that both particles are at the same distance from the original position.]

10(i)	Let $y=\ln(3+x)$ $\frac{dy}{dx}=(3+x)^{-1}$ $\frac{d^2y}{dx^2}=-(3+x)^{-2}$ $\frac{d^3y}{dx^3}=2(3+x)^{-3}$ When $x=0$, $y=\ln 3,\quad \frac{dy}{dx}=\frac{1}{3},\quad \frac{d^2y}{dx^2}=-\frac{1}{9},\quad \frac{d^3y}{dx^3}=\frac{2}{27}$ $\therefore y=\ln 3+\frac{x}{3}-\frac{x^2}{18}+\frac{x^3}{81}+\ldots$
(ii)	Given that θ is small, $\left(2-\cos 5\theta^2\right)^{\frac{1}{2}}=\left[2-\left(1-\frac{\left(5\theta^2\right)^2}{2}\right)+\ldots\right]^{\frac{1}{2}}$ $=\left(1+\frac{25}{2}\theta^4+\ldots\right)^{\frac{1}{2}}$ $=1+\left(\frac{1}{2}\right)\frac{25}{2}\theta^4+\ldots$ $=1+\frac{25}{4}\theta^4+\ldots$
(iii)	The solution (x value) denotes the distance in units where both particles produce the same number of units of energy.
(iv)	$\left\|\ln 3+\frac{x}{3}-\frac{x^2}{18}+\frac{x^3}{81}-\left(1+\frac{25}{4}x^4\right)\right\|\le 0.4$ Or $-0.4\le \ln 3+\frac{x}{3}-\frac{x^2}{18}+\frac{x^3}{81}-\left(1+\frac{25}{4}x^4\right)\le 0.4$ From GC, $x\le 0.57298752$ (given $x\ge 0$) An estimate for the maximum distance is 0.572 units. (3 s.f.)

11 **(i)** Find a vector equation of the line through the points A and B with position vectors $3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ and $-\mathbf{i}+12\mathbf{j}+9\mathbf{k}$ respectively. [2]

(ii) The perpendicular to this line from the point C with position vector $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ meets the line at the point N. Find the position vector of N. [3]

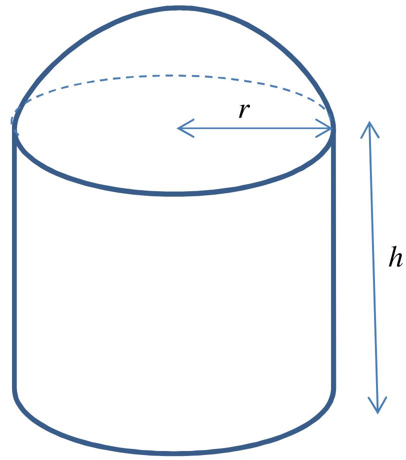
(iii) Find a Cartesian equation of the line AC. [2]

(iv) Use a vector product to find the exact area of triangle OAB. [3]

11(i)	$\overrightarrow{AB}=\begin{pmatrix}-1\\12\\9\end{pmatrix}-\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}-4\\8\\4\end{pmatrix}=4\begin{pmatrix}-1\\2\\1\end{pmatrix}$ $l_{AB}:\mathbf{r}=\begin{pmatrix}3\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix},\lambda\in\mathbb{R}$ or $l_{AB}:\mathbf{r}=\begin{pmatrix}-1\\12\\9\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix},\lambda\in\mathbb{R}$
(ii)	Since N lies on line AB, $\overrightarrow{ON}=\begin{pmatrix}3\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{CN}=\begin{pmatrix}3\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}-\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}1\\3\\7\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}$ Since $CN\perp AB$, $\overrightarrow{CN}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=0$ $\left[\begin{pmatrix}1\\3\\7\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}\right]\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=0$ $\begin{pmatrix}1\\3\\7\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=0$ $12+6\lambda=0$ $\lambda=-2$ $\overrightarrow{ON}=\begin{pmatrix}3\\4\\5\end{pmatrix}-2\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}5\\0\\3\end{pmatrix}$

11(iii)	$\overrightarrow{AC}=\begin{pmatrix}2\\1\\-2\end{pmatrix}-\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}-1\\-3\\-7\end{pmatrix}=-\begin{pmatrix}1\\3\\7\end{pmatrix}$ Cartesian eqn of line AC: $x-3=\frac{y-4}{3}=\frac{z-5}{7}$ or $x-2=\frac{y-1}{3}=\frac{z+2}{7}$
(iv)	Area of triangle OAB $=\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}3\\4\\5\end{pmatrix}\times\begin{pmatrix}-1\\12\\9\end{pmatrix}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}-24\\-32\\40\end{pmatrix}\right\|$ $=\frac{1}{2}\times 8\left\|\begin{pmatrix}-3\\-4\\5\end{pmatrix}\right\|$ $=4\sqrt{9+16+25}$ $=4\sqrt{50}$ $=20\sqrt{2}$

12 A container is made up of an open cylinder of varying height h cm and varying radius r cm, and a hollow hemispherical lid of varying radius r cm. It costs 5 cents per square centimetre to manufacture the base, 3 cents per square centimetre to manufacture the curved surface of the cylinder and 4 cents per square centimetre to manufacture the curved surface of the hemisphere.



(i) Given that the cylinder is of fixed volume V cm^3, show that the manufacturing cost of the container is minimum when r is $\left(\dfrac{3V}{13\pi}\right)^{\frac{1}{3}}$. [7]

(ii) Using the value of r in part **(i)** and taking V to be 30, find the maximum number of containers that a person can buy if he has \$22. [2]

[The surface area of a sphere is $4\pi r^2$.]

12(i)	$V = \pi r^2 h$ $\therefore h = \dfrac{V}{\pi r^2}$

	Let C cents be the manufacturing cost of the container. $C=4\left(2\pi r^2\right)+3\left(2\pi rh\right)+5\left(\pi r^2\right)$ $=13\pi r^2+6\pi r\left(\dfrac{V}{\pi r^2}\right)$ $=13\pi r^2+\dfrac{6V}{r}$ $\dfrac{\mathrm{d}C}{\mathrm{d}r}=13\pi\left(2r\right)+6V\left(-r^{-2}\right)$ $=26\pi r-\dfrac{6V}{r^2}$ Let $\dfrac{\mathrm{d}C}{\mathrm{d}r}=0$ $26\pi r-\dfrac{6V}{r^2}=0$ $26\pi r^3=6V$ $r^3=\dfrac{6V}{26\pi}$ $=\dfrac{3V}{13\pi}$ $r=\sqrt[3]{\dfrac{3V}{13\pi}}$ $\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2}=26\pi-6V\left(-2r^{-3}\right)$ $=26\pi+\dfrac{12V}{r^3}$ $=26\pi+\dfrac{12V}{\left(\dfrac{3V}{13\pi}\right)}$ $=26\pi+52\pi$ $=78\pi>0$ Hence, the manufacturing cost is minimum when $r=\sqrt[3]{\dfrac{3V}{13\pi}}$. [Shown]

(ii)	$C = 13\pi r^2 + \frac{6V}{r}$ $= 13\pi\left(\sqrt[3]{\frac{3V}{13\pi}}\right)^2 + \frac{6V}{\sqrt[3]{\frac{3V}{13\pi}}}$ $= 13\pi\left(\frac{90}{13\pi}\right)^{\frac{2}{3}} + \frac{180}{\left(\frac{90}{13\pi}\right)^{\frac{1}{3}}}$ $= 207.48$ cents $= \$2.0748$ $\frac{22}{2.0748}$ $= 10.603$ $\therefore$ Maximum number of containers he can buy is 10.

13 The function f is defined as follows:

$$\mathrm{f} : x \mapsto \frac{1}{x^2 - 4} \quad \text{for } x \in \mathbb{R},\ x \neq -2,\ x \neq 2.$$

(i) Sketch the graph of $y = \mathrm{f}\left(x\right)$. [2]

The function g is defined as follows:

$$\mathrm{g} : x \mapsto \frac{1}{x - 3} \quad \text{for } x \in \mathbb{R},\ x \neq a,\ x \neq 3,\ x \neq b.$$

It is given that the function fg exists.

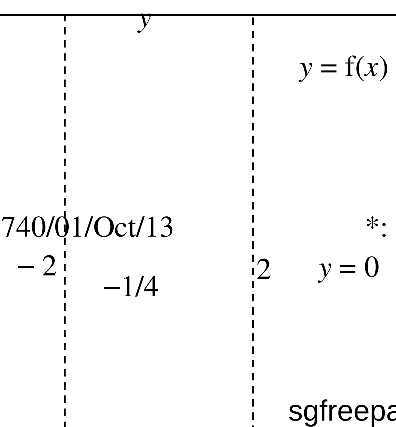
(ii) Find the values of a and b. [2]

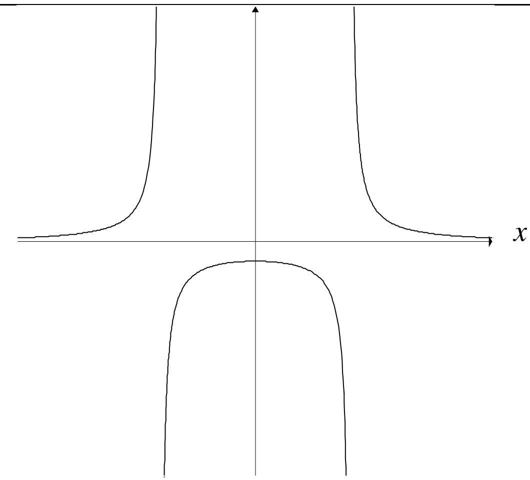
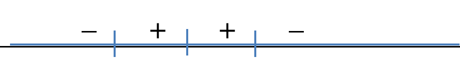
(iii) Show that $\mathrm{fg}\left(x\right) = \frac{\left(x-3\right)^2}{\left(2x-5\right)\left(7-2x\right)}$. [2]

(iv) Solve the inequality $\mathrm{fg}\left(x\right) > 0$. [3]

(v) Find the range of fg. [3]

13(i)	



	
(ii)	For fg to exist, $R_{\mathrm{g}} \subseteq D_{\mathrm{f}}$. Hence, $\mathrm{g}(x)$ cannot take the values – 2 and 2. $\frac{1}{x-3} = -2 \Rightarrow x = \frac{5}{2}$ $\frac{1}{x-3} = 2 \Rightarrow x = \frac{7}{2}$ The values of a and b are $\frac{5}{2}$ and $\frac{7}{2}$.
(iii)	$\mathrm{f\,g}(x) = \frac{1}{\left(\frac{1}{x-3}\right)^2 - 4}$ $= \frac{1}{\frac{1-4(x-3)^2}{(x-3)^2}}$ $= \frac{(x-3)^2}{1^2 - \left[2(x-3)\right]^2}$ $= \frac{(x-3)^2}{\left[1+2(x-3)\right]\left[1-2(x-3)\right]}$ $= \frac{(x-3)^2}{(2x-5)(7-2x)}$ (shown)
(iv)	$\frac{(x-3)^2}{(7-2x)(2x-5)} > 0$ 

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2.5 3 3.5

Solving,

$\frac{5}{2} < x < 3$ or $3 < x < \frac{7}{2}$

or $\frac{5}{2} < x < \frac{7}{2}$, $x \neq 3$

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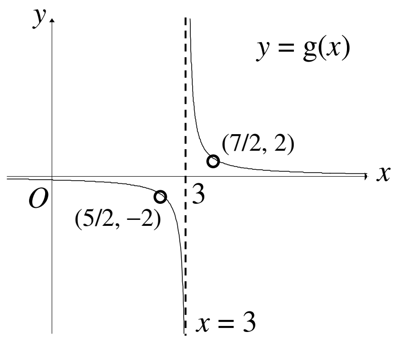
<tr><td>(v)</td><td>

Sketching the graph of $y = \mathrm{g}(x)$,



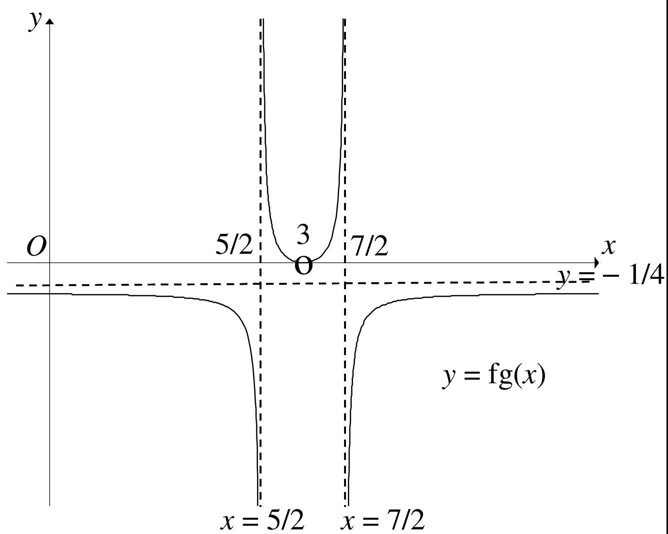
$R_{\mathrm{g}} = \{y \in \mathbb{R}: y \neq -2, 0, 2\}$

Referring to the graph of $y = \mathrm{f}(x)$ in part (i),

$R_{\mathrm{fg}} = \left\{y \in \mathbb{R}: y < -\frac{1}{4} \text{ or } y > 0\right\}$

OR

Sketch the graph of $y = \mathrm{fg}(x)$.



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	From the graph of $y = \mathrm{fg}(x)$, $R_{\mathrm{fg}} = \left\{ y \in \mathbb{R} : y < -\frac{1}{4} \text{ or } y > 0 \right\}$.

1* Expand

$$(1+2x)\sqrt{4+3x}$$

in ascending powers of x, up to and including the term in x^2. [3]

Determine the range of values of x for which the expansion is valid. [1]

2 **(i)** Given that $\dfrac{2n-1}{(n-1)^2 n^2}$ can be written in the form $\dfrac{A}{(n-1)^2}+\dfrac{B}{n^2}$, find the values of the constants A and B. [2]

(ii) Hence find $\displaystyle\sum_{r=2}^{N}\frac{2r-1}{(r-1)^2 r^2}$. [3]

(iii) Using your answer in (ii), find $\displaystyle\sum_{r=1}^{N}\frac{2r+1}{r^2(r+1)^2}$. [2]

3 Machines *A* and *B* are used to cut metal bars of length 30m into pieces of decreasing lengths.

(i) The lengths of all the pieces cut by machine *A* form an arithmetic progression with common difference *d* m. If the total length of the first 25 pieces cut is 25m and the length of the 25th piece is 0.5m, find the value of *d*. [3]

(ii) The length of the first piece cut by machine *B* is 2m and the lengths of all the pieces cut form a geometric progression. The 25th piece cut by machine *B* has length 0.5m. Find the maximum number of pieces of metal bars cut. [4]

4 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = 1 \text{ and } u_{n+1} = \frac{4+2u_n}{5} \quad \text{for } n \geq 1.$$

(i) Find the values of u_2 and u_3. [2]

(ii) It is given that $u_n \to l$ as $n \to \infty$. Showing your working, find the exact value of *l*. [2]

(iii) For this value of l, use the method of mathematical induction to prove that

$$u_n = l - \frac{1}{3}\left(\frac{2}{5}\right)^{n-1} \quad \text{for } n \geq 1.$$ [4]

*: Not in the topics tested in 2014 SRJC Promo

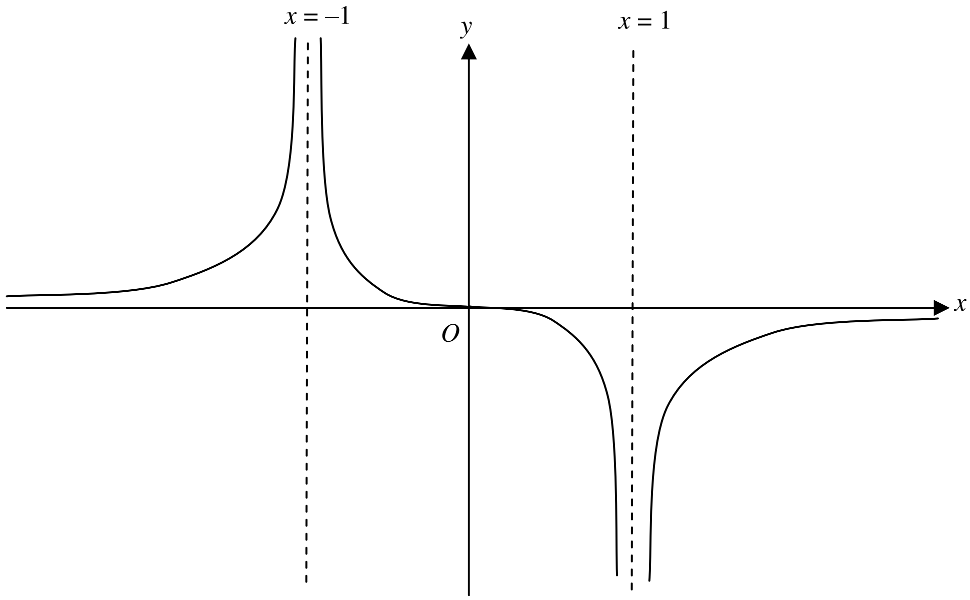
5 The curve C has equation $y = \dfrac{x^2 - 3x + 3}{1 - x}$.

(i) Find the equations of the asymptotes of C. [2]

(ii) Prove using an algebraic method, that y cannot lie between two certain values (to be determined). [3]

(iii) Sketch the curve C clearly labeling all asymptotes, turning points and axial intercepts. [3]

6 The diagram shows the graph of $y = \mathrm{f}(x)$. It has a vertical asymptotes at $x = 1$ and $x = -1$. It has a stationary point of inflexion at the origin.



Sketch on separate diagrams, the graphs of

(i) $y = \mathrm{f}(2 - x)$, [3]

(ii) $y = -\left|\mathrm{f}(x)\right|$, [2]

(iii) $y = \mathrm{f}'(x)$. [2]

*: Not in the topics tested in 2014 SRJC Promo

[Turn over

7 **(a)** Show that x^2-3x+5 is always positive and solve the inequality

$$\frac{x^2-3x+5}{(4-x)(x-2)}<0.$$ [4]

Hence find the solution for the inequality $\dfrac{(x+2)^2-3x-1}{x(2-x)}<0$. [2]

(b) A factory produces 3 brands of drinks, *A*, *B* and *C*. The total price of 1 litre of *A*, 1 litre of *B* and 2 litres of *C* is $9. The total price of 1 litre of *B* and 1 litre of *C* is $3.50. The total price of 2.5 litres of *B* and 2 litres of *C* is twice the price of 1 litre of *A*.
Write down and solve the equations to find the price of each litre of *A*, *B* and *C*. [4]

8 The functions f and g are defined by

$$\mathrm{f}: x \mapsto 3\ln\left(x^2+1\right), \quad 0 \le x \le 2,$$
$$\mathrm{g}: x \mapsto \mathrm{e}^x+1, \quad x \ge 0.$$

(i) Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. [3]

(ii) Sketch the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ on a single diagram. State the geometrical relationship between the graphs and hence state the number of solutions to $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [4]

(iii) Show that gf exists, define it in a similar form and find its range. [4]

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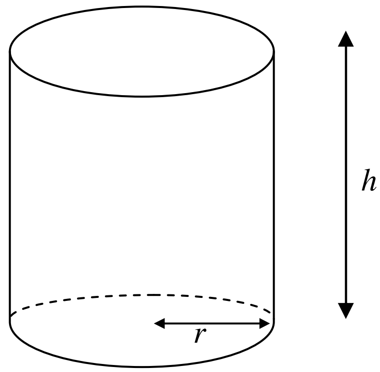
9 **(a)**



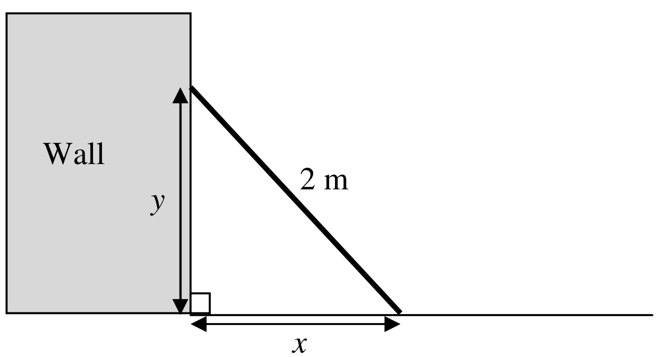
A closed cylindrical can with base radius r and height h has a fixed volume V.

(i) Show that the total surface area of the can, A, is given by

$$A = 2\pi r^2 + \frac{2V}{r}.$$ [1]

(ii) Find h in terms of r when the minimum surface area is achieved. [4]

(b)



A ladder of length 2 m, leaning against the wall, slips in such a way that x increases at a rate of 0.02 ms^{-1}. Find the rate of decrease of y at the instant when x is 1 m. [4]

*: Not in the topics tested in 2014 SRJC Promo

[Turn over

10 **(a)** The curve C is defined by

$$x = e^{3t}, \qquad y = t^2, \qquad \text{where } t \geq 0.$$

(i) Find $\frac{dy}{dx}$ in terms of t and determine the value of t for which $\frac{dy}{dx}$ is zero. [3]

(ii) Sketch the graph of C. [2]

(b) The equation of a curve C is $x^2 - 2xy + 2y^2 = k$, where k is a constant.

Find $\frac{dy}{dx}$ in terms of x and y. [3]

Given that C has two points for which the tangents are parallel to the line $y = x$, find the range of values of k. [3]

Given that $k = 4$, find the exact coordinates of each point on the curve C at which the tangent is parallel to the y-axis. [4]

11* **(a)** Find

(i) $\int x^2 e^x \, dx$, [3]

(ii) $\int_0^{\frac{\pi}{3}} \sin^2 2x \, dx$, leaving your answer in exact form. [3]

(b) Using the substitution $u = 3x - 1$, find

$$\int \frac{9x}{(3x-1)^2} \, dx.$$ [3]

(c) Given that $x + 1 = A(2x - 4) + B$ for all values of x, find the constants A and B.

Hence, find

$$\int \frac{x+1}{x^2 - 4x + 13} \, dx.$$ [5]

[End of Paper]

*: Not in the topics tested in 2014 SRJC Promo

Jurong Junior College
2013 JC1 H2 Mathematics Promo Solutions
(Qn 1 and 11 are not in topics tested for 2014 SRJC Promo)

Qn	Solution
1	$(1+2x)\sqrt{4+3x}$ $=(1+2x)2\left(1+\frac{3x}{4}\right)^{\frac{1}{2}}$ $=2(1+2x)\left(1+\frac{3x}{8}-\frac{9x^2}{128}+\ldots\right)$ $=2+\frac{19x}{4}+\frac{87x^2}{64}\ldots$
	Validity: $\left\|\frac{3x}{4}\right\|<1$ $-\frac{4}{3}<x<\frac{4}{3}$

Qn	Solution
2	(i) $\frac{2n-1}{(n-1)^2n^2}=\frac{A}{(n-1)^2}+\frac{B}{n^2}$ $=\frac{An^2+B(n-1)^2}{(n-1)^2n^2}$ $2n-1=An^2+B(n-1)^2$ When $n=0,\ B=-1.$ When $n=1,\ A=1.$ $\therefore \frac{2n-1}{(n-1)^2n^2}=\frac{1}{(n-1)^2}-\frac{1}{n^2}$
	(ii) $\sum_{r=2}^{N}\frac{2r-1}{(r-1)^2r^2}=\sum_{r=2}^{N}\left[\frac{1}{(r-1)^2}-\frac{1}{r^2}\right]$ $=\Big[\frac{1}{1^2}-\frac{1}{2^2}$ $+\frac{1}{2^2}-\frac{1}{3^2}$ $+\frac{1}{3^2}-\frac{1}{4^2}$ $+\ldots$ $+\frac{1}{(N-1)^2}-\frac{1}{N^2}\Big]=1-\frac{1}{N^2}$

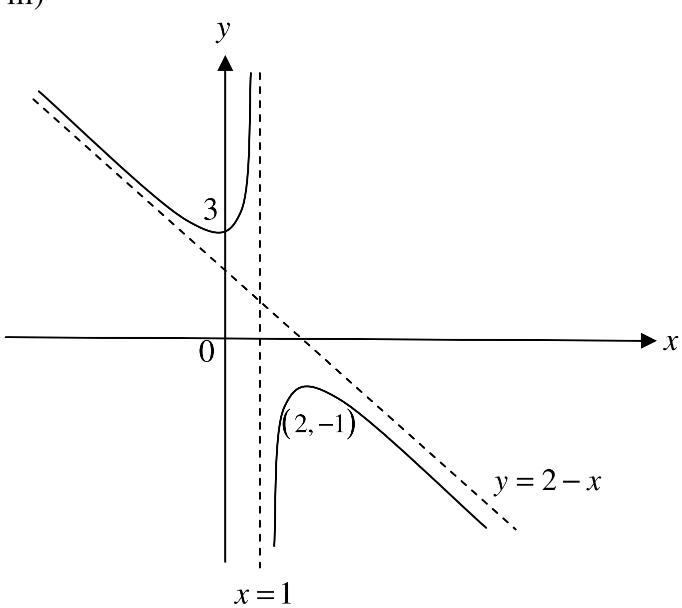
	$\sum_{r=1}^{N} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=2}^{N+1} \frac{2r-1}{(r-1)^2 r^2}$ $= 1 - \frac{1}{(N+1)^2}$

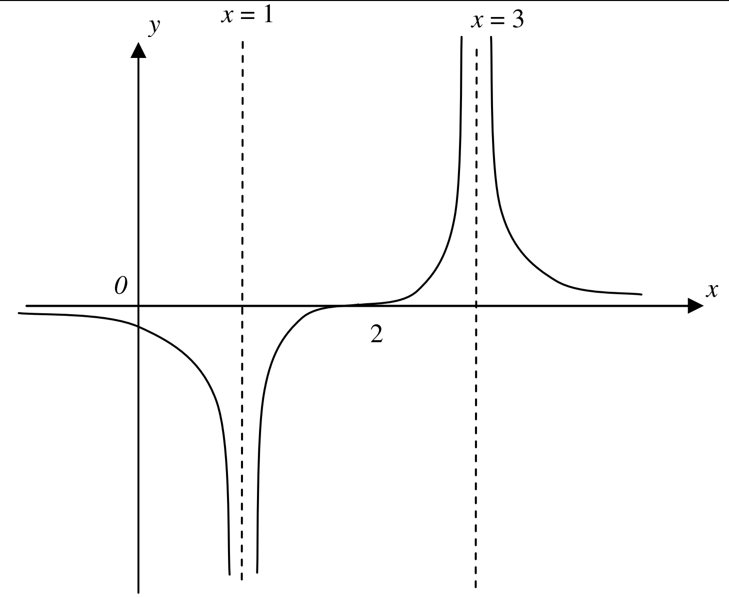
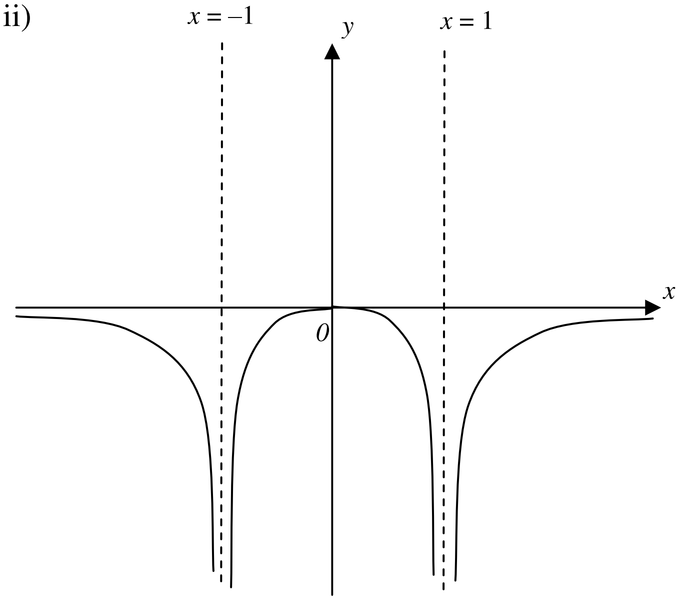
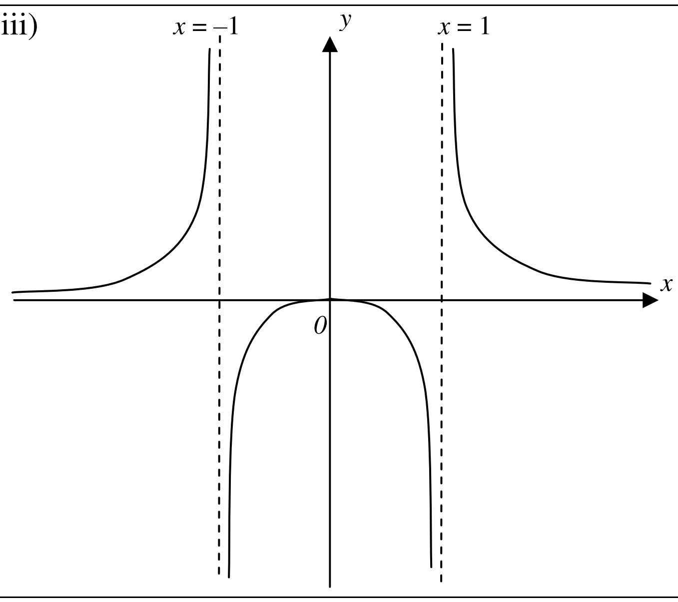
Qn	Solution
3	(i) $S_{25} = 25$ $\frac{25}{2}[a+0.5] = 25$ $\Rightarrow \quad a = 1.5$ $a + 24d = 0.5$ Subst $a = 1.5$, $d = -\frac{1}{24}$ = 0.0417 (to 3 s.f)
	(ii) GP $a = 2$ $ar^{24} = 0.5$ $2r^{24} = 0.5$ $r^{24} = \frac{1}{4}$ $r = \sqrt[24]{\frac{1}{4}}$ =0.94387 (to 5 s.f) $S_n \le 30$ $\frac{2\left[1-\left(\sqrt[24]{\frac{1}{4}}\right)^n\right]}{1-\left(\sqrt[24]{\frac{1}{4}}\right)} \le 30$ $1-\left(\sqrt[24]{\frac{1}{4}}\right)^n \le 0.84195$ $\left(\sqrt[24]{\frac{1}{4}}\right)^n \ge 0.15805$ $n \le \frac{\ln 0.15805}{\ln \sqrt[24]{\frac{1}{4}}}$ $n \le 31.931$ Therefore maximum number of pieces cut = 31.

Alternative Solution	
	$S_n \le 30$ $\dfrac{2\left[1-(0.94387)^n\right]}{1-(0.94387)} \le 30$ $1-(0.94387)^n \le 0.84195$ $(0.94387)^n \ge 0.15805$ $n \le \dfrac{\ln 0.15805}{\ln 0.94387}$ $n \le 31.9$
	Therefore maximum number of pieces cut = 31.

Qn	Solution
4	(i) $u_2 = \dfrac{4+2(1)}{5} = \dfrac{6}{5} = 1.2$ $u_3 = \dfrac{4+2(\frac{6}{5})}{5} = \dfrac{32}{25} = 1.28$
	(ii) As $n \to \infty$, $u_n \to l$, $u_{n+1} \to l$. $l = \dfrac{4+2l}{5}$ $l = \dfrac{4}{3}$
	(iii) Let P_n be the statement $u_n = \dfrac{4}{3} - \dfrac{1}{3}\left(\dfrac{2}{5}\right)^{n-1}$ for all $n \ge 1$. . LHS of $P_1 = u_1 = 1$ (by defn) RHS of $P_1 = \dfrac{4}{3} - \dfrac{1}{3}\left(\dfrac{2}{5}\right)^{1-1} = \dfrac{3}{3} = 1$ $\therefore P_1$ is true. Assume that P_k is true for some $k \ge 1$, ie $u_k = \dfrac{4}{3} - \dfrac{1}{3}\left(\dfrac{2}{5}\right)^{k-1}$ We want to prove P_{k+1}, ie $u_{k+1} = \dfrac{4}{3} - \dfrac{1}{3}\left(\dfrac{2}{5}\right)^{k}$ LHS of $P_{k+1} = u_{k+1}$ $= \dfrac{4+2u_k}{5}$ $= \dfrac{4}{5} + \dfrac{2}{5}\left[\dfrac{4}{3} - \dfrac{1}{3}\left(\dfrac{2}{5}\right)^{k-1}\right]$

	$=\dfrac{12}{15}+\dfrac{8}{15}-\dfrac{1}{3}\left(\dfrac{2}{5}\right)\left(\dfrac{2}{5}\right)^{k-1}$ $=\dfrac{4}{3}-\dfrac{1}{3}\left(\dfrac{2}{5}\right)^{k}$ $=\text{RHS of } P_{k+1}$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true. $\therefore$ By Mathematical Induction, P_n is true for all $n \geq 1$.

5	i) Asymptotes: By Long Division, $y=\dfrac{x^2-3x+3}{1-x}=2-x+\dfrac{1}{1-x}$ Asymptotes: $x=1$, $y=2-x$
	ii) $y=\dfrac{x^2-3x+3}{1-x}$ $y(1-x)=x^2-3x+3$ $x^2+(y-3)x+3-y=0$ For no solutions, Discriminant <0 $(y-3)^2-4(3-y)<0$ $(y^2-6y+9)-(12-4y)<0$ $y^2-2y-3<0$ $(y-3)(y+1)<0$ $\therefore -1<y<3$
	iii) 

6	i) 
	ii) 
	iii) 

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>7</td><td>

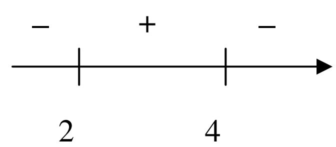
(a) $x^2-3x+5=\left(x-\frac{3}{2}\right)^2-\left(-\frac{3}{2}\right)^2+5$

$=\left(x-\frac{3}{2}\right)^2+\frac{11}{4}$

Since $\left(x-\frac{3}{2}\right)^2\geq 0$ for all real values of x, $\therefore$ x^2-3x+5 is always positive.

$\frac{x^2-3x+5}{(4-x)(x-2)}<0$

Since x^2-3x+5 is always positive, $(4-x)(x-2)<0$



$\therefore$ $x<2$ or $x>4$ -------(1)

$\frac{(x+2)^2-3x-1}{x(2-x)}<0$

Replace x in eqn (1) with $(x+2)$,

$\therefore$ $x+2<2$ or $x+2>4$

$\Rightarrow$ $x<0$ or $x>2$

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(b) Let the price of 1 litre of A, B and C be a, b and c respectively.

Given that

$a+b+2c=9$

$b+c=3.50$

$2.5b+2c=2a \Rightarrow 2a-2.5b-2c=0$

Using GC, $a=\$4$, $b=\$2$, $c=\$1.50$.

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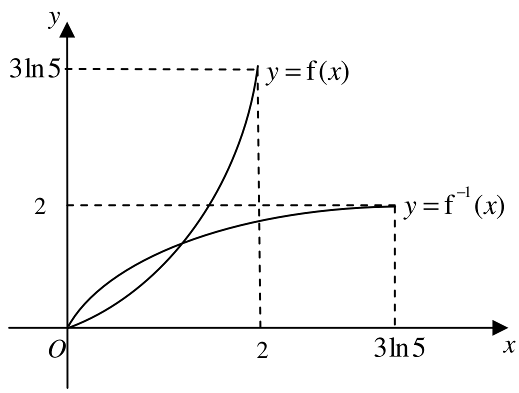
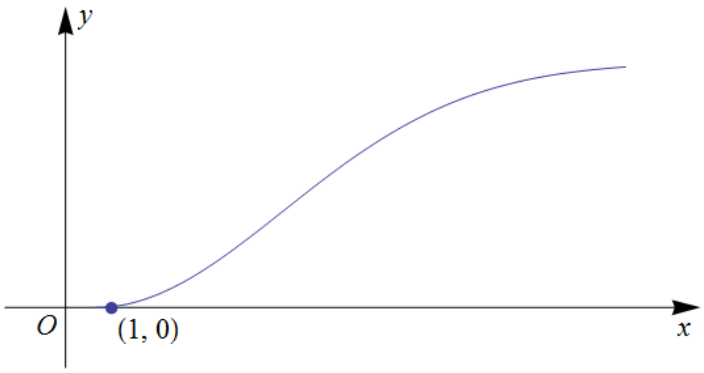
Qn	Solution
8	i) $y=3\ln\left(x^2+1\right)$ $x=\pm\sqrt{e^{\frac{y}{3}}-1}$ $x=\sqrt{e^{\frac{y}{3}}-1}$ since $0\le x\le 2$ $\therefore f^{-1}(x)=\sqrt{e^{\frac{x}{3}}-1},\quad 0\le x\le 3\ln 5$
	ii)  They are reflections about $y=x$ and there are 2 solutions.
	iii) $R_f=[0,3\ln 5]$ $D_g=[0,\infty)$ $R_f\subseteq D_g$ $\therefore$ gf exists $gf(x)=\left(x^2+1\right)^3+1,\qquad 0\le x\le 2$ $R_{gf}=[2,126]$

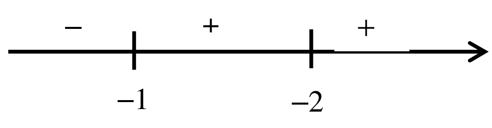
Qn	Solution
9 (a)	(i) $V = \pi r^2 h$ $h = \dfrac{V}{\pi r^2}$ $A = 2\pi r^2 + 2\pi r h$ $= 2\pi r^2 + 2\pi r\left(\dfrac{V}{\pi r^2}\right)$ $= 2\pi r^2 + \dfrac{2V}{r}$ (shown)
	(ii) For min A, $\dfrac{dA}{dr} = 4\pi r - \dfrac{2V}{r^2} = 0$ $4\pi r^3 = 2V$ $r = \left(\dfrac{V}{2\pi}\right)^{\frac{1}{3}}$ $\dfrac{d^2A}{dr^2} = 4\pi + \dfrac{4V}{r^3} > 0$ Thus, A is minimum. Substitute $V = \pi r^2 h$, $r = \left(\dfrac{\pi r^2 h}{2\pi}\right)^{\frac{1}{3}}$ $r^3 = \dfrac{r^2 h}{2}$ $h = 2r$
(b)	$y = \sqrt{2^2 - x^2}$ $= \sqrt{4 - x^2}$ $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{4-x^2}}(-2x)$ $= -\dfrac{x}{\sqrt{4-x^2}}$ $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$ $= -\dfrac{x}{\sqrt{4-x^2}} \times (0.02)$ $= -\dfrac{1}{\sqrt{4-1^2}} \times (0.02)$ $= -0.011547$ $= -0.0115$ $\therefore$ y decreases at a rate of 0.0115 ms^{-1}.

Qn	Solution
10(a) (i)	$x = e^{3t} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 3e^{3t}$ $y = t^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2t$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t}{3e^{3t}}$ When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\frac{2t}{3e^{3t}} = 0$ $t = 0$
(ii)	
(b)	$x^2 - 2xy + 2y^2 = k \quad \ldots (1)$ Differentiate throughout w.r.t. x. $2x - 2\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + 4y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-x}{2y-x}$
	For tangents which are parallel to the line $y = x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$. $\frac{y-x}{2y-x} = 1$ $y - x = 2y - x$ $y = 0$ Subst. $y = 0$ into (1): $x^2 - 2x(0) + 2(0)^2 = k$ $x^2 = k$ Given that there are 2 tangents parallel to the line $y = x$, $k > 0$

	For tangents which are parallel to the y-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined. $$2y-x=0$$ $$x=2y$$ Subst. $x=2y$ and $k=4$ into (1): $$(2y)^2-2(2y)y+2y^2=4$$ $$y=\pm\sqrt{2}$$ $$x=\pm 2\sqrt{2}$$ The coordinates are $\left(-2\sqrt{2},-\sqrt{2}\right)$ and $\left(2\sqrt{2},\sqrt{2}\right)$.

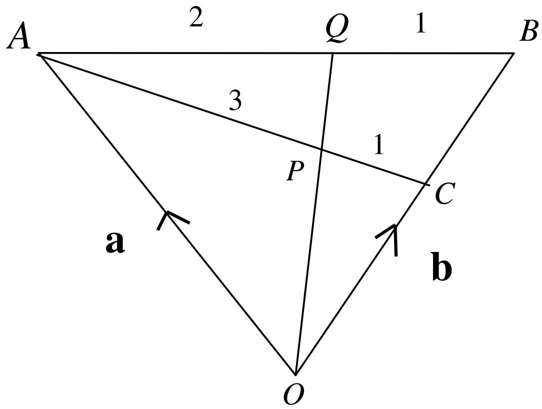
Qn	**Solution**
11(a) (i)	$\int x^2e^x\ \mathrm{d}x = x^2e^x - 2\int xe^x\ \mathrm{d}x$ $= x^2e^x - 2\left[xe^x - \int e^x \mathrm{d}x\right]$ $= x^2e^x - 2\left[xe^x - e^x\right] + c$ $= e^x\left(x^2-2x+2\right)+c$
(ii)	$\int_0^{\frac{\pi}{3}} \sin^2 2x\ \mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{3}} 1-\cos 4x\ \mathrm{d}x$ $=\frac{1}{2}\left[x-\frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{3}}$ $=\frac{1}{2}\left[\frac{\pi}{3}-\frac{1}{4}\sin\frac{4\pi}{3}\right]$ $=\frac{1}{2}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right]$
(b)	$\int \frac{9x}{(3x-1)^2}\ \mathrm{d}x = \int \frac{u+1}{u^2}\ \mathrm{d}u$ $=\int \frac{1}{u}+u^{-2}\mathrm{d}u$ $=\ln\lvert u\rvert - \frac{1}{u}+c$ $=\ln\lvert 3x-1\rvert - \frac{1}{3x-1}+c$

(c)	$x+1=A(2x-4)+B$ $\quad =2Ax-4A+B$ By comparing coefficients, $2A=1 \Rightarrow A=\dfrac{1}{2}$ $-4A+B=1 \Rightarrow B=3$
	$\displaystyle\int \frac{x+1}{x^2-4x+13}\,\mathrm{d}x$ $\displaystyle=\int \frac{\frac{1}{2}(2x-4)+3}{x^2-4x+13}\,\mathrm{d}x$ $\displaystyle=\frac{1}{2}\int \frac{2x-4}{x^2-4x+13}\,\mathrm{d}x+3\int \frac{1}{(x-2)^2+3^2}\,\mathrm{d}x$ $\displaystyle=\frac{1}{2}\ln\left\|x^2-4x+13\right\|+3\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{x-2}{3}\right)+c$ $\displaystyle=\frac{1}{2}\ln\left(x^2-4x+13\right)+\tan^{-1}\left(\frac{x-2}{3}\right)+c$

Qn	Solution
1	**Inequalities**
	$x^2-x+7=\left(x-\frac{1}{2}\right)^2+7-\left(\frac{1}{2}\right)^2$ $=\left(x-\frac{1}{2}\right)^2+\frac{27}{4}$ Since $\left(x-\frac{1}{2}\right)^2\geq 0$ for all real values of x, $\left(x-\frac{1}{2}\right)^2+\frac{27}{4}>0$ (shown). $\frac{3}{(x-2)^2}>\frac{-1}{x+1},\ x\neq -1,\ x\neq 2$ $\frac{3}{(x-2)^2}+\frac{1}{x+1}>0$ $\frac{3(x+1)+(x^2-4x+4)}{(x+1)(x-2)^2}>0$ $\frac{x^2-x+7}{(x+1)(x-2)^2}>0$ Since $x^2-x+7=\left(x-\frac{1}{2}\right)^2+\frac{27}{4}>0$ and $(x-2)^2>0$ for all $x\in\mathbb{R}\setminus\{2\}$ $\Rightarrow (x+1)>0$ $\therefore x>-1$, $x\neq 2$ **<u>Alternatively</u>** Since $x^2-x+7=\left(x-\frac{1}{2}\right)^2+\frac{27}{4}>0$ for all real values of x, $\frac{1}{(x+1)(x-2)^2}>0$  $\therefore x>-1$, $x\neq 2$

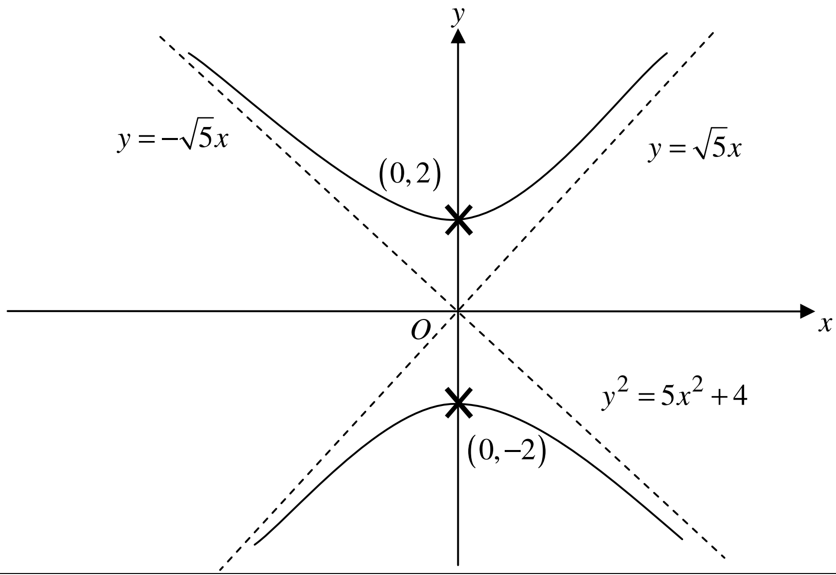
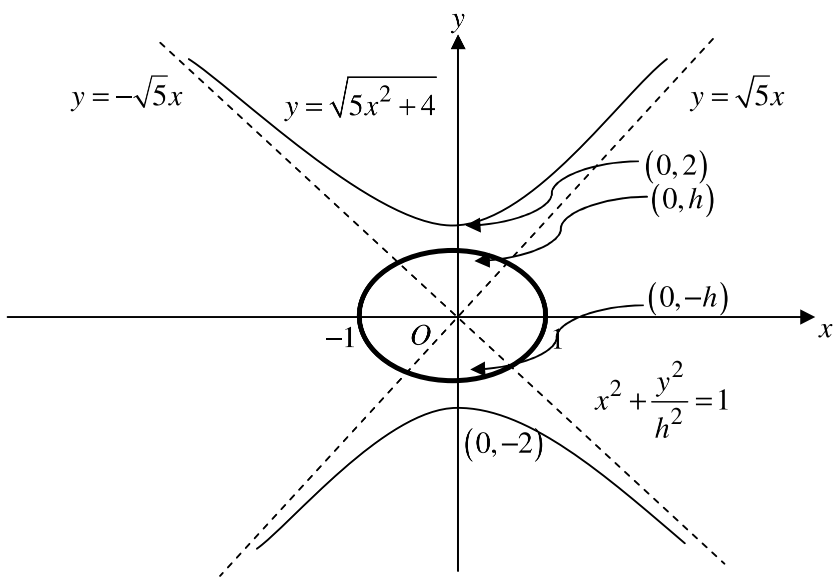
Qn	Solution
2	**Techniques of Differentiation**
	$x=\sin^{-1}(1-t)$ $\quad$ $y=\mathrm{e}^{\sqrt{2t-t^2}}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{\sqrt{1-(1-t)^2}}(-1)$ $\quad$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^{\sqrt{2t-t^2}}\frac{1}{2}\left(2t-t^2\right)^{-\frac{1}{2}}(2-2t)$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{\sqrt{2t-t^2}}$ $\quad$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{e}^{\sqrt{2t-t^2}}(1-t)}{\sqrt{2t-t^2}}$ $\therefore\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\mathrm{e}^{\sqrt{2t-t^2}}(t-1)$

Qn	Solution
3	**SLE**
(i)	At A, $b+c=a+d$. At B, $a+b+c=48$. At C, $a+c=2b$. At D, $d=b+2a$. After simplifying, $-a+b+c-d=0$. $a+b+c=48$. $a-2b+c=0$. $2a+b-d=0$. Using GC, $a=8, b=16, c=24$ and $d=32$.
(ii)	Total amount collected $=\$0.50(2c+b)$ $=\$0.50(48+16)$ $=\$32$

Qn	Solution
4	**Vectors I**
(i)	 $\overrightarrow{OC} = k\mathbf{b}$ Using Ratio Theorem, $\overrightarrow{OP} = \dfrac{\mathbf{a}+3k\mathbf{b}}{4}$ $\overrightarrow{OQ} = \dfrac{\mathbf{a}+2\mathbf{b}}{3}$
(ii)	Given that O, P and Q are collinear, $\overrightarrow{OP} = \lambda\overrightarrow{OQ}$ for some $\lambda \in \mathbb{R}$ $\frac{1}{4}\mathbf{a}+\frac{3k}{4}\mathbf{b} = \lambda\left(\frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{b}\right)$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors, $\frac{1}{4} = \frac{\lambda}{3}$ ------ (1) and $\frac{3}{4}k = \frac{2}{3}\lambda$ ------ (2) From (1): $\lambda = \frac{3}{4}$ ------ (3) Substitute (3) into (2) $k = \frac{2}{3}\left(\frac{3}{4}\right)\left(\frac{4}{3}\right)$ $= \frac{2}{3}$ $\therefore k = \frac{2}{3}$ **<u>Alternatively,</u>** Given that O, P and Q are collinear, $\overrightarrow{OQ} = \lambda\overrightarrow{OP}$ for some $\lambda \in \mathbb{R}$

	$\frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{b}=\lambda\left(\frac{1}{4}\mathbf{a}+\frac{3k}{4}\mathbf{b}\right)$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors, $\frac{1}{3}=\lambda\left(\frac{1}{4}\right)$ ------ (1) and $\frac{2}{3}=\lambda\left(\frac{3k}{4}\right)$ ------ (2) From (1): $\lambda=\frac{4}{3}$ ------ (3) Substitute (3) into (2) $k=\frac{2}{3}\left(\frac{4}{3\lambda}\right)$ $=\frac{2}{3}\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)=\frac{2}{3}$ $\therefore k=\frac{2}{3}$

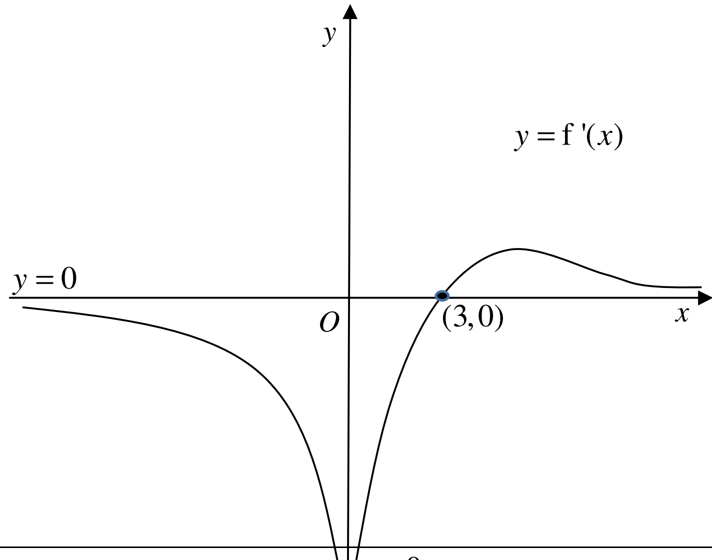
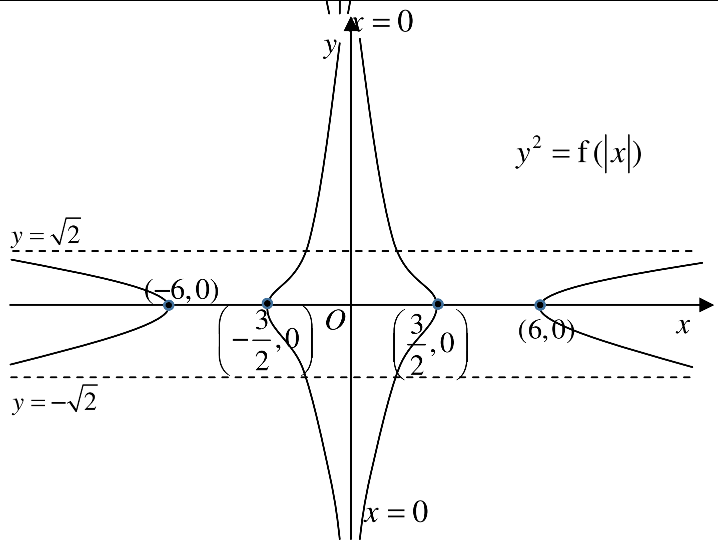
Qn	Solution
5*	**Maclaurin's Series and Binomial Theorem [Not in topics tested for SRJC 2014 Promo]**
(i)	$\mathrm{e}^x\sin 2x$ $=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\left(2x-\frac{(2x)^3}{3!}+\ldots\right)$ $=\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\right)\left(2x-\frac{8x^3}{6}+\ldots\right)$ $=2x-\frac{8x^3}{6}+2x^2+x^3+\ldots$ $=2x+2x^2-\frac{1}{3}x^3+\ldots$
(ii)	$\frac{\mathrm{e}^x\sin 2x}{\sqrt{4-x}}=\left[\mathrm{e}^x\sin 2x\right](4-x)^{-\frac{1}{2}}$ $=\left(2x+2x^2-\frac{1}{3}x^3+\ldots\right)(4)^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$ $=\frac{1}{2}\left(2x+2x^2-\frac{1}{3}x^3+\ldots\right)\left(1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\ldots\right)$ $=\frac{1}{2}\left(2x+2x^2-\frac{1}{3}x^3+\ldots\right)\left(1+\frac{x}{8}+\frac{3}{8}\left(\frac{x^2}{16}\right)+\ldots\right)$ $=\frac{1}{2}\left(2x+\frac{2x^2}{8}+2x^2+\frac{3x^3}{64}-\frac{x^3}{3}+\frac{2x^3}{8}\ldots\right)$ $=x+\frac{9x^2}{8}-\frac{7x^3}{384}+..$

Qn	Solution
6	**Graphing Techniques 1**
(i)	
(ii)	$5x^2+4=h^2\left(1-x^2\right)$ $y^2=h^2\left(1-x^2\right)$ $y^2+x^2h^2=h^2$ $\frac{y^2}{h^2}+x^2=1$  Graph to be inserted is $x^2+\frac{y^2}{h^2}=1$. From the graphs, $0<h<2$.

Qn	Solution
7	**Application of Differentiation (Tangent/ Normal)**

	$y=\dfrac{x^2}{x-1}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x(x-1)-x^2}{(x-1)^2}$ $=\dfrac{x^2-2x}{(x-1)^2}$ Since gradient of tangent at A is $\dfrac{8}{9}$ $\dfrac{x^2-2x}{(x-1)^2}=\dfrac{8}{9}$ Using GC, $x=4$ or $x=-2$ Since $x_2<x_1$, x coordinate at point B is $x_2=-2$ Sub $x_2=-2$ into C we have $y_2=-\dfrac{4}{3}$ $\therefore$ coordinates of B is $\left(-2,-\dfrac{4}{3}\right)$ Since gradient of normal at B is $-\dfrac{9}{8}$ $y-\left(-\dfrac{4}{3}\right)=-\dfrac{9}{8}\left(x-(-2)\right)----(*)$ $y=-\dfrac{9}{8}x-\dfrac{43}{12}$

Qn	**Solution**
8	**Transformation of graphs**
(a)	$y=\dfrac{x-1}{3x^2-5}$ $\downarrow$ 1. Replace y by $-y$ $-y=\dfrac{x-1}{3x^2-5}$ $\downarrow$ 2. Replace y by $y-1$ $1-y=\dfrac{x-1}{3x^2-5}$ $\downarrow$ 3. Replace x by $2x$ $1-y=\dfrac{2x-1}{12x^2-5}$ The transformations are in the following order: 1. Reflection in the x-axis. 2. Translation of 1 unit in the positive y-direction. 3. Scaling parallel to the x-axis by factor ½. (or 3-1-2, 1-3-2, 1-3-2) **Alternatively,** The transformations are in the following order:

	1. Translation of 1 unit in the negative y-direction. 2. Reflection in the x-axis. 3. Scaling parallel to the x-axis by factor ½. (or 1-3-2)
(b)(i)	
(b)(ii)	

Qn	Solution
9	**Mathematical Induction (RR) and MOD**
(i)	Let P_n be the statement $u_n = \dfrac{1}{2n^2}$ for $n \in \mathbb{Z}^+$. When $n = 1$, LHS = $u_1 = \dfrac{1}{2}$ RHS $= \dfrac{1}{2(1)^2} = \dfrac{1}{2} =$ LHS $\therefore \mathrm{P}_1$ is true. Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \dfrac{1}{2k^2}$ -------------------- (*)

	To prove P_{k+1} is also true, i.e. $u_{k+1}=\dfrac{1}{2(k+1)^2}$. LHS = $u_{k+1}=u_k-\dfrac{2(k+1)-1}{2k^2(k+1)^2}$ (from the recurrence relation) $=u_k-\dfrac{2k+1}{2k^2(k+1)^2}$ $=\dfrac{1}{2k^2}-\dfrac{2k+1}{2k^2(k+1)^2}$ from (*) $=\dfrac{(k+1)^2-2k-1}{2k^2(k+1)^2}$ $=\dfrac{k^2}{2k^2(k+1)^2}$ $=\dfrac{1}{2(k+1)^2}=\text{RHS}$ Thus P_k is true $\Rightarrow P_{k+1}$ is true. Since P_1 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n\in\mathbb{Z}^+$.
(ii)	$\sum_{n=1}^{N}\dfrac{2n+1}{2n^2(n+1)^2}=\sum_{n=1}^{N}(u_n-u_{n+1})$ $= u_1-u_2$ $+u_2-u_3$ $\vdots$ $+u_N-u_{N+1}$ $= u_1-u_{N+1}$ $=\dfrac{1}{2}-\dfrac{1}{2(N+1)^2}=\dfrac{1}{2}\left(1-\dfrac{1}{(N+1)^2}\right)$
(iii)	$\sum_{n=0}^{N}\dfrac{2n+3}{2(n+1)^2(n+2)^2}=\sum_{n=1}^{N+1}\dfrac{2n+1}{2n^2(n+1)^2}$ $=\dfrac{1}{2}-\dfrac{1}{2(N+2)^2}$

Qn	Solution
10	**Vectors**
(i)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$ $\Rightarrow$ a direction vector for the line is $\begin{pmatrix}2\\-1\\1\end{pmatrix}$ vector equation of the line AB : $r=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix},\quad \lambda\in\mathbb{R}$ To determine whether point C lies on the line: Let $\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$. Then $\begin{cases}2=1+2\lambda\Rightarrow\lambda=\frac{1}{2}\\1=1-\lambda\Rightarrow\lambda=0\\5=1+\lambda\Rightarrow\lambda=4\end{cases}$ Since the values of λ are inconsistent, i.e. no value of λ satisfies all the equations, hence shown that point C does not lie on the line AB .
(ii)	Let N be the foot of the perpendicular from C to line AB line AB : $r=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix},\quad \lambda\in\mathbb{R}$ Since N lies on line AB then $\overrightarrow{ON}=\begin{pmatrix}1+2\lambda\\1-\lambda\\1+\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. $\overrightarrow{CN}=\overrightarrow{ON}-\overrightarrow{OC}=\begin{pmatrix}1+2\lambda\\1-\lambda\\1+\lambda\end{pmatrix}-\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}-1+2\lambda\\-\lambda\\-4+\lambda\end{pmatrix}$ $\overrightarrow{CN}\perp$ line AB, $\overrightarrow{CN}\cdot\mathbf{d}=0\Rightarrow\begin{pmatrix}-1+2\lambda\\-\lambda\\-4+\lambda\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix}=0$ $\Rightarrow -2+4\lambda+\lambda-4+\lambda=0\Rightarrow\quad\lambda=1$ Therefore, the position vector of the foot of the perpendicular from point C to line AB. $\overrightarrow{ON}=\begin{pmatrix}1+2(1)\\1-(1)\\1+(1)\end{pmatrix}=\begin{pmatrix}3\\0\\2\end{pmatrix}$ Since $\overrightarrow{ON}=\overrightarrow{OB}$, the angle ABC is 90 degrees.
(iii)	The position vector of C', the reflection of point C in the line AB

	$\overrightarrow{ON} = \dfrac{\overrightarrow{OC} + \overrightarrow{OC'}}{2}$ $\overrightarrow{OC'} = 2\overrightarrow{ON} - \overrightarrow{OC}$ $= 2\begin{pmatrix}3\\0\\2\end{pmatrix} - \begin{pmatrix}2\\1\\5\end{pmatrix}$ $= \begin{pmatrix}4\\-1\\-1\end{pmatrix}$

Qn	Solution
11	**AP GP**
(a)	Let T_1, T_3, T_6, be the first, third and sixth term of an arithmetic series with first term a and common difference d. $T_1 = a, \quad T_3 = a + 2d\ , \ T_6 = a + 5d$ $\dfrac{a+5d}{a+2d} = \dfrac{a+2d}{a}$ $a(a+5d) = (a+2d)^2$ $a^2 + 5ad = a^2 + 4ad + 4d^2$ $ad = 4d^2$ Since $d \neq 0 \Rightarrow a = 4d$ Common ratio $r = \dfrac{T_3}{T_1} = \dfrac{a+2d}{a} = \dfrac{6d}{4d} = \dfrac{3}{2}$ Since $\lvert r \rvert > 1$, the geometric progression is not convergent. $S_{15} = \dfrac{15}{2}\left[2a + 14d\right]$ $= \dfrac{15}{2}\left[2(4d) + 14d\right]$ $= 165d$ $= \dfrac{165}{4}a$
(b)	$a = 2, r = \dfrac{9}{10}$ $S_\infty = \dfrac{a}{1-r}$ $= \dfrac{2}{1 - \dfrac{9}{10}}$ $= 20$

	$S_n \geq 15$ $\frac{2}{1-\frac{9}{10}}\left(1-\left(\frac{9}{10}\right)^n\right) \geq 15$ $\left(1-\left(\frac{9}{10}\right)^n\right) \geq 0.75$ $\left(\frac{9}{10}\right)^n \leq 0.25$ $n \geq 13.158$ The minimum number of days required is 14 days.

Qn	Solution
12	**Applications of Differentiation**
(i)	$\sin\alpha = \frac{h}{PQ} \quad \therefore PQ = h \text{ cosec } \alpha$ $QR = k - PQ - RS$ $= k - 2PQ$ $= k - 2h \text{ cosec } \alpha$ (shown) $A = \frac{h}{2}\left(QR + PS\right)$ $= \frac{h}{2}\left(2QR + 2\frac{h}{\tan\alpha}\right)$ $= h\left(k - 2h \text{ cosec } \alpha + h\cot\alpha\right)$ $= hk + h^2(\cot\alpha - 2 \text{ cosec } \alpha)$ (shown)
(ii)	$A = hk + h^2(\cot\alpha - 2 \text{ cosec } \alpha)$ $\frac{\mathrm{d}A}{\mathrm{d}\alpha} = h^2(-\text{cosec}^2\ \alpha + 2\text{cosec } \alpha \cot\alpha)$ $= h^2\text{cosec } \alpha(-\text{cosec } \alpha + 2\cot\alpha)$ When $\frac{\mathrm{d}A}{\mathrm{d}\alpha} = 0$, $h^2\text{cosec } \alpha(-\text{cosec } \alpha + 2\cot\alpha) = 0$ Since $h^2\text{cosec } \alpha \neq 0$, $-\text{cosec } \alpha + 2\cot\alpha = 0$ $\frac{-1+2\cos\alpha}{\sin\alpha} = 0$ $-1+2\cos\alpha = 0$ $\cos\alpha = \frac{1}{2}$ $\alpha = \frac{\pi}{3}$

α	$\left(\frac{\pi}{3}\right)^-$	$\frac{\pi}{3}$	$\left(\frac{\pi}{3}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}\alpha}$	/	—	\

Alternatively

$$\frac{\mathrm{d}A}{\mathrm{d}\alpha} = h^2(-\operatorname{cosec}^2\alpha + 2\operatorname{cosec}\alpha\cot\alpha)$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}\alpha^2} = h^2(2\operatorname{cosec}^2\alpha\cot\alpha - 2\operatorname{cosec}^3\alpha - 2\operatorname{cosec}\alpha\cot^2\alpha)$$

$$= 2h^2\operatorname{cosec}\alpha(\operatorname{cosec}\alpha\cot\alpha - \operatorname{cosec}^2\alpha - \cot^2\alpha)$$

When $\alpha = \frac{\pi}{3}$,

$$\frac{\mathrm{d}^2A}{\mathrm{d}\alpha^2} = 2h^2\operatorname{cosec}\frac{\pi}{3}(\operatorname{cosec}\frac{\pi}{3}\cot\frac{\pi}{3} - \operatorname{cosec}^2\frac{\pi}{3} - \cot^2\frac{\pi}{3})$$

$$= 2h^2\frac{2}{\sqrt{3}}\left(\frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) - \left(\frac{2}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right)$$

$$= \frac{4}{\sqrt{3}}h^2\left(\frac{2}{3} - \frac{4}{3} - \frac{1}{3}\right) < 0$$

$$= -\frac{4}{\sqrt{3}}h^2 < 0$$

$\alpha = \frac{\pi}{3}$ gives max A

When $\alpha = \frac{\pi}{3}$

$$\text{Max } A = hk + h^2(\cot\alpha - 2\operatorname{cosec}\alpha)$$

$$= hk + h^2(\cot\frac{\pi}{3} - 2\operatorname{cosec}\frac{\pi}{3})$$

$$= hk + h^2\left(\frac{1}{\sqrt{3}} - 2\left(\frac{2}{\sqrt{3}}\right)\right)$$

$$= hk - \sqrt{3}h^2$$

MERIDIAN JUNIOR COLLEGE
JC1 Promotional Examination
Higher 2

H2 Mathematics 9740/01

Paper 1 **08 October 2013**

2 Hours 30 Minutes

Additional Materials: Writing paper
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages and **2** blank page

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***: Not in topics tested for SRJC 2014 Promotional Exam**

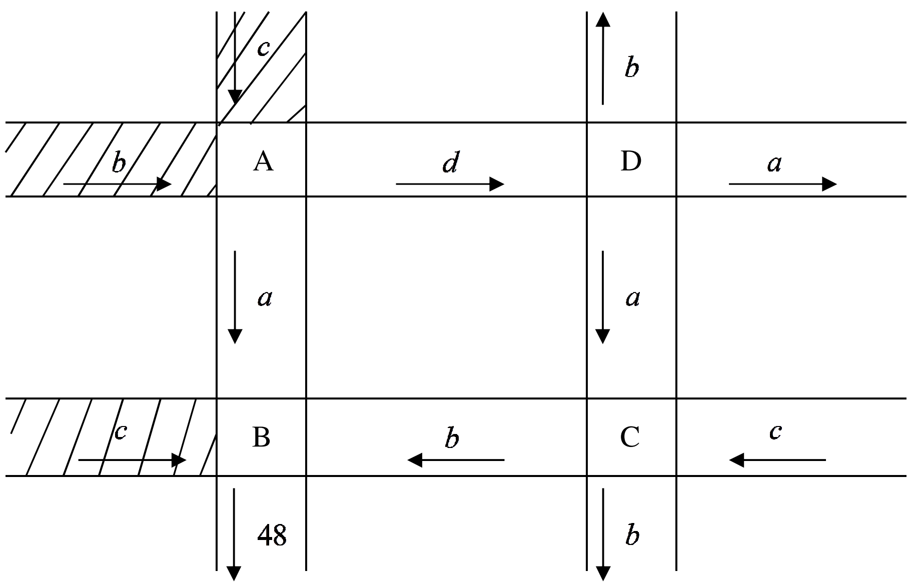
1 Show that $x^2 - x + 7$ is always positive for all real values of x. [1]

Hence, using an algebraic method, solve the inequality

$$\frac{3}{(x-2)^2} > -\frac{1}{x+1}.$$ [3]

2 The parametric equations of a curve C are $x = \sin^{-1}(1-t)$, $y = \mathrm{e}^{\sqrt{2t-t^2}}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [4]

3 The diagram below shows the traffic flow of vehicles in four traffic junctions A, B, C and D. Each arrow indicates the direction of the vehicles entering or leaving the junction. The unknown constants *a, b, c* and *d* indicate the number of vehicles entering or leaving a particular junction. It is given that the total number of vehicles entering a traffic junction must be equal to the total number of vehicles leaving that same junction. There are 48 vehicles leaving junction B.



(i) Determine the values of a, b, c and d. [3]

(ii) The shaded region indicates the presence of an Electronic Road Pricing (ERP) gantry located at that road. It is known that each gantry charges a fixed price of \$0.50 per vehicle. How much revenue will be collected in total by the gantries in these regions? [1]

[Turn Over

4 Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where **a** and **b** are non-zero and non-parallel vectors. The point C lies on OB such that $\overrightarrow{OC} = k\overrightarrow{OB}$, where k is a constant. P is on AC such that $AP : PC = 3 : 1$, and Q is on AB such that $AQ : AB = 2 : 3$.

(i) Find $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of **a**, **b** and k. [2]

(ii) Given that O, P and Q are collinear, find the value of k. [3]

5 **(i)*** Obtain the series expansion for $e^x \sin 2x$, up to and including the term in x^3. [3]

(ii)* Hence deduce the first three non-zero terms in the series expansion of $\dfrac{e^x \sin 2x}{\sqrt{4-x}}$. [3]

6 The curve C has equation $y^2 = 5x^2 + 4$.

(i) Sketch C, indicating clearly the axial intercepts, the equations of the asymptotes and the coordinates of the stationary points. [3]

(ii) Hence by inserting a suitable graph, determine the range of values of h, where h is a positive constant, such that the equation $5x^2 + 4 = h^2\left(1 - x^2\right)$ has no real roots. [3]

7 The curve C has equation

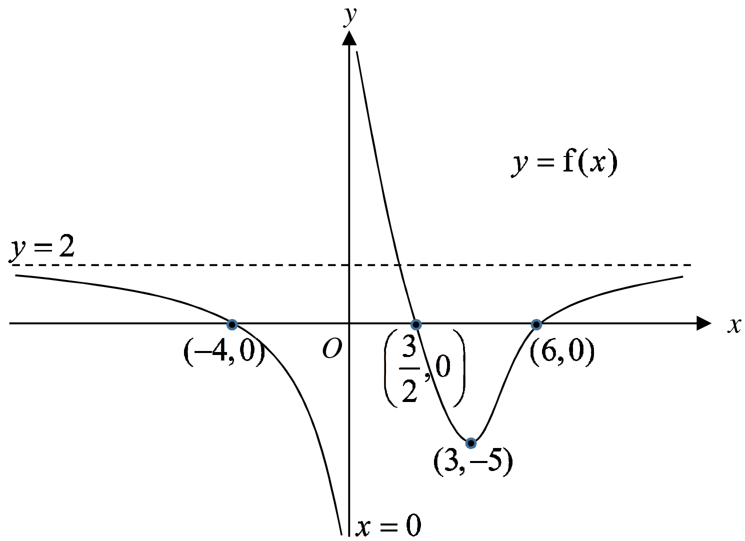
$$y = \frac{x^2}{x-1}.$$

Points $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ lie on curve C such that the tangent at A is parallel to tangent at B where $x_2 < x_1$. Given further that the equation of tangent at A is $y = \frac{8}{9}x + \frac{16}{9}$, find the coordinates of B, and hence find the equation of normal at point B. [6]

***: Not in topics tested for SRJC 2014 Promotional Exam**

8 **(a)** State a sequence of transformations which transform the graph of $y=\dfrac{x-1}{3x^2-5}$ to the graph of $1-y=\dfrac{2x-1}{12x^2-5}$. [3]

(b) The diagram below shows the graph of $y=\mathrm{f}(x)$.



Sketch, on separate clearly labeled diagrams, the graphs of

(i) $y=\mathrm{f}'(x)$, [2]

(ii) $y^2=\mathrm{f}(|x|)$. [3]

9 A sequence $u_1, u_2, u_3, \cdots$ is such that $u_1 = \frac{1}{2}$ and

$$u_{n+1} = u_n - \frac{2n+1}{2n^2(n+1)^2}, \text{ for all } n \geq 1.$$

(i) Use the method of mathematical induction to prove that $u_n = \frac{1}{2n^2}$ for $n \in \mathbb{Z}^+$. [4]

(ii) Hence find $\sum_{n=1}^{N} \frac{2n+1}{2n^2(n+1)^2}$. [3]

(iii) Use your answer to part **(ii)** to find $\sum_{n=0}^{N} \frac{2n+3}{2(n+1)^2(n+2)^2}$. [2]

10 Referred to the origin O, the position vectors of two points A and B are given by $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}+2\mathbf{k}$ respectively. Also, the position vector of C is given by $2\mathbf{i}+\mathbf{j}+5\mathbf{k}$.

(i) Find a vector equation of the line AB and show that point C does not lie on the line. [3]

(ii) Find the position vector of the foot of the perpendicular from point C to line AB.
Hence write down the size of angle ABC. [5]

(iii) Find the position vector of C', the reflection of point C in the line AB. [2]

***: Not in topics tested for SRJC 2014 Promotional Exam**

11 **(a)** The first, third and sixth terms of an arithmetic progression with non-zero common difference d and first term a, are three consecutive terms of a geometric progression. Determine if the geometric series is convergent, justifying your answer. Find also the sum of the first 15 terms of the arithmetic progression in terms of a. [5]

(b) A pile driver is used to drive piles into the soil at a new condominium site. On the first day, the depth piled into the soil is 2 m. On each subsequent day, the depth piled into the soil is $\frac{9}{10}$ of the depth piled into the soil on the previous day. Find the maximum theoretical depth that can possibly be piled into the soil. Find the minimum number of days required to drive the piles to a depth of at least 15m into the soil. [5]

12 A student wants to construct a model of a roof structure of fixed height h cm from a rectangular piece of cardboard of width k cm. The cardboard is to be bent in such a way that the cross-section $PQRS$ is as shown in the diagram, with $PQ + QR + RS = k$ and with PQ and RS each inclined to the horizontal at an angle α.

(i) Show that $QR = k - 2h \operatorname{cosec} \alpha$ and that the area $A\,\text{cm}^2$ of the cross-section $PQRS$ is given by $A = hk + h^2(\cot \alpha - 2 \operatorname{cosec} \alpha)$. [3]

(ii) Use differentiation to find, in terms of k and h, the maximum value of A as α varies. [5]

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NANYANG JUNIOR COLLEGE
JC1 PROMOTIONAL EXAMINATION

Higher 2

MATHEMATICS **9740/01**

Paper 1 **1st October 2013**

3 Hours

Additional Materials: Cover Sheet
Answer Papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

1 Solve the inequality $\left|x^2-2x-3\right|>x+1$. **[4]**

2 Differentiate the following expressions with respect to x, simplifying your answers as far as possible:

(a) $\tan^{-1}\left(\dfrac{2}{x}\right)$, **[3]**

(b) $\ln\sqrt{\dfrac{1+x}{1-x}}$. **[3]**

3 A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1=\dfrac{1}{4}$ and $u_{n+1}=u_n+\dfrac{1}{n(n+1)}+2^{-n}$, for $n\in\mathbb{Z}^+$.

(i) Prove by mathematical induction that $u_n=\dfrac{9}{4}-\dfrac{1}{n}-2^{-n+1}$ for $n\in\mathbb{Z}^+$. **[5]**

(ii) Explain why $\{u_n\}$ is convergent. **[1]**

(iii) Show that u_n is less than $\dfrac{9}{4}$ for $n\in\mathbb{Z}^+$. **[1]**

4 Show that $r!(r^2+1)=(r+2)!-3(r+1)!+2r!$ where $r\in\mathbb{Z}^+$. **[1]**

Hence, using method of difference, show that the sum of the first n terms of the series

$(5)(2!)+(10)(3!)+(17)(4!)+\cdots$ is $(n+2)!(n+1)-2$. **[4]**

Using the above result, explain why $\sum_{r=1}^{n} r!(r^2)$ is less than $(n+1)!n$. **[2]**

5 **(a)** The points A and B relative to the origin O have position vectors $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $-4\mathbf{i}+5\mathbf{j}+2\mathbf{k}$ respectively. The point P lies on line AB such that $\dfrac{AP}{PB}=\dfrac{\lambda}{1-\lambda}$.

(i) Show that $\overrightarrow{OP}=(1-5\lambda)\mathbf{i}+(2+3\lambda)\mathbf{j}+(4\lambda-2)\mathbf{k}$. **[1]**

(ii) Given further that C is a point with position vector $-5\mathbf{i}+\alpha\mathbf{j}-2\mathbf{k}$ and that O, P and C are collinear, find the values of λ and α. **[3]**

(b) The equations of three planes π_1, π_2, π_3 are

$$\pi_1 : 2x-2y+z=-4\ ,$$
$$\pi_2 : 2x+3y-4z=1\ ,$$
$$\pi_3 : \beta x-3y+z=\gamma\ ,$$

respectively.

(i) The planes π_1 and π_2 intersect in a line l. Find a vector equation of l. **[1]**

(ii) Hence, find the values of β and γ such that there are infinitely many points of intersection between π_1, π_2 and π_3. **[2]**

6 The curve C_1 has equation $x^2-\dfrac{y^2}{4}=1$. The curve C_2 has parametric equations

$$x=a\sin t,\quad y=a\cos t\ \text{, where } 0\le t\le 2\pi \text{ and } a>0\,.$$

(i) Write down the Cartesian equation of C_2. Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes. **[5]**

(ii) State the range of values of a such that there are 4 points of intersection between C_1 and C_2. Show algebraically, that the x-coordinates of the points of intersection satisfy the equation $5x^2=4+a^2$. **[2]**

(iii) Explain geometrically why there are only 2 values for the x-coordinates when there are 4 points of intersection between C_1 and C_2. Find the exact values of x if $a=3$. **[2]**

7 The function f is defined by

$$\mathrm{f}: x \to x^2 - \frac{1}{x}, \quad x \in \mathbb{R},\ 1 \le x < 2.$$

(i) Show, by differentiation, that f is strictly increasing. **[2]**

(ii) State the range of f. **[1]**

(iii) Solve the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, giving your answer to two decimal places. **[2]**

The function g is defined by

$$\mathrm{g}: x \to 1 + \sin x, \quad x \in \mathbb{R},\ 0 \le x < \frac{\pi}{2}.$$

(iv) Only one of the composite functions fg and gf exists. Give a definition (including the domain) of the composite that exists, and explain why the other composite does not exist. **[3]**

(v) For the composite function which exists, state its range. **[1]**

8 The equation of a curve is

$y(x+2)^2 + 2y^2(x+2) - 12x = 0$, where x and y are positive variables.

(i) Show that the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is $\frac{1}{16}$ when $x = 2$. **[5]**

(ii) Find the equation of the normal to the curve at the point where $x = 2$. **[2]**

(iii) Given that the normal in **(ii)** meets the line $x = 2$ at the point P and the line $x = 0$ at the point S. Find the exact area of triangle OSP, where O is the origin. **[2]**

9 There are 16 boys and 10 girls in a JC1 class. It so happens that within the class, the heights of all the girls form a geometric progression, while the heights of all the boys form an arithmetic progression. The two shortest students in the class, a boy and a girl, both have a height of 150.0 cm, while the tallest boy in the class has a height of 180.0 cm. The fourth shortest girl in the class has a height of 157.5 cm.

(i) Show that the common ratio r between the heights of the girls is $1.05^{\frac{1}{3}}$ and find the height of the tallest girl in the class, giving your answer in cm correct to one decimal place. **[2]**

(ii) Find the number of girls in the class taller than 164.0 cm. **[3]**

(iii) Find the average height of the girls in the class, giving your answer in cm correct to one decimal place. **[3]**

(iv) Find the average height of the entire class, giving your answer in cm correct to one decimal place. **[2]**

10 The position vectors of the points A, B and C with respect to the origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}-2\mathbf{b}$ respectively. Plane π contains the point A and has $\mathbf{b}$ as its normal vector. If the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ is 60° and $|\mathbf{a}| = 2|\mathbf{b}|$, find in terms of $\mathbf{b}$,

(i) the length of projection of $\mathbf{a}$ onto $\mathbf{b}$, **[2]**

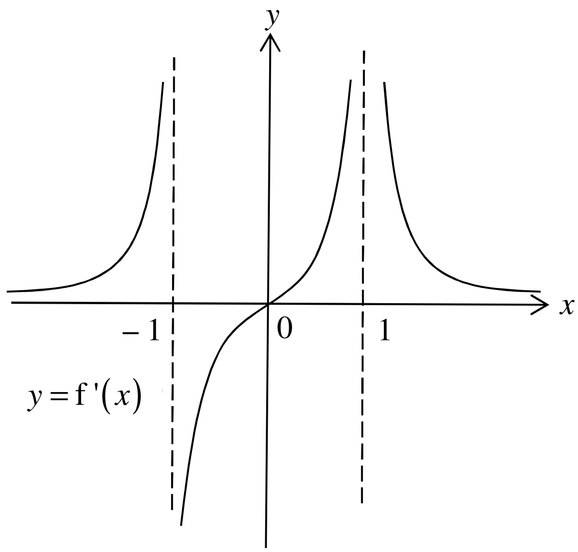
(ii) the distance between point C and the plane π. **[3]**

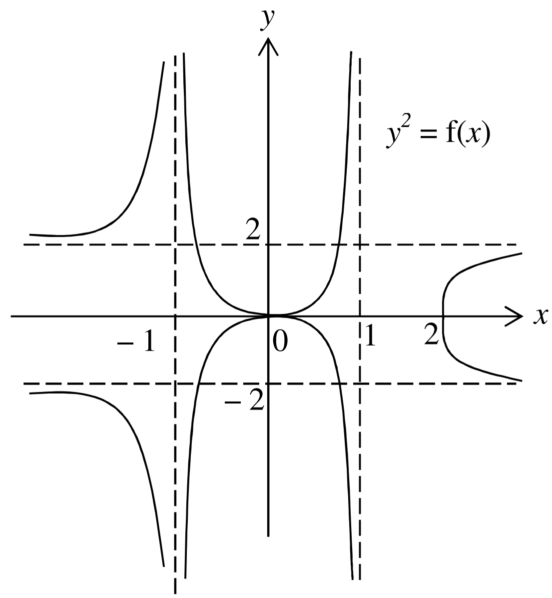
Given that $\mathbf{a} = \mathbf{i}+5\mathbf{j}+2\mathbf{k}$ and $\mathbf{b} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$,

(iii) find the position vector of the foot of perpendicular from point C to the plane π, **[5]**

(iv) show that the position vector of the point of the reflection of point C in the plane π is $3\mathbf{i}+9\mathbf{j}$. **[2]**

11 The graphs of $y = \text{f}'(x)$ and $y^2 = \text{f}(x)$ are shown in the diagrams below.





(a) On separate diagrams, sketch the graphs of

(i) $y = \text{f}'(1 - x)$, **[3]**

(ii) $y = \text{f}(x)$, **[4]**

showing clearly the x-intercepts and asymptotes (if any).

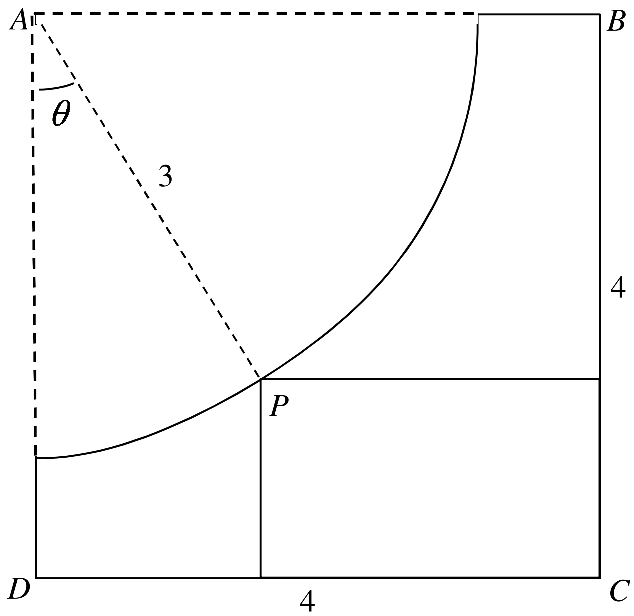
(b) State the set of values of x for which the graph of $y = \text{f}(x)$ is concave upwards. **[2]**

12 **(a)** The curve C has parametric equations

$$x=\theta^2+4\theta, \qquad y=\frac{2}{\theta}, \qquad \text{for } \theta>0.$$

A point $P(x, y)$ moves on the curve C in such a way that the x-coordinate of P decreases at a constant rate of 4 units per second. Find the rate at which the y-coordinate of P is changing when $x=4$. **[4]**

(b)



The diagram above shows the floor plan of a storeroom. The floor plan consists of a square $ABCD$ of side 4 units from which a quadrant of a circle with centre A and radius 3 units has been removed. The owner intends to store a rectangular crate with one corner of the base at C, and the opposite corner of the base at P against the curved wall. The base of the crate has area y unit^2 and angle DAP is θ radians, where $0\le\theta\le\frac{\pi}{4}$.

Show that $\frac{\mathrm{d}y}{\mathrm{d}\theta}=3(\sin\theta-\cos\theta)(4-3\sin\theta-3\cos\theta)$. **[2]**

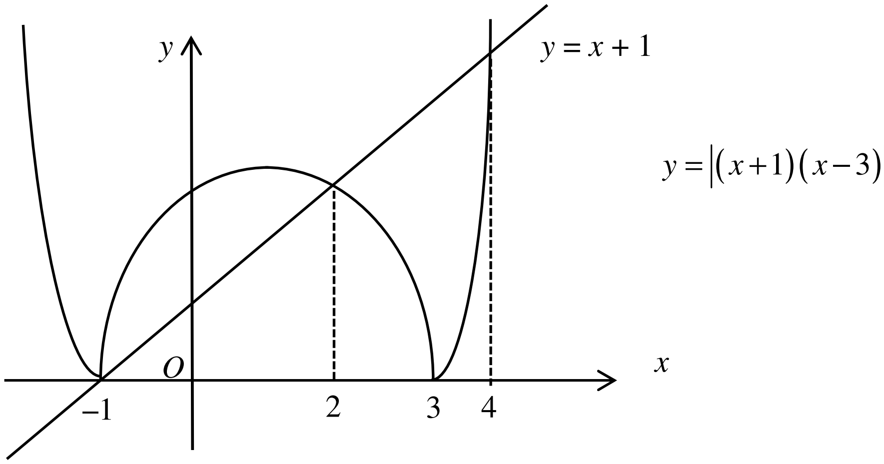
Hence, find the least possible value of y. **[5]**

-----END OF PAPER-----

2013 NYJC JC1 Promo 9740/1 Solutions

Qn

1



$x<-1$ or $-1<x<2$ or $x>4$.

2

(a) $\frac{\mathrm{d}}{\mathrm{dx}}[\tan^{-1}\left(\frac{2}{x}\right)]=\frac{2(-x^{-2})}{1+(\frac{2}{x})^2}=\frac{-2}{x^2+4}$

(b) $\frac{\mathrm{d}}{\mathrm{dx}}(\ln\sqrt{\frac{1+x}{1-x}})=\frac{\mathrm{d}}{\mathrm{dx}}[\frac{1}{2}\left(\ln(1+x)-\ln(1-x)\right)]$

$=\frac{1}{2}(\frac{1}{1+x}-\frac{-1}{1-x})=\frac{1}{1-x^2}$ or $\frac{1}{(1+x)(1-x)}$

Alternative Solution

$$\frac{d}{dx}\left(\text{In}\sqrt{\frac{1+x}{1-x}}\right)=\left(\frac{1}{\sqrt{\frac{1+x}{1-x}}}\right)\left(\frac{1}{2}.\sqrt{\frac{1-x}{1+x}}\left(\frac{1-x+1+x}{(1-x)^2}\right)\right)$$

$$=\frac{1}{2}\left(\frac{1-x}{1+x}\right)\left(\frac{2}{(1-x)^2}\right)$$

$$=\frac{1}{(1-x)(1+x)}$$

3

(i) Let P_n denote the proposition $u_n=\frac{9}{4}-\frac{1}{n}-2^{-n+1}$ for all $n\in\mathbb{Z}^+$.

For $n=1$, LHS $=u_1=\frac{1}{4}$

RHS $=\frac{9}{4}-\frac{1}{1}-2^{-1+1}=\frac{9}{4}-1-1=\frac{1}{4}=$ LHS.

$\therefore P_1$ is true.

Qn

Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e., $u_k = \frac{9}{4} - \frac{1}{k} - 2^{-k+1}$.

To prove that that P_{k+1} is true, i.e., $u_{k+1} = \frac{9}{4} - \frac{1}{k+1} - 2^{-(k+1)+1}$

For $n = k + 1$,

$$\begin{aligned}
\text{LHS} = u_{k+1} &= u_k + \frac{1}{k(k+1)} + 2^{-k} \\
&= \frac{9}{4} - \frac{1}{k} - 2^{-k+1} + \frac{1}{k(k+1)} + 2^{-k} \\
&= \frac{9}{4} - \left(\frac{1}{k} - \frac{1}{k(k+1)}\right) - \left(2^{-k}\right)(2-1) \\
&= \frac{9}{4} - \frac{k+1-1}{k(k+1)} - 2^{-k-1+1} \\
&= \frac{9}{4} - \frac{1}{k+1} - 2^{-(k+1)+1}
\end{aligned}$$

Hence P_{k+1} is true

Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, hence by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

(ii) As $n \to \infty$, $\frac{1}{n} \to 0$, $2^{-n} \to 0$, hence $u_n \to \frac{9}{4}$, i.e. $\{u_n\}$ is convergent

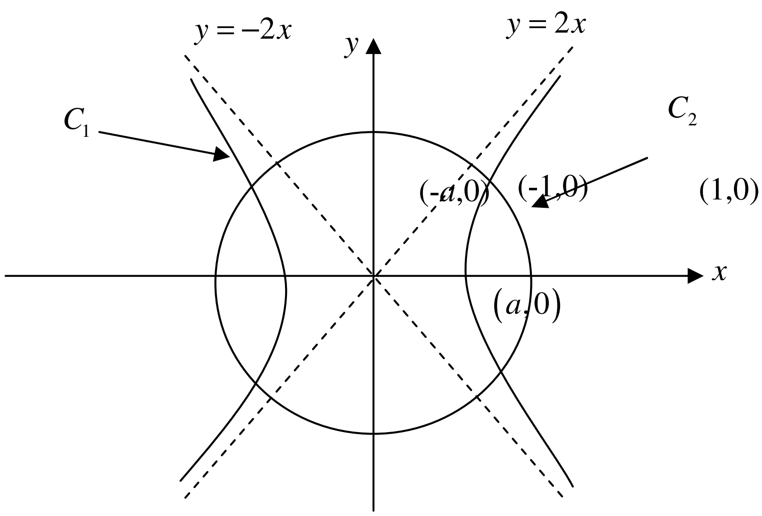
(iii) Since $\frac{1}{n} > 0$, $2^{-n} > 0$ for $n \geq 1$, $u_n = \frac{9}{4} - \frac{1}{n} - 2^{-n+1} < \frac{9}{4}$

4

$$\begin{aligned}
(r+2)! - 3(r+1)! + 2r! &= r!\left((r+2)(r+1) - 3(r+1) + 2\right) \\
&= r!(r^2 + 3r + 2 - 3r - 3 + 2) \\
&= r!\left(r^2 + 1\right) \quad \text{(Shown)}
\end{aligned}$$

$$\begin{aligned}
\sum_{r=2}^{n+1} r!(r^2+1) &= \sum_{r=2}^{n+1} \left[(r+2)! - 3(r+1)! + 2r!\right] \\
&= 4! - 3(3!) + 2(2!) \\
&\quad + 5! - 3(4!) + 2(3!) \\
&\quad + 6! - 3(5!) + 2(4!) \\
&\quad \vdots \\
&\quad + (n+1)! - 3(n)! + 2(n-1)! \\
&\quad + (n+2)! - 3(n+1)! + 2(n)!
\end{aligned}$$

Qn	
	$+(n+3)!-3(n+2)!+2(n+1)!$ $=(n+3)!-2(n+2)!-3!+2(2!)$ $=(n+2)!(n+3-2)-2$ $=(n+2)!(n+1)-2$ (Shown) $\sum_{r=1}^{n} r!(r^2+1)=(n+1)!(n)-2+(1!)(1^1+1)=(n+1)!n$ Since $r!(r^2)<r!(r^2+1)$ for $r\in\mathbb{Z}^+$ Therefore $\sum_{r=1}^{n} r!(r^2)<\sum_{r=1}^{n} r!(r^2+1)=(n+1)!n$
5a **b**	$\overrightarrow{OP}=\dfrac{(1-\lambda)\overrightarrow{OA}+\lambda\overrightarrow{OB}}{1-\lambda+\lambda}$ $=(1-\lambda)(\mathbf{i}+2\mathbf{j}-2\mathbf{k})+\lambda(-4\mathbf{i}+5\mathbf{j}+2\mathbf{k})$ $(1-5\lambda)\mathbf{i}+(2+3\lambda)\mathbf{j}+(4\lambda-2)\mathbf{k}$ $\overrightarrow{OP}=\mu\overrightarrow{OC}$ $\begin{pmatrix}1-5\lambda\\2+3\lambda\\4\lambda-2\end{pmatrix}=\mu\begin{pmatrix}-5\\\alpha\\-2\end{pmatrix}$ Solving, $\lambda=\frac{2}{5}$, $\mu=\frac{1}{5}$, $\alpha=16$ $\pi_1: 2x-2y+z=-4$, $\pi_2: 2x+3y-4z=1$, $\pi_3: \beta x-3y+z=\gamma$. Line of intersection of π_1 and π_2, l: $\mathbf{r}=\begin{pmatrix}-1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$, $\lambda\in\mathbb{R}$. For infinite points of intersection between 3 planes, l is on π_3.

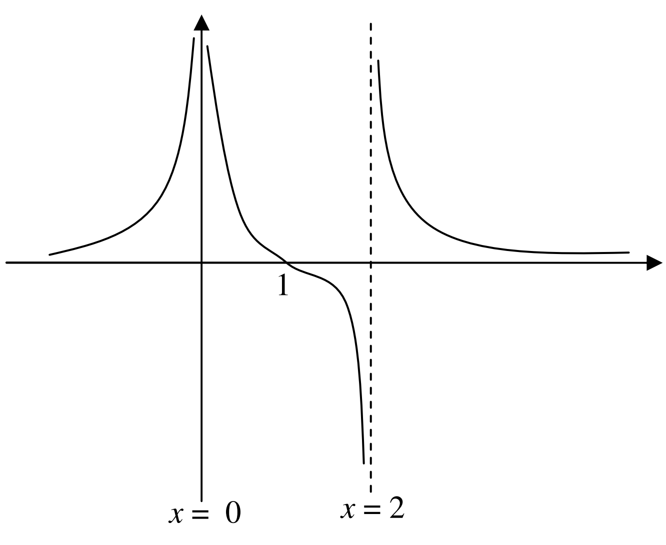
Qn	
	$\begin{pmatrix}1\\2\\2\end{pmatrix}\bullet\begin{pmatrix}\beta\\-3\\1\end{pmatrix}=0 \quad\Rightarrow \beta=4$ $\begin{pmatrix}-1\\1\\0\end{pmatrix}\bullet\begin{pmatrix}\beta\\-3\\1\end{pmatrix}=\gamma \quad\Rightarrow\ \gamma=-7$
6(i)	$x=a\sin t,\quad y=a\cos t$ $\sin t=\frac{x}{a},\quad \cos t=\frac{y}{a}$ $\sin^2 t+\cos^2 t=1$ $\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2=1$ $x^2+y^2=a^2$ 
(ii)	$a>1$ $x^2-\frac{y^2}{4}=1 \quad ....(1)$ $x^2+y^2=a^2$ $y^2=a^2-x^2 \quad ....(2)$

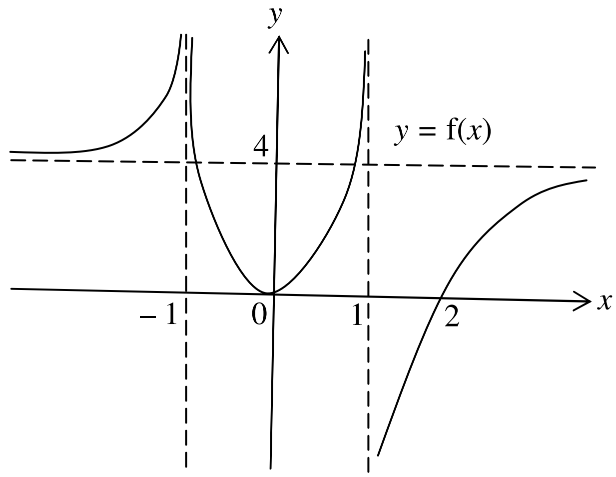
Qn	
(iii)	$x^2-\left[\dfrac{a^2-x^2}{4}\right]=1$ $4x^2-a^2+x^2=4$ $5x^2=4+a^2$ (shown) The points of intersection between the 2 curves are **symmetrical about the *x*-axis**, thus there are only 2 values for the *x*-coordinates. $5x^2=13$ $x=\pm\sqrt{\dfrac{13}{5}}$
7(i) (ii) (iii) (iv) (v)	$f'(x)=2x+\dfrac{1}{x^2}>0$ for $1\le x<2$ $\Rightarrow$ f is strictly increasing. Since f is strictly increasing, its minimum and maximum values correspond to the minimum and maximum *x* values. Thus $R_f=\left[1-1,4-\dfrac{1}{2}\right)=\left[0,\dfrac{7}{2}\right)$. $f(x)=f^{-1}(x)\Rightarrow f(x)=x$ $\Rightarrow x^2-\dfrac{1}{x}=x$ $\Rightarrow x^3-x^2-1=0$ $\Rightarrow x=1.47$. Since $R_g=[1,2)=D_f$, fg exists. Since $R_f=\left[0,\dfrac{7}{2}\right)\not\subset\left[0,\dfrac{\pi}{2}\right)=D_g$, gf does not exist. $fg(x)=f(\sin x+1)=(\sin x+1)^2-\dfrac{1}{\sin x+1}$. $D_{fg}=D_g=\left[0,\dfrac{\pi}{2}\right)$. $fg: x\to(\sin x+1)^2-\dfrac{1}{\sin x+1},\ x\in\mathbb{R},\ 0\le x<\dfrac{\pi}{2}$. $R_{fg}=\left[0,\dfrac{7}{2}\right)$.

Qn	
8(i)	$(x+2)^2 y+2(x+2)y^2-12x=0$ Differentiating wrt x, $\frac{dy}{dx}(x+2)^2+2y(x+2)+4y\frac{dy}{dx}(x+2)+2y^2-12=0$ -------(1) When $x=2$, $16y+8y^2-24=0$ $y^2+2y-3=0$ $(y+3)(y-1)=0$ $y=-3(\text{rejected} \because y>0)$ or $y=1$ Subst (2 , 1) into equation (1), $16\frac{dy}{dx}+8+16\frac{dy}{dx}+2-12=0$ $32\frac{dy}{dx}=2$ $\frac{dy}{dx}=\frac{1}{16}$
(ii)	Equation of normal: $y-1=-16(x-2)$ $y=-16x+33$
(iii)	Points P and S has coordinates (2 , 1) and (0 , 33) respectively Area of triangle OSP $=\frac{1}{2}\times33\times2=33$
9(i)	Let u_n denote the height of the nth shortest girl in the class in cm, and r denote the common ratio between the heights of the girls. Then $u_n=ar^{n-1}$ where $u_1=a=150.0$ and $u_4=ar^3=157.5$ $\Rightarrow r^3=\frac{157.5}{150.0}=1.05 \quad \Rightarrow \quad r=1.05^{\frac{1}{3}}$ Also, $u_{10}=ar^9=a(r^3)^3=(150.0)(1.05)^3=173.6$ (to 1 d.p.) $\therefore$ The height of the tallest girl is 173.6 cm.
ii	$u_n>164.0$ $\Rightarrow (150.0)(1.05)^{\frac{n-1}{3}}-164.0>0$ Using GC,

n	$(150.0)(1.05)^{\frac{n-1}{3}}-164.0$
6	-1.29
7	1.38
8	4.09

Qn	
	Hence $n \geq 7$. Since there are 10 girls in the class, the number of girls who are taller than 164.0 cm is 10-7+1=4. Thus there are 4 girls in the class taller than 164.0 cm.
iii	Average height of girls $=\frac{1}{10}S_{10}=\frac{1}{10}\frac{a(1-r^{10})}{1-r}$ $=\frac{(150.0)(1-1.05^{\frac{10}{3}})}{10(1-1.05^{\frac{1}{3}})}$ $=161.57$ $=161.6$ cm (to 1 d.p.)
iv	Average height of boys $=\frac{1}{16}S_{16}$ $=\frac{1}{16}\times\frac{16}{2}(150.0+180.0)$ $=165.0$ cm Average height of class $=\frac{16(165.0)+10(161.57)}{16+10}$ $=163.7$ cm (to 1 d.p.)
10(i)	length of projection $=\left\|\mathbf{a}\cdot\hat{\mathbf{b}}\right\|=\|\mathbf{a}\|\left\|\hat{\mathbf{b}}\right\|\cos 60°$ $=2\|\mathbf{b}\|\left(\frac{1}{2}\right)=\|\mathbf{b}\|$
(ii)	distance between C and the plane = $\left\|\frac{(\mathbf{a}-2\mathbf{b}-\mathbf{a})\cdot(\mathbf{b})}{\|\mathbf{b}\|}\right\|=\left\|\frac{-2\mathbf{b}\cdot\mathbf{b}}{\|\mathbf{b}\|}\right\|$ $=\left\|\frac{-2\|\mathbf{b}\|^2}{\|\mathbf{b}\|}\right\|$ $=2\|\mathbf{b}\|$
(iii)	$\mathbf{c}=\begin{pmatrix}1\\5\\2\end{pmatrix}-2\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\\4\end{pmatrix}$

Qn	
	$\pi : \mathbf{r}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}1\\5\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}=9,$ $l : \mathbf{r}=\begin{pmatrix}-1\\1\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$ $\begin{pmatrix}-1+\lambda\\1+2\lambda\\4-\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}=9$ $-1+\lambda+2+4\lambda-4+\lambda=9\Rightarrow\lambda=2$
(iv)	position vector of the foot of perpendicular from **c** to plane = $\begin{pmatrix}-1\\1\\4\end{pmatrix}+2\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}1\\5\\2\end{pmatrix}$ position vector of point of reflection of C in plane $=2\begin{pmatrix}1\\5\\2\end{pmatrix}-\begin{pmatrix}-1\\1\\4\end{pmatrix}=\begin{pmatrix}3\\9\\0\end{pmatrix}$
11a	

Qn	
	
b	$(-\infty,-1)\cup(-1,1)$
12a	$x=\theta^2+4\theta$, $\quad y=\dfrac{2}{\theta}$ $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\mathrm{d}y}{\mathrm{d}x}\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t}$ $=\left(\dfrac{\mathrm{d}y}{\mathrm{d}\theta}\cdot\dfrac{\mathrm{d}\theta}{\mathrm{d}x}\right)\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t}$ $=\dfrac{-2}{\theta^2}\cdot\dfrac{1}{2\theta+4}\cdot(-4)$ $=\dfrac{4}{\theta^2(\theta+2)}$ When $x=4$, $\theta^2+4\theta=4\Rightarrow\theta=0.82843$ since $\theta>0$ $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{4}{\theta^2(\theta+2)}=2.06$ units/sec Rate of change of y-coordinate is 2.06 units/sec.

Qn	
12b	$y=(4-3\cos\theta)(4-3\sin\theta)$ $\frac{dy}{d\theta}=(4-3\sin\theta)(3\sin\theta)+(4-3\cos\theta)(-3\cos\theta)$ $=3\left[4\sin\theta-3\sin^2\theta-4\cos\theta+3\cos^2\theta\right]$ $=3\left[3(\cos^2\theta-\sin^2\theta)+4\sin\theta-4\cos\theta\right]$ $=3\left[3(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)+4(\sin\theta-\cos\theta)\right]$ $=3(\sin\theta-\cos\theta)(4-3\sin\theta-3\cos\theta)$ $\frac{dy}{d\theta}=0$ $3(\sin\theta-\cos\theta)(4-3\sin\theta-3\cos\theta)=0$ $\sin\theta-\cos\theta=0$ or $4-3\sin\theta-3\cos\theta=0$ $\sin\theta+\cos\theta=\frac{4}{3}$ $\theta=\frac{\pi}{4}$ or $\theta=0.44556$ θ=1.1252 (rej $0\le\theta\le\frac{\pi}{4}$) $\frac{d^2y}{d\theta^2}=3(\sin\theta-\cos\theta)(3\sin\theta-3\cos\theta)+3(\sin\theta+\cos\theta)(4-3\sin\theta-3\cos\theta)$ When $\theta=\frac{\pi}{4}$, $\frac{d^2y}{d\theta^2}<0\Rightarrow y$ is max When $\theta=0.44556$, $\frac{d^2y}{d\theta^2}>0\Rightarrow y$ is min Min $y=(4-3\cos 0.44556)(4-3\sin 0.44556)$=3.50

Name	( )	Class	

RIVER VALLEY HIGH SCHOOL
2013 Year 5 Promotional Examination
Higher 2

MATHEMATICS **9740/01**

Paper 1 **19 September 2013**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)
Cover Page

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, class and index number in the space at the top of this page.

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
Up to **2 marks may be deducted** for poor presentation in your answers.

At the end of the examination, place the cover page on top of your answer paper and fasten all your work securely together.

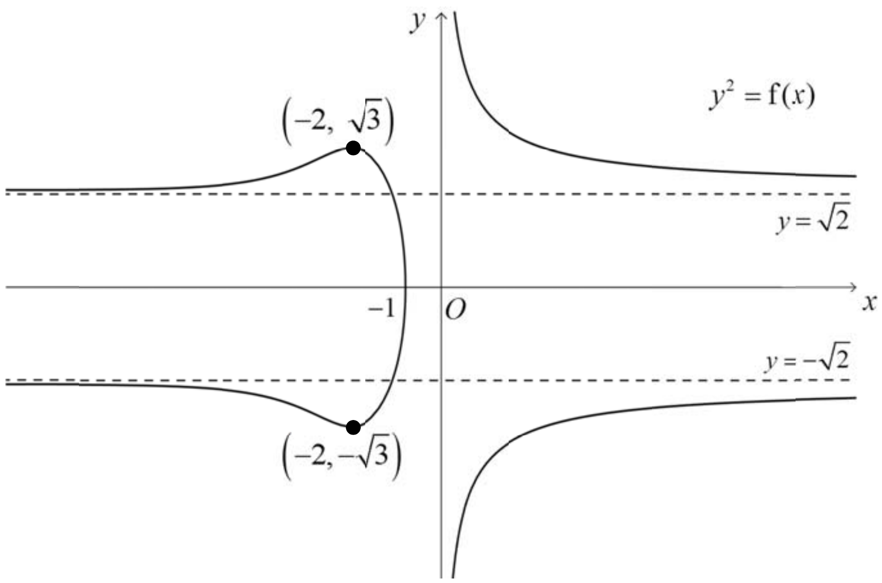
The number of marks is given in brackets [ ] at the end of each question or part question.

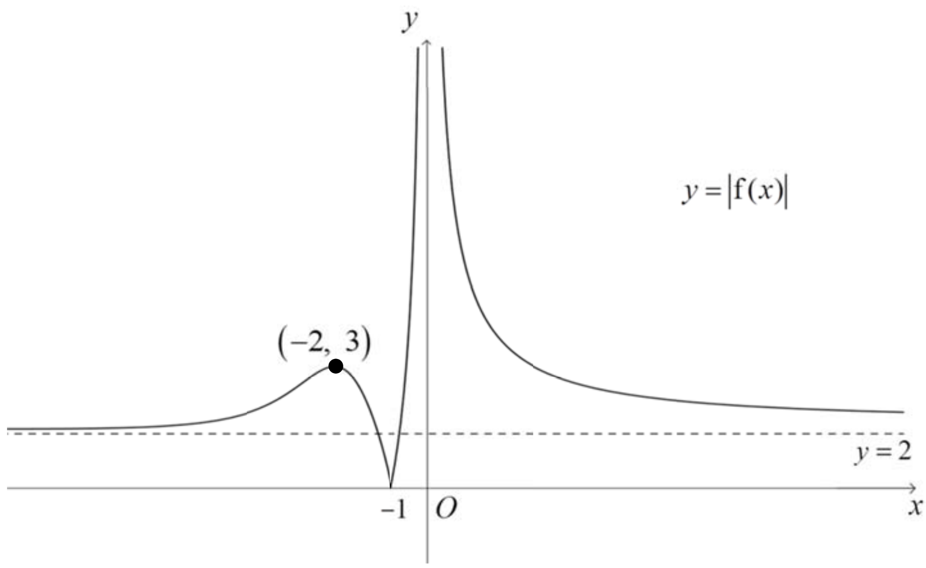
This document consists of **7** printed pages and **1** blank page.

1. (i) Let $f(x)=(x+3)(9-4x)^{-\frac{1}{2}}$. Find the series expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [3]

(ii) Denote the answer to part **(i)** by $g(x)$. Find, for $-\frac{9}{4}\le x\le\frac{9}{4}$, the set of values of x for which the value of $g(x)$ is within ± 0.2 of $f(x)$. [2]

2. The graphs of $y^2=f(x)$ and $y=|f(x)|$ are given below.





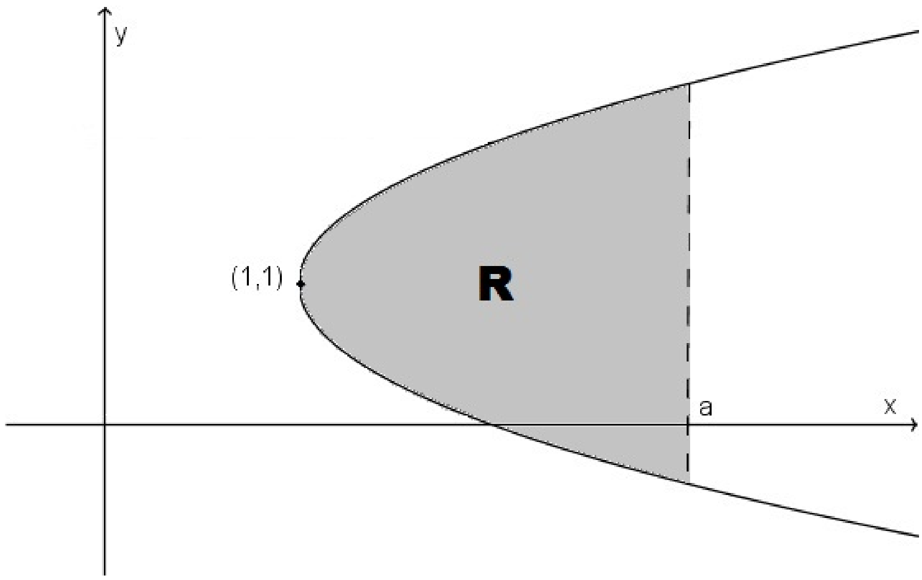
Deduce the graphs of

(i) $y=f(x)$, [3]

(ii) $y=f'(x)$, [2]

clearly indicating any asymptotes, intersections with the axes and stationary points.

3. The diagram shows the sketch of the curve $C, (y-1)^2 = x\sqrt{x^2-1}$, with the vertex at $(1,1)$.



(i) Write down the equation of the graph when C is translated 1 unit in the negative y-direction. [1]

(ii) The shaded region R, bounded by C and the vertical line, $x = a$, is rotated through π radians about the line $y = 1$. By using the substitution $u = \sqrt{x^2-1}$, or otherwise, find the exact volume obtained in terms of a. [5]

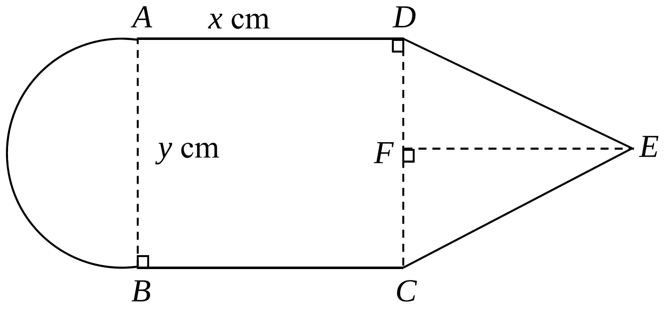
4. (a) A theme park sells day passes at different prices depending on the age of the customer. The age categories are senior citizens (ages 60 and above), adult (ages 13 to 59) and child (ages 4 to 12). Three tour groups visited the theme park on the same day. The numbers in each category for each group together with the total cost of the day passes for each group are given as follows.

Group	Senior Citizens	Adult	Child	Total Cost
1	2	19	9	$196.40
2	0	10	3	$90.20
3	1	7	4	$77.00

Write down and solve equations to find the cost of a day pass for each of the age category. [3]

(b) Without using a GC, solve $\dfrac{4x^2-4|x|+1}{x^2-2|x|-8} \geq 0$. [4]

5. The cross section of an open container consists of a semicircle, a rectangle $ABCD$ and an isosceles triangle CED as shown in the diagram below. Given that $AD = BC = x$ cm, $AB = DC = FE = y$ cm, $DE = CE$ and the height of the container is $\frac{5}{3}$ cm.



The interior vertical walls of this container, $ADECB$, need to be painted. The time needed to paint the walls will be 1 minute per 10 cm^2 for the straight parts and 1 minute per 8 cm^2 for the semicircular part. Given that a total time of 200 minutes is required to paint all the walls, find, by differentiation, the values of x and y which gives a maximum cross-sectional area, giving your answers correct to the nearest integers. [7]

6. It is given that the curve $y^3 + \tan^{-1} y = \ln(\cos x)$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$, passes through the origin.

(i) Show that $\left(3y^4 + 3y^2 + 1\right)\frac{dy}{dx} = -\left(1 + y^2\right)\tan x$. [2]

(ii) Find the Maclaurin series for y, up to and including the term x^2. [3]

(iii) Hence, find an approximation to the value of $\int_0^{\pi/4} \frac{dy}{dx}\,dx$, in terms of π. [2]

7. In a particular river in Brazil, a sudden surge in the number of piranhas (a type of fish known for their sharp teeth and a voracious appetite for meat) is observed and has affected the livelihood of the villagers living along the river. A group of fishermen is engaged to catch these piranhas and the piranhas are caught at a rate inversely proportional to the number of piranhas left. Furthermore, due to aggressive nature, the number of piranhas is reduced at a rate of one-tenth of the piranhas remaining.

(i) If x (in thousands) is the number of piranhas remaining at time t (in days) after the group of fishermen is deployed to catch the piranhas, show that $x^2 + 10k = Ae^{-0.2t}$ where k is a positive constant. [4]

(ii) If there are 5000 piranhas at the start of the deployment of the fishermen and after 5 days, the number of piranhas remaining is 3000. Calculate the number of days required to remove all the piranhas. [3]

8. (a) Five out of the six digits, 0, 1, 2, 3, 4 and 5 are chosen and arranged randomly to form a five-digit number. No digit is repeated.

Find the number of five-digit numbers that are

(i) greater than 10000, [2]

(ii) greater than 10000 and even. [3]

(b) An ice-cream shop has 4 different flavours of ice-cream, vanilla, chocolate, strawberry and durian and 3 different toppings containing peanuts, raisins and berries. Assuming Peter decides to visit the ice-cream shop and make a selection of at least 1 flavour and at least 1 topping, find how many different selections can he make? [3]

9. (a) The function f and g are defined by

$$\mathrm{f}: x \mapsto x^2 - 6x + 11, \quad x > 3$$

$$\mathrm{g}: x \mapsto \frac{1}{x^2}, \quad x \geq k, \text{ where } k \text{ is a positive constant.}$$

(i) Show that the inverse function of f exists. [1]

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(iii) State the greatest value of k for which the composite function gf exists and find the range of gf for this value of k. [3]

(b) Given that h is a one-one function, determine, with reasons, if hh^{-1} exists. [2]

10. (a) The sum, S_n of the first n terms of a sequence $u_1,\ u_2,\ u_3,\ \ldots$ is given by

$$S_n = \ln a^n b^{\frac{1}{2}\left(n^2 - n\right)}, \text{ where } 0 < a < 1,\ b > 1.$$

(i) Find u_n in terms of a and b. [2]

(ii) Prove that the sequence is an arithmetic progression. [2]

(iii) Given that $0 < ab^{n-1} < 1$ when $n < 7$, find the sum of the negative terms of the sequence. [1]

(b) By considering $\sin(n\theta)\sin\left(\frac{1}{2}\theta\right)$, show, using the method of differences,

$$\sum_{n=1}^{N} \sin(n\theta) = \frac{1}{2}\cot\left(\frac{1}{2}\theta\right) - \frac{\cos\left[\left(N + \frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{1}{2}\theta\right)}.$$ [4]

11. (a) A and B are events such that $\mathrm{P}(B)=0.3$, $\mathrm{P}(A' \cup B')=0.9$ and $\mathrm{P}(A \cap B')=0.45$.

(i) $\mathrm{P}(A)$, [2]

(ii) $\mathrm{P}(A' \cap B)$. [2]

(b) In a cooking school, all students must take a theory and practical test. It is reported that 95% of the students pass the theory test. Of those who pass, 85% also pass the practical test. Of those who fail the theory test, 60% pass the practical test.

Draw a tree diagram to show the above information. [2]

Find the probability that a student, randomly chosen from the cooking school,

(i) passes the practical test, [1]

(ii) passes the theory test, given that he fails the practical test. [2]

12. A curve C has parametric equations

$$x=\mathrm{e}^t,\ y=t^2.$$

(i) Sketch the curve C. [2]

The normal to C at point A with coordinates $\left(\mathrm{e}^2, 4\right)$ is denoted by l.

(ii) Find the Cartesian equation of l, expressing y in terms of x. [3]

(iii) Find the exact area of the region bounded by l, C and the x-axis. Express your answer in the form $\frac{a}{\mathrm{e}^2}+b\mathrm{e}^2+c$ where a, b and c are constants to be determined.

[5]

13. It is thought that the pH value of water may affect the size of pearl in pearl oyster farming. A pearl farmer wished to investigate whether there was any correlation between the pH value of the water and the size of the pearl cultivated. The size of the pearls and the pH value of the water where the oysters are cultivated are shown in the table below.

pH value of water, x	7.7	7.8	7.9	8.0	8.1	8.2	8.3
Size of pearl, y (in cm)	6.82	7.28	7.61	7.79	7.91	8.02	8.05

(i) Draw a scatter diagram to illustrate the data, labeling the axes clearly. [2]

(ii) Comment on whether a linear model would be appropriate. [1]

It is thought that the size of pearl can be modeled by one of the formulae

$$y = a + bx^2 \quad \text{or} \quad y^2 = c + dx$$

where a, b, c and d are constants.

(iii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) x^2 and y,

(b) x and y^2. [2]

(iv) Use your answer to parts **(i)** and **(iii)** to explain which of $y = a + bx^2$ or $y^2 = c + dx$ is the better model. [2]

(v) The pearl farmer will like to have pearls which are exactly 8.00 cm. Find the equation of a suitable regression line, and use it to find the required pH value of the water, correct to 1 decimal place. Comment on the reliability of your answer. [4]

END OF PAPER

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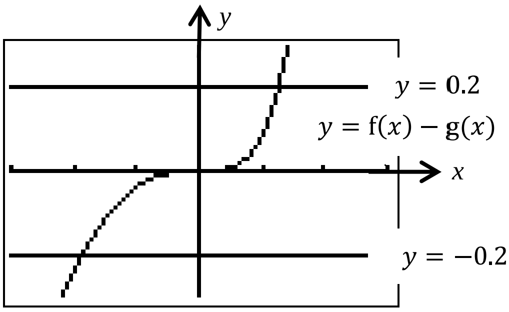
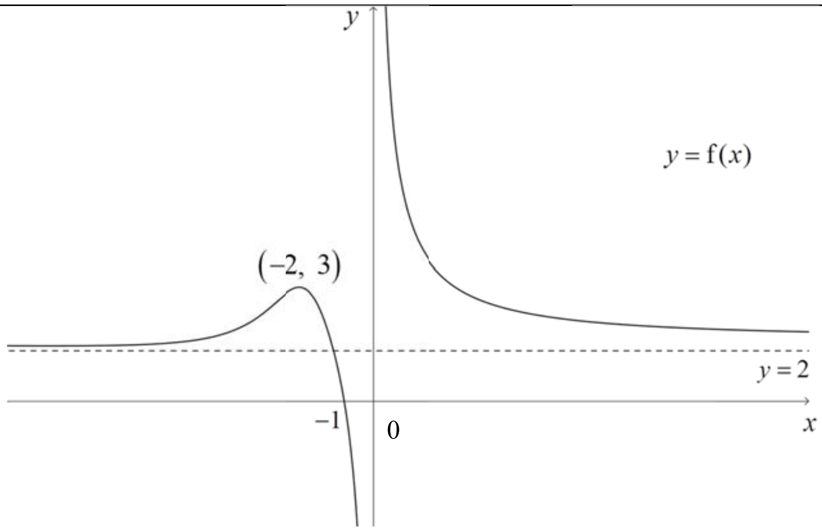
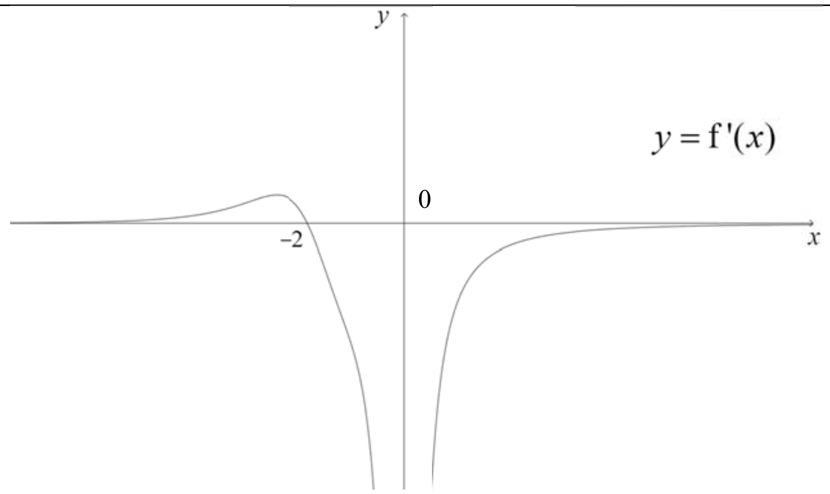
2013 Year 5 H2 Maths Promotional Examination Marking Scheme

1(i)	$\mathrm{f}(x)=(x+3)(9-4x)^{-\frac{1}{2}}$ $=(x+3)9^{-\frac{1}{2}}\left(1-\frac{4}{9}x\right)^{-\frac{1}{2}}$ $=\frac{1}{3}(x+3)\left[1+\frac{\left(-\frac{1}{2}\right)}{1}\left(-\frac{4}{9}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{4}{9}x\right)^{2}+\ldots\right]$ $=\frac{1}{3}(x+3)\left(1+\frac{2}{9}x+\frac{2}{27}x^{2}+\ldots\right)$ $\approx 1+\frac{5}{9}x+\frac{4}{27}x^{2}$	
(ii)	$-0.2<\mathrm{f}(x)-\mathrm{g}(x)<0.2$ or $\lvert\mathrm{f}(x)-\mathrm{g}(x)\rvert<0.2$  Using GC, $\{x\in R, -1.87<x<1.25\}$	

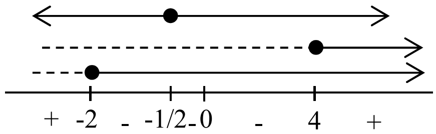
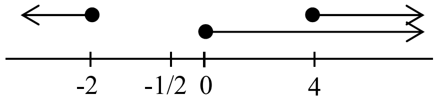
2(i)	 	
(ii)		

3 (i)	Graph to be translated 1 unit in negative y- direction $\Rightarrow y = \mathrm{f}(x) - 1 \Rightarrow y + 1 = \mathrm{f}(x)$ Replace y with $y+1$, $(y+1-1)^2 = x\sqrt{x^2-1}$ $y^2 = x\sqrt{x^2-1}$	
(ii)	Volume obtained $= \pi \int_1^a x\sqrt{x^2-1}\ \mathrm{d}x$ $= \pi \int_0^{\sqrt{a^2-1}} xu\left(\frac{u}{x}\right)\ \mathrm{d}u$ $u = \sqrt{x^2-1}$ $= \pi \int_0^{\sqrt{a^2-1}} u^2\ \mathrm{d}u$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x}{u}$ $= \pi \left[\frac{u^3}{3}\right]_0^{\sqrt{a^2-1}}$ $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{u}{x}$ $= \frac{\pi}{3}\left(a^2-1\right)^{\frac{3}{2}}$	

4(a)	Let x, y and z be the cost of a day pass for a senior, adult and child respectively. $2x + 19y + 9z = 196.4$ $10y + 3z = 90.2$ $x + 7y + 4z = 77$ Using GC,	

	$x=3.60$ $y=7.40$ $z=5.40$ Thus, the cost of a day pass for a senior is \$3.60, for an adult is \$7.40 and for a child is \$5.40.	
(b)	$\dfrac{4x^2-4\lvert x\rvert+1}{x^2-2\lvert x\rvert-8}\geq 0$ Let $y=\lvert x\rvert$ $\dfrac{4y^2-4y+1}{y^2-2y-8}\geq 0$ $\dfrac{(2y-1)^2}{(y+2)(y-4)}\geq 0$ Since $(2y-1)^2\geq 0$, $(2y-1)^2=0$ satisfy the inequality $y=\dfrac{1}{2}$ $\lvert x\rvert=\dfrac{1}{2}$ $x=\dfrac{1}{2}$ or $x=-\dfrac{1}{2}$ $(y+2)(y-4)>0$ $y<-2$ or $y>4$ $\lvert x\rvert<-2$ or $\lvert x\rvert>4$ (no solution) $x>4$ or $x<-4$ Answer: $x<-4$ or $x>4$ <u>Alternatively(Method 2),</u> $\dfrac{4\lvert x\rvert^2-4\lvert x\rvert+1}{\lvert x\rvert^2-2\lvert x\rvert-8}\geq 0$	

	When $x \geq 0$, $\dfrac{4x^2-4x+1}{x^2-2x-8} \geq 0$ and $x \geq 0$  $\dfrac{(2x-1)^2}{(x+2)(x-4)} \geq 0$ $x<-2$ or $x>4$ and $x \geq 0$  $x=\dfrac{1}{2}$ or $x>4$ Or when $x<0$, $\dfrac{4x^2+4x+1}{x^2+2x-8} \geq 0$ and $x<0$ $\dfrac{(2x+1)^2}{(x-2)(x+4)} \geq 0$ $x>2$ or $x<-4$ and $x<0$ $x=-\dfrac{1}{2}$ or $x<-4$ Answer: $x=\dfrac{1}{2}$ or $x=-\dfrac{1}{2}$ or $x<-4$ or $x>4$ <u>Alternatively(Method 3),</u> $\dfrac{4\|x\|^2-4\|x\|+1}{\|x\|^2-2\|x\|-8} \geq 0$ $\dfrac{(2\|x\|-1)^2}{(\|x\|+2)(\|x\|-4)} \geq 0$ $(2\|x\|-1)^2=0$ satisfy the inequality $\Rightarrow \|x\|=\dfrac{1}{2} \Rightarrow x=\pm\dfrac{1}{2}$ $(\|x\|+2)>0$ for all values of x, $(\|x\|-4)>0 \Rightarrow \|x\|>4 \Rightarrow x<-4$ or $x>4$ Answer: $x=\dfrac{1}{2}$ or $x=-\dfrac{1}{2}$ or $x<-4$ or $x>4$	

5	side of triangle $=\sqrt{\left(\dfrac{y}{2}\right)^2+y^2}=\dfrac{\sqrt{5}y}{2}$	

	$\left[\dfrac{x+x+2\left(\dfrac{y\sqrt{5}}{2}\right)}{10}+\dfrac{\frac{\pi y}{2}}{8}\right]\left(\dfrac{5}{3}\right)=200$ $\dfrac{x}{5}+\dfrac{\sqrt{5}y}{10}+\dfrac{\pi y}{16}=120$ $\dfrac{x}{5}=120-\dfrac{\sqrt{5}y}{10}-\dfrac{\pi y}{16}$ $x=600-\dfrac{\sqrt{5}y}{2}-\dfrac{5\pi y}{16}$ Cross sectional area, W $=\pi\left(\dfrac{y}{2}\right)^2\left(\dfrac{1}{2}\right)+xy+\dfrac{1}{2}y^2$ $=\dfrac{\pi y^2}{8}+\dfrac{y^2}{2}+y\left(600-\dfrac{\sqrt{5}y}{2}-\dfrac{5\pi y}{16}\right)$ $=\dfrac{\pi y^2}{8}+\dfrac{y^2}{2}+600y-\dfrac{\sqrt{5}y^2}{2}-\dfrac{5\pi y^2}{16}$ $=600y-\dfrac{3\pi y^2}{16}+\dfrac{y^2}{2}-\dfrac{\sqrt{5}y^2}{2}$ For maximum W, $\dfrac{\mathrm{d}W}{\mathrm{d}y}=0$ $600-\dfrac{3\pi y}{8}+y-\sqrt{5}y=0$ $y\left(-\dfrac{3\pi}{8}+1-\sqrt{5}\right)=-60$ $y=248.533\approx 249$ $x=78.1347\approx 78$ $\dfrac{\mathrm{d}^2W}{\mathrm{d}y^2}=-\dfrac{3\pi}{8}+1-\sqrt{5}=-2.414$ $y=249$ and $x=78$ will result in a maximum cross sectional area.	

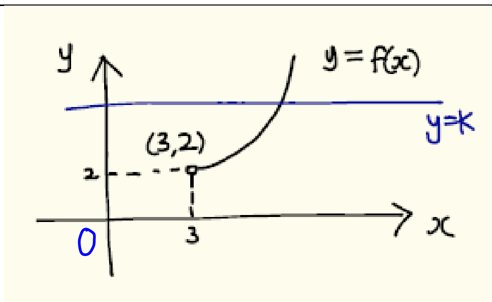
6(i)	$y^3+\tan^{-1}y=\ln(\cos x)$ Differentiating both sides w.r.t x, $3y^2\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sin x}{\cos x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}\left(3y^2\left(1+y^2\right)+1\right)=-\left(1+y^2\right)\tan x$ $\left(3y^4+3y^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(1+y^2\right)\tan x$ (shown)	
(ii)	Differentiating both sides w.r.t x, $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\left(12y^3+6y\right)+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(3y^4+3y^2+1\right)$ $=-\left(1+y^2\right)\sec^2 x-\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)\tan x$ When $x=0$, $y=0,\ \frac{\mathrm{d}y}{\mathrm{d}x}=0,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1$ $\therefore y=0+\frac{0}{1!}x+\frac{-1}{2!}x^2+\ldots=-\frac{1}{2}x^2+\ldots$	
(iii)	$\int_0^{\pi/4}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x\approx\left[-\frac{x^2}{2}\right]_0^{\pi/4}$ $=-\frac{\left(\pi/4\right)^2}{2}-0$ $=-\frac{\pi^2}{32}$ or $-0.03125\pi^2$	

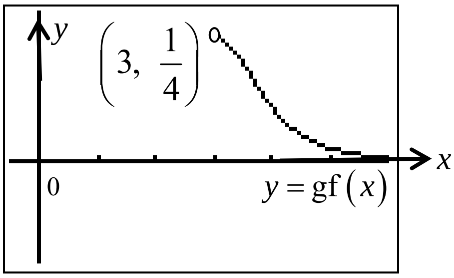
7 (i)	Due to the fishermen catching the fishes, $\frac{\mathrm{d}x}{\mathrm{d}t}\propto\frac{1}{x}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{k}{x}$,where k is a positive constant Due to the aggressive nature of the fishes, $\frac{\mathrm{d}x}{\mathrm{d}t}=-0.1x$ Rate of change of fishes,	

	$\frac{dx}{dt}=-\frac{k}{x}-0.1x$ $=-\frac{k+0.1x^2}{x}$ $\int \frac{x}{k+0.1x^2}\ dx=\int -1\ dt$ $\frac{1}{0.2}\int \frac{0.2x}{k+0.1x^2}\ dx=\int -1\ dt$ $\frac{1}{0.2}\ln\left\|k+0.1x^2\right\|=-t+c$ $\ln\left\|k+0.1x^2\right\|=-0.2t+c_1$ $\left\|k+0.1x^2\right\|=e^{-0.2t+c_1}$ $k+0.1x^2=\pm e^{c_1}e^{-0.2t}$ $x^2+10k=\pm 10e^{c_1}e^{-0.2t}$ $x^2+10k=Ae^{-0.2t}$ Alternatively, $\int \frac{x}{k+0.1x^2}\ dx=\int -1\ dt$ $\frac{1}{0.2}\int \frac{0.2x}{k+0.1x^2}\ dx=\int -1\ dt$ $\frac{1}{0.2}\ln\left(k+0.1x^2\right)=-t+c$ since $k+0.1x^2>0$ $\ln\left(k+0.1x^2\right)=-0.2t+c_1$ $k+0.1x^2=e^{-0.2t+c_1}$ $x^2+10k=Ae^{-0.2t}$	
(ii)	When $t=0$, $x=5$ $25+10k=A$ When $t=5$, $x=3$ $9+10k=Ae^{-1}$ Solving, $A=25.3116$ and $k=0.0311627$ When $x=0$, $t=21.986$ Number of days required = 22	

8(a) (i)	No. of five-digit numbers greater than 10000 $=5\times5\times4\times3\times2$ $=600$ Alternatively, No restrictions – case where 0 is the first digit ${}^{6}P_{5}-{}^{5}P_{4}=600$	
(ii)	<u>Method 1</u> Case 1: First digit is 1 or 3 or 5 (odd) $3\times4\times3\times2\times3=216$ Case 2: First digit is 2 or 4 (even) $2\times4\times3\times2\times2=96$ No. of five-digit numbers greater than 10000 and even $=216+96$ $=312$ <u>Method 2</u> Case 1: Last digit is 2 or 4 $4\times4\times3\times2\times2=192$ Case 2: Last digit is 0 $5\times4\times3\times2\times1=120$ No. of five-digit numbers greater than 10000 and even $=192+120$ $=312$ <u>Method 3</u> No. of five digit numbers greater than 10000 – No. of five digit numbers greater than 10000 that are odd $=600-4\times4\times3\times2\times3$ $=312$	
(b)	Total number of selections $=\left(2^{4}-1\right)\times\left(2^{3}-1\right)=105$ Alternatively, <u>Method 2:</u> Listing 12 Cases $\left({}^{4}C_{1}+{}^{4}C_{2}+{}^{4}C_{3}+{}^{4}C_{4}\right)\times\left({}^{3}C_{1}+{}^{3}C_{2}+{}^{3}C_{3}\right)=105$ <u>Method 3:</u> Complement	

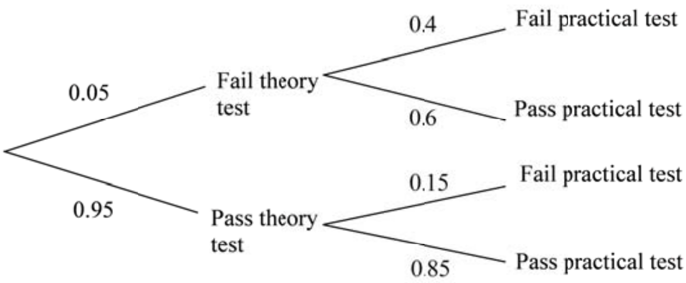
	No restrictions – 0 flavors or 0 toppings $=2^4\times2^3-\left(2^3+2^4-1\right)$ $=128-23$ $=105$	

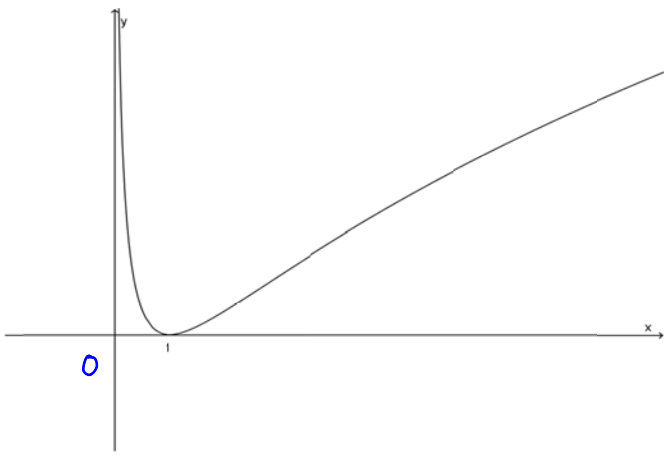
9(i)	 Any horizontal line, $y=k$, $k>2$ cuts the graph of $y=\mathrm{f}(x)$ at most once.	
(ii)	Let $y=\mathrm{f}(x)=x^2-6x+11$ $y=(x-3)^2+2$ $(x-3)^2=y-2$ $x=3\pm\sqrt{y-2}$ Since $x>3$, $x=3+\sqrt{y-2}$ $\therefore \mathrm{f}^{-1}(x)=3+\sqrt{x-2},\ x>2$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=\left(2,\ \infty\right)$	
(iii)	For gf to exist, $R_{\mathrm{f}}\subseteq D_{\mathrm{g}}$ i.e. $\left(2,\ \infty\right)\subseteq\left[k,\ \infty\right)$ $\therefore$ greatest value of $k=2$ $\underset{(3,\infty)}{D_{\mathrm{f}}} \rightarrow \underset{(2,\infty)}{R_{\mathrm{f}}} \rightarrow \underset{\left(0,\ \frac{1}{4}\right)}{R_{\mathrm{gf}}}$ $R_{\mathrm{gf}}=\left(0,\ \dfrac{1}{4}\right)$ Alternative method, $\mathrm{gf}\left(x\right)=g\left(x^2-6x+11\right)$ $=\dfrac{1}{\left(x^2-6x+11\right)^2},x>3$	

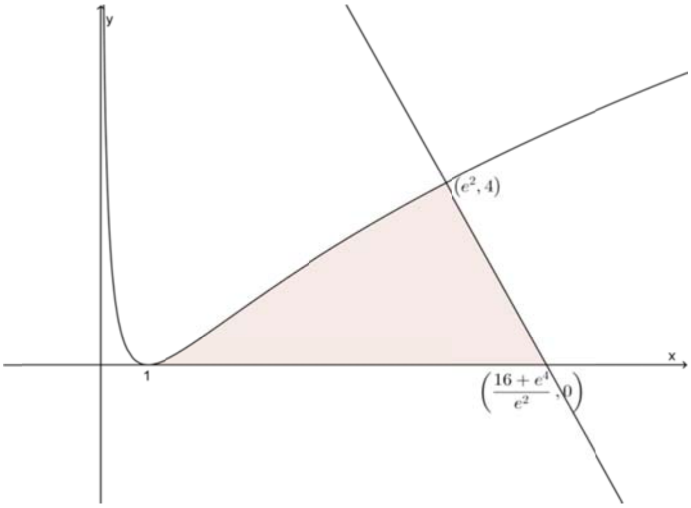
	 $R_{\mathrm{gf}}=\left(0,\ \frac{1}{4}\right)$	
(b)	Since h is a one-one function, h^{-1} exists. Since $R_{\mathrm{h}^{-1}}=D_{\mathrm{h}}$, the rule for composite function, $R_{\mathrm{h}^{-1}}\subseteq D_{\mathrm{h}}$ is fulfilled. Therefore hh^{-1} exists.	

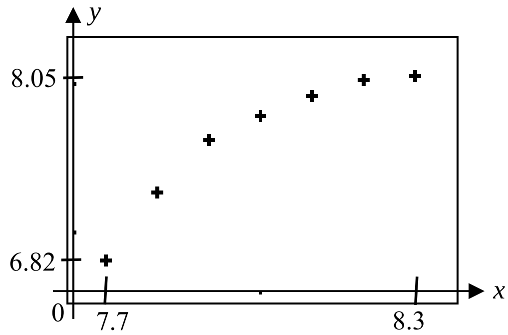
10 (a)(i)	$u_n=S_n-S_{n-1}$ $=\ln a^n b^{\frac{1}{2}\left(n^2-n\right)}-\ln a^{n-1}b^{\frac{1}{2}\left((n-1)^2-(n-1)\right)}$ $=\ln ab^{\frac{1}{2}\left(n^2-n-\left(n^2-3n+2\right)\right)}$ $=\ln ab^{n-1}$	
(ii)	$u_n-u_{n-1}=\ln ab^{n-1}-\ln ab^{n-1-1}$ $=\ln b$ Since $\ln b$ is a constant, the sequence is an AP.	
(iii)	For $n<7$, $0<ab^{n-1}<1\Rightarrow \ln ab^{n-1}<0$ Therefore, sum of negative terms is $S_6=\ln a^6 b^{\frac{1}{2}\left(6^2-6\right)}=\ln a^6 b^{15}$	
(b)	Using factor formula, $\sin(n\theta)\sin\left(\frac{1}{2}\theta\right)=\frac{-1}{2}\left(\cos\left[\left(n+\frac{1}{2}\right)\theta\right]-\cos\left[\left(n-\frac{1}{2}\right)\theta\right]\right)$ $\sin(n\theta)=\frac{-1}{2\sin\left(\frac{1}{2}\theta\right)}\left(\cos\left[\left(n+\frac{1}{2}\right)\theta\right]-\cos\left[\left(n-\frac{1}{2}\right)\theta\right]\right)$	

	$\sum_{n=1}^{N}\sin\left(n\theta\right)=\frac{-1}{2\sin\left(\frac{1}{2}\theta\right)}\sum_{n=1}^{N}\left(\cos\left[\left(n+\frac{1}{2}\right)\theta\right]-\cos\left[\left(n-\frac{1}{2}\right)\theta\right]\right)$ $=\frac{-1}{2\sin\left(\frac{1}{2}\theta\right)}\left(\begin{array}{l}\cancel{\cos\frac{3\theta}{2}}-\cos\frac{\theta}{2}\\ \cancel{+\cos\frac{5\theta}{2}}\cancel{-\cos\frac{3\theta}{2}}\\ \cancel{+\cos\frac{7\theta}{2}}\cancel{-\cos\frac{5\theta}{2}}\\ +\cdots\\ \cancel{+\cos\left[\left(N-\frac{1}{2}\right)\theta\right]}\cancel{-\cos\left[\left(N-\frac{3}{2}\right)\theta\right]}\\ +\cos\left[\left(N+\frac{1}{2}\right)\theta\right]\cancel{-\cos\left[\left(N-\frac{1}{2}\right)\theta\right]}\end{array}\right)$ $=\frac{-1}{2\sin\left(\frac{1}{2}\theta\right)}\left(\cos\left[\left(N+\frac{1}{2}\right)\theta\right]-\cos\frac{\theta}{2}\right)$ $=\frac{\cos\frac{\theta}{2}}{2\sin\left(\frac{1}{2}\theta\right)}-\frac{\cos\left[\left(N+\frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{1}{2}\theta\right)}$ $=\frac{1}{2}\cot\left(\frac{1}{2}\theta\right)-\frac{\cos\left[\left(N+\frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{1}{2}\theta\right)}$ (shown)	

11 (a)(i)	$\mathrm{P}(A\cap B)=1-\mathrm{P}(A'\cup B')=1-0.9=0.1$ $\mathrm{P}(A)=\mathrm{P}(A\cap B')+\mathrm{P}(A\cap B)=0.45+0.1=0.55$	
(ii)	$\mathrm{P}(A'\cap B)=\mathrm{P}(B)-\mathrm{P}(A\cap B)$ $=0.3-0.1$ $=0.2$	
(b)		
(i)	$\mathrm{P}(\text{passes the practical test})$ $=0.05\times0.6+0.95\times0.85=0.8375$	
(ii)	$\mathrm{P}(\text{passes the theory test} \mid \text{he fails the practical test})$ $=\dfrac{0.95\times0.15}{1-0.8375}$ $=0.877$	

12 (i)		

(ii)	$\frac{dy}{dt} = 2t, \ \frac{dx}{dt} = e^t$ $\frac{dy}{dx} = 2te^{-t}$ At point A, $t = 2$, gradient of normal $= \frac{-1}{2(2)e^{-2}} = \frac{-e^2}{4}$ Equation of line l, $y - 4 = -\frac{e^2}{4}\left(x - e^2\right)$ $y = -\frac{e^2}{4}x + \frac{16+e^4}{4}$	
(iii)	 Required area = area of triangle + area under curve C $= \frac{1}{2}(4)\left(\frac{16+e^4}{e^2} - e^2\right) + \int_{t=0}^{t=2} y\frac{dx}{dt}\,dt$ $= 2\left(\frac{16}{e^2}\right) + \int_0^2 t^2e^t\,dt$ $= \frac{32}{e^2} + \left\{\left[t^2e^t\right]_0^2 - 2\int_0^2 te^t\,dt\right\}$ $= \frac{32}{e^2} + 4e^2 - 2\left\{\left[te^t\right]_0^2 - \int_0^2 e^t\,dt\right\}$ $= \frac{32}{e^2} + 4e^2 - 2\left\{2e^2 - \left[e^t\right]_0^2\right\}$ $= \frac{32}{e^2} + 4e^2 - 4e^2 + 2e^2 - 2$ $= \frac{32}{e^2} + 2e^2 - 2$	

13(i)		
(ii)	The scatter diagram shows y is increasing at a decreasing rate and hence a linear model is not appropriate.	
(iii)	(a) $r \approx 0.9358$ (b) $r \approx 0.9464$	
(iv)	Since the product moment correlation coefficient between x and y^2 is closer to 1 compared to that between x^2 and y and y increases as x increases but at a decreasing rate, hence $y^2 = c + dx$ is the better model.	
(v)	Using the GC, $y^2 = -176.23 + 29.347x$ When $y = 8.00$, $x \approx 8.2$ From (iii), $r \approx 0.9464$ is close to 1. Since $y = 8.00$ is within the data range of y and x is the independent variable, hence the answer is reliable.	

TEMASEK JUNIOR COLLEGE, SINGAPORE
JC One
Promotion Examination 2013
Higher 2

MATHEMATICS **9740**

Solutions

1 Find the general solution of the following differential equation

$$\frac{1}{1+x}\frac{dy}{dx}+\frac{1}{1+x^2}=0, \quad \text{where } x \neq -1.$$ **[4]**

Solution:

$$\left(\frac{1}{1+x}\right)\frac{dy}{dx}+\frac{1}{1+x^2}=0$$

$$\frac{dy}{dx}=-\frac{1+x}{1+x^2}$$

$$\frac{dy}{dx}=-\frac{1}{1+x^2}-\frac{x}{1+x^2}$$

$$y=-\int\frac{1}{1+x^2}\,dx-\frac{1}{2}\int\frac{2x}{1+x^2}\,dx$$

$$=-\tan^{-1}x-\frac{1}{2}\ln\left(1+x^2\right)+c$$

(or $-\tan^{-1}x-\frac{1}{2}\ln\left|1+x^2\right|+c$)

2 **(i)** The first three terms of a sequence are given by $u_1 = 19$, $u_2 = 34$, $u_3 = 52$. Given that u_n is a quadratic polynomial in n, find u_n in terms of n. **[4]**

(ii) Find the smallest value of n for which u_n is greater than 200. **[2]**

Solution:

(i) Let $u_n = an^2 + bn + c$ where a, b, c are constants.

When $n = 1$, $a + b + c = 19$ ----- (1)
When $n = 2$, $4a + 2b + c = 34$ ----- (2)
When $n = 3$, $9a + 3b + c = 52$ ----- (3)

Using GC to solve the system of equations, we get

$$a = \frac{3}{2}, \quad b = \frac{21}{2}, \quad c = 7$$

$$\therefore \quad u_n = \frac{3}{2}n^2 + \frac{21}{2}n + 7$$

(ii)

Method I: For $u_n > 200$,

$$\frac{3}{2}n^2 + \frac{21}{2}n + 7 > 200$$

$\Rightarrow \quad n < -15.4 \quad$ or $\quad n > 8.37 \quad$ (3sf)

$\therefore$ the smallest value of n is 9.

Method II:

For $u_n > 200$,

By GC

$U_8 = 187 < 200$

$U_9 = 223 > 200$

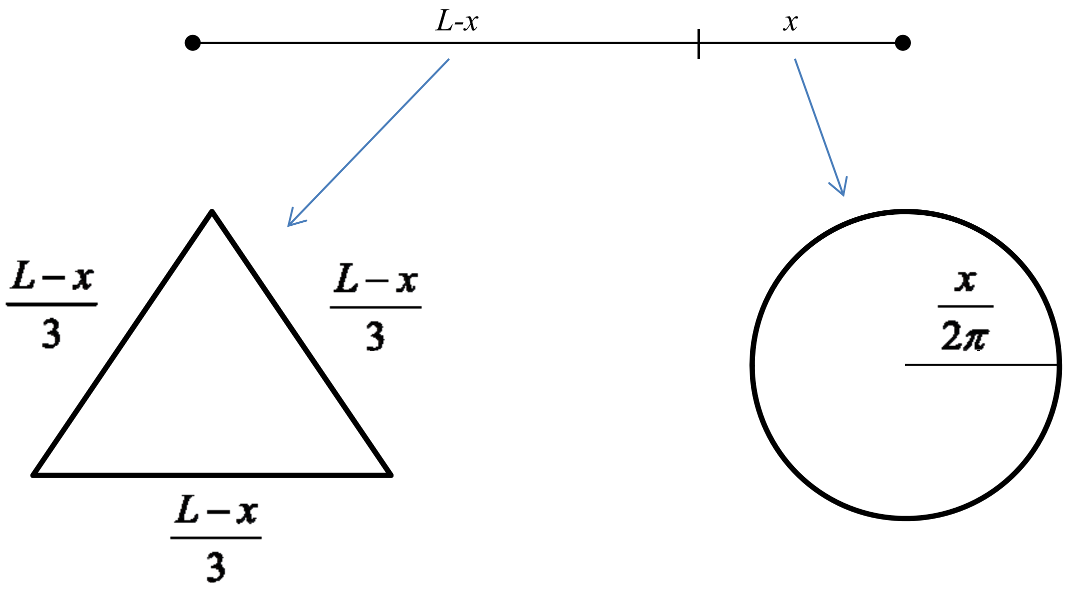
$\therefore$ The smallest value of n is 9.

3 A wire of length L cm is cut into two pieces. One piece is used to form a circle while the other piece is used to form an equilateral triangle.
Show that, with the total area of the circle and triangle being the smallest, the proportion of the length of the smaller piece to the length of the bigger piece is $\dfrac{\sqrt{3}\pi}{9}$.

[6]

Solution:

Let one of the pieces be x cm and use it for form the circle.
So the other piece is L-x and it's used to for the equilateral triangle.



For area of circle (radius r): $2\pi r = x \Rightarrow r = \dfrac{x}{2\pi}$

Therefore area is $\pi\left(\dfrac{x}{2\pi}\right)^2 = \dfrac{x^2}{4\pi}$

For area of equilateral triangle:

Area = $\dfrac{1}{2}\left(\dfrac{L-x}{3}\right)^2 \sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{36}(L-x)^2$

Hence total area, A = $\dfrac{x^2}{4\pi} + \dfrac{\sqrt{3}}{36}(L-x)^2$ [the other form $\dfrac{(L-x)^2}{4\pi} + \dfrac{\sqrt{3}}{36}x^2$ also accepted]

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{x}{2\pi} - \frac{\sqrt{3}}{18}(L-x)$$

Method I:

For max/min, $\frac{\mathrm{d}A}{\mathrm{d}x}=0$,

$$\Rightarrow \frac{1x}{2\pi}-\frac{2\sqrt{3}}{36}\left(L-x\right)=0 \Rightarrow \frac{x}{2\pi}=\frac{\sqrt{3}}{18}\left(L-x\right) \Rightarrow \frac{x}{\left(L-x\right)}=\frac{\sqrt{3}}{9}\pi<1$$

Hence the ratio of the length of the smaller piece to the length of the bigger piece is $\frac{\sqrt{3}\pi}{9}$

(shown)

And $\frac{\mathrm{d}^2 A}{\mathrm{d}x^2}=\frac{1}{2\pi}+\frac{\sqrt{3}}{18}>0 \Rightarrow A$ is minimum.

Method II:

For max/min, $\frac{\mathrm{d}A}{\mathrm{d}x}=0$,

$$\Rightarrow \frac{2x}{4\pi}-\frac{2\sqrt{3}}{36}\left(L-x\right)=0 \Rightarrow \frac{x}{2\pi}-\frac{\sqrt{3}}{18}\left(L-x\right)=0 \text{ --- } (*)$$

$$\Rightarrow x\left(\frac{1}{2\pi}+\frac{\sqrt{3}}{18}\right)=\frac{\sqrt{3}}{18}L \Rightarrow x=\frac{\sqrt{3}\pi L}{9+\sqrt{3}\pi}$$

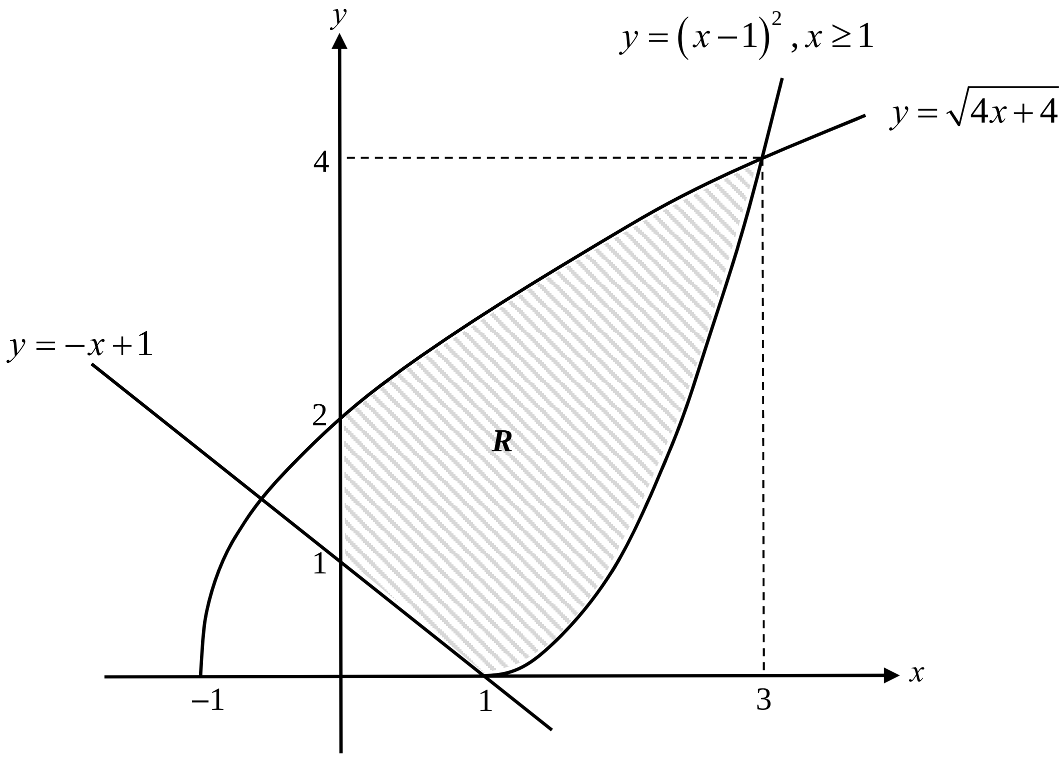
$\frac{\mathrm{d}^2 A}{\mathrm{d}x^2}=\frac{1}{2\pi}+\frac{\sqrt{3}}{18}>0 \Rightarrow A$ is minimum at $x=\frac{\sqrt{3}\pi L}{9+\sqrt{3}\pi}$

From (*) $\frac{x}{2\pi}-\frac{\sqrt{3}}{18}(L-x)=0 \Rightarrow \frac{x}{2\pi}=\frac{\sqrt{3}}{18}(L-x) \Rightarrow \frac{x}{L-x}=\frac{\sqrt{3}\pi}{9}(<1)$

Hence the ratio of the length of the smaller piece to the length of the bigger piece is $\frac{\sqrt{3}\pi}{9}$

(shown)

4



The shaded region R in the diagram above is bounded by the y-axis, the line $y=-x+1$ and the curves $y=(x-1)^2$ for $x \geq 1$ and $y=\sqrt{4x+4}$.

Find the volume of the solid of revolution formed when R is rotated completely about the y-axis. **[6]**

Solution:

$$\text{Required volume} = \pi\int_0^4 \left(1+\sqrt{y}\right)^2 \text{ dy} - \pi\int_2^4 \left(\frac{y^2-4}{4}\right)^2 \text{ dy} - \frac{\pi}{3}(1)^2(1)$$

$$\approx 17.26666709\pi \approx 54.24483447 \approx 54.2 \text{ unit}^2$$

5 Given that $y=\ln\left(2+\tan^{-1}x\right)$, show that

$$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1+x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0\,. \qquad \textbf{[3]}$$

Hence find the Maclaurin's expansion for y, up to and including the term in x^2. **[3]**

Solution:

$y=\ln\left(2+\tan^{-1}x\right)\Rightarrow e^y=2+\tan^{-1}x$

Differentiate wrt x

$\Rightarrow e^y\frac{dy}{dx}=\frac{1}{1+x^2}\Rightarrow\left(1+x^2\right)\frac{dy}{dx}=e^{-y}---(1)$

Differentiate (1) wrt x

$\Rightarrow\left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=-e^{-y}\frac{dy}{dx}=-\left(1+x^2\right)\left(\frac{dy}{dx}\right)^2$ [From (1)]

$\Rightarrow\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1+x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$

When $x=0, y=\ln 2, \frac{dy}{dx}=\frac{1}{2}, \frac{d^2y}{dx^2}=-\frac{1}{4}$

$\Rightarrow y=\ln 2+\frac{\left(\frac{1}{2}\right)}{1!}x+\frac{\left(-\frac{1}{4}\right)}{2!}x^2+\cdots\approx\ln 2+\frac{1}{2}x-\frac{1}{8}x^2$

6 Prove by mathematical induction $\sum_{r=1}^{n} \frac{r}{3^{r-1}} = \frac{9}{4} - \frac{1}{3^{n-1}}\left(\frac{3}{4}+\frac{n}{2}\right)$ for all positive integers of n. **[5]**

Hence show that

$$\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\frac{4}{4^4}\cdots<\frac{9}{16}.$$ **[2]**

Solution:

Let $P(n)$ be the statement $\sum_{r=1}^{n} \frac{r}{3^{r-1}} = \frac{9}{4} - \frac{1}{3^{n-1}}\left(\frac{3}{4}+\frac{n}{2}\right)$ for $n = 1, 2, 3, 4, \ldots$

When n = 1, LHS = $\sum_{r=1}^{1} \frac{r}{3^{r-1}}$ = 1 ; RHS = $\frac{9}{4} - \left(\frac{3}{4}+\frac{1}{2}\right) = 1$

So $P(1)$ is true.

Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e. $\sum_{r=1}^{k} \frac{r}{3^{r-1}} = \frac{9}{4} - \frac{1}{3^{k-1}}\left(\frac{3}{4}+\frac{k}{2}\right)$

To show $P(k + 1)$ is true i.e. $\sum_{r=1}^{k+1} \frac{r}{3^{r-1}} = \frac{9}{4} - \frac{1}{3^{k}}\left(\frac{3}{4}+\frac{k+1}{2}\right)$

$$\text{LHS} = \sum_{r=1}^{k+1} \frac{r}{3^{r-1}} = \sum_{r=1}^{k} \frac{r}{3^{r-1}} + \frac{k+1}{3^k}$$

$$= \left[\frac{9}{4} - \frac{1}{3^{k-1}}\left(\frac{3}{4}+\frac{k}{2}\right)\right] + \frac{k+1}{3^k}$$

$$= \frac{9}{4} - \frac{1}{3^{k}}\left(\frac{9}{4}+\frac{3k}{2}-k-1\right)$$

$$= \frac{9}{4} - \frac{1}{3^{k}}\left(\frac{9}{4}+\frac{3k-2k-2}{2}\right) = \frac{9}{4} - \frac{1}{3^{k}}\left(\frac{9}{4}+\frac{k-2}{2}\right) = \frac{9}{4} - \frac{1}{3^{k}}\left(\frac{5}{4}+\frac{k}{2}\right)$$

$$= \frac{9}{4} - \frac{1}{3^{k}}\left(\frac{3}{4}+\frac{k+1}{2}\right) = \text{RHS}$$

So $P(k+1)$ is true.

Since $P(1)$ is true, and $P(k)$ is true $\Rightarrow$ $P(k + 1)$ is true.

$\therefore$ By mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$, ie. $\sum_{r=1}^{n} \frac{r}{3^{r-1}} = \frac{9}{4} - \frac{1}{3^{n-1}}\left(\frac{3}{4}+\frac{n}{2}\right)$ $n \in \mathbb{Z}^+$

Since $\sum_{r=1}^{\infty} \frac{r}{3^{r-1}} = \frac{9}{4}$

Hence $\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\cdots = \sum_{r=1}^{\infty} \frac{r}{4^r} = \frac{1}{4}\sum_{r=1}^{\infty} \frac{r}{4^{r-1}} < \frac{1}{4}\sum_{r=1}^{\infty} \frac{r}{3^{r-1}} = \frac{9}{16}$ (deduced)

7 Functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{2x-2}{x-2}, \quad \text{for } x \in \mathbb{R}, x < 1,$$

$$\mathrm{g}: x \mapsto \sqrt{2-x}, \quad \text{for } x \in \mathbb{R}, x \leq 2.$$

(i) Given that f has an inverse, show that the composite function gf^{-1} exists. Find gf^{-1} and state its range. **[5]**

(ii) Find the value(s) of x such that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. **[2]**

Solution:

(i) $\mathrm{R}_{\mathrm{f}^{-1}} = \mathrm{D}_{\mathrm{f}} = (-\infty, 1)$

$\mathrm{D}_g = (-\infty, 2]$

Since $\mathrm{R}_{\mathrm{f}^{-1}} \subset \mathrm{D}_g$, the composite function gf^{-1} exists. (Shown)

Let $y = \dfrac{2x-2}{x-2}$.

$\Rightarrow \quad xy - 2y = 2x - 2$

$\Rightarrow \quad xy - 2x = 2y - 2$

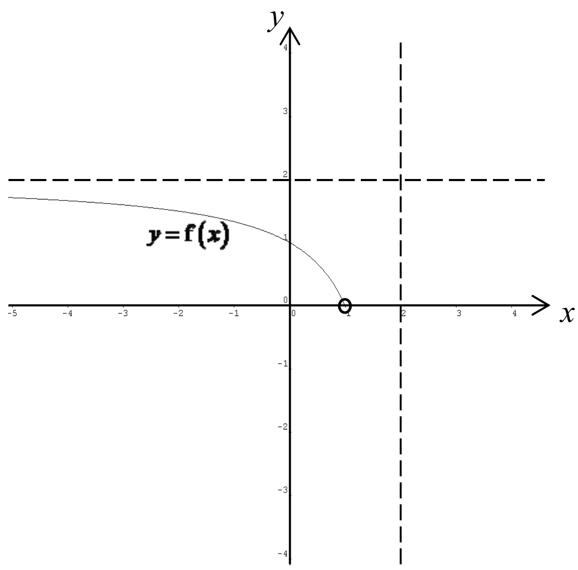
$\Rightarrow \quad x(y-2) = 2y - 2$

$\Rightarrow \quad x = \dfrac{2y-2}{y-2}$

$\Rightarrow \quad \mathrm{f}^{-1}(x) = \dfrac{2x-2}{x-2}$.

$$\mathrm{gf}^{-1}(x) = \mathrm{g}\left(\frac{2x-2}{x-2}\right)$$

$$= \sqrt{2 - \left(\frac{2x-2}{x-2}\right)} = \sqrt{2 - \left(2 + \frac{2}{x-2}\right)} = \sqrt{-\frac{2}{x-2}}$$



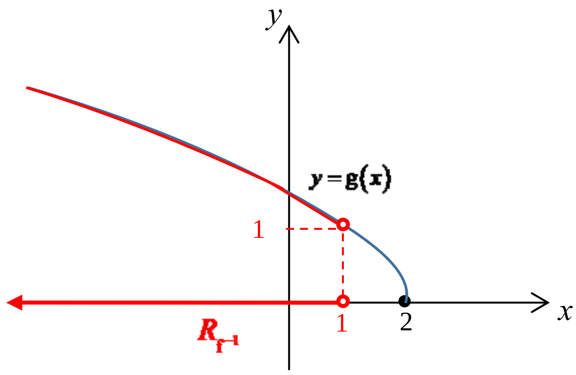
$\mathrm{D}_{\mathrm{gf}^{-1}} = \mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = (0, 2)$

So, $\mathrm{gf}^{-1}: x \mapsto \sqrt{-\dfrac{2}{x-2}}$, $x \in \mathbb{R}, 0 < x < 2$

For range of gf^{-1}:

M1 - By mapping method

$$\underset{D_{gf^{-1}}=D_{f^{-1}}}{(0,2)} \xrightarrow{\quad f^{-1}\quad} \underset{R_{f^{-1}}}{(-\infty,1)} \xrightarrow{\quad g\quad} \underset{R_{gf^{-1}}}{(1,\infty)}$$

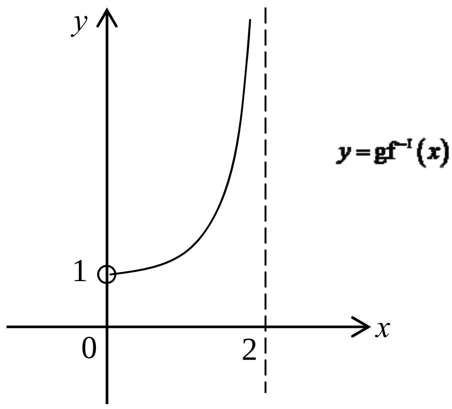


Thus, $R_{gf^{-1}}=(1,\infty)$.

M2 - By direct sketching method

$D_{gf^{-1}}=D_{f^{-1}}=R_f=(0,2)$

Therefore $R_{gf^{-1}}=(1,\infty)$

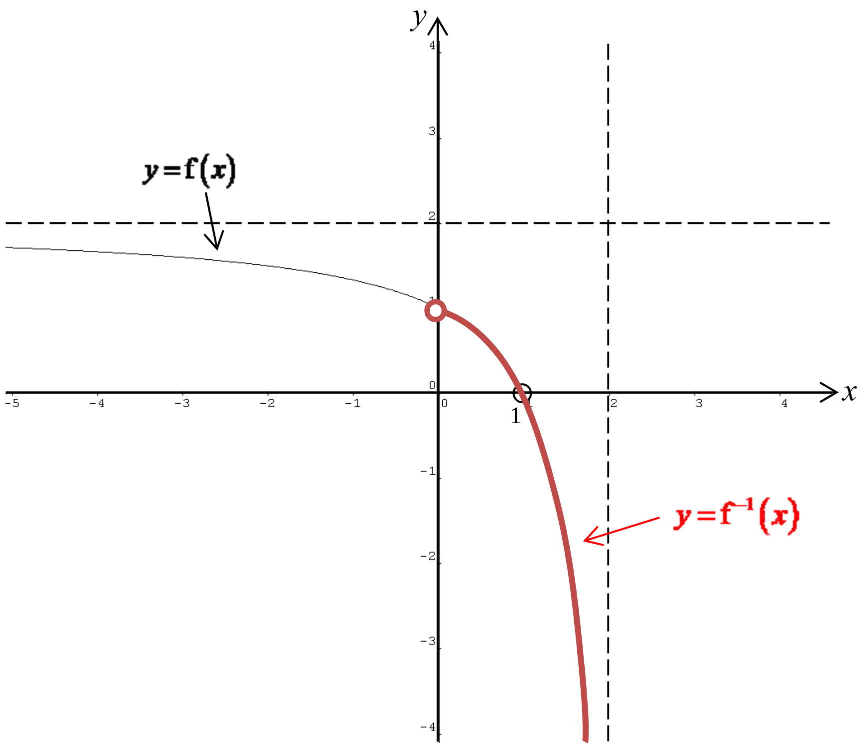


(ii)

From the graph,

$f(x)=f^{-1}(x)$

$\Rightarrow$ $0<x<1$



8 Prove that

$$\ln\left(\frac{(r-1)(r+2)}{r(r+1)}\right) \equiv \ln\left((r-1)(r)\right) - 2\ln\left((r)(r+1)\right) + \ln\left((r+1)(r+2)\right).$$ **[2]**

Hence, find in terms of n,

$$\ln\left(\frac{1\times4}{2\times3}\right)+\ln\left(\frac{2\times5}{3\times4}\right)+\ln\left(\frac{3\times6}{4\times5}\right)+\cdots+\ln\left(\frac{(n-1)(n+2)}{(n)(n+1)}\right)+\ln\left(\frac{(n)(n+3)}{(n+1)(n+2)}\right),$$

leaving your answer as a single logarithmic function. **[5]**

Solution:

(i)

$$\begin{aligned} RHS &\equiv \ln\left((r-1)(r)\right) - 2\ln\left((r)(r+1)\right) + \ln\left((r+1)(r+2)\right) \\ &\equiv \ln\left(\frac{(r-1)(r)(r+1)(r+2)}{(r)^2(r+1)^2}\right) \\ &\equiv \ln\left(\frac{(r-1)(r+2)}{(r)(r+1)}\right) \equiv LHS \end{aligned}$$

(ii)

$$\ln\left(\frac{1\times4}{2\times3}\right)+\ln\left(\frac{2\times5}{3\times4}\right)+\ln\left(\frac{3\times6}{4\times5}\right)+\cdots+\ln\left(\frac{(n-1)(n+2)}{(n)(n+1)}\right)+\ln\left(\frac{(n)(n+3)}{(n+1)(n+2)}\right)$$

$$=\sum_{r=2}^{n+1}\left[\ln\frac{(r-1)(r+2)}{r(r+1)}\right]=\sum_{r=2}^{n+1}\left[\ln(r-1)r-2\ln\left[r(r+1)\right]+\ln(r+1)(r+2)\right]$$

$$\begin{aligned} &= \ln(1)(2) - 2\ln(2)(3) + \ln(3)(4) \\ &+\ln(2)(3) - 2\ln(3)(4) + \ln(4)(5) \\ &+\ln(3)(4) - 2\ln(4)(5) + \ln(5)(6) \\ &+\ln(4)(5) - 2\ln(5)(6) + \ln(6)(7) \\ &+\cdots \\ &+\ln(n-2)(n-1) - 2\ln(n-1)(n) + \ln(n)(n+1) \\ &+\ln(n-1)(n) - 2\ln(n)(n+1) + \ln(n+1)(n+2) \\ &+\ln(n)(n+1) - 2\ln(n+1)(n+2) + \ln(n+2)(n+3) \\ &= \ln(1)(2) - \ln(2)(3) - \ln(n+1)(n+2) + \ln(n+2)(n+3) \end{aligned}$$ **[**

$$= \ln\left[\frac{2(n+2)(n+3)}{6(n+1)(n+2)}\right] = \ln\left[\frac{n+3}{3(n+1)}\right]$$

9 Jessie wishes to take up a loan of \$20,000 on the 1st day of the Year 2014. She intends to pay an instalment of \$300 on the 1st day of each month, beginning from February 2014. She sources out two banks, *XYZ* Bank and *ABC* Bank, which offer such loans. The two banks have different ways of charging interest. *XYZ* Bank charges a monthly interest of 0.5% on the outstanding amount owed at the end of each month, while *ABC* Bank charges a fixed interest of \$60 at the end of each month until the loan is repaid.

(a) If Jessie takes up the loan from *XYZ* Bank, show that the outstanding loan at the end of February 2014 after the interest has been added will be \$19899. **[2]**

Hence, find the number of months Jessie will take to repay her loan. **[4]**

(b) Which bank should Jessie take a loan from if she wishes to clear her loan as soon as possible? Justify your answers. **[3]**

Solution:

k^th^ month	**Outstanding loan at the beginning of k^th^ month from 2014**	**Outstanding loan at the end of k^th^ month from 2014**
1	20000	$1.005(20000)$
2	$1.005(20000)-300$	$1.005^2(20000)-300(1.005)$
3	$1.005^2(20000)-300(1.005)-300$	
$\vdots$	$\vdots$	
n	$1.005^{n-1}(20000)-300(1.005)^{n-2}-300(1.005)^{n-3}$ $-\cdots-300(1.005)^2-300(1.005)-300$	

(a) Outstanding loan at the end of February 2014 $=1.005^2(20000)-300(1.005)=\$19899$

[Shown]

Hence

Let $1.005^{n-1}(20000)-300(1.005)^{n-2}-\cdots-300(1.005)-300\le 0$

$$\Rightarrow 1.005^{n-1}(20000)-300\left[1+(1.005)+(1.005)^2+\cdots+(1.005)^{n-2}\right]\le 0$$

$$\Rightarrow 1.005^{n-1}(20000)-300\left[\frac{1\bullet(1.005)^{n-1}-1}{1.005-1}\right]\le 0$$

$$\Rightarrow 1.005^{n-1}(20000)-60000\left[(1.005)^{n-1}-1\right]\le 0$$

$$\Rightarrow 40000(1.005)^{n-1}\ge 60000$$

$$\Rightarrow (n-1)\ge\frac{\ln\frac{60000}{40000}}{\ln(1.005)}\Rightarrow n\ge 82.29558565$$

$\Rightarrow$ Jessie will repay her loan on the 1st day of 83rd month. Therefore, she will take 82 months to repay her loan.

(b)

Method I:

For Bank *ABC*,

k^{th} month	**Outstanding loan at the beginning of k^{th} month from 2014**	**Outstanding loan at the end of k^{th} month from 2014**
1	20000	$20000+60$
2	$20000+60-300$	$20000+60-300+60$
3	$20000+60-300+60-300=20000+60(2)-300(2)$ $=20000-240(2)$	
$\vdots$	$\vdots$	
n	$20000-240(n-1)$	

For $20000-240(n-1)\le 0 \Rightarrow n \ge 84.33333$

$\Rightarrow$ Jessie will repay her loan on the 1st day of 85th month if she takes up bank *ABC*.
Hence, she should take the loan from bank *XYZ*.

Method II:

When n = 83, $20000-240(83-1)=320>0$

$\Rightarrow$ Jessie will not be able to clear her loan by the 83rd month if she takes up bank *ABC*.
Hence, she should take the loan from bank *XYZ*.

10 A curve C is given parametrically by the equations

$$x = 2\cos^3\theta, \quad y = 2\sin^3\theta$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Show that the normal at the point with parameter θ has equation

$$y\sin\theta = x\cos\theta + 2\left(\sin^4\theta - \cos^4\theta\right).$$ **[4]**

The normal at the point Q where $\theta = \frac{\pi}{6}$, cuts C again at the point P, where $\theta = p$.

Show that $\sin^3 p - \sqrt{3}\cos^3 p + 1 = 0$ and hence find the coordinates of P. **[5]**

Solution:

$x = 2\cos^3\theta,$ $\qquad y = 2\sin^3\theta$

$\frac{dx}{dt} = 3(2)\cos^2\theta(-\sin\theta)$ $\qquad \frac{dy}{dt} = 3(2)\sin^2\theta\cos\theta$

$= -6\sin\theta\cos^2\theta$ $\qquad = 6\sin^2\theta\cos\theta$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\tan\theta$

$\Rightarrow$ Gradient of normal to the curve $= \cot\theta$

Eqn. of normal to the curve at $\left(2\cos^3\theta, 2\sin^3\theta\right)$:

$$\frac{y - 2\sin^3\theta}{x - 2\cos^3\theta} = \frac{\cos\theta}{\sin\theta}$$

$\Rightarrow \quad y\sin\theta - 2\sin^4\theta = x\cos\theta - 2\cos^4\theta$

$\Rightarrow \quad y\sin\theta = x\cos\theta + 2\left(\sin^4\theta - \cos^4\theta\right)$ (shown)

Eqn. of normal to the curve at Q, i.e. $\theta = \frac{\pi}{6}$:

$$y\left(\frac{1}{2}\right) = x\left(\frac{\sqrt{3}}{2}\right) + 2\left(\left(\frac{1}{2}\right)^4 - \left(\frac{\sqrt{3}}{2}\right)^4\right)$$

$\Rightarrow \quad y = \sqrt{3}x - 2$

When the normal to the curve at Q cuts C again at P, i.e. $\theta = p$,

$$2\sin^3 p = \sqrt{3}\left(2\cos^3 p\right) - 2$$

$\Rightarrow \quad \sin^3 p - \sqrt{3}\cos^3 p + 1 = 0$ (shown)

$\Rightarrow \quad p = -0.7445633$ or 0.52359878 $\left(\text{rejected}, \because \text{point } Q\right)$

$\therefore$ The coordinates of P is $(0.795, -0.622)$. (3sf)

11 A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \sqrt{\frac{2(x_n^2 - x_n)}{3}} + 1, \quad x_1 = k, \text{ where } k \geq 1.$$

(a) When $k = 5$, state the value of x_9 and describe the behavior of the sequence. **[2]**

(b) Prove algebraically that, if the sequence converges, then it converges to either 1 or 3. **[3]**

(c) State a value of k such that the sequence converges to 1. **[1]**

(d) When $k = 2$, state the integer m such that $m \leq x_n < m+1$ for all integers $n \geq 1$. **[1]**

Hence, by considering $\dfrac{x_{n+1}-1}{x_n - 1}$, show that $x_{n+1} > x_n$ for all integers $n \geq 1$. **[3]**

Solution:

(a) $x_9 = 3.44$

The sequence converges to 3 decreasingly.

(b) If the sequence converges to l. So when $n \to \infty$, $x_{n+1} \to l$ and $x_n \to l$.

Solving, we have

$$l = \sqrt{\frac{2(l^2 - l)}{3}} + 1 \Rightarrow 3(l-1)^2 = 2l^2 - 2l \Rightarrow l^2 - 4l + 3 = 0 \Rightarrow l = 1 \text{ or } l = 3.$$

Hence, if the sequence converges, then it converges to either 1 or 3. [Proven]

(c) The sequence converges to 1 when k = 1

(d) From GC, $m = 2$.

Method I:

$$\frac{x_{n+1}-1}{x_n-1} = \frac{\sqrt{\frac{2}{3}\left(x_n{}^2 - x_n\right)}}{x_n - 1} = \sqrt{\frac{2x_n}{3\left(x_n - 1\right)}} = \sqrt{\frac{2}{3}}\sqrt{1 + \frac{1}{x_n - 1}}$$

$$2 \leq x_n < 3 \Rightarrow \frac{1}{x_n - 1} > \frac{1}{2} \quad .$$

$$\Rightarrow \sqrt{\frac{2}{3}}\sqrt{1 + \frac{1}{x_n - 1}} > \sqrt{\frac{2}{3}}\sqrt{\frac{3}{2}} = 1$$

$$\Rightarrow \frac{x_{n+1}-1}{x_n-1} > 1 \Rightarrow x_{n+1} > x_n$$

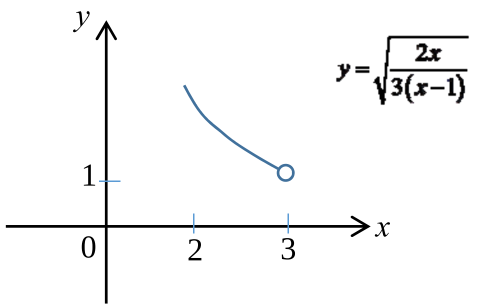
Or $\dfrac{x_{n+1}-1}{x_n-1} = \dfrac{\sqrt{\frac{2}{3}\left(x_n{}^2 - x_n\right)}}{x_n - 1} = \sqrt{\dfrac{2x_n}{3\left(x_n - 1\right)}}$

Now $x_n < 3 \Rightarrow -2x_n < 3 - 3x_n \Rightarrow \dfrac{-2x_n}{3 - 3x_n} > 1 \left(\because x_n > 1\right) \Rightarrow \sqrt{\dfrac{2x_n}{3\left(x_n - 1\right)}} > 1$

Method II:

$$\frac{x_{n+1}-1}{x_n-1}=\frac{\sqrt{\frac{2}{3}\left(x_n{}^2-x_n\right)}}{x_n-1}=\sqrt{\frac{2x_n}{3\left(x_n-1\right)}}$$

From the graph of $y=\sqrt{\frac{2x}{3\left(x-1\right)}}$, when $2\le x<3, y>1$



Since $2\le x_n<3$

$$\frac{x_{n+1}-1}{x_n-1}=\sqrt{\frac{2x_n}{3\left(x_n-1\right)}}>1\Rightarrow x_{n+1}>x_n$$

12 **(a)** Find $\displaystyle\int_1^e \frac{1}{x^2}\ln\left(\frac{1}{x^2}\right)\,dx$, leaving your answer in exact form. **[4]**

(b) Using the substitution $u=\sqrt{t}$, find $\displaystyle\int \frac{\sqrt{t}}{t-1}\,dt$. **[6]**

Solution:

(a) Method I (simplify using Laws of Log before integration):

$$\int_1^e \frac{1}{x^2}\ln\left(\frac{1}{x^2}\right)\,dx$$

$$=-2\int_1^e x^{-2}\ln x\,dx$$

$$=-2\left\{\left[-x^{-1}\ln x\right]_1^e-\int_1^e -x^{-1}\frac{1}{x}\,dx\right\}$$

$$=-2\left\{\left[-e^{-1}-0\right]-\int_1^e x^{-2}\,dx\right\}$$

$$=-2\left\{e^{-1}-\left[-x^{-1}\right]_1^e\right\}$$

$$=-2\left\{e^{-1}-\left[-e^{-1}+1\right]\right\}=-2\left\{2e^{-1}-1\right\}$$

$$=4e^{-1}-2$$

Method II (apply By Parts formula without simplification):

$$\int_1^e \frac{1}{x^2}\ln\left(\frac{1}{x^2}\right)\,dx$$

$$=\left[-\frac{1}{x}\ln\left(\frac{1}{x^2}\right)\right]_1^e-\int_1^e -\frac{1}{x}\left(\frac{1}{\frac{1}{x^2}}\right)\left(-\frac{2}{x^3}\right)\,dx$$

$$=\left[-\frac{1}{e}\ln\left(\frac{1}{e^2}\right)+\ln 1\right]-\int_1^e \frac{2}{x^2}\,dx$$

$$=\left[-\frac{1}{e}(-2)+0\right]-\int_1^e \frac{2}{x^2}\,dx$$

$$=\frac{2}{e}-\left[-\frac{2}{x}\right]_1^e$$

$$=\frac{2}{e}-\left[-\frac{2}{e}+2\right]$$

$$=\frac{4}{e}-2$$

(b) $u=\sqrt{t} \quad \Rightarrow \quad t=u^2$

Diff. wrt u, $\dfrac{\mathrm{d}t}{\mathrm{d}u}=2u$

$$\int \frac{\sqrt{t}}{t-1}\,\mathrm{d}t$$

$$=\int \frac{u}{u^2-1}(2u)\,\mathrm{d}u=2\int \frac{u^2}{u^2-1}\,\mathrm{d}u$$

$$=2\int\left(1+\frac{1}{u^2-1}\right)\mathrm{d}u$$

$$=2\left[u+\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right]+C$$

$$=2\sqrt{t}+\ln\left|\frac{\sqrt{t}-1}{\sqrt{t}+1}\right|+C$$

13 It is given that $\mathrm{f}(x) = -x - 1 + \dfrac{k^2 - 1}{x - 1}$ where $k > 1$.

(i) Show by differentiation that the graph of $y = \mathrm{f}(x)$ has no turning points. **[3]**

(ii) On separate diagrams, draw sketches of the graphs of

(a) $y = \mathrm{f}(x)$, **[4]**

(b) $y = \mathrm{f}'(x)$. **[2]**

You should indicate where possible, numerically or in terms of *k*, any asymptotes and axial intercepts for each of the curves.

(iii) Find in terms of *k*, the range of *x* that satisfies the inequality

$$k\,\mathrm{f}(x) \le (x - k)^2 (x + k)$$ **[4]**

Solution:

(i) $\mathrm{f}(x) = -x - 1 + \dfrac{k^2 - 1}{x - 1} \Rightarrow \mathrm{f}'(x) = -1 - \dfrac{k^2 - 1}{(x - 1)^2}$

Since $k > 1$, $\therefore k^2 - 1 < 0$

Since $(x - 1)^2$ is also always > 0, $-1 - \dfrac{k^2 - 1}{(x - 1)^2} < 0$

$\therefore \mathrm{f}'(x) \ne 0$ for all $x \in \mathbb{R}$

$\therefore$ $y = \mathrm{f}(x)$ has no turning points.

Hence $y = \mathrm{f}(x)$ has no turning point.

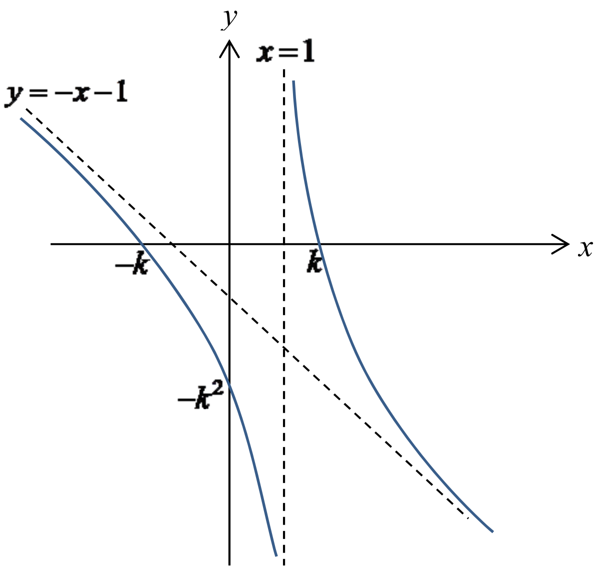
(ii)(a) When $x = 0$, $y=-1+\frac{k^2-1}{-1}=-k^2$

When $y = 0$, $-x-1+\frac{k^2-1}{x-1}=0$

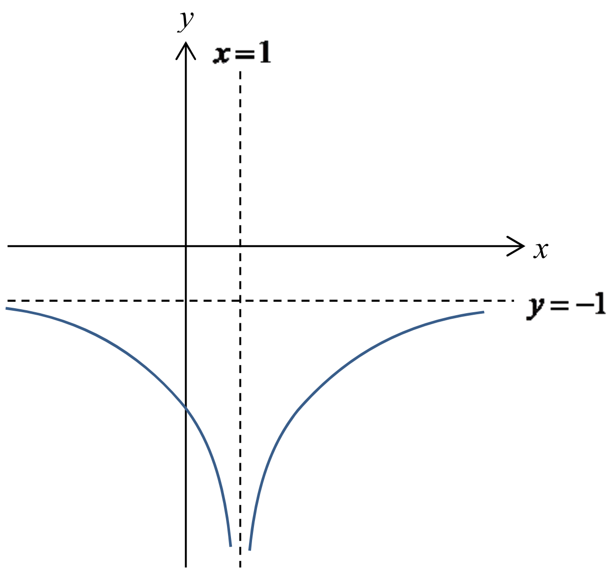
$k^2-1=(x+1)(x-1)$

$k^2-1=x^2-1$

$x=\pm k$



(ii)(b)



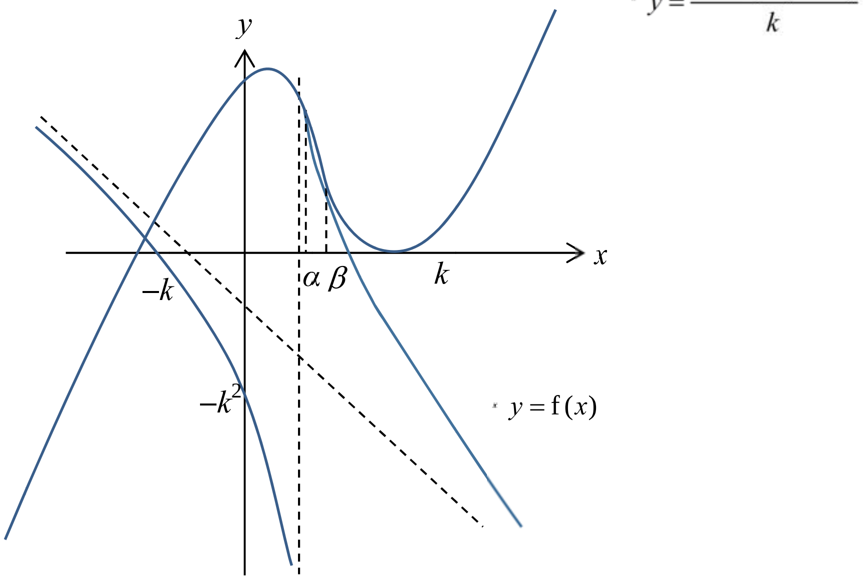
(iii)
Method 1:

$$k\,\mathrm{f}(x) \le (x-k)^2(x+k)$$

$$\Rightarrow \quad \mathrm{f}(x) \le \frac{(x-k)^2(x+k)}{k}$$

$\therefore$ Sketch the curves $y=\mathrm{f}(x)$ and $y=\dfrac{(x-k)^2(x+k)}{k}$

Case 1:



To find α and β, set

$$-x-1+\frac{k^2-1}{x-1}=\frac{(x-k)^2(x+k)}{k}$$

$$\Rightarrow \quad \frac{-x^2+k^2}{x-1}=\frac{(x-k)^2(x+k)}{k}$$

$$\Rightarrow \quad (x-k)(x+k)\left[\frac{(x-k)}{k}+\frac{1}{x-1}\right]=0$$

$$\Rightarrow \quad (x-k)(x+k)\left[\frac{x^2-(k+1)x+2k}{k(x-1)}\right]=0$$

$$\Rightarrow \quad x=\pm k \quad \text{or} \quad x^2-(k+1)x+2k=0$$

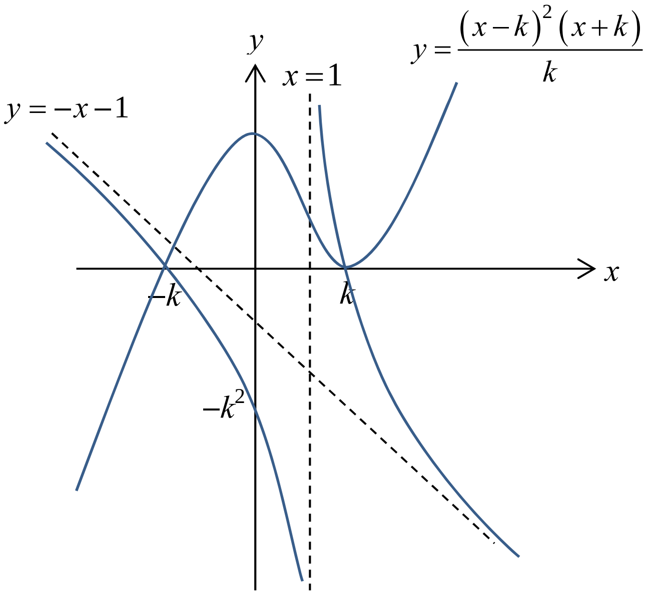
$$\Rightarrow \quad x=\pm k \quad \text{or} \quad x=\frac{(k+1)\pm\sqrt{(k+1)^2-8k}}{2}$$

$$\therefore \alpha=\frac{(k+1)-\sqrt{k^2-6k+1}}{2} \text{ and } \beta=\frac{(k+1)+\sqrt{k^2-6k+1}}{2}$$

$$\therefore -k\le x<1 \text{ or } \frac{(k+1)-\sqrt{k^2-6k+1}}{2}\le x\le\frac{(k+1)+\sqrt{k^2-6k+1}}{2} \text{ or } x\ge k$$

This case is valid if $k^2-6k+1\ge 0$, i.e. $(k-3)^2-8\ge 0$, i.e. $k\ge 3+2\sqrt{2}$ (since $k>1$)

Case 2 ($1 < k < 3+2\sqrt{2}$):



From the diagram, we have

$$-k \le x < 1 \quad \text{or} \quad x \ge k \ .$$

Method 2:

$$k\left(-x-1+\frac{k^2-1}{x-1}\right)\le (x-k)^2(x+k)$$

$$k\left(\frac{-x^2+k^2}{x-1}\right)\le (x-k)^2(x+k)$$

$$(x-k)(x+k)\left(\frac{-k}{x-1}-(x-k)\right)\le 0$$

$$(x-k)(x+k)\left(\frac{x^2-(k+1)+2k}{x-1}\right)\ge 0$$

$$(x-k)(x+k)(x-1)\left(x^2-(k+1)+2k\right)\ge 0 \quad , \quad x\ne 1$$

Case 1($\left(x^2-(k+1)+2k\right)$ can be factorized, i.e. when $(k+1)^2-4(1)(2k)\ge 0$,

i.e. $k^2-6k+1\ge 0$,

i.e. $k\ge\dfrac{6+\sqrt{36-4}}{2}$,

i.e. $k\ge 3+2\sqrt{2}$)

We have

$$(x-k)(x+k)(x-1)\left(x-\left(\frac{-(k+1)-\sqrt{(k+1)^2-8k}}{2}\right)\right)\left(x-\left(\frac{-(k+1)+\sqrt{(k+1)^2-8k}}{2}\right)\right)\ge 0$$

$$\therefore -k\le x<1 \text{ or } \frac{(k+1)-\sqrt{k^2-6k+1}}{2}\le x\le\frac{(k+1)+\sqrt{k^2-6k+1}}{2} \text{ or } x\ge k$$

Case 2 ($1<k<3+2\sqrt{2}$)

Since $\left(x^2-(k+1)+2k\right)>0$,

$\therefore (x-k)(x+k)(x-1)\ge 0$

$\therefore -k\le x<1$ or $x\ge k$

H

TEMASEK JUNIOR COLLEGE, SINGAPORE
JC One
Promotion Examination 2013
Higher 2

MATHEMATICS 9740

4 October 2013

Additional Materials: Answer paper
List of Formulae (MF15)

3 hours

READ THESE INSTRUCTIONS FIRST

Write your *Civics Group* and *Name* on all the work that you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, fasten all your work securely together.

This document consists of 5 printed pages.

TEMASEK JUNIOR COLLEGE, SINGAPORE
PASSION PURPOSE DRIVE

[Turn over

1 Find the general solution of the following differential equation

$$\frac{1}{1+x}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{1+x^2}=0, \quad \text{where } x \neq -1.$$ **[4]**

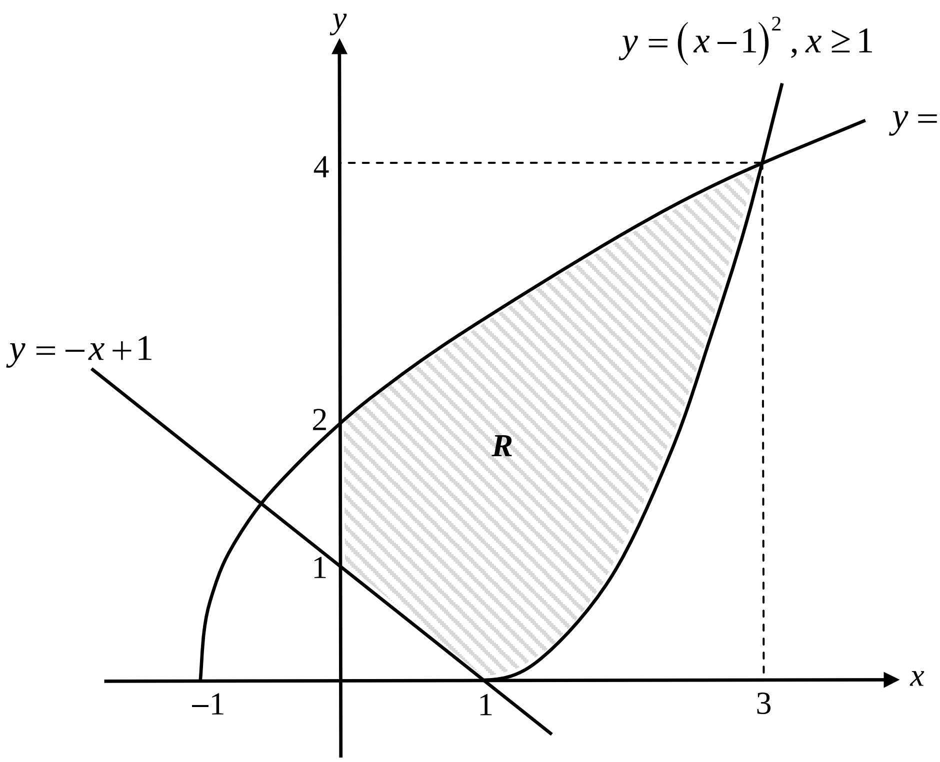
2 **(i)** The first three terms of a sequence are given by $u_1=19$, $u_2=34$, $u_3=52$. Given that u_n is a quadratic polynomial in n, find u_n in terms of n. **[4]**

(ii) Find the smallest value of n for which u_n is greater than 200. **[2]**

3 A wire of length L cm is cut into two pieces. One piece is used to form a circle while the other piece is used to form an equilateral triangle.
Show that, with the total area of the circle and triangle being the smallest, the ratio of the length of the smaller piece to the length of the bigger piece is $\dfrac{\sqrt{3}\pi}{9}$.

[6]

4



The shaded region R in the diagram above is bounded by the y-axis, the line $y=-x+1$ and the curves $y=(x-1)^2$ for $x \geq 1$ and $y=\sqrt{4x+4}$.

Find the volume of the solid of revolution formed when R is rotated completely about the y-axis. **[6]**

5 Given that $y=\ln\left(2+\tan^{-1}x\right)$, show that

$$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1+x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0.$$ **[3]**

Hence find the Maclaurin's expansion for y, up to and including the term in x^2. **[3]**

6 Prove by mathematical induction $\sum_{r=1}^{n}\frac{r}{3^{r-1}}=\frac{9}{4}-\frac{1}{3^{n-1}}\left(\frac{3}{4}+\frac{n}{2}\right)$ for all positive integers of n. **[5]**

Hence show that

$$\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\frac{4}{4^4}\cdots<\frac{9}{16}.$$ **[2]**

7 Functions f and g are defined by

$$\mathrm{f}:x\mapsto\frac{2x-2}{x-2},\quad \text{for } x\in\mathbb{R},\ x<1,$$

$$\mathrm{g}:x\mapsto\sqrt{2-x},\quad \text{for } x\in\mathbb{R},\ x\le 2.$$

(i) Given that f has an inverse, show that the composite function gf^{-1} exists. Find gf^{-1} and state its range. **[5]**

(ii) Find the value(s) of x such that $\mathrm{f}\left(x\right)=\mathrm{f}^{-1}\left(x\right)$. **[2]**

8 Prove that

$$\ln\left(\frac{(r-1)(r+2)}{r(r+1)}\right)\equiv\ln\left((r-1)(r)\right)-2\ln\left((r)(r+1)\right)+\ln\left((r+1)(r+2)\right).$$ **[2]**

Hence, find in terms of n,

$$\ln\left(\frac{1\times4}{2\times3}\right)+\ln\left(\frac{2\times5}{3\times4}\right)+\ln\left(\frac{3\times6}{4\times5}\right)+\cdots+\ln\left(\frac{(n-1)(n+2)}{(n)(n+1)}\right)+\ln\left(\frac{(n)(n+3)}{(n+1)(n+2)}\right),$$

leaving your answer as a single logarithmic function. **[5]**

9 Jessie wishes to take up a loan of \$20,000 on the 1st day of the Year 2014. She intends to pay an instalment of \$300 on the 1st day of each month, beginning from February 2014. She sources out two banks, *XYZ* Bank and *ABC* Bank, which offer such loans. The two banks have different ways of charging interest. *XYZ* Bank charges a monthly interest of 0.5% on the outstanding amount owed at the end of each month, while *ABC* Bank charges a fixed interest of \$60 at the end of each month until the loan is repaid.

(a) If Jessie takes up the loan from *XYZ* Bank, show that the outstanding loan at the end of February 2014 after the interest has been added will be \$19899. **[2]**

Hence, find the number of months Jessie will take to repay her loan. **[4]**

(b) Which bank should Jessie take a loan from if she wishes to clear her loan as soon as possible? Justify your answers. **[3]**

10 A curve C is given parametrically by the equations

$$x = 2\cos^3\theta, \quad y = 2\sin^3\theta$$

where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

Show that the normal at the point with parameter θ has equation

$$y\sin\theta = x\cos\theta + 2\left(\sin^4\theta - \cos^4\theta\right) \text{ .} \qquad \textbf{[4]}$$

The normal at the point Q where $\theta = \dfrac{\pi}{6}$, cuts C again at the point P, where $\theta = p$. Show that $\sin^3 p - \sqrt{3}\cos^3 p + 1 = 0$ and hence find the coordinates of P. **[5]**

11 A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \sqrt{\frac{2(x_n^2 - x_n)}{3} + 1}\,, \quad x_1 = k \text{, where } k \geq 1.$$

(a) When $k = 5$, state the value of x_9 and describe the behavior of the sequence. **[2]**

(b) Prove algebraically that, if the sequence converges, then it converges to either 1 or 3. **[3]**

(c) State a value of k such that the sequence converges to 1. **[1]**

(d) When $k = 2$, state the integer m such that $m \leq x_n < m+1$ for all integers $n \geq 1$. **[1]**

Hence, by considering $\dfrac{x_{n+1} - 1}{x_n - 1}$, show that $x_{n+1} > x_n$ for all integers $n \geq 1$. **[3]**

12 **(a)** Find $\int_1^e \frac{1}{x^2}\ln\left(\frac{1}{x^2}\right)\,\mathrm{d}x$, leaving your answer in exact form. **[4]**

(b) Using the substitution $u=\sqrt{t}$, find $\int \frac{\sqrt{t}}{t-1}\,\mathrm{d}t$. **[6]**

13 It is given that $\mathrm{f}\left(x\right)=-x-1+\frac{k^2-1}{x-1}$ where $k>1$.

(i) Show by differentiation that the graph of $y=\mathrm{f}\left(x\right)$ has no turning points. **[3]**

(ii) On separate diagrams, draw sketches of the graphs of

(a) $y=\mathrm{f}\left(x\right)$, **[4]**

(b) $y=\mathrm{f}'\left(x\right)$. **[2]**

You should indicate where possible, numerically or in terms of *k*, any asymptotes and axial intercepts for each of the curves.

(iii) Find in terms of *k*, the range of *x* that satisfies the inequality

$$k\,\mathrm{f}(x)\le\left(x-k\right)^2\left(x+k\right)$$ **[4]**

End of Paper

VICTORIA JUNIOR COLLEGE
PROMOTIONAL EXAMINATION

MATHEMATICS **9740**
(HIGHER 2)

Monday 8 am -11 am
23 September 2013 3 hours

Additional materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed, you are required to present the mathematical steps using mathematical notations and not calculator commands.
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At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 6 printed pages

 VICTORIA JUNIOR COLLEGE **[Turn over**

1 A sequence with its first four terms given is shown below.

$$1,\ (1+2),\ \ (1+2+2^2),\ \ (1+2+2^2+2^3),\ \ \cdots$$

Show that the n^{th} term of this sequence is $2^n - 1$. **[2]**

Find the sum of the first n terms of the sequence. **[3]**

2 A sequence of positive real numbers $x_1, x_2, x_3, \ldots$ satisfies the relation

$$x_{n+1} = \frac{3 - x_n}{2x_n + 3} \quad \text{for } n \ge 1.$$

(i) Given that the sequence converges to α, find the exact value of α. **[3]**

(ii) By using a graphical approach, prove that

$$x_{n+1} > x_n \text{ if } 0 < x_n < \alpha.$$

[2]

3 A curve is defined by the parametric equations

$$x = 2at^2, \qquad y = 3at,$$

where a is a non–zero constant.

Given that B is the point $\left(\frac{17a}{4}, 0\right)$, find the coordinates of the points on the curve which are nearest to B. **[5]**

4 **(i)** Given that $f(r) = (r-1)r^2$, show that $f(r+1) - f(r) = r(3r+1)$. **[1]**

(ii) Use the method of differences to find $\sum_{r=1}^{N} r(3r+1)$ in terms of N. Hence find the limit of $\sum_{r=1}^{N} \frac{r(3r+1)}{N^3}$ as N approaches infinity. **[3]**

(iii) Use your first answer in **part (ii)** to find $\sum_{r=3}^{N} (r-1)(3r-2)$ in the form $aN^3 + bN^2 + cN + d$, where a, b, c and d are constants to be found. **[2]**

5 **(a)** **(i)** Prove that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x}{x^2+1}\right)=\dfrac{2}{\left(x^2+1\right)^2}-\dfrac{1}{x^2+1}$. **[2]**

(ii) Find the exact value of $\displaystyle\int_0^1 \frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x$. **[3]**

(b) Find the constant A such that $\dfrac{1}{1-\mathrm{e}^{2x}}=A+\dfrac{\mathrm{e}^{2x}}{1-\mathrm{e}^{2x}}$. Hence find $\displaystyle\int \frac{1}{1-\mathrm{e}^{2x}}\,\mathrm{d}x$. **[3]**

6 **(i)** Find the expansion of $\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{(1+x)^2}$ in ascending powers of x, up to and including the term in x^2. **[3]**

Let $y=\sin^{-1}(x)+\dfrac{1}{(1+x)}$.

(ii) By successively differentiating y, find the Maclaurin's series for y, up to and including the term in x^3. **[4]**

(iii) Show that the same result in **part (i)** can be obtained by using your answer in **part (ii)**. **[2]**

7 A sequence $u_0, u_1, u_2, \ldots$ is such that $u_0=b$ and $u_{n+1}=ru_n+a$, for all $n\geq 0$, where a, b and r are constants.

(a) For the case where $r\neq 1$,

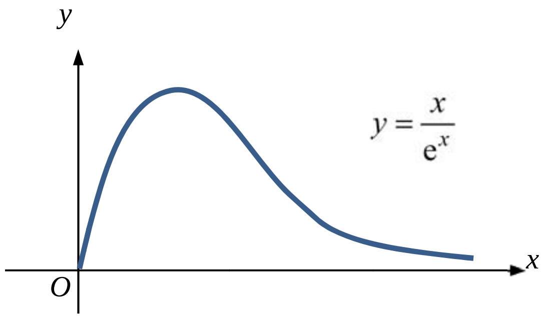
(i) prove by induction that $u_n=r^n b+a\dfrac{1-r^n}{1-r}$ for $n\geq 0$, **[4]**

(ii) write down the set of values of r for which the sequence $u_0, u_1, u_2, \ldots$ converges, and state the limit of this sequence. **[2]**

(**b**) For the case where $r=1$, find u_1, u_2, u_3, and hence find $\displaystyle\sum_{n=0}^{N}u_n$ in terms of a, b, N. Give your answer in the form $\dfrac{N+1}{k_1}\left(k_2 b+Na\right)$, where k_1 and k_2 are integers to be determined. **[3]**

[Turn over

8



The above diagram shows a sketch of the curve C with equation $y = \frac{x}{e^x}$, $x \geq 0$.

(a) **(i)** Find the exact coordinates of the maximum point on C. **[3]**

(ii) Hence show that $\ln x \leq x - 1$ for all $x > 0$. **[2]**

(b) A particle is constrained to move along C, starting from the origin O, such that its x-coordinate increases at a constant rate. The particle took 2 seconds to reach the point $\left(4, \frac{4}{e^4}\right)$. When it is at the point $\left(a, \frac{a}{e^a}\right)$, the y-coordinate of the particle is decreasing at a rate of 0.25 unit per second. Find a given that $a < 2$. **[4]**

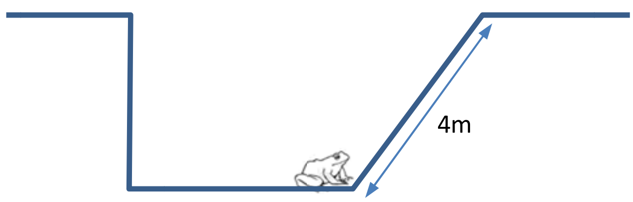
9 **(a)** The sum, S_{n-1}, of the first $n - 1$ terms of a sequence $u_1, u_2, u_3, \ldots$ is given by

$$S_{n-1} = 8n^2 - 19n + 11.$$

(i) Find u_n and show that the sequence is an arithmetic progression. **[4]**

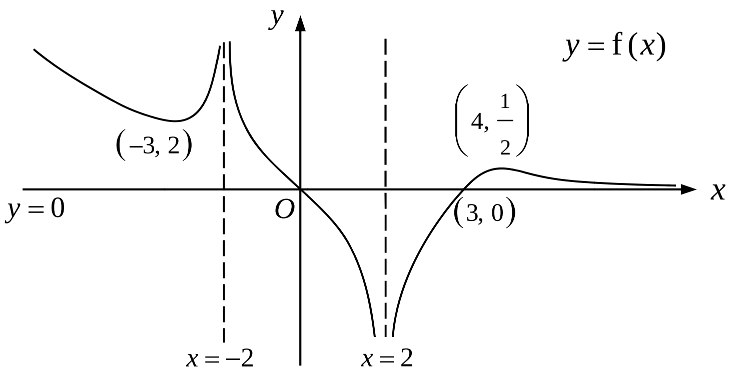
(ii) Find the least value of n, such that sum of the first n terms is at least 4000 less than the sum of the next n terms. **[3]**

(b)



A frog falls into a muddy drain with a slant wall measuring 4m in length. It tries to escape from the drain by leaping successively on the slant wall. Though it can cover 0.7 m in its first leap, the wall is so slippery that for subsequent attempts it can only cover 4/5 the distance of its previous leap. Determine if the frog will be able to escape form the drain, justifying your answer. **[3]**

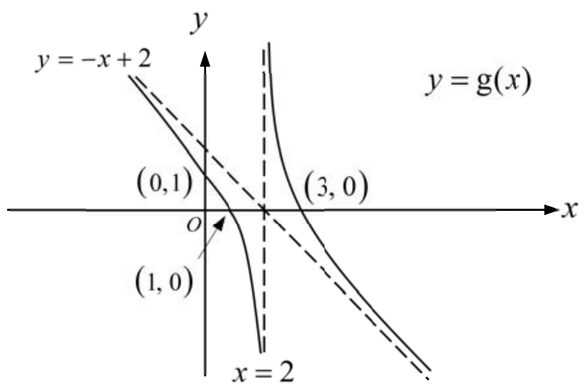
10 **(i)**



The diagram above shows the graph of $y = \mathrm{f}\left(x\right)$. It has a non-stationary point of inflexion $\left(0, 0\right)$, an intersection with the x-axis at $\left(3, 0\right)$, a minimum point $\left(-3, 2\right)$ and a maximum point $\left(4, \frac{1}{2}\right)$. The vertical asymptotes of the graph are $x = -2$ and $x = 2$. The horizontal asymptote is $y = 0$.

Sketch the graph of $y = \sqrt{\mathrm{f}\left(2x\right)}$, making clear the main relevant features and the shape of the graph near the points where $y = 0$. **[3]**

(ii)



The diagram above shows the graph of $y = \mathrm{g}\left(x\right)$. The intersections of the graph with the axes have coordinates $\left(0,1\right)$, $\left(1,0\right)$ and $\left(3,0\right)$. The asymptotes of the graph are the lines $x = 2$ and $y = -x + 2$.

Sketch the graph of $y = \mathrm{g}'\left(x\right)$, making clear the main relevant features. **[3]**

(iii) The function h is defined as

$$\mathrm{h}(x) = \begin{cases} \mathrm{g}(x) & \text{for} \quad x \le 2, \\ \mathrm{f}(x) & \text{for} \quad x > 2. \end{cases}$$

Sketch the graphs of

(a) $y = \mathrm{h}(x)$, **[1]**

(b) $y = \dfrac{1}{\mathrm{h}(x)}$, making clear the main relevant features. **[4]**

[Turn over

11 The function f is defined as follows.

$$f : x \mapsto x - \frac{4}{x} \quad \text{for } x \in \mathbb{R},\ x < 0.$$

(i) Find $f^{-1}(x)$. **[3]**

(ii) Show that $f'(x) > 0$. **[1]**

(iii) Solve the inequality $f^{-1}(x) < -6$, giving your answer in exact form. **[2]**

(iv) Sketch the graph of $y = f^{-1}f(x)$. **[1]**

Functions h and g are defined by

$$h : x \mapsto x - \frac{4}{x} \qquad \text{for } x \in \mathbb{R},\ x \neq -2,\ x \neq 0,\ x \neq 2,$$

$$g : x \mapsto \frac{1}{x} - 1 \qquad \text{for } x \in \mathbb{R},\ x \neq 0.$$

(v) Show that $gh(x) = -\dfrac{(x^2 - x - 4)}{(x^2 - 4)}$. **[1]**

(vi) Solve the inequality $gh(x) \geq 0$, giving your answer in an exact form. **[3]**

12 The curve C_1 has equation $\dfrac{(x-1)^2}{4} = \dfrac{y^2}{9} + 4$.

Sketch C_1, making clear the main relevant features, and state the set of values that x can take. **[4]**

Another curve C_2 is defined by the parametric equations

$$x = \frac{2}{t^2 + 1}, \qquad y = 3\sqrt{t}\ln t, \qquad \text{where } t > 1.$$

Use a non-graphical method to determine the set of possible values of x. **[2]**

Sketch the curve C_2, labelling all axial intercepts and asymptotes (if any) clearly. **[2]**

Hence, **without solving the equation**, state the number of real roots to the equation

$$9\left(\frac{2}{t^2+1} - 1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144,$$

explaining your reason(s) clearly. **[2]**

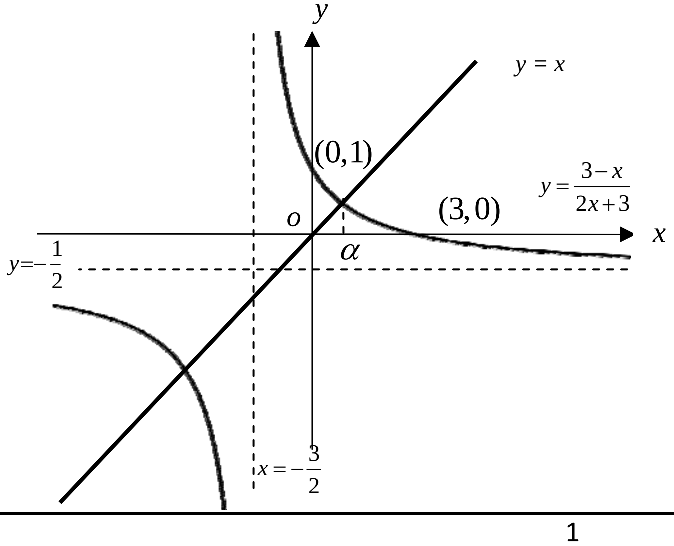
Given that $k > 0$, state the smallest integer value of k such that the equation

$$9\left(\frac{2}{t^2+1} + k - 1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144$$

has exactly one real root which is positive. **[2]**

Victoria Junior College

Mathematics H2 (9740) – JC 1 Promotional Examination 2013

Solutions	
1. (i) The nth term $=1+2+2^2+...+2^{n-1}$ $=\dfrac{1-2^n}{1-2}$ $=2^n-1$ (ii) $S_n=\sum_{r=1}^{n}\left(2^r-1\right)=\sum_{r=1}^{n}2^r-\sum_{r=1}^{n}1$ $=\dfrac{2(1-2^n)}{1-2}-n$ $=2^{n+1}-n-2$	
2. (i) As $n\to\infty$, $x_n\to\alpha$ and $x_{n+1}\to\alpha$. $\alpha=\dfrac{3-\alpha}{2\alpha+3}$ $2\alpha^2+4\alpha-3=0$ $\alpha=\dfrac{-4\pm\sqrt{16+24}}{4}$ $\alpha=-1\pm\frac{1}{2}\sqrt{10}$ Since $x_n>0$ for all *n*, $\alpha=-1+\frac{1}{2}\sqrt{10}$. (ii) Sketch $y=\dfrac{3-x}{2x+3}=-\dfrac{1}{2}+\dfrac{9}{2(2x+3)}$ and $y=x$. 	

2. When $0 < x < \alpha$, the graph of $y = \dfrac{3-x}{2x+3}$ is above the graph of $y = x$. $\therefore \dfrac{3-x}{2x+3} > x$. Hence for $0 < x_n < \alpha$, $\dfrac{3-x_n}{2x_n+3} > x_n$ $\Rightarrow x_{n+1} > x_n$.	
3 (i) Let A be a point on the curve. $AB^2 = \left(\dfrac{17a}{4} - 2at^2\right)^2 + (0-3at)^2$ $= \dfrac{289a^2}{16} + 4a^2t^4 - 17a^2t^2 + 9a^2t^2$ $= 4a^2t^4 - 8a^2t^2 + \dfrac{289a^2}{16}$ $AB = \sqrt{4a^2t^4 - 8a^2t^2 + \dfrac{289a^2}{16}}$ Let $S = AB$. $\dfrac{dS}{dt} = \dfrac{16a^2t^3 - 16a^2t}{2\sqrt{4a^2t^4 - 8a^2t^2 + \dfrac{289a^2}{16}}}$ Let $\dfrac{dS}{dt} = 0$, then $\dfrac{16a^2t^3 - 16a^2t}{2\sqrt{4a^2t^4 - 8a^2t^2 + \dfrac{289a^2}{16}}} = 0$ $16a^2t^3 - 16a^2t = 0 \Rightarrow t(t^2-1) = 0$ $\Rightarrow t = 0$ or $t = 1$ or $t = -1$ At $t = 0$, $S = AB = \dfrac{17a}{4}$. At $t = \pm 1$, $S = AB = \dfrac{15a}{4}$ (nearer) Hence, substitute $t = \pm 1$(which correspond to points nearest to B) into x and y. The coordinates are: (2a, 3a) and (2a, -3a).	

4. (i) $\mathrm{f}(r+1)-\mathrm{f}(r)$ $=r(r+1)^2-(r-1)r^2$ $=r\left[(r+1)^2-(r-1)r\right]$ $=r\left(r^2+2r+1-r^2+r\right)$ $=r(3r+1)$ (ii) $\sum_{r=1}^{N} r(3r+1)$ $=\sum_{r=1}^{N}\left(\mathrm{f}(r+1)-\mathrm{f}(\mathrm{r})\right)$ $=\ \mathrm{f}(2)\ -\mathrm{f}(1)\ +$ $\mathrm{f}(3)\ -\mathrm{f}(2)\ +$ $\vdots$ $\mathrm{f}(N)\ -\mathrm{f}(N-1)\ +$ $\mathrm{f}(N+1)\ -\mathrm{f}(N)\ +$ $=\ \mathrm{f}(N+1)-\mathrm{f}(1)$ $=\ N(N+1)^2-0$ $=\ N(N+1)^2$ $\sum_{r=1}^{N}\frac{r(3r+1)}{N^3}=\frac{N(N+1)^2}{N^3}=\left(\frac{N+1}{N}\right)^2=\left(1+\frac{1}{N}\right)^2.$ As $N\to\infty$, $\frac{1}{N}\to 0$. $\therefore$ the limit of $\sum_{r=1}^{N}\frac{r(3r+1)}{N^3}$ is 1. (iii) $\sum_{r=3}^{N}(r-1)(3r-2)$ $=2\times 7+3\times 10+\ldots+(N-1)(3N-2)$ $\sum_{r=1}^{N} r(3r+1)$ $=1\times 4+\left[2\times 7+\ldots+(N-1)(3N-2)\right]+N(3N+1)$ $\therefore \sum_{r=3}^{N}(r-1)(3r-2)=\sum_{r=1}^{N} r(3r+1)-4-N(3N+1)$ $=N(N+1)^2-4-N(3N+1)$ $=N^3+2N^2+N-4-3N^2-N$ $=N^3-N^2-4$	

5. (a)(i) $\dfrac{d}{dx}\left(\dfrac{x}{x^2+1}\right)=\dfrac{x^2+1-x(2x)}{\left(x^2+1\right)^2}$ $=\dfrac{1-x^2}{\left(x^2+1\right)^2}$ $=\dfrac{2-1-x^2}{\left(x^2+1\right)^2}$ $=\dfrac{2}{\left(x^2+1\right)^2}-\dfrac{1+x^2}{\left(x^2+1\right)^2}$ $=\dfrac{2}{\left(x^2+1\right)^2}-\dfrac{1}{x^2+1}$ (ii) $\int_0^1\left[\dfrac{2}{\left(x^2+1\right)^2}-\dfrac{1}{x^2+1}\right]dx=\left[\dfrac{x}{x^2+1}\right]_0^1$ $2\int_0^1\dfrac{1}{\left(x^2+1\right)^2}\,dx-\left[\tan^{-1}x\right]_0^1=\dfrac{1}{2}$ $2\int_0^1\dfrac{1}{\left(x^2+1\right)^2}\,dx=\dfrac{1}{2}+\dfrac{\pi}{4}$ $\int_0^1\dfrac{1}{\left(x^2+1\right)^2}\,dx=\dfrac{1}{4}+\dfrac{\pi}{8}$ (b) RHS $=A+\dfrac{e^{2x}}{1-e^{2x}}$ $=\dfrac{A-Ae^{2x}+e^{2x}}{1-e^{2x}}$ Comparing the numerator to that of the LHS, $A-Ae^{2x}+e^{2x}=1$ $\Rightarrow A=1$ $\int\dfrac{1}{1-e^{2x}}dx=\int\left(1+\dfrac{e^{2x}}{1-e^{2x}}\right)dx$ $=x-\dfrac{1}{2}\ln\left\|1-e^{2x}\right\|+C$	

6 (i)

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{(1+x)^2}=(1-x^2)^{-1/2}-(1+x)^{-2}$$

$$=\left(1+\frac{1}{2}x^2+\cdots\right)-\left(1-2x+\frac{(-2)(-3)}{2!}x^2+\cdots\right)$$

$$=2x-\frac{5}{2}x^2+\cdots$$

(ii) $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-x^2}}-(1+x)^{-2}$

$$\frac{d^2y}{dx^2}=\left(-\frac{1}{2}\right)(1-x^2)^{-3/2}(-2x)+2(1+x)^{-3}$$

$$=x(1-x^2)^{-3/2}+2(1+x)^{-3}$$

$$\frac{d^3y}{dx^3}=(1-x^2)^{-3/2}+x\left(-\frac{3}{2}\right)(1-x^2)^{-5/2}(-2x)-6(1+x)^{-4}$$

When $x = 0$,

$$y=1$$

$$\frac{dy}{dx}=1-1=0$$

$$\frac{d^2y}{dx^2}=0+2=2$$

$$\frac{d^3y}{dx^3}=1+0-6=-5$$

Hence, $y=1+x^2-\dfrac{5}{6}x^3+\cdots$

(iii) $y=\sin^{-1}(x)+\dfrac{1}{(1+x)}=1+x^2-\dfrac{5}{6}x^3+\cdots$

Differentiating both sides w.r.t x,

$\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{(1+x)^2}=2x-\dfrac{5}{2}x^2+\cdots$ (verified).

7. (a)(i) Let P_n be the statement: $u_n = r^n b + a\dfrac{1-r^n}{1-r}$ for $n \geq 0$. Consider P_0 : L.H.S. of $P_0 = u_0 = b$ R.H.S. of $P_0 = r^0 b + a\dfrac{1-r^0}{1-r} = b$ $\therefore$ P_0 is true. Assume P_k is true for some $k \geq 0$. i.e. $u_k = r^k b + a\dfrac{1-r^k}{1-r}$. Consider P_{k+1} : R.H.S. of $P_{k+1} = r^{k+1} b + a\dfrac{1-r^{k+1}}{1-r}$ L.H.S. of $P_{k+1} = u_{k+1}$ $= ru_k + a$ $= r\left(r^k b + a\dfrac{1-r^k}{1-r}\right) + a$ $= r^{k+1} b + \dfrac{ar\left(1-r^k\right)}{1-r} + \dfrac{a(1-r)}{1-r}$ $= r^{k+1} b + \dfrac{ar - ar^{k+1} + a - ar}{1-r}$ $= r^{k+1} b + \dfrac{a\left(1-r^{k+1}\right)}{1-r}$ $\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true. Hence, $\begin{cases} P_0 \text{ is true} \\ P_k \text{ is true} \Rightarrow P_{k+1} \text{ is true.} \end{cases}$ By induction, $u_n = r^n b + a\dfrac{1-r^n}{1-r}$ for $n \geq 0$.	

7(ii) The sequence converges for $\{r \in \mathbb{R} : -1 < r < 1\}$. The limit of the sequence is $\dfrac{a}{1-r}$. (b) $u_0 = b$ $u_1 = b + a$ $u_2 = b + 2a$ $u_3 = b + 3a$ $\vdots \quad \vdots$ $u_N = b + Na$ $\therefore \sum_{n=0}^{N} u_n = (N+1)b + \dfrac{N}{2}(a + Na)$ $= (N+1)b + \dfrac{N}{2}(1+N)a$ $= \dfrac{N+1}{2}(2b + Na)$	
8(a)(i) $y = \dfrac{x}{e^x}$ $\dfrac{dy}{dx} = \dfrac{e^x - xe^x}{e^{2x}}$ $= \dfrac{1-x}{e^x}$ $\dfrac{dy}{dx} = 0 \Rightarrow x = 1$ Substitute $x = 1$ into y. Maximum point is $\left(1, \dfrac{1}{e}\right)$. (ii) For $x > 0$, $y \le \dfrac{1}{e}$ i.e. $\dfrac{x}{e^x} \le \dfrac{1}{e}$ Since ln is an increasing function, $\ln\left(\dfrac{x}{e^x}\right) \le \ln\left(e^{-1}\right)$ $\Rightarrow \ln x - \ln e^x \le -1$ $\Rightarrow \ln x - x \le -1$ $\Rightarrow \ln x \le x - 1$	

8(*b*) The particle took 2 seconds to move from $x=0$ to $x=4$,

so $\dfrac{\mathrm{d}x}{\mathrm{d}t}=2$.

At $x=a$,

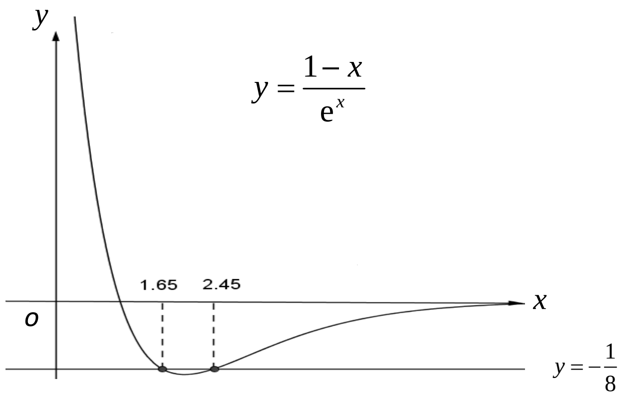
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}$$
$$=-0.25\times\frac{1}{2}=-\frac{1}{8}$$

At $\left(a,\dfrac{a}{\mathrm{e}^a}\right)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1-a}{\mathrm{e}^a}$

$$\therefore \frac{1-a}{\mathrm{e}^a}=-\frac{1}{8}$$

$y=\dfrac{1-x}{\mathrm{e}^x}$

From GC, $a=1.65$ (reject 2.45 as $a<2$).



9(a)(i) Replacing n with $n+1$,

$$\begin{aligned}
S_n &= 8(n+1)^2-19(n+1)+11\\
&= 8n^2+16n+8-19n-19+11\\
&= 8n^2-3n
\end{aligned}$$

$$\begin{aligned}
u_n &= S_n-S_{n-1}\\
&= \left(8n^2-3n\right)-\left(8n^2-19n+11\right)\\
&= 16n-11\\
u_n-u_{n-1} &= \left(16n-11\right)-\left(16(n-1)-11\right)\\
&= 16
\end{aligned}$$

Since the difference between 2 consecutive terms is a constant, the sequence is an AP.

(ii)
$$\left(S_{2n} - S_n\right) - S_n \geq 4000$$
$$\left(8(2n)^2 - 3(2n)\right) - 2\left(8n^2 - 3n\right) \geq 4000$$
$$32n^2 - 6n - 16n^2 + 6n \geq 4000$$
$$n^2 \geq 250$$
$$\Rightarrow n \leq -15.8 \text{ (reject as } n \in \mathbb{Z}^+) \quad or \quad n \geq 15.8$$

Thus, least n is 16.

(b) The distance covered by frog is a GP with a =0.7 and r = 0.8

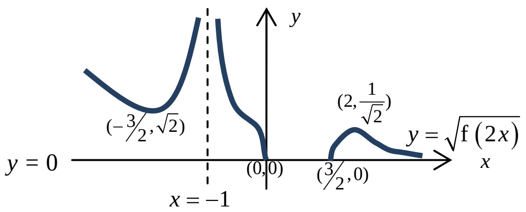
Total distance covered after n leaps is given by

$$S_n = \frac{0.7\left(1-0.8^n\right)}{1-0.8}$$
$$= 3.5\left(1-0.8^n\right)$$

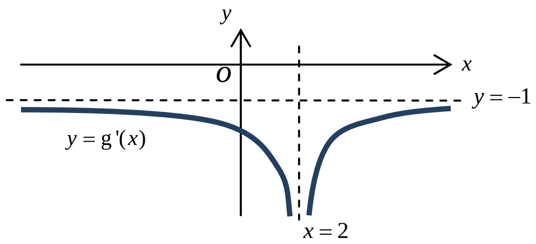
As $n \to \infty$, $(0.8)^n \to 0 \quad \Rightarrow S_n \to 3.5$, that is, $S_\infty = 3.5$

Since $S_\infty < 4$, the frog will never be able to escape from the drain.

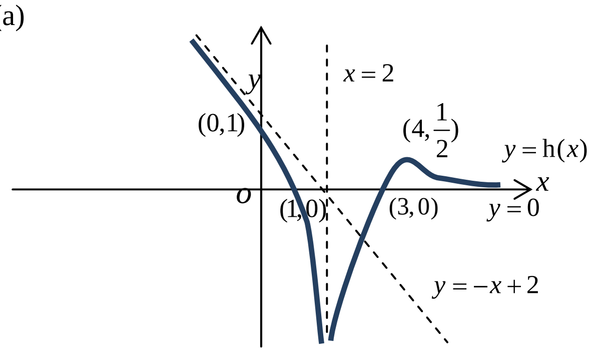
10 (i)



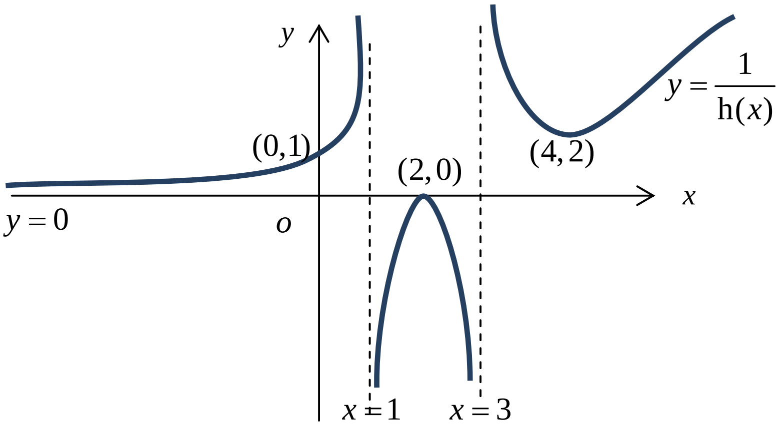
(ii)



(iii) (a)



10(b)



11 (i) $y = x - \frac{4}{x} \Rightarrow y = \frac{x^2-4}{x}$

$x^2 - xy - 4 = 0$

$$x = \frac{y \pm \sqrt{y^2+16}}{2}$$

Since $x<0$, $x = \frac{y-\sqrt{y^2+16}}{2}$

$\Rightarrow f^{-1}(y) = \frac{1}{2}y - \frac{1}{2}\sqrt{y^2+16} \Rightarrow f^{-1}(x) = \frac{1}{2}x - \frac{1}{2}\sqrt{x^2+16}$.

(ii) $f'(x) = 1 + \frac{4}{x^2}$. Since $\frac{4}{x^2} > 0$ for all real $x<0$, $f'(x) > 1$

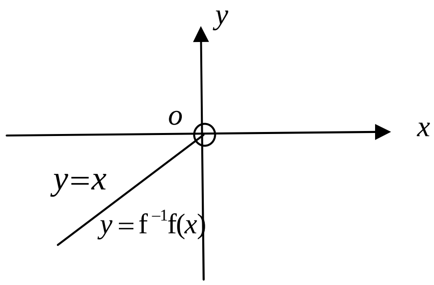
Hence $f'(x) > 0$.

(iii) Since f is an increasing function,

$$f^{-1}(x) < -6 \Rightarrow f\left(f^{-1}(x)\right) < f(-6)$$

$$x < -6 - \frac{4}{-6} \Rightarrow x < -\frac{16}{3}$$

(iv)

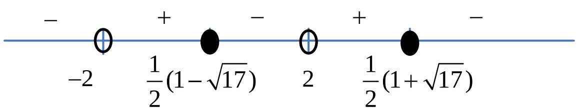


(v) $gh(x) = g[h(x)] = \frac{1}{\frac{x^2-4}{x}} - 1$

$$= \frac{x}{x^2-4} - 1 = \frac{x-(x^2-4)}{x^2-4} \quad = -\frac{(x^2-x-4)}{x^2-4}$$

11(vi) **Test Point method:**

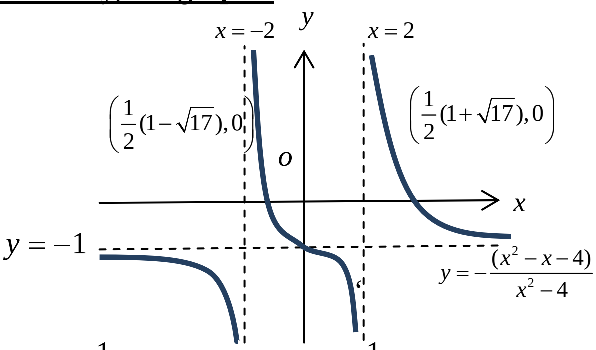
$$x^2 - x - 4 = 0 \Rightarrow x = \frac{1}{2}\left(1 \pm \sqrt{17}\right)$$



Sign of $-\frac{(x^2-x-4)}{x^2-4}$

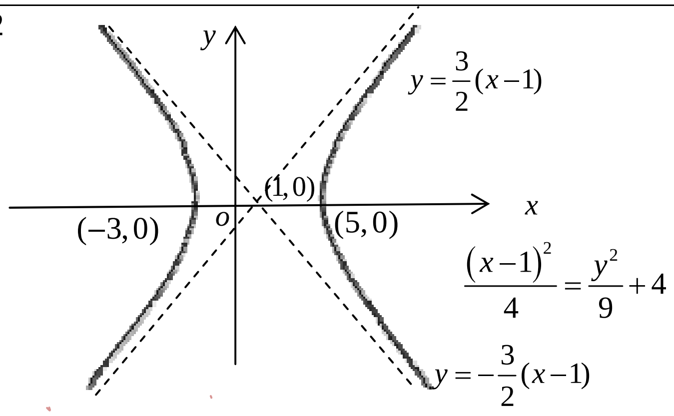
$$\therefore -2 < x \le \frac{1}{2}(1-\sqrt{17}) \text{ or } 2 < x \le \frac{1}{2}(1+\sqrt{17})$$

Alternatively, use graphs:



$$\therefore -2 < x \le \frac{1}{2}(1-\sqrt{17}) \text{ or } 2 < x \le \frac{1}{2}(1+\sqrt{17})$$

12



$$\frac{(x-1)^2}{4} = \frac{y^2}{9} + 4 \Rightarrow \frac{(x-1)^2}{4^2} - \frac{y^2}{6^2} = 1$$

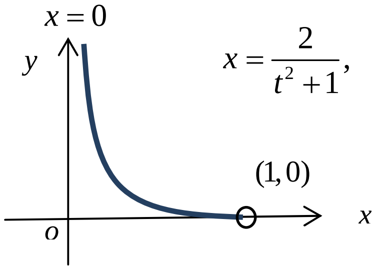
$\therefore$ the set of values of x = $\{x \in \mathbb{R} : x \le -3 \text{ or } x \ge 5\}$

$$t^2 > 1 \Rightarrow t^2 + 1 > 2 \Rightarrow 0 < \frac{1}{t^2+1} < \frac{1}{2}$$

$0 < \frac{2}{t^2+1} < 1$, that is, $0 < x < 1$

$\therefore$ the set of values of x = $\{x \in \mathbb{R} : 0 < x < 1\}$

12



$x = \dfrac{2}{t^2+1}, \quad y = 3\sqrt{t}\ln t, \quad t > 1.$

$$9\left(\frac{2}{t^2+1}-1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144 \quad - \quad (1)$$

$$\frac{\left(\frac{2}{t^2+1}-1\right)^2}{4} = \frac{\left(3\sqrt{t}\ln t\right)^2}{9} + 4$$

Since $C_1 : \dfrac{(x-1)^2}{4} = \dfrac{y^2}{9} + 4$ and $C_2 : x = \dfrac{2}{t^2+1}, \quad y = 3\sqrt{t}\ln t,$

the number of roots of the above equation can then be found by the number of intersections between C_1 and C_2. However, since C_1 is only defined for $x \le -3$ or $x \ge 5$ and C_2 is defined for $0 < x < 1$, there is no point of intersection.

Hence $9\left(\dfrac{2}{t^2+1}-1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144$ has no real root.

$$9\left(\frac{2}{t^2+1}+k-1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144$$

Since x is replaced with $x + k$ in the equation of C_1, C_1 is translated k units in the negative x-direction. Hence smallest integer value of k is 5.

<u>OR</u>

Since x is replaced with $x - k$ in the equation of C_2, C_2 is translated k units in the positive x-direction. Hence smallest integer value of k is 5.