ANDERSON JUNIOR COLLEGE
2016 Preliminary Examination
H2 Mathematics Paper 1 (9740/01)

Duration: 3 hours

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1 Poh, Kee and Mong work for GO! taxi firm. The firm pays them different rates for different periods of time in a day. During off-peak periods, the firm pays $\$a$ for each km driven. For peak and super-peak periods, the firm pays $\$k$ and $\$m$ more per km respectively. The table below shows the distance (in km) covered by Poh, Kee and Mong during the three different periods, together with the amount they were paid, on a particular day.

	Off-peak	Peak	Super-peak	Total
Poh	63	26	10	\$165.10
Kee	59	34	11	\$176.15
Mong	31	52	28	\$205.70

Write down and solve equations to find the values of a, k and m. [4]

2 Without using the calculator, solve the inequality $\dfrac{2}{x+1} \geq \dfrac{x-2}{x}$. [4]

3 **(i)** Prove by induction that $\displaystyle\sum_{r=1}^{n} r(3^r) = \frac{3}{4} + \frac{1}{4}(2n-1)3^{n+1}$. [4]

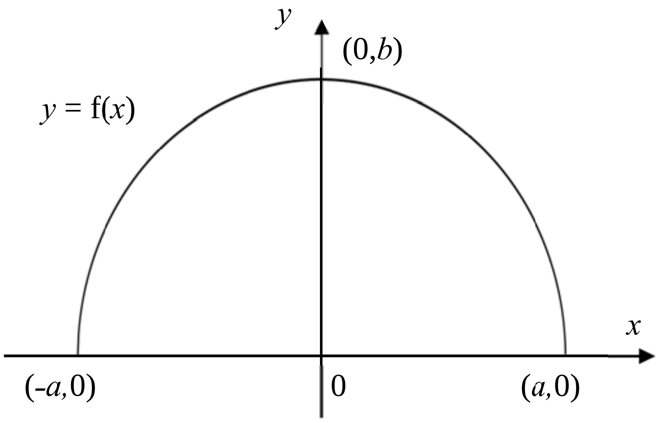
(ii) Hence, find $\displaystyle\sum_{r=2}^{n} (2r-1)3^r$. Give your answer in simplified form. [4]

4 Given that $\mathrm{f}(r) = \dfrac{1}{r}$, where r is a positive integer, express $\mathrm{f}(r) - \mathrm{f}(r+1)$ as a single fraction.

Hence, find the sum of the series $\dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{12} + \dfrac{1}{20} + \dfrac{1}{30} ..... + \dfrac{1}{2n(2n+1)}$.

Deduce that the sum to $2n$ terms of the series $\dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \dfrac{1}{5^2} + \dfrac{1}{6^2} ...$ is less than 1. [7]

5



The diagram shows the curve C_1 with equation $y = \mathrm{f}(x)$, where C_1 is symmetrical about the y-axis and passes through the points $(-a, 0)$, $(a, 0)$ and $(0, b)$.

The curve C_2 has equation given by $y = \dfrac{1}{\mathrm{f}(x)}$.

(i) Sketch the curve C_2, showing clearly all the features of the curve. [2]

(ii) Write down the range of values of b such that C_1 and C_2 do not intersect. [1]

It is given that C_1 is part of an ellipse with centre (0, 0).

(iii) Sketch, on a different diagram, the curve with equation $y = \mathrm{f}'(x)$, showing clearly all the features of the curve. [2]

(iv) State a sequence of transformations that will transform C_1 into a curve with equation $(x+b)^2 + y^2 = b^2$, $y \leq 0$. [3]

6 **Do not use a calculator in answering this question.**

The complex number w is such that $w^3 + 8 = 0$.

(i) Find exactly the possible values of w, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

(ii) Hence, solve the equation $\dfrac{1}{z^4} + \dfrac{8}{z} = \dfrac{3}{z^3} + 24$, where $z \neq 0$. [4]

7 The function f is defined by $\quad \mathrm{f}: x \mapsto \mathrm{e}^{|4-x|}, \ x \le a$.

(i) State the greatest value of a for which the function f^{-1} exists. [1]

Use this value of a for the rest of this question.

(ii) Define the inverse function f^{-1} in a similar form. [3]

(iii) Solve the inequality $\mathrm{f}^{-1}\mathrm{f}\left(|x|\right) > 1$. [3]

The function g is defined by $\quad \mathrm{g}: x \mapsto b + \sqrt{3-x}, \ x \le 3$.

(iv) Find the smallest value of b for the composite function $\mathrm{f}^{-1}\mathrm{g}$ to exist. [2]

(v) If $b = 2$, find the range of $\mathrm{f}^{-1}\mathrm{g}$ in exact form. [2]

8 **(a)** Given that $\ln(2+y^3) = x$, show that $3y^2\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 + y^3$.

By further differentiation of this result, find the series expansion of y in ascending powers of x up to and including the term in x^2. [4]

(b) In a parallelogram $ABCD$, $AB = \sqrt{2}$, $BC = 1$ and $\angle DAB = \dfrac{\pi}{4} - \theta$. Show that the length of the diagonal AC is $\sqrt{3 + 2\cos\theta + 2\sin\theta}$. Given that θ is small, show that $AC \approx \sqrt{5}\left(1 + a\theta + b\theta^2\right)$, where constants a and b are to be determined. [6]

9 **(a)** The curve C_1 has parametric equations $\quad x = t + \tan^{-1} t, \quad y = t^2, \quad$ where $t \ge 0$.

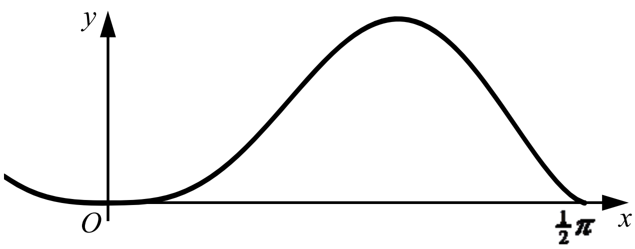
Given that C_1 meets the line $y = 2 - \left(\dfrac{4}{4+\pi}\right)x$ at the point where $t = 1$, find the exact area of the finite region bounded by C_1, the line $y = 2 - \left(\dfrac{4}{4+\pi}\right)x$ and the x-axis. [5]

(b) The curve C_2 has parametric equations $x = 2t^2, \quad y = t^3$.

(i) Find the equation of the tangent to C_2 at point $P\left(2p^2, p^3\right)$. [2]

(ii) This tangent meets the y-axis at the point A. The point Q on the line segment AP is such that $AQ : QP = 1 : 3$. Find the cartesian equation of the locus of Q as p varies. [4]

10 **(i)** Using the sul [2]



The diagram sh ion

bounded by the c

(ii) Find the exa is. [3]

(iii) By differentiating $y^2 = \sin^6 x \cos^3 x$ with respect to *x*, show that the *x*-coordinate of the maximum point of the above curve is $\tan^{-1}\sqrt{2}$, and find the exact *y*-coordinate of the maximum point, expressing your answer in the form $2^a \cdot 3^b$, where *a* and *b* are rational numbers to be determined. (You do not need to show that it is a maximum point.) [5]

11 With reference to the origin *O*, the point *R* has position vector $2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$. The plane p_1 has vector equation $\mathbf{r} = (2 + 2\lambda + 5\mu)\,\mathbf{i} + (1 + 2\lambda - \mu)\,\mathbf{j} + (6 + \lambda - 5\mu)\,\mathbf{k}$, where $\lambda, \mu \in \mathbb{R}$.

The point *Q*, which is on *OR* produced, is such that $RQ = 2OR$ and p_2 is the plane through *Q* which is parallel to p_1.

(i) Show that *R* lies in p_1. [1]

(ii) Find a cartesian equation of p_2. [3]

(iii) Find the position vector of the point *L* in p_2 such that *OL* is perpendicular to p_2. [3]

(iv) The point *M* is on the line segment *RQ*. Given that *h* is the perpendicular distance from *M* to the line passing through *R* and *L*, show that

$$h = \frac{\left|\overrightarrow{RM} \times \overrightarrow{RL}\right|}{\left|\overrightarrow{RL}\right|}.$$ [2]

12 [It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]

Figure 1 shows a frustum (a truncated cone) with radii 2 cm and 10 cm and height 20 cm. A conical flask (Figure 2) is made up of the frustum and a cylinder. The cylinder has radius 2 cm and height 6 cm. Water is poured into the empty flask at a constant rate of π cm^{3}s^{-1}. At time t seconds, the depth of the water is h cm and the radius of the water surface is r cm.

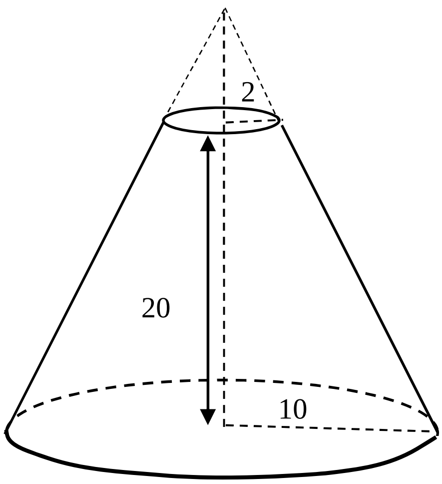


Figure 1

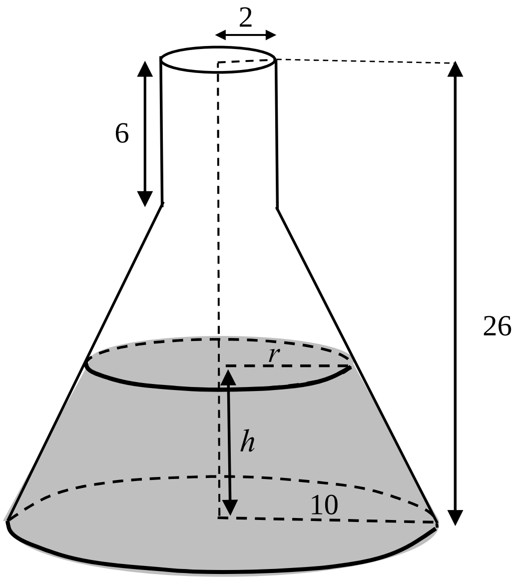


Figure 2

(i) For $0 \leq h \leq 20$,

(a) by using similar triangles, find r in terms of h; [3]

(b) show that the volume of water in the flask is $V = \dfrac{2500\pi}{3} - \dfrac{4\pi}{75}(25-h)^3$.

Hence, find $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ in terms of h. [4]

(ii) Find $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ for $20 < h \leq 26$. [2]

(iii) Sketch a graph to illustrate how $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ varies with h for $0 \leq h \leq 26$. [2]

End of paper

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2016 Preliminary Examination

H2 Mathematics Paper 2 (9740/02)

Duration: 3 hours

Section A: Pure Mathematics [40 marks]

1 With reference to the origin *O*, the points *A*, *B*, *C* have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. The mid-points of OA and BC are *S* and *T* respectively.

(i) Find $\overrightarrow{OS}$ and $\overrightarrow{OT}$ in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. [2]

$OABC$ is a tetrahedron where angle OBA and angle OCA are right angles.

(ii) Show that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{c}$. [2]

(iii) By considering $\overrightarrow{BC} \cdot \overrightarrow{BC}$, show that $BC^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2 - 2\mathbf{b} \cdot \mathbf{c}$.

Hence show that $ST = \frac{1}{2}\sqrt{OA^2 - BC^2}$. [4]

2 Anson is closing down his chicken farm and he decides to sell exactly *k* chickens in the market at the end of every week, where *k* is a divisor of 1000. Anson has 1000 chickens and the cost of rearing each chicken per week is \$0.50. Show that, when he has sold all his chickens, the total cost of rearing the chickens is \$ $\frac{250}{k}(1000 + k)$. [3]

At the end of the first week, Anson sells each chicken at a price of \$12. At the end of every subsequent week, he sells the chickens at a price of 5% less than that of the previous week.

Find the smallest value of *k* in order for Anson to make a profit when he has sold all his chickens. [6]

3 The population of fish in a pond is n thousand at time t years. It is known that n and t are related by the differential equation

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{1}{5}\left(k - n + n^2\right),$$

where $0 \le n \le \frac{3}{2}$ and k is a constant. When $t = 0,$ the population size is 1250.

It is also known that $\frac{\mathrm{d}n}{\mathrm{d}t} = -0.15$ when the population size is 1000.

(i) Show that $k = -\frac{3}{4}$. [1]

(ii) Find the population size at which the population is shrinking at the highest rate. [2]

(iii) Find an expression for n in terms of t. [5]

(iv) Find the exact value of t when the fish population size reaches zero. [1]

(v) Sketch the part of the solution curve which is relevant in this context. [2]

4 The complex number z satisfies the equation $|z| = \left|z + 6 - 2\sqrt{3}\,\mathrm{i}\right|$.

(i) On an Argand diagram sketch the locus of z. Find the exact values of the axial intercepts of the locus. [5]

The complex number z also satisfies the equation $|z+1|^2 = a$, where $a > 0$.

(ii) Find the range of values of a such that there are two possible values of z. [3]

Another complex number w satisfies the relations

$$|w| \ge \left|w + 6 - 2\sqrt{3}\,\mathrm{i}\right| \text{ , } \quad |w+1|^2 \le 49 \quad \text{and} \quad \mathrm{Im}(w) > 0 \text{ .}$$

(iii) Illustrate the region in which the point representing w can lie. [2]

(iv) Given that $\arg(w - 4\mathrm{i}) = \theta$ where $-\pi < \theta \le \pi$. State, in exact form, the range of values of θ. [2]

Section B: Statistics [60 marks]

5 The management of ABC Country Club would like to conduct a survey to determine if members are satisfied with the recreational facilities in the club. The club has 10 000 members, where more than 60% are below 30 years old and less than 10% are above 60 years old.

(i) Describe how a sample of 100 members can be chosen using systematic sampling. [2]

(ii) State a possible disadvantage of using this sampling method in this context. [1]

6 An experiment is carried out with three coins. Two of the coins are fair. The third coin is biased such that the probability of obtaining a head is $\frac{1}{5}$. The three coins are tossed.

A is the event: All the three coins show the same result.

B is the event: The biased coin shows a tail.

Find **(i)** $\mathrm{P}\left(A \cup B\right)$, [3]

(ii) $\mathrm{P}\left(B' \mid A'\right)$. [3]

7 A group of ten friends, comprising six women and four men, wishes to go out to a restaurant for lunch. Two of the men own a car each. They decide to go to the restaurant in two cars driven by their respective owners, with the same number of passengers in each car.

(i) Find the number of ways in which the remaining eight people may be allocated to the two cars if two of the women, Helen and Karen, must take the same car. [The arrangement of people within a particular car is not relevant.] [2]

At the restaurant, the group sits at a round table. Find the number of ways of arranging the ten people if

(ii) no two men are seated together, [2]

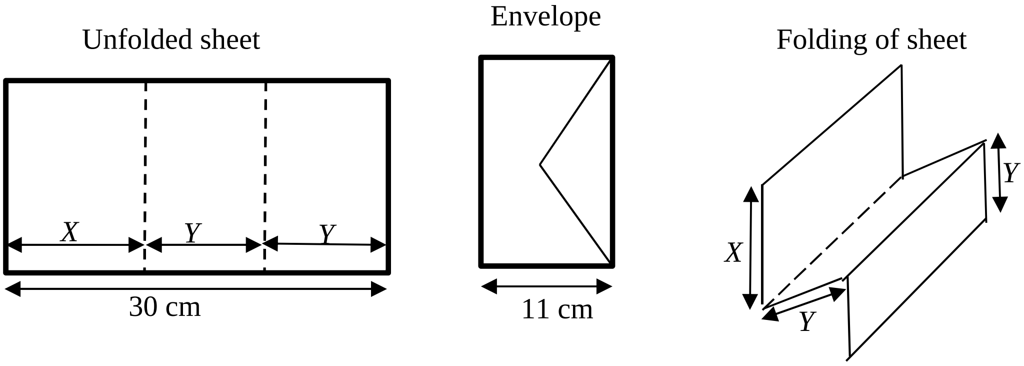
(iii) neither Helen nor Karen is seated beside a driver. [4]

8 A large number of participants take part in a shooting contest. Each participant is given 10 attempts at hitting a target. The probability that a participant hits the target in an attempt is 0.32. It is assumed that the outcome of an attempt is independent of the outcome of any other attempts.

(i) Find the probability that a participant hits the target more than half of the time. [2]

(ii) A random sample of 50 participants is taken. Using a suitable approximation, find the probability that not more than 44 participants hit the target at most 5 times each. [4]

9 A secretary types letters onto sheets of paper 30 cm long and folds the letters as shown.



The first fold is X cm from one edge. The second fold, Y cm from the other edge, is exactly in the middle of the remaining part of the paper, so that $Y = \frac{1}{2}(30 - X)$.

The length X cm is normally distributed with mean 10.2 cm and standard deviation 1.2 cm. The letters have to fit into envelopes 11 cm wide.

(i) Find $\mathrm{P}(11 < Y < 15)$. [2]

(ii) Find the probability that a randomly chosen letter will fit into the envelope. [3]

(iii) By expressing $X - Y$ in terms of X, verify that

$$\mathrm{Var}(X - Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y) .$$

Explain why the rule $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$ does not apply in this case. [4]

10 Travellers arrive at a railway station, to catch a train, either individually or in groups. On a Saturday morning, the number of travellers who arrive individually during a one-minute interval may be modelled by a Poisson distribution with mean 7.5. The number of groups who arrive during a one-minute interval may be modelled by an independent Poisson distribution with mean 2.0.

(i) The probability that no more than one traveller arrives individually during a period of t seconds is less than 0.1. Write down an inequality in terms of t to represent this information, and hence find the range of values of t, giving your answer correct to one decimal place. [3]

(ii) Find the probability that at most 4 groups arrive during the time interval from 0945 to 0950 with at least 3 of the groups arriving within the first 3 minutes. [3]

(iii) Using a suitable approximation, find the probability that, from 1045 to 1055, the number of travellers arriving individually exceeds three times the number of groups arriving by more than 8. [4]

11 A study of the relationship between the amount of advertising time (in minutes) on radio for a product and the quantity of the product sold (in thousands) was carried out. The results of the study are given below:

Advertising time (minutes), t	2	4	6	8	10	12	14	16
Quantity of the product sold (thousands), S	1.11	1.62	k	1.91	2.11	2.15	2.20	2.23

(i) Given that the regression line of S on t is $S = 0.069940t + 1.2718$, find k, correct to 2 decimal places. Hence find the product moment correlation coefficient between S and t. [3]

(ii) It is thought that the relationship between S and t can also be modelled by $S = a + b\ln t$ where a and b are constants. Find the least squares estimates of a and b and the product moment correlation coefficient r for this new model. [2]

(iii) Use the regression line for the better model to estimate the quantity of the product sold when the advertising time is 9 minutes, giving your answer to the nearest integer.

Give two reasons why this estimate is reliable. [3]

12 Bank A claims that the average monthly salary of a fresh graduate working in the banking industry is \$3000. However, a human resource manager from another bank suspects that this is not the correct average monthly salary. A random sample of 50 fresh graduates is surveyed. The monthly salaries, \$$x$, of these 50 fresh graduates are summarised by

$$\sum(x-3000)=-505 \qquad \sum(x-3000)^2 = 100580$$

(i) Calculate the unbiased estimates of the mean and variance of the monthly salary of a fresh graduate, giving your answers to 2 decimal places. [2]

(ii) Test at 10% significance level whether the manager's suspicion is supported. [4]

The manager believes that the average monthly salary of a fresh Mathematics graduate in the banking industry is more than \$3000. He decides to conduct investigations into this by surveying a random sample of 15 fresh Mathematics graduates and the mean of this sample is found to be \$3050 and the sample variance is k^2. Find the range of values for k^2 (to the nearest integer) if it is found that, at 5% level, there is sufficient evidence to justify his belief. State an assumption necessary to carry out the test. [4]

End of Paper

2016 Prelim H2 Mathematics Paper 1 Solution

1

$$63a+26(a+k)+10(a+m)=165.10$$
$$59a+34(a+k)+11(a+m)=176.15$$
$$31a+52(a+k)+28(a+m)=205.70$$

Simplifying,

$$99a+26k+10m=165.10$$
$$104a+34k+11m=176.15$$
$$111a+52k+28m=205.70$$

Solving,

$$a=\$1.50,\ k=\$0.35,\ m=\$0.75$$

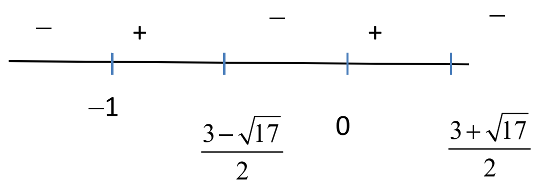
2

$$\frac{2}{x+1}-\frac{x-2}{x}\geq 0 \quad x\neq 0,-1$$

$$\frac{2x-(x-2)(x+1)}{x\left(x+1\right)}\geq 0$$

$$\frac{-x^2+3x+2}{x(x+1)}\geq 0$$

$$\frac{-(x-\frac{3}{2})^2+\frac{17}{4}}{x(x+1)}\geq 0$$



$$-1<x\leq\frac{3-\sqrt{17}}{2} \quad \text{or} \quad 0<x\leq\frac{3+\sqrt{17}}{2}$$

3

Let P_n be the statement $\displaystyle\sum_{r=1}^{n} r(3)^r=\frac{3}{4}+\frac{1}{4}\left(2n-1\right)3^{n+1},\ n\geq 1.$

When $n=1$, $\quad \text{LHS}=\displaystyle\sum_{r=1}^{1} r(3)^r=1(3)=3$

$$\text{RHS}=\frac{3}{4}+\frac{1}{4}(1)3^2=3$$

Since LHS = RHS, P_1 is true

Assume P_k is true for **some** $k\in\mathbb{Z}^+$, i.e $\displaystyle\sum_{r=1}^{k} r(3)^r=\frac{3}{4}+\frac{1}{4}\left(2k-1\right)3^{k+1}$

To prove that P_{k+1} is true,

i.e. to prove : $\displaystyle\sum_{r=1}^{k+1} r(3)^r=\frac{3}{4}+\frac{1}{4}\left(2(k+1)-1\right)3^{(k+1)+1}=\frac{3}{4}+\frac{1}{4}(2k+1)3^{k+2}$

$$\text{LHS}=\sum_{r=1}^{k+1} r(3)^r \quad =\sum_{r=1}^{k} r(3)^r+(k+1)3^{k+1}$$

$$=\frac{3}{4}+\frac{1}{4}(2k-1)3^{k+1}+(k+1)3^{k+1} \quad =\frac{3}{4}+\frac{1}{4}(3)^{k+1}(2k-1+4k+4)$$

$$=\frac{3}{4}+\frac{1}{4}(3)^{(k+1)+1}(2k+1) \quad =\text{RHS}. \quad \mathrm{P}_k \text{ is true} \Rightarrow \mathrm{P}_{k+1} \text{ is true}$$

Since P_1 is true, and P_k is true implies P_{k+1} is true, by Mathematical Induction, P_n is true for all $n\in\mathbb{Z}^+, n\geq 1$.

(ii) $\displaystyle\sum_{r=2}^{n}(2r-1)3^r=\sum_{r=2}^{n}2r(3)^r-\sum_{r=2}^{n}3^r$

$$\sum_{r=2}^{n}2r(3)^r=\left(\frac{3}{2}+\frac{1}{2}(2n-1)3^{n+1}\right)-6=-\frac{9}{2}+\frac{1}{2}(2n-1)3^{n+1}$$

$$\sum_{r=2}^{n}3^r=\frac{9(3^{n-1}-1)}{3-1}=\frac{9}{2}\left(3^{n-1}-1\right)$$

$$\sum_{r=2}^{n}(2r-1)3^r=-\frac{9}{2}+\frac{1}{2}(2n-1)3^{n+1}-\frac{9}{2}(3^{n-1}-1)$$

$$=\frac{1}{2}(2n-1)3^{n+1}-\frac{9}{2}(3^{n-1})$$

OR $\quad =3^{n+1}\left[\frac{1}{2}(2n-1)-\frac{1}{2}\right]\quad=3^{n+1}\left(n-1\right)$

OR $\quad =3^{n-1}\left[\frac{9}{2}(2n-1)-\frac{9}{2}\right]\quad=3^{n-1}\left(9n-9\right)$

4

$$\mathrm{f}(r)-\mathrm{f}(r+1)=\frac{1}{r}-\frac{1}{r+1}=\frac{1}{r(r+1)}.$$

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}.....+\frac{1}{2n\left(2n+1\right)}=\sum_{r=1}^{2n}\frac{1}{r(r+1)}=\sum_{r=1}^{2n}[\mathrm{f}(r)-\mathrm{f}(r+1)]$$

$$\begin{aligned}=[\ &\mathrm{f}(1)-\cancel{\mathrm{f}(2)}\\ +\ &\cancel{\mathrm{f}(2)}-\cancel{\mathrm{f}(3)}\\ +\ &\ldots\\ +\ &\cancel{\mathrm{f}(2n-1)}-\cancel{\mathrm{f}(2n)}\\ +\ &\cancel{\mathrm{f}(2n)}-\mathrm{f}(2n+1)\ ]\end{aligned}$$

$$=\mathrm{f}(1)-\mathrm{f}(2n+1)$$

$$=1-\frac{1}{2n+1}$$

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}...(2n)^{\text{th}}\text{ term }=\sum_{r=1}^{2n}\frac{1}{(r+1)^2}$$

Consider $\displaystyle\frac{1}{(r+1)^2}=\frac{1}{(r+1)(r+1)}<\frac{1}{r(r+1)}$

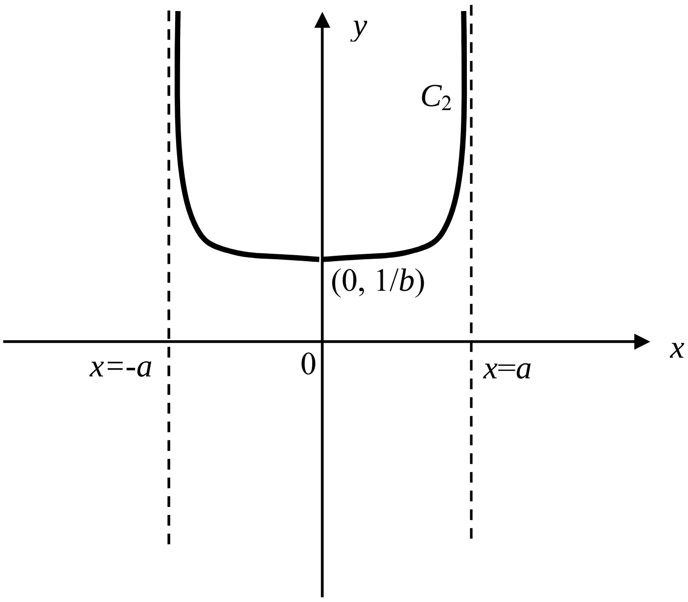
$$\Rightarrow\sum_{r=1}^{2n}\frac{1}{(r+1)^2}<\sum_{r=1}^{2n}\frac{1}{r(r+1)}$$

$$\Rightarrow\sum_{r=1}^{2n}\frac{1}{(r+1)^2}<1-\frac{1}{2n+1}$$

Since $\displaystyle\frac{1}{2n+1}>0,\ \sum_{r=1}^{2n}\frac{1}{(r+1)^2}<1$ (shown)

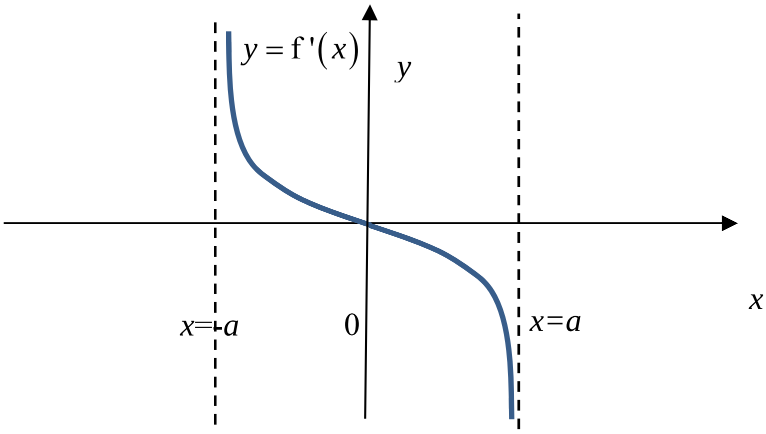
5

(i)



(ii) $0 < b < 1$

(iii)



(iv) The sequence is:

(1) Scale parallel to the x-axis by a factor of $\frac{b}{a}$.

(2) Translate by b units in the negative x-direction

(3) Reflect about the x-axis

Or

(1) Reflect about the x-axis

(2) Scale parallel to the x-axis by a factor of $\frac{b}{a}$.

(3) Translate by b units in the negative x-direction

Or

(1) Translate by a units in the negative x-direction

(2) Scale parallel to the x-axis by a factor of $\frac{b}{a}$

(3) Reflect about the x-axis.

6

(i)

$$w^3 = -8$$

$$= 8e^{i\pi}.e^{i2k\pi} \quad = 8e^{i\pi(1+2k)}$$

$$w = 2e^{i\frac{\pi}{3}(1+2k)} \qquad ,k = 0,1,-1$$

$$w = 2e^{i\frac{\pi}{3}}, 2e^{-i\frac{\pi}{3}}, 2e^{i\pi}$$

(ii)

$$\frac{1}{z^4}+\frac{8}{z}=\frac{3}{z^3}+24$$

$$\frac{1}{z}\left(\frac{1}{z^3}+8\right)=3\left(\frac{1}{z^3}+8\right)$$

$$\left(\frac{1}{z^3}+8\right)\left(\frac{1}{z}-3\right)=0$$

$$\left(\frac{1}{z^3}+8\right)=0 \qquad \text{or} \qquad \left(\frac{1}{z}-3\right)=0$$

Replace $w=\dfrac{1}{z}$ from part (i)

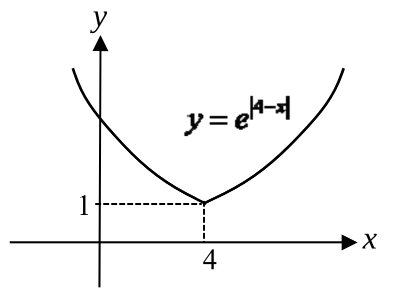
$$\frac{1}{z}=2e^{i\frac{\pi}{3}},2e^{i\pi},2e^{-i\frac{\pi}{3}} \qquad \text{or} \qquad \frac{1}{z}=3$$

$$z=\frac{1}{2}e^{-i\frac{\pi}{3}},\frac{1}{2}e^{-i\pi},\frac{1}{2}e^{i\frac{\pi}{3}} \qquad \text{or} \qquad z=\frac{1}{3}$$

$$z=\frac{1}{2}e^{-i\frac{\pi}{3}},\ -\frac{1}{2}\ ,\ \frac{1}{2}e^{i\frac{\pi}{3}} \qquad \text{or} \qquad z=\frac{1}{3}$$

7

(i) For the function f^{-1} to exist, f must be one-one,
Greatest value of $a = 4$.



(ii) Let $y=e^{|4-x|}$

Since $x\le 4$, $\quad y=e^{4-x}$

$\ln y = 4-x$

$x=4-\ln y$

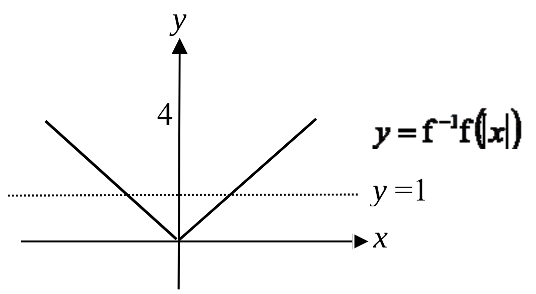
$f^{-1}:x\mapsto 4-\ln x\,,\ x\ge 1$

(iii) $f^{-1}f(x)=x\,,\quad x\le 4$

$f^{-1}f(|x|)=|x|\,,\quad |x|\le 4$

For $\quad f^{-1}f\ (|x|)>1$,

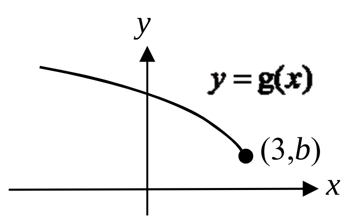
$-4\le x<-1$ or $1<x\le 4$



(iv) For the composite function $f^{-1}g$ to exist,

$R_g\subseteq D_{f^{-1}}$, that is $[b,\infty)\subseteq[1,\infty)$.

Least value of $b = 1$.



(v) Given $\quad R_g=[2,\infty)$

$f^{-1}(2)=4-\ln 2$

$R_{f^{-1}g}=(-\infty,4-\ln 2]$

8

(a) $\ln(2+y^3)=x$ --- (1)

Differentiate w.r.t. x,

$$\Rightarrow \quad \frac{3y^2}{2+y^3}\frac{\mathrm{d}y}{\mathrm{d}x}=1$$

$$\Rightarrow \quad 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=2+y^3 \quad \text{(shown)} \qquad \text{--- (2)}$$

Differentiate again w.r.t. x,

$$3y^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+6y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=3y^2\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{--- (3)}$$

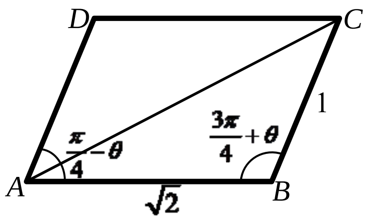
When $x=0$, $\ln(2+y^3)=0 \Rightarrow y=-1$, [from (1)]

$$3(1)\frac{\mathrm{d}y}{\mathrm{d}x}=2-1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{3}, \quad \text{[from (2)]}$$

$$3(1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+6(-1)\left(\frac{1}{3}\right)^2=3(1)\left(\frac{1}{3}\right) \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{5}{9}. \quad \text{[from (3)]}$$

$$\therefore \quad y=-1+\frac{1}{3}x+\frac{5}{18}x^2+\ldots$$

(b) For a parallelogram, $\angle DAB+\angle ABC=\pi$. So $\angle ABC=\pi-\left(\frac{\pi}{4}-\theta\right)=\frac{3\pi}{4}+\theta.$



Using cosine rule, $AC^2=2+1-2\sqrt{2}(1)\cos\left(\frac{3\pi}{4}+\theta\right)$

$$=3-2\sqrt{2}\left[\cos\frac{3\pi}{4}\cos\theta-\sin\frac{3\pi}{4}\sin\theta\right]$$

$$=3-2\sqrt{2}\left[-\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta\right]$$

$$=3+2\cos\theta+2\sin\theta$$

$$\therefore \quad AC=\sqrt{3+2\cos\theta+2\sin\theta} \quad \text{(shown)}$$

Given that θ is small, $\cos\theta\approx 1-\frac{1}{2}\theta^2$ and $\sin\theta\approx\theta$.

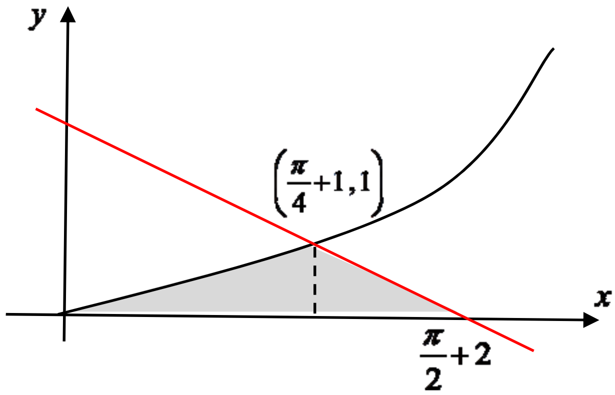
$$AC\approx\left(3+2-\theta^2+2\theta\right)^{\frac{1}{2}}=\left(5+(2\theta-\theta^2)\right)^{\frac{1}{2}}$$

$$=\sqrt{5}\left(1+\left(\frac{2\theta-\theta^2}{5}\right)\right)^{\frac{1}{2}}$$

$$=\sqrt{5}\left(1+\frac{1}{2}\left(\frac{2\theta-\theta^2}{5}\right)+\frac{\frac{1}{2}(-\frac{1}{2})}{2}\left(\frac{2\theta-\theta^2}{5}\right)^2+\ldots\right)$$

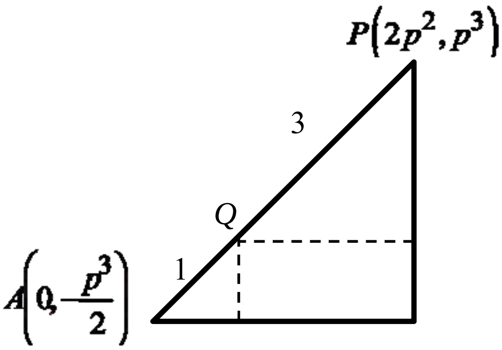
$$=\sqrt{5}\left(1+\frac{1}{5}\theta-\frac{1}{10}\theta^2-\frac{1}{8}\left(\frac{2}{5}\right)^2\theta^2+\ldots\right)$$

$$\approx\sqrt{5}\left(1+\frac{1}{5}\theta-\frac{3}{25}\theta^2\right) \qquad [a=\frac{1}{5},\ b=-\frac{3}{25}]$$

9	(a) When $t=1$, $x=\frac{\pi}{4}+1$ and $y=1$ Line cuts $x-$axis at $x=\frac{\pi}{2}+2$ Area under graph  Area of region $= \text{Area of triangle} + \int_0^{\frac{\pi}{4}+1} y\,\mathrm{d}x$ $=\frac{1}{2}(1)\left(\frac{\pi}{4}+1\right)+\int_0^1 t^2\left(1+\frac{1}{1+t^2}\right)\mathrm{d}t$ $=\frac{\pi}{8}+\frac{1}{2}+\int_0^1\left(t^2+1-\frac{1}{1+t^2}\right)\mathrm{d}t$ $=\frac{\pi}{8}+\frac{1}{2}+\left[\frac{t^3}{3}+t-\tan^{-1}t\right]_0^1$ $=\frac{\pi}{8}+\frac{1}{2}-\frac{\pi}{4}+\frac{4}{3}=\frac{11}{6}-\frac{\pi}{8}$ (b) (i) $\frac{\mathrm{d}x}{\mathrm{d}t}=4t$ and $\frac{\mathrm{d}y}{\mathrm{d}t}=3t^2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{3}{4}t$ Equation of Tangent: $y-p^3=\frac{3p}{4}\left(x-2p^2\right)$ $y=\frac{3p}{4}x-\frac{3}{2}p^3+p^3$ $y=\frac{3p}{4}x-\frac{1}{2}p^3$ (ii) On $y-$axis, $x=0$, $y=-\frac{1}{2}p^3$ $A\left(0,-\frac{1}{2}p^3\right)$ and $P\left(2p^2,p^3\right)$ and Q divide AP in a ratio of 1:3

$$\overrightarrow{OQ}=\frac{3\overrightarrow{OA}+\overrightarrow{OP}}{4}=\frac{\begin{pmatrix}0\\-\frac{3}{2}p^3\end{pmatrix}+\begin{pmatrix}2p^2\\p^3\end{pmatrix}}{4}=\begin{pmatrix}\frac{1}{2}p^2\\-\frac{1}{8}p^3\end{pmatrix}$$

Alternative Method



x – coordinate of Q

$$=\frac{1}{4}\left(2p^2\right)\;=\frac{1}{2}p^2$$

y – coordinate of Q

$$=-\frac{p^3}{2}+\frac{1}{4}\left(p^3+\frac{p^3}{2}\right)=-\frac{p^3}{8}$$

The parametric equation is $x=\frac{1}{2}p^2\quad,\quad y=-\frac{1}{8}p^3$

$$\therefore y=\pm\frac{1}{2\sqrt{2}}x^{3/2}\quad\text{or}\quad y^2=\frac{1}{8}x^3$$

10

(i) $u=\sin x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\cos x$

$$\int\sin^6 x\cos^3 x\;\mathrm{d}x=\int\sin^6 x(1-\sin^2 x)\cos x\;\mathrm{d}x$$
$$=\int u^6(1-u^2)\;\mathrm{d}u$$
$$=\int\left(u^6-u^8\right)\;\mathrm{d}u\quad\text{(shown)}$$

(ii) When $x=0,\;\;u=0.\quad$ When $x=\tfrac{1}{2}\pi,\;\;u=1.$

Volume of revolution $=\pi\int_0^{\frac{1}{2}\pi}\sin^6 x\cos^3 x\;\mathrm{d}x$

$$=\pi\int_0^1(u^6-u^8)\;\mathrm{d}u$$
$$=\pi\left[\frac{u^7}{7}-\frac{u^9}{9}\right]_0^1$$
$$=\frac{2}{63}\pi\text{ units}^3$$

(iii) $\qquad y^2=\sin^6 x\cos^3 x$

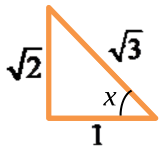
$$\Rightarrow\quad 2y\frac{\mathrm{d}y}{\mathrm{d}x}=6\sin^5 x\cos x\cdot\cos^3 x+\sin^6 x\cdot3\cos^2 x\cdot(-\sin x)$$
$$=6\sin^5 x\cos^4 x-3\sin^7 x\cos^2 x$$

When $\frac{\mathrm{d}y}{\mathrm{d}x}=0,\quad 3\sin^5 x\cos^2 x(2\cos^2 x-\sin^2 x)=0$

$$\Rightarrow \sin x = 0 \quad \text{or} \quad \cos x = 0 \quad \text{or} \quad \sin^2 x = 2\cos^2 x$$
$$x = 0 \quad \text{or} \quad x = \tfrac{1}{2}\pi \quad \text{or} \quad \tan^2 x = 2$$

From the sketch of the curve, maximum point does not occur at $x = 0$ or $x = \frac{1}{2}\pi$.

$$\therefore \quad \tan^2 x = 2 \Rightarrow \tan x = \sqrt{2} \quad (\text{since } 0 < x < \tfrac{1}{2}\pi)$$
$$\Rightarrow x = \tan^{-1}\sqrt{2} \ \ (\text{shown})$$



$$\sin x = \sqrt{\frac{2}{3}}$$
$$\cos x = \frac{1}{\sqrt{3}}$$

$$y\text{-coordinate} = \sin^3 x\sqrt{\cos^3 x}$$
$$= \left(\sqrt{\frac{2}{3}}\right)^3 \sqrt{\left(\frac{1}{\sqrt{3}}\right)^3}$$
$$= \frac{2^{\frac{3}{2}}}{3^{\frac{3}{2}}} \cdot \frac{1}{3^{\frac{3}{4}}}$$
$$= 2^{\frac{3}{2}} \cdot 3^{-\frac{9}{4}} \qquad \left(a = \frac{3}{2}, \quad b = -\frac{9}{4}\right)$$

11

i) $\mathrm{r} = (2 + 2\lambda + 5\mu)\mathrm{i} + (1 + 2\lambda - \mu)\,\mathrm{j} + (6 + \lambda - 5\mu)\mathrm{k}$

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -1 \\ -5 \end{pmatrix}, \qquad \lambda, \mu \in \mathbb{R}$$

When $\lambda = 0$ and $\mu = 0$, $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} = \overrightarrow{OR}$, hence R lies in p_1.

ii) Cartesian equation of p_2:

$$\overrightarrow{OQ} = 3\overrightarrow{OR} = \begin{pmatrix} 6 \\ 3 \\ 18 \end{pmatrix}; \qquad \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 5 \\ -1 \\ -5 \end{pmatrix} = \begin{pmatrix} -9 \\ 15 \\ -12 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$$

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 18 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = 75, \quad \text{hence} \quad p_2:\ 3x - 5y + 4z = 75$$

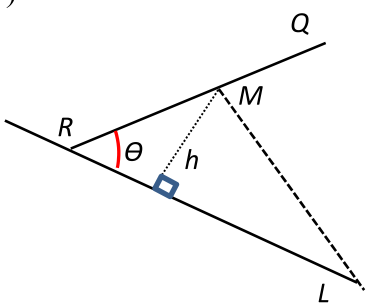
iii) <u>Method 1</u> Let $\overrightarrow{OL} = \alpha \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$;

since point L *lies* in p_2, $3(3\alpha) - 5(-5\alpha) + 4(4\alpha) = 75 \Rightarrow \quad \alpha = \dfrac{3}{2}$

$$\Rightarrow \overrightarrow{OL} = \frac{3}{2}\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ -\frac{15}{2} \\ 6 \end{pmatrix}$$

<u>Method 2</u> $\overrightarrow{OL} = \left|OL\right| \cdot \hat{n} = \frac{75}{\sqrt{50}} \cdot \frac{1}{\sqrt{50}}\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

iv)



$h = \left|\overrightarrow{RM}\right| \sin\theta$ sine ratio

$= \left|\overrightarrow{RM}\right| \left|\frac{\overrightarrow{RM} \times \overrightarrow{RL}}{\left|\overrightarrow{RM}\right| . \left|\overrightarrow{RL}\right|}\right|$ Def. of cross product

$$= \frac{\left|\overrightarrow{RM} \times \overrightarrow{RL}\right|}{\left|\overrightarrow{RL}\right|}$$

OR From definition:

$$\left|\overrightarrow{RM} \times \overrightarrow{RL}\right| = \left|\frac{1}{2}\left|\overrightarrow{RM}\right|\left|\overrightarrow{RL}\right| \sin(\angle MRL)\,\underset{\sim}{\hat{n}}\right|$$

$$= \frac{1}{2}\left|\overrightarrow{RM}\right|\left|\overrightarrow{RL}\right| \frac{h}{\left|\overrightarrow{RM}\right|} .1$$

$$\Rightarrow h = \frac{\left|\overrightarrow{RM} \times \overrightarrow{RL}\right|}{\left|\overrightarrow{RL}\right|}$$

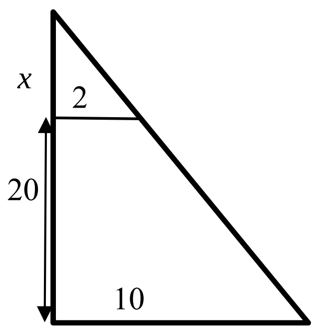
12 Solutions

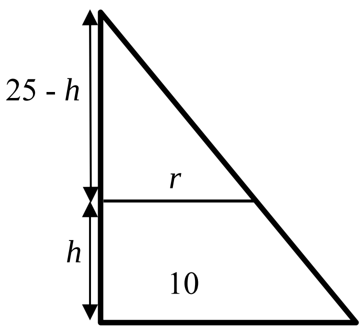
(i) a)

$$\frac{x}{2} = \frac{x+20}{10}$$

$$10x = 2x + 40$$

$$x = 5$$





$\therefore$ Height of cone is 25.

$$\frac{25}{10} = \frac{25-h}{r}$$

$$r = 10 - \frac{2}{5}h$$

b) For $0 \le h \le 20$,

Volume of water in flask, $V = \frac{\pi}{3}10^2(25) - \frac{\pi}{3}r^2(25-h)$

$$= \frac{\pi}{3}\left(10^2(25)-\left(10-\frac{2}{5}h\right)^2(25-h)\right)$$

$$= \frac{2500\pi}{3}-\frac{4\pi}{75}(25-h)^3 \text{ (Shown)}$$

Diff V w.r.t. t, $\frac{dV}{dt}=0-3\left[\frac{4\pi}{75}(25-h)^2\right]\left(-\frac{dh}{dt}\right)$

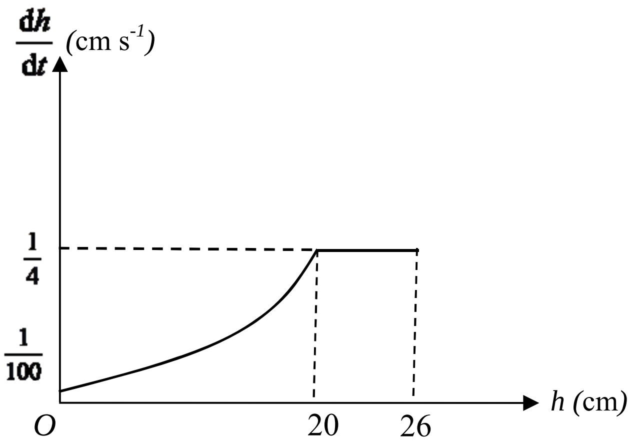
$$\Rightarrow \frac{dh}{dt}=\frac{1}{\frac{4\pi}{25}(25-h)^2}(\pi)=\frac{25}{4(25-h)^2}$$

(ii) For $20<h\le 26$, $V=V_{frustum}+V_{cylinder}$

$$\Rightarrow \frac{dV}{dt}=\frac{dV_f}{dt}+\frac{dV_c}{dt}=0+\frac{d}{dt}\left(\pi(2)^2\ (h-20)\right)=4\pi\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt}=\frac{\pi}{4\pi}=\frac{1}{4}$$

(iii) When $h=0$, $\frac{dh}{dt}=\frac{1}{100}$



2016 Preliminary Examination H2 Mathematics Paper 2 (9740/02) Solutions

1

i) $\overrightarrow{OS} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\mathbf{a}$

By Ratio Theorem, $\overrightarrow{OT} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$

ii) $\overrightarrow{OB} \perp \overrightarrow{AB} \quad \Rightarrow \quad \mathbf{b}\bullet(\mathbf{b}-\mathbf{a}) = 0 \quad \Rightarrow \quad \mathbf{a}\bullet\mathbf{b} = \mathbf{b}\bullet\mathbf{b}$

$\overrightarrow{OC} \perp \overrightarrow{AC} \quad \Rightarrow \quad \mathbf{c}\bullet(\mathbf{c}-\mathbf{a}) = 0 \quad \Rightarrow \quad \mathbf{a}\bullet\mathbf{c} = \mathbf{c}\bullet\mathbf{c}$

iii) $BC^2 = \overrightarrow{BC}\bullet\overrightarrow{BC} = (\mathbf{c}-\mathbf{b})\bullet(\mathbf{c}-\mathbf{b})$

$$= |\mathbf{b}|^2 + |\mathbf{c}|^2 - 2\mathbf{b}\bullet\mathbf{c}$$

$\overrightarrow{ST} = \frac{1}{2}(\mathbf{b}+\mathbf{c}-\mathbf{a})$; $\quad \overrightarrow{ST}\bullet\overrightarrow{ST} = \frac{1}{4}(\mathbf{b}+\mathbf{c}-\mathbf{a})\bullet(\mathbf{b}+\mathbf{c}-\mathbf{a})$

$$\Rightarrow \left|\overrightarrow{ST}\right|^2 = \frac{1}{4}\left(\mathbf{b}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{c}-\mathbf{b}\bullet\mathbf{a}+\mathbf{c}\bullet\mathbf{b}+\mathbf{c}\bullet\mathbf{c}-\mathbf{c}\bullet\mathbf{a}-(\mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{c}-\mathbf{a}\bullet\mathbf{a})\right)$$

$$= \frac{1}{4}\left(2\mathbf{b}\bullet\mathbf{c}+|\mathbf{a}|^2-|\mathbf{b}|^2-|\mathbf{c}|^2\right)$$

$$= \frac{1}{4}\left(|\mathbf{a}|^2+2\mathbf{b}\bullet\mathbf{c}-|\mathbf{b}|^2-|\mathbf{c}|^2\right) = \frac{1}{4}\left(OA^2-BC^2\right)$$

$$\Rightarrow ST = \frac{1}{2}\sqrt{OA^2-BC^2}$$

2

(a) Number of weeks needed to sell off all the chickens $n = \frac{1000}{k}$

Total cost $= (0.5)\left[(1000)+(1000-k)+(1000-2k)+\cdots+(k)\right]$

$$= (0.5)\left\{\frac{1}{2}\left(\frac{1000}{k}\right)[1000+k]\right\}$$

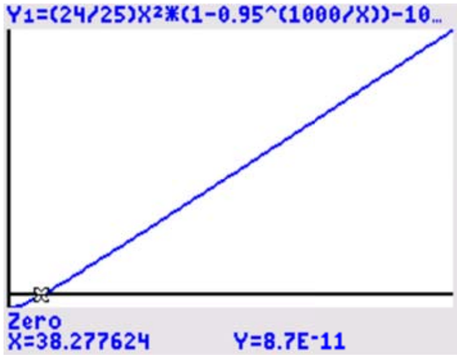
$$= \$\,\frac{250}{k}(1000+k) \text{ (shown)}$$

(b) Total earnings $= 12k\left[1+(0.95)+(0.95)^2+\cdots+(0.95)^{\frac{1000}{k}-1}\right]$

$$= 12k\left[\frac{1-(0.95)^{\frac{1000}{k}}}{1-0.95}\right]$$

$$= 240k\left(1-(0.95)^{\frac{1000}{k}}\right)$$

For Anson to make a profit,



Total earnings $>$ Total cost

$$240k\left(1-(0.95)^{\frac{1000}{k}}\right) > \frac{250}{k}(1000+k)$$

$$\frac{24k^2}{25}\left(1-(0.95)^{\frac{1000}{k}}\right) - 1000 - k > 0$$

Plot the graph of $y = \dfrac{24x^2}{25}\left(1-(0.95)^{\frac{1000}{x}}\right) - 1000 - x$.

From GC, $k > 38.2$

If k is a positive integer and 1000 is divisible by k,

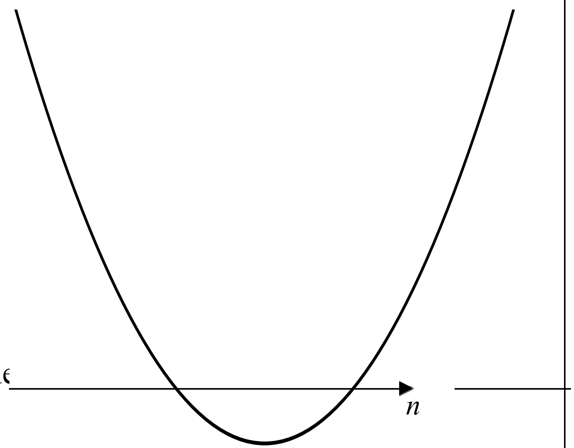
Smallest $k = 40$.

3

(i) Since $\dfrac{dn}{dt} = -0.15$ when $n = 1$,

$$-0.15 = \frac{1}{5}\left(k - 1 + 1^2\right) \quad \Rightarrow \quad k = -\frac{3}{4} \text{ (shown)}$$

(ii) $\dfrac{dn}{dt} = \dfrac{1}{5}\left(n^2 - n - \dfrac{3}{4}\right) = \dfrac{1}{5}\left[\left(n - \dfrac{1}{2}\right)^2 - 1\right]$



$\dfrac{dn}{dt}$ is the most negative when $n = \dfrac{1}{2}$.

$\therefore$ Population is shrinking at the highest rate whe

population size is <u>500</u>.

(iii) $$\frac{dn}{dt} = \frac{1}{5}\left[\left(n - \frac{1}{2}\right)^2 - 1\right]$$

$$\int \frac{1}{\left(n - \frac{1}{2}\right)^2 - 1}\, dn = \int \frac{1}{5}\, dt$$

$$\frac{1}{2}\ln\left|\frac{\left(n - \frac{1}{2}\right) - 1}{\left(n - \frac{1}{2}\right) + 1}\right| = \frac{1}{5}t + c$$

$$\frac{1}{2}\ln\left(\frac{\frac{3}{2} - n}{n + \frac{1}{2}}\right) = \frac{1}{5}t + c$$

[Given: $0 \le n \le \frac{3}{2} \Rightarrow n - \frac{3}{2} \le 0$. So $\left|n - \frac{3}{2}\right| = -(n - \frac{3}{2})$.]

$$\frac{\frac{3}{2} - n}{n + \frac{1}{2}} = e^{\frac{2}{5}t + 2c} = Ae^{\frac{2}{5}t}, \quad \text{where } A = e^{2c}$$

$$\frac{3}{2} - n = \left(n + \frac{1}{2}\right)Ae^{\frac{2}{5}t}$$

$$\frac{3}{2}-\frac{1}{2}Ae^{\frac{2}{5}t}=n\left(1+Ae^{\frac{2}{5}t}\right)$$

$$n=\frac{3-Ae^{\frac{2}{5}t}}{2\left(1+Ae^{\frac{2}{5}t}\right)}$$

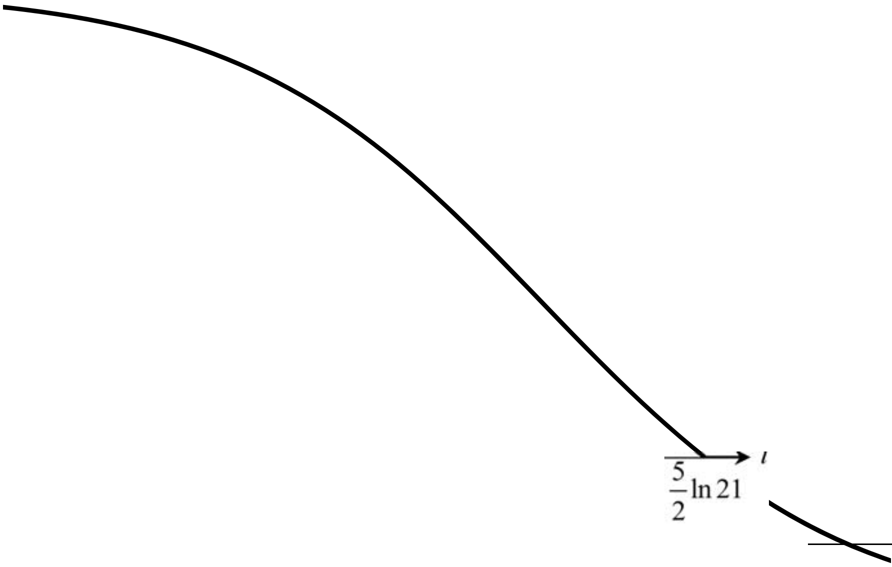
When $t = 0$, $n=\frac{1250}{1000}=\frac{5}{4}$. So $\frac{5}{4}=\frac{3-A}{2(1+A)}\Rightarrow A=\frac{1}{7}$.

$$\therefore\quad n=\frac{3-\frac{1}{7}e^{\frac{2}{5}t}}{2\left(1+\frac{1}{7}e^{\frac{2}{5}t}\right)}=\frac{21-e^{\frac{2}{5}t}}{14+2e^{\frac{2}{5}t}}$$

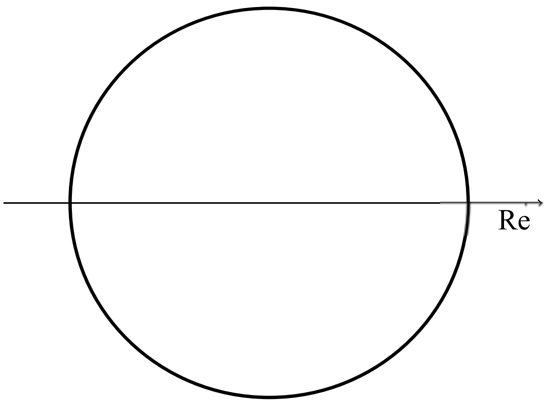
(iv) When $n=0$, $\quad 21-e^{\frac{2}{5}t}=0$

$$\Rightarrow\frac{2}{5}t=\ln 21 \qquad \Rightarrow t=\frac{5}{2}\ln 21$$

(v) Sketch of n (in thousands) against t:



$|z|=|z-(-6+2\sqrt{3}i)|$



[B1] – Identifying the 2 points (0, 0) and $\left(-6,2\sqrt{3}\right)$ with the correct line.

To fi… es:

M1: Finding Cartesian equation

$\sqrt{x^2+y^2}=\sqrt{(x+6)^2+\left(y-2\sqrt{3}\right)^2}$

$y=\sqrt{3}x+4\sqrt{3}$

Sub $x = 0$ and $y = 0$ to find the intersection with axes.

Intersection with x-axis = $(-4, 0)$

Intersection with y-axis = $(0, 4\sqrt{3})$

M2: Using Midpoint

Midpoint = $(-3, \sqrt{3})$

Gradient between (0,0) and (-6, $2\sqrt{3}$) = $\frac{2\sqrt{3}}{-6}=-\frac{\sqrt{3}}{3}$

Gradient of locus = $\frac{3}{\sqrt{3}}=\sqrt{3}$

Intersection with x-axis = $(-4, 0)$

$$\frac{y}{4}=\sqrt{3}\ \Rightarrow y=4\sqrt{3}$$

Intersection with y-axis = $(0, 4\sqrt{3})$

(ii) $|z+1|^2=a \quad\Rightarrow\quad |z+1|=\sqrt{a}$

For only one value of z, locus in (i) ≡ tangent of the circle.

To find radius $\sqrt{a}$:

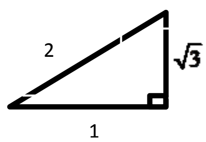
M1 using gradient of perpendicular bisector = $\sqrt{3}$, $\theta=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$

$\sin\frac{\pi}{3}=\frac{\sqrt{a}}{3}$

$\sqrt{a}=3\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2}$

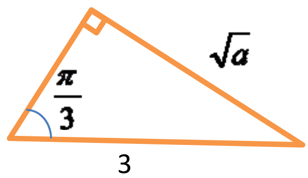
M2

Or right angled triangle , $\tan\theta=\frac{\sqrt{3}}{1}$



$\therefore \sin\theta=\frac{\sqrt{3}}{2}$

Using another right angled triangle, $\sin\theta=\frac{\sqrt{a}}{3}$



$\sin\theta=\frac{\sqrt{a}}{3}=\frac{\sqrt{3}}{2}\quad\Rightarrow\sqrt{a}=\frac{3\sqrt{3}}{2}$

$a=\frac{27}{4}$ for one value of z,

$\therefore$ for 2 values of z, $a>\frac{27}{4}$

	(iii) Identifying $\lvert w+1\rvert \le 7$ as the area inside a circle with centre (-1, 0) and radius 7 Shade the correct region (iv) angle from (0, 4) to (-4, 0) = $-\pi+\tan^{-1}\frac{4}{4}=-\frac{3\pi}{4}$ hence $\frac{\pi}{2}\le\theta\le\pi$ or $-\pi<\theta<-\frac{3\pi}{4}$

Section B: Statistics [60 marks]

5	(i) Number the members in alphabetical order from 1 to 10, 000. Randomly select a member from the first interval of $\frac{10\ 000}{100}=100$, eg 5th member was randomly selected. Select every 100th member from every subsequent interval of 100, ie .105th, 205th etc.. until a sample of 100 members have been selected. (ii) A possible disadvantage is that members of the age group above 60 years (or below 30 years) may be under (or over) represented and thus the results of the survey may not truly reflect the opinions of the members of the Club.
6	(a) $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ $\mathrm{P}(A)=$ P(all heads) + P(all tails) $=\left(\frac{1}{2}\right)^2\left(\frac{1}{5}\right)+\left(\frac{1}{2}\right)^2\left(\frac{4}{5}\right)=\frac{1}{4}$ $\mathrm{P}(A\cap B)=$ P(all tails) $=\left(\frac{1}{2}\right)^2\left(\frac{4}{5}\right)=\frac{1}{5}$ $\therefore\ \mathrm{P}(A\cup B)==\frac{1}{4}+\frac{4}{5}-\frac{1}{5}=0.85$ (or $\frac{17}{20}$) (b) $\mathrm{P}(B'\mid A')=\frac{\mathrm{P}(A'\cap B')}{\mathrm{P}(A')}$ $=\frac{1-\mathrm{P}(A\cup B)}{1-\mathrm{P}(A)}=\frac{1-\frac{17}{20}}{1-\frac{1}{4}}=\frac{1}{5}$ Alternative: $\frac{\mathrm{P}(A'\cap B')}{\mathrm{P}(A')}=\frac{\mathrm{P}(THH)\times 2+\mathrm{P}(TTH)}{\mathrm{P}(TTH)+\mathrm{P}(THH)\times 2+\mathrm{P}(THT)\times 2+\mathrm{P}(HHT)}$ $=\frac{2\left(\frac{1}{2}\right)^2\left(\frac{1}{5}\right)+\left(\frac{1}{2}\right)^2\left(\frac{1}{5}\right)}{2\left(\frac{1}{2}\right)^2\left(\frac{1}{5}\right)+\left(\frac{1}{2}\right)^2\left(\frac{1}{5}\right)+2\left(\frac{1}{2}\right)^2\left(\frac{4}{5}\right)+\left(\frac{1}{2}\right)^2\left(\frac{4}{5}\right)}=\frac{\frac{3}{20}}{\frac{15}{20}}=\frac{1}{5}$

7

(i) Total number of ways of allocating the remaining 8 people in the 2 cars driven by its driver with Helen and Karen in the same car

$= {}^6C_2 \times {}^4C_4 \times 2 = 30$

Comments: H & K have 2 choices of which car to join and there are ${}^6C_2$ ways of selecting people to join them

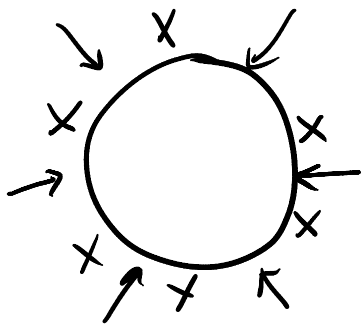
(ii) No of ways of arranging the 10 people if the men must be separated

$= (6-1)! \times {}^6P_4 = 43200$

Comments: To separate the men, the easiest way is to arrange the women first then slot the men into the gaps between the women.

Case 1 : H K D_1 D_2 + 6 others (X)

No of arrangements $= (6-1)! \times {}^6\mathrm{P}_4 = 43200$



Case 2 : HK D_1 D_2 + 6 others

No of arrangements $= (6-1)! \times {}^6\mathrm{P}_3 \times 2 = 28800$

Case 3 : D_1D_2 H K + 6 others

No of arrangements $= (6-1)! \times {}^6\mathrm{P}_3 \times 2 = 28800$

Case 4 : D_1D_2 HK + 6 others

No of arrangements $= (6-1)! \times {}^6\mathrm{P}_2 \times 2 \times 2 = 14400$

Total no of arrangements = 43200 + 28800 + 28800 + 14400
= 115200

8

(i) Let X = number of attempts that hit the target out of 10.

$X \sim B(10, 0.32)$

$\mathrm{P}(X > 5) = 1 - \mathrm{P}(X \le 5)$

$= 1 - 0.93628 = 0.063715 = 0.0637$ (to 3 sig. figs)

(ii) Let Y= number of participants who hit the target more than 5 times, out of 50 participants. Then $Y \sim \mathrm{B}(50, 0.063715)$

Since n = 50 and $np = 50 \times 0.063715 = 3.18575 < 5$

Therefore $Y \sim Po(3.18575)$ approximately

P(not more than 44 participants fire at most 5 successful rounds)

$= \mathrm{P}(Y \ge 6) = 1 - \mathrm{P}(Y \le 5) = 0.10379... = 0.104$

9

(i) $X \sim N\left(10.2, 1.2^2\right)$

$$\mathrm{P}(11 < Y < 15) = \mathrm{P}\left(11 < \frac{1}{2}(30 - X) < 15\right)$$
$$= \mathrm{P}(0 < X < 8) = 0.03376 = 0.0334 \text{(to 3 sf)}$$

<u>Alternative Method</u>:

$$Y \sim N\left(\frac{1}{2}(30 - 10.2), \left(\frac{1}{2}\right)^2 1.2^2\right) \Rightarrow Y \sim N(9.9, 0.36)$$

$\mathrm{P}(11 < Y < 15) = 0.03376 = 0.0334$ (to 3 sf)

(ii) P(letter will fit into envelope)

$$= \mathrm{P}(X < 11 \text{ and } Y < 11)$$
$$= \mathrm{P}\left(X < 11 \text{ and } \frac{1}{2}(30 - X) < 11\right)$$
$$= \mathrm{P}(8 < X < 11)$$
$$= 0.714$$

(iii) $X - Y = X - \frac{1}{2}(30 - X) = \frac{3}{2}X - 15$

$$\mathrm{Var}(X - Y) = \mathrm{Var}\left(\frac{3}{2}X - 15\right)$$
$$= \left(\frac{3}{2}\right)^2 \mathrm{Var}(X) = \frac{9}{4}(1.2)^2 = 3.24$$
$$\mathrm{Var}(X) + \mathrm{Var}(Y) = 1.2^2 + \mathrm{Var}\left(15 - \frac{1}{2}X\right)$$
$$= 1.2^2 + \left(\frac{1}{2}\right)^2 \mathrm{Var}(X) = 1.8$$

Therefore $\mathrm{Var}(X - Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y)$

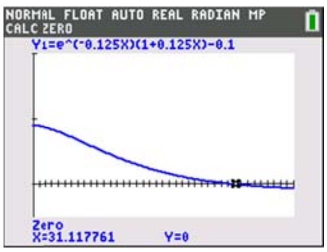
The rule does not hold since X and Y are not independent.

10

(i) Let X = number of travellers arriving individually in t seconds.

$$X \sim Po\left(\frac{7.5}{60}t\right) \Rightarrow X \sim Po(0.125t)$$

$\mathrm{P}(X \leq 1) < 0.1 \quad \Rightarrow e^{-0.125t}(1 + 0.125t) < 0.1$



From GC, $t > 31.118 \Rightarrow t \geq 31.2$

(ii) Let S = number of groups arriving in the first 3minutes

Let T = number of groups arriving in the last 2 minutes

$S \sim Po(6)$ and $T \sim Po(4)$

Required probability = $\mathrm{P}(S \geq 3 \cap S + T \leq 4)$

$= \mathrm{P}(S = 3)\mathrm{P}(T \leq 1) + \mathrm{P}(S = 4)\mathrm{P}(T = 0)$

$= 0.089235 \times 0.091578 + 0.13385 \times 0.018316$

$= 0.0106$ (to 3 sig. figs)

	(iii) Let Y = number of travellers arriving individually in 10 minutes $Y \sim Po(75)$ Since $\lambda = 75 > 10$, $Y \sim N(75,75)$ approximately Let W= number of groups arriving in 10 minutes. $W \sim Po(20)$ Since $\lambda = 20 > 10$, $W \sim N(20,20)$ approximately $Y-3W \sim N\left(75-3(20), 75+3^2(20)\right) \Rightarrow Y-3W \sim N(15,255)$ $\mathrm{P}(Y-3W>8) \xrightarrow{c.c.} \mathrm{P}(Y-3W>8.5) \quad = 0.658$
11	(i) $\bar{x}=9 \quad \bar{y}=\dfrac{13.33+k}{8}$ $S = 0.069940t + 1.1319$ $\dfrac{13.33+k}{8} = 0.069940(9) + 1.2718$ $k = 1.88008 = 1.88$ $r = 0.903$ (ii) From the G.C., using the formula $S = a + b\ln t$, we get $S = 0.82848 + 0.53141\ln t$, $r = 0.98479 = 0.985.$ Thus $a = 0.828, \quad b = 0.531$ (iii) When $t = 9$, $S = 0.53141\ln 9 + 0.82848 = 1.9961$ (iv) Quantity of the product = 1996 (to nearest integer). This estimate for S is reliable because: (1) t = 9 is within the data range of 2 to 16, and (2) the r value between S and ln t is closer to 1 than the r value between S and t, hence there is a stronger relationship between S and ln t .
12	(i) unbiased estimate of the mean monthly salary $= \dfrac{\sum(x-3000)}{50} + 3000 = \dfrac{-505}{50} + 3000 = 2989.90$ Unbiased estimate of the variance of monthly salary, s^2 $= \dfrac{1}{49}\left(\sum(x-3000)^2 - \dfrac{\left(\sum(x-3000)\right)^2}{50}\right)$ $= \dfrac{1}{49}\left(100580 - \dfrac{505^2}{50}\right) = \dfrac{95479.5}{49} = 1948.56$ (ii) Let μ_x be the mean monthly salary

$H_0 : \mu_x = 3000$

$H_1 : \mu_x \neq 3000$

Since population variance is unknown but sample size (= 50) is large, use a two-tailed Z-test at 10% level, ie reject H_0 if p-value < 0.1

Under H_0, test statistic, $Z = \dfrac{\overline{X} - 3000}{\sqrt{\dfrac{1948.56}{50}}} \sim N(0,1)$.

From GC, p-value = 0.106 > 0.1, we do not reject H_0 .
There is insufficient evidence at 10% level to support the manager's suspicion.
(or There is insufficient evidence at 10% level to conclude that the average monthly salary of a fresh graduate is not \$3000).

Let Y be the r.v. denoting the monthly salary of a fresh Mathematics graduate and μ_y be the mean monthly salary.

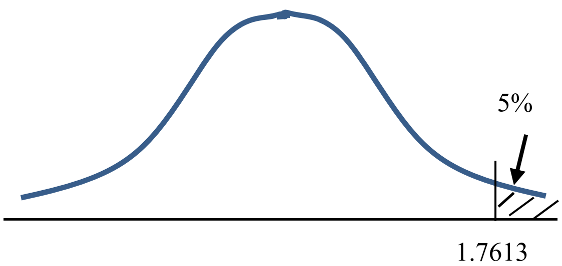
$H_0 : \mu_y = 3000$

$H_1 : \mu_y > 3000$

Since population variance is unknown but sample size (=15) is small, use a one-tailed T-test at 5% level, ie reject H_0 if p-value < 0.05

Given $\overline{y} = 3050, \quad n = 15, \quad {s_y}^2 = \dfrac{15}{14}k^2$

Test statistic $T = \dfrac{\overline{Y} - \mu_y}{S_y / \sqrt{15}} \sim \text{t}(14)$



Since H_0 is rejected, $t_{cal} > t_{crit}$

$$\Rightarrow \frac{3050 - 3000}{\sqrt{\dfrac{\frac{15}{14}k^2}{15}}} > 1.7613$$

$$\Rightarrow \frac{50}{\sqrt{\dfrac{k^2}{14}}} > 1.7613$$

$$\Rightarrow \sqrt{\frac{k^2}{14}} < \frac{50}{1.7613} \Rightarrow k^2 < 11282.40 \quad \Rightarrow \quad 0 < k^2 \leq 11282$$

Assumption: The monthly salary of a fresh Mathematics graduate follows a Normal distribution.